



**HAESE MATHEMATICS**

# **Mathematics**

**Analysis and  
Approaches HL**

# **2**



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for use with

**IB Diploma Programme**

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**WORKED SOLUTIONS**



# MATHEMATICS: ANALYSIS AND APPROACHES HL WORKED SOLUTIONS

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## FOREWORD

This book gives you fully worked solutions for every question in Exercises, Review Sets, Activities, and Investigations (which do not involve student experimentation) in each chapter of our textbook *Mathematics: Analysis and Approaches HL*.

Correct answers can sometimes be obtained by different methods. In this book, where applicable, each worked solution is modelled on the worked example in the textbook.

Be aware of the limitations of calculators and computer modelling packages. Understand that when your calculator gives an answer that is different from the answer you find in the book, you have not necessarily made a mistake, but the book may not be wrong either.

We have a list of errata for our books on our website. Please contact us if you notice any errors in this book.

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# Chapter 1

## FURTHER TRIGONOMETRY

### EXERCISE 1A

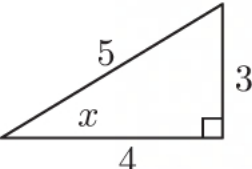
**1 a**  $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$   
 $\therefore \operatorname{cosec} \frac{\pi}{3} = \frac{2}{\sqrt{3}}$

**c**  $\cos \frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$   
 $\therefore \sec \frac{5\pi}{6} = -\frac{2}{\sqrt{3}}$

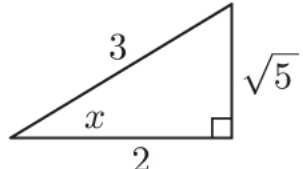
**b**  $\tan \frac{2\pi}{3} = -\sqrt{3}$   
 $\therefore \cot \frac{2\pi}{3} = -\frac{1}{\sqrt{3}}$

**d**  $\tan \pi = 0$   
 $\therefore \cot \pi$  is undefined.

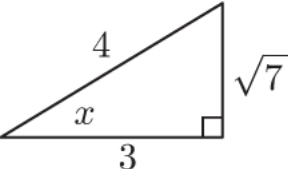
**2 a**  $\sin x = \frac{3}{5}$  and  $0 \leq x \leq \frac{\pi}{2}$   
 $x$  is in quadrant 1  
 $\therefore \cos x > 0$  and  $\tan x > 0$   
 $\therefore \cos x = \frac{4}{5}$  and  $\tan x = \frac{3}{4}$   
 $\therefore \operatorname{cosec} x = \frac{1}{\sin x} = \frac{5}{3}$   
 $\sec x = \frac{1}{\cos x} = \frac{5}{4}$   
 $\cot x = \frac{1}{\tan x} = \frac{4}{3}$



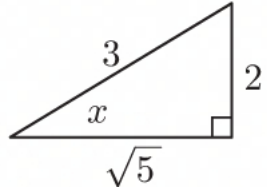
**b**  $\cos x = \frac{2}{3}$  and  $\frac{3\pi}{2} < x < 2\pi$   
 $x$  is in quadrant 4  
 $\therefore \sin x < 0$  and  $\tan x < 0$   
 $\therefore \sin x = -\frac{\sqrt{5}}{3}$  and  $\tan x = -\frac{\sqrt{5}}{2}$   
 $\therefore \operatorname{cosec} x = \frac{1}{\sin x} = -\frac{3}{\sqrt{5}}$   
 $\sec x = \frac{1}{\cos x} = \frac{3}{2}$   
 $\cot x = \frac{1}{\tan x} = -\frac{2}{\sqrt{5}}$



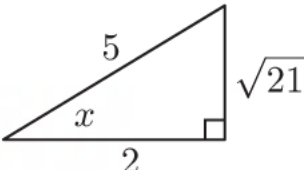
**3 a**  $\cos x = \frac{3}{4}$  and  $\frac{3\pi}{2} < x < 2\pi$   
 $\therefore \sec x = \frac{4}{3}$   
 $x$  is in quadrant 4  
 $\therefore \sin x < 0$  and  $\tan x < 0$   
 $\therefore \sin x = -\frac{\sqrt{7}}{4}$  and  $\tan x = -\frac{\sqrt{7}}{3}$   
 $\therefore \operatorname{cosec} x = -\frac{4}{\sqrt{7}}$  and  $\cot x = -\frac{3}{\sqrt{7}}$



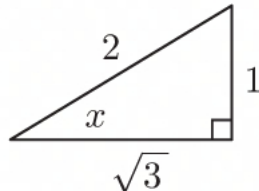
**b**  $\sin x = -\frac{2}{3}$  and  $\pi < x < \frac{3\pi}{2}$   
 $\therefore \operatorname{cosec} x = -\frac{3}{2}$   
 $x$  is in quadrant 3  
 $\therefore \cos x < 0$  and  $\tan x > 0$   
 $\therefore \cos x = -\frac{\sqrt{5}}{3}$  and  $\tan x = \frac{2}{\sqrt{5}}$   
 $\therefore \sec x = -\frac{3}{\sqrt{5}}$  and  $\cot x = \frac{\sqrt{5}}{2}$



**c**  $\sec x = 2\frac{1}{2} = \frac{5}{2}$  and  $0 < x < \frac{\pi}{2}$   
 $\therefore \cos x = \frac{2}{5}$   
 $x$  is in quadrant 1  
 $\therefore \sin x > 0$  and  $\tan x > 0$   
 $\therefore \sin x = \frac{\sqrt{21}}{5}$  and  $\tan x = \frac{\sqrt{21}}{2}$   
 $\therefore \operatorname{cosec} x = \frac{5}{\sqrt{21}}$  and  $\cot x = \frac{2}{\sqrt{21}}$



**d**  $\operatorname{cosec} x = 2$  and  $\frac{\pi}{2} < x < \pi$   
 $\therefore \sin x = \frac{1}{2}$   
 $x$  is in quadrant 2  
 $\therefore \cos x < 0$  and  $\tan x < 0$   
 $\therefore \cos x = -\frac{\sqrt{3}}{2}$  and  $\tan x = -\frac{1}{\sqrt{3}}$   
 $\therefore \sec x = -\frac{2}{\sqrt{3}}$  and  $\cot x = -\sqrt{3}$





**e**  $\tan \beta = \frac{1}{2}$  and  $\pi < \beta < \frac{3\pi}{2}$

$\therefore \cot \beta = 2$

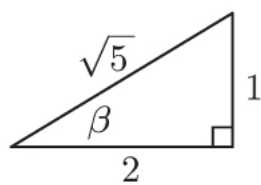
$\beta$  is in quadrant 3

$\therefore \sin \beta < 0$  and

$\cos \beta < 0$

$\therefore \sin \beta = -\frac{1}{\sqrt{5}}$  and  $\cos \beta = -\frac{2}{\sqrt{5}}$

$\therefore \operatorname{cosec} \beta = -\sqrt{5}$  and  $\sec \beta = -\frac{\sqrt{5}}{2}$



**f**  $\cot \theta = \frac{4}{3}$  and  $\pi < \theta < \frac{3\pi}{2}$

$\therefore \tan \theta = \frac{3}{4}$

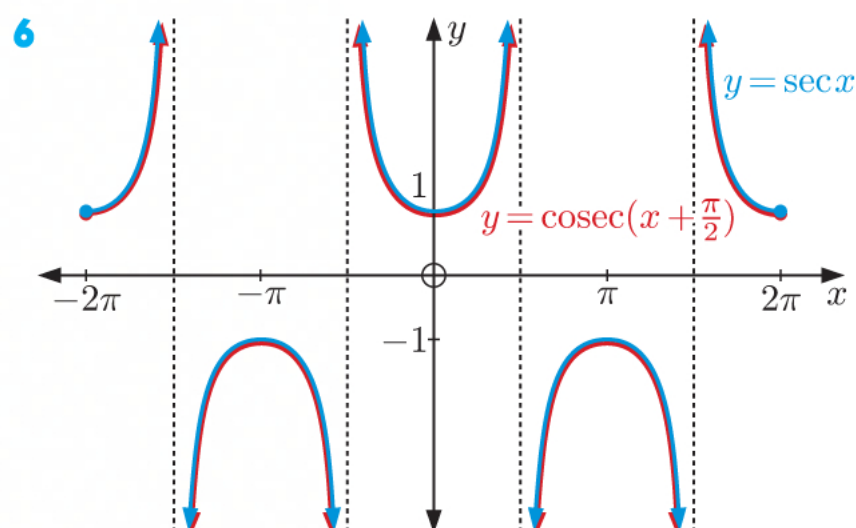
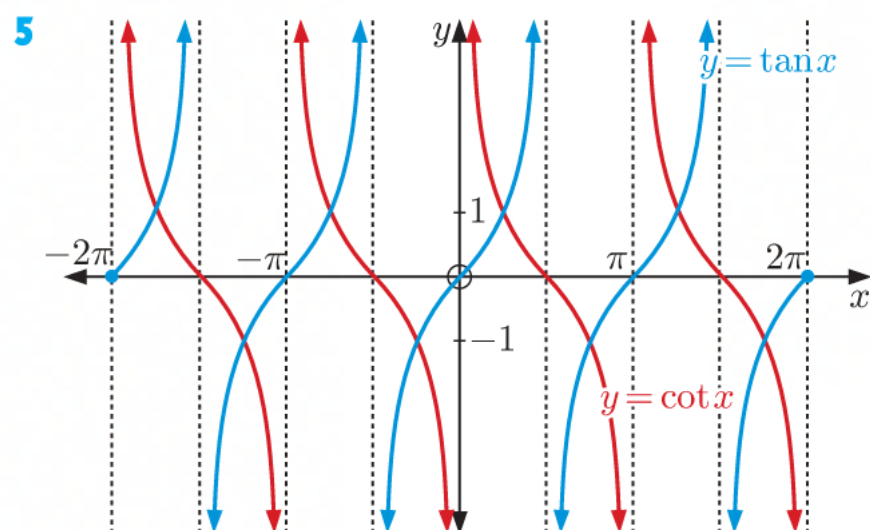
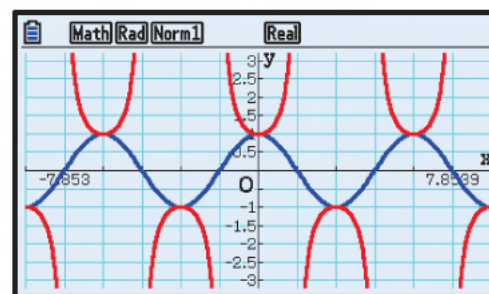
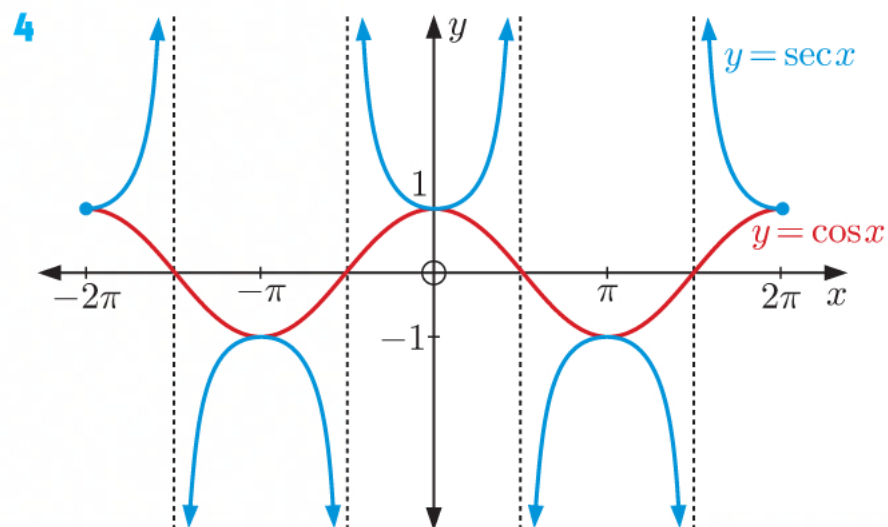
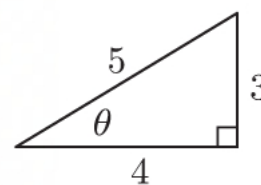
$\theta$  is in quadrant 3

$\therefore \sin \theta < 0$  and

$\cos \theta < 0$

$\therefore \sin \theta = -\frac{3}{5}$  and  $\cos \theta = -\frac{4}{5}$

$\therefore \operatorname{cosec} \theta = -\frac{5}{3}$  and  $\sec \theta = -\frac{5}{4}$



The graphs coincide as

$$\sin\left(x + \frac{\pi}{2}\right) = \cos x \quad \text{for all } x$$

$$\therefore \frac{1}{\sin\left(x + \frac{\pi}{2}\right)} = \frac{1}{\cos x}$$

$$\therefore \operatorname{cosec}\left(x + \frac{\pi}{2}\right) = \sec x$$



**7 a**  $\sec x = 2, \quad 0 \leq x \leq 2\pi$

$$\therefore \frac{1}{\cos x} = 2$$

$$\therefore \cos x = \frac{1}{2}$$

$$\therefore x = \frac{\pi}{3} \text{ or } \frac{5\pi}{3}$$

**c** The domain is  $0 \leq x \leq 2\pi$   
 $\therefore 0 \leq 2x \leq 4\pi$

$$\sqrt{3} \sec 2x = -2$$

$$\therefore \sec 2x = -\frac{2}{\sqrt{3}}$$

$$\therefore \frac{1}{\cos 2x} = -\frac{2}{\sqrt{3}}$$

$$\therefore \cos 2x = -\frac{\sqrt{3}}{2}$$

$$\therefore 2x = \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{17\pi}{6}, \text{ or } \frac{19\pi}{6}$$

$$\therefore x = \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{17\pi}{12}, \text{ or } \frac{19\pi}{12}$$

**e**  $\cot x + 1 = 0, \quad 0 \leq x \leq 2\pi$

$$\therefore \cot x = -1$$

$$\therefore \frac{1}{\tan x} = -1$$

$$\therefore \tan x = -1$$

$$\therefore x = \frac{3\pi}{4} \text{ or } \frac{7\pi}{4}$$

**b**  $\operatorname{cosec} x = -\sqrt{2}, \quad 0 \leq x \leq 2\pi$

$$\therefore \frac{1}{\sin x} = -\sqrt{2}$$

$$\therefore \sin x = -\frac{1}{\sqrt{2}}$$

$$\therefore x = \frac{5\pi}{4} \text{ or } \frac{7\pi}{4}$$

**d** The domain is  $0 \leq x \leq 2\pi$   
 $\therefore \frac{\pi}{6} \leq x + \frac{\pi}{6} \leq \frac{13\pi}{6}$

$$\operatorname{cosec}\left(x + \frac{\pi}{6}\right) + \sqrt{2} = 0$$

$$\therefore \operatorname{cosec}\left(x + \frac{\pi}{6}\right) = -\sqrt{2}$$

$$\therefore \frac{1}{\sin\left(x + \frac{\pi}{6}\right)} = -\sqrt{2}$$

$$\therefore \sin\left(x + \frac{\pi}{6}\right) = -\frac{1}{\sqrt{2}}$$

$$\therefore x + \frac{\pi}{6} = \frac{5\pi}{4} \text{ or } \frac{7\pi}{4}$$

$$\therefore x = \frac{13\pi}{12} \text{ or } \frac{19\pi}{12}$$

**f** The domain is  $0 \leq x \leq 2\pi$

$$\therefore 0 \leq 2x \leq 4\pi$$

$$\therefore -\frac{\pi}{4} \leq 2x - \frac{\pi}{4} \leq \frac{15\pi}{4}$$

$$\cot\left(2x - \frac{\pi}{4}\right) - \sqrt{3} = 0$$

$$\therefore \cot\left(2x - \frac{\pi}{4}\right) = \sqrt{3}$$

$$\therefore \frac{1}{\tan\left(2x - \frac{\pi}{4}\right)} = \sqrt{3}$$

$$\therefore \tan\left(2x - \frac{\pi}{4}\right) = \frac{1}{\sqrt{3}}$$

$$\therefore 2x - \frac{\pi}{4} = \frac{\pi}{6}, \frac{7\pi}{6}, \frac{13\pi}{6}, \text{ or } \frac{19\pi}{6}$$

$$\therefore 2x = \frac{5\pi}{12}, \frac{17\pi}{12}, \frac{29\pi}{12}, \text{ or } \frac{41\pi}{12}$$

$$\therefore x = \frac{5\pi}{24}, \frac{17\pi}{24}, \frac{29\pi}{24}, \text{ or } \frac{41\pi}{24}$$

**8 a**  $2 \sin x + \operatorname{cosec} x = 3, \quad 0 \leq x \leq 2\pi$

$$\therefore 2 \sin x + \frac{1}{\sin x} = 3$$

$$\therefore 2 \sin^2 x + 1 = 3 \sin x$$

$$\therefore 2 \sin^2 x - 3 \sin x + 1 = 0$$

$$\therefore (2 \sin x - 1)(\sin x - 1) = 0 \quad \{2X^2 - 3X + 1 = (2X - 1)(X - 1)\}$$

$$\therefore 2 \sin x - 1 = 0 \quad \text{or} \quad \sin x - 1 = 0$$

$$\therefore 2 \sin x = 1 \quad \therefore \sin x = 1$$

$$\therefore \sin x = \frac{1}{2} \quad \therefore x = \frac{\pi}{2}$$

$$\therefore x = \frac{\pi}{6} \text{ or } \frac{5\pi}{6}$$

$$\therefore x = \frac{\pi}{6}, \frac{\pi}{2}, \text{ or } \frac{5\pi}{6}$$

**b**  $2 \cos x = \sec x, \quad 0 \leq x \leq 2\pi$

$$\therefore 2 \cos x = \frac{1}{\cos x}$$

$$\therefore 2 \cos^2 x = 1$$

$$\therefore \cos^2 x = \frac{1}{2}$$

$$\therefore \cos x = \pm \frac{1}{\sqrt{2}}$$

$$\therefore x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \text{ or } \frac{7\pi}{4}$$

**c**  $\tan^2 x - 2 - 3 \cot^2 x = 0, \quad 0 \leq x \leq 2\pi$

$$\therefore \tan^2 x - 2 - \frac{3}{\tan^2 x} = 0$$

$$\therefore \tan^4 x - 2 \tan^2 x - 3 = 0$$

$$\therefore (\tan^2 x + 1)(\tan^2 x - 3) = 0 \quad \{X^2 - 2X - 3 = (X + 1)(X - 3)\}$$

$$\therefore \tan^2 x + 1 = 0 \quad \text{or} \quad \tan^2 x - 3 = 0$$

$$\therefore \tan^2 x = -1 \quad \therefore \tan^2 x = 3$$

which is impossible

$$\therefore \tan x = \pm \sqrt{3}$$

$$\therefore x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \text{ or } \frac{5\pi}{3}$$

**d**  $4 \sin x = \sqrt{3} \operatorname{cosec} x + 2 - 2\sqrt{3}, \quad 0 \leq x \leq 2\pi$

$$\therefore 4 \sin x = \frac{\sqrt{3}}{\sin x} + 2 - 2\sqrt{3}$$

$$\therefore 4 \sin^2 x = \sqrt{3} + (2 - 2\sqrt{3}) \sin x$$

$$\therefore 4 \sin^2 x + (2\sqrt{3} - 2) \sin x - \sqrt{3} = 0$$

$$\therefore (2 \sin x - 1)(2 \sin x + \sqrt{3}) = 0 \quad \{4X^2 + (2\sqrt{3} - 2)X - \sqrt{3} = (2X - 1)(2X + \sqrt{3})\}$$

$$\therefore 2 \sin x - 1 = 0 \quad \text{or} \quad 2 \sin x + \sqrt{3} = 0$$

$$\therefore 2 \sin x = 1$$

$$\therefore 2 \sin x = -\sqrt{3}$$

$$\therefore \sin x = \frac{1}{2}$$

$$\therefore \sin x = -\frac{\sqrt{3}}{2}$$

$$\therefore x = \frac{\pi}{6} \text{ or } \frac{5\pi}{6}$$

$$\therefore x = \frac{4\pi}{3} \text{ or } \frac{5\pi}{3}$$

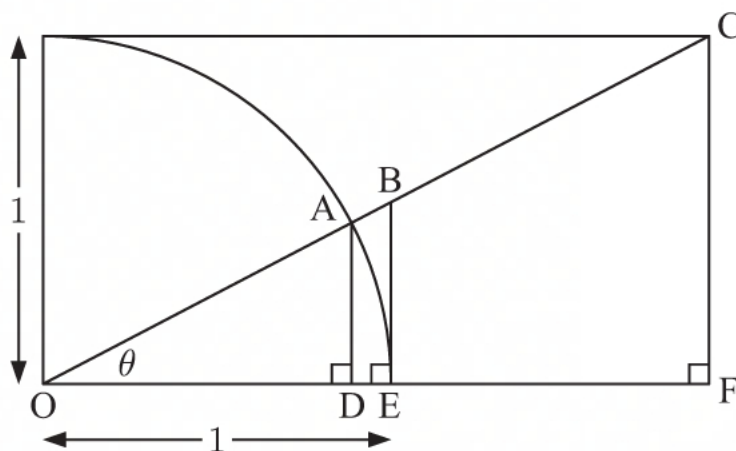
$$\therefore x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{4\pi}{3}, \text{ or } \frac{5\pi}{3}$$

**9 a**  $\cos \theta = \frac{OD}{OA}$   
 $= \frac{OD}{1}$   
 $= OD$

$\therefore [OD]$  has length  $\cos \theta$ .

**b**  $\sin \theta = \frac{AD}{OA}$   
 $= \frac{AD}{1}$   
 $= AD$

$\therefore [AD]$  has length  $\sin \theta$ .





$$\begin{aligned} \text{c} \quad \tan \theta &= \frac{BE}{OE} \\ &= \frac{BE}{1} \\ &= BE \end{aligned}$$

$\therefore$  [BE] has length  $\tan \theta$ .

$$\begin{aligned} \text{e} \quad \frac{CF}{AD} &= \frac{OC}{OA} \quad \{\text{similar triangles}\} \\ \therefore \frac{1}{\sin \theta} &= \frac{OC}{1} \quad \{\text{using b}\} \end{aligned}$$

$\therefore \operatorname{cosec} \theta = OC$

$\therefore$  [OC] has length  $\operatorname{cosec} \theta$ .

$$\text{d} \quad \frac{OE}{OD} = \frac{OB}{OA} \quad \{\text{similar triangles}\}$$

$$\therefore \frac{1}{\cos \theta} = \frac{OB}{1} \quad \{\text{using a}\}$$

$\therefore \sec \theta = OB$

$\therefore$  [OB] has length  $\sec \theta$ .

$$\text{f} \quad \frac{CF}{BE} = \frac{OF}{OE} \quad \{\text{similar triangles}\}$$

$$\therefore \frac{1}{\tan \theta} = \frac{OF}{1} \quad \{\text{using c}\}$$

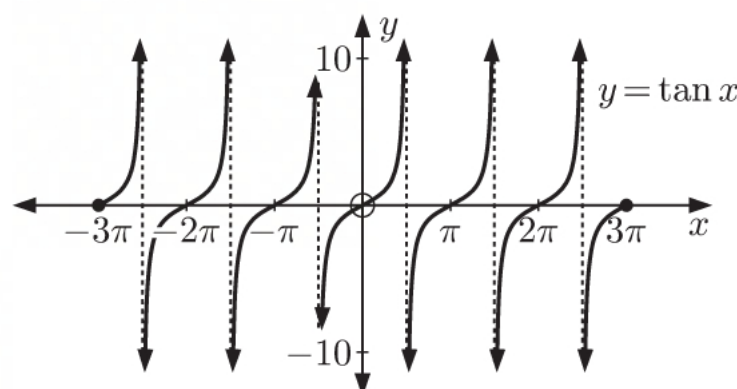
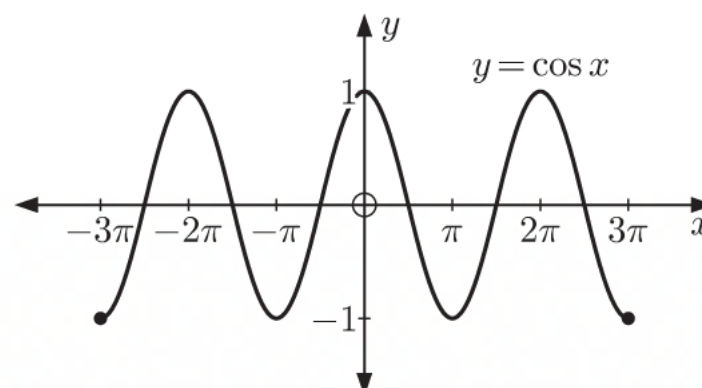
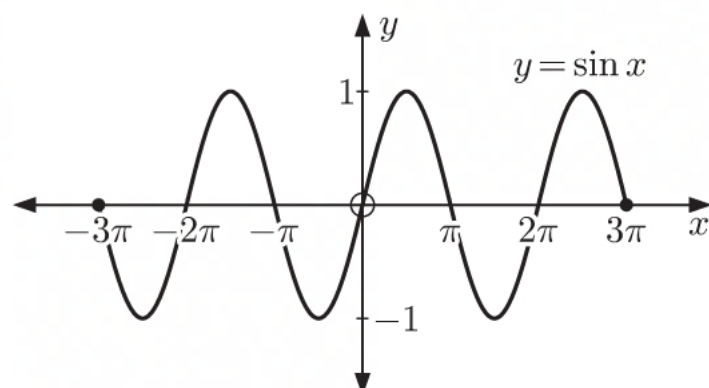
$\therefore \cot \theta = OF$

$\therefore$  [OF] has length  $\cot \theta$ .

## INVESTIGATION 1

## INVERSE TRIGONOMETRIC FUNCTIONS

1



2

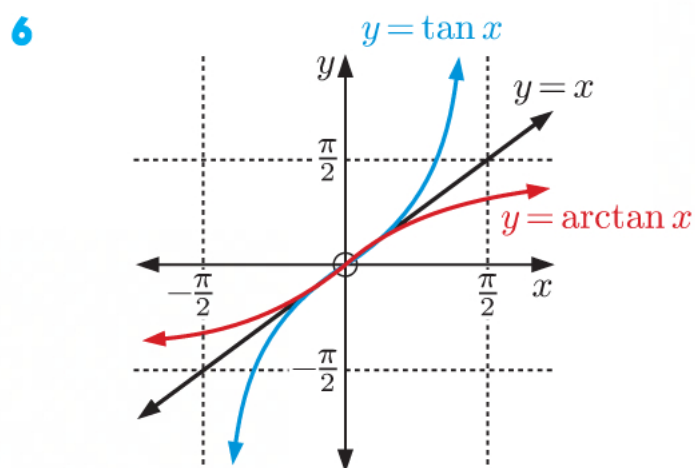
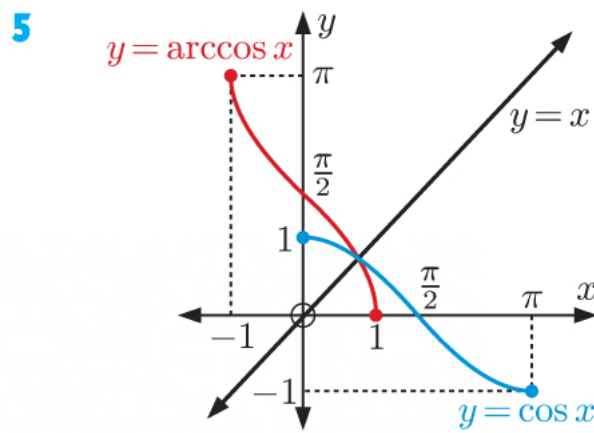
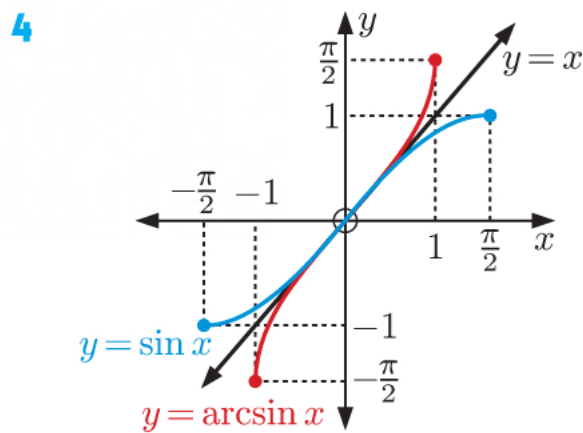
Restricted domain	$\sin x$	$\cos x$	$\tan x$
$0 \leq x \leq 2\pi$	✗	✗	✗
$-\pi \leq x \leq \pi$	✗	✗	✗
$-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$	✓	✗	✓
$0 \leq x \leq \pi$	✗	✓	✗
$-\frac{\pi}{2} \leq x \leq \frac{3\pi}{2}$	✗	✗	✗
$\pi \leq x \leq 2\pi$	✗	✓	✗

3

a The restricted domain  $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$  is the most suitable for the inverse function of  $\sin x$  since  $\sin x$  is one-to-one on this interval.

b The restricted domain  $0 \leq x \leq \pi$  is the most suitable for the inverse function of  $\cos x$  since it is the smallest positive interval where  $\cos x$  is one-to-one.

- c  $\tan x$  is one-to-one on the interval  $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ , but since  $\tan x$  is undefined when  $x = -\frac{\pi}{2}$  or  $\frac{\pi}{2}$ , the restricted domain  $-\frac{\pi}{2} < x < \frac{\pi}{2}$  is the most suitable for the inverse function of  $\tan x$ .



## EXERCISE 1B

- 1 a Let  $x = \arccos 1$   
 $\therefore \cos x = 1$   
 $\therefore x = 0 \quad \{0 \leq x \leq \pi\}$   
 $\therefore \arccos 1 = 0$
- b Let  $x = \arcsin(-1)$   
 $\therefore \sin x = -1$   
 $\therefore x = -\frac{\pi}{2} \quad \{-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}\}$   
 $\therefore \arcsin(-1) = -\frac{\pi}{2}$
- c Let  $x = \arctan 1$   
 $\therefore \tan x = 1$   
 $\therefore x = \frac{\pi}{4} \quad \{-\frac{\pi}{2} < x < \frac{\pi}{2}\}$   
 $\therefore \arctan 1 = \frac{\pi}{4}$
- d Let  $x = \arctan(-1)$   
 $\therefore \tan x = -1$   
 $\therefore x = -\frac{\pi}{4} \quad \{-\frac{\pi}{2} < x < \frac{\pi}{2}\}$   
 $\therefore \arctan(-1) = -\frac{\pi}{4}$
- e Let  $x = \arcsin \frac{1}{2}$   
 $\therefore \sin x = \frac{1}{2}$   
 $\therefore x = \frac{\pi}{6} \quad \{-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}\}$   
 $\therefore \arcsin \frac{1}{2} = \frac{\pi}{6}$
- f Let  $x = \arccos\left(-\frac{\sqrt{3}}{2}\right)$   
 $\therefore \cos x = -\frac{\sqrt{3}}{2}$   
 $\therefore x = \frac{5\pi}{6} \quad \{0 \leq x \leq \pi\}$   
 $\therefore \arccos\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$
- g Let  $x = \arctan \sqrt{3}$   
 $\therefore \tan x = \sqrt{3}$   
 $\therefore x = \frac{\pi}{3} \quad \{-\frac{\pi}{2} < x < \frac{\pi}{2}\}$   
 $\therefore \arctan \sqrt{3} = \frac{\pi}{3}$
- h Let  $x = \arccos\left(-\frac{1}{\sqrt{2}}\right)$   
 $\therefore \cos x = -\frac{1}{\sqrt{2}}$   
 $\therefore x = \frac{3\pi}{4} \quad \{0 \leq x \leq \pi\}$   
 $\therefore \arccos\left(-\frac{1}{\sqrt{2}}\right) = \frac{3\pi}{4}$



$$\begin{aligned}
 \text{i} \quad & \text{Let } x = \arctan\left(-\frac{1}{\sqrt{3}}\right) & \text{j} \quad \sin^{-1}(-0.767) \approx -0.874 \\
 & \therefore \tan x = -\frac{1}{\sqrt{3}} \\
 & \therefore x = -\frac{\pi}{6} \quad \left\{-\frac{\pi}{2} < x < \frac{\pi}{2}\right\} \\
 & \therefore \arctan\left(-\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}
 \end{aligned}$$

$$\text{k} \quad \cos^{-1} 0.327 \approx 1.24 \qquad \text{l} \quad \tan^{-1}(-50) \approx -1.55$$

- 2** **a** The inverse transformation from  $y = \sin x$  to  $y = \arcsin x$  has an invariant point where  $\sin x = \arcsin x \therefore$  at  $(0, 0)$ .
- b** The inverse transformation from  $y = \tan x$  to  $y = \arctan x$  has an invariant point where  $\tan x = \arctan x \therefore$  at  $(0, 0)$ .
- c** The inverse transformation from  $y = \cos x$  to  $y = \arccos x$  has an invariant point where  $\cos x = \arccos x \therefore$  at  $(0.739, 0.739)$ .
- 3** **a**  $y = \arctan x$  has horizontal asymptotes  $y = -\frac{\pi}{2}$  and  $y = \frac{\pi}{2}$ .
- b** The functions  $y = \arcsin x$  and  $y = \arccos x$  do not have vertical asymptotes as  $y = \sin x$  and  $y = \cos x$  do not have horizontal asymptotes.

$$\begin{aligned}
 \text{4} \quad \text{a} \quad & \text{Let } x = \arcsin\left(\sin \frac{\pi}{3}\right) & \text{b} \quad & \text{Let } x = \arccos\left(\cos\left(-\frac{\pi}{6}\right)\right) \\
 & \therefore \sin x = \sin \frac{\pi}{3} & & \therefore \cos x = \cos\left(-\frac{\pi}{6}\right) \\
 & \therefore x = \frac{\pi}{3} \quad \left\{-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}\right\} & & = \cos \frac{\pi}{6} \\
 & \therefore \arcsin\left(\sin \frac{\pi}{3}\right) = \frac{\pi}{3} & & \therefore x = \frac{\pi}{6} \quad \{0 \leq x \leq \pi\} \\
 & & & \therefore \arccos\left(\cos\left(-\frac{\pi}{6}\right)\right) = \frac{\pi}{6}
 \end{aligned}$$

$$\text{c} \quad \tan(\arctan(0.3)) = 0.3$$

$$\text{d} \quad \cos(\arccos(-\frac{1}{2})) = -\frac{1}{2}$$

$$\begin{aligned}
 \text{e} \quad & \text{Let } x = \arctan(\tan \pi) & \text{f} \quad & \text{Let } x = \arcsin\left(\sin \frac{4\pi}{3}\right) \\
 & \therefore \tan x = \tan \pi & & \therefore \sin x = \sin \frac{4\pi}{3} \\
 & = \tan 0 & & = \sin\left(-\frac{\pi}{3}\right) \\
 & \therefore x = 0 \quad \left\{-\frac{\pi}{2} < x < \frac{\pi}{2}\right\} & & \therefore x = -\frac{\pi}{3} \quad \left\{-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}\right\} \\
 & \therefore \arctan(\tan \pi) = 0 & & \therefore \arcsin\left(\sin \frac{4\pi}{3}\right) = -\frac{\pi}{3}
 \end{aligned}$$

- 5** **a** The range of  $y = \arctan x$  is  $-\frac{\pi}{2} < y < \frac{\pi}{2}$ .  $\frac{\pi}{4}$  is within the range.  
 $\therefore x = \tan \frac{\pi}{4} = 1$
- b** The range of  $y = \arcsin x$  is  $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ .  $-\frac{\pi}{3}$  is within the range.  
 $\therefore x = \sin\left(-\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$
- c** The range of  $y = \arccos x$  is  $0 \leq y \leq \pi$ .  $\frac{3\pi}{4}$  is within the range.  
 $\therefore x = \cos \frac{3\pi}{4} = -\frac{1}{\sqrt{2}}$
- d** The range of  $y = \arcsin(x + 1)$  is  $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ .  $\frac{\pi}{6}$  is within the range.  
 $\therefore x + 1 = \sin \frac{\pi}{6}$   
 $\therefore x + 1 = \frac{1}{2}$   
 $\therefore x = -\frac{1}{2}$

- e** The range of  $y = \arccos x$  is  $0 \leq y \leq \pi$ .  $-\frac{\pi}{4}$  is outside the range.  
 $\therefore$  there are no solutions.
- f** The range of  $y = \arctan(x - \sqrt{3})$  is  $-\frac{\pi}{2} < y < \frac{\pi}{2}$ .  $-\frac{\pi}{3}$  is within the range.  
 $\therefore x - \sqrt{3} = \tan\left(-\frac{\pi}{3}\right)$   
 $\therefore x - \sqrt{3} = -\sqrt{3}$   
 $\therefore x = 0$

**ACTIVITY 1** **$\arctan x$** 

$$1 \quad \frac{1}{1 + \frac{1}{3 + \frac{1}{5 + \frac{1}{7 + \frac{1}{9 + \frac{1}{11 + \frac{36}{13}}}}}}} \approx 0.78540$$

- 2**  $\arctan 1 \approx 0.78540$  which agrees with our result in **1**. The original continued fraction appears to represent  $\arctan x$ .

- 3** Let  $x = 2$ .

$$1 + \frac{2}{3 + \frac{4}{5 + \frac{16}{7 + \frac{36}{9 + \frac{64}{11 + \frac{100}{13 + \frac{144}{\dots + \frac{576}{25 + \frac{676}{27}}}}}}}}} \approx 1.10715$$

and  $\arctan 2 \approx 1.10715$

**EXERCISE 1C.1**

- |            |  |          |   |          |   |
|------------|--|----------|---|----------|---|
| <b>1 a</b> | $\sin \theta + \sin \theta$<br>$= 2 \sin \theta$   | <b>b</b> | $2 \cos \theta + \cos \theta$<br>$= 3 \cos \theta$  | <b>c</b> | $3 \sin \theta - \sin \theta$<br>$= 2 \sin \theta$          |
| <b>d</b>   | $3 \sin \theta - 2 \sin \theta$<br>$= \sin \theta$ | <b>e</b> | $\tan \theta - 3 \tan \theta$<br>$= -2 \tan \theta$ | <b>f</b> | $2 \cos^2 \theta - 5 \cos^2 \theta$<br>$= -3 \cos^2 \theta$ |



$$\begin{aligned}
 \text{2 a} \quad & 3 \tan x - \frac{\sin x}{\cos x} \\
 &= 3 \tan x - \tan x \\
 &= 2 \tan x
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad & \frac{\sin x}{\tan x} \\
 &= \sin x \div \frac{\sin x}{\cos x} \\
 &= \sin x \times \frac{\cos x}{\sin x} \\
 &= \cos x
 \end{aligned}$$

$$\begin{aligned}
 \text{3 a} \quad & \tan x \cot x \\
 &= \frac{\sin x}{\cos x} \times \frac{\cos x}{\sin x} \\
 &= 1
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad & \sin x \cot x \\
 &= \sin x \times \frac{\cos x}{\sin x} \\
 &= \cos x
 \end{aligned}$$

$$\begin{aligned}
 \text{4 a} \quad & 3 \cos \theta - \cos(-\theta) \\
 &= 3 \cos \theta - \cos \theta \\
 &= 2 \cos \theta
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad & \tan(\pi - \theta) \\
 &= \frac{\sin(\pi - \theta)}{\cos(\pi - \theta)} \\
 &= \frac{\sin \theta}{-\cos \theta} \\
 &= -\tan \theta
 \end{aligned}$$

$$\begin{aligned}
 \text{g} \quad & \frac{\sin(-\theta)}{\cos(\pi - \theta)} \\
 &= \frac{-\sin \theta}{-\cos \theta} \\
 &= \tan \theta
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & \frac{\sin^2 x}{\cos^2 x} \\
 &= \left( \frac{\sin x}{\cos x} \right)^2 \\
 &= \tan^2 x
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & 3 \sin x + 2 \cos x \tan x \\
 &= 3 \sin x + 2 \cos x \times \frac{\sin x}{\cos x} \\
 &= 3 \sin x + 2 \sin x \\
 &= 5 \sin x
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & \sin x \operatorname{cosec} x \\
 &= \sin x \times \frac{1}{\sin x} \\
 &= 1
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & \frac{\cot x}{\operatorname{cosec} x} \\
 &= \frac{\cos x}{\sin x} \div \frac{1}{\sin x} \\
 &= \frac{\cos x}{\sin x} \times \frac{\sin x}{1} \\
 &= \cos x
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & \tan(-\theta) \\
 &= \frac{\sin(-\theta)}{\cos(-\theta)} \\
 &= \frac{-\sin \theta}{\cos \theta} \\
 &= -\tan \theta
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & \tan\left(\frac{\pi}{2} - \theta\right) \\
 &= \frac{\sin\left(\frac{\pi}{2} - \theta\right)}{\cos\left(\frac{\pi}{2} - \theta\right)} \\
 &= \frac{\cos \theta}{\sin \theta} \\
 &= \cot \theta
 \end{aligned}$$

$$\begin{aligned}
 \text{h} \quad & \frac{\cos\left(\frac{\pi}{2} - \theta\right)}{\cos(-\theta)} \\
 &= \frac{\sin \theta}{\cos \theta} \\
 &= \tan \theta
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & \tan x \cos x \\
 &= \frac{\sin x}{\cos x} \times \cos x \\
 &= \sin x
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & \frac{2 \tan x}{\sin x} \\
 &= 2 \left( \frac{\sin x}{\cos x} \right) \times \frac{1}{\sin x} \\
 &= \frac{2}{\cos x} \\
 &= 2 \sec x
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & \operatorname{cosec} x \cot x \\
 &= \frac{1}{\sin x} \times \frac{\cos x}{\sin x} \\
 &= \frac{\cos x}{\sin^2 x}
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & \sin(-\theta) + \cos\left(\frac{\pi}{2} - \theta\right) \\
 &= -\sin \theta + \sin \theta \\
 &= 0
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & \sin\left(\frac{\pi}{2} - \theta\right) - \cos(\pi - \theta) \\
 &= \cos \theta - (-\cos \theta) \\
 &= \cos \theta + \cos \theta \\
 &= 2 \cos \theta
 \end{aligned}$$

$$\begin{aligned}
 \text{i} \quad & \frac{\sin(\pi - \theta) - \sin(-\theta)}{\cos(-\theta)} \\
 &= \frac{\sin \theta - (-\sin \theta)}{\cos \theta} \\
 &= \frac{2 \sin \theta}{\cos \theta} \\
 &= 2 \tan \theta
 \end{aligned}$$

**EXERCISE 1C.2**

$$\begin{aligned}
 \mathbf{1} \quad \mathbf{a} \quad & 3 \sin^2 \theta + 3 \cos^2 \theta \\
 &= 3(\sin^2 \theta + \cos^2 \theta) \\
 &= 3(1) \\
 &= 3
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & -2 \sin^2 \theta - 2 \cos^2 \theta \\
 &= -2(\sin^2 \theta + \cos^2 \theta) \\
 &= -2(1) \\
 &= -2
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & -\cos^2 \theta - \sin^2 \theta \\
 &= -(\cos^2 \theta + \sin^2 \theta) \\
 &= -(1) \\
 &= -1
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad & 3 - 3 \sin^2 \theta \\
 &= 3(1 - \sin^2 \theta) \\
 &= 3 \cos^2 \theta
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{e} \quad & 4 - 4 \cos^2 \theta \\
 &= 4(1 - \cos^2 \theta) \\
 &= 4 \sin^2 \theta
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{f} \quad & \cos^3 \theta + \cos \theta \sin^2 \theta \\
 &= \cos \theta (\cos^2 \theta + \sin^2 \theta) \\
 &= \cos \theta (1) \\
 &= \cos \theta
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{g} \quad & \cos^2 \theta - 1 \\
 &= 1 - \sin^2 \theta - 1 \\
 &= -\sin^2 \theta
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{h} \quad & \sin^2 \theta - 1 \\
 &= 1 - \cos^2 \theta - 1 \\
 &= -\cos^2 \theta
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{i} \quad & 2 \cos^2 \theta - 2 \\
 &= -2(1 - \cos^2 \theta) \\
 &= -2 \sin^2 \theta
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{j} \quad & \frac{1 - \sin^2 \theta}{\cos^2 \theta} \\
 &= \frac{\cos^2 \theta}{\cos^2 \theta} \\
 &= 1
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{k} \quad & \frac{1 - \cos^2 \theta}{\sin \theta} \\
 &= \frac{\sin^2 \theta}{\sin \theta} \\
 &= \sin \theta
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{l} \quad & \frac{\cos^2 \theta - 1}{-\sin \theta} \\
 &= \frac{1 - \sin^2 \theta - 1}{-\sin \theta} \\
 &= \frac{-\sin^2 \theta}{-\sin \theta} \\
 &= \sin \theta
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{2} \quad 1 + \cot^2 \theta &= 1 + \left( \frac{\cos \theta}{\sin \theta} \right)^2 \\
 &= 1 + \frac{\cos^2 \theta}{\sin^2 \theta} \\
 &= \frac{\sin^2 \theta + \cos^2 \theta}{\sin^2 \theta} \\
 &= \frac{1}{\sin^2 \theta} \\
 &= \operatorname{cosec}^2 \theta
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{3} \quad \mathbf{a} \quad & (1 + \sin \theta)^2 \\
 &= 1 + 2 \sin \theta + \sin^2 \theta
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & (\tan \alpha - 1)^2 \\
 &= \tan^2 \alpha - 2 \tan \alpha + 1 \\
 &= \tan^2 \alpha + 1 - 2 \tan \alpha \\
 &= \sec^2 \alpha - 2 \tan \alpha
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{e} \quad & (\sin \beta - \cos \beta)^2 \\
 &= \sin^2 \beta - 2 \sin \beta \cos \beta + \cos^2 \beta \\
 &= \sin^2 \beta + \cos^2 \beta - 2 \sin \beta \cos \beta \\
 &= 1 - 2 \sin \beta \cos \beta
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & (\sin \alpha - 2)^2 \\
 &= \sin^2 \alpha - 4 \sin \alpha + 4
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad & (\sin \alpha + \cos \alpha)^2 \\
 &= \sin^2 \alpha + 2 \sin \alpha \cos \alpha + \cos^2 \alpha \\
 &= \sin^2 \alpha + \cos^2 \alpha + 2 \sin \alpha \cos \alpha \\
 &= 1 + 2 \sin \alpha \cos \alpha
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{f} \quad & -(2 - \cos \alpha)^2 \\
 &= -(4 - 4 \cos \alpha + \cos^2 \alpha) \\
 &= -4 + 4 \cos \alpha - \cos^2 \alpha
 \end{aligned}$$

$$\begin{aligned}
 4 \quad a \quad & 1 - \sec^2 \beta \\
 &= 1 - \frac{1}{\cos^2 \beta} \\
 &= \frac{\cos^2 \beta - 1}{\cos^2 \beta} \\
 &= \frac{-\sin^2 \beta}{\cos^2 \beta} \\
 &= -\tan^2 \beta
 \end{aligned}$$

$$\begin{aligned}
 b \quad & \frac{\tan^2 \theta (\cot^2 \theta + 1)}{\tan^2 \theta + 1} \\
 &= \frac{\tan^2 \theta \cot^2 \theta + \tan^2 \theta}{\tan^2 \theta + 1} \\
 &= \frac{1 + \tan^2 \theta}{\tan^2 \theta + 1} \\
 &= 1
 \end{aligned}$$

$$\begin{aligned}
 c \quad & \cos^2 \alpha (\sec^2 \alpha - 1) \\
 &= \cos^2 \alpha \sec^2 \alpha - \cos^2 \alpha \\
 &= 1 - \cos^2 \alpha \\
 &= \sin^2 \alpha
 \end{aligned}$$

$$\begin{aligned}
 d \quad & (\sin x + \tan x)(\sin x - \tan x) = \sin^2 x - \tan^2 x \\
 &= \sin^2 x - \frac{\sin^2 x}{\cos^2 x} \\
 &= \sin^2 x \left(1 - \frac{1}{\cos^2 x}\right) \\
 &= \sin^2 x \left(\frac{\cos^2 x - 1}{\cos^2 x}\right) \\
 &= \sin^2 x \left(-\frac{\sin^2 x}{\cos^2 x}\right) \\
 &= -\sin^2 x \tan^2 x
 \end{aligned}$$

$$\begin{aligned}
 e \quad & (2 \sin \theta + 3 \cos \theta)^2 + (3 \sin \theta - 2 \cos \theta)^2 \\
 &= 4 \sin^2 \theta + 12 \sin \theta \cos \theta + 9 \cos^2 \theta + 9 \sin^2 \theta - 12 \sin \theta \cos \theta + 4 \cos^2 \theta \\
 &= 13 \sin^2 \theta + 13 \cos^2 \theta \\
 &= 13(\sin^2 \theta + \cos^2 \theta) \\
 &= 13
 \end{aligned}$$

$$\begin{aligned}
 f \quad & (1 + \operatorname{cosec} \theta)(\sin \theta - \sin^2 \theta) = \sin \theta - \sin^2 \theta + \operatorname{cosec} \theta \sin \theta - \operatorname{cosec} \theta \sin^2 \theta \\
 &= \sin \theta - \sin^2 \theta + 1 - \sin \theta \\
 &= 1 - \sin^2 \theta \\
 &= \cos^2 \theta
 \end{aligned}$$

$$\begin{aligned}
 g \quad & \sec A - \sin A \tan A - \cos A = \frac{1}{\cos A} - \frac{\sin A \sin A}{\cos A} - \frac{\cos^2 A}{\cos A} \\
 &= \frac{1 - \sin^2 A - \cos^2 A}{\cos A} \\
 &= \frac{1 - (\sin^2 A + \cos^2 A)}{\cos A} \\
 &= 0
 \end{aligned}$$

### EXERCISE 1C.3

$$\begin{aligned}
 1 \quad a \quad & 1 - \sin^2 \theta \\
 &= (1 + \sin \theta)(1 - \sin \theta)
 \end{aligned}$$

$$\begin{aligned}
 c \quad & \tan^2 \alpha - 1 \\
 &= (\tan \alpha + 1)(\tan \alpha - 1)
 \end{aligned}$$

$$\begin{aligned}
 b \quad & \sin^2 \alpha - \cos^2 \alpha \\
 &= (\sin \alpha + \cos \alpha)(\sin \alpha - \cos \alpha)
 \end{aligned}$$

$$\begin{aligned}
 d \quad & 2 \sin^2 \beta - \sin \beta \\
 &= \sin \beta (2 \sin \beta - 1)
 \end{aligned}$$



$$\begin{aligned} \text{e} \quad & 2 \cos \phi + 3 \cos^2 \phi \\ &= \cos \phi(2 + 3 \cos \phi) \end{aligned}$$

$$\begin{aligned} \text{g} \quad & \tan^2 \theta + 5 \tan \theta + 6 \\ &= (\tan \theta + 2)(\tan \theta + 3) \end{aligned}$$

$$\begin{aligned} \text{i} \quad & 6 \cos^2 \alpha - \cos \alpha - 1 \\ &= (3 \cos \alpha + 1)(2 \cos \alpha - 1) \end{aligned}$$

$$\begin{aligned} \text{k} \quad & \sec^2 \beta - \operatorname{cosec}^2 \beta \\ &= (\sec \beta + \operatorname{cosec} \beta)(\sec \beta - \operatorname{cosec} \beta) \end{aligned}$$

$$\begin{aligned} \text{m} \quad & 2 \sin^2 x + 7 \sin x \cos x + 3 \cos^2 x \\ &= (2 \sin x + \cos x)(\sin x + 3 \cos x) \end{aligned}$$

$$\begin{aligned} 2 \quad \text{a} \quad & \frac{1 - \sin^2 \alpha}{1 - \sin \alpha} \\ &= \frac{(1 + \sin \alpha)(\cancel{1 - \sin \alpha})}{\cancel{1 - \sin \alpha}} \\ &= 1 + \sin \alpha \end{aligned}$$

$$\begin{aligned} \text{c} \quad & \frac{\cos^2 \phi - \sin^2 \phi}{\cos \phi + \sin \phi} \\ &= \frac{(\cancel{\cos \phi + \sin \phi})(\cos \phi - \sin \phi)}{\cancel{\cos \phi + \sin \phi}} \\ &= \cos \phi - \sin \phi \end{aligned}$$

$$\begin{aligned} \text{e} \quad & \frac{\sin \alpha + \cos \alpha}{\sin^2 \alpha - \cos^2 \alpha} \\ &= \frac{\cancel{\sin \alpha + \cos \alpha}}{(\cancel{\sin \alpha + \cos \alpha})(\sin \alpha - \cos \alpha)} \\ &= \frac{1}{\sin \alpha - \cos \alpha} \end{aligned}$$

$$\begin{aligned} \text{g} \quad & 1 - \frac{\cos^2 \theta}{1 + \sin \theta} \\ &= \frac{(1 + \sin \theta) - (1 - \sin^2 \theta)}{1 + \sin \theta} \\ &= \frac{1 + \sin \theta - 1 + \sin^2 \theta}{1 + \sin \theta} \\ &= \frac{\sin \theta(\cancel{1 + \sin \theta})}{\cancel{1 + \sin \theta}} \\ &= \sin \theta \end{aligned}$$

$$\begin{aligned} \text{f} \quad & 3 \sin^2 \theta - 6 \sin \theta \\ &= 3 \sin \theta(\sin \theta - 2) \end{aligned}$$

$$\begin{aligned} \text{h} \quad & 2 \cos^2 \theta + 7 \cos \theta + 3 \\ &= (2 \cos \theta + 1)(\cos \theta + 3) \end{aligned}$$

$$\begin{aligned} \text{j} \quad & 3 \tan^2 \alpha - 2 \tan \alpha \\ &= \tan \alpha(3 \tan \alpha - 2) \end{aligned}$$

$$\begin{aligned} \text{l} \quad & 2 \cot^2 x - 3 \cot x + 1 \\ &= (2 \cot x - 1)(\cot x - 1) \end{aligned}$$

$$\begin{aligned} \text{b} \quad & \frac{\tan^2 \beta - 1}{\tan \beta + 1} \\ &= \frac{(\cancel{\tan \beta + 1})(\tan \beta - 1)}{\cancel{\tan \beta + 1}} \\ &= \tan \beta - 1 \end{aligned}$$

$$\begin{aligned} \text{d} \quad & \frac{\cos^2 \phi - \sin^2 \phi}{\cos \phi - \sin \phi} \\ &= \frac{(\cos \phi + \sin \phi)(\cancel{\cos \phi - \sin \phi})}{\cancel{\cos \phi - \sin \phi}} \\ &= \cos \phi + \sin \phi \end{aligned}$$

$$\begin{aligned} \text{f} \quad & \frac{3 - 3 \sin^2 \theta}{6 \cos \theta} \\ &= \frac{3(1 - \sin^2 \theta)}{6 \cos \theta} \\ &= \frac{3 \cos^2 \theta}{6 \cos \theta} \\ &= \frac{\cos \theta}{2} \end{aligned}$$

$$\begin{aligned} \text{h} \quad & \frac{1 + \cot \theta}{\operatorname{cosec} \theta} - \frac{\sec \theta}{\tan \theta + \cot \theta} \\ &= \sin \theta \left( 1 + \frac{\cos \theta}{\sin \theta} \right) - \frac{1}{\cos \theta \left( \frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta} \right)} \\ &= \sin \theta + \cos \theta - \frac{1}{\sin \theta + \frac{\cos^2 \theta}{\sin \theta}} \\ &= \sin \theta + \cos \theta - \frac{\sin \theta}{\sin^2 \theta + \cos^2 \theta} \\ &= \sin \theta + \cos \theta - \sin \theta \\ &= \cos \theta \end{aligned}$$

$$\begin{aligned}
 & \text{i} \quad \frac{\tan^2 \theta}{\sec \theta - 1} \\
 &= \frac{\sec^2 \theta - 1}{\sec \theta - 1} \\
 &= \frac{(\sec \theta + 1)(\sec \theta - 1)}{\sec \theta - 1} \\
 &= \sec \theta + 1
 \end{aligned}$$

$$\begin{aligned}
 & \text{3 a} \quad (\cos \theta + \sin \theta)^2 + (\cos \theta - \sin \theta)^2 \\
 &= \cos^2 \theta + \cancel{2 \cos \theta \sin \theta} + \sin^2 \theta + \cos^2 \theta - \cancel{2 \cos \theta \sin \theta} + \sin^2 \theta \\
 &= 2 \cos^2 \theta + 2 \sin^2 \theta \\
 &= 2(\cos^2 \theta + \sin^2 \theta) \\
 &= 2
 \end{aligned}$$

$$\begin{aligned}
 & \text{b} \quad (\sin \theta + 4 \cos \theta)^2 + (4 \sin \theta - \cos \theta)^2 \\
 &= \sin^2 \theta + \cancel{8 \sin \theta \cos \theta} + 16 \cos^2 \theta + 16 \sin^2 \theta - \cancel{8 \sin \theta \cos \theta} + \cos^2 \theta \\
 &= 17 \sin^2 \theta + 17 \cos^2 \theta \\
 &= 17(\sin^2 \theta + \cos^2 \theta) \\
 &= 17
 \end{aligned}$$

$$\begin{aligned}
 & \text{c} \quad (1 - \cos \theta) \left( 1 + \frac{1}{\cos \theta} \right) \\
 &= 1 + \frac{1}{\cos \theta} - \cos \theta - 1 \\
 &= \frac{1}{\cos \theta} - \cos \theta \\
 &= \frac{1 - \cos^2 \theta}{\cos \theta} \\
 &= \frac{\sin^2 \theta}{\cos \theta} \\
 &= \tan \theta \sin \theta
 \end{aligned}$$

$$\begin{aligned}
 & \text{e} \quad \sec A - \cos A \\
 &= \frac{1}{\cos A} - \cos A \\
 &= \frac{1 - \cos^2 A}{\cos A} \\
 &= \frac{\sin^2 A}{\cos A} \\
 &= \tan A \sin A
 \end{aligned}$$

$$\begin{aligned}
 & \text{d} \quad \left( 1 + \frac{1}{\sin \theta} \right) (\sin \theta - \sin^2 \theta) \\
 &= \cancel{\sin \theta} - \sin^2 \theta + 1 - \cancel{\sin \theta} \\
 &= 1 - \sin^2 \theta \\
 &= \cos^2 \theta
 \end{aligned}$$

$$\begin{aligned}
 & \text{f} \quad \frac{\cos \theta}{1 - \sin \theta} \\
 &= \frac{\cos \theta}{1 - \sin \theta} \times \frac{1 + \sin \theta}{1 + \sin \theta} \\
 &= \frac{\cos \theta + \cos \theta \sin \theta}{1 - \sin^2 \theta} \\
 &= \frac{\cos \theta + \cos \theta \sin \theta}{\cos^2 \theta} \\
 &= \frac{1}{\cos \theta} + \frac{\sin \theta}{\cos \theta} \\
 &= \sec \theta + \tan \theta
 \end{aligned}$$

$$\begin{aligned}
\text{g} \quad & \frac{\cos \alpha}{1 - \tan \alpha} + \frac{\sin \alpha}{1 - \cot \alpha} \\
&= \frac{\cos \alpha}{1 - \frac{\sin \alpha}{\cos \alpha}} + \frac{\sin \alpha}{1 - \frac{\cos \alpha}{\sin \alpha}} \\
&= \frac{\cos \alpha}{\frac{\cos \alpha - \sin \alpha}{\cos \alpha}} + \frac{\sin \alpha}{\frac{\sin \alpha - \cos \alpha}{\sin \alpha}} \\
&= \frac{\cos^2 \alpha}{\cos \alpha - \sin \alpha} + \frac{\sin^2 \alpha}{\sin \alpha - \cos \alpha} \\
&= \frac{\cos^2 \alpha - \sin^2 \alpha}{\cos \alpha - \sin \alpha} \\
&= \frac{(\cos \alpha + \sin \alpha)(\cancel{\cos \alpha} - \cancel{\sin \alpha})}{\cancel{\cos \alpha} - \cancel{\sin \alpha}} \\
&= \sin \alpha + \cos \alpha
\end{aligned}$$

$$\begin{aligned}
\text{i} \quad & \frac{\sin \theta}{1 - \cos \theta} - \frac{\sin \theta}{1 + \cos \theta} \\
&= \frac{\sin \theta(1 + \cos \theta) - \sin \theta(1 - \cos \theta)}{(1 - \cos \theta)(1 + \cos \theta)} \\
&= \frac{\cancel{\sin \theta} + \sin \theta \cos \theta - \cancel{\sin \theta} + \sin \theta \cos \theta}{1 - \cos^2 \theta} \\
&= \frac{2 \sin \theta \cos \theta}{\sin^2 \theta} \\
&= \frac{2 \cos \theta}{\sin \theta} \\
&= 2 \cot \theta
\end{aligned}$$

$$\begin{aligned}
\text{h} \quad & \frac{\sin \theta}{1 + \cos \theta} + \frac{1 + \cos \theta}{\sin \theta} \\
&= \frac{\sin^2 \theta + (1 + \cos \theta)^2}{\sin \theta(1 + \cos \theta)} \\
&= \frac{\sin^2 \theta + 1 + 2 \cos \theta + \cos^2 \theta}{\sin \theta(1 + \cos \theta)} \\
&= \frac{1 + 1 + 2 \cos \theta}{\sin \theta(1 + \cos \theta)} \\
&= \frac{\cancel{2(1 + \cos \theta)}}{\cancel{\sin \theta(1 + \cos \theta)}} \\
&= \frac{2}{\sin \theta} \\
&= 2 \operatorname{cosec} \theta
\end{aligned}$$

$$\begin{aligned}
\text{j} \quad & \frac{1}{1 - \sin \theta} + \frac{1}{1 + \sin \theta} \\
&= \frac{1 + \sin \theta + 1 - \sin \theta}{(1 - \sin \theta)(1 + \sin \theta)} \\
&= \frac{2}{1 - \sin^2 \theta} \\
&= \frac{2}{\cos^2 \theta} \\
&= 2 \sec^2 \theta
\end{aligned}$$

$$\begin{aligned}
\text{4 a} \quad & 2 \cos^2 x = \sin x + 1 \\
& \therefore 2(1 - \sin^2 x) = \sin x + 1 \\
& \therefore 2 - 2 \sin^2 x = \sin x + 1 \\
& \therefore 2 \sin^2 x + \sin x - 1 = 0 \\
& \therefore (2 \sin x - 1)(\sin x + 1) = 0 \\
& \therefore 2 \sin x - 1 = 0 \quad \text{or} \quad \sin x + 1 = 0 \\
& \therefore \sin x = \frac{1}{2} \quad \text{or} \quad \sin x = -1
\end{aligned}$$

On  $0 \leq x \leq 2\pi$ , the angles with sine  $\frac{1}{2}$  are  $\frac{\pi}{6}$  and  $\frac{5\pi}{6}$ ,  
and the angle with sine  $-1$  is  $\frac{3\pi}{2}$ .

$\therefore$  the solutions are  $x = \frac{\pi}{6}, \frac{5\pi}{6}, \text{ or } \frac{3\pi}{2}$ .



**b**

$$\sin^2 x = 2 - \cos x$$

$$\therefore 1 - \cos^2 x = 2 - \cos x$$

$$\therefore \cos^2 x - \cos x = -1$$

$$\therefore \cos^2 x - \cos x + \left(-\frac{1}{2}\right)^2 = -1 + \left(-\frac{1}{2}\right)^2 \quad \{\text{completing the square}\}$$

$$\therefore \left(\cos x - \frac{1}{2}\right)^2 = -\frac{3}{4}$$

which has no real solutions as  $\left(\cos x - \frac{1}{2}\right)^2$  cannot be negative.

**c**

$$2\cos^2 x = 3\sin x$$

$$\therefore 2(1 - \sin^2 x) = 3\sin x$$

$$\therefore 2 - 2\sin^2 x = 3\sin x$$

$$\therefore 2\sin^2 x + 3\sin x - 2 = 0$$

$$\therefore (2\sin x - 1)(\sin x + 2) = 0$$

$$\therefore 2\sin x - 1 = 0 \quad \text{or} \quad \sin x + 2 = 0$$

$$\therefore \sin x = \frac{1}{2} \quad \text{or} \quad \sin x = -2$$

Now  $-1 \leq \sin x \leq 1$  for all real values of  $x$ , so  $\sin x = -2$  has no solutions.

On  $0 \leq x \leq 2\pi$ , the angles with sine  $\frac{1}{2}$  are  $\frac{\pi}{6}$  and  $\frac{5\pi}{6}$ .

$\therefore$  the solutions are  $x = \frac{\pi}{6}$  or  $\frac{5\pi}{6}$ .

**d**

$$2\tan^2 x + 3\sec^2 x = 7$$

$$\therefore 2\tan^2 x + 3(\tan^2 x + 1) = 7$$

$$\therefore 2\tan^2 x + 3\tan^2 x + 3 = 7$$

$$\therefore 5\tan^2 x = 4$$

$$\therefore \tan^2 x = \frac{4}{5}$$

$$\therefore \tan x = \pm \frac{2}{\sqrt{5}}$$

On  $0 \leq x \leq 2\pi$ , the angles with tangent  $\frac{2}{\sqrt{5}}$  are  $\approx 0.730$  and  $3.87$ ,

and the angles with tangent  $-\frac{2}{\sqrt{5}}$  are  $\approx 2.41$  and  $5.55$ .

$\therefore$  the solutions are  $x \approx 0.730, 2.41, 3.87$ , or  $5.55$ .

- 5 The domain is  $-\pi \leq x \leq \pi$   
 $\therefore -2\pi \leq 2x \leq 2\pi$

$$3 \sec 2x = \cot 2x + 3 \tan 2x$$

$$\therefore \frac{3}{\cos 2x} = \frac{\cos 2x}{\sin 2x} + \frac{3 \sin 2x}{\cos 2x}$$

$$\therefore 3 \sin 2x = \cos^2 2x + 3 \sin^2 2x \quad \{\text{multiplying both sides by } \cos 2x \sin 2x\}$$

$$\therefore 3 \sin 2x = \cos^2 2x + \sin^2 2x + 2 \sin^2 2x$$

$$\therefore 3 \sin 2x = 1 + 2 \sin^2 2x$$

$$\therefore 2 \sin^2 2x - 3 \sin 2x + 1 = 0$$

$$\therefore (2 \sin 2x - 1)(\sin 2x - 1) = 0$$

$$\therefore 2 \sin 2x - 1 = 0$$

$$\therefore \sin 2x = \frac{1}{2}$$

$$\therefore 2x = -\frac{11\pi}{6}, -\frac{7\pi}{6}, \frac{\pi}{6}, \text{ or } \frac{5\pi}{6}$$

$$\therefore x = -\frac{11\pi}{12}, -\frac{7\pi}{12}, \frac{\pi}{12}, \text{ or } \frac{5\pi}{12}$$

$$\text{or } \sin 2x - 1 = 0$$

$$\therefore \sin 2x = 1$$

$$\therefore 2x = -\frac{3\pi}{2} \text{ or } \frac{\pi}{2}$$

$$\therefore x = -\frac{3\pi}{4} \text{ or } \frac{\pi}{4}$$

However, for  $x = -\frac{3\pi}{4}$  or  $\frac{\pi}{4}$ ,  $\sec 2x$  and  $\tan 2x$  are undefined.

$\therefore$  the solutions are  $x = -\frac{11\pi}{12}, -\frac{7\pi}{12}, \frac{\pi}{12}, \text{ or } \frac{5\pi}{12}$ .

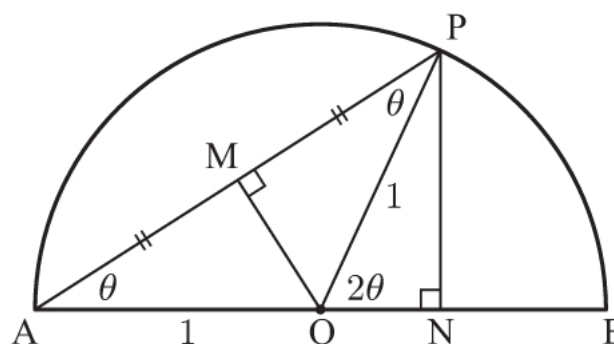
## INVESTIGATION 2

## DOUBLE ANGLE IDENTITIES

$\theta$	$\sin 2\theta$	$2 \sin \theta$	$2 \sin \theta \cos \theta$	$\cos 2\theta$	$2 \cos \theta$	$\cos^2 \theta - \sin^2 \theta$
0.631	0.953	1.180	0.953	0.304	1.615	0.304
$57.81^\circ$	0.902	1.693	0.902	-0.432	1.065	-0.432
-3.697	-0.896	1.055	-0.896	0.444	-1.699	0.444
1.234	0.624	1.888	0.624	-0.782	0.661	-0.782
2.236	-0.971	1.574	-0.971	-0.238	-1.234	-0.238

- 2 We can see that  $\sin 2\theta = 2 \sin \theta \cos \theta$  and  $\cos 2\theta = \cos^2 \theta - \sin^2 \theta$ .

- 3 a i  $\sin \theta = \frac{OM}{1}$   
 $= OM$   
 $\therefore OM = \sin \theta$
- ii  $\cos \theta = \frac{AM}{1}$   
 $= AM$   
 $\therefore AM = \cos \theta$
- iii  $\cos 2\theta = \frac{ON}{1}$   
 $= ON$   
 $\therefore ON = \cos 2\theta$
- iv  $\sin 2\theta = \frac{PN}{1}$   
 $= PN$   
 $\therefore PN = \sin 2\theta$



$$\begin{aligned}
 \text{b i In } \triangle ANP, \quad \sin \theta &= \frac{PN}{AP} \\
 &= \frac{PN}{AM + MP} \\
 &= \frac{\sin 2\theta}{\cos \theta + \cos \theta} \\
 \therefore \sin \theta &= \frac{\sin 2\theta}{2 \cos \theta} \\
 \therefore \cos \theta &= \frac{\sin 2\theta}{2 \sin \theta}
 \end{aligned}$$

$$\begin{aligned}
 \text{ii In } \triangle ANP, \quad \cos \theta &= \frac{AN}{AP} \\
 &= \frac{AO + ON}{AM + MP} \\
 &= \frac{1 + \cos 2\theta}{\cos \theta + \cos \theta} \\
 \therefore \cos \theta &= \frac{1 + \cos 2\theta}{2 \cos \theta}
 \end{aligned}$$

$$\begin{aligned}
 \text{c i From b i,} \quad \cos \theta &= \frac{\sin 2\theta}{2 \sin \theta} \\
 \therefore \sin 2\theta &= 2 \sin \theta \cos \theta
 \end{aligned}$$

$$\begin{aligned}
 \text{ii From b ii,} \quad \cos \theta &= \frac{1 + \cos 2\theta}{2 \cos \theta} \\
 \therefore 2 \cos^2 \theta &= 1 + \cos 2\theta \\
 \therefore \cos 2\theta &= 2 \cos^2 \theta - 1
 \end{aligned}$$

d Since  $\theta$  is the size of the base angles of an isosceles triangle, we have proven the identities for  $0 < \theta < \frac{\pi}{2}$ .

## EXERCISE 1D

$$\begin{aligned}
 \text{1 a If } \theta = 30^\circ, \quad \sin 2\theta &= \sin 60^\circ = \frac{\sqrt{3}}{2} \\
 \text{and } 2 \sin \theta \cos \theta &= 2 \times \sin 30^\circ \times \cos 30^\circ \\
 &= 2 \times \frac{1}{2} \times \frac{\sqrt{3}}{2} \\
 &= \frac{\sqrt{3}}{2}
 \end{aligned}$$

$$\therefore \sin 2\theta = 2 \sin \theta \cos \theta \quad \text{for } \theta = 30^\circ.$$

$$\begin{aligned}
 \text{b If } \theta = 30^\circ, \quad \cos 2\theta &= \cos 60^\circ = \frac{1}{2} \\
 \text{and } \cos^2 \theta - \sin^2 \theta &= \cos^2 30^\circ - \sin^2 30^\circ \\
 &= \left(\frac{\sqrt{3}}{2}\right)^2 - \left(\frac{1}{2}\right)^2 \\
 &= \frac{3}{4} - \frac{1}{4} \\
 &= \frac{1}{2}
 \end{aligned}$$

$$\therefore \cos 2\theta = \cos^2 \theta - \sin^2 \theta \quad \text{for } \theta = 30^\circ.$$



**c** If  $\theta = 30^\circ$ ,  $\tan 2\theta = \tan 60^\circ = \sqrt{3}$

$$\begin{aligned}\text{and } \frac{2 \tan \theta}{1 - \tan^2 \theta} &= \frac{2 \tan 30^\circ}{1 - \tan^2 30^\circ} \\ &= \frac{2 \times \frac{1}{\sqrt{3}}}{1 - \left(\frac{1}{\sqrt{3}}\right)^2} \\ &= \frac{\frac{2}{\sqrt{3}}}{1 - \frac{1}{3}} \\ &= \frac{\cancel{2}}{\sqrt{3}} \times \frac{3}{\cancel{2}} \times \frac{\sqrt{3}}{\sqrt{3}} \\ &= \sqrt{3}\end{aligned}$$

$$\therefore \tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta} \quad \text{for } \theta = 30^\circ.$$

**2**  $\sin \theta = \frac{4}{5}$  and  $\cos \theta = \frac{3}{5}$

**a**  $\sin 2\theta = 2 \sin \theta \cos \theta$   
 $= 2\left(\frac{4}{5}\right)\left(\frac{3}{5}\right)$   
 $= \frac{24}{25}$

**b**  $\cos 2\theta = \cos^2 \theta - \sin^2 \theta$   
 $= \left(\frac{3}{5}\right)^2 - \left(\frac{4}{5}\right)^2$   
 $= \frac{9}{25} - \frac{16}{25}$   
 $= -\frac{7}{25}$

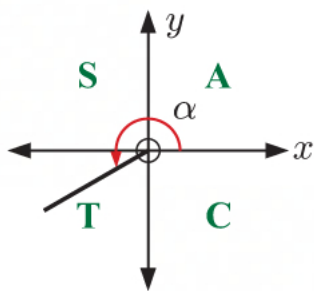
**c**  $\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta}$   
 $= \frac{\frac{24}{25}}{-\frac{7}{25}}$   
 {using **a** and **b**}  
 $= -\frac{24}{7}$

**3 a** If  $\cos A = \frac{1}{3}$ ,  $\cos 2A = 2 \cos^2 A - 1$   
 $= 2\left(\frac{1}{3}\right)^2 - 1$   
 $= 2 \times \frac{1}{9} - 1$   
 $= \frac{2}{9} - 1$   
 $= -\frac{7}{9}$

**b** If  $\sin \phi = -\frac{2}{3}$ ,  $\cos 2\phi = 1 - 2 \sin^2 \phi$   
 $= 1 - 2\left(-\frac{2}{3}\right)^2$   
 $= 1 - 2\left(\frac{4}{9}\right)$   
 $= 1 - \frac{8}{9}$   
 $= \frac{1}{9}$

**4**  $\sin \alpha = -\frac{2}{3}$  and  $\pi < \alpha < \frac{3\pi}{2}$

**a**  $\alpha$  is in quadrant 3, so  $\cos \alpha$  is negative.



Now  $\cos^2 \alpha + \sin^2 \alpha = 1$

$$\therefore \cos^2 \alpha + \frac{4}{9} = 1$$

$$\therefore \cos^2 \alpha = \frac{5}{9}$$

$$\therefore \cos \alpha = -\frac{\sqrt{5}}{3} \quad \{\cos \alpha < 0\}$$

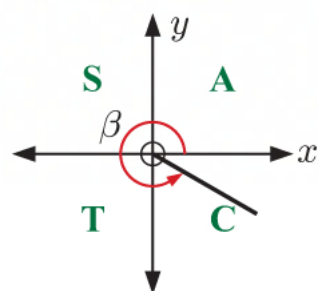
**b** Using the double angle identity,  $\sin 2\alpha = 2 \sin \alpha \cos \alpha$

$$\begin{aligned}&= 2\left(-\frac{2}{3}\right)\left(-\frac{\sqrt{5}}{3}\right) \quad \{\text{using **a**}\} \\ &= \frac{4\sqrt{5}}{9}\end{aligned}$$

$$\begin{aligned}
 \text{c } \tan 2\alpha &= \frac{\sin 2\alpha}{\cos 2\alpha} \\
 &= \frac{\sin 2\alpha}{2 \cos^2 \alpha - 1} \\
 &= \frac{\frac{4\sqrt{5}}{9}}{2 \left(-\frac{\sqrt{5}}{3}\right)^2 - 1} \quad \{\text{using a and b}\} \\
 &= \frac{\frac{4\sqrt{5}}{9}}{\frac{10}{9} - 1} \\
 &= \frac{\frac{4\sqrt{5}}{9}}{\frac{1}{9}} \\
 &= 4\sqrt{5}
 \end{aligned}$$

5  $\cos \beta = \frac{2}{5}$  and  $270^\circ < \beta < 360^\circ$

a  $\beta$  is in quadrant 4, so  $\sin \beta$  is negative.



Now  $\cos^2 \beta + \sin^2 \beta = 1$

$$\therefore \frac{4}{25} + \sin^2 \beta = 1$$

$$\therefore \sin^2 \beta = \frac{21}{25}$$

$$\therefore \sin \beta = -\frac{\sqrt{21}}{5} \quad \{\sin \beta < 0\}$$

b Using the double angle identity,  $\sin 2\beta = 2 \sin \beta \cos \beta$

$$= 2 \left(-\frac{\sqrt{21}}{5}\right) \left(\frac{2}{5}\right) \quad \{\text{using a}\}$$

$$= -\frac{4\sqrt{21}}{25}$$

$$\begin{aligned}
 \text{c } \tan 2\beta &= \frac{\sin 2\beta}{\cos 2\beta} \\
 &= \frac{\sin 2\beta}{2 \cos^2 \beta - 1} \\
 &= \frac{-\frac{4\sqrt{21}}{25}}{2 \left(\frac{2}{5}\right)^2 - 1} \quad \{\text{using b}\} \\
 &= \frac{-\frac{4\sqrt{21}}{25}}{\frac{8}{25} - 1} \\
 &= \frac{-\frac{4\sqrt{21}}{25}}{-\frac{17}{25}} \\
 &= \frac{4\sqrt{21}}{17}
 \end{aligned}$$

**6**  $\alpha$  is acute and  $\cos 2\alpha = -\frac{7}{9}$   $\therefore$   $\cos \alpha$  and  $\sin \alpha$  are positive

**a**  $\cos 2\alpha = 2\cos^2 \alpha - 1$

$$\therefore -\frac{7}{9} = 2\cos^2 \alpha - 1$$

$$\therefore 2\cos^2 \alpha = \frac{2}{9}$$

$$\therefore \cos^2 \alpha = \frac{1}{9}$$

$$\therefore \cos \alpha = \frac{1}{3} \quad \{\text{since } \cos \alpha > 0\}$$

**b**  $\sin^2 \alpha = 1 - \cos^2 \alpha$

$$\therefore \sin \alpha = \sqrt{1 - \cos^2 \alpha} \quad \{\text{since } \sin \alpha > 0\}$$

$$= \sqrt{1 - \frac{1}{9}} \quad \{\text{using a}\}$$

$$= \sqrt{\frac{8}{9}}$$

$$= \frac{2\sqrt{2}}{3}$$

**7**  $\theta$  is obtuse and  $\cos 2\theta = -\frac{1}{3}$   $\therefore$   $\cos \theta$  is negative and  $\sin \theta$  is positive

**a**  $\cos 2\theta = 2\cos^2 \theta - 1$

$$\therefore -\frac{1}{3} = 2\cos^2 \theta - 1$$

$$\therefore 2\cos^2 \theta = \frac{2}{3}$$

$$\therefore \cos^2 \theta = \frac{1}{3}$$

$$\therefore \cos \theta = -\frac{1}{\sqrt{3}} \quad \{\text{since } \cos \theta < 0\}$$

**b**  $\sin^2 \theta = 1 - \cos^2 \theta$

$$\therefore \sin \theta = \sqrt{1 - \cos^2 \theta} \quad \{\text{since } \sin \theta > 0\}$$

$$= \sqrt{1 - \frac{1}{3}} \quad \{\text{using a}\}$$

$$= \sqrt{\frac{2}{3}} = \frac{\sqrt{2}}{\sqrt{3}}$$

**8 a**  $\tan 2A = \frac{21}{20}$

$$\therefore \frac{2\tan A}{1 - \tan^2 A} = \frac{21}{20}$$

$$\therefore 40\tan A = 21 - 21\tan^2 A$$

$$\therefore 21\tan^2 A + 40\tan A - 21 = 0$$

$$\therefore (7\tan A - 3)(3\tan A + 7) = 0$$

$$\therefore \tan A = \frac{3}{7} \text{ or } -\frac{7}{3}$$

But  $A$  is obtuse  $\therefore \tan A < 0$

$$\therefore \tan A = -\frac{7}{3}$$

**b**  $\tan 2A = -\frac{12}{5}$

$$\therefore \frac{2\tan A}{1 - \tan^2 A} = -\frac{12}{5}$$

$$\therefore 10\tan A = -12 + 12\tan^2 A$$

$$\therefore 12\tan^2 A - 10\tan A - 12 = 0$$

$$\therefore 2(6\tan^2 A - 5\tan A - 6) = 0$$

$$\therefore 2(3\tan A + 2)(2\tan A - 3) = 0$$

$$\therefore \tan A = -\frac{2}{3} \text{ or } \frac{3}{2}$$

But  $A$  is acute  $\therefore \tan A > 0$

$$\therefore \tan A = \frac{3}{2}$$



9

$$\begin{aligned}
 \tan \frac{\pi}{4} &= 1 \\
 \therefore \tan \left( 2 \times \frac{\pi}{8} \right) &= 1 \\
 \therefore \frac{2 \tan \frac{\pi}{8}}{1 - \tan^2 \left( \frac{\pi}{8} \right)} &= 1 \\
 \therefore 2 \tan \frac{\pi}{8} &= 1 - \tan^2 \left( \frac{\pi}{8} \right) \\
 \therefore \tan^2 \left( \frac{\pi}{8} \right) + 2 \tan \frac{\pi}{8} - 1 &= 0 \\
 \therefore \tan \frac{\pi}{8} &= \frac{-2 \pm \sqrt{2^2 - 4(1)(-1)}}{2} \\
 &= \frac{-2 \pm 2\sqrt{2}}{2} \\
 &= -1 \pm \sqrt{2}
 \end{aligned}$$

But  $\frac{\pi}{8}$  is in quadrant 1  $\therefore \tan \frac{\pi}{8} > 0$

$$\therefore \tan \frac{\pi}{8} = \sqrt{2} - 1$$

**10 a**  $2 \sin \alpha \cos \alpha$   
 $= \sin 2\alpha$

**b**  $4 \cos \alpha \sin \alpha$   
 $= 2(2 \sin \alpha \cos \alpha)$   
 $= 2 \sin 2\alpha$

**c**  $\sin \alpha \cos \alpha$   
 $= \frac{1}{2}(2 \sin \alpha \cos \alpha)$   
 $= \frac{1}{2} \sin 2\alpha$

**d**  $2 \cos^2 \beta - 1$   
 $= \cos 2\beta$

**e**  $1 - 2 \cos^2 \phi$   
 $= -(2 \cos^2 \phi - 1)$   
 $= -\cos 2\phi$

**f**  $1 - 2 \sin^2 N$   
 $= \cos 2N$

**g**  $2 \sin^2 M - 1$   
 $= -(1 - 2 \sin^2 M)$   
 $= -\cos 2M$

**h**  $\cos^2 \alpha - \sin^2 \alpha$   
 $= \cos 2\alpha$

**i**  $\sin^2 \alpha - \cos^2 \alpha$   
 $= -(\cos^2 \alpha - \sin^2 \alpha)$   
 $= -\cos 2\alpha$

**j**  $2 \sin 2A \cos 2A$   
 $= \sin[2(2A)]$   
 $= \sin 4A$

**k**  $2 \cos 3\alpha \sin 3\alpha$   
 $= \sin[2(3\alpha)]$   
 $= \sin 6\alpha$

**l**  $2 \cos^2 4\theta - 1$   
 $= \cos[2(4\theta)]$   
 $= \cos 8\theta$

**m**  $1 - 2 \cos^2 3\beta$   
 $= -(2 \cos^2 3\beta - 1)$   
 $= -\cos[2(3\beta)]$   
 $= -\cos 6\beta$

**n**  $1 - 2 \sin^2 5\alpha$   
 $= \cos[2(5\alpha)]$   
 $= \cos 10\alpha$

**o**  $2 \sin^2 3D - 1$   
 $= -(1 - 2 \sin^2 3D)$   
 $= -\cos[2(3D)]$   
 $= -\cos 6D$

**p**  $\cos^2 2A - \sin^2 2A$   
 $= \cos[2(2A)]$   
 $= \cos 4A$

**q**  $\cos^2 \left( \frac{\alpha}{2} \right) - \sin^2 \left( \frac{\alpha}{2} \right)$   
 $= \cos[2 \left( \frac{\alpha}{2} \right)]$   
 $= \cos \alpha$

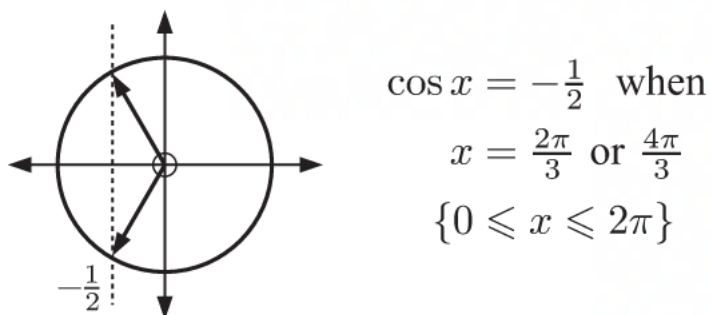
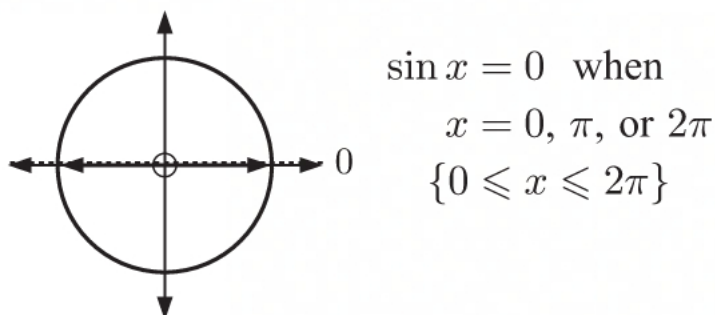
**r**  $2 \sin^2 3P - 2 \cos^2 3P$   
 $= -2(\cos^2 3P - \sin^2 3P)$   
 $= -2 \cos[2(3P)]$   
 $= -2 \cos 6P$

$$\begin{aligned}
 11 \quad & \left[ \cos \frac{\pi}{12} + \sin \frac{\pi}{12} \right]^2 \\
 &= \cos^2 \left( \frac{\pi}{12} \right) + 2 \cos \frac{\pi}{12} \sin \frac{\pi}{12} + \sin^2 \left( \frac{\pi}{12} \right) \\
 &= 1 + 2 \cos \frac{\pi}{12} \sin \frac{\pi}{12} \\
 &= 1 + \sin \frac{\pi}{6} \quad \{2 \cos A \sin A = \sin 2A\} \\
 &= 1 + \frac{1}{2} \\
 &= \frac{3}{2}
 \end{aligned}$$

$$\begin{aligned}
 12 \quad a \quad & (\sin \theta + \cos \theta)^2 \\
 &= \sin^2 \theta + 2 \sin \theta \cos \theta + \cos^2 \theta \\
 &= \sin^2 \theta + \cos^2 \theta + 2 \sin \theta \cos \theta \\
 &= 1 + \sin 2\theta
 \end{aligned}$$

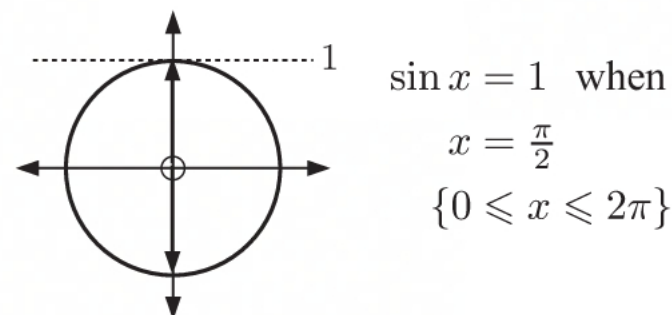
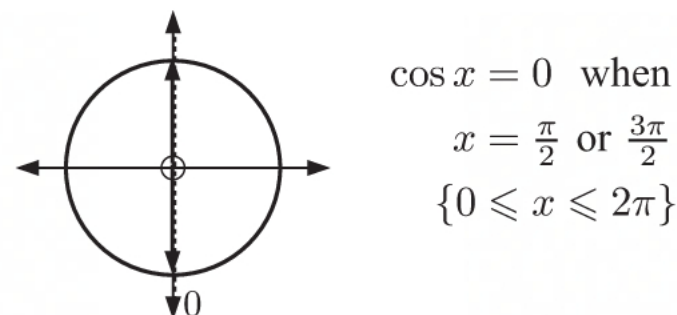
$$\begin{aligned}
 b \quad & \cos^4 \theta - \sin^4 \theta \\
 &= (\cos^2 \theta + \sin^2 \theta)(\cos^2 \theta - \sin^2 \theta) \\
 &= 1 \times \cos 2\theta \\
 &= \cos 2\theta
 \end{aligned}$$

$$\begin{aligned}
 13 \quad a \quad & \sin 2x + \sin x = 0 \\
 \therefore & 2 \sin x \cos x + \sin x = 0 \\
 \therefore & \sin x(2 \cos x + 1) = 0 \\
 \therefore & \sin x = 0 \quad \text{or} \quad \cos x = -\frac{1}{2}
 \end{aligned}$$



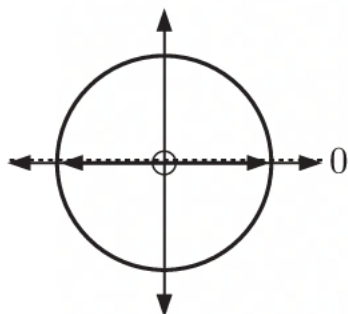
$$\therefore x = 0, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \text{ or } 2\pi$$

$$\begin{aligned}
 b \quad & \sin 2x - 2 \cos x = 0 \\
 \therefore & 2 \sin x \cos x - 2 \cos x = 0 \\
 \therefore & 2 \cos x(\sin x - 1) = 0 \\
 \therefore & \cos x = 0 \quad \text{or} \quad \sin x = 1
 \end{aligned}$$



$$\therefore x = \frac{\pi}{2} \text{ or } \frac{3\pi}{2}$$

$$\begin{aligned}
 c \quad & \sin 2x + 3 \sin x = 0 \\
 \therefore & 2 \sin x \cos x + 3 \sin x = 0 \\
 \therefore & \sin x(2 \cos x + 3) = 0 \\
 \therefore & \sin x = 0 \quad \{-1 \leq \cos x \leq 1\}
 \end{aligned}$$



$$\begin{aligned}
 14 \quad a \quad i \quad & \frac{1}{2} - \frac{1}{2} \cos 2\theta \\
 &= \frac{1}{2} - \frac{1}{2}(1 - 2\sin^2 \theta) \\
 &= \frac{1}{2} - \frac{1}{2} + \sin^2 \theta \\
 &= \sin^2 \theta
 \end{aligned}$$

$$\begin{aligned}
 b \quad i \quad & \text{Substituting } \theta = \frac{\theta}{2} \text{ in } a \quad i \text{ gives} \\
 & \frac{1}{2} - \frac{1}{2} \cos \theta = \sin^2 \left( \frac{\theta}{2} \right).
 \end{aligned}$$

$$\begin{aligned}
 ii \quad & \frac{1}{2} + \frac{1}{2} \cos 2\theta \\
 &= \frac{1}{2} + \frac{1}{2}(2\cos^2 \theta - 1) \\
 &= \frac{1}{2} + \cos^2 \theta - \frac{1}{2} \\
 &= \cos^2 \theta
 \end{aligned}$$

$$\begin{aligned}
 ii \quad & \text{Substituting } \theta = \frac{\theta}{2} \text{ in } a \quad ii \text{ gives} \\
 & \frac{1}{2} + \frac{1}{2} \cos \theta = \cos^2 \left( \frac{\theta}{2} \right).
 \end{aligned}$$

$$\begin{aligned}
 15 \quad & \sin \theta \cos \theta = \frac{1}{4}, \quad -\pi \leq \theta \leq \pi \\
 \therefore & \frac{1}{2}(2\sin \theta \cos \theta) = \frac{1}{4} \\
 \therefore & \frac{1}{2} \sin 2\theta = \frac{1}{4} \\
 \therefore & \sin 2\theta = \frac{1}{2}
 \end{aligned}$$

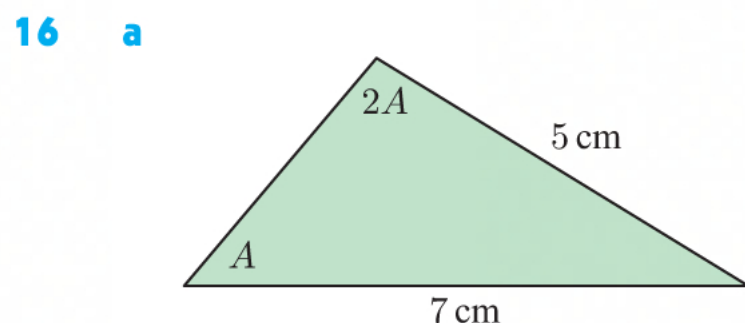
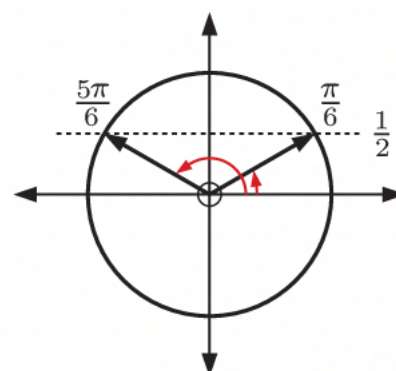
There are two points on the unit circle with sine  $\frac{1}{2}$ .  
They correspond to angles  $\frac{\pi}{6}$  and  $\frac{5\pi}{6}$ .

Since  $-\pi \leq \theta \leq \pi$

$$\therefore -2\pi \leq 2\theta \leq 2\pi$$

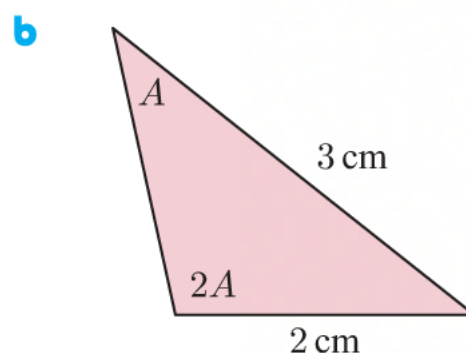
$$\text{So, } 2\theta = -\frac{11\pi}{6}, -\frac{7\pi}{6}, \frac{\pi}{6}, \text{ or } \frac{5\pi}{6}$$

$$\therefore \theta = -\frac{11\pi}{12}, -\frac{7\pi}{12}, \frac{\pi}{12}, \text{ or } \frac{5\pi}{12}$$



Using the sine rule,

$$\begin{aligned}
 & \frac{\sin 2A}{7} = \frac{\sin A}{5} \\
 \therefore & \frac{2\sin A \cos A}{7} = \frac{\sin A}{5} \\
 \therefore & \cos A = \frac{7}{10}
 \end{aligned}$$



Using the sine rule,

$$\begin{aligned}
 & \frac{\sin 2A}{3} = \frac{\sin A}{2} \\
 \therefore & \frac{2\sin A \cos A}{3} = \frac{\sin A}{2} \\
 \therefore & \cos A = \frac{3}{4}
 \end{aligned}$$

$$\begin{aligned}
 17 \quad a \quad & \frac{\sin 2\theta}{1 - \cos 2\theta} \\
 &= \frac{2\sin \theta \cos \theta}{1 - (1 - 2\sin^2 \theta)} \\
 &= \frac{2\sin \theta \cos \theta}{1 - 1 + 2\sin^2 \theta} \\
 &= \frac{2\sin \theta \cos \theta}{2\sin^2 \theta} \\
 &= \frac{\cos \theta}{\sin \theta} \\
 &= \cot \theta
 \end{aligned}$$

$$\begin{aligned}
 b \quad & \frac{\sin \theta + \sin 2\theta}{1 + \cos \theta + \cos 2\theta} \\
 &= \frac{\sin \theta + 2\sin \theta \cos \theta}{1 + \cos \theta + 2\cos^2 \theta - 1} \\
 &= \frac{\sin \theta(1 + 2\cos \theta)}{\cos \theta(1 + 2\cos \theta)} \\
 &= \frac{\sin \theta}{\cos \theta} \\
 &= \tan \theta
 \end{aligned}$$

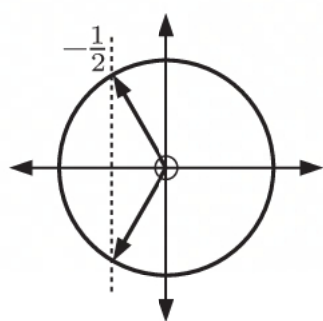


$$\begin{aligned}
 & \frac{\sin 2\theta}{1 + \cos 2\theta} \\
 &= \frac{2 \sin \theta \cos \theta}{\cancel{1} + (2 \cos^2 \theta - \cancel{1})} \\
 &= \frac{\cancel{2} \sin \theta \cos \theta}{\cancel{2} \cos^2 \theta} \\
 &= \frac{\sin \theta}{\cos \theta} \\
 &= \tan \theta
 \end{aligned}$$

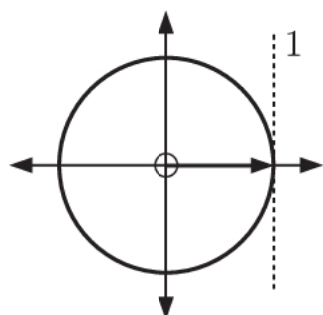
$$\begin{aligned}
 & \frac{\sin 2\theta}{\sin \theta} - \frac{\cos 2\theta}{\cos \theta} \\
 &= \frac{\cancel{2} \sin \theta \cos \theta}{\cancel{\sin \theta}} - \frac{2 \cos^2 \theta - 1}{\cos \theta} \\
 &= 2 \cos \theta - \frac{2 \cos^2 \theta - 1}{\cos \theta} \\
 &= \frac{\cancel{2 \cos^2 \theta} - \cancel{2 \cos^2 \theta} + 1}{\cos \theta} \\
 &= \frac{1}{\cos \theta} \\
 &= \sec \theta
 \end{aligned}$$

$$\begin{aligned}
 & \tan \theta + \cot 2\theta \\
 &= \frac{\sin \theta}{\cos \theta} + \frac{\cos 2\theta}{\sin 2\theta} \\
 &= \frac{\sin \theta}{\cos \theta} + \frac{1 - 2 \sin^2 \theta}{2 \sin \theta \cos \theta} \\
 &= \frac{\cancel{2 \sin^2 \theta} + 1 - \cancel{2 \sin^2 \theta}}{2 \sin \theta \cos \theta} \\
 &= \frac{1}{2 \sin \theta \cos \theta} \\
 &= \frac{1}{\sin 2\theta} \\
 &= \operatorname{cosec} 2\theta
 \end{aligned}$$

$$\begin{aligned}
 18 \quad & \text{a} \quad \cos 2x - \cos x = 0 \\
 & \therefore (2 \cos^2 x - 1) - \cos x = 0 \\
 & \therefore 2 \cos^2 x - \cos x - 1 = 0 \\
 & \therefore (2 \cos x + 1)(\cos x - 1) = 0 \\
 & \therefore \cos x = -\frac{1}{2} \text{ or } 1
 \end{aligned}$$



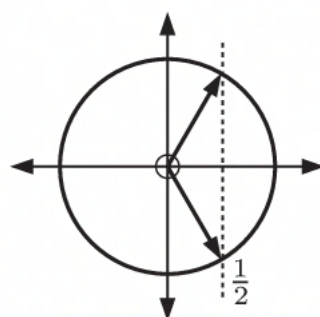
$$\begin{aligned}
 & \cos x = -\frac{1}{2} \text{ when} \\
 & \quad x = \frac{2\pi}{3} \text{ or } \frac{4\pi}{3} \\
 & \quad \{0 \leq x \leq 2\pi\}
 \end{aligned}$$



$$\begin{aligned}
 & \cos x = 1 \text{ when} \\
 & \quad x = 0 \text{ or } 2\pi \\
 & \quad \{0 \leq x \leq 2\pi\}
 \end{aligned}$$

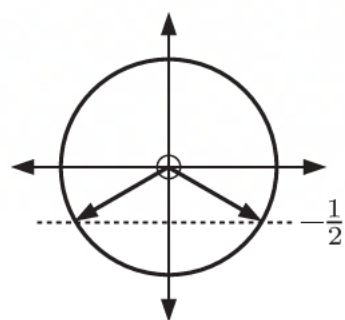
$$\therefore x = 0, \frac{2\pi}{3}, \frac{4\pi}{3}, \text{ or } 2\pi$$

$$\begin{aligned}
 & \text{b} \quad \cos 2x + 3 \cos x = 1 \\
 & \therefore (2 \cos^2 x - 1) + 3 \cos x = 1 \\
 & \therefore 2 \cos^2 x + 3 \cos x - 2 = 0 \\
 & \therefore (2 \cos x - 1)(\cos x + 2) = 0 \\
 & \therefore \cos x = \frac{1}{2} \\
 & \quad \{-1 \leq \cos x \leq 1\}
 \end{aligned}$$

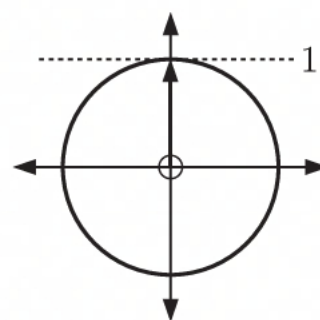


$$\therefore x = \frac{\pi}{3} \text{ or } \frac{5\pi}{3}$$

$$\begin{aligned}
 \text{c} \quad & \cos 2x + \sin x = 0 \\
 \therefore & (1 - 2\sin^2 x) + \sin x = 0 \\
 \therefore & -2\sin^2 x + \sin x + 1 = 0 \\
 \therefore & 2\sin^2 x - \sin x - 1 = 0 \\
 \therefore & (2\sin x + 1)(\sin x - 1) = 0 \\
 \therefore & \sin x = -\frac{1}{2} \text{ or } 1
 \end{aligned}$$



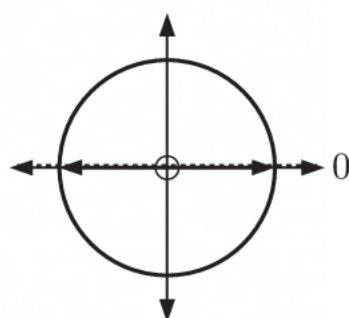
$$\begin{aligned}
 \sin x = -\frac{1}{2} \text{ when} \\
 x = \frac{7\pi}{6} \text{ or } \frac{11\pi}{6} \\
 \{0 \leq x \leq 2\pi\}
 \end{aligned}$$



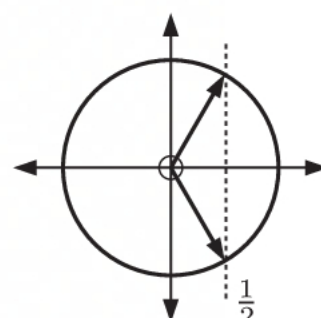
$$\begin{aligned}
 \sin x = 1 \text{ when} \\
 x = \frac{\pi}{2} \\
 \{0 \leq x \leq 2\pi\}
 \end{aligned}$$

$$\therefore x = \frac{\pi}{2}, \frac{7\pi}{6}, \text{ or } \frac{11\pi}{6}$$

$$\begin{aligned}
 \text{d} \quad & \sin 4x = \sin 2x \\
 \therefore & 2\sin 2x \cos 2x = \sin 2x \\
 \therefore & 2\sin 2x \cos 2x - \sin 2x = 0 \\
 \therefore & \sin 2x(2\cos 2x - 1) = 0 \\
 \therefore & \sin 2x = 0 \text{ or } \cos 2x = \frac{1}{2}
 \end{aligned}$$



$$\begin{aligned}
 \sin 2x = 0 \text{ when} \\
 2x = 0, \pi, 2\pi, 3\pi, \\
 \text{or } 4\pi \\
 \{0 \leq 2x \leq 4\pi\}
 \end{aligned}$$

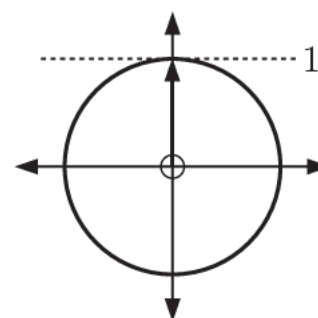


$$\begin{aligned}
 \cos 2x = \frac{1}{2} \text{ when} \\
 2x = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \\
 \text{or } \frac{11\pi}{3} \\
 \{0 \leq 2x \leq 4\pi\}
 \end{aligned}$$

$$\therefore 2x = 0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}, 2\pi, \frac{7\pi}{3}, 3\pi, \frac{11\pi}{3}, \text{ or } 4\pi$$

$$\therefore x = 0, \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \pi, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}, \text{ or } 2\pi$$

$$\begin{aligned}
 \text{e} \quad & 2\cos 2x + 9\sin x = 7 \\
 \therefore & 2(1 - 2\sin^2 x) + 9\sin x = 7 \\
 \therefore & 2 - 4\sin^2 x + 9\sin x = 7 \\
 \therefore & 4\sin^2 x - 9\sin x + 5 = 0 \\
 \therefore & (4\sin x - 5)(\sin x - 1) = 0 \\
 \therefore & \sin x = 1 \quad \{-1 \leq \sin x \leq 1\} \\
 \therefore & x = \frac{\pi}{2}
 \end{aligned}$$



**f**  $\sin x + \cos x = \sqrt{2}$

Squaring both sides gives:

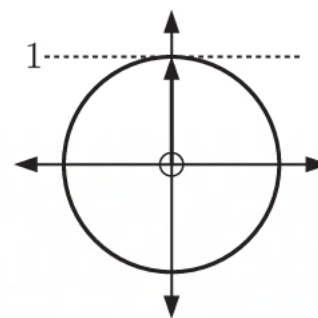
$$\sin^2 x + 2 \sin x \cos x + \cos^2 x = 2$$

$$\therefore \sin 2x + 1 = 2$$

$$\therefore \sin 2x = 1$$

$$\therefore 2x = \frac{\pi}{2} \text{ or } \frac{5\pi}{2} \quad \{0 \leq 2x \leq 4\pi\}$$

$$\therefore x = \frac{\pi}{4} \text{ or } \frac{5\pi}{4}$$



Since we squared the original equation, we must check our answers.

$$\sin \frac{\pi}{4} + \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2} \quad \checkmark$$

$$\sin \frac{5\pi}{4} + \cos \frac{5\pi}{4} = -\frac{1}{\sqrt{2}} + \left(-\frac{1}{\sqrt{2}}\right) = -\frac{2}{\sqrt{2}} = -\sqrt{2} \quad \times$$

$$\therefore x = \frac{\pi}{4} \text{ is the only solution}$$

**g**

$$2 \cos^2 x = 3 \sin x$$

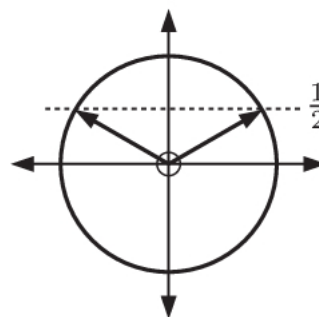
$$\therefore 2(1 - \sin^2 x) = 3 \sin x$$

$$\therefore 2 \sin^2 x + 3 \sin x - 2 = 0$$

$$\therefore (2 \sin x - 1)(\sin x + 2) = 0$$

$$\therefore \sin x = \frac{1}{2} \quad \{-1 \leq \sin x \leq 1\}$$

$$\therefore x = \frac{\pi}{6} \text{ or } \frac{5\pi}{6} \quad \{0 \leq x \leq 2\pi\}$$



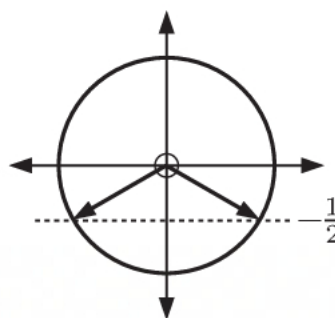
**h**

$$\sin 2x + \cos x - 2 \sin x - 1 = 0$$

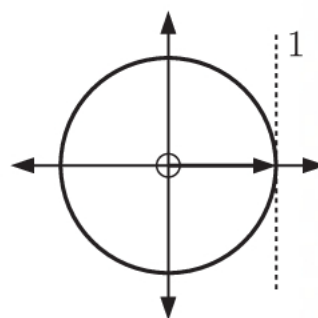
$$\therefore 2 \sin x \cos x + \cos x - 2 \sin x - 1 = 0$$

$$\therefore (2 \sin x + 1)(\cos x - 1) = 0$$

$$\therefore \sin x = -\frac{1}{2} \quad \text{or} \quad \cos x = 1$$



$$\begin{aligned} \sin x = -\frac{1}{2} \text{ when} \\ x = \frac{7\pi}{6} \text{ or } \frac{11\pi}{6} \\ \{0 \leq x \leq 2\pi\} \end{aligned}$$



$$\begin{aligned} \cos x = 1 \text{ when} \\ x = 0 \text{ or } 2\pi \\ \{0 \leq x \leq 2\pi\} \end{aligned}$$

$$\therefore x = 0, \frac{7\pi}{6}, \frac{11\pi}{6}, \text{ or } 2\pi$$

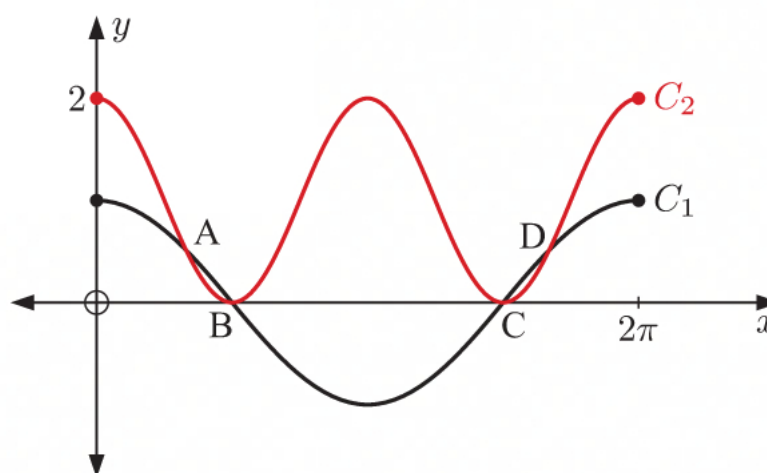
**19 a**  $-1 \leq \cos 2x \leq 1$

$$\therefore 0 \leq \cos 2x + 1 \leq 2$$

So, the graph of  $y = \cos 2x + 1$  lies entirely on or above the  $x$ -axis.

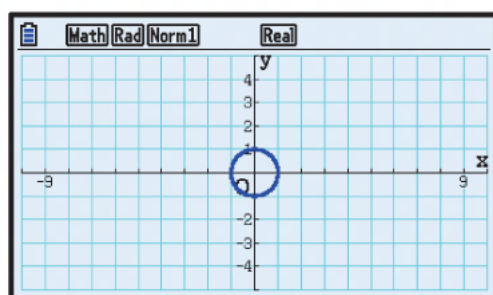
$$\therefore C_2 \text{ is } y = \cos 2x + 1$$

$$\text{and } C_1 \text{ is } y = \cos x$$





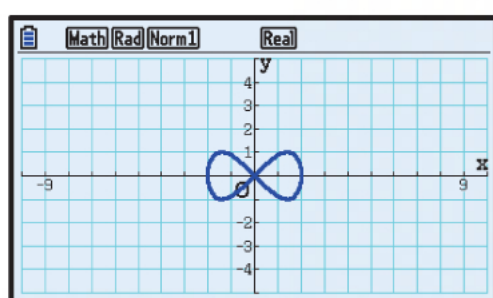
- b** We solve  $\cos x = \cos 2x + 1$ , for  $0 \leq x \leq 2\pi$   
 $\therefore \cos x = 2\cos^2 x - 1 + 1$   
 $\therefore 2\cos^2 x - \cos x = 0$   
 $\therefore \cos x(2\cos x - 1) = 0$   
 $\therefore \cos x = 0$  or  $\cos x = \frac{1}{2}$   
 $\therefore x = \frac{\pi}{2}, \frac{3\pi}{2}$  or  $x = \frac{\pi}{3}, \frac{5\pi}{3}$   
 $\therefore$  A is  $(\frac{\pi}{3}, \frac{1}{2})$ , B is  $(\frac{\pi}{2}, 0)$ , C is  $(\frac{3\pi}{2}, 0)$ , and D is  $(\frac{5\pi}{3}, \frac{1}{2})$ .

**ACTIVITY 2****PARAMETRIC EQUATIONS****1 a**

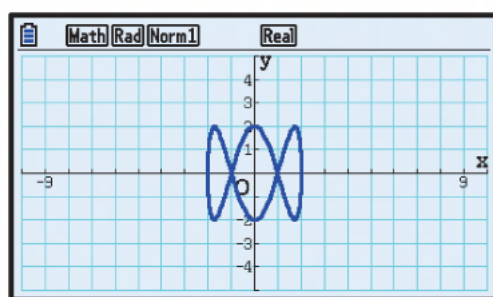
$$\{(x, y) \mid x = \cos t, y = \sin t, 0 \leq t \leq 2\pi\}$$

- b** The graph is not a function since the graph does not pass the vertical line test.

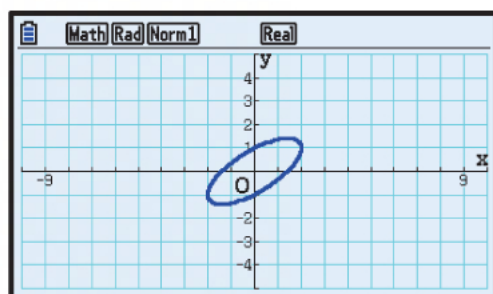
**c**  $x^2 + y^2 = \cos^2 t + \sin^2 t$   
 $\therefore x^2 + y^2 = 1$

**2 a**

$$\{(x, y) \mid x = 2 \cos t, y = \sin 2t, 0 \leq t \leq 2\pi\}$$

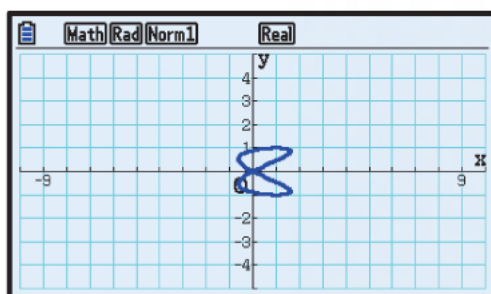
**b**

$$\{(x, y) \mid x = 2 \cos t, y = 2 \sin 3t, 0 \leq t \leq 2\pi\}$$

**c**

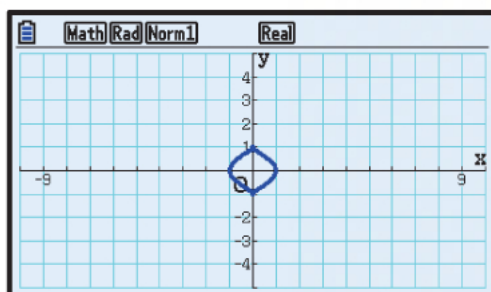
$$\{(x, y) \mid x = 2 \cos t, y = \cos t - \sin t, 0 \leq t \leq 2\pi\}$$

d



$$\{(x, y) \mid x = \cos^2 t + \sin 2t, y = \cos t, 0 \leq t \leq 2\pi\}$$

e



$$\{(x, y) \mid x = \cos^3 t, y = \sin t, 0 \leq t \leq 2\pi\}$$

### INVESTIGATION 3

### ANGLE SUM AND DIFFERENCE IDENTITIES

1	$A$	$B$	$\cos A$	$\cos B$	$\cos(A - B)$	$\cos A - \cos B$	$\cos A \cos B + \sin A \sin B$
	$47^\circ$	$24^\circ$	0.682	0.914	0.921	-0.232	0.921
	$138^\circ$	$49^\circ$	-0.743	0.656	0.0175	-1.40	0.0175
	$3^\circ$	$2^\circ$	-0.990	-0.416	0.540	-0.574	0.540
	$-2.1^\circ$	$0.65^\circ$	-0.505	0.796	-0.924	-1.30	-0.924
	$5.7^\circ$	$3.21^\circ$	0.835	-0.998	-0.795	1.83	-0.795

2  $\cos(A - B) = \cos A \cos B + \sin A \sin B$

3	$A$	$B$	$\sin A$	$\sin B$	$\sin(A + B)$	$\sin A + \sin B$	$\sin A \cos B + \cos A \sin B$
	$85^\circ$	$117^\circ$	0.996	0.891	-0.375	1.89	-0.375
	$12^\circ$	$98^\circ$	0.208	0.990	0.940	1.20	0.940
	$1.53^\circ$	$2.37^\circ$	0.999	0.697	-0.688	1.70	-0.688
	$-0.67^\circ$	$4.11^\circ$	-0.621	-0.824	-0.294	-1.44	-0.294

4  $\sin(A + B) = \sin A \cos B + \cos A \sin B$

### EXERCISE 1E

1 Let  $A = 57^\circ$  and  $B = 21^\circ$ .

$$\begin{aligned} \cos(A + B) &= \cos(57^\circ + 21^\circ) \\ &= \cos 78^\circ \\ &\approx 0.208 \end{aligned}$$

$$\begin{aligned} \cos A \cos B - \sin A \sin B &= (\cos 57^\circ)(\cos 21^\circ) - (\sin 57^\circ)(\sin 21^\circ) \\ &\approx 0.208 \end{aligned}$$

$$\therefore \cos(A + B) = \cos A \cos B - \sin A \sin B \quad \text{when } A = 57^\circ \text{ and } B = 21^\circ.$$

$$\begin{aligned}\cos(A - B) &= \cos(57^\circ - 21^\circ) \\ &= \cos 36^\circ \\ &\approx 0.809\end{aligned}$$

$$\begin{aligned}\cos A \cos B + \sin A \sin B \\ &= (\cos 57^\circ)(\cos 21^\circ) + (\sin 57^\circ)(\sin 21^\circ) \\ &\approx 0.809\end{aligned}$$

$$\therefore \cos(A - B) = \cos A \cos B + \sin A \sin B \quad \text{when } A = 57^\circ \text{ and } B = 21^\circ.$$

$$\begin{aligned}\sin(A + B) &= \sin(57^\circ + 21^\circ) \\ &= \sin 78^\circ \\ &\approx 0.978\end{aligned}$$

$$\begin{aligned}\sin A \cos B + \cos A \sin B \\ &= (\sin 57^\circ)(\cos 21^\circ) + (\cos 57^\circ)(\sin 21^\circ) \\ &\approx 0.978\end{aligned}$$

$$\therefore \sin(A + B) = \sin A \cos B + \cos A \sin B \quad \text{when } A = 57^\circ \text{ and } B = 21^\circ.$$

$$\begin{aligned}\sin(A - B) &= \sin(57^\circ - 21^\circ) \\ &= \sin 36^\circ \\ &\approx 0.588\end{aligned}$$

$$\begin{aligned}\sin A \cos B - \cos A \sin B \\ &= (\sin 57^\circ)(\cos 21^\circ) - (\cos 57^\circ)(\sin 21^\circ) \\ &\approx 0.588\end{aligned}$$

$$\therefore \sin(A - B) = \sin A \cos B - \cos A \sin B \quad \text{when } A = 57^\circ \text{ and } B = 21^\circ.$$

$$\begin{aligned}\tan(A + B) &= \tan(57^\circ + 21^\circ) \\ &= \tan 78^\circ \\ &\approx 4.70\end{aligned}$$

$$\frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{\tan 57^\circ + \tan 21^\circ}{1 - (\tan 57^\circ)(\tan 21^\circ)} \approx 4.70$$

$$\therefore \tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} \quad \text{when } A = 57^\circ \text{ and } B = 21^\circ.$$

$$\begin{aligned}\tan(A - B) &= \tan(57^\circ - 21^\circ) \\ &= \tan 36^\circ \\ &\approx 0.727\end{aligned}$$

$$\frac{\tan A - \tan B}{1 + \tan A \tan B} = \frac{\tan 57^\circ - \tan 21^\circ}{1 + (\tan 57^\circ)(\tan 21^\circ)} \approx 0.727$$

$$\therefore \tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B} \quad \text{when } A = 57^\circ \text{ and } B = 21^\circ.$$

**2 a**  $\sin(90^\circ + \theta)$   
 $= \sin 90^\circ \cos \theta + \cos 90^\circ \sin \theta$   
 $= 1 \times \cos \theta + 0 \times \sin \theta$   
 $= \cos \theta$

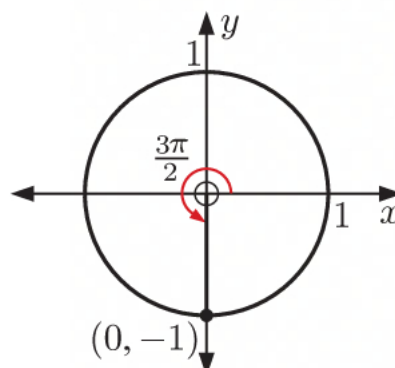
**c**  $\sin(180^\circ - \theta)$   
 $= \sin 180^\circ \cos \theta - \cos 180^\circ \sin \theta$   
 $= 0 \times \cos \theta - (-1) \times \sin \theta$   
 $= \sin \theta$

**e**  $\sin(2\pi - A)$   
 $= \sin 2\pi \cos A - \cos 2\pi \sin A$   
 $= 0 \times \cos A - 1 \times \sin A$   
 $= -\sin A$

**f**  $\cos\left(\frac{3\pi}{2} - \theta\right)$   
 $= \cos \frac{3\pi}{2} \cos \theta + \sin \frac{3\pi}{2} \sin \theta$   
 $= 0 \times \cos \theta + (-1) \times \sin \theta$   
 $= -\sin \theta$

**b**  $\cos(90^\circ + \theta)$   
 $= \cos 90^\circ \cos \theta - \sin 90^\circ \sin \theta$   
 $= 0 \times \cos \theta - 1 \times \sin \theta$   
 $= -\sin \theta$

**d**  $\cos(\pi + \alpha)$   
 $= \cos \pi \cos \alpha - \sin \pi \sin \alpha$   
 $= (-1) \times \cos \alpha - 0 \times \sin \alpha$   
 $= -\cos \alpha$



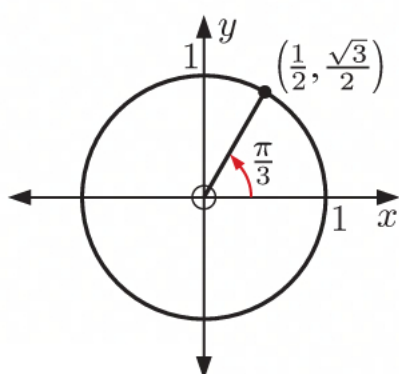


$$\begin{aligned}
 \text{g} \quad & \tan\left(\frac{\pi}{4} + \theta\right) \\
 &= \frac{\tan \frac{\pi}{4} + \tan \theta}{1 - \tan \frac{\pi}{4} \tan \theta} \\
 &= \frac{1 + \tan \theta}{1 - \tan \theta}
 \end{aligned}$$

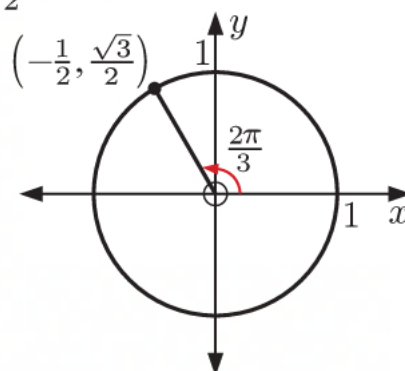
$$\begin{aligned}
 \text{h} \quad & \tan\left(\theta - \frac{3\pi}{4}\right) \\
 &= \frac{\tan \theta - \tan \frac{3\pi}{4}}{1 + \tan \theta \tan \frac{3\pi}{4}} \\
 &= \frac{\tan \theta - (-1)}{1 + \tan \theta(-1)} \\
 &= \frac{1 + \tan \theta}{1 - \tan \theta}
 \end{aligned}$$

$$\begin{aligned}
 \text{i} \quad & \tan(\pi + \theta) \\
 &= \frac{\tan \pi + \tan \theta}{1 - \tan \pi \tan \theta} \\
 &= \frac{0 + \tan \theta}{1 - (0) \tan \theta} \\
 &= \tan \theta
 \end{aligned}$$

$$\begin{aligned}
 \text{3 a} \quad & \sin\left(\theta + \frac{\pi}{3}\right) \\
 &= \sin \theta \cos \frac{\pi}{3} + \cos \theta \sin \frac{\pi}{3} \\
 &= \sin \theta \times \left(\frac{1}{2}\right) + \cos \theta \times \left(\frac{\sqrt{3}}{2}\right) \\
 &= \frac{1}{2} \sin \theta + \frac{\sqrt{3}}{2} \cos \theta
 \end{aligned}$$



$$\begin{aligned}
 \text{b} \quad & \cos\left(\frac{2\pi}{3} - \theta\right) \\
 &= \cos \frac{2\pi}{3} \cos \theta + \sin \frac{2\pi}{3} \sin \theta \\
 &= \left(-\frac{1}{2}\right) \times \cos \theta + \left(\frac{\sqrt{3}}{2}\right) \times \sin \theta \\
 &= -\frac{1}{2} \cos \theta + \frac{\sqrt{3}}{2} \sin \theta \\
 &= \frac{\sqrt{3}}{2} \sin \theta - \frac{1}{2} \cos \theta
 \end{aligned}$$



$$\begin{aligned}
 \text{c} \quad & \cos\left(\theta + \frac{\pi}{4}\right) \\
 &= \cos \theta \cos \frac{\pi}{4} - \sin \theta \sin \frac{\pi}{4} \\
 &= \cos \theta \times \left(\frac{1}{\sqrt{2}}\right) - \sin \theta \times \left(\frac{1}{\sqrt{2}}\right) \\
 &= -\frac{1}{\sqrt{2}} \sin \theta + \frac{1}{\sqrt{2}} \cos \theta
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad & \sin\left(\frac{\pi}{6} - \theta\right) \\
 &= \sin \frac{\pi}{6} \cos \theta - \cos \frac{\pi}{6} \sin \theta \\
 &= \frac{1}{2} \times \cos \theta - \frac{\sqrt{3}}{2} \times \sin \theta \\
 &= -\frac{\sqrt{3}}{2} \sin \theta + \frac{1}{2} \cos \theta
 \end{aligned}$$

$$\begin{aligned}
 \text{4 a} \quad & \cos 2\theta \cos \theta + \sin 2\theta \sin \theta \\
 &= \cos(2\theta - \theta) \\
 &= \cos \theta
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & \sin 2A \cos A + \cos 2A \sin A \\
 &= \sin(2A + A) \\
 &= \sin 3A
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & \cos A \sin B - \sin A \cos B \\
 &= \sin B \cos A - \cos B \sin A \\
 &= \sin(B - A)
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad & \sin \alpha \sin \beta + \cos \alpha \cos \beta \\
 &= \cos \alpha \cos \beta + \sin \alpha \sin \beta \\
 &= \cos(\alpha - \beta)
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & \sin \phi \sin \theta - \cos \phi \cos \theta \\
 &= -[\cos \phi \cos \theta - \sin \phi \sin \theta] \\
 &= -\cos(\phi + \theta)
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & 2 \sin \alpha \cos \beta - 2 \cos \alpha \sin \beta \\
 &= 2[\sin \alpha \cos \beta - \cos \alpha \sin \beta] \\
 &= 2 \sin(\alpha - \beta)
 \end{aligned}$$

$$\begin{aligned}
 \text{g} \quad & \frac{\tan 2\theta - \tan \theta}{1 + \tan 2\theta \tan \theta} = \tan(2\theta - \theta) \\
 &= \tan \theta
 \end{aligned}$$

$$\begin{aligned}
 \text{h} \quad & \frac{\tan 2A + \tan A}{1 - \tan 2A \tan A} = \tan(2A + A) \\
 &= \tan 3A
 \end{aligned}$$

$$\begin{aligned}
 \text{5 a} \quad & \sin 2\theta = \sin(\theta + \theta) \\
 &= \sin \theta \cos \theta + \cos \theta \sin \theta \\
 &= 2 \sin \theta \cos \theta
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & \cos 2\theta = \cos(\theta + \theta) \\
 &= \cos \theta \cos \theta - \sin \theta \sin \theta \\
 &= \cos^2 \theta - \sin^2 \theta
 \end{aligned}$$



$$\begin{aligned}
 \text{c } \tan 2\theta &= \tan(\theta + \theta) \\
 &= \frac{\tan \theta + \tan \theta}{1 - \tan \theta \tan \theta} \\
 &= \frac{2 \tan \theta}{1 - \tan^2 \theta}
 \end{aligned}$$

$$\begin{aligned}
 \text{6 a } \cos 75^\circ &= \cos(45^\circ + 30^\circ) \\
 &= \cos 45^\circ \cos 30^\circ - \sin 45^\circ \sin 30^\circ \\
 &= \left(\frac{1}{\sqrt{2}}\right) \left(\frac{\sqrt{3}}{2}\right) - \left(\frac{1}{\sqrt{2}}\right) \left(\frac{1}{2}\right) \\
 &= \left(\frac{\sqrt{3} - 1}{2\sqrt{2}}\right) \times \frac{\sqrt{2}}{\sqrt{2}} \\
 &= \frac{\sqrt{6} - \sqrt{2}}{4}
 \end{aligned}$$

$$\begin{aligned}
 \text{c } \cos \frac{13\pi}{12} &= \cos \left(\frac{10\pi}{12} + \frac{3\pi}{12}\right) \\
 &= \cos \left(\frac{5\pi}{6} + \frac{\pi}{4}\right) \\
 &= \cos \frac{5\pi}{6} \cos \frac{\pi}{4} - \sin \frac{5\pi}{6} \sin \frac{\pi}{4} \\
 &= \left(-\frac{\sqrt{3}}{2}\right) \left(\frac{1}{\sqrt{2}}\right) - \left(\frac{1}{2}\right) \left(\frac{1}{\sqrt{2}}\right) \\
 &= \left(\frac{-\sqrt{3} - 1}{2\sqrt{2}}\right) \times \frac{\sqrt{2}}{\sqrt{2}} \\
 &= \frac{-\sqrt{6} - \sqrt{2}}{4}
 \end{aligned}$$

$$\begin{aligned}
 \text{7 a } \tan \frac{5\pi}{12} &= \tan \left(\frac{5 \times 180^\circ}{12}\right) \\
 &= \tan 75^\circ \\
 &= \tan(45^\circ + 30^\circ) \\
 &= \frac{\tan 45^\circ + \tan 30^\circ}{1 - \tan 45^\circ \tan 30^\circ} \\
 &= \frac{1 + \frac{1}{\sqrt{3}}}{1 - (1) \left(\frac{1}{\sqrt{3}}\right)} \\
 &= \left(\frac{\sqrt{3} + 1}{\sqrt{3} - 1}\right) \left(\frac{\sqrt{3} + 1}{\sqrt{3} + 1}\right) \\
 &= \frac{3 + \sqrt{3} + \sqrt{3} + 1}{3 - 1} \\
 &= \frac{4 + 2\sqrt{3}}{2} \\
 &= 2 + \sqrt{3}
 \end{aligned}$$

$$\begin{aligned}
 \text{b } \sin 105^\circ &= \sin(60^\circ + 45^\circ) \\
 &= \sin 60^\circ \cos 45^\circ + \cos 60^\circ \sin 45^\circ \\
 &= \left(\frac{\sqrt{3}}{2}\right) \left(\frac{1}{\sqrt{2}}\right) + \left(\frac{1}{2}\right) \left(\frac{1}{\sqrt{2}}\right) \\
 &= \left(\frac{\sqrt{3} + 1}{2\sqrt{2}}\right) \times \frac{\sqrt{2}}{\sqrt{2}} \\
 &= \frac{\sqrt{6} + \sqrt{2}}{4}
 \end{aligned}$$

$$\begin{aligned}
 \text{b } \tan 105^\circ &= \tan(180^\circ - 75^\circ) \\
 &= \frac{\tan 180^\circ - \tan 75^\circ}{1 + \tan 180^\circ \tan 75^\circ} \\
 &= \frac{0 - (2 + \sqrt{3})}{1 + (0)(2 + \sqrt{3})} \quad \{\text{using a}\} \\
 &= -2 - \sqrt{3}
 \end{aligned}$$

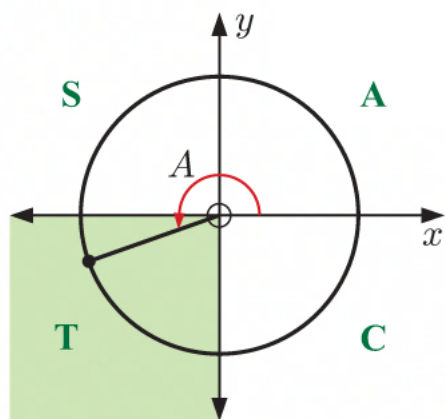
$$\begin{aligned}
 8 \quad \tan(A+B) &= \frac{\tan A + \tan B}{1 - \tan A \tan B} \\
 &= \frac{\frac{2}{3} - \frac{1}{5}}{1 - (\frac{2}{3})(-\frac{1}{5})} \quad \{\tan A = \frac{2}{3}, \tan B = -\frac{1}{5}\} \\
 &= \frac{\frac{10}{15} - \frac{3}{15}}{1 + \frac{2}{15}} \\
 &= \frac{\frac{7}{15}}{\frac{17}{15}} \\
 &= \frac{7}{17}
 \end{aligned}$$

$$\begin{aligned}
 9 \quad \tan(A + \frac{\pi}{4}) &= \frac{\tan A + \tan \frac{\pi}{4}}{1 - \tan A \tan \frac{\pi}{4}} \\
 &= \frac{\frac{3}{4} + 1}{1 - (\frac{3}{4})(1)} \quad \{\tan A = \frac{3}{4}\} \\
 &= \frac{\frac{7}{4}}{\frac{1}{4}} \\
 &= 7
 \end{aligned}$$

$$10 \quad \sin A = -\frac{1}{3} \quad \text{and} \quad \pi \leq A \leq \frac{3\pi}{2}$$

$A$  is in quadrant 3

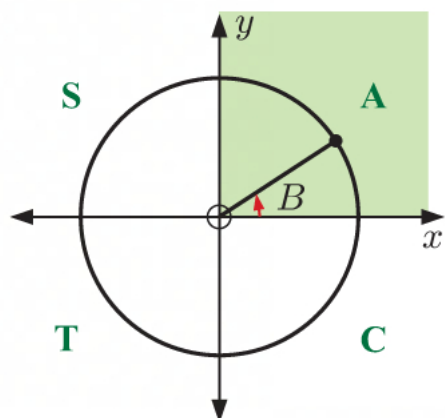
$$\therefore \cos A < 0$$



$$\cos B = \frac{1}{\sqrt{5}} \quad \text{and} \quad 0 \leq B \leq \frac{\pi}{2}$$

$B$  is in quadrant 1

$$\therefore \sin B > 0$$



$$\cos^2 A + \sin^2 A = 1$$

$$\therefore \cos^2 A + \frac{1}{9} = 1$$

$$\therefore \cos^2 A = \frac{8}{9}$$

$$\therefore \cos A = -\frac{2\sqrt{2}}{3} \quad \{\text{since } \cos A < 0\}$$

$$\begin{aligned}
 \text{Now } \tan A &= \frac{\sin A}{\cos A} \\
 &= \frac{-\frac{1}{3}}{-\frac{2\sqrt{2}}{3}} \\
 &= \frac{1}{2\sqrt{2}}
 \end{aligned}$$

$$\cos^2 B + \sin^2 B = 1$$

$$\therefore \frac{1}{5} + \sin^2 B = 1$$

$$\therefore \sin^2 B = \frac{4}{5}$$

$$\therefore \sin B = \frac{2}{\sqrt{5}} \quad \{\text{since } \sin B > 0\}$$

$$\begin{aligned}
 \text{Now } \tan B &= \frac{\sin B}{\cos B} \\
 &= \frac{\frac{2}{\sqrt{5}}}{\frac{1}{\sqrt{5}}} \\
 &= 2
 \end{aligned}$$

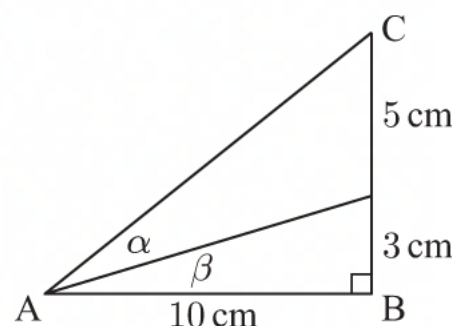
$$\begin{aligned}
\tan(A+B) &= \frac{\tan A + \tan B}{1 - \tan A \tan B} \\
&= \frac{\frac{1}{2\sqrt{2}} + 2}{1 - (\frac{1}{2\sqrt{2}})(2)} \times \left( \frac{2\sqrt{2}}{2\sqrt{2}} \right) \\
&= \frac{1 + 4\sqrt{2}}{2\sqrt{2} - 2} \times \left( \frac{2\sqrt{2} + 2}{2\sqrt{2} + 2} \right) \\
&= \frac{2\sqrt{2} + 2 + 16 + 8\sqrt{2}}{8 - 4} \\
&= \frac{18 + 10\sqrt{2}}{4} \\
&= \frac{9 + 5\sqrt{2}}{2}
\end{aligned}$$

$$\begin{aligned}
11 \quad \frac{\tan 80^\circ - \tan 20^\circ}{1 + \tan 80^\circ \tan 20^\circ} &= \tan(80^\circ - 20^\circ) \\
&= \tan 60^\circ \\
&= \sqrt{3}
\end{aligned}$$

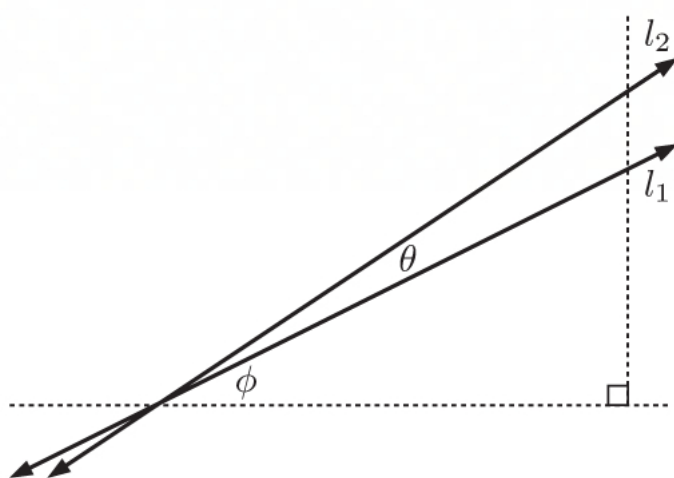
$$\begin{aligned}
12 \quad \tan(A+B) &= \frac{3}{5} \\
\therefore \frac{\tan A + \tan B}{1 - \tan A \tan B} &= \frac{3}{5} \\
\therefore \frac{\tan A + \frac{2}{3}}{1 - \tan A \times (\frac{2}{3})} &= \frac{3}{5} \quad \{ \tan B = \frac{2}{3} \} \\
\therefore 5 \tan A + \frac{10}{3} &= 3 - 2 \tan A \\
\therefore 7 \tan A &= -\frac{1}{3} \\
\therefore \tan A &= -\frac{1}{21}
\end{aligned}$$

13 We mark angle  $\beta$  on the diagram.

$$\begin{aligned}
\tan \beta &= \frac{3}{10} \quad \text{and} \quad \tan(\alpha + \beta) = \frac{8}{10} \\
\therefore \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} &= \frac{8}{10} \\
\therefore \frac{\tan \alpha + \frac{3}{10}}{1 - \tan \alpha \times (\frac{3}{10})} &= \frac{4}{5} \\
\therefore 5 \tan \alpha + \frac{15}{10} &= 4 - \frac{12}{10} \tan \alpha \\
\therefore \frac{31}{5} \tan \alpha &= \frac{5}{2} \\
\therefore \tan \alpha &= \frac{25}{62}
\end{aligned}$$



14

Let  $\theta$  be the acute angle between the lines  $l_1$  and  $l_2$ .

$$\tan \phi = \frac{1}{2} \quad \{\text{gradient of } l_1\}$$

$$\tan(\theta + \phi) = \frac{2}{3} \quad \{\text{gradient of } l_2\}$$

$$\therefore \frac{\tan \theta + \tan \phi}{1 - \tan \theta \tan \phi} = \frac{2}{3}$$

$$\therefore \frac{\tan \theta + \frac{1}{2}}{1 - \tan \theta \times (\frac{1}{2})} = \frac{2}{3}$$

$$\therefore 3 \tan \theta + \frac{3}{2} = 2 - \tan \theta$$

$$\therefore 4 \tan \theta = \frac{1}{2}$$

$$\therefore \tan \theta = \frac{1}{8}$$

 $\therefore$  the tangent of the acute angle is  $\frac{1}{8}$ .

15 a  $\cos(\alpha + \beta) \cos(\alpha - \beta) - \sin(\alpha + \beta) \sin(\alpha - \beta)$

$$= \cos [(\alpha + \beta) + (\alpha - \beta)]$$

$$= \cos 2\alpha$$

b  $\sin(\theta - 2\phi) \cos(\theta + \phi) - \cos(\theta - 2\phi) \sin(\theta + \phi)$

$$= \sin [(\theta - 2\phi) - (\theta + \phi)]$$

$$= \sin(-3\phi)$$

$$= -\sin 3\phi$$

c  $\cos \alpha \cos(\beta - \alpha) - \sin \alpha \sin(\beta - \alpha)$

$$= \cos [\alpha + (\beta - \alpha)]$$

$$= \cos \beta$$

d  $\tan(A + \frac{\pi}{4}) \tan(A - \frac{\pi}{4})$

$$= \frac{\tan A + \tan \frac{\pi}{4}}{1 - \tan A \tan \frac{\pi}{4}} \times \frac{\tan A - \tan \frac{\pi}{4}}{1 + \tan A \tan \frac{\pi}{4}}$$

$$= \left( \frac{\tan A + 1}{1 - \tan A} \right) \left( \frac{\tan A - 1}{1 + \tan A} \right)$$

$$= \frac{\tan^2 A - 1}{1 - \tan^2 A}$$

$$= -1$$

e  $\frac{\tan(A + B) + \tan(A - B)}{1 - \tan(A + B) \tan(A - B)}$

$$= \tan [(A + B) + (A - B)]$$

$$= \tan 2A$$



**16**  $\sin A = \frac{2}{3}$  and  $\frac{\pi}{2} \leq A \leq \pi$   $\cos^2 A + \sin^2 A = 1$   $\therefore \tan A = \frac{\frac{2}{3}}{-\frac{\sqrt{5}}{3}}$   
 $A$  is in quadrant 2  $\therefore \cos^2 A + \frac{4}{9} = 1$   $= -\frac{2}{\sqrt{5}}$   
 $\therefore \cos A < 0$   $\therefore \cos^2 A = \frac{5}{9}$   $\therefore \cos A = -\frac{\sqrt{5}}{3}$

$\cos B = -\frac{4}{5}$  and  $\pi \leq B \leq \frac{3\pi}{2}$   $\cos^2 B + \sin^2 B = 1$   $\therefore \tan B = \frac{-\frac{3}{5}}{-\frac{4}{5}}$   
 $B$  is in quadrant 3  $\therefore \frac{16}{25} + \sin^2 B = 1$   $= \frac{3}{4}$   
 $\therefore \sin B < 0$   $\therefore \sin^2 B = \frac{9}{25}$   $\therefore \sin B = -\frac{3}{5}$

**a**  $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$   
 $= \frac{-\frac{2}{\sqrt{5}} + \frac{3}{4}}{1 - \left(-\frac{2}{\sqrt{5}}\right)\left(\frac{3}{4}\right)}$   
 $= \frac{\frac{-8+3\sqrt{5}}{4\sqrt{5}}}{\frac{4\sqrt{5}+6}{4\sqrt{5}}}$   
 $= \left(\frac{-8+3\sqrt{5}}{6+4\sqrt{5}}\right) \times \left(\frac{6-4\sqrt{5}}{6-4\sqrt{5}}\right)$   
 $= \frac{-48+18\sqrt{5}+32\sqrt{5}-60}{36-80}$   
 $= \frac{-108+50\sqrt{5}}{-44}$   
 $= \frac{54-25\sqrt{5}}{22}$

**b**  $\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$   
 $= \frac{2\left(-\frac{2}{\sqrt{5}}\right)}{1 - \left(-\frac{2}{\sqrt{5}}\right)^2}$   
 $= \frac{-\frac{4}{\sqrt{5}}}{1 - \frac{4}{5}}$   
 $= \frac{-\frac{4}{\sqrt{5}}}{\frac{1}{5}}$   
 $= -\frac{4}{\sqrt{5}} \times \frac{5}{1} \times \frac{\sqrt{5}}{\sqrt{5}}$   
 $= -4\sqrt{5}$

**17**  $\tan(A-B) \tan(A+B) = 1$   
 $\therefore \left(\frac{\tan A - \tan B}{1 + \tan A \tan B}\right) \left(\frac{\tan A + \tan B}{1 - \tan A \tan B}\right) = 1$   
 $\therefore \frac{\tan^2 A - \tan^2 B}{1 - \tan^2 A \tan^2 B} = 1$   
 $\therefore \tan^2 A - \tan^2 B = 1 - \tan^2 A \tan^2 B$   
 $\therefore \tan^2 A (\tan^2 B + 1) = 1 + \tan^2 B$   
 $\therefore \tan^2 A = 1 \quad \{\tan^2 B + 1 \neq 0\}$   
 $\therefore \tan A = \pm 1$

$$\begin{aligned}
18 \quad \tan(A+B+C) &= \tan[(A+B)+C] \\
&= \frac{\tan(A+B) + \tan C}{1 - \tan(A+B)\tan C} \\
&= \left( \frac{\frac{\tan A + \tan B}{1 - \tan A \tan B} + \tan C}{1 - \frac{\tan A + \tan B}{1 - \tan A \tan B} \times \tan C} \right) \times \left( \frac{1 - \tan A \tan B}{1 - \tan A \tan B} \right) \\
&= \frac{\tan A + \tan B + \tan C(1 - \tan A \tan B)}{1 - \tan A \tan B - (\tan A + \tan B) \times \tan C} \\
&= \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan A \tan C - \tan B \tan C}
\end{aligned}$$

If  $A$ ,  $B$ , and  $C$  are the angles of a triangle then  $A+B+C = 180^\circ$

$$\therefore \tan(A+B+C) = 0$$

$$\therefore \tan A + \tan B + \tan C - \tan A \tan B \tan C = 0$$

$$\therefore \tan A + \tan B + \tan C = \tan A \tan B \tan C$$

$$\begin{aligned}
19 \quad \text{a} \quad & \sqrt{2} \cos\left(\theta + \frac{\pi}{4}\right) \\
&= \sqrt{2} \left[ \cos \theta \cos \frac{\pi}{4} - \sin \theta \sin \frac{\pi}{4} \right] \\
&= \sqrt{2} \left( \frac{1}{\sqrt{2}} \cos \theta - \frac{1}{\sqrt{2}} \sin \theta \right) \\
&= \cos \theta - \sin \theta \\
\text{b} \quad & 2 \cos\left(\theta - \frac{\pi}{3}\right) \\
&= 2 \left[ \cos \theta \cos \frac{\pi}{3} + \sin \theta \sin \frac{\pi}{3} \right] \\
&= 2 \left( \frac{1}{2} \cos \theta + \frac{\sqrt{3}}{2} \sin \theta \right) \\
&= \cos \theta + \sqrt{3} \sin \theta \\
\text{c} \quad & \cos(\alpha + \beta) - \cos(\alpha - \beta) \\
&= \cos \alpha \cos \beta - \sin \alpha \sin \beta - (\cos \alpha \cos \beta + \sin \alpha \sin \beta) \\
&= \cancel{\cos \alpha \cos \beta} - \sin \alpha \sin \beta - \cancel{\cos \alpha \cos \beta} - \sin \alpha \sin \beta \\
&= -2 \sin \alpha \sin \beta \\
\text{d} \quad & \cos(\alpha + \beta) \cos(\alpha - \beta) \\
&= (\cos \alpha \cos \beta - \sin \alpha \sin \beta)(\cos \alpha \cos \beta + \sin \alpha \sin \beta) \\
&= \cos^2 \alpha \cos^2 \beta - \sin^2 \alpha \sin^2 \beta \\
&= \cos^2 \alpha (1 - \sin^2 \beta) - (1 - \cos^2 \alpha) \sin^2 \beta \\
&= \cos^2 \alpha - \cancel{\cos^2 \alpha \sin^2 \beta} - \sin^2 \beta + \cancel{\cos^2 \alpha \sin^2 \beta} \\
&= \cos^2 \alpha - \sin^2 \beta
\end{aligned}$$

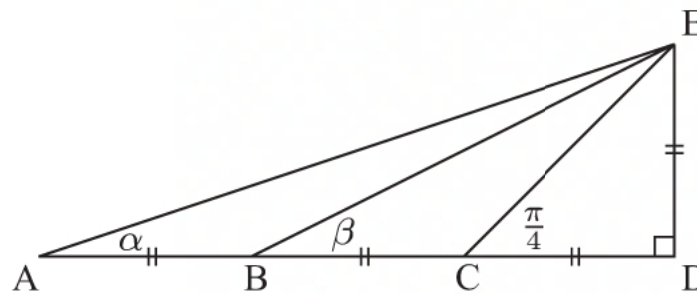
$$20 \quad \tan \alpha = \frac{1}{3} \quad \text{and} \quad \tan \beta = \frac{1}{2}$$

$$\begin{aligned}
\therefore \tan(\alpha + \beta) &= \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} \\
&= \frac{\frac{1}{3} + \frac{1}{2}}{1 - \left(\frac{1}{3}\right)\left(\frac{1}{2}\right)} \\
&= \frac{\frac{5}{6}}{\frac{5}{6}} = 1
\end{aligned}$$

$$\therefore \alpha + \beta = \frac{\pi}{4} + k\pi, \quad k \in \mathbb{Z} \quad \{\text{since } \tan \frac{\pi}{4} = 1\}$$

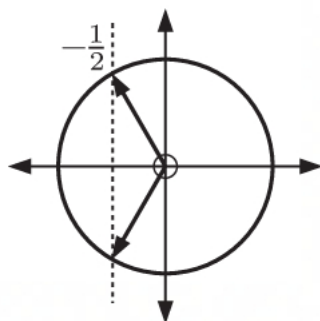
But clearly  $0 < \alpha, \beta < \frac{\pi}{4}$  so  $0 < \alpha + \beta < \frac{\pi}{2}$

$$\therefore \alpha + \beta = \frac{\pi}{4} \quad \{\text{choosing } k = 1\}$$



$$\begin{aligned}
 21 \quad a \quad \cos 3\theta &= \cos(2\theta + \theta) \\
 &= \cos 2\theta \cos \theta - \sin 2\theta \sin \theta \\
 &= (2\cos^2 \theta - 1)\cos \theta - (2\sin \theta \cos \theta)\sin \theta \\
 &= 2\cos^3 \theta - \cos \theta - 2\sin^2 \theta \cos \theta \\
 &= 2\cos^3 \theta - \cos \theta - 2(1 - \cos^2 \theta)\cos \theta \\
 &= 2\cos^3 \theta - \cos \theta - 2\cos \theta + 2\cos^3 \theta \\
 &= 4\cos^3 \theta - 3\cos \theta
 \end{aligned}$$

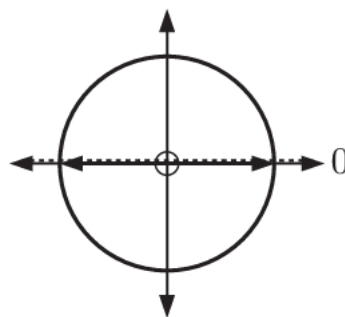
$$\begin{aligned}
 b \quad 8\cos^3 \theta - 6\cos \theta + 1 &= 0 \\
 \therefore 8\cos^3 \theta - 6\cos \theta &= -1 \\
 \therefore 4\cos^3 \theta - 3\cos \theta &= -\frac{1}{2} \\
 \therefore \cos 3\theta &= -\frac{1}{2}
 \end{aligned}$$



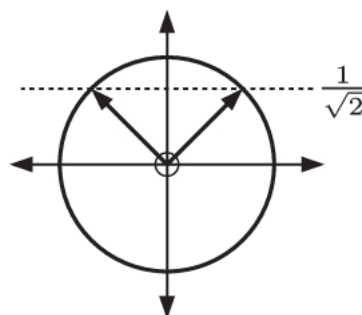
$$\begin{aligned}
 \therefore 3\theta &= -\frac{8\pi}{3}, -\frac{4\pi}{3}, -\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \text{ or } \frac{8\pi}{3} \quad \{-\pi \leq \theta \leq \pi \quad \therefore -3\pi \leq 3\theta \leq 3\pi\} \\
 \therefore \theta &= -\frac{8\pi}{9}, -\frac{4\pi}{9}, -\frac{2\pi}{9}, \frac{2\pi}{9}, \frac{4\pi}{9}, \text{ or } \frac{8\pi}{9}
 \end{aligned}$$

$$\begin{aligned}
 22 \quad a \quad \sin 3\theta &= \sin(2\theta + \theta) \\
 &= \sin 2\theta \cos \theta + \cos 2\theta \sin \theta \\
 &= (2\sin \theta \cos \theta)\cos \theta + (1 - 2\sin^2 \theta)\sin \theta \\
 &= 2\sin \theta \cos^2 \theta + \sin \theta - 2\sin^3 \theta \\
 &= 2\sin \theta(1 - \sin^2 \theta) + \sin \theta - 2\sin^3 \theta \\
 &= 2\sin \theta - 2\sin^3 \theta + \sin \theta - 2\sin^3 \theta \\
 &= -4\sin^3 \theta + 3\sin \theta
 \end{aligned}$$

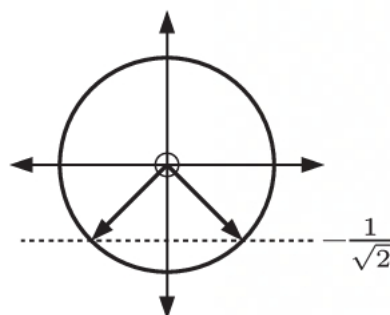
$$\begin{aligned}
 b \quad \sin 3\theta &= \sin \theta \\
 \therefore -4\sin^3 \theta + 3\sin \theta &= \sin \theta \\
 \therefore 4\sin^3 \theta - 2\sin \theta &= 0 \\
 \therefore 2\sin \theta(2\sin^2 \theta - 1) &= 0 \\
 \therefore \sin \theta = 0 \text{ or } \sin^2 \theta &= \frac{1}{2} \\
 \therefore \sin \theta = 0 \text{ or } \sin \theta &= \pm \frac{1}{\sqrt{2}}
 \end{aligned}$$



$$\begin{aligned}
 \sin \theta = 0 \text{ when} \\
 \theta = 0, \pi, 2\pi, \text{ or } 3\pi \\
 0 \leq \theta \leq 3\pi
 \end{aligned}$$



$$\begin{aligned}
 \sin \theta = \frac{1}{\sqrt{2}} \text{ when} \\
 \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{4}, \text{ or } \frac{11\pi}{4} \\
 0 \leq \theta \leq 3\pi
 \end{aligned}$$



$$\begin{aligned}
 \sin \theta = -\frac{1}{\sqrt{2}} \text{ when} \\
 \theta = \frac{5\pi}{4} \text{ or } \frac{7\pi}{4} \\
 0 \leq \theta \leq 3\pi
 \end{aligned}$$

$$\therefore \theta = 0, \frac{\pi}{4}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{7\pi}{4}, 2\pi, \frac{9\pi}{4}, \frac{11\pi}{4}, \text{ or } 3\pi \quad \{0 \leq \theta \leq 3\pi\}$$

$$\begin{aligned}
 \text{23 } \sqrt{3} \sin x + \cos x &= k \sin(x + a), \quad k > 0 \text{ and } 0 < a < 2\pi \\
 &= k(\sin x \cos a + \cos x \sin a) \\
 &= k \cos a \sin x + k \sin a \cos x
 \end{aligned}$$

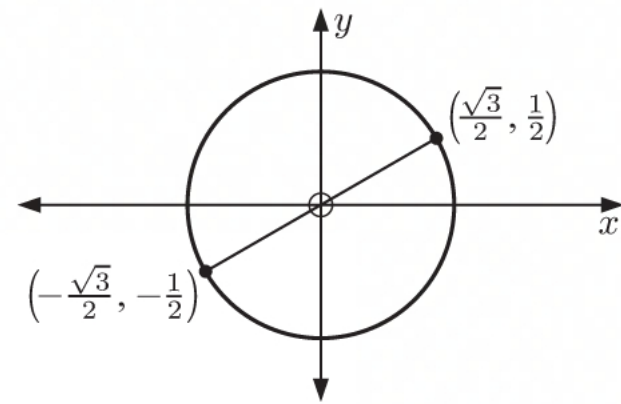
Equating coefficients of  $\sin x$  and  $\cos x$ ,

$$k \cos a = \sqrt{3} \quad \dots (1) \quad \text{and} \quad k \sin a = 1 \quad \dots (2)$$

$$\frac{k \sin a}{k \cos a} = \frac{1}{\sqrt{3}} \quad \{\text{dividing (2) by (1)}\}$$

$$\therefore \tan a = \frac{1}{\sqrt{3}}$$

$$\therefore a = \frac{\pi}{6} \text{ or } \frac{7\pi}{6}$$



$$\text{Substituting } a = \frac{\pi}{6} \text{ into (1) gives } k \cos \frac{\pi}{6} = \sqrt{3}$$

$$\therefore k \times \frac{\sqrt{3}}{2} = \sqrt{3}$$

$$\therefore k = 2$$

$$\text{Substituting } a = \frac{7\pi}{6} \text{ into (1) gives } k \cos \frac{7\pi}{6} = \sqrt{3}$$

$$\therefore k \times \left(-\frac{\sqrt{3}}{2}\right) = \sqrt{3}$$

$$\therefore k = -2$$

We reject this solution as  $k > 0$ .

So,  $k = 2$  and  $a = \frac{\pi}{6}$ .

$$\begin{aligned}
 \text{24 a } 2 \cos x + 2 \sin x &= k \cos(x + a), \quad k > 0 \text{ and } 0 < a < 2\pi \\
 &= k(\cos x \cos a - \sin x \sin a) \\
 &= k \cos a \cos x - k \sin a \sin x
 \end{aligned}$$

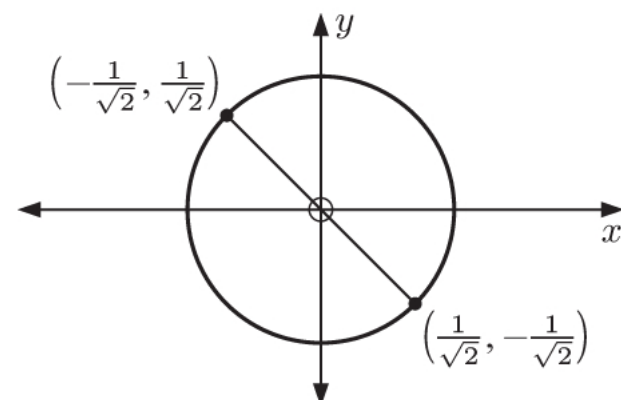
Equating coefficients of  $\cos x$  and  $\sin x$ ,

$$k \cos a = 2 \quad \dots (1) \quad \text{and} \quad k \sin a = -2 \quad \dots (2)$$

$$\frac{k \sin a}{k \cos a} = \frac{-2}{2} \quad \{\text{dividing (2) by (1)}\}$$

$$\therefore \tan a = -1$$

$$\therefore a = \frac{3\pi}{4} \text{ or } \frac{7\pi}{4}$$



$$\text{Substituting } a = \frac{3\pi}{4} \text{ into (1) gives } k \cos \frac{3\pi}{4} = 2$$

$$\therefore k \times \left(-\frac{1}{\sqrt{2}}\right) = 2$$

$$\therefore k = -2\sqrt{2}$$

We reject this solution as  $k > 0$ .

$$\text{Substituting } a = \frac{7\pi}{4} \text{ into (1) gives } k \cos \frac{7\pi}{4} = 2$$

$$\therefore k \times \frac{1}{\sqrt{2}} = 2$$

$$\therefore k = 2\sqrt{2}$$

So,  $k = 2\sqrt{2}$  and  $a = \frac{7\pi}{4}$ .

$$\therefore 2 \cos x + 2 \sin x = 2\sqrt{2} \cos\left(x + \frac{7\pi}{4}\right)$$



**b**  $2 \cos x + 2 \sin x = \sqrt{2}$

$$\therefore 2\sqrt{2} \cos\left(x + \frac{7\pi}{4}\right) = \sqrt{2} \quad \{\text{using a}\}$$

$$\therefore \cos\left(x + \frac{7\pi}{4}\right) = \frac{1}{2}$$

$$\therefore x + \frac{7\pi}{4} = \frac{7\pi}{3} \text{ or } \frac{11\pi}{3} \quad \{0 \leq x \leq 2\pi \quad \therefore \frac{7\pi}{4} \leq x + \frac{7\pi}{4} \leq \frac{15\pi}{4}\}$$

$$\therefore x = \frac{7\pi}{12} \text{ or } \frac{23\pi}{12}$$

**25 a** Let  $\cos x + 3 \sin x = k \sin(x + a)$ ,  $k > 0$  and  $0 < a < 2\pi$

$$= k(\sin x \cos a + \cos x \sin a)$$

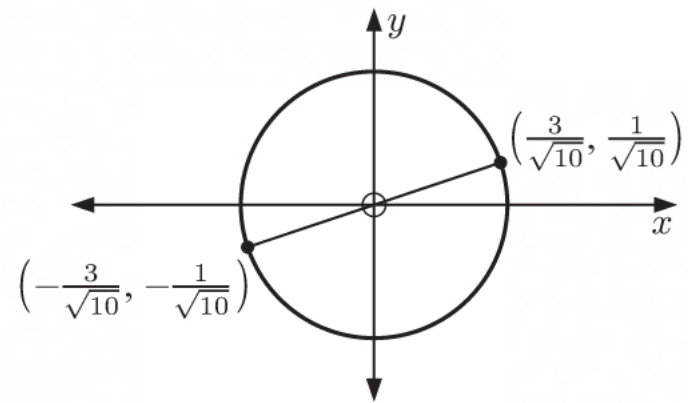
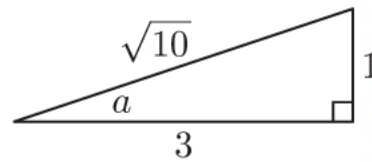
$$= k \cos a \sin x + k \sin a \cos x$$

Equating coefficients of  $\sin x$  and  $\cos x$ ,

$$k \cos a = 3 \quad \dots (1) \quad \text{and} \quad k \sin a = 1 \quad \dots (2)$$

$$\frac{k \sin a}{k \cos a} = \frac{1}{3} \quad \{\text{dividing (2) by (1)}\}$$

$$\therefore \tan a = \frac{1}{3}$$



If  $0 < a < \frac{\pi}{2}$ ,  $\cos a = \frac{3}{\sqrt{10}}$

Substituting into (1) gives  $k \times \frac{3}{\sqrt{10}} = 3$

$$\therefore k = \sqrt{10}$$

If  $\pi < a < \frac{3\pi}{2}$ ,  $\cos a = -\frac{3}{\sqrt{10}}$

Substituting into (1) gives  $k \times \left(-\frac{3}{\sqrt{10}}\right) = 3$

$$\therefore k = -\sqrt{10}$$

We reject this solution as  $k > 0$ .

So,  $k = \sqrt{10}$  and  $\tan a = \frac{1}{3}$

$$\therefore a \approx 0.322 \quad \{0 < a < \frac{\pi}{2}\}$$

$$\therefore \cos x + 3 \sin x \approx \sqrt{10} \sin(x + 0.322)$$

$\therefore y = \sin x$  is mapped onto the graph of  $y = \cos x + 3 \sin x$  using a vertical stretch with scale factor  $\sqrt{10}$ , then a translation of about 0.322 units left.

**b**  $(\cos x + 3 \sin x)^2 + 2 \approx (\sqrt{10} \sin(x + 0.322))^2 + 2 \quad \{\text{using a}\}$

$$\approx 10 \sin^2(x + 0.322) + 2$$

$\therefore$  the greatest value of  $(\cos x + 3 \sin x)^2 + 2$  is  $10 + 2 = 12$  (when  $\sin^2(x + 0.322) = 1$ )  
and the least value is 2 (when  $\sin^2(x + 0.322) = 0$ ).

**26** The period of a function  $f(x)$  is the smallest  $p > 0$  such that

$$f(x+p) = f(x) \quad \text{for all } x$$

$$\therefore \sin[n(x+p)] = \sin(nx)$$

$$\therefore \sin(nx+np) = \sin(nx)$$

$$\therefore \sin(nx)\cos(np) + \cos(nx)\sin(np) = \sin(nx)$$

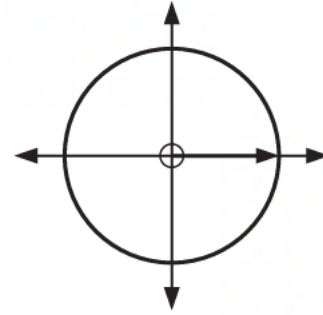
Equating coefficients of  $\sin(nx)$  and  $\cos(nx)$ ,

$$\cos(np) = 1 \quad \text{and} \quad \sin(np) = 0$$

$$\therefore np = 2k\pi, \quad k \in \mathbb{Z}$$

$$\therefore p = \frac{2k\pi}{n}, \quad k \in \mathbb{Z}$$

The smallest  $p > 0$  occurs when  $k = 1 \therefore p = \frac{2\pi}{n}$



**27 a**  $2\cos x - 5\sin x = k\cos(x+b)$ ,  $k > 0$  and  $0 < b < 2\pi$

$$= k(\cos x \cos b - \sin x \sin b)$$

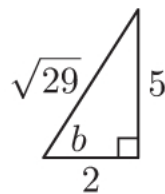
$$= k\cos b \cos x - k\sin b \sin x$$

Equating coefficients of  $\cos x$  and  $\sin x$ ,

$$k\cos b = 2 \quad \dots (1) \quad \text{and} \quad k\sin b = 5 \quad \dots (2)$$

$$\frac{k\sin b}{k\cos b} = \frac{5}{2} \quad \{\text{dividing (2) by (1)}\}$$

$$\therefore \tan b = \frac{5}{2}$$



$$\text{If } 0 < b < \frac{\pi}{2}, \quad \cos b = \frac{2}{\sqrt{29}}$$

$$\text{Substituting into (1) gives } k \times \frac{2}{\sqrt{29}} = 2$$

$$\therefore k = \sqrt{29}$$

$$\text{If } \pi < b < \frac{3\pi}{2}, \quad \cos b = -\frac{2}{\sqrt{29}}$$

$$\text{Substituting into (1) gives } k \times \left(-\frac{2}{\sqrt{29}}\right) = 2$$

$$\therefore k = -\sqrt{29}$$

We reject this solution as  $k > 0$ .

$$\text{So, } k = \sqrt{29} \quad \text{and} \quad \tan b = \frac{5}{2}$$

$$\therefore b \approx 1.19 \quad \{0 < b < \frac{\pi}{2}\}$$

$$\therefore 2\cos x - 5\sin x \approx \sqrt{29}\cos(x+1.19)$$

**b**  $2\cos x - 5\sin x = -2$

$$\therefore \sqrt{29}\cos(x+1.19) \approx -2$$

$$\therefore \cos(x+1.19) \approx -\frac{2}{\sqrt{29}}$$

$$\therefore x+1.19 \approx 1.951, 4.33 \quad \{1.19 \leq x+1.19 \leq \pi+1.19\}$$

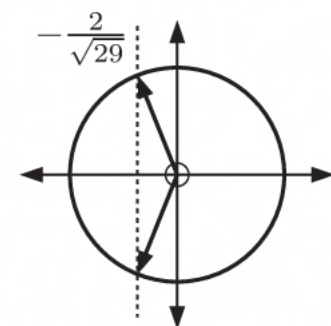
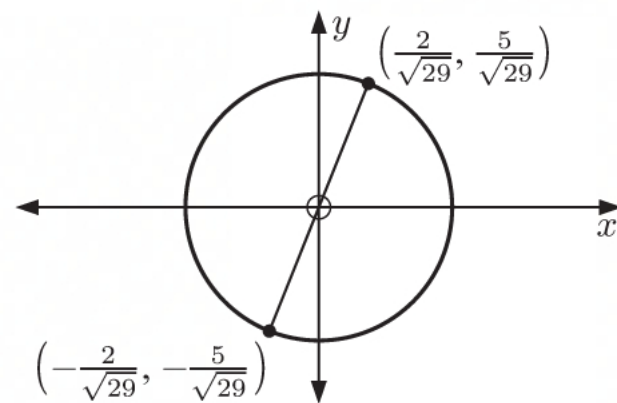
$$\therefore x \approx 0.761, 3.14$$

**Note:**  $3.14 \approx \pi$ , so we suspect the second solution is exactly  $\pi$ .

$$\text{Substituting } x = \pi \text{ into the original equation: } 2\cos \pi - 5\sin \pi = 2(-1) - 5(0)$$

$$= -2 \quad \checkmark$$

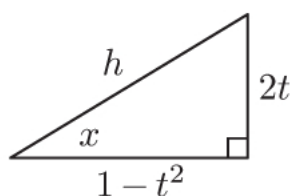
$\therefore$  the second solution is exactly  $\pi$ .



$$\text{c } \tan x = \tan\left(2 \times \frac{x}{2}\right)$$

$$= \frac{2 \tan \frac{x}{2}}{1 - \tan^2\left(\frac{x}{2}\right)}$$

$$= \frac{2t}{1 - t^2} \quad \{t = \tan \frac{x}{2}\}$$



$$\text{Now } h^2 = (1 - t^2)^2 + (2t)^2 \quad \{\text{Pythagoras}\}$$

$$= 1 - 2t^2 + t^4 + 4t^2$$

$$= 1 + 2t^2 + t^4$$

$$= (1 + t^2)^2$$

$$\therefore h = 1 + t^2 \quad \{\text{since } h > 0\}$$

$$\therefore \sin x = \frac{2t}{1 + t^2} \quad \text{and} \quad \cos x = \frac{1 - t^2}{1 + t^2}$$

$$\text{d } 2 \cos x - 5 \sin x = -2$$

$$\therefore 2 \left( \frac{1 - t^2}{1 + t^2} \right) - 5 \left( \frac{2t}{1 + t^2} \right) = -2 \quad \{\text{using c}\}$$

$$\therefore \frac{2 - 2t^2 - 10t}{1 + t^2} = -2$$

$$\therefore 2 - \cancel{2t^2} - 10t = -2 - \cancel{2t^2}$$

$$\therefore 4 = 10t$$

$$\therefore t = \frac{2}{5}$$

$$\text{So } \tan \frac{x}{2} = \frac{2}{5}$$

$$\therefore \frac{x}{2} \approx 0.3805 \quad \{0 \leq x \leq \pi \quad \therefore 0 \leq \frac{x}{2} \leq \frac{\pi}{2}\}$$

$$\therefore x \approx 0.761$$

The  $x = \pi$  solution has been lost since  $t$  is undefined when  $x = \pi$ .

$$\text{28 a Let } \arctan 5 = \theta \quad \text{and} \quad \arctan \frac{2}{3} = \phi$$

$$\therefore 5 = \tan \theta$$

$$\therefore \frac{2}{3} = \tan \phi$$

$$\text{Now } \tan(\theta - \phi) = \frac{\tan \theta - \tan \phi}{1 + \tan \theta \tan \phi}$$

$$\therefore \tan(\theta - \phi) = \frac{5 - \frac{2}{3}}{1 + 5 \times \frac{2}{3}}$$

$$= \frac{\frac{13}{3}}{\frac{13}{3}}$$

$$= 1$$

$$\therefore \theta - \phi = \frac{\pi}{4} + k\pi, \quad k \in \mathbb{Z}$$

$$\therefore \theta - \phi = \frac{\pi}{4} \quad \{\text{since } 0 < \theta, \phi < \frac{\pi}{2}\}$$

$$\therefore \arctan 5 - \arctan \frac{2}{3} = \frac{\pi}{4}$$

$$\begin{aligned} \text{b Let } \arctan \frac{1}{5} = \theta \quad \text{and} \quad \arctan \frac{2}{3} = \phi \\ \therefore \frac{1}{5} = \tan \theta \quad \therefore \frac{2}{3} = \tan \phi \\ \therefore -\frac{2}{3} = \tan(-\phi) \end{aligned}$$

$$\text{Now } \tan(\theta - (-\phi)) = \frac{\tan \theta - \tan(-\phi)}{1 + \tan \theta \tan(-\phi)}$$

$$\therefore \tan(\theta + \phi) = \frac{\tan \theta + \tan \phi}{1 - \tan \theta \tan \phi}$$

$$\therefore \tan(\theta + \phi) = \frac{\frac{1}{5} + \frac{2}{3}}{1 - \frac{1}{5} \times \frac{2}{3}}$$

$$\begin{aligned} &= \frac{\frac{13}{15}}{\frac{13}{15}} \\ &= 1 \end{aligned}$$

$$\therefore \theta + \phi = \frac{\pi}{4} + k\pi, \quad k \in \mathbb{Z}$$

$$\therefore \theta + \phi = \frac{\pi}{4} \quad \{\text{since } 0 < \theta, \phi < \frac{\pi}{2}\}$$

$$\therefore \arctan \frac{1}{5} + \arctan \frac{2}{3} = \frac{\pi}{4}$$

$$\text{29 a Let } \arctan \frac{4}{3} = \theta \quad \text{and} \quad \arctan \frac{1}{2} = \phi$$

$$\therefore \frac{4}{3} = \tan \theta \quad \therefore \frac{1}{2} = \tan \phi$$

$$\therefore \tan 2\phi = \frac{2(\frac{1}{2})}{1 - (\frac{1}{2})^2} = \frac{1}{\frac{3}{4}} = \frac{4}{3}$$

$$\text{Now } \tan(\theta - 2\phi) = \frac{\tan \theta - \tan 2\phi}{1 + \tan \theta \tan 2\phi}$$

$$\begin{aligned} \therefore \tan(\theta - 2\phi) &= \frac{\frac{4}{3} - \frac{4}{3}}{1 + \frac{4}{3} \times \frac{4}{3}} \\ &= 0 \end{aligned}$$

$$\therefore \theta - 2\phi = 0 + k\pi, \quad k \in \mathbb{Z}$$

$$\therefore \theta - 2\phi = 0 \quad \{\text{since } 0 < \theta, \phi < \frac{\pi}{2}\}$$

$$\therefore \arctan \frac{4}{3} - 2 \arctan \frac{1}{2} = 0$$



**b** Let  $\arctan \frac{1}{5} = \theta$  and  $\arctan \frac{1}{239} = \phi$   
 $\therefore \frac{1}{5} = \tan \theta$   $\therefore \frac{1}{239} = \tan \phi$

$$\begin{aligned} \therefore \tan 2\theta &= \frac{2(\frac{1}{5})}{1 - (\frac{1}{5})^2} & \text{and} & \tan 4\theta = \frac{2(\frac{5}{12})}{1 - (\frac{5}{12})^2} \\ &= \frac{\frac{2}{5}}{\frac{24}{25}} & & = \frac{\frac{5}{6}}{1 - \frac{25}{144}} \\ &= \frac{5}{12} & & = \frac{\frac{5}{6}}{\frac{119}{144}} \\ & & & = \frac{120}{119} \end{aligned}$$

$$\begin{aligned} \therefore \tan(4\theta - \phi) &= \frac{\tan 4\theta - \tan \phi}{1 + \tan 4\theta \tan \phi} \\ &= \frac{\frac{120}{119} - \frac{1}{239}}{1 + \frac{120}{119} \times \frac{1}{239}} \\ &= \frac{\frac{28\,680 - 119}{28\,441}}{\frac{28\,441 + 120}{28\,441}} \\ &= \frac{28\,561}{28\,441} \times \frac{28\,441}{28\,561} \\ &= 1 \end{aligned}$$

$$\therefore 4\theta - \phi = \frac{\pi}{4} + k\pi, \quad k \in \mathbb{Z}$$

$$\therefore 4\theta - \phi = \frac{\pi}{4} \quad \{\text{since } 0 < \theta, \phi < \frac{\pi}{4}\}$$

$$\therefore 4 \arctan \frac{1}{5} - \arctan \frac{1}{239} = \frac{\pi}{4}$$

**30 a**  $\sin(A+B) + \sin(A-B) = \sin A \cos B + \cancel{\cos A \sin B} + \sin A \cos B - \cancel{\cos A \sin B}$   
 $= 2 \sin A \cos B$

**b**  $2 \sin A \cos B = \sin(A+B) + \sin(A-B)$

$$\therefore \sin A \cos B = \frac{1}{2} \sin(A+B) + \frac{1}{2} \sin(A-B) \quad \{\text{dividing both sides by 2}\}$$

**c i**  $\sin 3\theta \cos \theta$   
 $= \frac{1}{2} \sin(3\theta + \theta) + \frac{1}{2} \sin(3\theta - \theta)$   
 $= \frac{1}{2} \sin 4\theta + \frac{1}{2} \sin 2\theta$

**ii**  $\sin 6\alpha \cos \alpha$   
 $= \frac{1}{2} \sin(6\alpha + \alpha) + \frac{1}{2} \sin(6\alpha - \alpha)$   
 $= \frac{1}{2} \sin 7\alpha + \frac{1}{2} \sin 5\alpha$

**iii**  $2 \sin 5\beta \cos \beta$   
 $= 2 \left[ \frac{1}{2} \sin(5\beta + \beta) + \frac{1}{2} \sin(5\beta - \beta) \right]$   
 $= \sin 6\beta + \sin 4\beta$

**iv**  $4 \cos \theta \sin 4\theta$   
 $= 4 [\sin 4\theta \cos \theta]$   
 $= 4 \left[ \frac{1}{2} \sin 5\theta + \frac{1}{2} \sin 3\theta \right]$   
 $= 2 \sin 5\theta + 2 \sin 3\theta$

**v**  $6 \cos 4\alpha \sin 3\alpha$   
 $= 6 \sin 3\alpha \cos 4\alpha$   
 $= 6 \left[ \frac{1}{2} \sin 7\alpha + \frac{1}{2} \sin(-\alpha) \right]$   
 $= 3 \sin 7\alpha + 3 \sin(-\alpha)$   
 $= 3 \sin 7\alpha - 3 \sin \alpha$

**vi**  $\frac{1}{3} \cos 5A \sin 3A$   
 $= \frac{1}{3} \sin 3A \cos 5A$   
 $= \frac{1}{3} \left[ \frac{1}{2} \sin 8A + \frac{1}{2} \sin(-2A) \right]$   
 $= \frac{1}{6} \sin 8A - \frac{1}{6} \sin 2A$

$$\begin{aligned} \mathbf{31} \quad \mathbf{a} \quad \cos(A+B) + \cos(A-B) &= \cos A \cos B - \cancel{\sin A \sin B} + \cos A \cos B + \cancel{\sin A \sin B} \\ &= 2 \cos A \cos B \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad 2 \cos A \cos B &= \cos(A+B) + \cos(A-B) \\ \therefore \cos A \cos B &= \frac{1}{2} \cos(A+B) + \frac{1}{2} \cos(A-B) \quad \{\text{dividing both sides by 2}\} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad \mathbf{i} \quad \cos 4\theta \cos \theta &= \frac{1}{2} \cos 5\theta + \frac{1}{2} \cos 3\theta \end{aligned}$$

$$\begin{aligned} \mathbf{iii} \quad 2 \cos 3\beta \cos \beta &= 2 \left[ \frac{1}{2} \cos 4\beta + \frac{1}{2} \cos 2\beta \right] \\ &= \cos 4\beta + \cos 2\beta \end{aligned}$$

$$\begin{aligned} \mathbf{v} \quad 3 \cos P \cos 4P &= 3 \cos 4P \cos P \\ &= 3 \left[ \frac{1}{2} \cos 5P + \frac{1}{2} \cos 3P \right] \\ &= \frac{3}{2} \cos 5P + \frac{3}{2} \cos 3P \end{aligned}$$

$$\begin{aligned} \mathbf{ii} \quad \cos 7\alpha \cos \alpha &= \frac{1}{2} \cos 8\alpha + \frac{1}{2} \cos 6\alpha \end{aligned}$$

$$\begin{aligned} \mathbf{iv} \quad 6 \cos x \cos 7x &= 6 \cos 7x \cos x \\ &= 6 \left[ \frac{1}{2} \cos 8x + \frac{1}{2} \cos 6x \right] \\ &= 3 \cos 8x + 3 \cos 6x \end{aligned}$$

$$\begin{aligned} \mathbf{vi} \quad \frac{1}{4} \cos 4x \cos 2x &= \frac{1}{4} \left[ \frac{1}{2} \cos 6x + \frac{1}{2} \cos 2x \right] \\ &= \frac{1}{8} \cos 6x + \frac{1}{8} \cos 2x \end{aligned}$$

$$\begin{aligned} \mathbf{32} \quad \mathbf{a} \quad \cos(A-B) - \cos(A+B) &= \cancel{\cos A \cos B} + \sin A \sin B - (\cancel{\cos A \cos B} - \sin A \sin B) \\ &= \sin A \sin B + \sin A \sin B \\ &= 2 \sin A \sin B \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad 2 \sin A \sin B &= \cos(A-B) - \cos(A+B) \\ \therefore \sin A \sin B &= \frac{1}{2} \cos(A-B) - \frac{1}{2} \cos(A+B) \quad \{\text{dividing both sides by 2}\} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad \mathbf{i} \quad \sin 3\theta \sin \theta &= \frac{1}{2} \cos 2\theta - \frac{1}{2} \cos 4\theta \end{aligned}$$

$$\begin{aligned} \mathbf{iii} \quad 2 \sin 5\beta \sin \beta &= 2 \left[ \frac{1}{2} \cos 4\beta - \frac{1}{2} \cos 6\beta \right] \\ &= \cos 4\beta - \cos 6\beta \end{aligned}$$

$$\begin{aligned} \mathbf{v} \quad 10 \sin 2A \sin 8A &= 10 \sin 8A \sin 2A \\ &= 10 \left[ \frac{1}{2} \cos 6A - \frac{1}{2} \cos 10A \right] \\ &= 5 \cos 6A - 5 \cos 10A \end{aligned}$$

$$\begin{aligned} \mathbf{ii} \quad \sin 6\alpha \sin \alpha &= \frac{1}{2} \cos 5\alpha - \frac{1}{2} \cos 7\alpha \end{aligned}$$

$$\begin{aligned} \mathbf{iv} \quad 4 \sin \theta \sin 4\theta &= 4 \sin 4\theta \sin \theta \\ &= 4 \left[ \frac{1}{2} \cos 3\theta - \frac{1}{2} \cos 5\theta \right] \\ &= 2 \cos 3\theta - 2 \cos 5\theta \end{aligned}$$

$$\begin{aligned} \mathbf{vi} \quad \frac{1}{5} \sin 3M \sin 7M &= \frac{1}{5} \sin 7M \sin 3M \\ &= \frac{1}{5} \left[ \frac{1}{2} \cos 4M - \frac{1}{2} \cos 10M \right] \\ &= \frac{1}{10} \cos 4M - \frac{1}{10} \cos 10M \end{aligned}$$

$$\begin{aligned} \mathbf{33} \quad \mathbf{a} \quad (1) \quad \text{becomes} \quad \sin A \cos A &= \frac{1}{2} \sin(A+A) + \frac{1}{2} \sin(A-A) \\ &= \frac{1}{2} \sin(2A) + \frac{1}{2} \sin(0) \\ &= \frac{1}{2} \sin 2A \end{aligned}$$

$$\begin{aligned} (2) \quad \text{becomes} \quad \cos A \cos A &= \frac{1}{2} \cos(A+A) + \frac{1}{2} \cos(A-A) \\ \therefore \cos^2 A &= \frac{1}{2} \cos(2A) + \frac{1}{2} \cos(0) \\ &= \frac{1}{2} \cos 2A + \frac{1}{2} \end{aligned}$$

$$\begin{aligned}
 (3) \quad & \text{becomes } \sin A \sin A = \frac{1}{2} \cos(A - A) - \frac{1}{2} \cos(A + A) \\
 & \therefore \sin^2 A = \frac{1}{2} \cos(0) - \frac{1}{2} \cos(2A) \\
 & = \frac{1}{2} - \frac{1}{2} \cos 2A
 \end{aligned}$$

**b**  $A + B = S$  and  $A - B = D$

$$\begin{aligned}
 \text{i} \quad & S + D = (A + B) + (A - B) \quad \text{and} \quad S - D = (A + B) - (A - B) \\
 & = 2A \qquad \qquad \qquad = 2B \\
 & \therefore A = \frac{S+D}{2} \qquad \qquad \qquad \therefore B = \frac{S-D}{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{ii} \quad & \sin A \cos B = \frac{1}{2} \sin(A + B) + \frac{1}{2} \sin(A - B) \quad \text{becomes} \\
 & \sin\left(\frac{S+D}{2}\right) \cos\left(\frac{S-D}{2}\right) = \frac{1}{2} \sin S + \frac{1}{2} \sin D \\
 & \text{or } \sin S + \sin D = 2 \sin\left(\frac{S+D}{2}\right) \cos\left(\frac{S-D}{2}\right) \quad \dots (4)
 \end{aligned}$$

$$\begin{aligned}
 \text{iii} \quad & \text{Replacing } D \text{ by } (-D) \text{ in (4) gives } \sin S + \sin(-D) = 2 \sin\left(\frac{S-D}{2}\right) \cos\left(\frac{S+D}{2}\right) \\
 & \text{or } \sin S - \sin D = 2 \cos\left(\frac{S+D}{2}\right) \sin\left(\frac{S-D}{2}\right)
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & \cos A \cos B = \frac{1}{2} \cos(A + B) + \frac{1}{2} \cos(A - B) \quad \text{becomes} \\
 & \cos\left(\frac{S+D}{2}\right) \cos\left(\frac{S-D}{2}\right) = \frac{1}{2} \cos S + \frac{1}{2} \cos D \\
 & \text{or } \cos S + \cos D = 2 \cos\left(\frac{S+D}{2}\right) \cos\left(\frac{S-D}{2}\right)
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad & \sin A \sin B = \frac{1}{2} \cos(A - B) - \frac{1}{2} \cos(A + B) \quad \text{becomes} \\
 & \sin\left(\frac{S+D}{2}\right) \sin\left(\frac{S-D}{2}\right) = \frac{1}{2} \cos D - \frac{1}{2} \cos S \\
 & \text{or } \cos D - \cos S = 2 \sin\left(\frac{S+D}{2}\right) \sin\left(\frac{S-D}{2}\right)
 \end{aligned}$$

**34 a**  $\sin 5x + \sin x$   
 $= 2 \sin\left(\frac{5x+x}{2}\right) \cos\left(\frac{5x-x}{2}\right)$   
 $= 2 \sin 3x \cos 2x$

**c**  $\cos 3\alpha - \cos \alpha$   
 $= -2 \sin\left(\frac{3\alpha+\alpha}{2}\right) \sin\left(\frac{3\alpha-\alpha}{2}\right)$   
 $= -2 \sin 2\alpha \sin \alpha$

**e**  $\cos 7\alpha - \cos \alpha$   
 $= -2 \sin\left(\frac{7\alpha+\alpha}{2}\right) \sin\left(\frac{7\alpha-\alpha}{2}\right)$   
 $= -2 \sin 4\alpha \sin 3\alpha$

**g**  $\cos 2B - \cos 4B$   
 $= -[\cos 4B - \cos 2B]$   
 $= -\left[-2 \sin\left(\frac{4B+2B}{2}\right) \sin\left(\frac{4B-2B}{2}\right)\right]$   
 $= 2 \sin 3B \sin B$

**b**  $\cos 8A + \cos 2A$   
 $= 2 \cos\left(\frac{8A+2A}{2}\right) \cos\left(\frac{8A-2A}{2}\right)$   
 $= 2 \cos 5A \cos 3A$

**d**  $\sin 5\theta - \sin 3\theta$   
 $= 2 \cos\left(\frac{5\theta+3\theta}{2}\right) \sin\left(\frac{5\theta-3\theta}{2}\right)$   
 $= 2 \cos 4\theta \sin \theta$

**f**  $\sin 3\alpha + \sin 7\alpha$   
 $= \sin 7\alpha + \sin 3\alpha$   
 $= 2 \sin\left(\frac{7\alpha+3\alpha}{2}\right) \cos\left(\frac{7\alpha-3\alpha}{2}\right)$   
 $= 2 \sin 5\alpha \cos 2\alpha$

**h**  $\sin(x+h) - \sin x$   
 $= 2 \cos\left(\frac{x+h+x}{2}\right) \sin\left(\frac{x+h-x}{2}\right)$   
 $= 2 \cos\left(\frac{2x+h}{2}\right) \sin \frac{h}{2}$   
 $= 2 \cos\left(x + \frac{h}{2}\right) \sin \frac{h}{2}$

$$\begin{aligned}
& \text{i} \quad \cos(x+h) - \cos x \\
&= -2 \sin\left(\frac{x+h+x}{2}\right) \sin\left(\frac{x+h-x}{2}\right) \\
&= -2 \sin\left(\frac{2x+h}{2}\right) \sin \frac{h}{2} \\
&= -2 \sin\left(x + \frac{h}{2}\right) \sin \frac{h}{2}
\end{aligned}$$

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$$\sin A = \cos B + \cos C$$

$$\therefore \sin(\pi - [B + C]) = \cos B + \cos C$$

$$\therefore \sin[B + C] = \cos B + \cos C$$

$$\therefore \sin\left(2\left[\frac{B+C}{2}\right]\right) = \cos B + \cos C$$

$$\therefore 2 \sin\left(\frac{B+C}{2}\right) \cos\left(\frac{B+C}{2}\right) = 2 \cos\left(\frac{B+C}{2}\right) \cos\left(\frac{B-C}{2}\right) \quad \{\text{using 34}\}$$

$$\therefore 2 \cos\left(\frac{B+C}{2}\right) \left[\sin\left(\frac{B+C}{2}\right) - \cos\left(\frac{B-C}{2}\right)\right] = 0$$

$$\therefore \cos\left(\frac{B+C}{2}\right) = 0 \quad \dots (1) \quad \text{or} \quad \sin\left(\frac{B+C}{2}\right) = \cos\left(\frac{B-C}{2}\right) \quad \dots (2)$$

$$\text{In (1), } \frac{B+C}{2} = \frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z}$$

$$\therefore B+C = \pi + k2\pi, \quad k \in \mathbb{Z}$$

$$\therefore B+C = \pi, 3\pi, -\pi, \text{ and so on, all of which are impossible.}$$

$$\text{In (2), } \sin\left(\frac{B+C}{2}\right) = \sin\left(\frac{\pi}{2} - \frac{B-C}{2}\right) \quad \{\cos \theta = \sin\left(\frac{\pi}{2} - \theta\right)\}$$

$$\therefore \frac{B+C}{2} = \frac{\pi - B + C}{2} \quad \text{or} \quad \frac{B+C}{2} = \pi - \left(\frac{\pi - B + C}{2}\right)$$

$$\therefore B + \cancel{C} = \pi - B + \cancel{C} \quad \text{or} \quad \cancel{B} + C = 2\pi - \pi + \cancel{B} - C$$

$$\therefore B = \frac{\pi}{2} \quad \text{or} \quad C = \frac{\pi}{2}$$

$\therefore$  the triangle is right angled at B or C.

### ACTIVITY 3

$$1 \quad f(x) = \sin x + \frac{\sin 3x}{3}$$

$$\begin{aligned}
\text{a} \quad f(x) &= \sin x + \frac{\sin(2x+x)}{3} \\
&= \sin x + \frac{\sin 2x \cos x + \cos 2x \sin x}{3} \\
&= \sin x + \frac{2 \sin x \cos^2 x + (2 \cos^2 x - 1) \sin x}{3} \\
&= \sin x + \frac{2 \sin x \cos^2 x + 2 \sin x \cos^2 x - \sin x}{3} \\
&= \sin x - \frac{1}{3} \sin x + \frac{4}{3} \sin x \cos^2 x \\
&= \frac{2}{3} \sin x + \frac{4}{3} \sin x \cos^2 x \\
&= \frac{2}{3} \sin x (2 \cos^2 x + 1)
\end{aligned}$$



**b**  $f(x) = 0$  when  $\frac{2}{3} \sin x (2 \cos^2 x + 1) = 0$

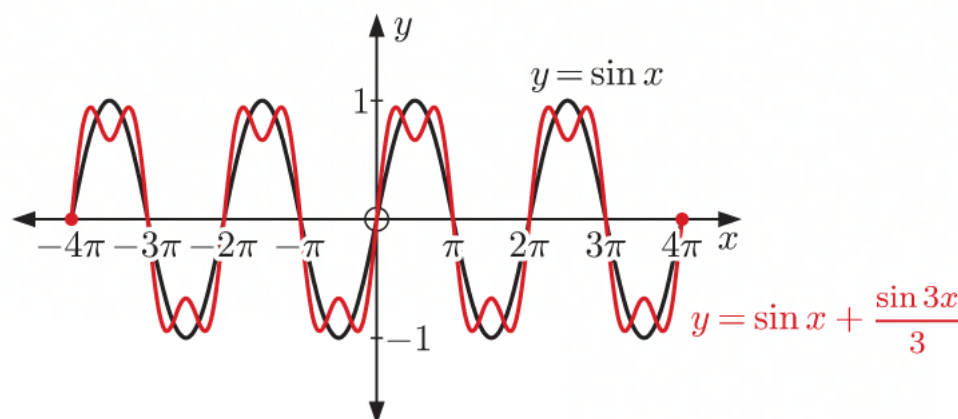
$$\therefore \sin x = 0 \quad \text{or} \quad 2 \cos^2 x + 1 = 0$$

$$\therefore \cos^2 x = -\frac{1}{2}$$

which is impossible

$\therefore$  the  $x$ -intercepts of  $y = f(x)$  are  $-4\pi, -3\pi, -2\pi, -\pi, 0, \pi, 2\pi, 3\pi$ , and  $4\pi$  on the domain  $-4\pi \leq x \leq 4\pi$ .

**c**



Both graphs are positive for the same values of  $x$ , and negative for the same values of  $x$ . Both graphs have the same  $x$ -intercepts. Both graphs have period  $2\pi$ .

$y = \sin x + \frac{\sin 3x}{3}$  oscillates close to the peaks and troughs of  $y = \sin x$ , but does not reach the maximum of 1 or the minimum of  $-1$ .

**2**  $f(x) = \sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5}$

**a**

$$f(x) = \frac{2}{3} \sin x + \frac{4}{3} \sin x \cos^2 x + \frac{\sin(4x + x)}{5} \quad \{\text{using 1 a}\}$$

$$= \frac{2}{3} \sin x + \frac{4}{3} \sin x \cos^2 x + \frac{\sin 4x \cos x + \cos 4x \sin x}{5}$$

$$= \frac{2}{3} \sin x + \frac{4}{3} \sin x \cos^2 x + \frac{2 \sin 2x \cos 2x \cos x + (2 \cos^2(2x) - 1) \sin x}{5}$$

$$= \frac{2}{3} \sin x + \frac{4}{3} \sin x \cos^2 x + \frac{4 \sin x \cos^2 x (2 \cos^2 x - 1) + (2(2 \cos^2 x - 1)^2 - 1) \sin x}{5}$$

$$= \frac{2}{3} \sin x + \frac{4}{3} \sin x \cos^2 x + \frac{8 \sin x \cos^4 x - 4 \sin x \cos^2 x + (2(4 \cos^4 x - 4 \cos^2 x + 1) - 1) \sin x}{5}$$

$$= \frac{2}{3} \sin x + \frac{4}{3} \sin x \cos^2 x + \frac{8 \sin x \cos^4 x - 4 \sin x \cos^2 x + 8 \sin x \cos^4 x - 8 \sin x \cos^2 x + \sin x}{5}$$

$$= \frac{13}{15} \sin x - \frac{16}{15} \sin x \cos^2 x + \frac{16}{5} \sin x \cos^4 x$$

$$= \frac{1}{15} \sin x (48 \cos^4 x - 16 \cos^2 x + 13)$$

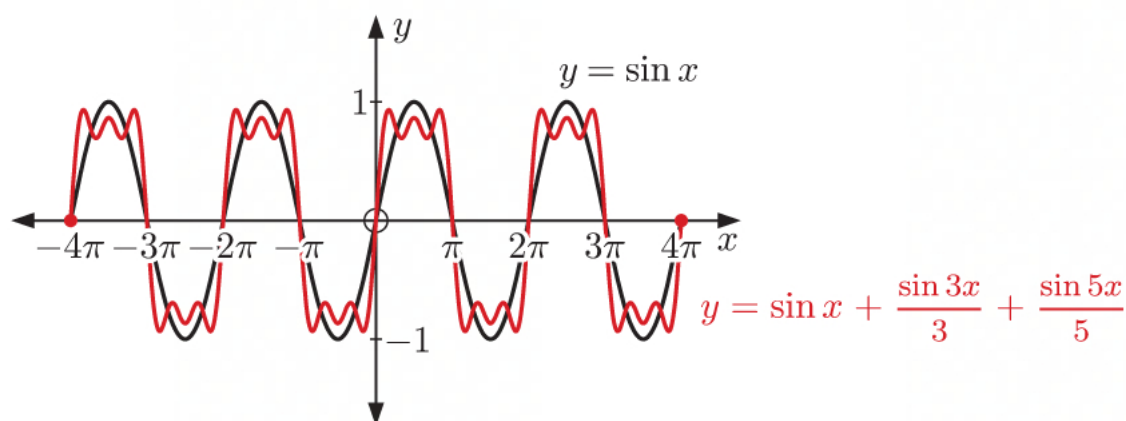
**b**  $f(x) = 0$  when  $\frac{1}{15} \sin x (48 \cos^4 x - 16 \cos^2 x + 13) = 0$

$$> 0 \quad \text{for all } x$$

$$\therefore \sin x = 0$$

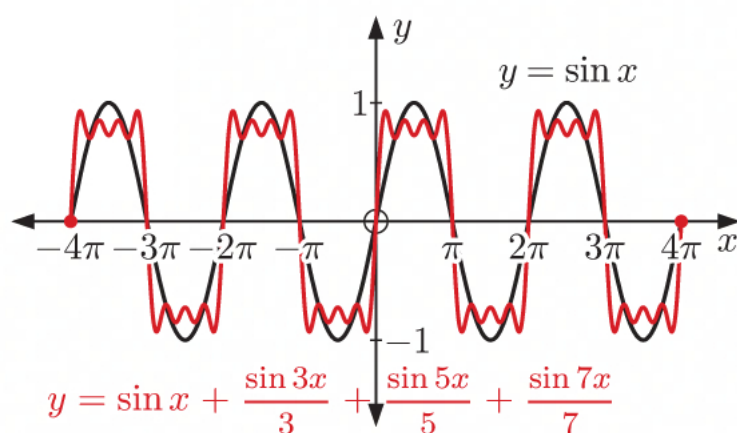
$\therefore$  the  $x$ -intercepts of  $y = f(x)$  are  $-4\pi, -3\pi, -2\pi, -\pi, 0, \pi, 2\pi, 3\pi$ , and  $4\pi$  on the domain  $-4\pi \leq x \leq 4\pi$ .

c

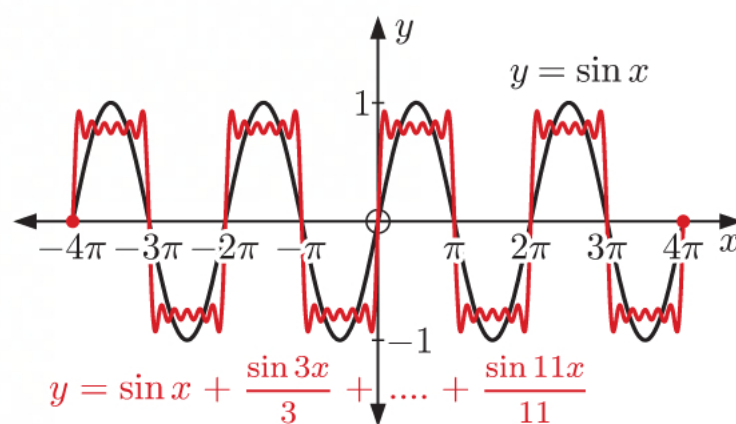


This graph is very similar to those in 1.  $y = \sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5}$  oscillates a greater number of times near the peaks and troughs of  $y = \sin x$ . The change as the graph moves from being positive to negative is becoming more pronounced.

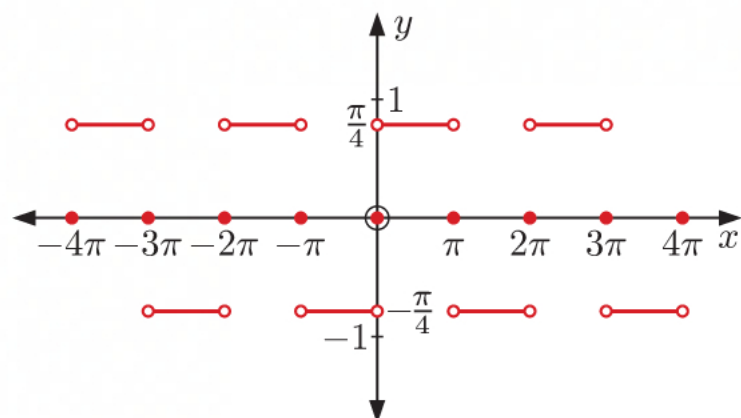
3 a



b



4 We predict that the graph of  $f(x) = \sum_{k=1}^{\infty} \frac{\sin[(2k-1)x]}{2k-1}$  will look like:



Note that  $f(x) = 0$  at  $x = k\pi$ ,  $k \in \mathbb{Z}$ .  
This is known as a **square wave** function.

## REVIEW SET 1A

1 a  $\operatorname{cosec} \frac{\pi}{4} = \frac{1}{\sin \frac{\pi}{4}}$

$$= \frac{1}{\frac{1}{\sqrt{2}}}$$

$$= \sqrt{2}$$

b  $\cot \frac{5\pi}{6} = \frac{\cos \frac{5\pi}{6}}{\sin \frac{5\pi}{6}}$

$$= \frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}}$$

$$= -\sqrt{3}$$

c  $\sec \frac{5\pi}{3} = \frac{1}{\cos \frac{5\pi}{3}}$

$$= \frac{1}{\frac{1}{2}}$$

$$= 2$$

**2 a** Let  $x = \arccos \frac{1}{\sqrt{2}}$   
 $\therefore \cos x = \frac{1}{\sqrt{2}}$   
 $\therefore x = \frac{\pi}{4} \quad \{0 \leq x \leq \pi\}$   
 $\therefore \arccos \frac{1}{\sqrt{2}} = \frac{\pi}{4}$

**c** Let  $x = \arcsin(-\frac{1}{2})$   
 $\therefore \sin x = -\frac{1}{2}$   
 $\therefore x = -\frac{\pi}{6} \quad \{-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}\}$   
 $\therefore \arcsin(-\frac{1}{2}) = -\frac{\pi}{6}$

**3 a**  $\sec x = \sqrt{2}, \quad 0 \leq x \leq 2\pi$   
 $\therefore \frac{1}{\cos x} = \sqrt{2}$   
 $\therefore \cos x = \frac{1}{\sqrt{2}}$   
 $\therefore x = \frac{\pi}{4} \text{ or } \frac{7\pi}{4}$

**4 a**  $3 \cos(-\theta) - 2 \cos \theta$   
 $= 3 \cos \theta - 2 \cos \theta$   
 $= \cos \theta$

**c**  $\sin(\theta + \frac{\pi}{2})$   
 $= \sin \theta \cos \frac{\pi}{2} + \cos \theta \sin \frac{\pi}{2}$   
 $= \sin \theta(0) + \cos \theta(1)$   
 $= \cos \theta$

**e**  $\frac{\sin \alpha - \cos \alpha}{\sin^2 \alpha - \cos^2 \alpha}$   
 $= \frac{\cancel{\sin \alpha} - \cancel{\cos \alpha}}{(\sin \alpha + \cos \alpha)(\cancel{\sin \alpha} - \cancel{\cos \alpha})}$   
 $= \frac{1}{\sin \alpha + \cos \alpha}$

**b** Let  $x = \arctan \frac{1}{\sqrt{3}}$   
 $\therefore \tan x = \frac{1}{\sqrt{3}}$   
 $\therefore x = \frac{\pi}{6} \quad \{-\frac{\pi}{2} < x < \frac{\pi}{2}\}$   
 $\therefore \arctan \frac{1}{\sqrt{3}} = \frac{\pi}{6}$

**b**  $\sqrt{3} \cos x \operatorname{cosec} x + 1 = 0, \quad 0 \leq x \leq 2\pi$   
 $\therefore \sqrt{3} \frac{\cos x}{\sin x} + 1 = 0$   
 $\therefore \frac{\cos x}{\sin x} = -\frac{1}{\sqrt{3}}$   
 $\therefore \frac{\sin x}{\cos x} = -\sqrt{3}$   
 $\therefore \tan x = -\sqrt{3}$   
 $\therefore x = \frac{2\pi}{3} \text{ or } \frac{5\pi}{3}$

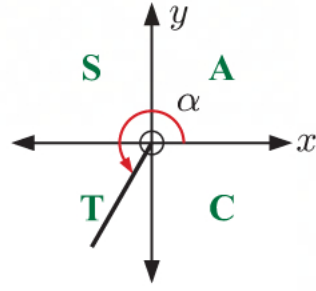
**b**  $\cos(\frac{3\pi}{2} - \theta)$   
 $= \cos \frac{3\pi}{2} \cos \theta + \sin \frac{3\pi}{2} \sin \theta$   
 $= (0) \cos \theta + (-1) \sin \theta$   
 $= -\sin \theta$

**d**  $\frac{1 - \cos^2 \theta}{1 + \cos \theta}$   
 $= \frac{(1 + \cancel{\cos \theta})(1 - \cancel{\cos \theta})}{1 + \cancel{\cos \theta}}$   
 $= 1 - \cos \theta$

**f**  $\frac{4 \sin^2 \alpha - 4}{8 \cos \alpha}$   
 $= \frac{-4(1 - \sin^2 \alpha)}{8 \cos \alpha}$   
 $= \frac{-4 \cos^2 \alpha}{8 \cos \alpha}$   
 $= \frac{-\cos \alpha}{2}$

**5**  $\sin \alpha = -\frac{3}{4}$  and  $\pi \leq \alpha \leq \frac{3\pi}{2}$

**a**  $\alpha$  is in quadrant 3, so  $\cos \alpha$  is negative.



Now  $\cos^2 \alpha + \sin^2 \alpha = 1$

$$\therefore \cos^2 \alpha + \frac{9}{16} = 1$$

$$\therefore \cos^2 \alpha = \frac{7}{16}$$

$$\therefore \cos \alpha = -\frac{\sqrt{7}}{4} \quad \{\cos \alpha < 0\}$$

**b** Using the double angle identity,  $\sin 2\alpha = 2 \sin \alpha \cos \alpha$

$$= 2\left(-\frac{3}{4}\right)\left(-\frac{\sqrt{7}}{4}\right) \quad \{\text{using a}\}$$

$$= \frac{3\sqrt{7}}{8}$$

**c** Using the double angle identity,  $\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha$

$$= \left(-\frac{\sqrt{7}}{4}\right)^2 - \left(-\frac{3}{4}\right)^2 \quad \{\text{using a}\}$$

$$= \frac{7}{16} - \frac{9}{16}$$

$$= -\frac{2}{16}$$

$$= -\frac{1}{8}$$

**d**  $\tan 2\alpha = \frac{\sin 2\alpha}{\cos 2\alpha}$

$$= \frac{\frac{3\sqrt{7}}{8}}{-\frac{1}{8}} \quad \{\text{using b and c}\}$$

$$= -3\sqrt{7}$$

**e**  $\pi \leq \alpha \leq \frac{3\pi}{2}$

$$\therefore \frac{\pi}{2} \leq \frac{\alpha}{2} \leq \frac{3\pi}{4}$$

$\therefore \frac{\alpha}{2}$  is in quadrant 2

$$\cos 2\alpha = 2 \cos^2 \alpha - 1$$

$$\therefore \cos \alpha = 2 \cos^2 \left(\frac{\alpha}{2}\right) - 1$$

$$\therefore \cos^2 \left(\frac{\alpha}{2}\right) = \frac{\cos \alpha + 1}{2}$$

$$= \frac{-\frac{\sqrt{7}}{4} + 1}{2} \quad \{\text{using a}\}$$

$$= \frac{-\sqrt{7} + 4}{8}$$

$$\therefore \cos \frac{\alpha}{2} = -\sqrt{\frac{4 - \sqrt{7}}{8}}$$

$\{\cos \frac{\alpha}{2} < 0 \text{ in quadrant 2}\}$



**f**  $\frac{\alpha}{2}$  is in quadrant 2 {from **e**}

$$\sin^2 \alpha + \cos^2 \alpha = 1$$

$$\therefore \sin^2\left(\frac{\alpha}{2}\right) + \cos^2\left(\frac{\alpha}{2}\right) = 1$$

$$\therefore \sin^2\left(\frac{\alpha}{2}\right) + \left(-\sqrt{\frac{4-\sqrt{7}}{8}}\right)^2 = 1 \quad \{\text{using **e**}\}$$

$$\therefore \sin^2\left(\frac{\alpha}{2}\right) + \frac{4-\sqrt{7}}{8} = 1$$

$$\therefore \sin^2\left(\frac{\alpha}{2}\right) = \frac{4+\sqrt{7}}{8}$$

$$\therefore \sin \frac{\alpha}{2} = \sqrt{\frac{4+\sqrt{7}}{8}} \quad \{\sin \frac{\alpha}{2} > 0 \text{ in quadrant 2}\}$$

**6 a**

$$\begin{aligned} & \sqrt{2} \sin\left(\theta - \frac{\pi}{4}\right) \\ &= \sqrt{2} \left(\sin \theta \cos \frac{\pi}{4} - \cos \theta \sin \frac{\pi}{4}\right) \\ &= \sqrt{2} \left(\sin \theta \left(\frac{1}{\sqrt{2}}\right) - \cos \theta \left(\frac{1}{\sqrt{2}}\right)\right) \\ &= \frac{\sqrt{2}}{\sqrt{2}} (\sin \theta - \cos \theta) \\ &= \sin \theta - \cos \theta \end{aligned}$$

**c**

$$\begin{aligned} & \frac{\sin 2\alpha - \sin \alpha}{\cos 2\alpha - \cos \alpha + 1} \\ &= \frac{2 \sin \alpha \cos \alpha - \sin \alpha}{2 \cos^2 \alpha - 1 - \cos \alpha + 1} \\ &= \frac{\sin \alpha (2 \cos \alpha - 1)}{\cos \alpha (2 \cos \alpha - 1)} \\ &= \frac{\sin \alpha}{\cos \alpha} \\ &= \tan \alpha \end{aligned}$$

**7 a**

$$\begin{aligned} & \cos 165^\circ \\ &= \cos(120^\circ + 45^\circ) \\ &= \cos 120^\circ \cos 45^\circ - \sin 120^\circ \sin 45^\circ \\ &= \left(-\frac{1}{2}\right) \left(\frac{1}{\sqrt{2}}\right) - \left(\frac{\sqrt{3}}{2}\right) \left(\frac{1}{\sqrt{2}}\right) \\ &= -\frac{1}{2\sqrt{2}} - \frac{\sqrt{3}}{2\sqrt{2}} \\ &= \frac{-1 - \sqrt{3}}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} \\ &= \frac{-\sqrt{2} - \sqrt{6}}{4} \end{aligned}$$

**b**

$$\begin{aligned} & \sin \theta \cos(\phi - \theta) + \cos \theta \sin(\phi - \theta) \\ &= \sin[\theta + (\phi - \theta)] \\ &= \sin(\theta + \phi - \theta) \\ &= \sin \phi \end{aligned}$$

**d**

$$\begin{aligned} \operatorname{cosec} 2x + \cot 2x &= \frac{1}{\sin 2x} + \frac{\cos 2x}{\sin 2x} \\ &= \frac{1 + \cos 2x}{\sin 2x} \\ &= \frac{\cancel{x} + 2 \cos^2 x - \cancel{x}}{2 \sin x \cos x} \\ &= \frac{2 \cos^2 x}{2 \sin x \cos x} \\ &= \frac{\cos x}{\sin x} \\ &= \cot x \end{aligned}$$

**b**

$$\begin{aligned} \tan \frac{\pi}{12} &= \tan\left(\frac{4\pi}{12} - \frac{3\pi}{12}\right) \\ &= \tan\left(\frac{\pi}{3} - \frac{\pi}{4}\right) \\ &= \frac{\tan \frac{\pi}{3} - \tan \frac{\pi}{4}}{1 + \tan \frac{\pi}{3} \tan \frac{\pi}{4}} \\ &= \frac{\sqrt{3} - 1}{1 + (\sqrt{3})(1)} \\ &= \frac{\sqrt{3} - 1}{\sqrt{3} + 1} \times \frac{\sqrt{3} - 1}{\sqrt{3} - 1} \\ &= \frac{3 - 2\sqrt{3} + 1}{3 - 1} \\ &= \frac{4 - 2\sqrt{3}}{2} \\ &= 2 - \sqrt{3} \end{aligned}$$

- 8 Let the shooter be  $x$  m from the wall.

$$\therefore \tan \alpha = \frac{20}{x}, \quad \tan 2\alpha = \frac{45}{x}$$

$$\therefore \frac{2 \tan \alpha}{1 - \tan^2 \alpha} = \frac{45}{x}$$

$$\therefore 2x \tan \alpha = 45 - 45 \tan^2 \alpha$$

$$\therefore 2x \left( \frac{20}{x} \right) = 45 - 45 \left( \frac{20}{x} \right)^2$$

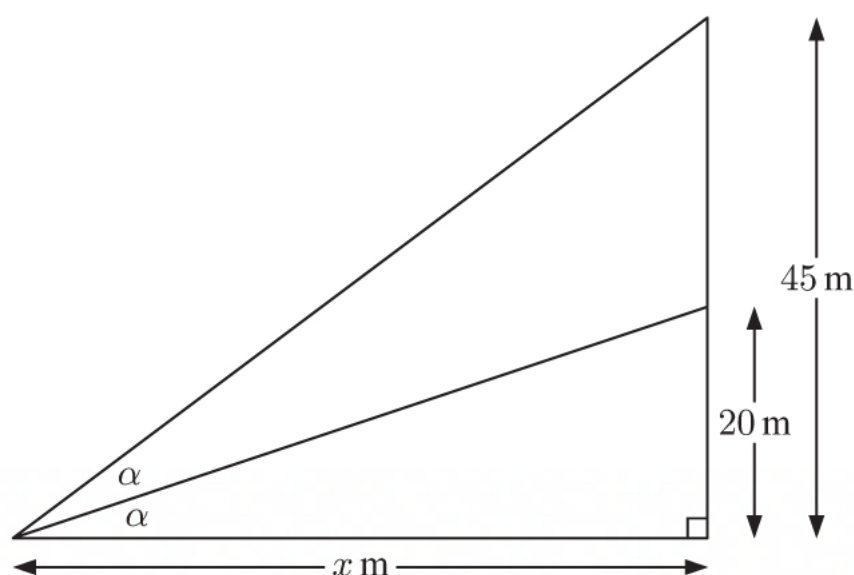
$$\therefore 40 = 45 - \frac{18\,000}{x^2}$$

$$\therefore \frac{18\,000}{x^2} = 5$$

$$\therefore x^2 = 3600$$

$$\therefore x = 60 \quad \{x > 0\}$$

So, the shooter is 60 m from the wall.



- 9  $\sec^2 x = \tan x + 1, \quad 0 \leq x \leq 2\pi$

$$\therefore \tan^2 x + 1 = \tan x + 1$$

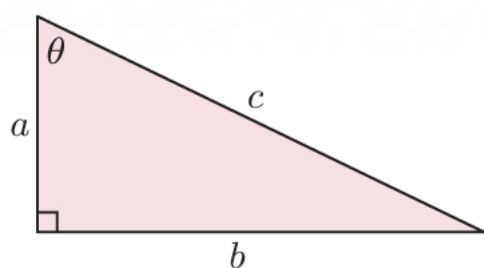
$$\therefore \tan^2 x - \tan x = 0$$

$$\therefore \tan x(\tan x - 1) = 0$$

$$\therefore \tan x = 0 \text{ or } 1$$

$$\therefore x = 0, \frac{\pi}{4}, \pi, \frac{5\pi}{4}, \text{ or } 2\pi$$

10



$$\sin \theta = \frac{b}{c} \quad \text{and} \quad \cos \theta = \frac{a}{c}$$

a  $\sin 2\theta = 2 \sin \theta \cos \theta$

$$= 2 \left( \frac{b}{c} \right) \left( \frac{a}{c} \right)$$

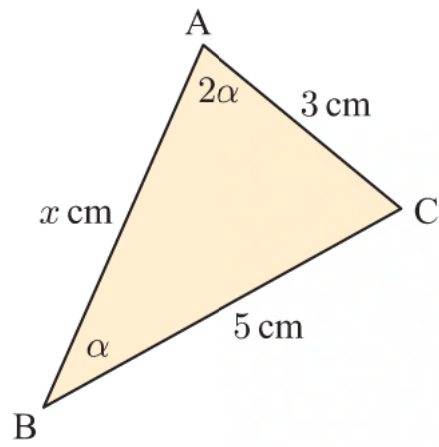
$$= \frac{2ab}{c^2}$$

b  $\cos 2\theta = \cos^2 \theta - \sin^2 \theta$

$$= \left( \frac{a}{c} \right)^2 - \left( \frac{b}{c} \right)^2$$

$$= \frac{a^2 - b^2}{c^2}$$

11



**a** By the sine rule,  $\frac{\sin 2\alpha}{5} = \frac{\sin \alpha}{3}$

$$\therefore \frac{2 \sin \alpha \cos \alpha}{\sin \alpha} = \frac{5}{3}$$

$$\therefore 2 \cos \alpha = \frac{5}{3}$$

$$\{\sin \alpha \neq 0 \text{ as } 0 < \alpha < \pi\}$$

$$\therefore \cos \alpha = \frac{5}{6}$$

**b** Using the cosine rule,

$$3^2 = x^2 + 5^2 - 2 \times x \times 5 \times \cos \alpha$$

$$\therefore 9 = x^2 + 25 - 10x\left(\frac{5}{6}\right) \quad \{\text{using a}\}$$

$$\therefore x^2 - \frac{25}{3}x + 16 = 0$$

$$\therefore 3x^2 - 25x + 48 = 0$$

**c**  $(3x - 16)(x - 3) = 0$

$$\therefore x = \frac{16}{3} \text{ or } 3$$

For  $x = 3$ , triangle ABC is isosceles.

$$\therefore \widehat{ACB} = \widehat{BAC} = \alpha \quad \{\text{base angles}\}$$

$$\therefore 2\alpha + \alpha + \alpha = \pi \quad \{\text{angles in a triangle}\}$$

$$\therefore 4\alpha = \pi$$

$$\therefore 2\alpha = \frac{\pi}{2}$$

So, triangle ABC is right angled at B.

$$\text{However, } 3^2 + 3^2 = 9 + 9 = 18 \neq 25 = 5^2.$$

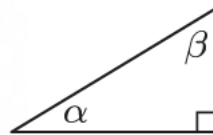
$$\therefore x = 3 \text{ is not a valid solution}$$

$$\therefore x = \frac{16}{3} \text{ is the only solution}$$

**12**  $\alpha + \beta = \frac{\pi}{2} \quad \{\text{angles in a triangle}\}$

$$\therefore \beta = \frac{\pi}{2} - \alpha$$

$$\therefore 2\beta = \pi - 2\alpha$$



So,  $\sin 2\beta = \sin(\pi - 2\alpha)$

$$= \sin \pi \cos 2\alpha - \cos \pi \sin 2\alpha$$

$$= (0) \cos 2\alpha - (-1) \sin 2\alpha$$

$$= \sin 2\alpha$$

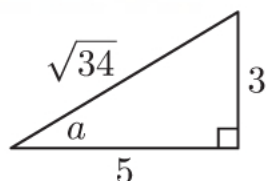
$$\begin{aligned}
 \mathbf{13} \quad 3 \sin x - 5 \cos x &= k \cos(x + a), \quad k > 0 \quad \text{and} \quad 0 < a < 2\pi \\
 &= k(\cos x \cos a - \sin x \sin a) \\
 &= k \cos a \cos x - k \sin a \sin x
 \end{aligned}$$

Equating coefficients of  $\cos x$  and  $\sin x$ ,

$$k \cos a = -5 \quad \dots (1) \quad \text{and} \quad k \sin a = -3 \quad \dots (2)$$

$$\frac{k \sin a}{k \cos a} = \frac{-3}{-5} \quad \{\text{dividing (2) by (1)}\}$$

$$\therefore \tan a = \frac{3}{5}$$



$$\text{If } 0 < a < \frac{\pi}{2}, \quad \cos a = \frac{5}{\sqrt{34}}$$

$$\begin{aligned}
 \text{Substituting into (1) gives} \quad k \times \frac{5}{\sqrt{34}} &= -5 \\
 \therefore k &= -\sqrt{34}
 \end{aligned}$$

We reject this solution as  $k > 0$ .

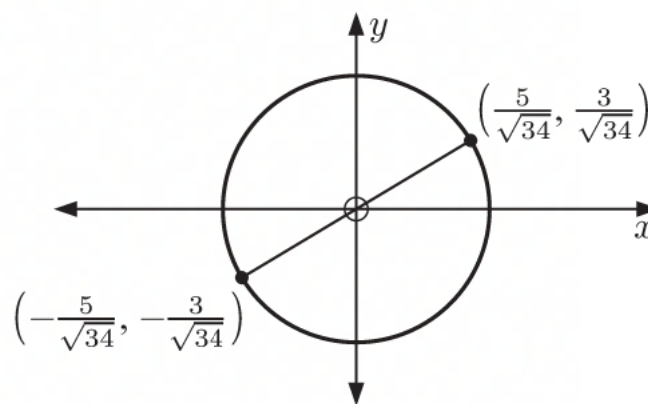
$$\text{If } \pi < a < \frac{3\pi}{2}, \quad \cos a = -\frac{5}{\sqrt{34}}$$

$$\begin{aligned}
 \text{Substituting into (1) gives} \quad k \times \left(-\frac{5}{\sqrt{34}}\right) &= -5 \\
 \therefore k &= \sqrt{34}
 \end{aligned}$$

$$\text{So, } k = \sqrt{34} \quad \text{and} \quad \tan a = \frac{3}{5}$$

$$\therefore a \approx 3.68 \quad \left\{ \pi < a < \frac{3\pi}{2} \right\}$$

$$\therefore 3 \sin x - 5 \cos x \approx \sqrt{34} \cos(x + 3.68)$$



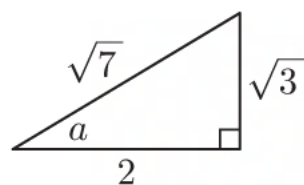
$$\begin{aligned}
 \mathbf{14} \quad \mathbf{a} \quad 2 \sin x + \sqrt{3} \cos x &= k \sin(x + a), \quad k > 0 \quad \text{and} \quad 0 < a < 2\pi \\
 &= k(\sin x \cos a + \cos x \sin a) \\
 &= k \cos a \sin x + k \sin a \cos x
 \end{aligned}$$

Equating coefficients of  $\sin x$  and  $\cos x$ ,

$$k \cos a = 2 \quad \dots (1) \quad \text{and} \quad k \sin a = \sqrt{3} \quad \dots (2)$$

$$\frac{k \sin a}{k \cos a} = \frac{\sqrt{3}}{2} \quad \{\text{dividing (2) by (1)}\}$$

$$\therefore \tan a = \frac{\sqrt{3}}{2}$$



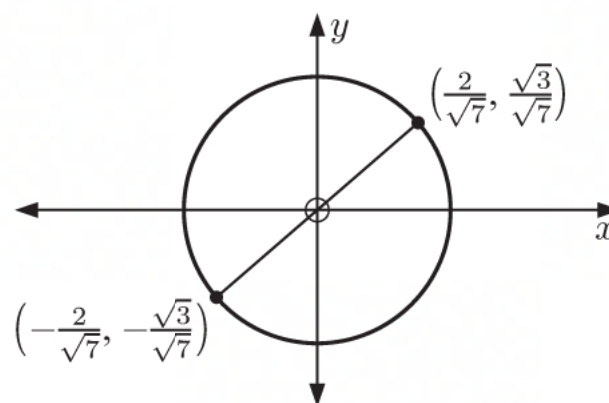
$$\text{If } 0 < a < \frac{\pi}{2}, \quad \cos a = \frac{2}{\sqrt{7}}$$

$$\begin{aligned}
 \text{Substituting into (1) gives} \quad k \times \frac{2}{\sqrt{7}} &= 2 \\
 \therefore k &= \sqrt{7}
 \end{aligned}$$

$$\text{If } \pi < a < \frac{3\pi}{2}, \quad \cos a = -\frac{2}{\sqrt{7}}$$

$$\begin{aligned}
 \text{Substituting into (1) gives} \quad k \times \left(-\frac{2}{\sqrt{7}}\right) &= 2 \\
 \therefore k &= -\sqrt{7}
 \end{aligned}$$

We reject this solution as  $k > 0$ .

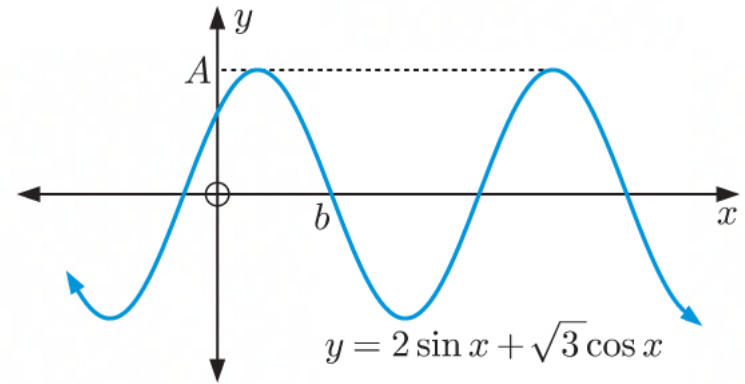




So,  $k = \sqrt{7}$  and  $\tan a = \frac{\sqrt{3}}{2}$   
 $\therefore a \approx 0.714 \quad \{0 < a < \frac{\pi}{2}\}$   
 $\therefore 2 \sin x + \sqrt{3} \cos x \approx \sqrt{7} \sin(x + 0.714)$

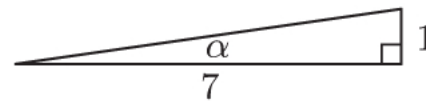
**b i**  $A = \text{amplitude}$   
 $= \sqrt{7}$

**ii**  $b$  is the smallest positive  $x$ -intercept of  
 $y = 2 \sin x + \sqrt{3} \cos x$   
 $\therefore 2 \sin b + \sqrt{3} \cos b = 0$   
 $\therefore \sqrt{7} \sin(b + 0.714) \approx 0 \quad \{\text{using a}\}$   
 $\therefore b + 0.714 \approx \pi \quad \{b > 0\}$   
 $\therefore b \approx \pi - 0.714$   
 $\therefore b \approx 2.43$

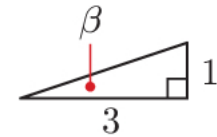


**15** Let  $\arctan \frac{1}{7} = \alpha$  and  $\arctan \frac{1}{3} = \beta$  where  $-\frac{\pi}{2} \leq \alpha, \beta \leq \frac{\pi}{2}$   
 $\therefore \tan \alpha = \frac{1}{7}$  and  $\tan \beta = \frac{1}{3}$

Since  $\tan \alpha, \tan \beta > 0$ , from the diagrams we see that  $0 < \alpha, \beta < \frac{\pi}{4}$ .



We need to find  $\tan(\alpha + 2\beta)$  where  $\tan 2\beta = \frac{2 \tan \beta}{1 - \tan^2 \beta} = \frac{\frac{2}{3}}{1 - \frac{1}{9}} = \frac{3}{4}$



Now  $\tan(\alpha + 2\beta) = \frac{\tan \alpha + \tan 2\beta}{1 - \tan \alpha \tan 2\beta} = \frac{\frac{1}{7} + \frac{3}{4}}{1 - \frac{3}{28}} = 1$

$\therefore \alpha + 2\beta = \frac{\pi}{4} + k\pi, \quad k \in \mathbb{Z}$

$\therefore \alpha + 2\beta = \frac{\pi}{4}$ , the only solution satisfying  $0 < \alpha, \beta < \frac{\pi}{4}$

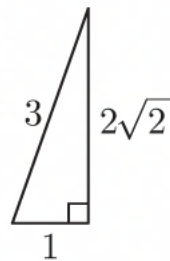
$\therefore \arctan \frac{1}{7} + 2 \arctan \frac{1}{3} = \frac{\pi}{4}$

## REVIEW SET 1B

**1**  $\cos x = -\frac{1}{3}$  and  $\pi < x < \frac{3\pi}{2}$

$x$  is in quadrant 3

$\therefore \sin x < 0$  and  
 $\tan x > 0$

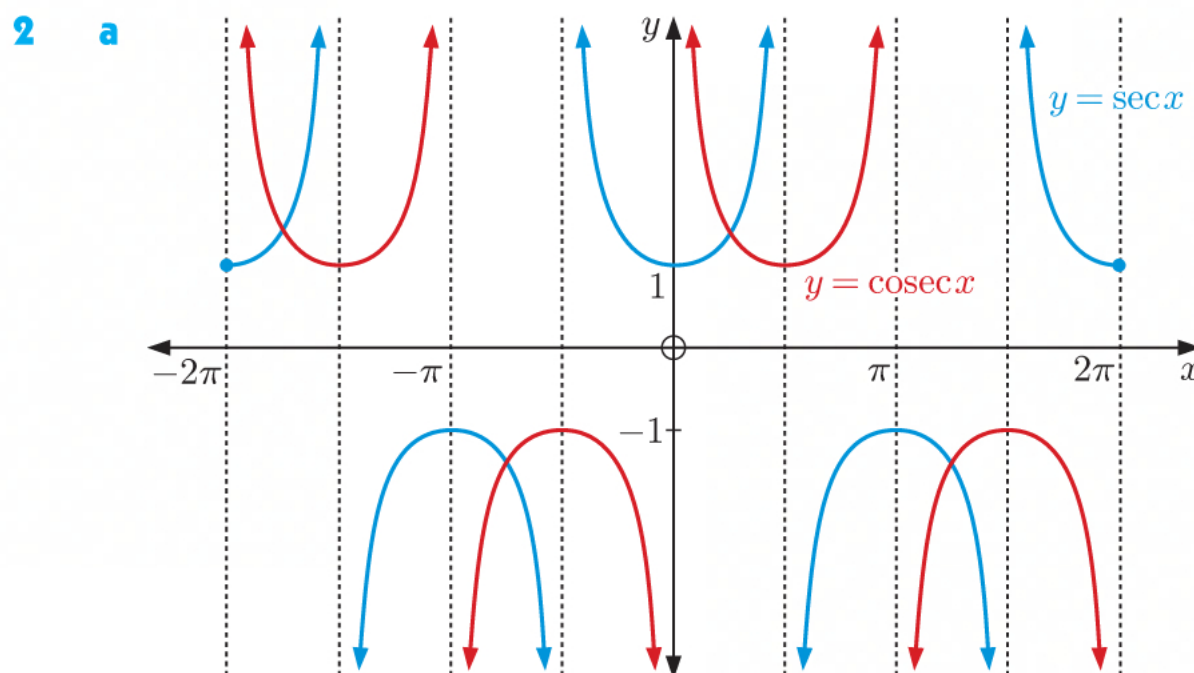


$\therefore \sin x = -\frac{2\sqrt{2}}{3}$  and  $\tan x = 2\sqrt{2}$

$\therefore \operatorname{cosec} x = \frac{1}{\sin x} = -\frac{3}{2\sqrt{2}}$

$\sec x = \frac{1}{\cos x} = -3$

$\cot x = \frac{1}{\tan x} = \frac{1}{2\sqrt{2}}$



**b**

$$\sec x = \frac{1}{\cos x}$$

$$\therefore \sec\left(x - \frac{\pi}{2}\right) = \frac{1}{\cos\left(x - \frac{\pi}{2}\right)}$$

$$= \frac{1}{\sin x}$$

$$= \operatorname{cosec} x$$

$\therefore$  a translation  $\frac{\pi}{2}$  units right maps  $y = \sec x$  onto  $y = \operatorname{cosec} x$  for all  $x \in \mathbb{R}$ .

**3**

$$\cot x = \sqrt{3}, \quad -\pi \leq x \leq \pi$$

$$\therefore \frac{1}{\tan x} = \sqrt{3}$$

$$\therefore \tan x = \frac{1}{\sqrt{3}}$$

$$\therefore x = -\frac{5\pi}{6} \text{ or } \frac{\pi}{6}$$

**4 a**

$$\arcsin x = \frac{\pi}{3}$$

$$\therefore x = \sin \frac{\pi}{3}$$

$$\therefore x = \frac{\sqrt{3}}{2}$$

**b**

$$\arctan(x - 2) = \frac{\pi}{6}$$

$$\therefore x - 2 = \tan \frac{\pi}{6}$$

$$\therefore x - 2 = \frac{1}{\sqrt{3}}$$

$$\therefore x = 2 + \frac{1}{\sqrt{3}}$$

**5 a**

$$\operatorname{cosec} x \tan x$$

$$= \frac{1}{\sin x} \times \frac{\sin x}{\cos x}$$

$$= \frac{1}{\cos x}$$

$$= \sec x$$

**b**

$$\frac{\tan x}{\sec x}$$

$$= \frac{\sin x}{\cos x} \times \frac{\cos x}{1}$$

$$= \sin x$$

**c**

$$\sec x - \tan x \sin x$$

$$= \frac{1}{\cos x} - \frac{\sin x}{\cos x} \times \sin x$$

$$= \frac{1 - \sin^2 x}{\cos x}$$

$$= \frac{\cos^2 x}{\cos x}$$

$$= \cos x$$

$$\begin{aligned}
 \text{6 a } & \cos^3 \theta + \sin^2 \theta \cos \theta \\
 &= \cos \theta (\cos^2 \theta + \sin^2 \theta) \\
 &= \cos \theta
 \end{aligned}$$

$$\begin{aligned}
 \text{b } & \frac{\cos^2 \theta - 1}{\sin \theta} \\
 &= \frac{-(1 - \cos^2 \theta)}{\sin \theta} \\
 &= -\frac{\sin^2 \theta}{\sin \theta} \\
 &= -\sin \theta
 \end{aligned}$$

$$\begin{aligned}
 \text{c } & 5 - 5 \sin^2 \theta \\
 &= 5(1 - \sin^2 \theta) \\
 &= 5 \cos^2 \theta
 \end{aligned}$$

$$\begin{aligned}
 \text{d } & \frac{\sin^2 \theta - 1}{\cos \theta} \\
 &= -\frac{(1 - \sin^2 \theta)}{\cos \theta} \\
 &= -\frac{\cos^2 \theta}{\cos \theta} \\
 &= -\cos \theta
 \end{aligned}$$

$$\begin{aligned}
 \text{e } & \frac{\tan \theta + \cot \theta}{\sec \theta} = \frac{\frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta}}{\frac{1}{\cos \theta}} \\
 &= \cos \theta \left( \frac{\sin^2 \theta + \cos^2 \theta}{\cos \theta \sin \theta} \right) \\
 &= \frac{\cancel{\cos \theta}}{\cancel{\cos \theta} \sin \theta} \quad \{\text{as } \sin^2 \theta + \cos^2 \theta = 1\} \\
 &= \frac{1}{\sin \theta} \\
 &= \operatorname{cosec} \theta
 \end{aligned}$$

$$\begin{aligned}
 \text{f } & \cos^2 \theta (\tan \theta + 1)^2 - 1 = \cos^2 \theta (\tan^2 \theta + 2 \tan \theta + 1) - 1 \\
 &= \cos^2 \theta \left( \frac{\sin^2 \theta}{\cos^2 \theta} + \frac{2 \sin \theta}{\cos \theta} + 1 \right) - 1 \\
 &= \sin^2 \theta + 2 \sin \theta \cos \theta + \cos^2 \theta - 1 \\
 &= 2 \sin \theta \cos \theta \quad \{\text{as } \sin^2 \theta + \cos^2 \theta = 1\} \\
 &= \sin 2\theta
 \end{aligned}$$

$$\text{7 } \sin A = \frac{5}{13} \quad \text{and} \quad \cos A = \frac{12}{13}$$

$$\begin{aligned}
 \text{a } & \sin 2A \\
 &= 2 \sin A \cos A \\
 &= 2 \left( \frac{5}{13} \right) \left( \frac{12}{13} \right) \\
 &= \frac{120}{169}
 \end{aligned}$$

$$\begin{aligned}
 \text{b } & \cos 2A \\
 &= \cos^2 A - \sin^2 A \\
 &= \left( \frac{12}{13} \right)^2 - \left( \frac{5}{13} \right)^2 \\
 &= \frac{144 - 25}{169} \\
 &= \frac{119}{169}
 \end{aligned}$$

$$\begin{aligned}
 \text{c } & \tan 2A = \frac{\sin 2A}{\cos 2A} \\
 &= \frac{\frac{120}{169}}{\frac{119}{169}} \\
 &= \frac{120}{119} \quad \{\text{using a and b}\}
 \end{aligned}$$

$$\begin{aligned}
 \text{8 a } & \frac{\cos \theta}{1 + \sin \theta} + \frac{1 + \sin \theta}{\cos \theta} \\
 &= \frac{\cos^2 \theta + (1 + \sin \theta)^2}{(1 + \sin \theta) \cos \theta} \\
 &= \frac{\cos^2 \theta + 1 + 2 \sin \theta + \sin^2 \theta}{(1 + \sin \theta) \cos \theta} \\
 &= \frac{2 + 2 \sin \theta}{(1 + \sin \theta) \cos \theta} \quad \{\cos^2 \theta + \sin^2 \theta = 1\} \\
 &= \frac{2(1 + \sin \theta)}{(1 + \sin \theta) \cos \theta} \\
 &= \frac{2}{\cos \theta} \\
 &= 2 \sec \theta
 \end{aligned}$$

$$\begin{aligned}
 \text{b } & \left( 1 + \frac{1}{\cos \theta} \right) (\cos \theta - \cos^2 \theta) \\
 &= \cancel{\cos \theta} - \cos^2 \theta + 1 - \cancel{\cos \theta} \\
 &= 1 - \cos^2 \theta \\
 &= \sin^2 \theta
 \end{aligned}$$

$$\begin{aligned}
 9 \quad & \cos 2\theta = 1 - 2\sin^2 \theta \\
 & \therefore \cos \frac{\pi}{4} = 1 - 2\sin^2\left(\frac{\pi}{8}\right) \quad \{\text{letting } \theta = \frac{\pi}{8}\} \\
 & \therefore \frac{1}{\sqrt{2}} = 1 - 2\sin^2\left(\frac{\pi}{8}\right) \\
 & \therefore 2\sin^2\left(\frac{\pi}{8}\right) = 1 - \frac{1}{\sqrt{2}} = \frac{\sqrt{2}-1}{\sqrt{2}} \\
 & \therefore \sin^2\left(\frac{\pi}{8}\right) = \frac{\sqrt{2}-1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} \\
 & \therefore \sin^2\left(\frac{\pi}{8}\right) = \frac{2-\sqrt{2}}{4} \\
 & \therefore \sin \frac{\pi}{8} = \pm \frac{\sqrt{2-\sqrt{2}}}{2}
 \end{aligned}$$

But  $\sin \frac{\pi}{8}$  is positive as  $\frac{\pi}{8}$  is in quadrant 1.

$$\therefore \sin \frac{\pi}{8} = \frac{1}{2} \sqrt{2-\sqrt{2}}$$

$$\begin{aligned}
 10 \quad a \quad & \sqrt{3} \cos x + \sin 2x = 0, \quad -\pi \leq x \leq \pi \\
 & \therefore \sqrt{3} \cos x + 2 \sin x \cos x = 0 \\
 & \therefore \cos x (\sqrt{3} + 2 \sin x) = 0 \\
 & \therefore \cos x = 0 \quad \text{or} \quad \sin x = -\frac{\sqrt{3}}{2} \\
 & \therefore x = -\frac{2\pi}{3}, -\frac{\pi}{2}, -\frac{\pi}{3}, \text{ or } \frac{\pi}{2}
 \end{aligned}$$

$$\begin{aligned}
 b \quad & \frac{1 - \cos 2\theta}{\sin 2\theta} = \sqrt{3}, \quad 0 < \theta < \frac{\pi}{2} \\
 & \therefore \frac{1 - (1 - 2\sin^2 \theta)}{2 \sin \theta \cos \theta} = \sqrt{3} \\
 & \therefore \frac{2\sin^2 \theta}{2 \sin \theta \cos \theta} = \sqrt{3} \\
 & \therefore \frac{\sin \theta}{\cos \theta} = \sqrt{3} \quad \{\sin \theta \neq 0 \text{ on } 0 < \theta < \frac{\pi}{2}\} \\
 & \therefore \tan \theta = \sqrt{3} \\
 & \therefore \theta = \frac{\pi}{3}
 \end{aligned}$$



**11**  $\sin \theta = \frac{3}{4}$  and  $\frac{\pi}{2} < \theta < \pi$

$$\begin{aligned}\sin\left(\theta + \frac{\pi}{6}\right) &= \sin \theta \cos \frac{\pi}{6} + \cos \theta \sin \frac{\pi}{6} \\ &= \frac{3}{4} \times \left(\frac{\sqrt{3}}{2}\right) + \cos \theta \times \left(\frac{1}{2}\right) \\ &= \frac{3\sqrt{3}}{8} + \frac{\cos \theta}{2}\end{aligned}$$

Now  $\cos^2 \theta + \sin^2 \theta = 1$

$$\therefore \cos^2 \theta + \left(\frac{3}{4}\right)^2 = 1$$

$$\therefore \cos^2 \theta + \frac{9}{16} = 1$$

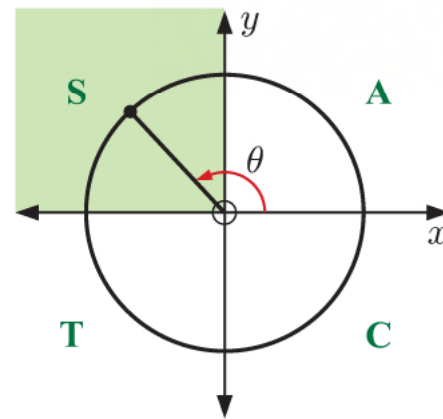
$$\therefore \cos^2 \theta = \frac{7}{16}$$

$$\therefore \cos \theta = \pm \frac{\sqrt{7}}{4}$$

But  $\frac{\pi}{2} < \theta < \pi$  where  $\cos \theta$  is negative

$$\therefore \cos \theta = -\frac{\sqrt{7}}{4}$$

$$\begin{aligned}\therefore \sin\left(\theta + \frac{\pi}{6}\right) &= \frac{3\sqrt{3}}{8} + \frac{\left(-\frac{\sqrt{7}}{4}\right)}{2} \\ &= \frac{3\sqrt{3} - \sqrt{7}}{8}\end{aligned}$$



**12**  $\tan \phi = \frac{2}{3}$  and  $\tan(\phi + \theta) = \frac{2+3}{3}$

$$\therefore \frac{\tan \phi + \tan \theta}{1 - \tan \phi \tan \theta} = \frac{5}{3}$$

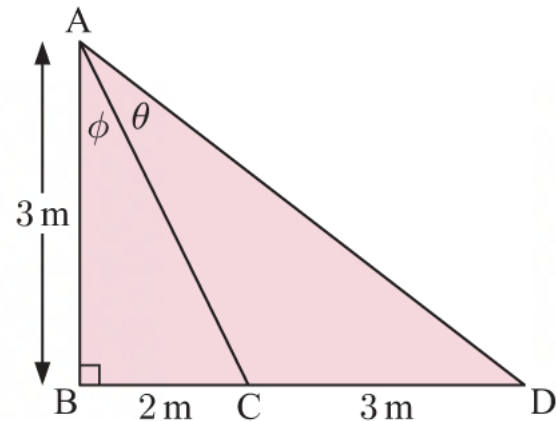
$$\therefore \frac{3}{3} \left( \frac{\frac{2}{3} + \tan \theta}{1 - \frac{2}{3} \tan \theta} \right) = \frac{5}{3}$$

$$\therefore \frac{2 + 3 \tan \theta}{3 - 2 \tan \theta} = \frac{5}{3}$$

$$\therefore 6 + 9 \tan \theta = 15 - 10 \tan \theta$$

$$\therefore 19 \tan \theta = 9$$

$$\therefore \tan \theta = \frac{9}{19}$$



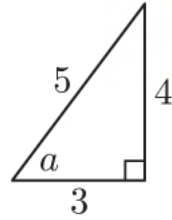
$$\begin{aligned}
 13 \quad 3 \sin x + 4 \cos x &= k \sin(x + a), \quad k > 0 \quad \text{and} \quad 0 < a < 2\pi \\
 &= k(\sin x \cos a + \cos x \sin a) \\
 &= k \cos a \sin x + k \sin a \cos x
 \end{aligned}$$

Equating coefficients of  $\sin x$  and  $\cos x$ ,

$$k \cos a = 3 \quad \dots (1) \quad \text{and} \quad k \sin a = 4 \quad \dots (2)$$

$$\frac{k \sin a}{k \cos a} = \frac{4}{3} \quad \{\text{dividing (2) by (1)}\}$$

$$\therefore \tan a = \frac{4}{3}$$



$$\text{If } 0 < a < \frac{\pi}{2}, \quad \cos a = \frac{3}{5}$$

$$\begin{aligned}
 \text{Substituting into (1) gives} \quad k \times \frac{3}{5} &= 3 \\
 \therefore k &= 5
 \end{aligned}$$

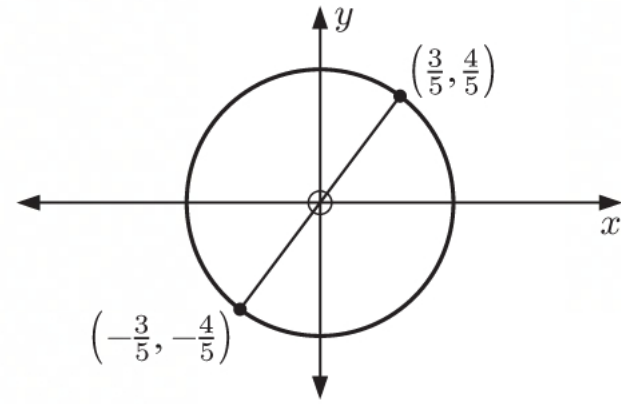
$$\text{If } \pi < a < \frac{3\pi}{2}, \quad \cos a = -\frac{3}{5}$$

$$\begin{aligned}
 \text{Substituting into (1) gives} \quad k \times \left(-\frac{3}{5}\right) &= 3 \\
 \therefore k &= -5
 \end{aligned}$$

We reject this solution as  $k > 0$ .

$$\begin{aligned}
 \text{So, } k = 5 \quad \text{and} \quad \tan a &= \frac{4}{3} \\
 \therefore a &\approx 0.927 \quad \{0 < a < \frac{\pi}{2}\}
 \end{aligned}$$

$$\therefore 3 \sin x + 4 \cos x \approx 5 \sin(x + 0.927)$$



$$\begin{aligned}
 14 \quad \tan 2\theta &= \frac{4}{3} \\
 \therefore \frac{2 \tan \theta}{1 - \tan^2 \theta} &= \frac{4}{3} \\
 \therefore 6 \tan \theta &= 4 - 4 \tan^2 \theta \\
 \therefore 2(2 \tan^2 \theta + 3 \tan \theta - 2) &= 0 \\
 \therefore 2(2 \tan \theta - 1)(\tan \theta + 2) &= 0 \\
 \therefore \tan \theta &= \frac{1}{2} \quad \text{or} \quad -2
 \end{aligned}$$

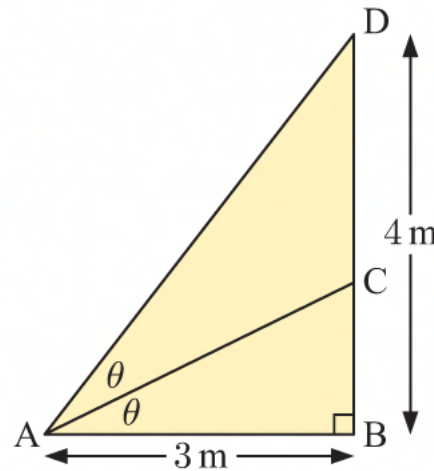
But  $\theta$  is clearly acute, so  $\tan \theta > 0$

$$\therefore \tan \theta = \frac{1}{2}$$

$$\therefore \frac{BC}{3} = \frac{1}{2}$$

$$\therefore BC = 1.5$$

So, [BC] is 1.5 m long.



$$\begin{aligned}
 15 \quad \mathbf{a} \quad \frac{1}{1 + \sqrt{2} \sin x} + \frac{1}{1 - \sqrt{2} \sin x} &= \frac{1 - \sqrt{2} \sin x + 1 + \sqrt{2} \sin x}{(1 + \sqrt{2} \sin x)(1 - \sqrt{2} \sin x)} \\
 &= \frac{2}{1 - 2 \sin^2 x} \\
 &= \frac{2}{\cos 2x} \quad \{\cos 2x = 1 - 2 \sin^2 x\} \\
 &= 2 \sec 2x
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad y = \cos 2x \quad &\text{has range } \{y \mid -1 \leq y \leq 1\} \\
 \therefore y = \sec 2x \quad &\text{has range } \{y \mid y \leq -1 \text{ or } y \geq 1\} \\
 \therefore y = 2 \sec 2x \quad &\text{has range } \{y \mid y \leq -2 \text{ or } y \geq 2\}
 \end{aligned}$$

$$\begin{aligned}
 \text{Now } \frac{1}{1 + \sqrt{2} \sin x} + \frac{1}{1 - \sqrt{2} \sin x} &= 2 \sec 2x \quad \{\text{from } \mathbf{a}\} \\
 \therefore \frac{1}{1 + \sqrt{2} \sin x} + \frac{1}{1 - \sqrt{2} \sin x} &= 1 \quad \text{has no solutions.}
 \end{aligned}$$

$$\begin{aligned}
 16 \quad &\sin 2A + \sin 2B + \sin 2C \\
 &= \sin 2A + \sin 2B + \sin(2\pi - 2A - 2B) && \{\text{as } A + B + C = \pi\} \\
 &= \sin 2A + \sin 2B - \sin(2A + 2B) && \{\sin(2\pi - \theta) = -\sin \theta\} \\
 &= \sin 2A + \sin 2B - [\sin 2A \cos 2B + \cos 2A \sin 2B] \\
 &= \sin 2A(1 - \cos 2B) + \sin 2B(1 - \cos 2A) \\
 &= 2 \sin A \cos A(2 \sin^2 B) + 2 \sin B \cos B(2 \sin^2 A) && \{\cos 2\theta = 1 - 2 \sin^2 \theta\} \\
 &= 4 \sin A \cos A \sin^2 B + 4 \sin^2 A \sin B \cos B \\
 &= 4 \sin A \sin B [\sin B \cos A + \cos B \sin A] \\
 &= 4 \sin A \sin B \sin(A + B) \\
 &= 4 \sin A \sin B \sin(\pi - C) \\
 &= 4 \sin A \sin B \sin C && \{\sin(\pi - \theta) = \sin \theta\}
 \end{aligned}$$

# Chapter 2

## EXPONENTIAL FUNCTIONS

### EXERCISE 2A

1 a  $\sqrt[5]{2} = 2^{\frac{1}{5}}$

b  $\frac{1}{\sqrt[5]{2}} = \frac{1}{2^{\frac{1}{5}}}$   
 $= 2^{-\frac{1}{5}}$

c  $2\sqrt{2} = 2^1 \times 2^{\frac{1}{2}}$   
 $= 2^{1+\frac{1}{2}}$   
 $= 2^{\frac{3}{2}}$

d  $4\sqrt{2} = 2^2 \times 2^{\frac{1}{2}}$   
 $= 2^{2+\frac{1}{2}}$   
 $= 2^{\frac{5}{2}}$

e  $\frac{1}{\sqrt[3]{2}} = \frac{1}{2^{\frac{1}{3}}}$   
 $= 2^{-\frac{1}{3}}$

f  $2 \times \sqrt[3]{2} = 2^1 \times 2^{\frac{1}{3}}$   
 $= 2^{1+\frac{1}{3}}$   
 $= 2^{\frac{4}{3}}$

g  $\frac{4}{\sqrt{2}} = \frac{2^2}{2^{\frac{1}{2}}}$   
 $= 2^{2-\frac{1}{2}}$   
 $= 2^{\frac{3}{2}}$

h  $(\sqrt{2})^3 = (2^{\frac{1}{2}})^3$   
 $= 2^{3 \times \frac{1}{2}}$   
 $= 2^{\frac{3}{2}}$

i  $\frac{1}{\sqrt[3]{16}} = \frac{1}{\sqrt[3]{2^4}}$   
 $= \frac{1}{2^{\frac{4}{3}}}$   
 $= 2^{-\frac{4}{3}}$

j  $\frac{1}{\sqrt{8}} = \frac{1}{\sqrt{2^3}}$   
 $= \frac{1}{2^{\frac{3}{2}}}$   
 $= 2^{-\frac{3}{2}}$

2 a  $\sqrt[3]{3} = 3^{\frac{1}{3}}$

b  $\frac{1}{\sqrt[3]{3}} = \frac{1}{3^{\frac{1}{3}}}$   
 $= 3^{-\frac{1}{3}}$

c  $\sqrt[4]{3} = 3^{\frac{1}{4}}$

d  $3\sqrt{3} = 3^1 \times 3^{\frac{1}{2}}$   
 $= 3^{\frac{3}{2}}$

e  $\frac{1}{9\sqrt{3}} = \frac{1}{3^2 \times 3^{\frac{1}{2}}}$   
 $= \frac{1}{3^{\frac{5}{2}}}$   
 $= 3^{-\frac{5}{2}}$

3 a  $\sqrt[3]{7} = 7^{\frac{1}{3}}$

b  $\sqrt[4]{27} = \sqrt[4]{3^3}$   
 $= (3^3)^{\frac{1}{4}}$   
 $= 3^{\frac{3}{4}}$

c  $\sqrt[5]{16} = \sqrt[5]{2^4}$   
 $= (2^4)^{\frac{1}{5}}$   
 $= 2^{\frac{4}{5}}$

d  $\sqrt[3]{32} = \sqrt[3]{2^5}$   
 $= (2^5)^{\frac{1}{3}}$   
 $= 2^{\frac{5}{3}}$

e  $\sqrt[7]{49} = \sqrt[7]{7^2}$   
 $= (7^2)^{\frac{1}{7}}$   
 $= 7^{\frac{2}{7}}$

f  $\frac{1}{\sqrt[3]{7}} = \frac{1}{7^{\frac{1}{3}}}$   
 $= 7^{-\frac{1}{3}}$

g  $\frac{1}{\sqrt[4]{27}} = \frac{1}{3^{\frac{3}{4}}}$   
 $= 3^{-\frac{3}{4}}$

h  $\frac{1}{\sqrt[5]{16}} = \frac{1}{2^{\frac{4}{5}}}$   
 $= 2^{-\frac{4}{5}}$



$$\begin{aligned} \text{i} \quad \frac{1}{\sqrt[3]{32}} &= \frac{1}{2^{\frac{5}{3}}} \\ &= 2^{-\frac{5}{3}} \end{aligned}$$

$$\begin{aligned} \text{j} \quad \frac{1}{\sqrt[7]{49}} &= \frac{1}{7^{\frac{2}{7}}} \\ &= 7^{-\frac{2}{7}} \end{aligned}$$

$$4 \quad \text{a} \quad \sqrt{x} = x^{\frac{1}{2}}$$

$$\begin{aligned} \text{b} \quad x\sqrt{x} &= x^1 \times x^{\frac{1}{2}} \\ &= x^{\frac{3}{2}} \end{aligned}$$

$$\begin{aligned} \text{c} \quad \frac{1}{\sqrt{x}} &= \frac{1}{x^{\frac{1}{2}}} \\ &= x^{-\frac{1}{2}} \end{aligned}$$

$$\begin{aligned} \text{d} \quad x^2\sqrt{x} &= x^2 \times x^{\frac{1}{2}} \\ &= x^{\frac{5}{2}} \end{aligned}$$

$$\begin{aligned} \text{e} \quad \frac{1}{x\sqrt{x}} &= \frac{1}{x^{\frac{3}{2}}} \\ &= x^{-\frac{3}{2}} \end{aligned}$$

$$5 \quad \text{a} \quad 3^{\frac{3}{4}} \approx 2.28$$

$$\text{b} \quad 4^{-\frac{3}{5}} \approx 0.435$$

$$\text{c} \quad \sqrt[4]{8} \approx 1.68$$

$$\text{d} \quad \sqrt[5]{27} \approx 1.93$$

$$\text{e} \quad \frac{1}{\sqrt[3]{7}} \approx 0.523$$

$$6 \quad \text{a} \quad 5^{\frac{1}{3}} = \sqrt[3]{5}$$

$$\begin{aligned} \text{b} \quad 3^{-\frac{1}{2}} &= \frac{1}{3^{\frac{1}{2}}} \\ &= \frac{1}{\sqrt{3}} \end{aligned}$$

$$\begin{aligned} \text{c} \quad 3^{\frac{5}{2}} &= 3^2 \times 3^{\frac{1}{2}} \\ &= 9\sqrt{3} \end{aligned}$$

$$\begin{aligned} \text{d} \quad m^{\frac{3}{2}} &= m \times m^{\frac{1}{2}} \\ &= m\sqrt{m} \end{aligned}$$

$$\begin{aligned} \text{e} \quad x^{\frac{7}{2}} &= x^3 \times x^{\frac{1}{2}} \\ &= x^3\sqrt{x} \end{aligned}$$

$$\begin{aligned} 7 \quad \text{a} \quad 4^{\frac{3}{2}} &= (2^2)^{\frac{3}{2}} \\ &= 2^3 \\ &= 8 \end{aligned}$$

$$\begin{aligned} \text{b} \quad 8^{\frac{5}{3}} &= (2^3)^{\frac{5}{3}} \\ &= 2^5 \\ &= 32 \end{aligned}$$

$$\begin{aligned} \text{c} \quad 16^{\frac{3}{4}} &= (2^4)^{\frac{3}{4}} \\ &= 2^3 \\ &= 8 \end{aligned}$$

$$\begin{aligned} \text{d} \quad 25^{\frac{3}{2}} &= (5^2)^{\frac{3}{2}} \\ &= 5^3 \\ &= 125 \end{aligned}$$

$$\begin{aligned} \text{e} \quad 32^{\frac{2}{5}} &= (2^5)^{\frac{2}{5}} \\ &= 2^2 \\ &= 4 \end{aligned}$$

$$\begin{aligned} \text{f} \quad 4^{-\frac{1}{2}} &= (2^2)^{-\frac{1}{2}} \\ &= 2^{-1} \\ &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} \text{g} \quad 9^{-\frac{3}{2}} &= (3^2)^{-\frac{3}{2}} \\ &= 3^{-3} \\ &= \frac{1}{27} \end{aligned}$$

$$\begin{aligned} \text{h} \quad 8^{-\frac{4}{3}} &= (2^3)^{-\frac{4}{3}} \\ &= 2^{-4} \\ &= \frac{1}{16} \end{aligned}$$

$$\begin{aligned} \text{i} \quad 27^{-\frac{4}{3}} &= (3^3)^{-\frac{4}{3}} \\ &= 3^{-4} \\ &= \frac{1}{81} \end{aligned}$$

$$\begin{aligned} \text{j} \quad 125^{-\frac{2}{3}} &= (5^3)^{-\frac{2}{3}} \\ &= 5^{-2} \\ &= \frac{1}{25} \end{aligned}$$

## EXERCISE 2B

$$\begin{aligned} 1 \quad a \quad & x^{\frac{1}{2}} \times x^{-\frac{1}{2}} \\ &= x^0 \\ &= 1 \end{aligned}$$

$$\begin{aligned} b \quad & x^{\frac{3}{2}} \times x^{-\frac{1}{2}} \\ &= x^1 \\ &= x \end{aligned}$$

$$\begin{aligned} c \quad & x^2 \times x^{-\frac{3}{2}} \\ &= x^{\frac{1}{2}} \quad \text{or} \quad \sqrt{x} \end{aligned}$$

$$\begin{aligned} 2 \quad a \quad & x^2(x^3 + 2x^2 + 1) \\ &= x^2 \times x^3 + x^2 \times 2x^2 + x^2 \times 1 \\ &= x^5 + 2x^4 + x^2 \end{aligned}$$

$$\begin{aligned} b \quad & 2^x(2^x + 1) \\ &= 2^x \times 2^x + 2^x \times 1 \\ &= 2^{2x} + 2^x \end{aligned}$$

$$\begin{aligned} c \quad & x^{\frac{1}{2}}(x^{\frac{1}{2}} + x^{-\frac{1}{2}}) \\ &= x^{\frac{1}{2}} \times x^{\frac{1}{2}} + x^{\frac{1}{2}} \times x^{-\frac{1}{2}} \\ &= x^1 + x^0 \\ &= x + 1 \end{aligned}$$

$$\begin{aligned} d \quad & 7^x(7^x + 2) \\ &= 7^x \times 7^x + 7^x \times 2 \\ &= 7^{2x} + 2(7^x) \end{aligned}$$

$$\begin{aligned} e \quad & 3^x(2 - 3^{-x}) \\ &= 3^x \times 2 - 3^x \times 3^{-x} \\ &= 2(3^x) - 3^0 \\ &= 2(3^x) - 1 \end{aligned}$$

$$\begin{aligned} f \quad & x^{\frac{1}{2}}(x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + 3x^{-\frac{1}{2}}) \\ &= x^{\frac{1}{2}} \times x^{\frac{3}{2}} + x^{\frac{1}{2}} \times 2x^{\frac{1}{2}} + x^{\frac{1}{2}} \times 3x^{-\frac{1}{2}} \\ &= x^2 + 2x^1 + 3x^0 \\ &= x^2 + 2x + 3 \end{aligned}$$

$$\begin{aligned} g \quad & 2^{-x}(2^x + 5) \\ &= 2^{-x} \times 2^x + 2^{-x} \times 5 \\ &= 2^0 + 5(2^{-x}) \\ &= 1 + 5(2^{-x}) \end{aligned}$$

$$\begin{aligned} h \quad & 5^{-x}(5^{2x} + 5^x) \\ &= 5^{-x} \times 5^{2x} + 5^{-x} \times 5^x \\ &= 5^x + 5^0 \\ &= 5^x + 1 \end{aligned}$$

$$\begin{aligned} i \quad & x^{-\frac{1}{2}}(x^2 + x + x^{\frac{1}{2}}) \\ &= x^{-\frac{1}{2}} \times x^2 + x^{-\frac{1}{2}} \times x^1 + x^{-\frac{1}{2}} \times x^{\frac{1}{2}} \\ &= x^{\frac{3}{2}} + x^{\frac{1}{2}} + x^0 \\ &= x^{\frac{3}{2}} + x^{\frac{1}{2}} + 1 \end{aligned}$$

$$\begin{aligned} j \quad & 3^x(3^x + 5 + 3^{-x}) \\ &= 3^x \times 3^x + 3^x \times 5 + 3^x \times 3^{-x} \\ &= 3^{2x} + 5(3^x) + 3^0 \\ &= 3^{2x} + 5(3^x) + 1 \end{aligned}$$

$$\begin{aligned} k \quad & x^{-\frac{1}{2}}(2x^2 - x + 5x^{\frac{1}{2}}) \\ &= x^{-\frac{1}{2}} \times 2x^2 - x^{-\frac{1}{2}} \times x^1 + x^{-\frac{1}{2}} \times 5x^{\frac{1}{2}} \\ &= 2x^{\frac{3}{2}} - x^{\frac{1}{2}} + 5x^0 \\ &= 2x^{\frac{3}{2}} - x^{\frac{1}{2}} + 5 \end{aligned}$$

$$\begin{aligned} l \quad & 2^{2x}(2^x - 3 - 2^{-2x}) \\ &= 2^{2x} \times 2^x - 2^{2x} \times 3 - 2^{2x} \times 2^{-2x} \\ &= 2^{3x} - 3(2^{2x}) - 2^0 \\ &= 2^{3x} - 3(2^{2x}) - 1 \end{aligned}$$

$$\begin{aligned} 3 \quad a \quad & (2^x - 1)(2^x + 3) \\ &= 2^x \times 2^x + 2^x \times 3 - 1 \times 2^x - 3 \\ &= 2^{2x} + 2(2^x) - 3 \\ &= 2^{2x} + 2^{x+1} - 3 \end{aligned}$$

$$\begin{aligned} b \quad & (3^x + 2)(3^x + 5) \\ &= 3^x \times 3^x + 3^x \times 5 + 2 \times 3^x + 10 \\ &= 3^{2x} + 7(3^x) + 10 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & (5^x - 2)(5^x - 4) \\
 &= 5^x \times 5^x - 5^x \times 4 - 2 \times 5^x + 8 \\
 &= 5^{2x} - 6(5^x) + 8
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & (3^x - 1)^2 \\
 &= (3^x)^2 - 2 \times 3^x \times 1 + 1^2 \\
 &= 3^{2x} - 2(3^x) + 1
 \end{aligned}$$

$$\begin{aligned}
 \text{g} \quad & (x^{\frac{1}{2}} + 2)(x^{\frac{1}{2}} - 2) \\
 &= (x^{\frac{1}{2}})^2 - 2^2 \\
 &= x - 4
 \end{aligned}$$

$$\begin{aligned}
 \text{i} \quad & (x^{\frac{1}{2}} + x^{-\frac{1}{2}})(x^{\frac{1}{2}} - x^{-\frac{1}{2}}) \\
 &= (x^{\frac{1}{2}})^2 - (x^{-\frac{1}{2}})^2 \\
 &= x^1 - x^{-1} \\
 &= x - \frac{1}{x}
 \end{aligned}$$

$$\begin{aligned}
 \text{k} \quad & (7^x - 7^{-x})^2 \\
 &= (7^x)^2 - 2 \times 7^x \times 7^{-x} + (7^{-x})^2 \\
 &= 7^{2x} - 2 \times 7^0 + 7^{-2x} \\
 &= 7^{2x} - 2 + 7^{-2x}
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad & (2^x + 3)^2 \\
 &= (2^x)^2 + 2 \times 2^x \times 3 + 3^2 \\
 &= 2^{2x} + 6(2^x) + 9
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & (4^x + 7)^2 \\
 &= (4^x)^2 + 2 \times 4^x \times 7 + 7^2 \\
 &= 4^{2x} + 14(4^x) + 49
 \end{aligned}$$

$$\begin{aligned}
 \text{h} \quad & (2^x + 3)(2^x - 3) \\
 &= (2^x)^2 - 3^2 \\
 &= 2^{2x} - 9 \\
 &= 4^x - 9
 \end{aligned}$$

$$\begin{aligned}
 \text{j} \quad & \left(x + \frac{2}{x}\right)^2 \\
 &= x^2 + 2 \times x \times \left(\frac{2}{x}\right) + \left(\frac{2}{x}\right)^2 \\
 &= x^2 + 4 + \frac{4}{x^2}
 \end{aligned}$$

$$\begin{aligned}
 \text{l} \quad & (5 - 2^{-x})^2 \\
 &= 5^2 - 2 \times 5 \times 2^{-x} + (2^{-x})^2 \\
 &= 25 - 10(2^{-x}) + 2^{-2x}
 \end{aligned}$$

$$\begin{aligned}
 4 \quad \text{a} \quad & 5^{2x} + 5^x \\
 &= 5^x \times 5^x + 5^x \\
 &= 5^x(5^x + 1)
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & 3^{n+2} + 3^n \\
 &= 3^n \times 3^2 + 3^n \\
 &= 3^n(3^2 + 1) \\
 &= 10(3^n)
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & 7^n + 7^{3n} \\
 &= 7^n + 7^n \times 7^{2n} \\
 &= 7^n(1 + 7^{2n})
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad & 5^{n+1} - 5 \\
 &= 5 \times 5^n - 5 \\
 &= 5(5^n - 1)
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & 6^{n+2} - 6 \\
 &= 6 \times 6^{n+1} - 6 \\
 &= 6(6^{n+1} - 1)
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & 4^{n+2} - 16 \\
 &= 4^2 \times 4^n - 16 \\
 &= 16 \times 4^n - 16 \\
 &= 16(4^n - 1)
 \end{aligned}$$

$$\begin{aligned}
 \text{g} \quad & 2^{2n} - 2^{n+3} \\
 &= 2^n \times 2^n - 2^n \times 2^3 \\
 &= 2^n \times 2^n - 2^n \times 8 \\
 &= 2^n(2^n - 8)
 \end{aligned}$$

$$\begin{aligned}
 \text{h} \quad & 2^{n+1} + 2^{n-1} \\
 &= 2^{n-1} \times 2^2 + 2^{n-1} \\
 &= 2^{n-1} \times 4 + 2^{n-1} \\
 &= 5(2^{n-1}) \\
 &= \frac{5}{2}(2^n)
 \end{aligned}$$

$$\begin{aligned}
 \text{i} \quad & 4^{n+1} + 2^{2n-1} \\
 &= (2^2)^{n+1} + 2^{2n-1} \\
 &= 2^{2n+2} + 2^{2n-1} \\
 &= 2^{2n-1} \times 2^3 + 2^{2n-1} \\
 &= 2^{2n-1} \times 8 + 2^{2n-1} \\
 &= 9(2^{2n-1}) \\
 &= \frac{9}{2}(2^{2n})
 \end{aligned}$$

$$\begin{aligned}
 5 \quad \text{a} \quad & 9^x - 4 \\
 &= (3^x)^2 - 2^2 \\
 &= (3^x + 2)(3^x - 2)
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & 4^x - 25 \\
 &= (2^x)^2 - 5^2 \\
 &= (2^x + 5)(2^x - 5)
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & 16 - 9^x \\
 &= 4^2 - (3^x)^2 \\
 &= (4 + 3^x)(4 - 3^x)
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad & 25 - 4^x \\
 &= 5^2 - (2^x)^2 \\
 &= (5 + 2^x)(5 - 2^x)
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & 9^x - 4^x \\
 &= (3^x)^2 - (2^x)^2 \\
 &= (3^x + 2^x)(3^x - 2^x)
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & 4^x + 6(2^x) + 9 \\
 &= (2^x)^2 + 6(2^x) + 9 \\
 &= (2^x + 3)^2 \\
 &\{a^2 + 6a + 9 = (a + 3)^2\}
 \end{aligned}$$

$$\begin{aligned}
 \text{g} \quad & 9^x + 10(3^x) + 25 \\
 &= (3^x)^2 + 10(3^x) + 25 \\
 &= (3^x + 5)^2 \\
 &\{a^2 + 10a + 25 = (a + 5)^2\}
 \end{aligned}$$

$$\begin{aligned}
 \text{h} \quad & 4^x - 14(2^x) + 49 \\
 &= (2^x)^2 - 14(2^x) + 49 \\
 &= (2^x - 7)^2 \\
 &\{a^2 - 14a + 49 = (a - 7)^2\}
 \end{aligned}$$

$$\begin{aligned}
 \text{i} \quad & 25^x - 4(5^x) + 4 \\
 &= (5^x)^2 - 4(5^x) + 4 \\
 &= (5^x - 2)^2 \\
 &\{a^2 - 4a + 4 = (a - 2)^2\}
 \end{aligned}$$

$$\begin{aligned}
 \text{6 a} \quad & (2^x)^2 - 2^x - 2 \\
 &= (2^x + 1)(2^x - 2) \\
 &\{a^2 - a - 2 = (a + 1)(a - 2)\}
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & (3^x)^2 + 3^x - 6 \\
 &= (3^x + 3)(3^x - 2) \\
 &\{a^2 + a - 6 = (a + 3)(a - 2)\}
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & 4^x - 7(2^x) + 12 \\
 &= (2^x)^2 - 7(2^x) + 12 \\
 &= (2^x - 3)(2^x - 4) \\
 &\{a^2 - 7a + 12 = (a - 3)(a - 4)\}
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad & 4^x + 9(2^x) + 18 \\
 &= (2^x)^2 + 9(2^x) + 18 \\
 &= (2^x + 3)(2^x + 6) \\
 &\{a^2 + 9a + 18 = (a + 3)(a + 6)\}
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & 4^x - 2^x - 20 \\
 &= (2^x)^2 - 2^x - 20 \\
 &= (2^x + 4)(2^x - 5) \\
 &\{a^2 - a - 20 = (a + 4)(a - 5)\}
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & 9^x + 9(3^x) + 14 \\
 &= (3^x)^2 + 9(3^x) + 14 \\
 &= (3^x + 2)(3^x + 7) \\
 &\{a^2 + 9a + 14 = (a + 2)(a + 7)\}
 \end{aligned}$$

$$\begin{aligned}
 \text{g} \quad & 9^x + 4(3^x) - 5 \\
 &= (3^x)^2 + 4(3^x) - 5 \\
 &= (3^x + 5)(3^x - 1) \\
 &\{a^2 + 4a - 5 = (a + 5)(a - 1)\}
 \end{aligned}$$

$$\begin{aligned}
 \text{h} \quad & 25^x + 5^x - 2 \\
 &= (5^x)^2 + 5^x - 2 \\
 &= (5^x + 2)(5^x - 1) \\
 &\{a^2 + a - 2 = (a + 2)(a - 1)\}
 \end{aligned}$$

$$\begin{aligned}
 \text{i} \quad & 49^x - 7^{x+1} + 12 \\
 &= (7^x)^2 - 7(7^x) + 12 \\
 &= (7^x - 4)(7^x - 3) \\
 &\{a^2 - 7a + 12 = (a - 4)(a - 3)\}
 \end{aligned}$$

$$\begin{aligned}
 \text{7 a} \quad & \frac{12^n}{6^n} \quad \text{or} \quad \frac{12^n}{6^n} \\
 &= \frac{2^n 6^n}{6^n} \quad = \left(\frac{12}{6}\right)^n \\
 &= 2^n \quad = 2^n
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & \frac{20^a}{2^a} \quad \text{or} \quad \frac{20^a}{2^a} \\
 &= \frac{2^a 10^a}{2^a} \quad = \left(\frac{20}{2}\right)^a \\
 &= 10^a \quad = 10^a
 \end{aligned}$$



$$\begin{aligned}
 \text{c} \quad & \frac{6^b}{2^b} \quad \text{or} \quad \frac{6^b}{2^b} \\
 & = \frac{2^b 3^b}{2^b} = \left(\frac{6}{2}\right)^b \\
 & = 3^b = 3^b
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad & \frac{4^n}{20^n} \quad \text{or} \quad \frac{4^n}{20^n} \\
 & = \frac{4^n}{4^n 5^n} = \left(\frac{4}{20}\right)^n \\
 & = \frac{1}{5^n} = \left(\frac{1}{5}\right)^n
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & \frac{35^x}{7^x} \quad \text{or} \quad \frac{35^x}{7^x} \\
 & = \frac{5^x 7^x}{7^x} = \left(\frac{35}{7}\right)^x \\
 & = 5^x = 5^x
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & \frac{6^a}{8^a} \quad \text{or} \quad \frac{6^a}{8^a} \\
 & = \frac{2^a 3^a}{2^a 4^a} = \left(\frac{6}{8}\right)^a \\
 & = \frac{3^a}{4^a} = \left(\frac{3}{4}\right)^a
 \end{aligned}$$

$$\begin{aligned}
 \text{g} \quad & \frac{24^k}{9^k} \quad \text{or} \quad \frac{24^k}{9^k} \\
 & = \frac{3^k 8^k}{3^k 3^k} = \left(\frac{24}{9}\right)^k \\
 & = \frac{8^k}{3^k} = \left(\frac{8}{3}\right)^k
 \end{aligned}$$

$$\begin{aligned}
 \text{h} \quad & \frac{5^{n+1}}{5^n} \\
 & = \frac{5^n 5^1}{5^n} \\
 & = 5
 \end{aligned}$$

$$\begin{aligned}
 \text{i} \quad & \frac{5^{n+1}}{5} \\
 & = \frac{5^n 5^1}{5} \\
 & = 5^n
 \end{aligned}$$

$$\begin{aligned}
 8 \quad \text{a} \quad & \frac{6^m + 2^m}{2^m} \\
 & = \frac{2^m 3^m + 2^m}{2^m} \\
 & = \frac{2^m(3^m + 1)}{2^m} \\
 & = 3^m + 1
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & \frac{2^n + 12^n}{2^n} \\
 & = \frac{2^n + 2^n 6^n}{2^n} \\
 & = \frac{2^n(1 + 6^n)}{2^n} \\
 & = 1 + 6^n
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & \frac{8^n + 4^n}{2^n} \\
 & = \frac{2^n 4^n + 2^n 2^n}{2^n} \\
 & = \frac{2^n(4^n + 2^n)}{2^n} \\
 & = 4^n + 2^n
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad & \frac{12^x - 3^x}{3^x} \\
 & = \frac{3^x 4^x - 3^x}{3^x} \\
 & = \frac{3^x(4^x - 1)}{3^x} \\
 & = 4^x - 1
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & \frac{6^n + 12^n}{1 + 2^n} \\
 & = \frac{6^n + 6^n 2^n}{1 + 2^n} \\
 & = \frac{6^n(1 + 2^n)}{1 + 2^n} \\
 & = 6^n
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & \frac{5^{n+1} - 5^n}{4} \\
 & = \frac{5^n \times 5 - 5^n}{4} \\
 & = \frac{5^n(5 - 1)}{4} \\
 & = 5^n
 \end{aligned}$$

$$\begin{aligned}
 \text{g} \quad & \frac{5^{n+1} - 5^n}{5^n} \\
 & = \frac{5^n \times 5 - 5^n}{5^n} \\
 & = \frac{5^n(5 - 1)}{5^n} \\
 & = 4
 \end{aligned}$$

$$\begin{aligned}
 \text{h} \quad & \frac{4^n - 2^n}{2^n} \\
 & = \frac{2^n 2^n - 2^n}{2^n} \\
 & = \frac{2^n(2^n - 1)}{2^n} \\
 & = 2^n - 1
 \end{aligned}$$

$$\begin{aligned}
 \text{i} \quad & \frac{2^n - 2^{n-1}}{2^n} \\
 & = \frac{2^{n-1} \times 2 - 2^{n-1}}{2^{n-1} \times 2} \\
 & = \frac{2^{n-1}(2 - 1)}{2^{n-1} \times 2} \\
 & = \frac{1}{2}
 \end{aligned}$$

$$\begin{aligned}
 9 \quad a \quad & 2^n(n+1) + 2^n(n-1) \\
 &= 2^n(n+1+n-1) \\
 &= 2^n(2n) \\
 &= n2^{n+1}
 \end{aligned}$$

$$\begin{aligned}
 b \quad & 3^n\left(\frac{n-1}{6}\right) - 3^n\left(\frac{n+1}{6}\right) \\
 &= 3^n\left(\frac{n-1}{6} - \frac{n+1}{6}\right) \\
 &= 3^n\left(-\frac{1}{3}\right) \\
 &= 3^n \times -3^{-1} \\
 &= -3^{n-1}
 \end{aligned}$$

## EXERCISE 2C

$$\begin{aligned}
 1 \quad a \quad & 2^x = 32 \\
 \therefore 2^x &= 2^5 \\
 \therefore x &= 5
 \end{aligned}$$

$$\begin{aligned}
 d \quad & 7^x = 1 \\
 \therefore 7^x &= 7^0 \\
 \therefore x &= 0
 \end{aligned}$$

$$\begin{aligned}
 g \quad & 5^x = \frac{1}{125} \\
 \therefore 5^x &= 5^{-3} \\
 \therefore x &= -3
 \end{aligned}$$

$$\begin{aligned}
 j \quad & 3^{x+1} = \frac{1}{27} \\
 \therefore 3^{x+1} &= 3^{-3} \\
 \therefore x+1 &= -3 \\
 \therefore x &= -4
 \end{aligned}$$

$$\begin{aligned}
 2 \quad a \quad & 8^x = 32 \\
 \therefore (2^3)^x &= 2^5 \\
 \therefore 2^{3x} &= 2^5 \\
 \therefore 3x &= 5 \\
 \therefore x &= \frac{5}{3}
 \end{aligned}$$

$$\begin{aligned}
 d \quad & 25^x = \frac{1}{5} \\
 \therefore (5^2)^x &= 5^{-1} \\
 \therefore 5^{2x} &= 5^{-1} \\
 \therefore 2x &= -1 \\
 \therefore x &= -\frac{1}{2}
 \end{aligned}$$

$$\begin{aligned}
 b \quad & 5^x = 25 \\
 \therefore 5^x &= 5^2 \\
 \therefore x &= 2
 \end{aligned}$$

$$\begin{aligned}
 e \quad & 3^x = \frac{1}{3} \\
 \therefore 3^x &= 3^{-1} \\
 \therefore x &= -1
 \end{aligned}$$

$$\begin{aligned}
 h \quad & 4^{x+1} = 64 \\
 \therefore 4^{x+1} &= 4^3 \\
 \therefore x+1 &= 3 \\
 \therefore x &= 2
 \end{aligned}$$

$$\begin{aligned}
 k \quad & 7^{x+1} = 343 \\
 \therefore 7^{x+1} &= 7^3 \\
 \therefore x+1 &= 3 \\
 \therefore x &= 2
 \end{aligned}$$

$$\begin{aligned}
 b \quad & 4^x = \frac{1}{8} \\
 \therefore (2^2)^x &= 2^{-3} \\
 \therefore 2^{2x} &= 2^{-3} \\
 \therefore 2x &= -3 \\
 \therefore x &= -\frac{3}{2}
 \end{aligned}$$

$$\begin{aligned}
 e \quad & 27^x = \frac{1}{9} \\
 \therefore (3^3)^x &= 3^{-2} \\
 \therefore 3^{3x} &= 3^{-2} \\
 \therefore 3x &= -2 \\
 \therefore x &= -\frac{2}{3}
 \end{aligned}$$

$$\begin{aligned}
 c \quad & 3^x = 81 \\
 \therefore 3^x &= 3^4 \\
 \therefore x &= 4
 \end{aligned}$$

$$\begin{aligned}
 f \quad & 2^x = \sqrt{2} \\
 \therefore 2^x &= 2^{\frac{1}{2}} \\
 \therefore x &= \frac{1}{2}
 \end{aligned}$$

$$\begin{aligned}
 i \quad & 2^{x-2} = \frac{1}{32} \\
 \therefore 2^{x-2} &= 2^{-5} \\
 \therefore x-2 &= -5 \\
 \therefore x &= -3
 \end{aligned}$$

$$\begin{aligned}
 l \quad & 5^{1-2x} = \frac{1}{\sqrt{5}} \\
 \therefore 5^{1-2x} &= 5^{-\frac{1}{2}} \\
 \therefore 1-2x &= -\frac{1}{2} \\
 \therefore -2x &= -\frac{3}{2} \\
 \therefore x &= \frac{3}{4}
 \end{aligned}$$

$$\begin{aligned}
 c \quad & 9^x = \frac{1}{27} \\
 \therefore (3^2)^x &= 3^{-3} \\
 \therefore 3^{2x} &= 3^{-3} \\
 \therefore 2x &= -3 \\
 \therefore x &= -\frac{3}{2}
 \end{aligned}$$

$$\begin{aligned}
 f \quad & 16^x = \frac{1}{32} \\
 \therefore (2^4)^x &= 2^{-5} \\
 \therefore 2^{4x} &= 2^{-5} \\
 \therefore 4x &= -5 \\
 \therefore x &= -\frac{5}{4}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{g} \quad & 4^{x+2} = 128 \\
 \therefore & (2^2)^{x+2} = 2^7 \\
 \therefore & 2^{2(x+2)} = 2^7 \\
 \therefore & 2(x+2) = 7 \\
 \therefore & 2x + 4 = 7 \\
 \therefore & 2x = 3 \\
 \therefore & x = \frac{3}{2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{j} \quad & 9^{x-3} = 27 \\
 \therefore & (3^2)^{x-3} = 3^3 \\
 \therefore & 3^{2(x-3)} = 3^3 \\
 \therefore & 2(x-3) = 3 \\
 \therefore & 2x - 6 = 3 \\
 \therefore & 2x = 9 \\
 \therefore & x = \frac{9}{2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{m} \quad & 81^x = 27^{-x} \\
 \therefore & (3^4)^x = (3^3)^{-x} \\
 \therefore & 3^{4x} = 3^{-3x} \\
 \therefore & 4x = -3x \\
 \therefore & 7x = 0 \\
 \therefore & x = 0
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{p} \quad & \left(\frac{1}{3}\right)^{x+1} = 243 \\
 \therefore & (3^{-1})^{x+1} = 3^5 \\
 \therefore & 3^{-(x+1)} = 3^5 \\
 \therefore & -(x+1) = 5 \\
 \therefore & -x - 1 = 5 \\
 \therefore & -x = 6 \\
 \therefore & x = -6
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{3} \quad \mathbf{a} \quad & 4^{2x+1} = 8^{1-x} \\
 \therefore & (2^2)^{2x+1} = (2^3)^{1-x} \\
 \therefore & 2^{2(2x+1)} = 2^{3(1-x)} \\
 \therefore & 2(2x+1) = 3(1-x) \\
 \therefore & 4x + 2 = 3 - 3x \\
 \therefore & 7x = 1 \\
 \therefore & x = \frac{1}{7}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{h} \quad & 25^{1-x} = \frac{1}{125} \\
 \therefore & (5^2)^{1-x} = 5^{-3} \\
 \therefore & 5^{2(1-x)} = 5^{-3} \\
 \therefore & 2(1-x) = -3 \\
 \therefore & 2 - 2x = -3 \\
 \therefore & -2x = -5 \\
 \therefore & x = \frac{5}{2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{k} \quad & \left(\frac{1}{2}\right)^{x+1} = 8 \\
 \therefore & (2^{-1})^{x+1} = 2^3 \\
 \therefore & 2^{-(x+1)} = 2^3 \\
 \therefore & -(x+1) = 3 \\
 \therefore & -x - 1 = 3 \\
 \therefore & -x = 4 \\
 \therefore & x = -4
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{n} \quad & \left(\frac{1}{4}\right)^{1-x} = 32 \\
 \therefore & (2^{-2})^{1-x} = 2^5 \\
 \therefore & 2^{-2(1-x)} = 2^5 \\
 \therefore & -2(1-x) = 5 \\
 \therefore & -2 + 2x = 5 \\
 \therefore & 2x = 7 \\
 \therefore & x = \frac{7}{2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{i} \quad & 4^{4x-1} = \frac{1}{2} \\
 \therefore & (2^2)^{4x-1} = 2^{-1} \\
 \therefore & 2^{2(4x-1)} = 2^{-1} \\
 \therefore & 2(4x-1) = -1 \\
 \therefore & 8x - 2 = -1 \\
 \therefore & 8x = 1 \\
 \therefore & x = \frac{1}{8}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{l} \quad & \left(\frac{1}{3}\right)^{x+2} = \sqrt{27} \\
 \therefore & (3^{-1})^{x+2} = (3^3)^{\frac{1}{2}} \\
 \therefore & 3^{-(x+2)} = 3^{\frac{3}{2}} \\
 \therefore & -(x+2) = \frac{3}{2} \\
 \therefore & -x - 2 = \frac{3}{2} \\
 \therefore & -x = \frac{7}{2} \\
 \therefore & x = -\frac{7}{2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{o} \quad & \left(\frac{1}{7}\right)^x = \sqrt[3]{49} \\
 \therefore & (7^{-1})^x = (7^2)^{\frac{1}{3}} \\
 \therefore & 7^{-x} = 7^{\frac{2}{3}} \\
 \therefore & -x = \frac{2}{3} \\
 \therefore & x = -\frac{2}{3}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & 9^{2-x} = \left(\frac{1}{3}\right)^{2x+1} \\
 \therefore & (3^2)^{2-x} = (3^{-1})^{2x+1} \\
 \therefore & 3^{2(2-x)} = 3^{-(2x+1)} \\
 \therefore & 2(2-x) = -(2x+1) \\
 \therefore & 4 - 2x = -2x - 1 \\
 \therefore & 4 = -1
 \end{aligned}$$

which is clearly false, so no solutions exist.

$$\begin{aligned}
 \text{c} \quad & 2^x \times 8^{1-x} = \frac{1}{4} \\
 \therefore & 2^x \times (2^3)^{1-x} = 2^{-2} \\
 \therefore & 2^x \times 2^{3(1-x)} = 2^{-2} \\
 \therefore & 2^{x+3(1-x)} = 2^{-2} \\
 \therefore & x + 3(1-x) = -2 \\
 \therefore & x + 3 - 3x = -2 \\
 \therefore & -2x = -5 \\
 \therefore & x = \frac{5}{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad & 3^{x+2} \times 9^x = 27 \\
 \therefore & 3^{x+2} \times (3^2)^x = 3^3 \\
 \therefore & 3^{x+2} \times 3^{2x} = 3^3 \\
 \therefore & 3^{x+2+2x} = 3^3 \\
 \therefore & x + 2 + 2x = 3 \\
 \therefore & 3x + 2 = 3 \\
 \therefore & 3x = 1 \\
 \therefore & x = \frac{1}{3}
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & \left(\frac{1}{2}\right)^{x-1} \times 8^x = 4^{-x} \\
 \therefore & (2^{-1})^{x-1} \times (2^3)^x = (2^2)^{-x} \\
 \therefore & 2^{-(x-1)} \times 2^{3x} = 2^{-2x} \\
 \therefore & 2^{-(x-1)+3x} = 2^{-2x} \\
 \therefore & -(x-1) + 3x = -2x \\
 \therefore & -x + 1 + 3x = -2x \\
 \therefore & 2x + 1 = -2x \\
 \therefore & 4x = -1 \\
 \therefore & x = -\frac{1}{4}
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & \left(\frac{1}{5}\right)^{x^2} \times 25^x = \frac{1}{125} \\
 \therefore & (5^{-1})^{x^2} \times (5^2)^x = 5^{-3} \\
 \therefore & 5^{-x^2} \times 5^{2x} = 5^{-3} \\
 \therefore & 5^{-x^2+2x} = 5^{-3} \\
 \therefore & -x^2 + 2x = -3 \\
 \therefore & -x^2 + 2x + 3 = 0 \\
 \therefore & -(x^2 - 2x - 3) = 0 \\
 \therefore & -(x+1)(x-3) = 0 \\
 \therefore & x = -1 \text{ or } 3
 \end{aligned}$$

$$\begin{aligned}
 \text{4 a} \quad & 3 \times 2^x = 24 \\
 \therefore & 2^x = 8 \\
 \therefore & 2^x = 2^3 \\
 \therefore & x = 3
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & 7 \times 2^x = 28 \\
 \therefore & 2^x = 4 \\
 \therefore & 2^x = 2^2 \\
 \therefore & x = 2
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & 4 \times 3^{x+2} = 12 \\
 \therefore & 3^{x+2} = 3 \\
 \therefore & x + 2 = 1 \\
 \therefore & x = -1
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad & 12 \times 3^{-x} = \frac{4}{3} \\
 \therefore & 3^{-x} = \frac{4}{3} \times \frac{1}{12} \\
 \therefore & 3^{-x} = \frac{1}{9} \\
 \therefore & 3^{-x} = 3^{-2} \\
 \therefore & -x = -2 \\
 \therefore & x = 2
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & 4 \times \left(\frac{1}{3}\right)^x = 36 \\
 \therefore & \left(\frac{1}{3}\right)^x = 9 \\
 \therefore & (3^{-1})^x = 3^2 \\
 \therefore & 3^{-x} = 3^2 \\
 \therefore & -x = 2 \\
 \therefore & x = -2
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & 5 \times \left(\frac{1}{2}\right)^x = 20 \\
 \therefore & \left(\frac{1}{2}\right)^x = 4 \\
 \therefore & (2^{-1})^x = 2^2 \\
 \therefore & 2^{-x} = 2^2 \\
 \therefore & -x = 2 \\
 \therefore & x = -2
 \end{aligned}$$

$$\begin{aligned}
 \text{5 a} \quad & 4^x - 6(2^x) + 8 = 0 \\
 \therefore & (2^x)^2 - 6(2^x) + 8 = 0 \\
 \therefore & (2^x - 2)(2^x - 4) = 0 \quad \{a^2 - 6a + 8 = (a-2)(a-4)\} \\
 \therefore & 2^x = 2 \text{ or } 2^x = 4 \\
 \therefore & 2^x = 2^1 \text{ or } 2^x = 2^2 \\
 \therefore & x = 1 \text{ or } 2
 \end{aligned}$$



$$\begin{aligned}
 \text{b} \quad & 4^x - 2^x - 2 = 0 \\
 & \therefore (2^x)^2 - 2^x - 2 = 0 \\
 & \therefore (2^x - 2)(2^x + 1) = 0 \quad \{a^2 - a - 2 = (a - 2)(a + 1)\} \\
 & \therefore 2^x = 2 \text{ or } 2^x = -1 \\
 & \therefore 2^x = 2^1 \quad \{\text{since } 2^x \text{ cannot be negative}\} \\
 & \therefore x = 1
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & 9^x - 12(3^x) + 27 = 0 \\
 & \therefore (3^x)^2 - 12(3^x) + 27 = 0 \\
 & \therefore (3^x - 3)(3^x - 9) = 0 \quad \{a^2 - 12a + 27 = (a - 3)(a - 9)\} \\
 & \therefore 3^x = 3 \text{ or } 3^x = 9 \\
 & \therefore 3^x = 3^1 \text{ or } 3^x = 3^2 \\
 & \therefore x = 1 \text{ or } 2
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad & 9^x = 3^x + 6 \\
 & \therefore (3^x)^2 - 3^x - 6 = 0 \\
 & \therefore (3^x - 3)(3^x + 2) = 0 \quad \{a^2 - a - 6 = (a - 3)(a + 2)\} \\
 & \therefore 3^x = 3 \text{ or } 3^x = -2 \\
 & \therefore 3^x = 3^1 \quad \{\text{since } 3^x \text{ cannot be negative}\} \\
 & \therefore x = 1
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & 25^x - 23(5^x) - 50 = 0 \\
 & \therefore (5^x)^2 - 23(5^x) - 50 = 0 \\
 & \therefore (5^x - 25)(5^x + 2) = 0 \quad \{a^2 - 23a - 50 = (a - 25)(a + 2)\} \\
 & \therefore 5^x = 25 \text{ or } 5^x = -2 \\
 & \therefore 5^x = 5^2 \quad \{\text{since } 5^x \text{ cannot be negative}\} \\
 & \therefore x = 2
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & 49^x + 1 = 2(7^x) \\
 & \therefore (7^x)^2 - 2(7^x) + 1 = 0 \\
 & \therefore (7^x - 1)^2 = 0 \quad \{a^2 - 2a + 1 = (a - 1)^2\} \\
 & \therefore 7^x = 1 \\
 & \therefore 7^x = 7^0 \\
 & \therefore x = 0
 \end{aligned}$$

**g**

$$\begin{aligned}
3^x - 1 &= 6(3^{-x}) \\
\therefore 3^x(3^x - 1) &= 3^x(6(3^{-x})) \\
\therefore 3^{2x} - 3^x &= 6 \times 3^x \times 3^{-x} \\
\therefore 3^{2x} - 3^x &= 6 \times 3^0 \\
\therefore 3^{2x} - 3^x - 6 &= 0 \\
\therefore (3^x + 2)(3^x - 3) &= 0 \quad \{a^2 - a - 6 = (a + 2)(a - 3)\} \\
\therefore 3^x &= -2 \quad \text{or} \quad 3^x = 3 \\
\therefore 3^x &= 3 \quad \{\text{since } 3^x \text{ cannot be negative}\} \\
\therefore 3^x &= 3^1 \\
\therefore x &= 1
\end{aligned}$$

**h**

$$\begin{aligned}
2(4^x) - 5(2^x) + 2 &= 0 \\
\therefore 2(2^x)^2 - 5(2^x) + 2 &= 0 \\
\therefore (2(2^x) - 1)(2^x - 2) &= 0 \quad \{2a^2 - 5a + 2 = (2a - 1)(a - 2)\} \\
\therefore 2(2^x) &= 1 \quad \text{or} \quad 2^x = 2 \\
\therefore 2^x &= \frac{1}{2} \quad \text{or} \quad 2^x = 2^1 \\
\therefore 2^x &= 2^{-1} \quad \text{or} \quad 2^x = 2^1 \\
\therefore x &= -1 \quad \text{or} \quad x = 1
\end{aligned}$$

**i**

$$\begin{aligned}
4(9^x) - 35(3^x) &= 9 \\
\therefore 4((3^x)^2) - 35(3^x) - 9 &= 0 \\
\therefore (4(3^x) + 1)(3^x - 9) &= 0 \quad \{4a^2 - 35a - 9 = (4a + 1)(a - 9)\} \\
\therefore 4(3^x) &= -1 \quad \text{or} \quad 3^x = 9 \\
\therefore 3^x &= 9 \quad \{\text{since } 4(3^x) \text{ cannot be negative}\} \\
\therefore 3^x &= 3^2 \\
\therefore x &= 2
\end{aligned}$$

**j**

$$\begin{aligned}
4^{x+1} + 2 &= 9(2^x) \\
\therefore 4 \times 4^x + 2 - 9(2^x) &= 0 \\
\therefore 4(2^x)^2 - 9(2^x) + 2 &= 0 \\
\therefore (4(2^x) - 1)(2^x - 2) &= 0 \quad \{4a^2 - 9a + 2 = (4a - 1)(a - 2)\} \\
\therefore 4(2^x) &= 1 \quad \text{or} \quad 2^x = 2 \\
\therefore 2^x &= \frac{1}{4} \quad \text{or} \quad 2^x = 2^1 \\
\therefore 2^x &= 2^{-2} \quad \text{or} \quad 2^x = 2^1 \\
\therefore x &= -2 \quad \text{or} \quad x = 1
\end{aligned}$$

$$\begin{aligned}
\mathbf{k} \quad & 3^{2x-1} = 3^x + 18 \\
& \therefore 3^{2x} \times 3^{-1} = 3^x + 18 \\
& \therefore (3^x)^2 \times \frac{1}{3} - 3^x - 18 = 0 \\
& \therefore \frac{1}{3}(3^x)^2 - 3^x - 18 = 0 \\
& \therefore \frac{1}{3}((3^x)^2 - 3(3^x) - 54) = 0 \\
& \therefore \frac{1}{3}(3^x + 6)(3^x - 9) = 0 \quad \{a^2 - 3a - 54 = (a + 6)(a - 9)\} \\
& \therefore 3^x = -6 \quad \text{or} \quad 3^x = 9 \\
& \therefore 3^x = 9 \quad \{\text{since } 3^x \text{ cannot be negative}\} \\
& \therefore 3^x = 3^2 \\
& \therefore x = 2
\end{aligned}$$

$$\begin{aligned}
\mathbf{l} \quad & 4^x + 2^{x+\frac{1}{2}} = 4 \\
& \therefore (2^x)^2 + (2^{\frac{1}{2}} \times 2^x) = 4 \\
& \therefore (2^x)^2 + \sqrt{2}(2^x) - 4 = 0 \\
& \therefore (2^x + 2\sqrt{2})(2^x - \sqrt{2}) = 0 \quad \{a^2 + \sqrt{2}a - 4 = (a + 2\sqrt{2})(a - \sqrt{2})\} \\
& \therefore 2^x = -2\sqrt{2} \quad \text{or} \quad 2^x = \sqrt{2} \\
& \therefore 2^x = \sqrt{2} \quad \{\text{since } 2^x \text{ cannot be negative}\} \\
& \therefore 2^x = 2^{\frac{1}{2}} \\
& \therefore x = \frac{1}{2}
\end{aligned}$$

$$\begin{aligned}
\mathbf{6} \quad & 4^x = 8^y \\
& \therefore (2^x)^2 = (2^y)^3 \\
& \therefore 2^{2x} = 2^{3y} \\
& \therefore 2x = 3y \\
& \therefore x = \frac{3}{2}y \quad \dots (*)
\end{aligned}$$

$$\begin{aligned}
\text{Also, } & 9^y = \frac{243}{3^x} \\
& \therefore (3^y)^2 = \frac{3^5}{3^x} \\
& \therefore 3^{2y} = 3^{5-x} \\
& \therefore 2y = 5 - x \\
& \therefore 2y = 5 - \frac{3}{2}y \quad \{\text{using } (*)\} \\
& \therefore \frac{7}{2}y = 5 \\
& \therefore y = \frac{10}{7} \\
& \therefore x = \frac{3}{2}\left(\frac{10}{7}\right) \quad \{\text{using } (*) \text{ again}\} \\
& \therefore x = \frac{15}{7}
\end{aligned}$$

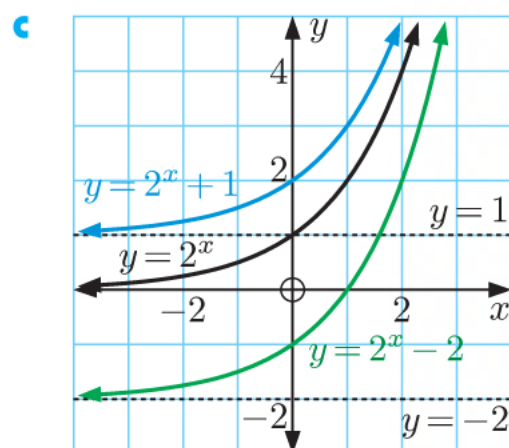
So,  $x = \frac{15}{7}$  and  $y = \frac{10}{7}$ .

## INVESTIGATION 1

## GRAPHS OF EXPONENTIAL FUNCTIONS

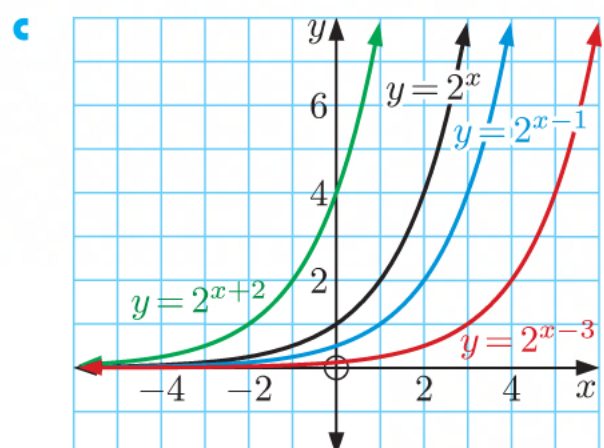
1 a A vertical translation through  $k$  units maps  $y = a^x$  to  $y = a^x + k$ .

- b i The shape of the graph will remain the same.  
 ii The graph is translated vertically through  $k$  units.  
 iii The horizontal asymptote  $y = 0$  is transformed to  $y = k$ .



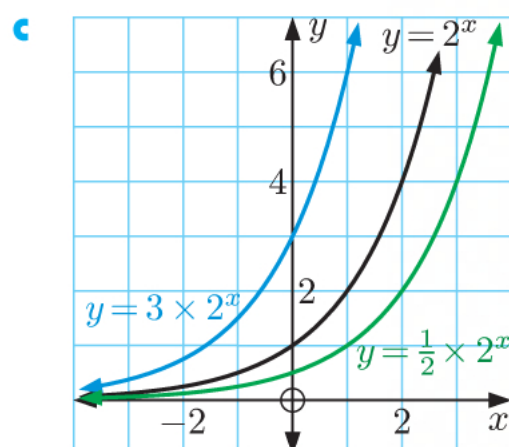
2 a A horizontal translation through  $h$  units maps  $y = a^x$  to  $y = a^{x-h}$ .

- b i The shape of the graph will remain the same.  
 ii The graph is translated horizontally through  $h$  units.  
 iii The horizontal asymptote  $y = 0$  will remain the same.



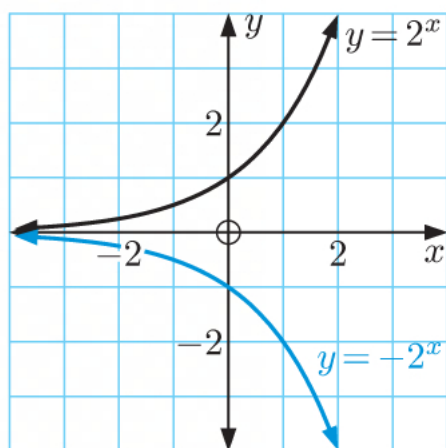
3 a A vertical stretch with scale factor  $p > 0$  will map  $y = a^x$  to  $y = p \times a^x$ .

- b i Each point will become  $p$  times its previous distance from the  $x$ -axis.  
 ii The graph is stretched vertically with invariant  $x$ -axis and scale factor  $p$ .  
 iii The horizontal asymptote  $y = 0$  will remain the same.

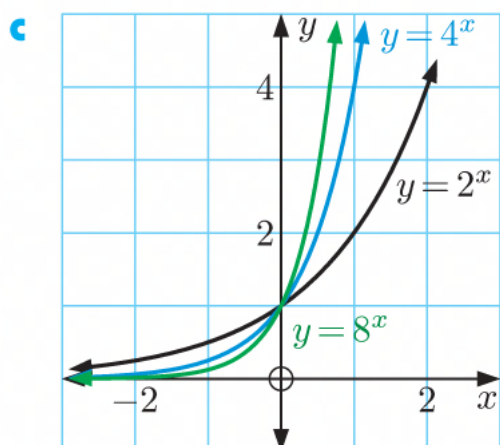




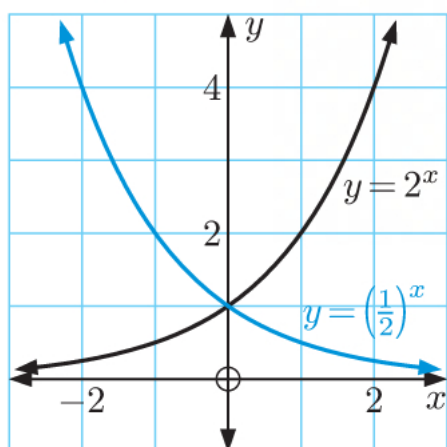
- 4 a** A reflection in the  $x$ -axis maps  $y = a^x$  to  $y = -a^x$ .  
**b**  $y = -2^x$  is a reflection of  $y = 2^x$  in the  $x$ -axis.



- 5 a** A horizontal stretch with scale factor  $\frac{1}{q}$  maps  $y = a^x$  to  $y = a^{qx}$ ,  $q > 0$ .  
**b i** Each point will become  $\frac{1}{q}$  times its previous distance from the  $y$ -axis.  
 If  $0 < q < 1$ , the graph becomes flatter, if  $q > 1$ , the graph becomes steeper, and if  $q = 1$ , the graph remains unchanged.  
**ii** The graph is stretched horizontally with invariant  $y$ -axis and scale factor  $\frac{1}{q}$ .  
**iii** The horizontal asymptote  $y = 0$  will remain the same.



- 6 a** A reflection in the  $y$ -axis will map  $y = a^x$  to  $y = a^{-x}$ .  
**b**  $y = \left(\frac{1}{2}\right)^x = 2^{-x}$  is a reflection of  $y = 2^x$  in the  $y$ -axis.



## EXERCISE 2D

1 a i When  $x = \frac{1}{2}$ ,  $y = 2^{\frac{1}{2}}$

From point A,  $y \approx 1.4$

$$\therefore 2^{\frac{1}{2}} \approx 1.4$$

ii When  $x = 0.8$ ,  $y = 2^{0.8}$

From point B,  $y \approx 1.7$

$$\therefore 2^{0.8} \approx 1.7$$

iii When  $x = 1.5$ ,  $y = 2^{1.5}$

From point C,  $y \approx 2.8$

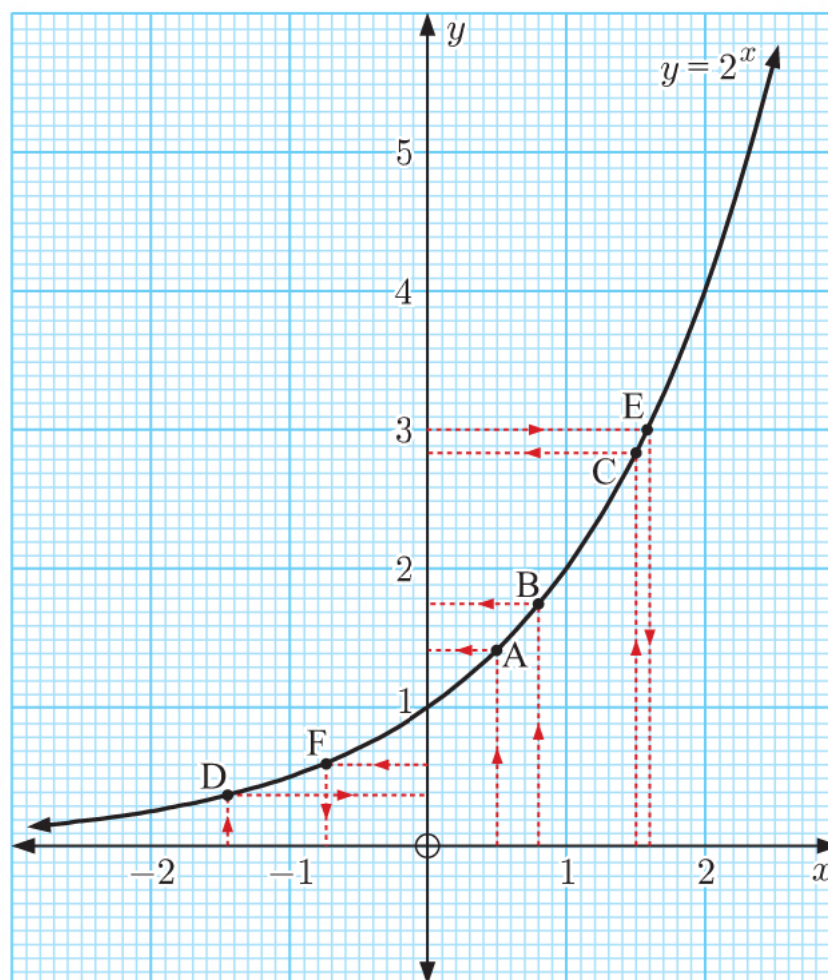
$$\therefore 2^{1.5} \approx 2.8$$

iv When  $x = -\sqrt{2}$ ,  $y = 2^{-\sqrt{2}}$

Using i we know  $x \approx -1.4$

From point D,  $y \approx 0.4$

$$\therefore 2^{-\sqrt{2}} \approx 0.4$$



b i When  $2^x = 3$ ,  $x \approx 1.6$  from point E.

ii When  $2^x = 0.6$ ,  $x \approx -0.7$  from point F.

c The graph of  $y = 2^x$  has horizontal asymptote  $y = 0$ .

$\therefore$  there is no value of  $x$  such that  $2^x = 0$ .

$\therefore 2^x = 0$  has no solutions.

2 a, b Both  $y = 2^x$  and  $y = 10^x$  have  $p > 0$ ,  $a > 0$ , and shape

$y = 10^x$  is steeper than  $y = 2^x$  as  $10 > 2$ .

$\therefore y = 2^x$  corresponds to **C**, and  $y = 10^x$  corresponds to **B**.

c  $y = -5^x$  has  $p < 0$ ,  $a > 1$ , and shape

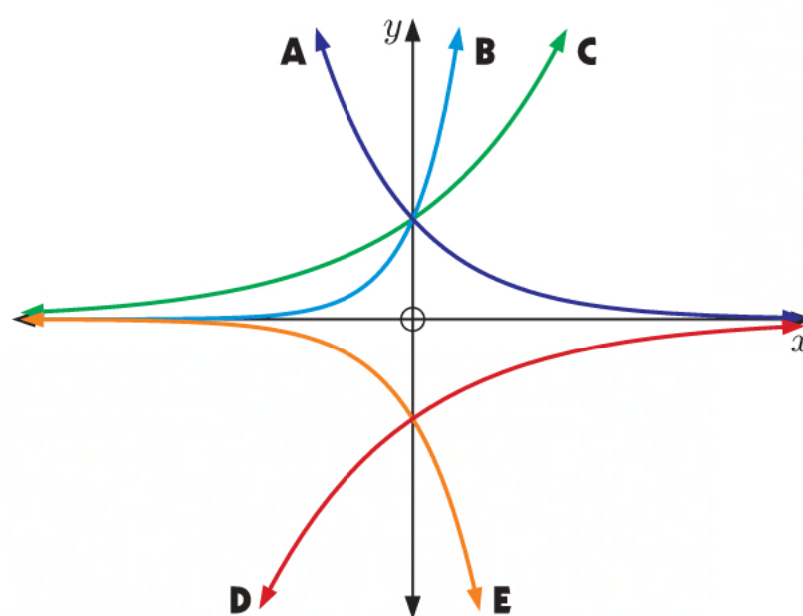
$\therefore y = -5^x$  corresponds to **E**.

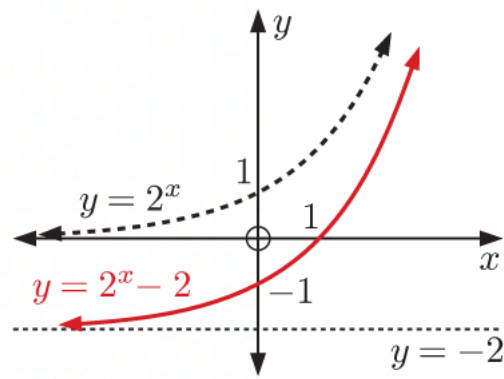
d  $y = \left(\frac{1}{3}\right)^x$  has  $p > 0$ ,  $0 < a < 1$ , and shape

$\therefore y = \left(\frac{1}{3}\right)^x$  corresponds to **A**.

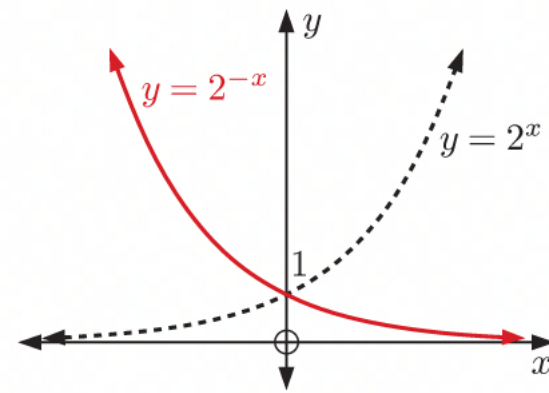
e  $y = -\left(\frac{1}{2}\right)^x$  has  $p < 0$ ,  $0 < a < 1$ , and shape

$\therefore y = -\left(\frac{1}{2}\right)^x$  corresponds to **D**.

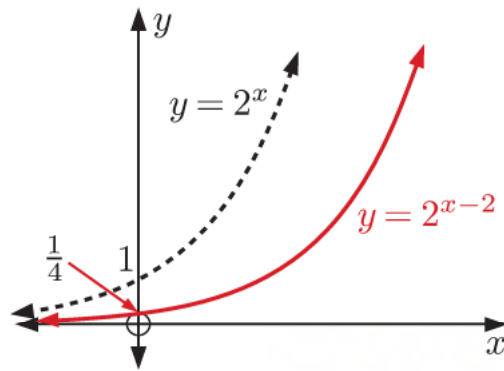


**3 a**

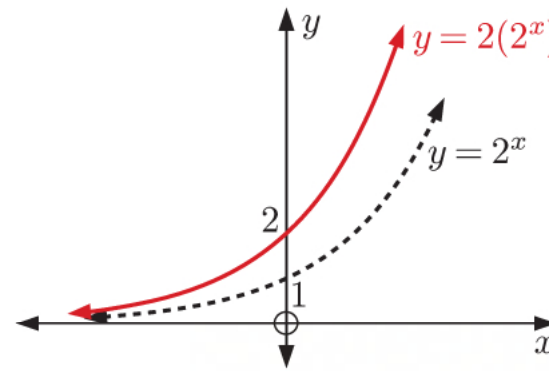
a vertical translation 2 units downwards  
 $y = -2$  is the horizontal asymptote

**b**

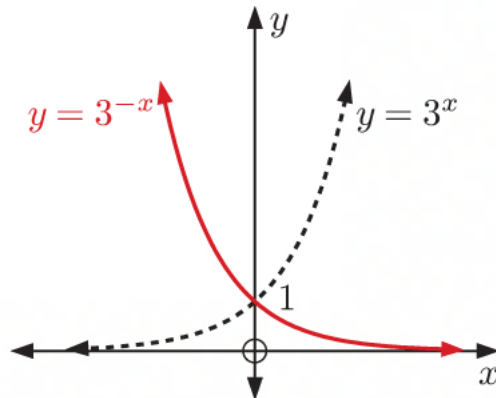
a reflection in the  $y$ -axis

**c**

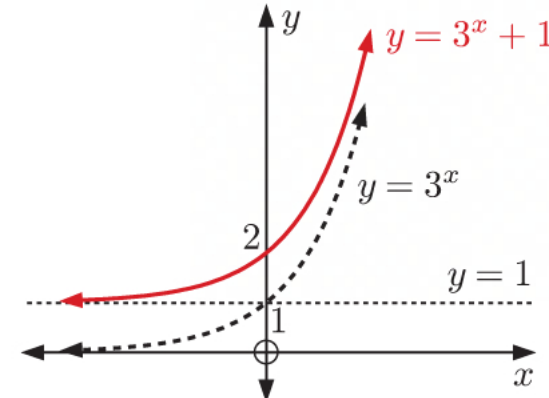
a horizontal translation 2 units right

**d**

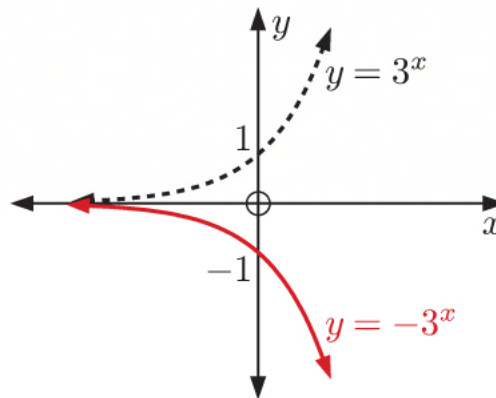
a vertical stretch with scale factor 2

**4 a**

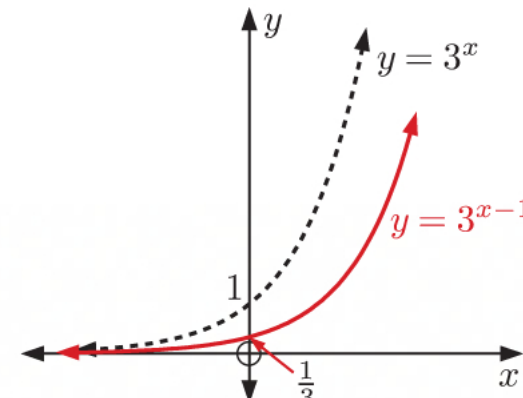
a reflection in the  $y$ -axis

**b**

a vertical translation 1 unit upwards  
 $y = 1$  is the horizontal asymptote

**c**

a reflection in the  $x$ -axis

**d**

a horizontal translation 1 unit right

**5**

- a** The graph of  $y = 3^{-x}$  has horizontal asymptote  $y = 0$ .
- b** The graph of  $y = 2^x - 1$  has horizontal asymptote  $y = -1$ .
- c** The graph of  $y = 3 - 2^{-x}$  has horizontal asymptote  $y = 3$ .
- d** The graph of  $y = 4 \times 2^x + 2$  has horizontal asymptote  $y = 2$ .
- e** The graph of  $y = 5 \times 3^{x+2}$  has horizontal asymptote  $y = 0$ .
- f** The graph of  $y = -2 \times 3^{1-x} - 4$  has horizontal asymptote  $y = -4$ .

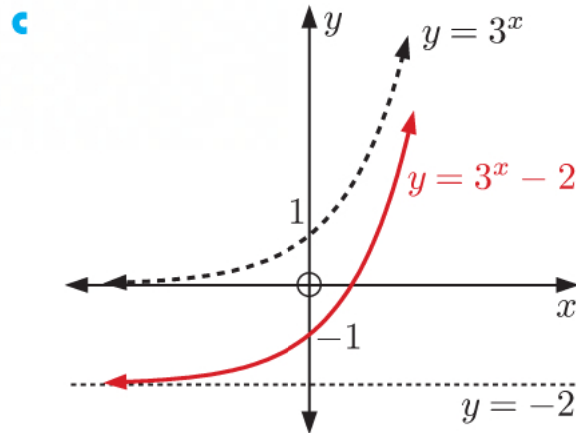
**6**  $f(x) = 3^x - 2$

**a i**  $f(0) = 3^0 - 2$   
 $= 1 - 2$   
 $= -1$

**ii**  $f(2) = 3^2 - 2$   
 $= 9 - 2$   
 $= 7$

**iii**  $f(-2) = 3^{-2} - 2$   
 $= \frac{1}{9} - 2$   
 $= -\frac{17}{9} = -1\frac{8}{9}$

**b** The graph of  $y = 3^x - 2$  has horizontal asymptote  $y = -2$ .



**d** The domain is  $\{x \mid x \in \mathbb{R}\}$ .  
 The range is  $\{y \mid y > -2\}$ .

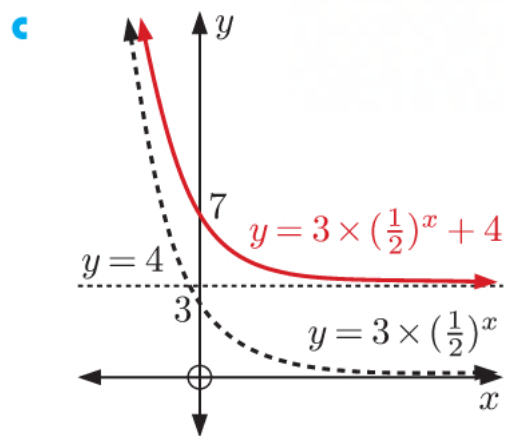
**7**  $g(x) = 3 \times \left(\frac{1}{2}\right)^x + 4$

**a i**  $g(0) = 3 \times \left(\frac{1}{2}\right)^0 + 4$   
 $= 3 \times 1 + 4$   
 $= 7$

**ii**  $g(2) = 3 \times \left(\frac{1}{2}\right)^2 + 4$   
 $= 3 \times \frac{1}{4} + 4$   
 $= \frac{19}{4} = 4\frac{3}{4}$

**iii**  $g(-2) = 3 \times \left(\frac{1}{2}\right)^{-2} + 4$   
 $= 3 \times 2^2 + 4$   
 $= 3 \times 4 + 4$   
 $= 16$

**b** The graph of  $y = 3 \times \left(\frac{1}{2}\right)^x + 4$  has horizontal asymptote  $y = 4$ .



**d** The domain is  $\{x \mid x \in \mathbb{R}\}$ .  
 The range is  $\{y \mid y > 4\}$ .

**8**  $h(x) = -2^{x-3} + 1$

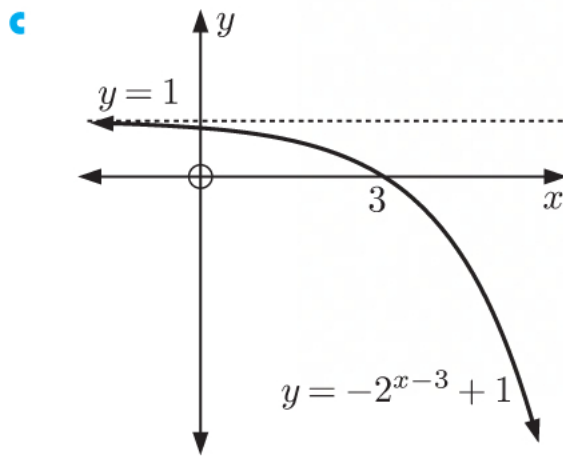
**a i**  $h(0) = -2^{-3} + 1$   
 $= -\frac{1}{2^3} + 1$   
 $= -\frac{1}{8} + 1$   
 $= \frac{7}{8}$

**ii**  $h(3) = -2^0 + 1$   
 $= -1 + 1$   
 $= 0$

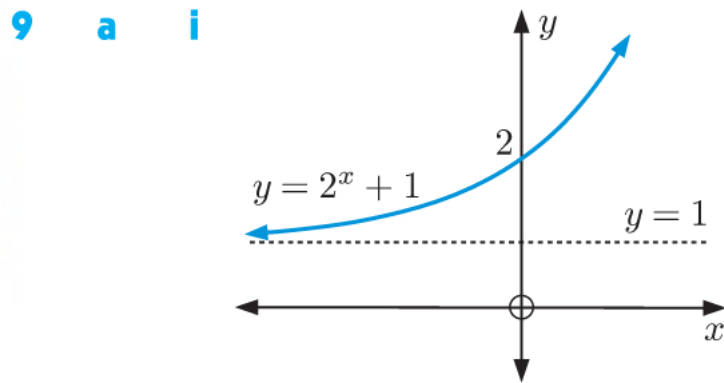
**iii**  $h(6) = -2^3 + 1$   
 $= -8 + 1$   
 $= -7$

**b** The graph of  $y = -2^{x-3} + 1$  has horizontal asymptote  $y = 1$ .



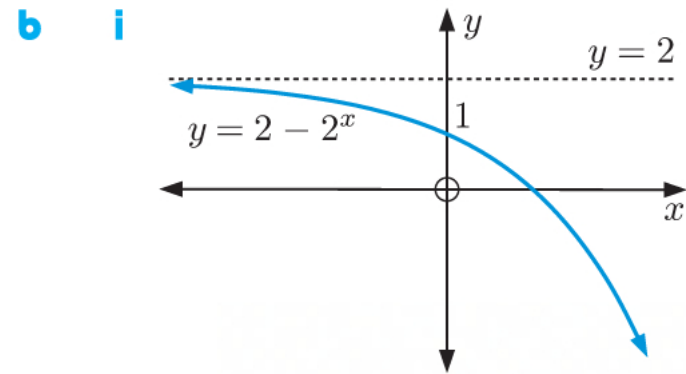


- d** The domain is  $\{x \mid x \in \mathbb{R}\}$ .  
The range is  $\{y \mid y < 1\}$ .



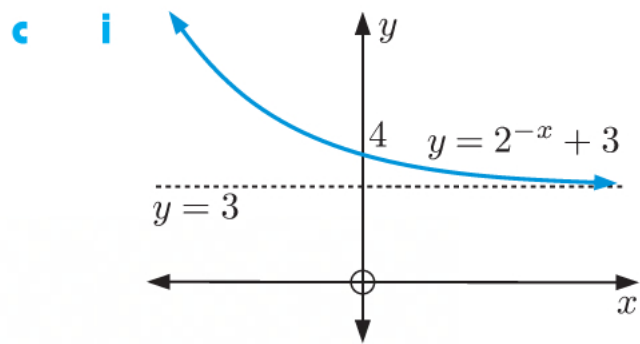
When  $x = 0$ ,  $y = 1 + 1 = 2$   
 When  $x = 2$ ,  $y = 4 + 1 = 5$   
 When  $x = -2$ ,  $y = \frac{1}{4} + 1 = 1\frac{1}{4}$

- ii** The domain is  $\{x \mid x \in \mathbb{R}\}$ .  
The range is  $\{y \mid y > 1\}$ .
- iii** Using technology, when  $x = \sqrt{2}$ ,  $y \approx 3.67$
- iv** As  $x \rightarrow \infty$ ,  $y \rightarrow \infty$   
As  $x \rightarrow -\infty$ ,  $y \rightarrow 1^+$
- v** The horizontal asymptote is  $y = 1$ .



When  $x = 0$ ,  $y = 2 - 1 = 1$   
 When  $x = 1$ ,  $y = 2 - 2 = 0$   
 When  $x = 2$ ,  $y = 2 - 4 = -2$   
 When  $x = -2$ ,  $y = 2 - \frac{1}{4} = 1\frac{3}{4}$

- ii** The domain is  $\{x \mid x \in \mathbb{R}\}$ .  
The range is  $\{y \mid y < 2\}$ .
- iii** Using technology, when  $x = \sqrt{2}$ ,  $y \approx -0.665$
- iv** As  $x \rightarrow \infty$ ,  $y \rightarrow -\infty$   
As  $x \rightarrow -\infty$ ,  $y \rightarrow 2^-$
- v** The horizontal asymptote is  $y = 2$ .



When  $x = 0$ ,  $y = 1 + 3 = 4$

When  $x = 2$ ,  $y = \frac{1}{4} + 3 = 3\frac{1}{4}$

When  $x = -2$ ,  $y = 2^2 + 3 = 7$

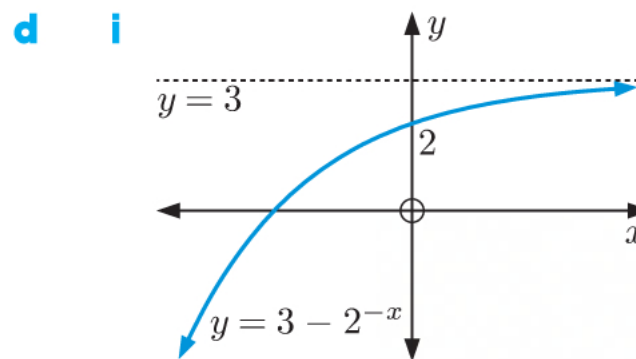
**ii** The domain is  $\{x \mid x \in \mathbb{R}\}$ .

The range is  $\{y \mid y > 3\}$ .

**iii** Using technology, when  
 $x = \sqrt{2}$ ,  $y \approx 3.38$

**iv** As  $x \rightarrow \infty$ ,  $y \rightarrow 3^+$   
As  $x \rightarrow -\infty$ ,  $y \rightarrow \infty$

**v** The horizontal asymptote is  $y = 3$ .



When  $x = 0$ ,  $y = 3 - 1 = 2$

When  $x = 2$ ,  $y = 3 - \frac{1}{4} = 2\frac{3}{4}$

When  $x = -2$ ,  $y = 3 - 4 = -1$

**ii** The domain is  $\{x \mid x \in \mathbb{R}\}$ .

The range is  $\{y \mid y < 3\}$ .

**iii** Using technology, when  
 $x = \sqrt{2}$ ,  $y \approx 2.62$

**iv** As  $x \rightarrow \infty$ ,  $y \rightarrow 3^-$   
As  $x \rightarrow -\infty$ ,  $y \rightarrow -\infty$

**v** The horizontal asymptote is  $y = 3$ .

**10 a**  $y = a \times 2^x + b$

When  $x = 0$ ,  $y = -5$

$\therefore a \times 2^0 + b = -5$

$\therefore a + b = -5$

$\therefore a = -5 - b \quad \dots (*)$

When  $y = 0$ ,  $x = 1$

$\therefore a \times 2^1 + b = 0$

$\therefore 2a + b = 0$

$\therefore 2(-5 - b) + b = 0 \quad \{\text{using } (*)\}$

$\therefore -10 - 2b + b = 0$

$\therefore -10 - b = 0$

$\therefore b = -10$

Substituting  $b = -10$  into  $(*)$ ,  $a = -5 - (-10)$

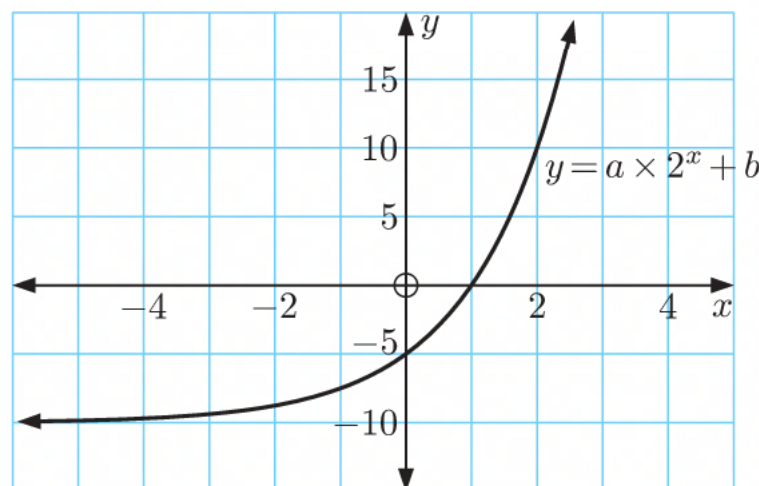
$\therefore a = 5$

So,  $a = 5$ ,  $b = -10$ .

**b** When  $x = 6$ ,  $y = 5 \times 2^6 - 10$

$\therefore y = 5 \times 64 - 10$

$\therefore y = 310$



$$\begin{aligned}
 11 \quad a \quad f(0) &= 3.5 - a^0 \\
 &= 3.5 - 1 \\
 &= 2.5
 \end{aligned}$$

$\therefore$  the  $y$ -intercept is 2.5.

$\therefore$  P is (0, 2.5).

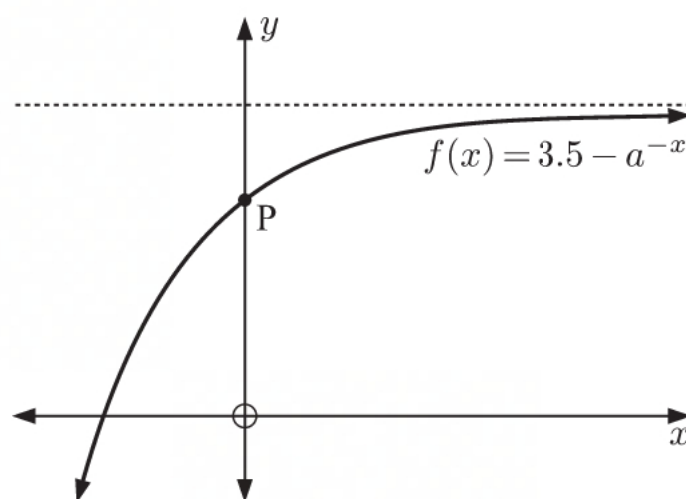
b The point  $(-1, 2)$  lies on the graph.

$$\therefore f(-1) = 2$$

$$\therefore 3.5 - a^1 = 2$$

$$\therefore a = 1.5$$

c  $f(x) = 3.5 - 1.5^{-x}$  has horizontal asymptote  $y = 3.5$ .



12 a  $y = 2^{x^2+1}$  is defined for all  $x \in \mathbb{R}$ .  
 $\therefore$  the domain is  $\{x \mid x \in \mathbb{R}\}$ .

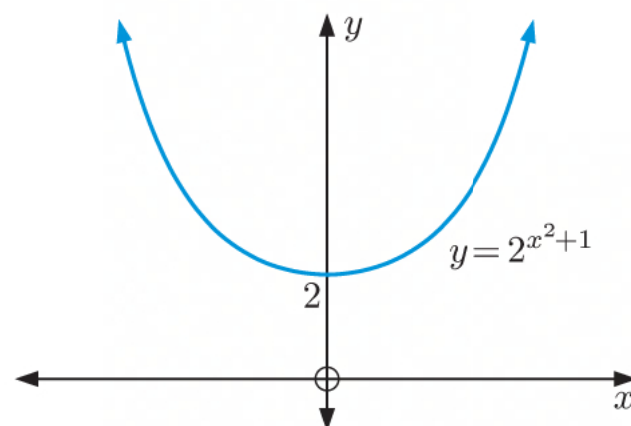
$$x^2 \geq 0 \text{ for all } x \in \mathbb{R}$$

$$\therefore x^2 + 1 \geq 1$$

$$\therefore 2^{x^2+1} \geq 2^1$$

$$\therefore 2^{x^2+1} \geq 2$$

$\therefore$  the range is  $\{y \mid y \geq 2\}$ .



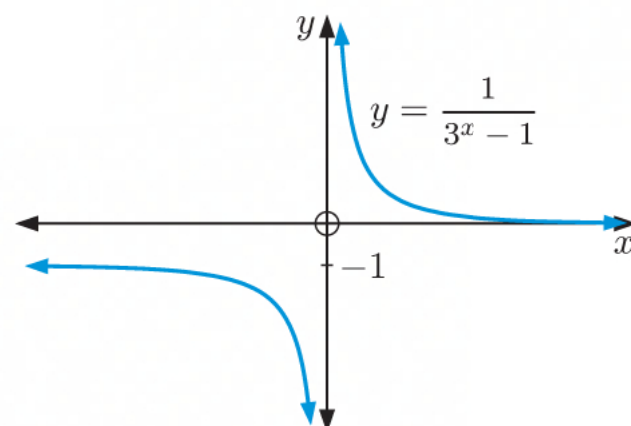
b  $y = \frac{1}{3^x - 1}$  is defined when  $3^x - 1 \neq 0$   
 $\therefore 3^x \neq 1$   
 $\therefore 3^x \neq 3^0$   
 $\therefore x \neq 0$

$\therefore$  the domain is  $\{x \mid x \neq 0\}$ .

$$\frac{1}{3^x - 1} < -1 \text{ for } x < 0$$

and  $\frac{1}{3^x - 1} > 0$  for  $x > 0$

$\therefore$  the range is  $\{y \mid y > 0 \text{ or } y < -1\}$ .

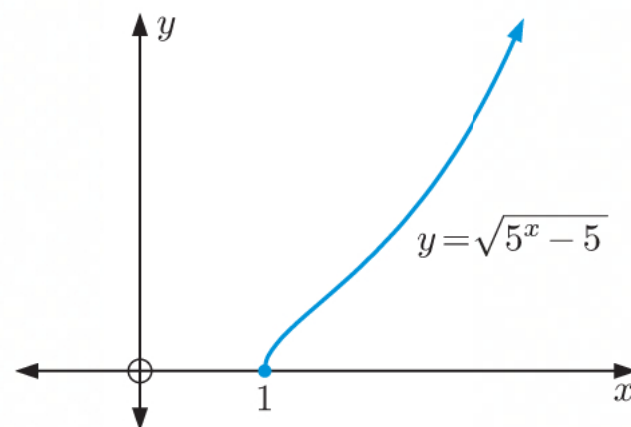


c  $y = \sqrt{5^x - 5}$  is defined when  $5^x - 5 \geq 0$   
 $\therefore 5^x \geq 5$   
 $\therefore 5^x \geq 5^1$   
 $\therefore x \geq 1$

$\therefore$  the domain is  $\{x \mid x \geq 1\}$ .

A square root cannot be negative.

$\therefore$  the range is  $\{y \mid y \geq 0\}$ .



**13**  $f(x) = 3^x - 9$  and  $g(x) = \sqrt{x}$

**a**  $(f \circ g)(x) = f(g(x))$   
 $= f(\sqrt{x})$   
 $= 3^{\sqrt{x}} - 9$

$\sqrt{x}$  is defined when  $x \geq 0$ , so  $3^{\sqrt{x}} - 9$  is defined when  $x \geq 0$ .

$\therefore$  the domain is  $\{x \mid x \geq 0\}$ .

$$3^{\sqrt{x}} \geq 1$$

$$\therefore 3^{\sqrt{x}} - 9 \geq -8$$

$\therefore$  the range is  $\{y \mid y \geq -8\}$ .

**b**  $(g \circ f)(x) = g(f(x))$   
 $= g(3^x - 9)$   
 $= \sqrt{3^x - 9}$

$$\begin{aligned} \sqrt{3^x - 9} \text{ is defined when } 3^x - 9 &\geq 0 \\ \therefore 3^x &\geq 9 \\ \therefore 3^x &\geq 3^2 \\ \therefore x &\geq 2 \end{aligned}$$

$\therefore$  the domain is  $\{x \mid x \geq 2\}$ .

A square root cannot be negative.

$\therefore$  the range is  $\{y \mid y \geq 0\}$ .

**c i**  $(f \circ g)(x) = 0$   
 $\therefore 3^{\sqrt{x}} - 9 = 0$   
 $\therefore 3^{\sqrt{x}} = 9$   
 $\therefore 3^{\sqrt{x}} = 3^2$   
 $\therefore \sqrt{x} = 2$   
 $\therefore x = 4$

**ii**  $(g \circ f)(x) = 3\sqrt{2}$   
 $\therefore \sqrt{3^x - 9} = \sqrt{18}$   
 $\therefore 3^x - 9 = 18$   
 $\therefore 3^x = 27$   
 $\therefore 3^x = 3^3$   
 $\therefore x = 3$

**14**  $f(x) = 2^x - 3$  and  $g(x) = 1 + 2^{-x}$

**a i** The graph of  $y = 2^x - 3$  has horizontal asymptote  $y = -3$ .  
The graph of  $y = 1 + 2^{-x}$  has horizontal asymptote  $y = 1$ .

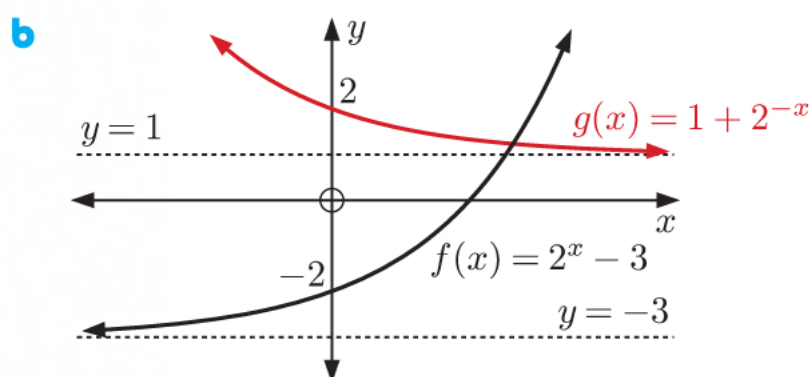
**ii**  $2^x > 0$  for all  $x \in \mathbb{R}$   
 $\therefore 2^x - 3 > -3$   
 $\therefore$  the range of  $y = 2^x - 3$  is  $\{y \mid y > -3\}$ .

$$\begin{aligned} 2^{-x} &> 0 \text{ for all } x \in \mathbb{R} \\ \therefore 1 + 2^{-x} &> 1 \\ \therefore \text{the range of } y = 1 + 2^{-x} &\text{ is } \{y \mid y > 1\}. \end{aligned}$$

**iii** For  $y = 2^x - 3$ :  
When  $x = 0$ ,  $y = 2^0 - 3$   
 $= 1 - 3$   
 $= -2$   
 $\therefore$  the  $y$ -intercept is  $-2$ .

For  $y = 1 + 2^{-x}$ :  
When  $x = 0$ ,  $y = 1 + 2^0$   
 $= 1 + 1$   
 $= 2$   
 $\therefore$  the  $y$ -intercept is  $2$ .





**c** The graphs intersect where  $2^x - 3 = 1 + 2^{-x}$

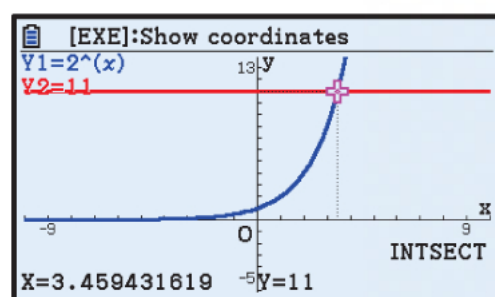
$$\begin{aligned} \therefore 2^x - 4 - 2^{-x} &= 0 \\ \therefore 2^x(2^x - 4 - 2^{-x}) &= 0 \\ \therefore (2^x)^2 - 4(2^x) - 2^0 &= 0 \\ \therefore (2^x)^2 - 4(2^x) &= 1 \\ \therefore (2^x)^2 - 4(2^x) + (-2)^2 &= 1 + (-2)^2 \\ \therefore (2^x - 2)^2 &= 5 \\ \therefore 2^x - 2 &= \pm\sqrt{5} \\ \therefore 2^x - 2 - 1 &= -1 \pm \sqrt{5} \\ \therefore 2^x - 3 &= -1 \pm \sqrt{5} \\ \therefore y &= -1 \pm \sqrt{5} \quad \{\text{since } y = 2^x - 3\} \end{aligned}$$

From the graph in **b**,  $y$  is positive where the graphs intersect.

$\therefore$  the  $y$ -coordinate of the point where the graphs intersect is  $-1 + \sqrt{5}$ .

**15 a**  $2^x = 11$

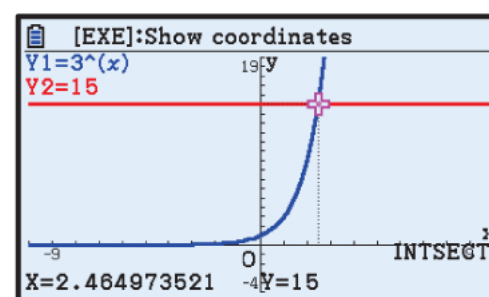
We graph  $Y_1 = 2^x$  and  $Y_2 = 11$  on the same set of axes and find their point of intersection.



The solution is  $x \approx 3.46$ .

**b**  $3^x = 15$

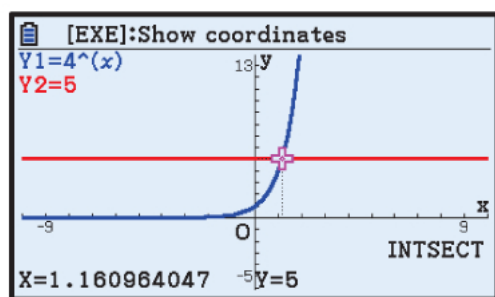
We graph  $Y_1 = 3^x$  and  $Y_2 = 15$  on the same set of axes and find their point of intersection.



The solution is  $x \approx 2.46$ .

c  $4^x + 5 = 10$   
 $\therefore 4^x = 5$

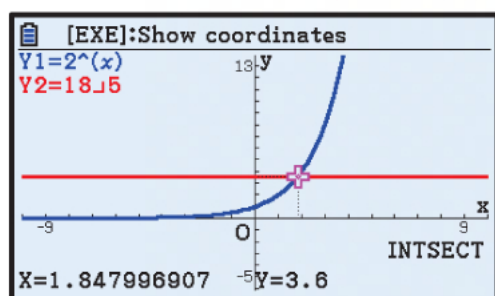
We graph  $Y_1 = 4^x$  and  $Y_2 = 5$  on the same set of axes and find their point of intersection.



The solution is  $x \approx 1.16$ .

e  $5 \times 2^x = 18$   
 $\therefore 2^x = \frac{18}{5}$

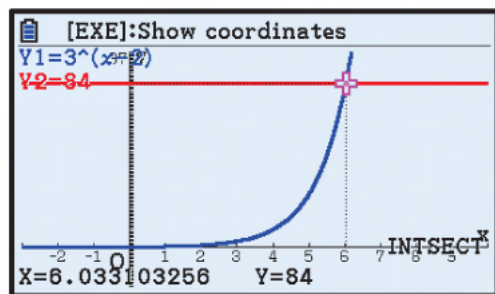
We graph  $Y_1 = 2^x$  and  $Y_2 = \frac{18}{5}$  on the same set of axes and find their point of intersection.



The solution is  $x \approx 1.85$ .

g  $2 \times 3^{x-2} = 168$   
 $\therefore 3^{x-2} = 84$

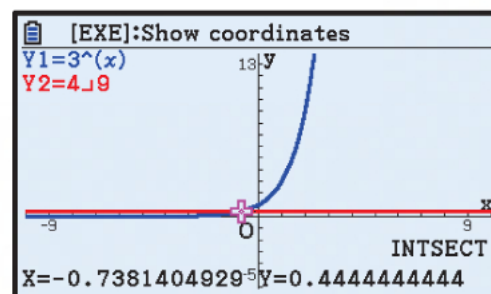
We graph  $Y_1 = 3^{x-2}$  and  $Y_2 = 84$  on the same set of axes and find their point of intersection.



The solution is  $x \approx 6.03$ .

d  $3^{x+2} = 4$   
 $\therefore 3^x \times 3^2 = 4$   
 $\therefore 3^x = \frac{4}{9}$

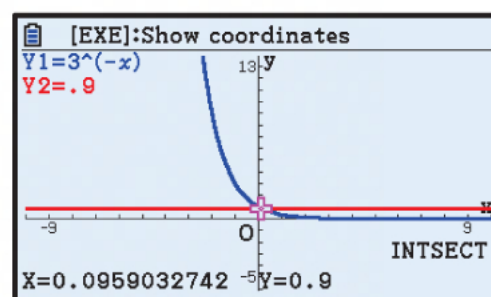
We graph  $Y_1 = 3^x$  and  $Y_2 = \frac{4}{9}$  on the same set of axes and find their point of intersection.



The solution is  $x \approx -0.738$ .

f  $3^{-x} = 0.9$

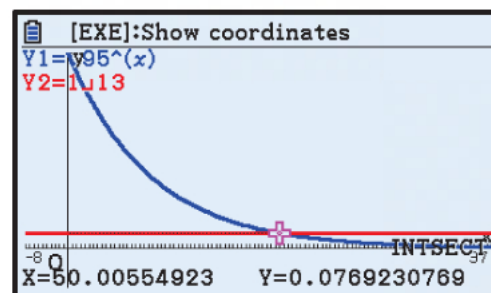
We graph  $Y_1 = 3^{-x}$  and  $Y_2 = 0.9$  on the same set of axes and find their point of intersection.



The solution is  $x \approx 0.0959$ .

h  $26 \times (0.95)^x = 2$   
 $\therefore (0.95)^x = \frac{1}{13}$

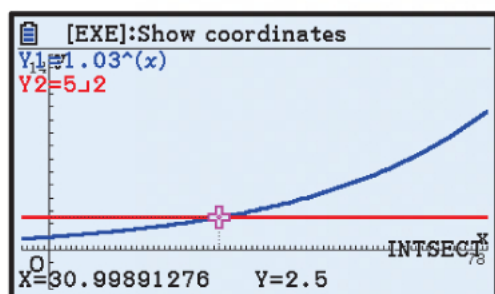
We graph  $Y_1 = (0.95)^x$  and  $Y_2 = \frac{1}{13}$  on the same set of axes and find their point of intersection.



The solution is  $x \approx 50.0$ .

$$\begin{aligned} \text{i } 2000 \times (1.03)^x &= 5000 \\ \therefore (1.03)^x &= \frac{5}{2} \end{aligned}$$

We graph  $Y_1 = (1.03)^x$  and  $Y_2 = \frac{5}{2}$  on the same set of axes and find their point of intersection.

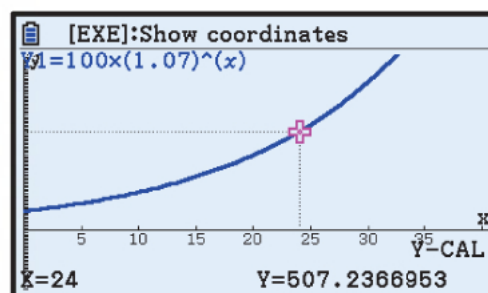
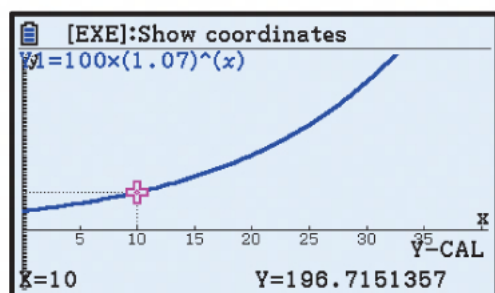
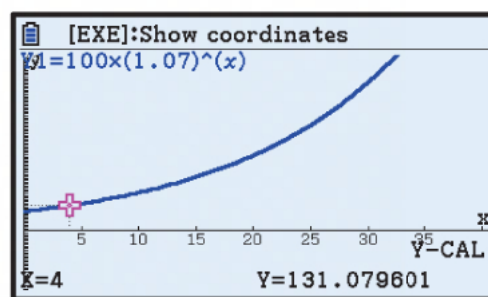
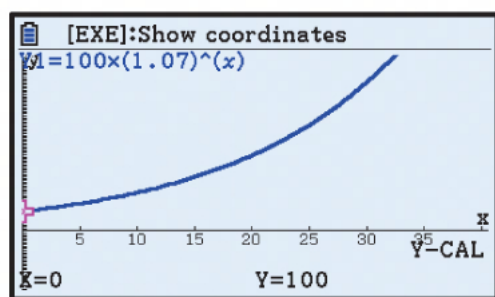
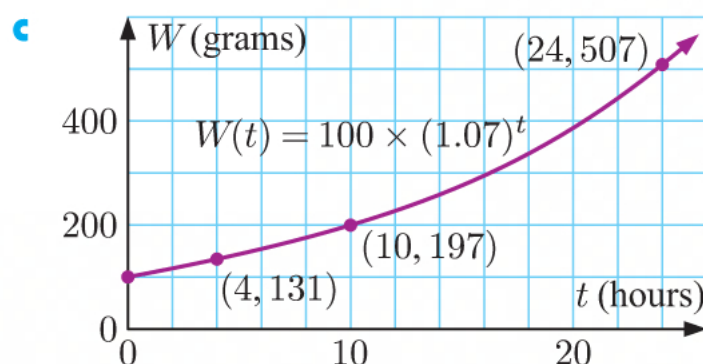


The solution is  $x \approx 31.0$ .

## EXERCISE 2E.1

- 1 a  $W(0) = 100 \times (1.07)^0$   
 $= 100 \times 1$   
 $= 100$   
 $\therefore$  the initial weight was 100 grams.

- b i  $W(4) = 100 \times (1.07)^4$   
 $\approx 131$   
 After 4 hours, the weight is about 131 g.
- ii  $W(10) = 100 \times (1.07)^{10}$   
 $\approx 197$   
 After 10 hours, the weight is about 197 g.
- iii  $W(24) = 100 \times (1.07)^{24}$   
 $\approx 507$   
 After 24 hours, the weight is about 507 g.



**2 a**  $P_0 = 50$  possums

**b**  $P(n) = 50 \times (1.23)^n$

**i**  $P(2) = 50 \times (1.23)^2$   
 $\approx 75.6$

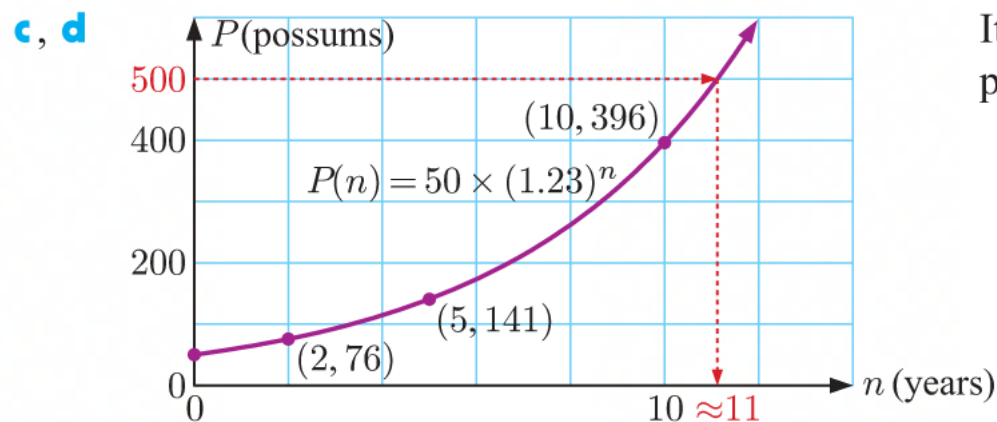
After 2 years, the expected population is about 76 possums.

**ii**  $P(5) = 50 \times (1.23)^5$   
 $\approx 141$

After 5 years, the expected population is about 141 possums.

**iii**  $P(10) = 50 \times (1.23)^{10}$   
 $\approx 396$

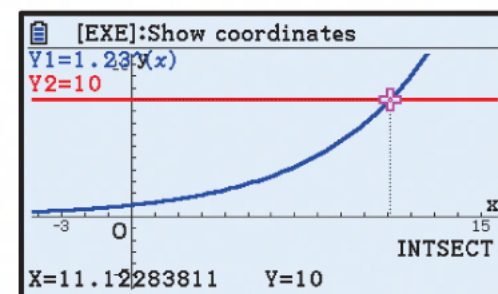
After 10 years, the expected population is about 396 possums.



It will take about 11 years for the population to reach 500.

**e**  $P(n) = 500$   
 $\therefore 50 \times (1.23)^n = 500$   
 $\therefore (1.23)^n = 10$

We graph  $Y_1 = (1.23)^x$  and  $Y_2 = 10$  on the same set of axes and find their point of intersection.



The solution is  $n \approx 11.1$ .

It will take about 11.1 years for the population to reach 500.

**3 a**  $N = 4 \times 1.332^t, t \geq 0$

When  $t = 0$ ,  $N = 4 \times 1.332^0$   
 $= 4 \times 1$   
 $= 4$

$\therefore$  the number of people initially infected was 4.

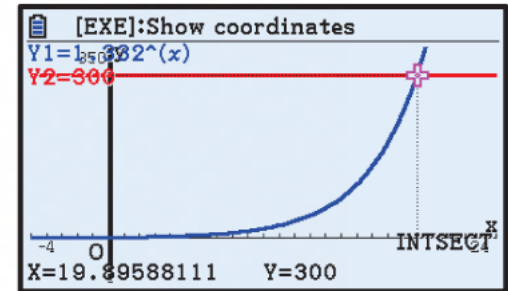
**b** When  $t = 16$ ,  $N = 4 \times 1.332^{16}$   
 $\approx 393$

$\therefore$  the number of people infected after 16 days was about 393.



$$\begin{aligned} \text{c} \quad N &= 1200 \\ \therefore 4 \times 1.332^t &= 1200 \\ \therefore 1.332^t &= 300 \end{aligned}$$

We graph  $Y_1 = 1.332^x$  and  $Y_2 = 300$  on the same set of axes and find their intersection.



The solution is  $t \approx 19.9$ .

It will take about 19.9 days for everybody in the school to catch the flu.

- 4 a  $B_0 = 200$  bears  
b 2000 is 2 years after 1998, so  $t = 2$ .

$$\begin{aligned} B(t) &= 200 \times a^t \\ B(2) &= 200 \times a^2 \\ \therefore 242 &= 200 \times a^2 \\ \therefore \frac{242}{200} &= a^2 \\ \therefore a &= \sqrt{\frac{242}{200}} = 1.1 \quad \{a > 0\} \end{aligned}$$

The bear population is increasing by 10% every year.

- c 2018 is 20 years after 1998, so  $t = 20$ .

$$\begin{aligned} B(20) &= 200 \times (1.1)^{20} \\ &\approx 1350 \end{aligned}$$

The expected bear population in 2018 is about 1350 bears.

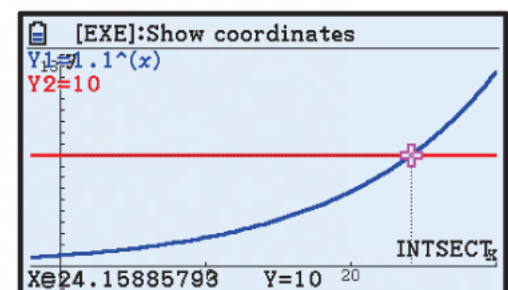
- d 2008 is 10 years after 1998, so  $t = 10$ .

$$B(10) = 200 \times (1.1)^{10}$$

$$\begin{aligned} \text{Percentage increase from 2008 to 2018} &= \left( \frac{B(20) - B(10)}{B(10)} \right) \times 100\% \\ &= \left( \frac{200 \times (1.1)^{20} - 200 \times (1.1)^{10}}{200 \times (1.1)^{10}} \right) \times 100\% \\ &\approx 159\% \end{aligned}$$

$$\begin{aligned} \text{e} \quad B(t) &= 2000 \\ \therefore 200 \times (1.1)^t &= 2000 \\ \therefore (1.1)^t &= 10 \end{aligned}$$

We graph  $Y_1 = (1.1)^x$  and  $Y_2 = 10$  on the same set of axes and find their point of intersection.



The solution is  $t \approx 24.2$ .

It will take about 24.2 years for the population to reach 200.

$$\begin{aligned}
 \text{5 a i } V(0) &= V_0 \times 2^{0.05(0)} \\
 &= V_0 \times 2^0 \\
 &= V_0 \times 1 \\
 &= V_0
 \end{aligned}$$

So, the reaction speed at  $0^\circ\text{C}$  is  $V_0$ .

$$\begin{aligned}
 \text{ii } V(20) &= V_0 \times 2^{0.05(20)} \\
 &= V_0 \times 2^1 \\
 &= 2V_0
 \end{aligned}$$

So, the reaction speed at  $20^\circ\text{C}$  is  $2V_0$ .

$$\begin{aligned}
 \text{b Percentage increase at } 20^\circ\text{C compared with } 0^\circ\text{C} &= \left( \frac{V(20) - V(0)}{V(0)} \right) \times 100\% \\
 &= \left( \frac{2V_0 - V_0}{V_0} \right) \times 100\% \\
 &= \left( \frac{V_0}{V_0} \right) \times 100\% \\
 &= 100\%
 \end{aligned}$$

So, there is a 100% increase in reaction speed at  $20^\circ\text{C}$  compared with  $0^\circ\text{C}$ .

$$\begin{aligned}
 \text{c } \left( \frac{V(50) - V(20)}{V(20)} \right) \times 100\% &= \left( \frac{V_0 \times 2^{0.05(50)} - 2V_0}{2V_0} \right) \times 100\% \\
 &= \left( \frac{V_0 \times 2^{2.5} - 2V_0}{2V_0} \right) \times 100\% \\
 &= \left( \frac{V_0(2^{2.5} - 2)}{V_0(2)} \right) \times 100\% \\
 &= \left( \frac{2^{2.5} - 2}{2} \right) \times 100\% \\
 &\approx 183\%
 \end{aligned}$$

This means that there is about a 183% increase in reaction speed at  $50^\circ\text{C}$  compared with  $20^\circ\text{C}$ .

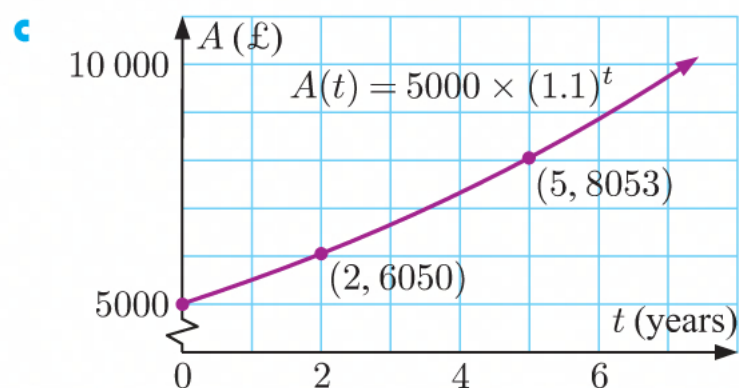
$$\text{6 a } A(t) = 5000 \times (1.1)^t, \text{ where } t \text{ is the number of years since Kayla deposited } \pounds 5000.$$

$$\begin{aligned}
 \text{b i } A(2) &= 5000 \times (1.1)^2 \\
 &= 6050
 \end{aligned}$$

So there was  $\pounds 6050$  in the account after 2 years.

$$\begin{aligned}
 \text{ii } A(5) &= 5000 \times (1.1)^5 \\
 &= 8052.55
 \end{aligned}$$

So there was  $\pounds 8052.55$  in the account after 5 years.

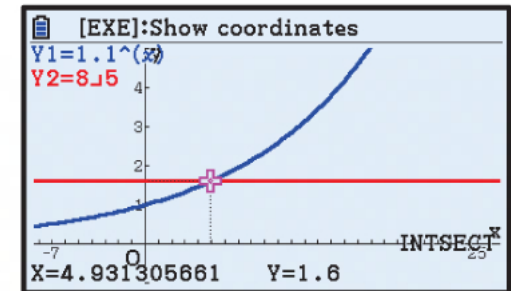


**d**  $A(t) = 8000$

$$\therefore 5000 \times (1.1)^t = 8000$$

$$\therefore (1.1)^t = \frac{8}{5}$$

We graph  $Y_1 = (1.1)^x$  and  $Y_2 = \frac{8}{5}$  on the same set of axes and find their point of intersection.



The solution is  $t \approx 4.93$ .

It will take about 4.93 years for the amount in the account to reach £8000.

**7 a**  $V = k \times a^t$  dollars,  $t \geq 0$

When  $t = 1$ ,  $V = 405\,000$

$$\therefore 405\,000 = k \times a^1$$

$$\therefore k = \frac{405\,000}{a} \quad \dots (*)$$

When  $t = 3$ ,  $V = 472\,392$

$$\therefore 472\,392 = k \times a^3$$

$$\therefore 472\,392 = \frac{405\,000}{a} \times a^3$$

{using (\*)}

$$\therefore 472\,392 = 405\,000 \times a^2$$

$$\therefore \frac{472\,392}{405\,000} = a^2$$

$$\therefore a = \sqrt{\frac{472\,392}{405\,000}} \quad \{a > 0\}$$

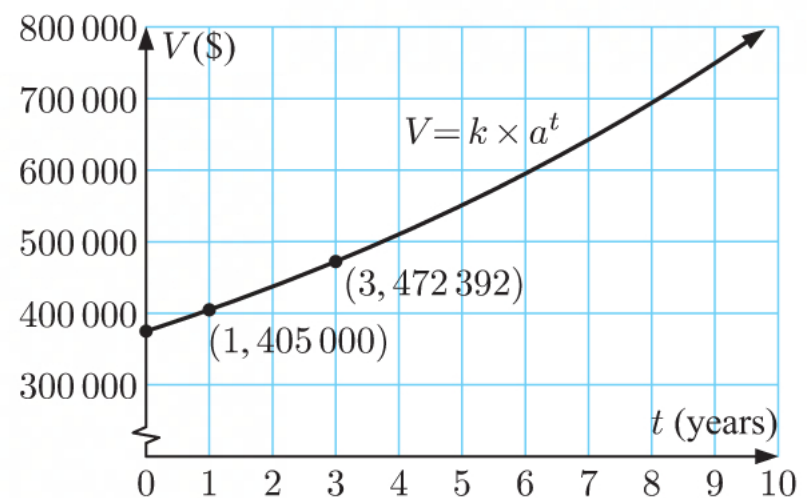
$$\therefore a = 1.08$$

$$\text{Now, } k = \frac{405\,000}{1.08} \quad \{\text{using (*) again}\}$$

$$\therefore k = 375\,000$$

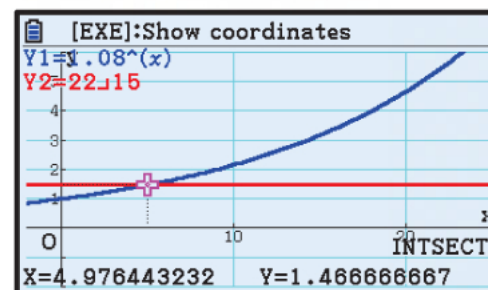
$a = 1.08$ , the expected value of the house is increasing by 8% per year.

$k = 375\,000$ , the original value of the house was \$375 000.



$$\begin{aligned}
 \text{b} \quad & V = 375\,000 \times (1.08)^t \\
 \therefore & 550\,000 = 375\,000 \times (1.08)^t \\
 \therefore & \frac{550\,000}{375\,000} = (1.08)^t \\
 \therefore & (1.08)^t = \frac{550}{375} = \frac{22}{15}
 \end{aligned}$$

We graph  $Y_1 = (1.08)^x$  and  $Y_2 = \frac{22}{15}$  on the same set of axes and find their point of intersection.



The solution is  $t \approx 4.98$ .

It will take about 4.98 years for the house's value to reach \$550 000.

$$8 \quad V = c - 60 \times 2^{kt} \text{ m s}^{-1}$$

$$\begin{aligned}
 \text{a} \quad & \text{When } t = 0, \quad V = c - 60 \times 2^{k(0)} \\
 & \quad \quad \quad = c - 60 \times 2^0 \\
 & \quad \quad \quad = c - 60
 \end{aligned}$$

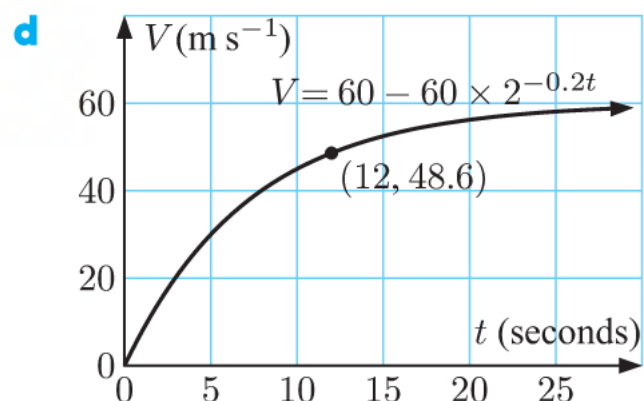
Now,  $V = 0$  at time  $t = 0$

$$\begin{aligned}
 \therefore & 0 = c - 60 \\
 \therefore & c = 60
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & V = 60 - 60 \times 2^{kt} \\
 \text{When } t = 5, & \quad V = 60 - 60 \times 2^{5k} \\
 \therefore & 30 = 60 - 60 \times 2^{5k} \\
 \therefore & -30 = -60 \times 2^{5k} \\
 \therefore & \frac{1}{2} = 2^{5k} \\
 \therefore & 2^{-1} = 2^{5k} \\
 \therefore & -1 = 5k \\
 \therefore & k = -\frac{1}{5} = -0.2
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & V = 60 - 60 \times 2^{-0.2t} \\
 \text{When } t = 12, & \quad V = 60 - 60 \times 2^{-0.2(12)} \\
 & \quad \quad \quad = 60 - 60 \times 2^{-2.4} \\
 & \quad \quad \quad \approx 48.6
 \end{aligned}$$

After 12 seconds, the speed of the parachutist is about  $48.6 \text{ m s}^{-1}$ .





- e From the graph, the parachutist accelerates rapidly until he approaches his terminal velocity of  $60 \text{ m s}^{-1}$ .

- 9 Let the number of microorganisms in the culture be  $N = k \times a^t$  where  $k$  is the initial population,  $a$  is the rate at which they increase, and  $t$  is the number of hours since their introduction.

After 6 hours,  $N$  has doubled, so  $N = k \times a^6$

$$\therefore 2 \times k = k \times a^6$$

$$\therefore 2 = a^6$$

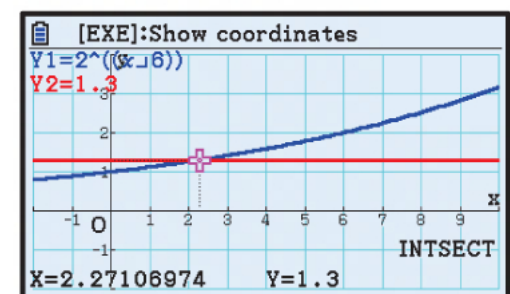
$$\therefore a = \sqrt[6]{2} \quad \{a > 0\}$$

For  $N$  to have increased by 30%,  $N = 1.3 \times k$

$$\therefore 1.3 \times k = k \times (\sqrt[6]{2})^t$$

$$\therefore 1.3 = 2^{\frac{t}{6}}$$

We graph  $Y_1 = 2^{\frac{x}{6}}$  and  $Y_2 = 1.3$  on the same set of axes and find their point of intersection.



The solution is  $t \approx 2.27$ .

It will take about 2.27 hours for the number of microorganisms to increase by 30%.

## EXERCISE 2E.2

- 1  $W(t) = 250 \times (0.998)^t$  grams

a  $W(0) = 250 \times (0.998)^0$   
 $= 250 \times 1$   
 $= 250$

$\therefore$  there was initially 250 grams of radioactive substance set aside.

b i  $W(400) = 250 \times (0.998)^{400}$   
 $\approx 112$

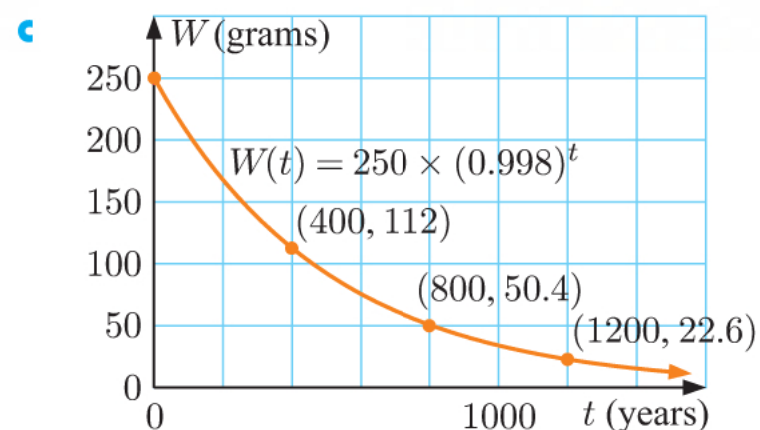
The weight was about 112 grams after 400 years.

ii  $W(800) = 250 \times (0.998)^{800}$   
 $\approx 50.4$

The weight was about 50.4 grams after 800 years.

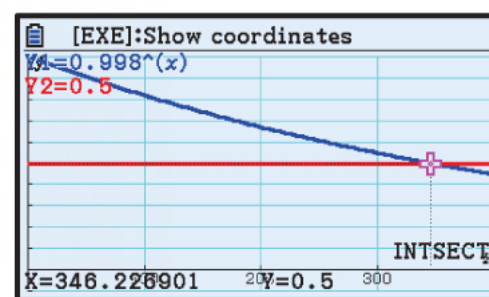
iii  $W(1200) = 250 \times (0.998)^{1200}$   
 $\approx 22.6$

The weight was about 22.6 grams after 1200 years.



$$\begin{aligned}
 \text{d} \quad & W(t) = 125 \\
 \therefore & 250 \times (0.998)^t = 125 \\
 \therefore & (0.998)^t = 0.5 \\
 \therefore & t \approx 346.2 \quad \{\text{using technology}\}
 \end{aligned}$$

It takes approximately 346 years for the substance to decay to 125 grams.



$$2 \quad T(t) = 100 \times (0.986)^t \text{ } ^\circ\text{C}$$

$$\begin{aligned}
 \text{a} \quad & T(0) = 100 \times (0.986)^0 \\
 & = 100 \times 1 \\
 & = 100
 \end{aligned}$$

The initial temperature of the liquid was  $100^\circ\text{C}$ .

$$\begin{aligned}
 \text{b} \quad \text{i} \quad & T(15) = 100 \times (0.986)^{15} \\
 & \approx 80.9
 \end{aligned}$$

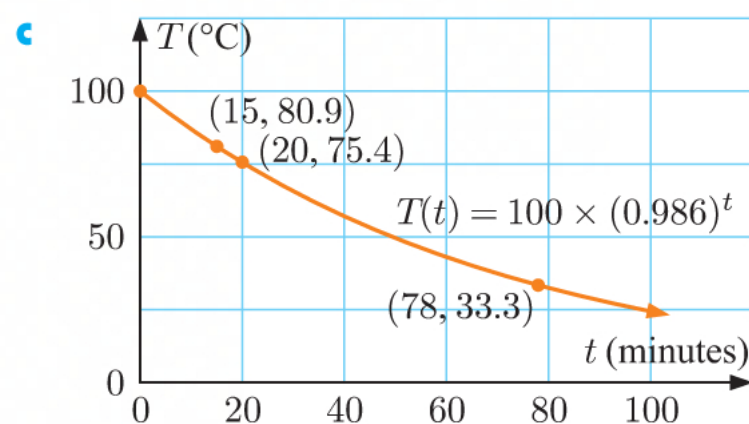
The temperature was about  $80.9^\circ\text{C}$  after 15 minutes.

$$\begin{aligned}
 \text{ii} \quad & T(20) = 100 \times (0.986)^{20} \\
 & \approx 75.4
 \end{aligned}$$

The temperature was about  $75.4^\circ\text{C}$  after 20 minutes.

$$\begin{aligned}
 \text{iii} \quad & T(78) = 100 \times (0.986)^{78} \\
 & \approx 33.3
 \end{aligned}$$

The temperature was about  $33.3^\circ\text{C}$  after 78 minutes.



$$3 \quad W(t) = 1000 \times 2^{-0.03t} \text{ grams}$$

$$\begin{aligned}
 \text{a} \quad & W(0) = 1000 \times 2^{-0.03(0)} \\
 & = 1000 \times 1 \\
 & = 1000
 \end{aligned}$$

The initial weight of the radioactive substance was 1000 grams.

$$\begin{aligned}
 \text{b} \quad \text{i} \quad & W(10) = 1000 \times 2^{-0.03(10)} \\
 & \approx 812
 \end{aligned}$$

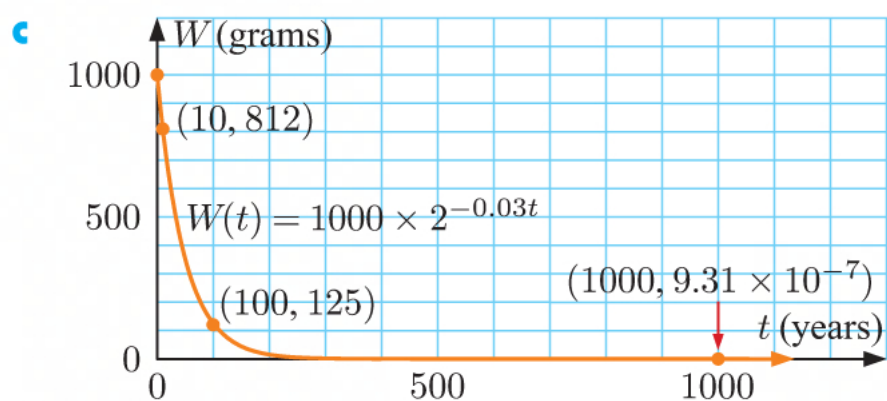
The weight remaining after 10 years was about 812 grams.

$$\begin{aligned}
 \text{ii} \quad & W(100) = 1000 \times 2^{-0.03(100)} \\
 & = 125
 \end{aligned}$$

The weight remaining after 100 years was 125 grams.

$$\begin{aligned}
 \text{iii} \quad & W(1000) = 1000 \times 2^{-0.03(1000)} \\
 & \approx 9.31 \times 10^{-7}
 \end{aligned}$$

The weight remaining after 1000 years was about  $9.31 \times 10^{-7}$  grams.



**d**

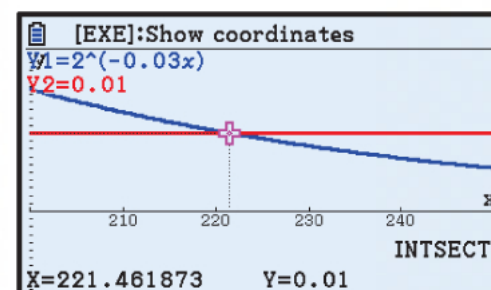
$$W(t) = 10$$

$$\therefore 1000 \times 2^{-0.03t} = 10$$

$$\therefore 2^{-0.03t} = 0.01$$

$$\therefore t \approx 221 \quad \{\text{using technology}\}$$

10 g of the substance remains after about 221 years.



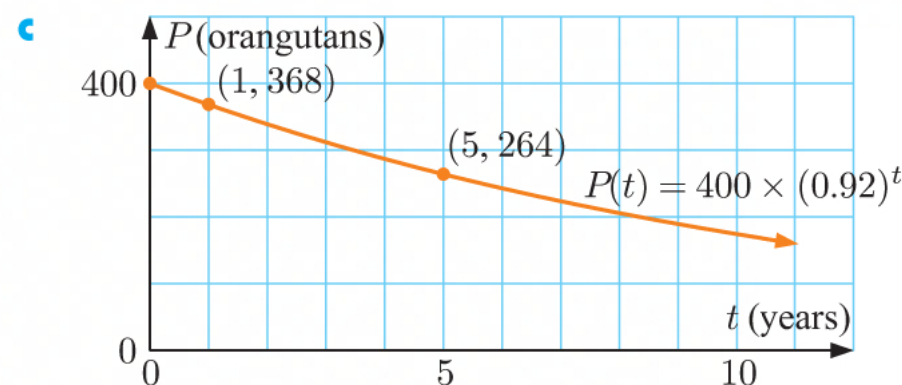
**e** Amount remaining after  $t$  years  $= W(t) = 1000 \times 2^{-0.03t}$  grams.

$$\begin{aligned} \text{Amount that has decayed after } t \text{ years} &= W(0) - W(t) \\ &= 1000 - 1000 \times 2^{-0.03t} \\ &= 1000(1 - 2^{-0.03t}) \text{ grams} \end{aligned}$$

**4 a**  $P(t) = 400 \times (0.92)^t$  orangutans

**b i**  $P(1) = 400 \times (0.92)^1$   
 $= 368$   
 There were 368 orangutans after 1 year.

**ii**  $P(5) = 400 \times (0.92)^5$   
 $\approx 264$   
 There were about 264 orangutans after 5 years.



**d**

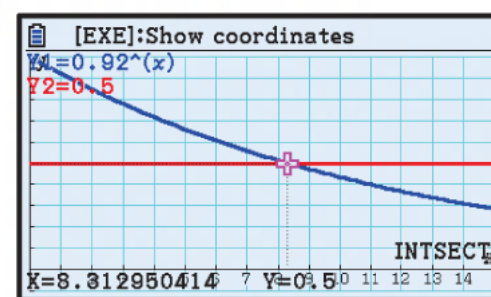
$$P(t) = 200$$

$$\therefore 400 \times (0.92)^t = 200$$

$$\therefore (0.92)^t = 0.5$$

$$\therefore t \approx 8.31 \quad \{\text{using technology}\}$$

The population will fall to 200 after about 8.31 years, or about 8 years and 4 months.



**5**  $L = L_0 \times (0.95)^d$  units

**a** When  $d = 0$ ,  $L = 10$   
 $\therefore 10 = L_0 \times (0.95)^0$   
 $\therefore L_0 = 10$  units

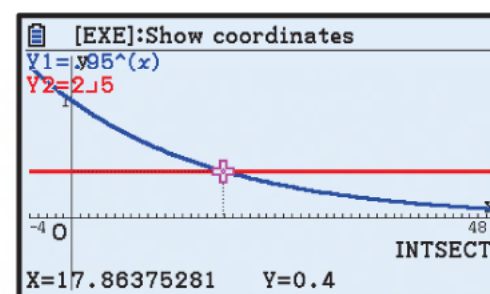
**b** When  $d = 25$ ,  $L = 10 \times (0.95)^{25}$   
 $\approx 2.77$

$\therefore$  the intensity of light 25 m below the surface is about 2.77 units.

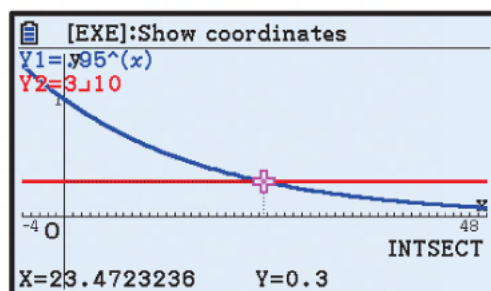


**c**  $L = 4$   
 $\therefore 10 \times (0.95)^d = 4$   
 $\therefore (0.95)^d = \frac{2}{5}$   
 $\therefore d \approx 17.9$  {using technology}

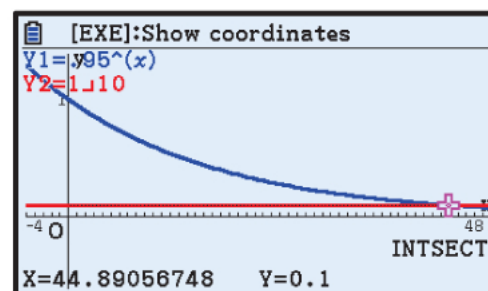
A depth of about 17.9 m has a light intensity of 4 units.



**d**  $L = 3$   
 $\therefore 10 \times (0.95)^d = 3$   
 $\therefore (0.95)^d = \frac{3}{10}$   
 $\therefore d \approx 23.5$  {using technology}



$L = 1$   
 $\therefore 10 \times (0.95)^d = 1$   
 $\therefore (0.95)^d = \frac{1}{10}$   
 $\therefore d \approx 44.9$  {using technology}



A depth between approximately 23.5 m and 44.9 m will have a light intensity between 1 and 3 units.

**6**  $V = 24\,000 \times r^t$  dollars

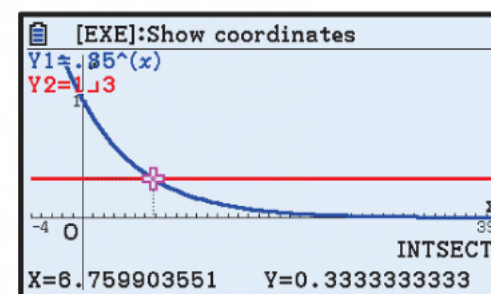
**a** When  $t = 0$ ,  $V = 24\,000 \times r^0$   
 $= 24\,000 \times 1$   
 $= 24\,000$

The value of the car when it was first purchased was \$24 000.

**b** When  $t = 2$ ,  $V = 17\,340$   
 $\therefore 24\,000 \times r^2 = 17\,340$   
 $\therefore r^2 = 0.7225$   
 $\therefore r = \sqrt{0.7225}$   $\{r > 0\}$   
 $\therefore r = 0.85$

**c**  $V = 8000$   
 $\therefore 24\,000 \times (0.85)^t = 8000$   
 $\therefore (0.85)^t = \frac{1}{3}$   
 $\therefore t \approx 6.76$  {using technology}

To the nearest year, it will take about 7 years for the value of the car to reduce to \$8000.



**7**  $T(t) = -10 + 32 \times 2^{-0.2t}$  °C

**a i**  $T(0) = -10 + 32 \times 2^{-0.2(0)}$   
 $= -10 + 32 \times 2^0$   
 $= -10 + 32$   
 $= 22$

The temperature of the peas was 22°C when placed in the freezer.

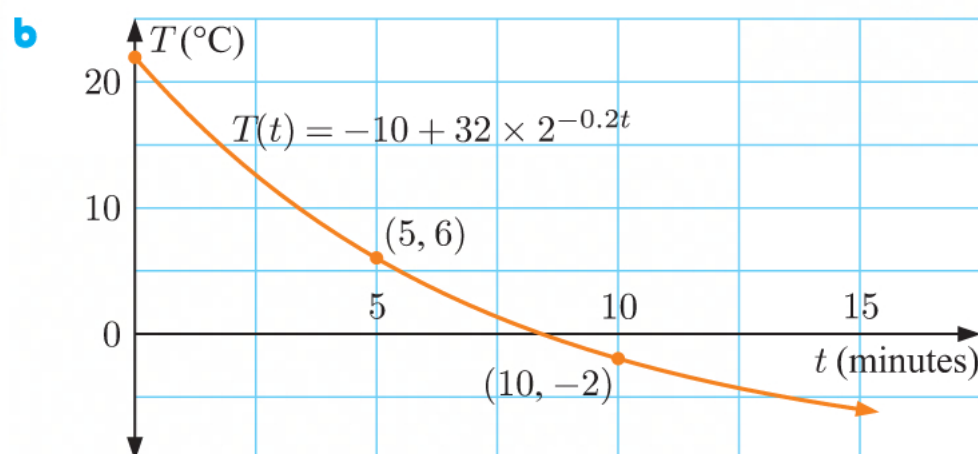
**ii**  $T(5) = -10 + 32 \times 2^{-0.2(5)}$   
 $= -10 + 32 \times 2^{-1}$   
 $= -10 + 16$   
 $= 6$

The temperature of the peas was 6°C after 5 minutes.

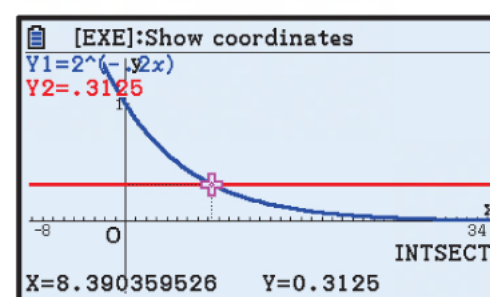


$$\begin{aligned}
 \text{iii } T(10) &= -10 + 32 \times 2^{-0.2(10)} \\
 &= -10 + 32 \times 2^{-2} \\
 &= -10 + 8 \\
 &= -2
 \end{aligned}$$

The temperature of the peas was  $-2^\circ\text{C}$  after 10 minutes.



$$\begin{aligned}
 \text{c } T(t) &= 0 \\
 \therefore -10 + 32 \times 2^{-0.2t} &= 0 \\
 \therefore 32 \times 2^{-0.2t} &= 10 \\
 \therefore 2^{-0.2t} &= 0.3125 \\
 \therefore t &\approx 8.39 \quad \{\text{using technology}\}
 \end{aligned}$$



It takes about 8.39 minutes, or about 8 minutes and 23 seconds, for the temperature of the peas to fall to  $0^\circ\text{C}$ .

**d**  $T(t) = -10 + 32 \times 2^{-0.2t}$

Now,  $32 \times 2^{-0.2t} > 0$  for all  $t$  since  $2^{-0.2t} > 0$  for all  $t$ .

$\therefore -10 + 32 \times 2^{-0.2t} > -10$  for all  $t$ .

$\therefore$  the temperature of the packet of peas will never reach  $-10^\circ\text{C}$ .

**8**  $W_t = W_0 \times 2^{-0.0002t}$  grams

**a** When  $t = 0$ ,  $W_0 = W_0 \times 2^0$

$$= W_0$$

$\therefore$  the original weight was  $W_0$  grams.

**b**

$$\begin{aligned}
 \left( \frac{W_{1000} - W_0}{W_0} \right) \times 100\% &= \left( \frac{W_0 \times 2^{-0.0002(1000)} - W_0}{W_0} \right) \times 100\% \\
 &= \left( \frac{W_0 \times 2^{-0.2} - W_0}{W_0} \right) \times 100\% \\
 &= \left( \frac{W_0(2^{-0.2} - 1)}{W_0} \right) \times 100\% \\
 &= (2^{-0.2} - 1) \times 100\% \\
 &\approx -12.9\%
 \end{aligned}$$

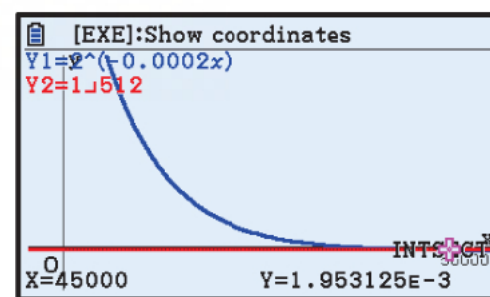
The percentage weight loss after 1000 years was about 12.9%.

$$c \quad W_0 \times 2^{-0.0002t} = \frac{1}{512} W_0$$

$$\therefore (2^{-0.0002})^t = \frac{1}{512}$$

$$\therefore t = 45\,000 \quad \{\text{using technology}\}$$

It will take 45 000 years until  $\frac{1}{512}$  of the sample remains.



9 a

$$A(t) = k_1 \times a^t$$

From the graph,  $A(0) = 150$

$$\therefore 150 = k_1 \times a^0$$

$$\therefore k_1 = 150$$

$$\text{So, } A(t) = 150 \times a^t$$

Also from the graph,  $A(3) = 222$

$$\therefore 150 \times a^3 = 222$$

$$\therefore a^3 = \frac{222}{150}$$

$$\therefore a^3 = 1.48$$

$$\therefore a = \sqrt[3]{1.48}$$

$$\text{So, } A(t) = 150 \times (1.48)^{\frac{t}{3}}$$

$$B(t) = k_2 \times (0.8)^t + c$$

From the graph,  $B(0) = 500$

$$\therefore 500 = k_2 \times (0.8)^0 + c$$

$$\therefore k_2 = 500 - c$$

$$\text{So, } B(t) = (500 - c) \times (0.8)^t + c$$

Also from the graph,  $B(2) = 356$

$$\therefore 356 = (500 - c) \times (0.8)^2 + c$$

$$\therefore 356 = 0.64(500 - c) + c$$

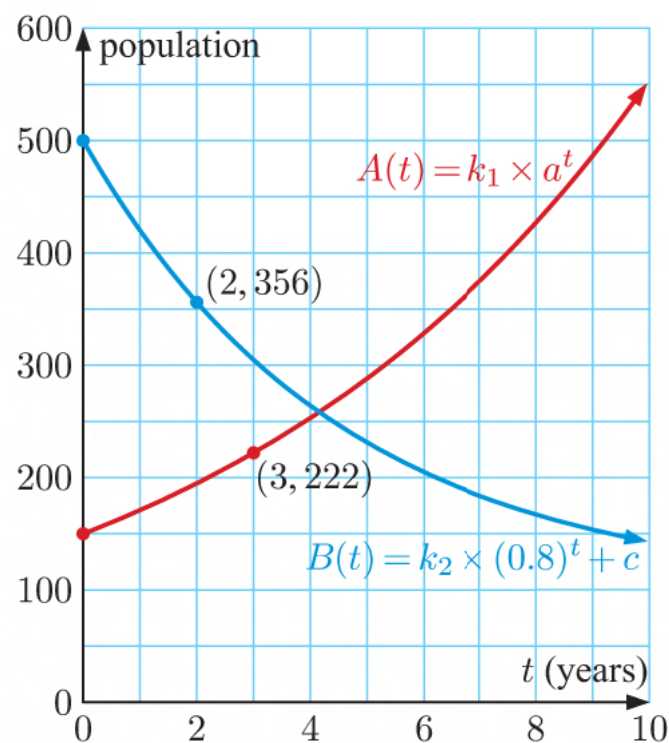
$$\therefore 356 = 320 - 0.64c + c$$

$$\therefore 36 = 0.36c$$

$$\therefore c = 100$$

$$\text{So, } B(t) = (500 - 100) \times (0.8)^t + 100$$

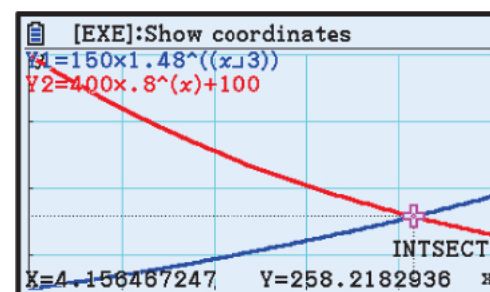
$$\therefore B(t) = 400 \times (0.8)^t + 100$$



b i The same number of group A and group B animals occurs where the graphs intersect.

We graph  $Y_1 = 150 \times (1.48)^{\frac{x}{3}}$  and

$Y_2 = 400 \times (0.8)^x + 100$  on the same set of axes and find their point of intersection.



The solution is  $t \approx 4.16$ .

There are the same number of group A and group B animals after about 4.16 years.

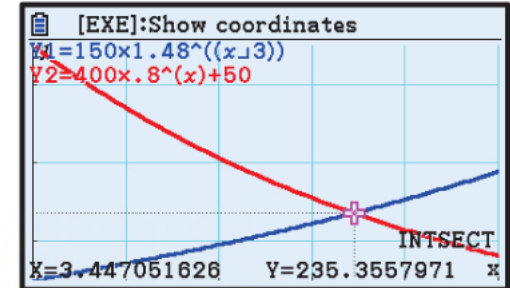
- ii There are 50 less group A than group B animals when

$$A(t) = B(t) - 50$$

$$\therefore 150 \times (1.48)^{\frac{t}{3}} = 400 \times (0.8)^t + 100 - 50$$

$$\therefore 150 \times (1.48)^{\frac{t}{3}} = 400 \times (0.8)^t + 50$$

We graph  $Y_1 = 150 \times (1.48)^{\frac{x}{3}}$  and  $Y_2 = 400 \times (0.8)^x + 50$  on the same set of axes and find their point of intersection.



The solution is  $t \approx 3.45$ .

There are 50 less group A than group B animals after about 3.45 years.

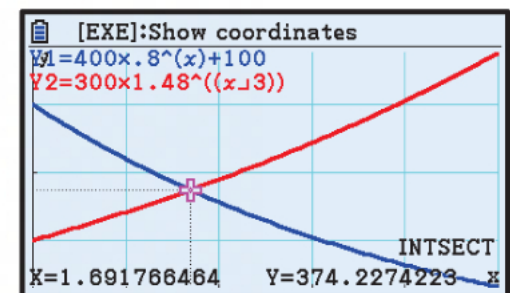
- iii There are twice as many group B animals compared to group A animals when

$$B(t) = 2 \times A(t)$$

$$\therefore 400 \times (0.8)^t + 100 = 2 \times 150 \times (1.48)^{\frac{t}{3}}$$

$$\therefore 400 \times (0.8)^t + 100 = 300 \times (1.48)^{\frac{t}{3}}$$

We graph  $Y_1 = 400 \times (0.8)^x + 100$  and  $Y_2 = 300 \times (1.48)^{\frac{x}{3}}$  on the same set of axes and find their point of intersection.



The solution is  $t \approx 1.69$ .

There are twice as many group B animals compared to group A animals after about 1.69 years.

10  $W(t) = 10 \times a^t$  mg

- a The initial weight of the isotope is 10 mg.

- b Fermium-253 has a half-life of 3 days.

So, after 3 days, its weight is  $\frac{1}{2} \times 10 = 5$  mg.

Now,  $W(3) = 10 \times a^3$

$$\therefore 5 = 10 \times a^3$$

$$\therefore \frac{5}{10} = a^3$$

$$\therefore a = \sqrt[3]{\frac{1}{2}}$$

$$\therefore a \approx 0.7937$$

Each day the isotope's weight is decreasing by about  $(1 - 0.7937) \times 100\% \approx 20.63\%$ .

c  $W(t) = 10 \times (0.5)^{\frac{t}{3}}$

$$\therefore W(2) = 10 \times (0.5)^{\frac{2}{3}} \approx 6.30$$

The weight of fermium-253 after 2 days is about 6.30 mg.

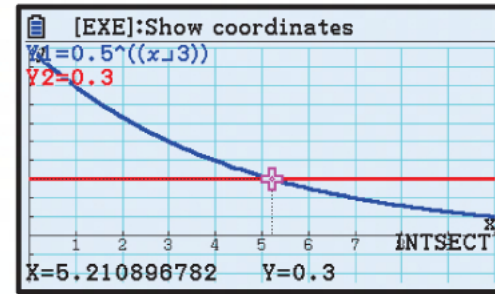


**d i**  $W(t) = 3$

$$\therefore 10 \times (0.5)^{\frac{t}{3}} = 3$$

$$\therefore (0.5)^{\frac{t}{3}} = 0.3$$

We graph  $Y_1 = (0.5)^{\frac{x}{3}}$  and  $Y_2 = 0.3$  on the same set of axes and find their point of intersection.



The solution is  $t \approx 5.21$ .

It will take about 5.21 days for the weight of fermium-253 to fall to 3 mg.

**ii**  $W(t) = 1.25$

$$\therefore 10 \times (0.5)^{\frac{t}{3}} = 1.25$$

$$\therefore (0.5)^{\frac{t}{3}} = 0.125$$

$$\therefore \left(\frac{1}{2}\right)^{\frac{t}{3}} = \frac{1}{8}$$

$$\therefore \left(\frac{1}{2}\right)^{\frac{t}{3}} = \left(\frac{1}{2}\right)^3$$

$$\therefore \frac{t}{3} = 3$$

$$\therefore t = 9$$

It will take 9 days for the weight of fermium-253 to fall to 1.25 mg.

- 11** Let the weight of nitrogen-13 be  $W(t) = k \times a^t$  mg, where  $k$  is the initial weight,  $a$  is the rate of decay, and  $t$  is the time in minutes.

Nitrogen-13 has a half-life of 10 minutes.

So, after 10 minutes, its weight is  $\frac{1}{2} \times k$ .

Now,  $W(10) = k \times a^{10}$

$$\therefore \frac{1}{2} \times k = k \times a^{10}$$

$$\therefore \frac{1}{2} = a^{10}$$

$$\therefore a = \sqrt[10]{0.5}$$

So,  $W(t) = k \times (0.5)^{\frac{t}{10}}$

A mass of nitrogen-13 will fall to 10% of its original value when

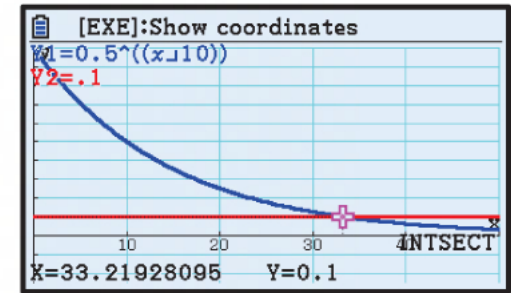
$$W(t) = 0.1 \times k$$

$$\therefore k \times (0.5)^{\frac{t}{10}} = 0.1 \times k$$

$$\therefore (0.5)^{\frac{t}{10}} = 0.1$$



We graph  $Y_1 = (0.5)^{\frac{x}{10}}$  and  $Y_2 = 0.1$  on the same set of axes and find their point of intersection.



The solution is  $t \approx 33.2$ .

It will take about 33.2 minutes, or about 33 minutes and 13.2 seconds for the mass of nitrogen-13 to fall to 10% of its original value.

## INVESTIGATION 2

## CONTINUOUS COMPOUND INTEREST

1  $u_n = u_0(1+i)^n$ ,  $u_0 = 1000$

a Interest paid annually:

$$n = 1, i = 6\% = 0.06$$

$$\begin{aligned}\therefore u_1 &= 1000(1 + 0.06)^1 \\ &= 1060\end{aligned}$$

The final amount is \$1060.

c Interest paid monthly:

$$n = 12, i = \frac{6\%}{12} = 0.005$$

$$\begin{aligned}\therefore u_{12} &= 1000(1 + 0.005)^{12} \\ &\approx 1061.68\end{aligned}$$

The final amount is \$1061.68.

e Interest paid by the second:

$$n = 365.25 \times 24 \times 60 \times 60 = 31\,557\,600, i = \frac{6\%}{31\,557\,600}$$

$$\begin{aligned}\therefore u_{31\,557\,600} &= 1000 \left( 1 + \frac{0.06}{31\,557\,600} \right)^{31\,557\,600} \\ &\approx 1061.84\end{aligned}$$

The final amount is \$1061.84.

f Interest paid by the millisecond:

$$n = 31\,557\,600 \times 1000 = 31\,557\,600\,000, i = \frac{6\%}{31\,557\,600\,000}$$

$$\begin{aligned}\therefore u_{31\,557\,600\,000} &= 1000 \left( 1 + \frac{0.06}{31\,557\,600\,000} \right)^{31\,557\,600\,000} \\ &\approx 1061.84\end{aligned}$$

The final amount is \$1061.84.

b Interest paid quarterly:

$$n = 4, i = \frac{6\%}{4} = 0.015$$

$$\begin{aligned}\therefore u_4 &= 1000(1 + 0.015)^4 \\ &\approx 1061.36\end{aligned}$$

The final amount is \$1061.36.

d Interest paid daily:

$$n = 365.25, i = \frac{6\%}{365.25}$$

$$\begin{aligned}\therefore u_{365.25} &= 1000 \left( 1 + \frac{0.06}{365.25} \right)^{365.25} \\ &\approx 1061.83\end{aligned}$$

The final amount is \$1061.83.

$$\begin{aligned}
 2 \quad u_n &= u_0(1+i)^n \\
 &= u_0 \left(1 + \frac{r}{N}\right)^{Nt} \quad \left\{i = \frac{r}{N}, n = Nt\right\} \\
 &= u_0 \left(1 + \frac{1}{a}\right)^{art} \quad \left\{a = \frac{N}{r}, N = ar\right\} \\
 &= u_0 \left[\left(1 + \frac{1}{a}\right)^a\right]^{rt}
 \end{aligned}$$

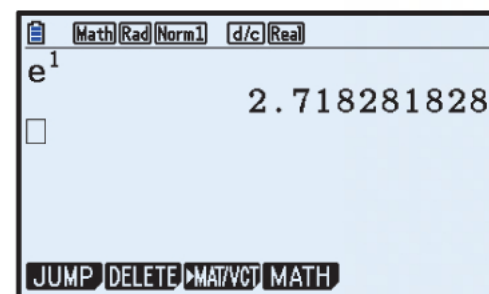
$$3 \quad a = \frac{N}{r}, \text{ so as } N \rightarrow \infty, a \rightarrow \infty$$

**b**

$a$	$\left(1 + \frac{1}{a}\right)^a$
10	2.593 724 46
100	2.704 813 829
1000	2.716 923 932
10 000	2.718 145 927
100 000	2.718 268 237
1 000 000	2.718 280 469
10 000 000	2.718 281 693

$$4 \quad e^1 \approx 2.718 281 828$$

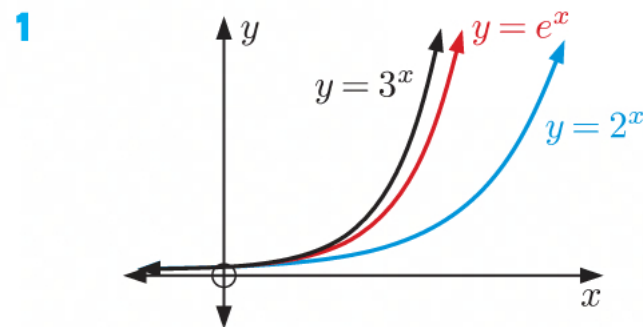
This appears to be the value of  $\left(1 + \frac{1}{a}\right)^a$  as  $a \rightarrow \infty$ .



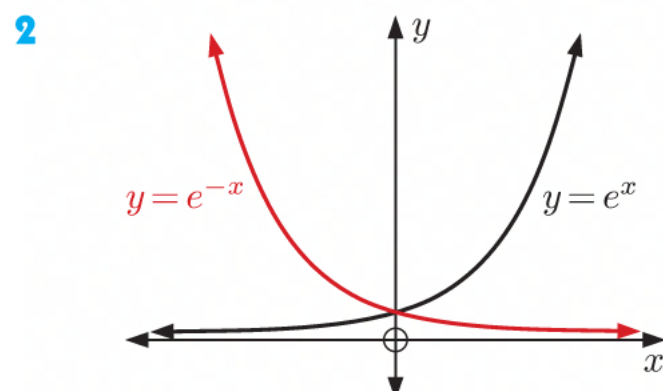
$$\begin{aligned}
 5 \quad u_n &= u_0 e^{rt}, \quad u_0 = 1000, \quad r = 0.06, \quad t = 4 \\
 \therefore u_n &= 1000 \times e^{0.06 \times 4} \\
 &\approx 1271.25
 \end{aligned}$$

The final amount is \$1271.25.

## EXERCISE 2F



The graph of  $y = e^x$  lies between  $y = 2^x$  and  $y = 3^x$ .



One is the other reflected in the  $y$ -axis.

- 3** When  $x = 0$ ,  $y = pe^0 = p \times 1 = p$   
 $\therefore$  the  $y$ -intercept is  $p$ .

- 4 a**  $e^x > 0$  for all  $x$   
 $\therefore 2e^x > 0$  for all  $x$   
 $\therefore y = 2e^x$  cannot be negative.

**b i** When  $x = -20$ ,  $y = 2e^{-20}$   
 $\approx 4.12 \times 10^{-9}$   
 $\approx 0.000\,000\,004\,12$

**ii** When  $x = 20$ ,  $y = 2e^{20}$   
 $\approx 9.70 \times 10^8$   
 $\approx 970\,000\,000$

- 5 a**  $e^2 \approx 7.39$       **b**  $e^3 \approx 20.1$       **c**  $e^{0.7} \approx 2.01$       **d**  $\sqrt{e} \approx 1.65$   
**e**  $e^{-1} \approx 0.368$

- 6 a**  $\sqrt{e} = e^{\frac{1}{2}}$       **b**  $\frac{1}{\sqrt{e}} = \frac{1}{e^{\frac{1}{2}}} = e^{-\frac{1}{2}}$       **c**  $\frac{1}{e^2} = e^{-2}$       **d**  $e\sqrt{e} = e^1 \times e^{\frac{1}{2}} = e^{\frac{3}{2}}$

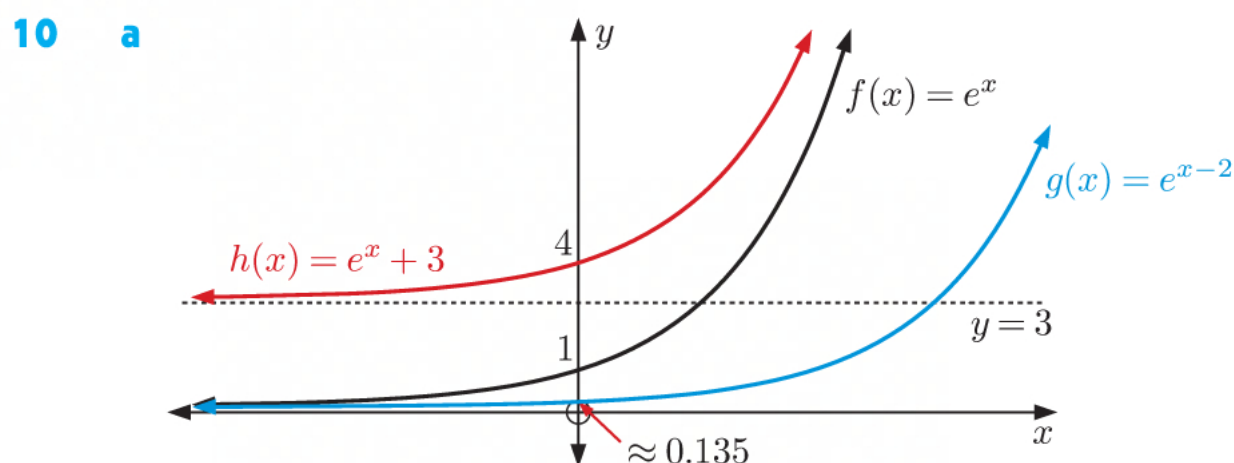
- 7 a**  $e^{2.31} \approx 10.074$       **b**  $e^{-2.31} \approx 0.099\,261$       **c**  $e^{4.829} \approx 125.09$   
**d**  $e^{-4.829} \approx 0.007\,994\,5$       **e**  $50e^{-0.1764} \approx 41.914$       **f**  $80e^{-0.6342} \approx 42.429$   
**g**  $1000e^{1.2642} \approx 3540.3$       **h**  $0.25e^{-3.6742} \approx 0.006\,342\,4$

- 8 a**  $(e^x + 1)^2 = (e^x)^2 + 2 \times e^x \times 1 + 1^2$   
 $= e^{2x} + 2e^x + 1$       **b**  $(1 + e^x)(1 - e^x) = 1^2 - (e^x)^2$   
 $= 1 - e^{2x}$

**c**  $e^x(e^{-x} - 3) = e^x \times e^{-x} - e^x \times 3$   
 $= e^0 - 3e^x$   
 $= 1 - 3e^x$

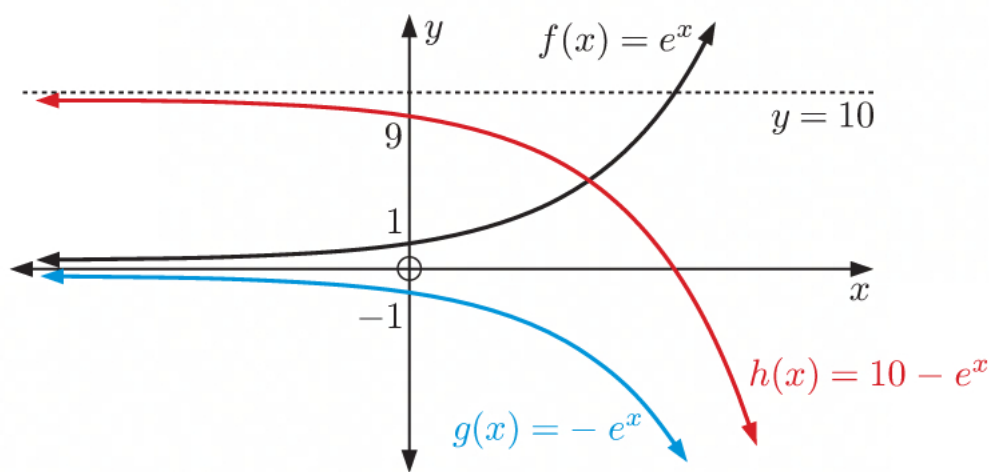
- 9 a**  $e^{2x} + e^x = e^x \times e^x + e^x$   
 $= e^x(e^x + 1)$       **b**  $e^{2x} - 16 = (e^x)^2 - 4^2$   
 $= (e^x + 4)(e^x - 4)$

**c**  $e^{2x} - 8e^x + 12 = (e^x - 2)(e^x - 6)$        $\{a^2 - 8a + 12 = (a - 2)(a - 6)\}$



- b** The domain of  $f$ ,  $g$ , and  $h$  is  $\{x \mid x \in \mathbb{R}\}$ .  
The range of  $f$  is  $\{y \mid y > 0\}$ . The range of  $g$  is  $\{y \mid y > 0\}$ .  
The range of  $h$  is  $\{y \mid y > 3\}$ .

11 a

b The domain of  $f$ ,  $g$ , and  $h$  is  $\{x \mid x \in \mathbb{R}\}$ .The range of  $f$  is  $\{y \mid y > 0\}$ . The range of  $g$  is  $\{y \mid y < 0\}$ .The range of  $h$  is  $\{y \mid y < 10\}$ .c For  $f$ : as  $x \rightarrow \infty$ ,  $y \rightarrow \infty$   
as  $x \rightarrow -\infty$ ,  $y \rightarrow 0^+$ For  $g$ : as  $x \rightarrow \infty$ ,  $y \rightarrow -\infty$   
as  $x \rightarrow -\infty$ ,  $y \rightarrow 0^-$ For  $h$ : as  $x \rightarrow \infty$ ,  $y \rightarrow -\infty$   
as  $x \rightarrow -\infty$ ,  $y \rightarrow 10^-$ 12  $f(x) = e^x - 1$  and  $g(x) = \frac{1}{x}$ 

$$\begin{aligned} \text{a } (f \circ g)(x) &= f(g(x)) \\ &= f\left(\frac{1}{x}\right) \\ &= e^{\frac{1}{x}} - 1 \end{aligned}$$

 $e^{\frac{1}{x}} - 1$  is defined when  $x \neq 0$ . $\therefore$  the domain of  $(f \circ g)(x)$  is  $\{x \mid x \neq 0\}$ .

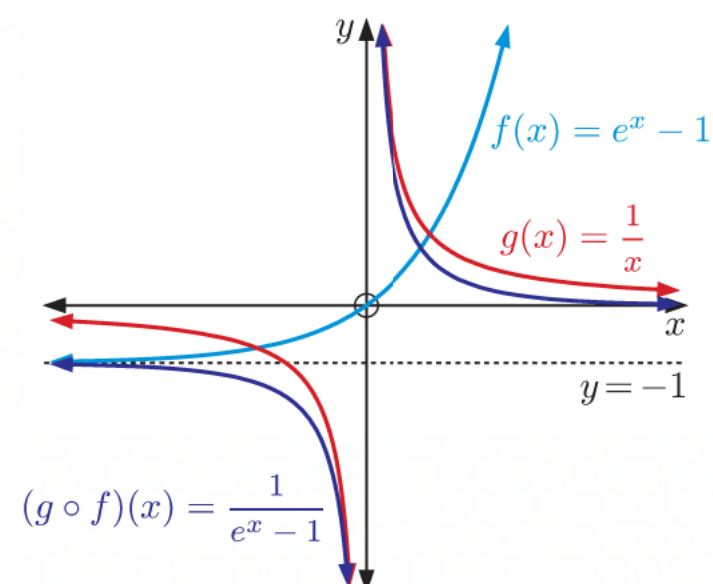
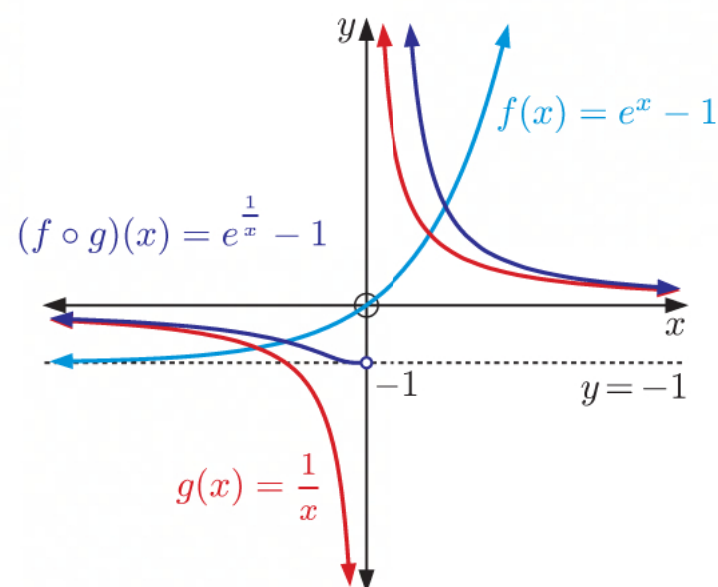
$$e^{\frac{1}{x}} - 1 > 0 \text{ when } x > 0$$

and  $-1 < e^{\frac{1}{x}} - 1 < 0$  when  $x < 0$  $\therefore$  the range of  $(f \circ g)(x)$  is  $\{y \mid -1 < y < 0 \text{ or } y > 0\}$ .

$$\begin{aligned} \text{b } (g \circ f)(x) &= g(f(x)) \\ &= g(e^x - 1) \\ &= \frac{1}{e^x - 1} \end{aligned}$$

 $\frac{1}{e^x - 1}$  is defined when  $e^x - 1 \neq 0$   
 $\therefore e^x \neq 1$   
 $\therefore x \neq 0$ 
 $\therefore$  the domain of  $(g \circ f)(x)$  is  $\{x \mid x \neq 0\}$ .

$$\frac{1}{e^x - 1} > 0 \text{ when } x > 0$$

and  $\frac{1}{e^x - 1} < -1$  when  $x < 0$  $\therefore$  the range of  $(g \circ f)(x)$  is  $\{y \mid y < -1 \text{ or } y > 0\}$ .



**13**  $W(t) = 2e^{\frac{t}{2}}$  grams

**a i**  $W(0) = 2e^0$   
 $= 2 \times 1$   
 $= 2$

The weight of the culture is 2 grams initially.

**iii**  $W(1\frac{1}{2}) = 2e^{\frac{3}{4}}$   
 $\approx 4.23$

The weight of the culture is about 4.23 grams after  $1\frac{1}{2}$  hours.

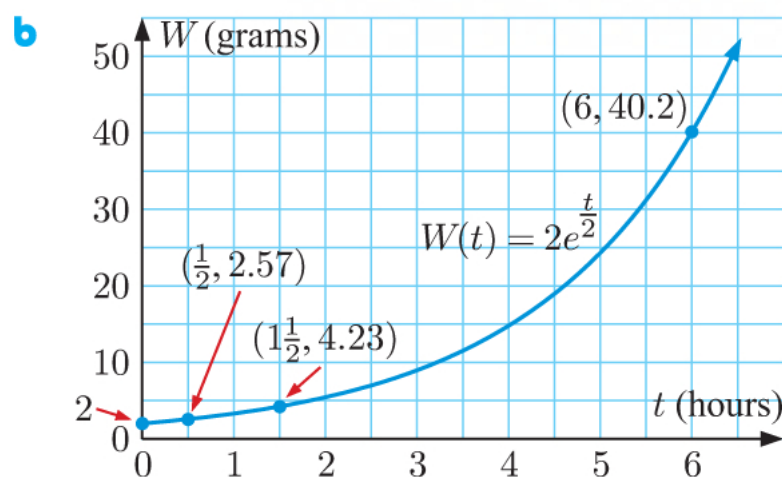
**ii**  $t = 30 \text{ min} = \frac{1}{2} \text{ hour}$

$W(\frac{1}{2}) = 2e^{\frac{1}{4}}$   
 $\approx 2.57$

The weight of the culture is about 2.57 grams after 30 minutes.

**iv**  $W(6) = 2e^3$   
 $\approx 40.2$

The weight of the culture is about 40.2 grams after 6 hours.



**14 a**  $e^x = \sqrt{e}$

$\therefore e^x = e^{\frac{1}{2}}$

$\therefore x = \frac{1}{2}$

**b**  $e^{\frac{1}{2}x} = \frac{1}{e^2}$

$\therefore e^{\frac{1}{2}x} = e^{-2}$

$\therefore \frac{1}{2}x = -2$

$\therefore x = -4$

**c**  $e^{2x} + e^x = 2$

$\therefore (e^x)^2 + e^x - 2 = 0$

$\therefore (e^x + 2)(e^x - 1) = 0$   $\{a^2 + a - 2 = (a + 2)(a - 1)\}$

$\therefore e^x = -2$  or  $e^x = 1$

$\therefore e^x = 1$   $\{\text{since } e^x \text{ cannot be negative}\}$

$\therefore x = 0$

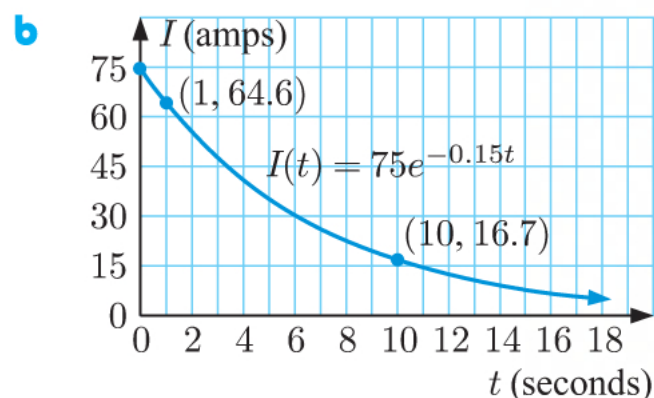
**15**  $I(t) = 75e^{-0.15t}$  amps

**a i**  $I(1) = 75e^{-0.15}$   
 $\approx 64.6$

About 64.6 amps of current is still flowing after 1 second.

**ii**  $I(10) = 75e^{-1.5}$   
 $\approx 16.7$

About 16.7 amps of current is still flowing after 10 seconds.

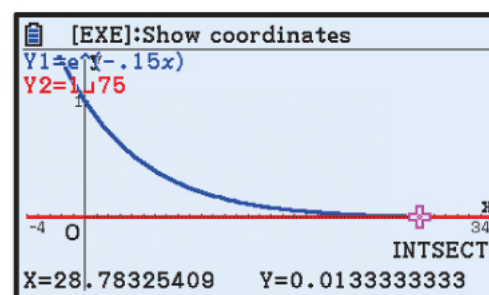


**c** We need to solve  $75e^{-0.15t} = 1$

$$\therefore e^{-0.15t} = \frac{1}{75}$$

$$\therefore t \approx 28.8 \quad \{\text{using technology}\}$$

It will take about 28.8 seconds for the current to fall to 1 amp.



**16**  $P(t) = \frac{800}{1 + ke^{-0.5t}}$

**a**  $P(0) = \frac{800}{1 + ke^{-0.5(0)}}$

$$\therefore 20 = \frac{800}{1 + ke^0}$$

$$\therefore 20 = \frac{800}{1 + k}$$

$$\therefore 20(1 + k) = 800$$

$$\therefore 1 + k = 40$$

$$\therefore k = 39$$

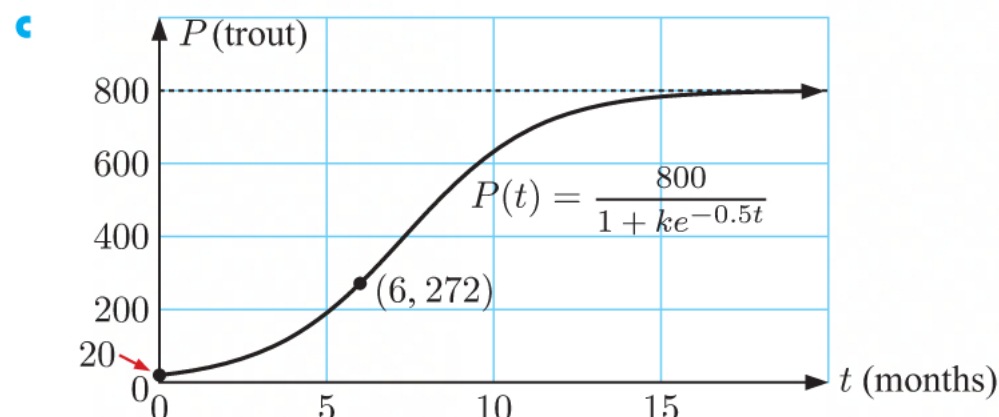
**b**  $P(t) = \frac{800}{1 + 39e^{-0.5t}}$

$$\therefore P(6) = \frac{800}{1 + 39e^{-0.5(6)}}$$

$$= \frac{800}{1 + 39e^{-3}}$$

$$\approx 272$$

So, the population after 6 months is about 272 trout.



**d** As  $t$  increases, the population approaches a limiting value of 800 trout.

**e**  $P(t) = 600$

$$\therefore \frac{800}{1 + 39e^{-0.5t}} = 600$$

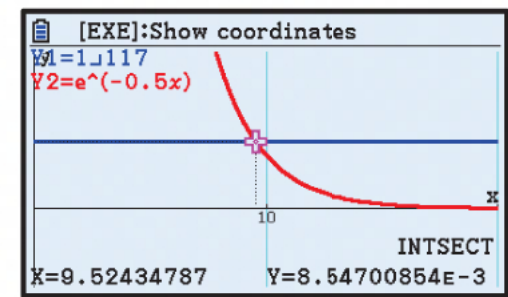
$$\therefore 800 = 600(1 + 39e^{-0.5t})$$

$$\therefore \frac{4}{3} = 1 + 39e^{-0.5t}$$

$$\therefore \frac{1}{3} = 39e^{-0.5t}$$

$$\therefore \frac{1}{117} = e^{-0.5t}$$

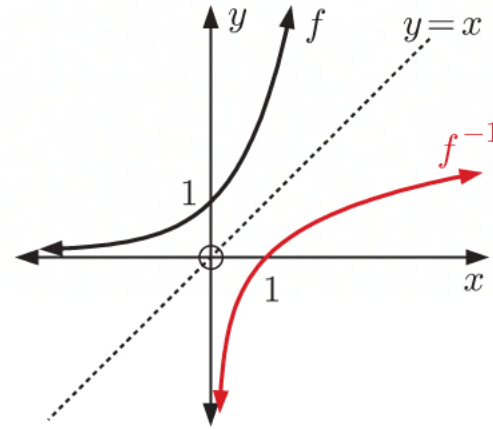
We graph  $Y_1 = \frac{1}{117}$  and  $Y_2 = e^{-0.5X}$  on the same set of axes and find their point of intersection.



The solution is  $t \approx 9.52$ .

It will take about 9.52 months for the population to reach 600.

- 17 a**  $f^{-1}$  is a reflection of  $f$  in the line  $y = x$



- b** The domain of  $f^{-1}$  is  $\{x \mid x > 0\}$ .  
The range of  $f^{-1}$  is  $\{y \mid y \in \mathbb{R}\}$ .

**18** 
$$e^1 \approx \sum_{k=0}^{19} \frac{1}{k!} 1^k \approx 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots + \frac{1}{19!} \approx 2.718\,281\,828$$

## REVIEW SET 2A

**1 a** 
$$8^{\frac{2}{3}} = (2^3)^{\frac{2}{3}}$$
$$= 2^2$$
$$= 4$$

**b** 
$$27^{-\frac{2}{3}} = (3^3)^{-\frac{2}{3}}$$
$$= 3^{-2}$$
$$= \frac{1}{3^2}$$
$$= \frac{1}{9}$$

**c** 
$$81^{-\frac{1}{4}} = (3^4)^{-\frac{1}{4}}$$
$$= 3^{-1}$$
$$= \frac{1}{3}$$

**2 a** 
$$2^{x-3} = \frac{1}{32}$$
$$\therefore 2^{x-3} = 2^{-5}$$
$$\therefore x-3 = -5$$
$$\therefore x = -2$$

**b** 
$$9^x = 27^{2-2x}$$
$$\therefore (3^2)^x = (3^3)^{2-2x}$$
$$\therefore 3^{2x} = 3^{3(2-2x)}$$
$$\therefore 2x = 3(2-2x)$$
$$\therefore 2x = 6-6x$$
$$\therefore 8x = 6$$
$$\therefore x = \frac{6}{8} = \frac{3}{4}$$

**c** 
$$e^{2x} = \frac{1}{\sqrt{e}}$$
$$\therefore e^{2x} = e^{-\frac{1}{2}}$$
$$\therefore 2x = -\frac{1}{2}$$
$$\therefore x = -\frac{1}{4}$$

**3 a** 
$$e^x(e^{-x} + e^x)$$
$$= e^x \times e^{-x} + e^x \times e^x$$
$$= e^0 + e^{2x}$$
$$= 1 + e^{2x}$$

**b** 
$$(2^x + 5)^2$$
$$= (2^x)^2 + 2 \times 2^x \times 5 + 5^2$$
$$= 2^{2x} + 10(2^x) + 25$$

**c** 
$$(x^{\frac{1}{2}} - 7)(x^{\frac{1}{2}} + 7)$$
$$= (x^{\frac{1}{2}})^2 - 7^2$$
$$= x^1 - 49$$
$$= x - 49$$

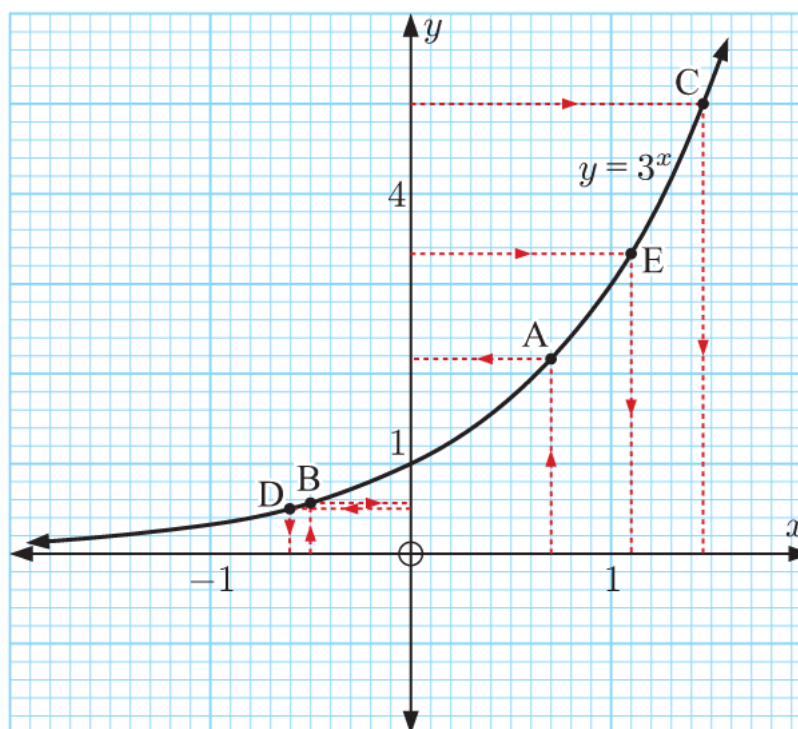
**4 a**  $6 \times 2^x = 192$   
 $\therefore 2^x = 32$   
 $\therefore 2^x = 2^5$   
 $\therefore x = 5$

**b**  $9^{x-1} \times \left(\frac{1}{27}\right)^x = \sqrt{3}$   
 $\therefore (3^2)^{x-1} \times (3^{-3})^x = 3^{\frac{1}{2}}$   
 $\therefore 3^{2x-2} \times 3^{-3x} = 3^{\frac{1}{2}}$   
 $\therefore 3^{2x-2+(-3x)} = 3^{\frac{1}{2}}$   
 $\therefore 2x - 2 - 3x = \frac{1}{2}$   
 $\therefore -x = \frac{5}{2}$   
 $\therefore x = -\frac{5}{2}$

**c**  $4^x - 32 = 4(2^x)$   
 $\therefore (2^2)^x - 32 - 4(2^x) = 0$   
 $\therefore (2^x)^2 - 4(2^x) - 32 = 0$   
 $\therefore (2^x + 4)(2^x - 8) = 0 \quad \{a^2 - 4a - 32 = (a + 4)(a - 8)\}$   
 $\therefore 2^x = -4 \quad \text{or} \quad 2^x = 8$   
 $\therefore 2^x = 8 \quad \{\text{since } 2^x \text{ cannot be negative}\}$   
 $\therefore 2^x = 2^3$   
 $\therefore x = 3$

**5** When  $x = 1$ ,  $y = \sqrt{8} \quad \therefore \sqrt{8} = 2^{k(1)}$   
 $\therefore (2^3)^{\frac{1}{2}} = 2^k$   
 $\therefore 2^{\frac{3}{2}} = 2^k$   
 $\therefore k = \frac{3}{2}$

- 6 a i** When  $x = 0.7$ ,  $y = 3^{0.7}$   
 From point A,  $y \approx 2.2$   
 $\therefore 3^{0.7} \approx 2.2$
- ii** When  $x = -0.5$ ,  $y = 3^{-0.5}$   
 From point B,  $y \approx 0.6$   
 $\therefore 3^{-0.5} \approx 0.6$
- b i** When  $3^x = 5$ ,  
 $x \approx 1.45$  from point C.
- ii** When  $3^x = \frac{1}{2}$ ,  
 $x \approx -0.6$  from point D.
- iii** When  $6 \times 3^x = 20$ ,  
 then  $3^x = \frac{20}{6}$   
 so  $x \approx 1.1$  from point E.





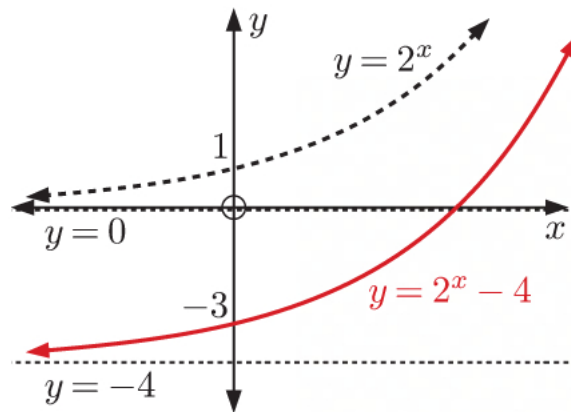
**7**  $f(x) = 3 \times 2^x$

**a**  $f(0) = 3 \times 2^0$   
 $= 3 \times 1$   
 $= 3$

**b**  $f(3) = 3 \times 2^3$   
 $= 3 \times 8$   
 $= 24$

**c**  $f(-2) = 3 \times 2^{-2}$   
 $= 3 \times \frac{1}{2^2}$   
 $= 3 \times \frac{1}{4}$   
 $= \frac{3}{4}$

**8**

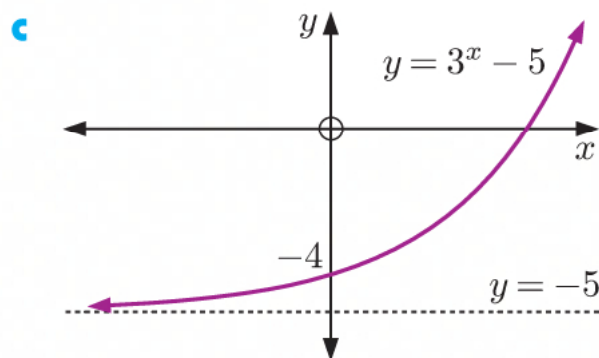


$y = 2^x$  has  $y$ -intercept 1 and horizontal asymptote  $y = 0$ .  
 $y = 2^x - 4$  has  $y$ -intercept  $-3$  and horizontal asymptote  $y = -4$ .

**9**  $y = 3^x - 5$

**a** When  $x = 0$ ,  $y = 3^0 - 5 = 1 - 5 = -4$   
 When  $x = 1$ ,  $y = 3^1 - 5 = 3 - 5 = -2$   
 When  $x = 2$ ,  $y = 3^2 - 5 = 9 - 5 = 4$   
 When  $x = -1$ ,  $y = 3^{-1} - 5 = \frac{1}{3} - 5 = -4\frac{2}{3}$   
 When  $x = -2$ ,  $y = 3^{-2} - 5 = \frac{1}{9} - 5 = -4\frac{8}{9}$

**b** As  $x \rightarrow \infty$ ,  $3^x \rightarrow \infty$   
 and so  $y \rightarrow \infty$   
 As  $x \rightarrow -\infty$ ,  $3^x \rightarrow 0^+$   
 and so  $y \rightarrow -5^+$

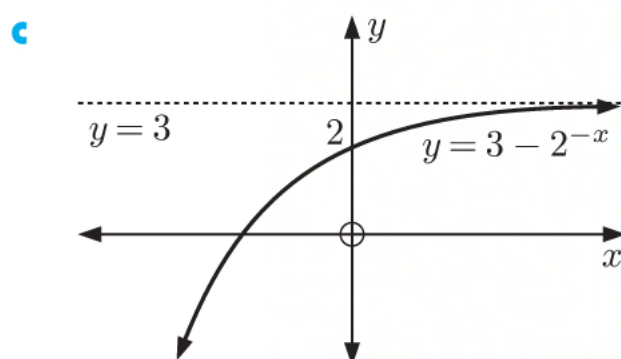


**d**  $y = -5$  is the horizontal asymptote.

**10**  $y = 3 - 2^{-x}$

**a** When  $x = 0$ ,  $y = 3 - 2^0 = 3 - 1 = 2$   
 When  $x = 1$ ,  $y = 3 - 2^{-1} = 3 - \frac{1}{2} = 2\frac{1}{2}$   
 When  $x = 2$ ,  $y = 3 - 2^{-2} = 3 - \frac{1}{4} = 2\frac{3}{4}$   
 When  $x = -1$ ,  $y = 3 - 2^1 = 3 - 2 = 1$   
 When  $x = -2$ ,  $y = 3 - 2^2 = 3 - 4 = -1$

**b** As  $x \rightarrow \infty$ ,  $2^{-x} \rightarrow 0^+$   
 and so  $y \rightarrow 3^-$   
 As  $x \rightarrow -\infty$ ,  $2^{-x} \rightarrow \infty$   
 and so  $y \rightarrow -\infty$



**d**  $y = 3$  is the horizontal asymptote.

**11**  $f(x) = 2^x$  and  $g(x) = 3 - x^2$

**a**  $(f \circ g)(x) = f(g(x))$   
 $= f(3 - x^2)$   
 $= 2^{3-x^2}$

$2^{3-x^2}$  is defined for all  $x \in \mathbb{R}$ .

$\therefore$  the domain is  $\{x \mid x \in \mathbb{R}\}$ .

$$x^2 \geq 0$$

$$\therefore -x^2 \leq 0$$

$$\therefore 3 - x^2 \leq 3$$

$$\therefore 2^{3-x^2} \leq 8 \quad \{\text{since } 2^3 = 8\}$$

Also,  $2^a > 0$  for all  $a \in \mathbb{R}$

$$\therefore 0 < 2^{3-x^2} \leq 8$$

$\therefore$  the range is  $\{y \mid 0 < y \leq 8\}$ .

**c i**  $(f \circ g)(x) = 2$   
 $\therefore 2^{3-x^2} = 2^1$   
 $\therefore 3 - x^2 = 1$   
 $\therefore 2 = x^2$   
 $\therefore x = \pm\sqrt{2}$

**b**  $(g \circ f)(x) = g(f(x))$   
 $= g(2^x)$   
 $= 3 - (2^x)^2$   
 $= 3 - 2^{2x}$   
 $= 3 - 4^x$

$3 - 4^x$  is defined for all  $x \in \mathbb{R}$ .

$\therefore$  the domain is  $\{x \mid x \in \mathbb{R}\}$ .

$$4^x > 0 \text{ for all } x \in \mathbb{R}$$

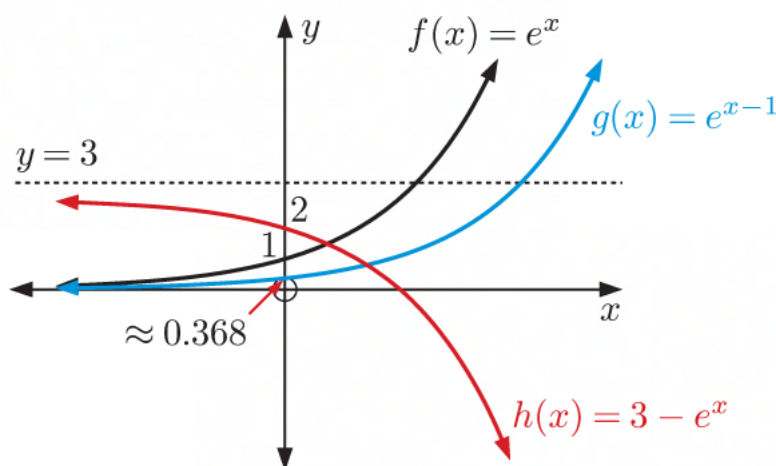
$$\therefore -4^x < 0$$

$$\therefore 3 - 4^x < 3$$

$\therefore$  the range is  $\{y \mid y < 3\}$ .

**ii**  $(g \circ f)(x) = -13$   
 $\therefore 3 - 4^x = -13$   
 $\therefore -4^x = -16$   
 $\therefore 4^x = 16$   
 $\therefore 2^{2x} = 2^4$   
 $\therefore 2x = 4$   
 $\therefore x = 2$

**12 a**



**b** For  $f(x)$ : the domain is  $\{x \mid x \in \mathbb{R}\}$ ,  
the range is  $\{y \mid y > 0\}$ .

For  $g(x)$ : the domain is  $\{x \mid x \in \mathbb{R}\}$ ,  
the range is  $\{y \mid y > 0\}$ .

For  $h(x)$ : the domain is  $\{x \mid x \in \mathbb{R}\}$ ,  
the range is  $\{y \mid y < 3\}$ .

**c** For  $f(x)$ : as  $x \rightarrow \infty$ ,  $f(x) \rightarrow \infty$   
as  $x \rightarrow -\infty$ ,  $f(x) \rightarrow 0^+$

For  $g(x)$ : as  $x \rightarrow \infty$ ,  $g(x) \rightarrow \infty$   
as  $x \rightarrow -\infty$ ,  $g(x) \rightarrow 0^+$

For  $h(x)$ : as  $x \rightarrow \infty$ ,  $h(x) \rightarrow -\infty$   
as  $x \rightarrow -\infty$ ,  $h(x) \rightarrow 3^-$

- 13** Let the size of the plant be  $S(t) = k \times a^t$  units, where  $k$  is the initial size of the plant,  $a$  is the rate of growth, and  $t$  is the time in days.

The plant doubles in size every 5 days.

So, after 5 days, its size is  $2k$  units.

$$\text{Now, } S(5) = k \times a^5$$

$$\therefore 2k = k \times a^5$$

$$\therefore 2 = a^5$$

$$\therefore a = \sqrt[5]{2}$$

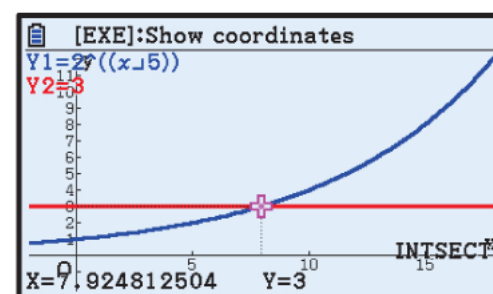
$$\text{So, } S(t) = k \times 2^{\frac{t}{5}}$$

The plant trebles in size when  $S(t) = 3k$

$$\therefore k \times 2^{\frac{t}{5}} = 3k$$

$$\therefore 2^{\frac{t}{5}} = 3$$

We graph  $Y_1 = 2^{\frac{x}{5}}$  and  $Y_2 = 3$  on the same set of axes and find their point of intersection.



The solution is  $t \approx 7.92$ .

It will take about 7.92 days for the plant to treble in size.

- 14**  $T(t) = 80 \times (0.913)^t$  °C

$$\begin{aligned} \text{a } T(0) &= 80 \times (0.913)^0 \\ &= 80 \times 1 \\ &= 80 \end{aligned}$$

$\therefore$  the initial temperature of the dish was 80°C.

$$\begin{aligned} \text{b i } T(12) &= 80 \times (0.913)^{12} \\ &\approx 26.8 \end{aligned}$$

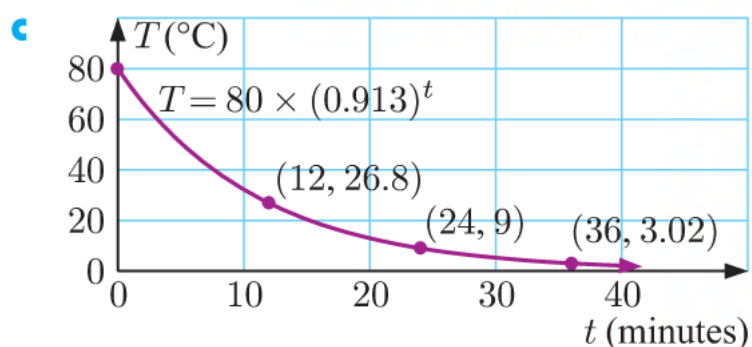
After 12 minutes, the temperature was about 26.8°C.

$$\begin{aligned} \text{ii } T(24) &= 80 \times (0.913)^{24} \\ &\approx 9.00 \end{aligned}$$

After 24 minutes, the temperature was about 9.00°C.

$$\begin{aligned} \text{iii } T(36) &= 80 \times (0.913)^{36} \\ &\approx 3.02 \end{aligned}$$

After 36 minutes, the temperature was about 3.02°C.



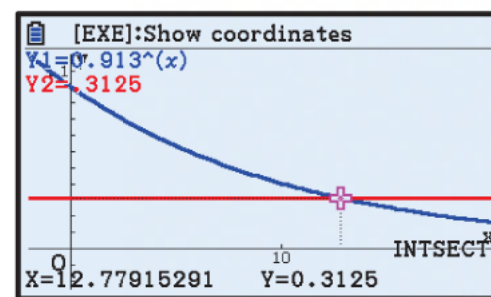
**d** When  $T(t) = 25$ ,

$$80 \times (0.913)^t = 25$$

$$\therefore (0.913)^t = 0.3125$$

$$\therefore t \approx 12.8 \quad \{\text{using technology}\}$$

It takes about 12.8 minutes for the temperature of the dish to fall to  $25^\circ\text{C}$ .



## REVIEW SET 2B

**1 a**  $3^{\frac{5}{4}} \approx 3.95$

**b**  $27^{-\frac{1}{5}} \approx 0.517$

**c**  $\sqrt[4]{100} \approx 3.16$

**2 a**  $(3 - e^x)^2 = 3^2 - 2 \times 3 \times e^x + (e^x)^2$   
 $= 9 - 6e^x + e^{2x}$

**b**  $x^{-\frac{1}{2}}(x^{\frac{3}{2}} - 2x^{\frac{1}{2}} - x^{-\frac{1}{2}})$   
 $= x^{-\frac{1}{2}} \times x^{\frac{3}{2}} - x^{-\frac{1}{2}} \times 2x^{\frac{1}{2}} - x^{-\frac{1}{2}} \times x^{-\frac{1}{2}}$   
 $= x^1 - 2x^0 - x^{-1}$   
 $= x - 2 - x^{-1}$

**c**  $2^{-x}(2^{2x} + 2^x) = 2^{-x} \times 2^{2x} + 2^{-x} \times 2^x$   
 $= 2^x + 2^0$   
 $= 2^x + 1$

**3 a**  $3^{x+2} - 3^x$   
 $= 3^x(3^2 - 1)$   
 $= 3^x(9 - 1)$   
 $= 8(3^x)$

**b**  $4^x - 2^x - 12$   
 $= (2^x)^2 - 2^x - 12$   
 $= (2^x + 3)(2^x - 4) \quad \{a^2 - a - 12 = (a + 3)(a - 4)\}$

**c**  $e^{2x} + 2e^x - 15$   
 $= (e^x)^2 + 2e^x - 15$   
 $= (e^x + 5)(e^x - 3) \quad \{a^2 + 2a - 15 = (a + 5)(a - 3)\}$

**4 a**  $3 \times \left(\frac{1}{7}\right)^{x+1} = 1029$   
 $\therefore (7^{-1})^{x+1} = 343$   
 $\therefore 7^{-(x+1)} = 7^3$   
 $\therefore -(x+1) = 3$   
 $\therefore -x - 1 = 3$   
 $\therefore -x = 4$   
 $\therefore x = -4$

**b**  $9^x - 10(3^x) + 9 = 0$   
 $\therefore (3^2)^x - 10(3^x) + 9 = 0$   
 $\therefore (3^x)^2 - 10(3^x) + 9 = 0$   
 $\therefore (3^x - 1)(3^x - 9) = 0 \quad \{a^2 - 10a + 9 = (a - 1)(a - 9)\}$   
 $\therefore 3^x = 1 \quad \text{or} \quad 3^x = 9$   
 $\therefore 3^x = 3^0 \quad \text{or} \quad 3^x = 3^2$   
 $\therefore x = 0 \quad \text{or} \quad x = 2$



$$\begin{aligned}
 & \text{c} \quad 2(4^{x+1}) + 1 = 6(2^x) \\
 & \therefore 2((2^2)^{x+1}) + 1 = 6(2^x) \\
 & \therefore 2(2^{2x+2}) + 1 = 6(2^x) \\
 & \therefore 2(2^{2x} \times 2^2) + 1 = 6(2^x) \\
 & \therefore 8(2^{2x}) + 1 = 6(2^x) \\
 & \therefore 8((2^x)^2) - 6(2^x) + 1 = 0 \\
 & \therefore (4(2^x) - 1)(2(2^x) - 1) = 0 \quad \{8x^2 - 6x + 1 = (4x - 1)(2x - 1)\} \\
 & \therefore 4(2^x) = 1 \quad \text{or} \quad 2(2^x) = 1 \\
 & \therefore 2^x = \frac{1}{4} \quad \text{or} \quad 2^x = \frac{1}{2} \\
 & \therefore 2^x = 2^{-2} \quad \text{or} \quad 2^x = 2^{-1} \\
 & \therefore x = -2 \quad \text{or} \quad x = -1
 \end{aligned}$$

5  $f(x) = 2^{-x} + 1$

a  $f(\frac{1}{2}) = 2^{-\frac{1}{2}} + 1$   
 $= \frac{1}{\sqrt{2}} + 1$   
 $\approx 1.71$

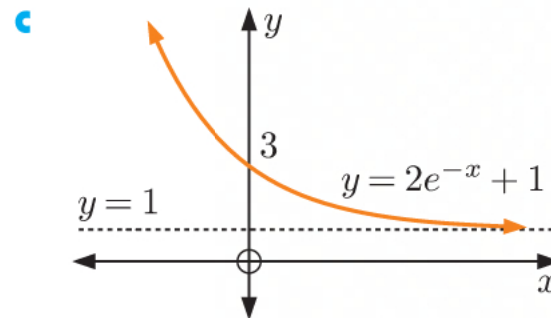
b  $f(a) = 3$   
 $\therefore 2^{-a} + 1 = 3$   
 $\therefore 2^{-a} = 2$   
 $\therefore 2^{-a} = 2^1$   
 $\therefore -a = 1$   
 $\therefore a = -1$

6  $y = 2e^{-x} + 1$

a When  $x = 0$ ,  $y = 2e^0 + 1 = 3$   
 When  $x = 1$ ,  $y = 2e^{-1} + 1 \approx 1.74$   
 When  $x = 2$ ,  $y = 2e^{-2} + 1 \approx 1.27$   
 When  $x = -1$ ,  $y = 2e^1 + 1 \approx 6.44$   
 When  $x = -2$ ,  $y = 2e^2 + 1 \approx 15.8$

b As  $x \rightarrow \infty$ ,  $y \rightarrow 1^+$   
 As  $x \rightarrow -\infty$ ,  $y \rightarrow \infty$

d  $y = 1$  is a horizontal asymptote.



7 a The clock cost £500 and increases in value by 5% each year.

$$\begin{aligned}
 \therefore \text{the value of the clock 1 year after purchase} &= £500 \times 1.05 \\
 &= £525
 \end{aligned}$$

The vase cost £400 and increases in value by 7% each year.

$$\begin{aligned}
 \therefore \text{the value of the vase 1 year after purchase} &= £400 \times 1.07 \\
 &= £428
 \end{aligned}$$

b The clock will have value  $V(t) = 500 \times (1.05)^t$  pounds,  $t$  years after purchase.  
 The vase will have value  $V(t) = 400 \times (1.07)^t$  pounds,  $t$  years after purchase.

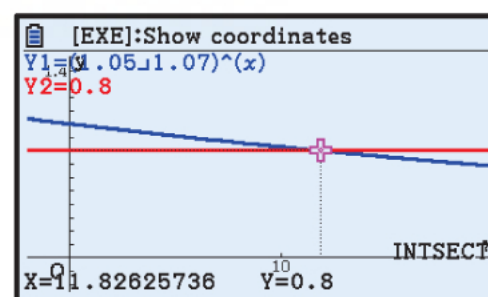
c For the clock,  $V(15) = 500 \times (1.05)^{15} \approx £1039.46$       For the vase,  $V(15) = 400 \times (1.07)^{15} \approx £1103.61$

$\therefore$  the vase is more valuable than the clock, 15 years after purchase.

- d** To find when the items are equal in value,  
we set  $500 \times (1.05)^t = 400 \times (1.07)^t$  and solve for  $t$ .

$$\begin{aligned}\therefore \frac{(1.05)^t}{(1.07)^t} &= \frac{400}{500} \\ \therefore \left(\frac{1.05}{1.07}\right)^t &= 0.8 \\ \therefore t &\approx 11.8 \quad \{\text{using technology}\}\end{aligned}$$

So, the items are equal in value after about 11.8 years.



- 8**  $y = 3^{\sqrt{x+1}}$  is defined when  $x + 1 \geq 0$   
 $\therefore x \geq -1$

$\therefore$  the domain is  $\{x \mid x \geq -1\}$ .

$$\sqrt{x+1} \geq 0 \quad \text{for all } x \in \mathbb{R}$$

$$\therefore 3^{\sqrt{x+1}} \geq 1 \quad \{\text{since } 3^0 = 1\}$$

$\therefore$  the range is  $\{y \mid y \geq 1\}$ .

- 9**  $f(x) = 3^x + 1$  and  $g(x) = 3^{-x} - 2$

$$\begin{aligned}\mathbf{a} \quad f(0) &= 3^0 + 1 \\ &= 1 + 1 \\ &= 2\end{aligned}$$

$\therefore$  the  $y$ -intercept of  $f(x)$  is 2.

$$\begin{aligned}g(0) &= 3^0 - 2 \\ &= 1 - 2 \\ &= -1\end{aligned}$$

$\therefore$  the  $y$ -intercept of  $g(x)$  is  $-1$ .

- b** Let the  $x$ -coordinate of A, B, and P be  $a$ .

$$f(a) = 3^a + 1 \quad \text{and} \quad g(a) = 3^{-a} - 2$$

So, A is  $(a, 3^a + 1)$ , B is  $(a, 3^{-a} - 2)$ , and P is  $(a, 0)$ .

Since  $AB = 4$ , the  $y$ -coordinate of B must be 4 units less than the  $y$ -coordinate of A.

$$\therefore 3^{-a} - 2 = 3^a + 1 - 4$$

$$\therefore 3^a - 1 - 3^{-a} = 0$$

$$\therefore 3^a(3^a - 1 - 3^{-a}) = 0$$

$$\therefore 3^{2a} - 3^a - 3^0 = 0$$

$$\therefore 3^{2a} - 3^a - 1 = 0$$

$$\therefore (3^a)^2 - 3^a = 1$$

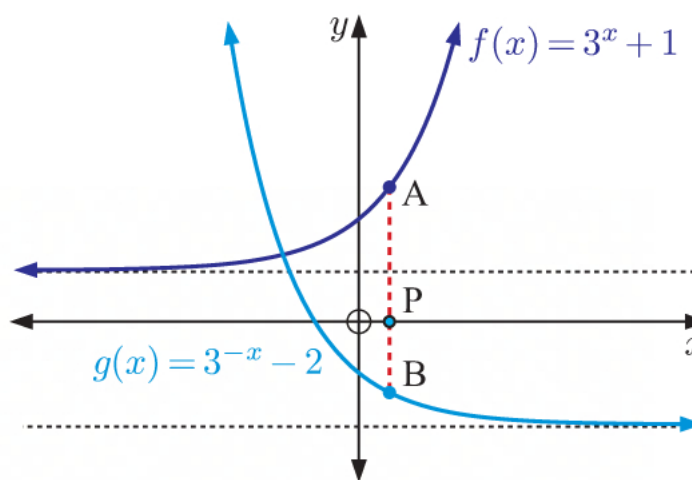
$$\therefore (3^a)^2 - 3^a + \left(\frac{1}{2}\right)^2 = 1 + \left(\frac{1}{2}\right)^2$$

$$\therefore \left(3^a - \frac{1}{2}\right)^2 = \frac{5}{4}$$

$$\therefore 3^a - \frac{1}{2} = \pm \sqrt{\frac{5}{4}}$$

$$\therefore 3^a = \frac{1}{2} \pm \frac{\sqrt{5}}{2}$$

$$\therefore 3^a = \frac{1}{2} + \frac{\sqrt{5}}{2} \quad \{\text{since } 3^a \text{ cannot be negative}\}$$



Now,  $f(a) = 3^a + 1$

$$\begin{aligned}\therefore f(a) &= \frac{1}{2} + \frac{\sqrt{5}}{2} + 1 \\ &= \frac{3}{2} + \frac{\sqrt{5}}{2}\end{aligned}$$

and since  $g(a) = f(a) - 4$ ,

$$\begin{aligned}g(a) &= \frac{3}{2} + \frac{\sqrt{5}}{2} - 4 \\ &= -\frac{5}{2} + \frac{\sqrt{5}}{2}\end{aligned}$$

So, B is  $\left(a, -\frac{5}{2} + \frac{\sqrt{5}}{2}\right)$  and P is  $(a, 0)$ .

$$\begin{aligned}\therefore PB &= \sqrt{(a-a)^2 + \left(-\frac{5}{2} + \frac{\sqrt{5}}{2} - 0\right)^2} \\ &= \sqrt{0^2 + \left(-\frac{5}{2} + \frac{\sqrt{5}}{2}\right)^2} \\ &= -\frac{5}{2} + \frac{\sqrt{5}}{2}\end{aligned}$$

So, PB is  $-\frac{5}{2} + \frac{1}{2}\sqrt{5}$  units in length.

**10**  $f(x) = 3^x$

**a i**  $f(4) = 3^4$   
 $= 81$

**ii**  $f(-1) = 3^{-1}$   
 $= \frac{1}{3}$

**b**  $f(x+2) = kf(x), k \in \mathbb{Z}$   
 $\therefore 3^{x+2} = k \times 3^x$   
 $\therefore 3^2 \times 3^x = k \times 3^x$   
 $\therefore k = 3^2 \quad \{\text{as } 3^x \neq 0\}$   
 $\therefore k = 9$

**11**  $y = a^x$

**a**  $a^{2x} = (a^x)^2$   
 $= y^2$

**b**  $a^{-x} = (a^x)^{-1}$   
 $= y^{-1}$

**c**  $\frac{1}{\sqrt{a^x}} = (a^x)^{-\frac{1}{2}}$   
 $= y^{-\frac{1}{2}} \quad \text{or} \quad \frac{1}{\sqrt{y}}$

**12**  $W = 1500 \times (0.993)^t$  grams

**a** When  $t = 0$ ,  $W = 1500 \times (0.993)^0$   
 $= 1500 \times 1$   
 $= 1500$

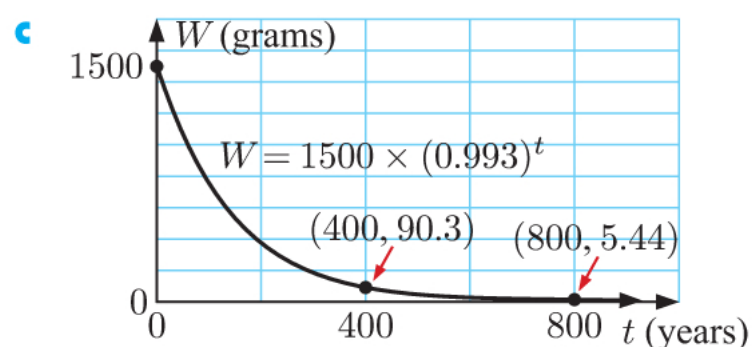
The initial weight of the radioactive substance was 1500 grams.

**b i** When  $t = 400$ ,  
 $W = 1500 \times (0.993)^{400}$   
 $\approx 90.3$

The weight remaining after 400 years was about 90.3 grams.

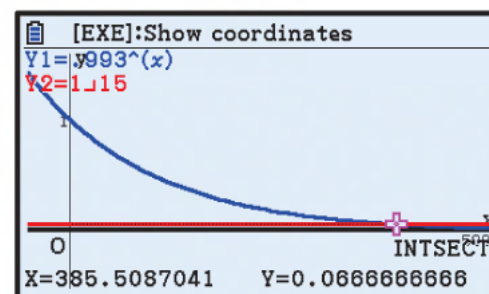
**ii** When  $t = 800$ ,  
 $W = 1500 \times (0.993)^{800}$   
 $\approx 5.44$

The weight remaining after 800 years was about 5.44 grams.



- d** When  $W = 100$ ,  
 $1500 \times (0.993)^t = 100$   
 $\therefore (0.993)^t = \frac{1}{15}$   
 $\therefore t \approx 386$  {using technology}

It takes about 386 years for the weight to reduce to 100 grams.



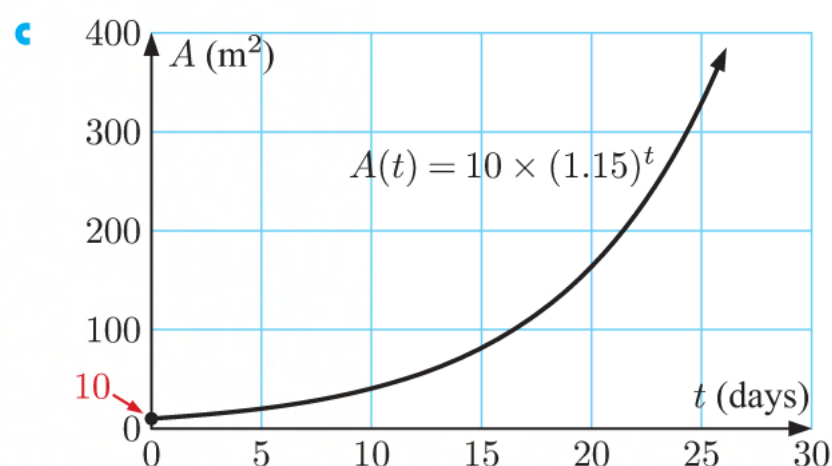
**13 a**  $A(t) = 10 \times (1.15)^t$

**b i**  $A(2) = 10 \times (1.15)^2$   
 $= 13.225$

After 2 days,  $13.225 \text{ m}^2$  is covered.

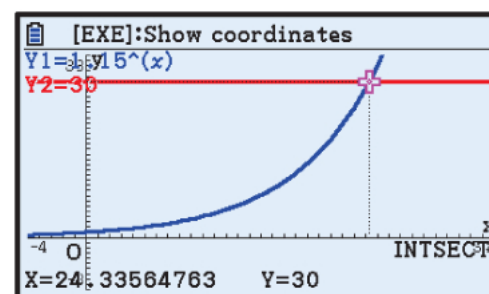
**ii**  $A(5) = 10 \times (1.15)^5$   
 $\approx 20.1$

After 5 days, about  $20.1 \text{ m}^2$  is covered.



**d**  $A(t) = 300$   
 $\therefore 10 \times (1.15)^t = 300$   
 $\therefore (1.15)^t = 30$   
 $\therefore t \approx 24.3$  {using technology}

$\therefore$  it will take about 24.3 days for the affected area to reach  $300 \text{ m}^2$ .





# Chapter 3

## LOGARITHMS

### EXERCISE 3A

- 1**
- a**  $\log 10\,000 = \log(10^4)$   
 $= 4$
- b**  $\log(0.001) = \log(10^{-3})$   
 $= -3$
- c**  $\log 10 = \log(10^1)$   
 $= 1$
- d**  $\log 1 = \log(10^0)$   
 $= 0$
- e**  $\log \sqrt{10} = \log(10^{\frac{1}{2}})$   
 $= \frac{1}{2}$
- f**  $\log \sqrt[3]{10} = \log(10^{\frac{1}{3}})$   
 $= \frac{1}{3}$
- g**  $\log\left(\frac{1}{\sqrt[4]{10}}\right) = \log(10^{-\frac{1}{4}})$   
 $= -\frac{1}{4}$
- h**  $\log(10\sqrt{10}) = \log(10^{\frac{3}{2}})$   
 $= \frac{3}{2} \text{ or } 1\frac{1}{2}$
- i**  $\log \sqrt[3]{100} = \log((10^2)^{\frac{1}{3}})$   
 $= \log(10^{\frac{2}{3}})$   
 $= \frac{2}{3}$
- j**  $\log\left(\frac{100}{\sqrt{10}}\right) = \log\left(\frac{10^2}{10^{\frac{1}{2}}}\right)$   
 $= \log(10^{\frac{3}{2}})$   
 $= \frac{3}{2} \text{ or } 1\frac{1}{2}$
- k**  $\log(10 \times \sqrt[3]{10}) = \log(10^1 \times 10^{\frac{1}{3}})$   
 $= \log(10^{\frac{4}{3}})$   
 $= \frac{4}{3} \text{ or } 1\frac{1}{3}$
- l**  $\log(1000\sqrt{10}) = \log(10^3 \times 10^{\frac{1}{2}})$   
 $= \log(10^{\frac{7}{2}})$   
 $= \frac{7}{2} \text{ or } 3\frac{1}{2}$
- 2**
- a**  $\log(10^n) = n$
- b**  $\log(10^a \times 100) = \log(10^a \times 10^2)$   
 $= \log(10^{a+2})$   
 $= a + 2$
- c**  $\log\left(\frac{10}{10^m}\right) = \log(10^{1-m})$   
 $= 1 - m$
- d**  $\log\left(\frac{10^a}{10^b}\right) = \log(10^{a-b})$   
 $= a - b$
- 3**
- a**  $100 < 237 < 1000$   
 $\therefore \log 100 < \log 237 < \log 1000$   
 $\therefore \log(10^2) < \log 237 < \log(10^3)$   
 $\therefore 2 < \log 237 < 3$
- b**  $\log 237 \approx 2.37$
- 4**
- a** We know that  $\log 1 = \log(10^0) = 0$  and  $\log(0.1) = \log(10^{-1}) = -1$ .  
 Also,  $0.1 < 0.6 < 1 \therefore \log(0.1) < \log(0.6) < \log 1$   
 $\therefore -1 < \log(0.6) < 0$
- b**  $\log(0.6) \approx -0.22$  which is between  $-1$  and  $0$ . ✓

- 5**   **a**  $\log 76 \approx 1.88$                       **b**  $\log 114 \approx 2.06$                       **c**  $\log 3 \approx 0.48$   
**d**  $\log 831 \approx 2.92$                       **e**  $\log(0.4) \approx -0.40$                       **f**  $\log 3247 \approx 3.51$   
**g**  $\log(0.008) \approx -2.10$                       **h**  $\log(-7)$  does not exist

- 6**   **a**  $\log x > 0$   
 $\therefore x > 10^0$   
 $\therefore x > 1$   
**c**  $\log x < 0$   
 $\therefore x < 10^0$   
 $\therefore x < 1$   
 but  $\log x$  is only defined for  $x > 0$   
 $\therefore \log x$  is negative when  $0 < x < 1$ .  
**b**  $\log x = 0$   
 $\therefore x = 10^0$   
 $\therefore x = 1$   
**d**  $\log x$  is undefined when  $x \leq 0$ .

- 7**   **a**      6  
 $= 10^{\log 6}$   
 $\approx 10^{0.7782}$                       **b**      60  
 $= 10^{\log 60}$   
 $\approx 10^{1.7782}$                       **c**      6000  
 $= 10^{\log 6000}$   
 $\approx 10^{3.7782}$                       **d**      0.6  
 $= 10^{\log(0.6)}$   
 $\approx 10^{-0.2218}$   
**e**      0.006  
 $= 10^{\log(0.006)}$   
 $\approx 10^{-2.2218}$                       **f**      15  
 $= 10^{\log 15}$   
 $\approx 10^{1.1761}$                       **g**      1500  
 $= 10^{\log 1500}$   
 $\approx 10^{3.1761}$                       **h**      1.5  
 $= 10^{\log(1.5)}$   
 $\approx 10^{0.1761}$   
**i**      0.15  
 $= 10^{\log(0.15)}$   
 $\approx 10^{-0.8239}$                       **j**      0.000 15  
 $= 10^{\log(0.000\ 15)}$   
 $\approx 10^{-3.8239}$

- 8**   **a**   **i**       $\log 3$   
 $\approx 0.477$                       **ii**       $\log 300$   
 $\approx 2.477$                       **b**       $300 = 3 \times 10^2$   
 $= 10^{\log 3} \times 10^2$   
 $= 10^{\log 3 + 2}$   
 $\therefore \log 300 = \log(10^{\log 3 + 2})$   
 $\therefore \log 300 = \log 3 + 2$

- 9**   **a**   **i**       $\log 5$   
 $\approx 0.699$                       **ii**       $\log(0.05)$   
 $\approx -1.301$                       **b**       $0.05 = 5 \times 10^{-2}$   
 $= 10^{\log 5} \times 10^{-2}$   
 $= 10^{\log 5 - 2}$   
 $\therefore \log(0.05) = \log(10^{\log 5 - 2})$   
 $\therefore \log(0.05) = \log 5 - 2$

- 10**   **a**       $\log x = 2$   
 $\therefore 10^{\log x} = 10^2$   
 $\therefore x = 10^2$   
 $\therefore x = 100$                       **b**       $\log x = 1$   
 $\therefore 10^{\log x} = 10^1$   
 $\therefore x = 10^1$   
 $\therefore x = 10$                       **c**       $\log x = 0$   
 $\therefore 10^{\log x} = 10^0$   
 $\therefore x = 10^0$   
 $\therefore x = 1$

$$\begin{aligned} \mathbf{d} \quad & \log x = -1 \\ \therefore & 10^{\log x} = 10^{-1} \\ \therefore & x = 10^{-1} \\ \therefore & x = \frac{1}{10} \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad & \log x = \frac{1}{2} \\ \therefore & 10^{\log x} = 10^{\frac{1}{2}} \\ \therefore & x = 10^{\frac{1}{2}} \\ \therefore & x = \sqrt{10} \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad & \log x = -\frac{1}{2} \\ \therefore & 10^{\log x} = 10^{-\frac{1}{2}} \\ \therefore & x = 10^{-\frac{1}{2}} \\ \therefore & x = \frac{1}{10^{\frac{1}{2}}} \\ \therefore & x = \frac{1}{\sqrt{10}} \end{aligned}$$

$$\begin{aligned} \mathbf{g} \quad & \log x = 4 \\ \therefore & 10^{\log x} = 10^4 \\ \therefore & x = 10^4 \\ \therefore & x = 10\,000 \end{aligned}$$

$$\begin{aligned} \mathbf{h} \quad & \log x = -5 \\ \therefore & 10^{\log x} = 10^{-5} \\ \therefore & x = 10^{-5} \\ \therefore & x = 0.000\,01 \end{aligned}$$

$$\begin{aligned} \mathbf{i} \quad & \log x \approx 0.8351 \\ \therefore & 10^{\log x} \approx 10^{0.8351} \\ \therefore & x \approx 10^{0.8351} \\ \therefore & x \approx 6.84 \end{aligned}$$

$$\begin{aligned} \mathbf{j} \quad & \log x \approx 2.1457 \\ \therefore & 10^{\log x} \approx 10^{2.1457} \\ \therefore & x \approx 10^{2.1457} \\ \therefore & x \approx 140 \end{aligned}$$

$$\begin{aligned} \mathbf{k} \quad & \log x \approx -1.378 \\ \therefore & 10^{\log x} \approx 10^{-1.378} \\ \therefore & x \approx 10^{-1.378} \\ \therefore & x \approx 0.0419 \end{aligned}$$

$$\begin{aligned} \mathbf{l} \quad & \log x \approx -3.1997 \\ \therefore & 10^{\log x} \approx 10^{-3.1997} \\ \therefore & x \approx 10^{-3.1997} \\ \therefore & x \approx 0.000\,631 \end{aligned}$$

### EXERCISE 3B

- 1
  - a** From  $\log_{10} 100 = 2$ , we deduce that  $10^2 = 100$ .
  - b** From  $\log_{10} 10\,000 = 4$ , we deduce that  $10^4 = 10\,000$ .
  - c** From  $\log_{10}(0.1) = -1$ , we deduce that  $10^{-1} = 0.1$ .
  - d** From  $\log_{10} \sqrt{10} = \frac{1}{2}$ , we deduce that  $10^{\frac{1}{2}} = \sqrt{10}$ .
  - e** From  $\log_2 8 = 3$ , we deduce that  $2^3 = 8$ .
  - f** From  $\log_3 9 = 2$ , we deduce that  $3^2 = 9$ .
  - g** From  $\log_2 \left(\frac{1}{4}\right) = -2$ , we deduce that  $2^{-2} = \frac{1}{4}$ .
  - h** From  $\log_3 \sqrt{27} = 1.5$ , we deduce that  $3^{1.5} = \sqrt{27}$ .
  - i** From  $\log_5 \left(\frac{1}{\sqrt{5}}\right) = -\frac{1}{2}$ , we deduce that  $5^{-\frac{1}{2}} = \frac{1}{\sqrt{5}}$ .
- 2
  - a** From  $4^3 = 64$ , we deduce that  $\log_4 64 = 3$ .
  - b** From  $5^2 = 25$ , we deduce that  $\log_5 25 = 2$ .
  - c** From  $7^2 = 49$ , we deduce that  $\log_7 49 = 2$ .
  - d** From  $2^6 = 64$ , we deduce that  $\log_2 64 = 6$ .
  - e** From  $2^{-3} = \frac{1}{8}$ , we deduce that  $\log_2 \left(\frac{1}{8}\right) = -3$ .
  - f** From  $10^{-2} = 0.01$ , we deduce that  $\log_{10}(0.01) = -2$ .
  - g** From  $2^{-1} = \frac{1}{2}$ , we deduce that  $\log_2 \left(\frac{1}{2}\right) = -1$ .
  - h** From  $3^{-3} = \frac{1}{27}$ , we deduce that  $\log_3 \left(\frac{1}{27}\right) = -3$ .

$$\begin{aligned} \mathbf{3} \quad \mathbf{a} \quad & \log_{10} 100\,000 \\ &= \log_{10}(10^5) \\ &= 5 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad & \log_{10}(0.01) \\ &= \log_{10}(10^{-2}) \\ &= -2 \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad & \log_3 \sqrt{3} \\ &= \log_3(3^{\frac{1}{2}}) \\ &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad & \log_2 4 \\ &= \log_2(2^2) \\ &= 2 \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad & \log_2 64 \\ &= \log_2(2^6) \\ &= 6 \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad & \log_2 128 \\ &= \log_2(2^7) \\ &= 7 \end{aligned}$$

$$\begin{aligned} \mathbf{g} \quad & \log_5 25 \\ &= \log_5(5^2) \\ &= 2 \end{aligned}$$

$$\begin{aligned} \mathbf{h} \quad & \log_5 125 \\ &= \log_5(5^3) \\ &= 3 \end{aligned}$$

$$\begin{aligned} \mathbf{i} \quad & \log_2(0.125) \\ &= \log_2\left(\frac{1}{8}\right) \\ &= \log_2(2^{-3}) \\ &= -3 \end{aligned}$$

$$\begin{aligned} \mathbf{j} \quad & \log_9 3 \\ &= \log_9(9^{\frac{1}{2}}) \\ &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} \mathbf{k} \quad & \log_4 16 \\ &= \log_4(4^2) \\ &= 2 \end{aligned}$$

$$\begin{aligned} \mathbf{l} \quad & \log_{36} 6 \\ &= \log_{36}(36^{\frac{1}{2}}) \\ &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} \mathbf{m} \quad & \log_3 243 \\ &= \log_3(3^5) \\ &= 5 \end{aligned}$$

$$\begin{aligned} \mathbf{n} \quad & \log_2 \sqrt[3]{2} \\ &= \log_2(2^{\frac{1}{3}}) \\ &= \frac{1}{3} \end{aligned}$$

$$\begin{aligned} \mathbf{o} \quad & \log_8 2 \\ &= \log_8(8^{\frac{1}{3}}) \\ &= \frac{1}{3} \end{aligned}$$

$$\begin{aligned} \mathbf{p} \quad & \log_6(6\sqrt{6}) \\ &= \log_6(6^1 \times 6^{\frac{1}{2}}) \\ &= \log_6(6^{\frac{3}{2}}) \\ &= \frac{3}{2} \end{aligned}$$

$$\begin{aligned} \mathbf{q} \quad & \log_4 1 \\ &= \log_4(4^0) \\ &= 0 \end{aligned}$$

$$\begin{aligned} \mathbf{r} \quad & \log_9 9 \\ &= \log_9(9^1) \\ &= 1 \end{aligned}$$

$$\begin{aligned} \mathbf{s} \quad & \log_3\left(\frac{1}{3}\right) \\ &= \log_3(3^{-1}) \\ &= -1 \end{aligned}$$

$$\begin{aligned} \mathbf{t} \quad & \log_{10} \sqrt[4]{1000} \\ &= \log_{10}((10^3)^{\frac{1}{4}}) \\ &= \log_{10}(10^{\frac{3}{4}}) \\ &= \frac{3}{4} \end{aligned}$$

$$\begin{aligned} \mathbf{u} \quad & \log_7\left(\frac{1}{\sqrt{7}}\right) \\ &= \log_7(7^{-\frac{1}{2}}) \\ &= -\frac{1}{2} \end{aligned}$$

$$\begin{aligned} \mathbf{v} \quad & \log_5(25\sqrt{5}) \\ &= \log_5(5^2 \times 5^{\frac{1}{2}}) \\ &= \log_5(5^{\frac{5}{2}}) \\ &= \frac{5}{2} \end{aligned}$$

$$\begin{aligned} \mathbf{w} \quad & \log_3\left(\frac{1}{\sqrt{27}}\right) \\ &= \log_3\left(\frac{1}{(3^3)^{\frac{1}{2}}}\right) \\ &= \log_3(3^{-\frac{3}{2}}) \\ &= -\frac{3}{2} \end{aligned}$$

$$\begin{aligned} \mathbf{x} \quad & \log_4\left(\frac{1}{2\sqrt{2}}\right) \\ &= \log_4(2^{-\frac{3}{2}}) \\ &= \log_4\left((2^2)^{-\frac{3}{4}}\right) \\ &= \log_4(4^{-\frac{3}{4}}) \\ &= -\frac{3}{4} \end{aligned}$$

$$\begin{aligned} \mathbf{4} \quad \mathbf{a} \quad & \log_x(x^2) \\ &= 2, \quad x > 0 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad & \log_t\left(\frac{1}{t}\right) \\ &= \log_t(t^{-1}) \\ &= -1, \quad t > 0 \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad & \log_x \sqrt{x} \\ &= \log_x(x^{\frac{1}{2}}) \\ &= \frac{1}{2}, \quad x > 0 \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad & \log_m(m^3) \\ &= 3, \quad m > 0 \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad & \log_k \sqrt[4]{k} \\ &= \log_k(k^{\frac{1}{4}}) \\ &= \frac{1}{4}, \quad k > 0 \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad & \log_x(x\sqrt{x}) \\ &= \log_x(x^1 \times x^{\frac{1}{2}}) \\ &= \log_x(x^{\frac{3}{2}}) \\ &= \frac{3}{2}, \quad x > 0 \end{aligned}$$



$$\begin{aligned} \mathbf{g} \quad & \log_a \left( \frac{1}{a^2} \right) \\ &= \log_a (a^{-2}) \\ &= -2, \quad a > 0 \end{aligned}$$

$$\begin{aligned} \mathbf{h} \quad & \log_a \left( \frac{1}{\sqrt{a}} \right) \\ &= \log_a (a^{-\frac{1}{2}}) \\ &= -\frac{1}{2}, \quad a > 0 \end{aligned}$$

$$\begin{aligned} \mathbf{i} \quad & \log_m \sqrt{m^5} \\ &= \log_m ((m^5)^{\frac{1}{2}}) \\ &= \log_m (m^{\frac{5}{2}}) \\ &= \frac{5}{2}, \quad m > 0 \end{aligned}$$

$$\begin{aligned} \mathbf{5} \quad \mathbf{a} \quad & \log_2 x = 3 \\ & \therefore x = 2^3 \\ & \therefore x = 8 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad & \log_4 x = \frac{1}{2} \\ & \therefore x = 4^{\frac{1}{2}} \\ & \therefore x = 2 \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad & \log_5 x = -3 \\ & \therefore x = 5^{-3} \\ & \therefore x = \frac{1}{125} \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad & \log_x 81 = 4 \\ & \therefore 81 = x^4 \\ & \therefore x = \pm \sqrt[4]{81} \\ & \therefore x = \pm 3 \\ & \therefore x = 3 \quad \{\text{as } x > 0\} \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad & \log_2 (x - 6) = 3 \\ & \therefore x - 6 = 2^3 \\ & \therefore x - 6 = 8 \\ & \therefore x = 14 \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad & \log_2 (\log_3 x) = -1 \\ & \therefore \log_3 x = 2^{-1} \\ & \therefore \log_3 x = \frac{1}{2} \\ & \therefore x = 3^{\frac{1}{2}} \\ & \therefore x = \sqrt{3} \end{aligned}$$

$$\begin{aligned} \mathbf{6} \quad & \log_a b = x \\ & \therefore b = a^x \\ & \therefore \log_b b = \log_b (a^x) \\ & \therefore 1 = x \log_b a \\ & \therefore \log_b a = \frac{1}{x} \end{aligned}$$

$$\begin{aligned} \mathbf{7} \quad & y = \log_2 \sqrt{5x - 1} \\ & \therefore 2^y = \sqrt{5x - 1} \\ & \therefore (2^y)^2 = 5x - 1 \\ & \therefore 2^{2y} = 5x - 1 \\ & \therefore 2^{2y} + 1 = 5x \\ & \therefore x = \frac{2^{2y} + 1}{5} \end{aligned}$$

## INVESTIGATION 1

## DISCOVERING THE LAWS OF LOGARITHMS

$$\begin{aligned} \mathbf{1} \quad \mathbf{a} \quad \mathbf{i} \quad & \log 2 + \log 3 \approx 0.778 \\ & \mathbf{iv} \quad \log 6 \approx 0.778 \\ \mathbf{b} \quad & \log m + \log n = \log(mn) \end{aligned}$$

$$\begin{aligned} \mathbf{ii} \quad & \log 3 + \log 7 \approx 1.32 \\ \mathbf{v} \quad & \log 21 \approx 1.32 \end{aligned}$$

$$\begin{aligned} \mathbf{iii} \quad & \log 4 + \log 20 \approx 1.90 \\ \mathbf{vi} \quad & \log 80 \approx 1.90 \end{aligned}$$

$$\begin{aligned} \mathbf{2} \quad \mathbf{a} \quad \mathbf{i} \quad & \log 6 - \log 2 \approx 0.477 \\ & \mathbf{iv} \quad \log 3 \approx 0.477 \\ \mathbf{b} \quad & \log m - \log n = \log \left( \frac{m}{n} \right) \end{aligned}$$

$$\begin{aligned} \mathbf{ii} \quad & \log 12 - \log 3 \approx 0.602 \\ \mathbf{v} \quad & \log 4 \approx 0.602 \end{aligned}$$

$$\begin{aligned} \mathbf{iii} \quad & \log 3 - \log 5 \approx -0.222 \\ \mathbf{vi} \quad & \log(0.6) \approx -0.222 \end{aligned}$$

$$\begin{aligned} \mathbf{3} \quad \mathbf{a} \quad \mathbf{i} \quad & 3 \log 2 \approx 0.903 \\ & \mathbf{iv} \quad \log(2^3) \approx 0.903 \\ \mathbf{b} \quad & m \log b = \log(b^m) \end{aligned}$$

$$\begin{aligned} \mathbf{ii} \quad & 2 \log 5 \approx 1.40 \\ \mathbf{v} \quad & \log(5^2) \approx 1.40 \end{aligned}$$

$$\begin{aligned} \mathbf{iii} \quad & -4 \log 3 \approx -1.91 \\ \mathbf{vi} \quad & \log(3^{-4}) \approx -1.91 \end{aligned}$$

## EXERCISE 3C

$$\begin{aligned} 1 \quad a \quad & \log 8 + \log 2 \\ &= \log(8 \times 2) \\ &= \log 16 \end{aligned}$$

$$\begin{aligned} d \quad & \log p - \log m \\ &= \log\left(\frac{p}{m}\right) \end{aligned}$$

$$\begin{aligned} g \quad & \log 250 + \log 4 \\ &= \log(250 \times 4) \\ &= \log 1000 \\ &= \log(10^3) \\ &= 3 \end{aligned}$$

$$\begin{aligned} j \quad & \log 5 + \log 4 - \log 2 \\ &= \log\left(\frac{5 \times 4}{2}\right) \\ &= \log 10 \\ &= 1 \end{aligned}$$

$$\begin{aligned} 2 \quad a \quad & \log 7 + 2 \\ &= \log 7 + \log(10^2) \\ &= \log(7 \times 100) \\ &= \log 700 \end{aligned}$$

$$\begin{aligned} d \quad & \log_3 5 - 2 \\ &= \log_3 5 - \log_3(3^2) \\ &= \log_3 5 - \log_3 9 \\ &= \log_3\left(\frac{5}{9}\right) \end{aligned}$$

$$\begin{aligned} g \quad & t + \log w \\ &= \log(10^t) + \log w \\ &= \log(10^t \times w) \end{aligned}$$

$$\begin{aligned} 3 \quad a \quad & 5 \log 2 + \log 3 \\ &= \log(2^5) + \log 3 \\ &= \log 32 + \log 3 \\ &= \log(32 \times 3) \\ &= \log 96 \end{aligned}$$

$$\begin{aligned} b \quad & \log 4 + \log 5 \\ &= \log(4 \times 5) \\ &= \log 20 \end{aligned}$$

$$\begin{aligned} e \quad & \log_4 8 - \log_4 2 \\ &= \log_4\left(\frac{8}{2}\right) \\ &= \log_4 4 \\ &= 1 \end{aligned}$$

$$\begin{aligned} h \quad & \log_5 100 - \log_5 4 \\ &= \log_5\left(\frac{100}{4}\right) \\ &= \log_5 25 \\ &= \log_5(5^2) \\ &= 2 \end{aligned}$$

$$\begin{aligned} k \quad & \log_3 6 - \log_3 2 - \log_3 3 \\ &= \log_3(6 \div 2 \div 3) \\ &= \log_3 1 \\ &= \log_3(3^0) \\ &= 0 \end{aligned}$$

$$\begin{aligned} b \quad & \log 4 - 1 \\ &= \log 4 - \log(10^1) \\ &= \log\left(\frac{4}{10}\right) \\ &= \log\left(\frac{2}{5}\right) \end{aligned}$$

$$\begin{aligned} e \quad & 2 + \log 2 \\ &= \log(10^2) + \log 2 \\ &= \log(100 \times 2) \\ &= \log 200 \end{aligned}$$

$$\begin{aligned} h \quad & \log_m 40 - 2 \\ &= \log_m 40 - \log_m(m^2) \\ &= \log_m\left(\frac{40}{m^2}\right) \end{aligned}$$

$$\begin{aligned} b \quad & 2 \log 3 + 3 \log 2 \\ &= \log(3^2) + \log(2^3) \\ &= \log 9 + \log 8 \\ &= \log(9 \times 8) \\ &= \log 72 \end{aligned}$$

$$\begin{aligned} c \quad & \log 40 - \log 5 \\ &= \log\left(\frac{40}{5}\right) \\ &= \log 8 \end{aligned}$$

$$\begin{aligned} f \quad & \log 5 + \log(0.4) \\ &= \log(5 \times 0.4) \\ &= \log 2 \end{aligned}$$

$$\begin{aligned} i \quad & \log 2 + \log 3 + \log 4 \\ &= \log(2 \times 3 \times 4) \\ &= \log 24 \end{aligned}$$

$$\begin{aligned} l \quad & \log\left(\frac{4}{3}\right) + \log 3 + \log 7 \\ &= \log\left(\frac{4}{3} \times 3 \times 7\right) \\ &= \log 28 \end{aligned}$$

$$\begin{aligned} c \quad & 1 + \log_2 3 \\ &= \log_2(2^1) + \log_2 3 \\ &= \log_2(2 \times 3) \\ &= \log_2 6 \end{aligned}$$

$$\begin{aligned} f \quad & \log 50 - 4 \\ &= \log 50 - \log(10^4) \\ &= \log 50 - \log 10\,000 \\ &= \log\left(\frac{50}{10\,000}\right) \\ &= \log(0.005) \end{aligned}$$

$$\begin{aligned} i \quad & 3 - \log_5 50 \\ &= \log_5(5^3) - \log_5 50 \\ &= \log_5 125 - \log_5 50 \\ &= \log_5\left(\frac{125}{50}\right) \\ &= \log_5\left(\frac{5}{2}\right) \end{aligned}$$

$$\begin{aligned} c \quad & 3 \log 4 - \log 8 \\ &= \log(4^3) - \log 8 \\ &= \log 64 - \log 8 \\ &= \log\left(\frac{64}{8}\right) \\ &= \log 8 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad & 2 \log_3 5 - 3 \log_3 2 \\
 &= \log_3(5^2) - \log_3(2^3) \\
 &= \log_3 25 - \log_3 8 \\
 &= \log_3\left(\frac{25}{8}\right)
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & \frac{1}{2} \log_6 4 + \log_6 3 \\
 &= \log_6(4^{\frac{1}{2}}) + \log_6 3 \\
 &= \log_6 2 + \log_6 3 \\
 &= \log_6(2 \times 3) \\
 &= \log_6 6 \\
 &= 1
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & \frac{1}{3} \log\left(\frac{1}{8}\right) \\
 &= \log\left(\left(\frac{1}{8}\right)^{\frac{1}{3}}\right) \\
 &= \log((2^{-3})^{\frac{1}{3}}) \\
 &= \log(2^{-1}) \\
 &= \log\left(\frac{1}{2}\right)
 \end{aligned}$$

$$\begin{aligned}
 \text{g} \quad & 3 - \log 2 - 2 \log 5 \\
 &= \log(10^3) - \log 2 - \log(5^2) \\
 &= \log 1000 - \log 2 - \log 25 \\
 &= \log(1000 \div 2 \div 25) \\
 &= \log 20
 \end{aligned}$$

$$\begin{aligned}
 \text{h} \quad & 1 - 3 \log 2 + \log 20 \\
 &= \log(10^1) - \log(2^3) + \log 20 \\
 &= \log 10 - \log 8 + \log 20 \\
 &= \log(10 \div 8 \times 20) \\
 &= \log 25
 \end{aligned}$$

$$\begin{aligned}
 \text{i} \quad & 2 - \frac{1}{2} \log_n 4 - \log_n 5 \\
 &= \log_n(n^2) - \log_n(4^{\frac{1}{2}}) - \log_n 5 \\
 &= \log_n(n^2) - \log_n 2 - \log_n 5 \\
 &= \log_n(n^2 \div 2 \div 5) \\
 &= \log_n\left(\frac{n^2}{10}\right)
 \end{aligned}$$

$$\begin{aligned}
 4 \quad \text{a} \quad & \frac{\log 4}{\log 2} \\
 &= \frac{\log(2^2)}{\log 2} \\
 &= \frac{2 \log 2}{\log 2} \\
 &= 2
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & \frac{\log_5 27}{\log_5 9} \\
 &= \frac{\log_5(3^3)}{\log_5(3^2)} \\
 &= \frac{3 \log_5 3}{2 \log_5 3} \\
 &= \frac{3}{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & \frac{\log 8}{\log 2} \\
 &= \frac{\log(2^3)}{\log 2} \\
 &= \frac{3 \log 2}{\log 2} \\
 &= 3
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad & \frac{\log 3}{\log 9} \\
 &= \frac{\log 3}{\log(3^2)} \\
 &= \frac{\log 3}{2 \log 3} \\
 &= \frac{1}{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & \frac{\log_3 25}{\log_3(0.2)} \\
 &= \frac{\log_3(5^2)}{\log_3(5^{-1})} \\
 &= \frac{2 \log_3 5}{-1 \log_3 5} \\
 &= -2
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & \frac{\log_4 8}{\log_4(0.25)} \\
 &= \frac{\log_4(2^3)}{\log_4(2^{-2})} \quad \{0.25 = \frac{1}{4} = \frac{1}{2^2}\} \\
 &= \frac{3 \log_4 2}{-2 \log_4 2} \\
 &= -\frac{3}{2}
 \end{aligned}$$

$$\begin{aligned}
 5 \quad \text{a} \quad & \log 9 = \log(3^2) \\
 &= 2 \log 3 \quad \checkmark
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & \log \sqrt{2} = \log(2^{\frac{1}{2}}) \\
 &= \frac{1}{2} \log 2 \quad \checkmark
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & \log\left(\frac{1}{8}\right) = \log\left(\frac{1}{2^3}\right) \\
 &= \log(2^{-3}) \\
 &= -3 \log 2 \quad \checkmark
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad & \log\left(\frac{1}{5}\right) = \log(5^{-1}) \\
 &= -1 \times \log 5 \\
 &= -\log 5 \quad \checkmark
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & \log 5 = \log\left(\frac{10}{2}\right) \\
 &= \log(10^1) - \log 2 \\
 &= 1 - \log 2 \quad \checkmark
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & \log 5000 \\
 &= \log\left(\frac{10\,000}{2}\right) \\
 &= \log(10^4) - \log 2 \\
 &= 4 - \log 2 \quad \checkmark
 \end{aligned}$$

$$\begin{aligned}
 6 \quad & \log(a \times 10^k) \\
 &= \log a + \log 10^k \\
 &= \log a + k
 \end{aligned}$$

$$\begin{aligned}
 7 \quad a \quad & \log_b 6 \\
 &= \log_b(2 \times 3) \\
 &= \log_b 2 + \log_b 3 \\
 &= p + q
 \end{aligned}$$

$$\begin{aligned}
 b \quad & \log_b 45 \\
 &= \log_b(9 \times 5) \\
 &= \log_b(3^2 \times 5) \\
 &= \log_b(3^2) + \log_b 5 \\
 &= 2\log_b 3 + \log_b 5 \\
 &= 2q + r
 \end{aligned}$$

$$\begin{aligned}
 c \quad & \log_b 108 \\
 &= \log_b(4 \times 27) \\
 &= \log_b(2^2 \times 3^3) \\
 &= \log_b(2^2) + \log_b(3^3) \\
 &= 2\log_b 2 + 3\log_b 3 \\
 &= 2p + 3q
 \end{aligned}$$

$$\begin{aligned}
 d \quad & \log_b\left(\frac{5\sqrt{3}}{2}\right) \\
 &= \log_b(5 \times 3^{\frac{1}{2}}) - \log_b 2 \\
 &= \log_b 5 + \log_b(3^{\frac{1}{2}}) - \log_b 2 \\
 &= \log_b 5 + \frac{1}{2}\log_b 3 - \log_b 2 \\
 &= r + \frac{1}{2}q - p
 \end{aligned}$$

$$\begin{aligned}
 e \quad & \log_b\left(\frac{5}{32}\right) \\
 &= \log_b 5 - \log_b 32 \\
 &= \log_b 5 - \log_b(2^5) \\
 &= \log_b 5 - 5\log_b 2 \\
 &= r - 5p
 \end{aligned}$$

$$\begin{aligned}
 f \quad & \log_b\left(\frac{2}{9}\right) \\
 &= \log_b 2 - \log_b 9 \\
 &= \log_b 2 - \log_b(3^2) \\
 &= \log_b 2 - 2\log_b 3 \\
 &= p - 2q
 \end{aligned}$$

$$\begin{aligned}
 8 \quad a \quad & \log_2(PR) \\
 &= \log_2 P + \log_2 R \\
 &= x + z
 \end{aligned}$$

$$\begin{aligned}
 b \quad & \log_2(RQ^2) \\
 &= \log_2 R + \log_2(Q^2) \\
 &= \log_2 R + 2\log_2 Q \\
 &= z + 2y
 \end{aligned}$$

$$\begin{aligned}
 c \quad & \log_2\left(\frac{PR}{Q}\right) \\
 &= \log_2(PR) - \log_2 Q \\
 &= \log_2 P + \log_2 R - \log_2 Q \\
 &= x + z - y
 \end{aligned}$$

$$\begin{aligned}
 d \quad & \log_2(P^2\sqrt{Q}) \\
 &= \log_2(P^2) + \log_2(Q^{\frac{1}{2}}) \\
 &= 2\log_2 P + \frac{1}{2}\log_2 Q \\
 &= 2x + \frac{1}{2}y
 \end{aligned}$$

$$\begin{aligned}
 e \quad & \log_2\left(\frac{Q^3}{\sqrt{R}}\right) \\
 &= \log_2(Q^3) - \log_2(R^{\frac{1}{2}}) \\
 &= 3\log_2 Q - \frac{1}{2}\log_2 R \\
 &= 3y - \frac{1}{2}z
 \end{aligned}$$

$$\begin{aligned}
 f \quad & \log_2\left(\frac{R^2\sqrt{Q}}{P^3}\right) \\
 &= \log_2(R^2) + \log_2(Q^{\frac{1}{2}}) - \log_2(P^3) \\
 &= 2\log_2 R + \frac{1}{2}\log_2 Q - 3\log_2 P \\
 &= 2z + \frac{1}{2}y - 3x
 \end{aligned}$$

$$\begin{aligned}
 9 \quad a \quad & \log_t(N^2) = 1.72 \\
 & \therefore 2\log_t N = 1.72 \\
 & \therefore \log_t N = 0.86
 \end{aligned}$$

$$\begin{aligned}
 b \quad & \log_t(MN) \\
 &= \log_t M + \log_t N \\
 &= 1.29 + 0.86 \\
 &= 2.15
 \end{aligned}$$

$$\begin{aligned}
 c \quad & \log_t\left(\frac{N^2}{\sqrt{M}}\right) \\
 &= \log_t(N^2) - \log_t(M^{\frac{1}{2}}) \\
 &= 1.72 - \frac{1}{2}\log_t M \\
 &= 1.72 - \frac{1}{2}(1.29) \\
 &= 1.075
 \end{aligned}$$



**10**  $\log_a(x+2) = \log_a x + 2, \quad a > 1$

$$\therefore \log_a(x+2) = \log_a x + \log_a(a^2)$$

$$\therefore \log_a(x+2) = \log_a(x \times a^2)$$

$$\therefore x+2 = a^2 x$$

$$\therefore x - a^2 x = -2$$

$$\therefore x(1 - a^2) = -2$$

$$\therefore x = \frac{-2}{1 - a^2} \quad \{1 - a^2 \neq 0 \text{ as } a > 1\}$$

$$\therefore x = \frac{2}{a^2 - 1} \quad \{a > 1\}$$

**11 a**  $\log(8!) - \log(7!) + \log(6!) - \log(5!) + \log(4!) - \log(3!) + \log(2!) - \log(1!)$

$$= \log\left(\frac{8!}{7!}\right) + \log\left(\frac{6!}{5!}\right) + \log\left(\frac{4!}{3!}\right) + \log\left(\frac{2!}{1!}\right)$$

$$= \log\left(\frac{8 \times 7!}{7!}\right) + \log\left(\frac{6 \times 5!}{5!}\right) + \log\left(\frac{4 \times 3!}{3!}\right) + \log\left(\frac{2 \times 1!}{1!}\right)$$

$$= \log 8 + \log 6 + \log 4 + \log 2$$

$$= \log(8 \times 6 \times 4 \times 2)$$

$$= \log 384$$

**b** We consider all of the possible ways of writing  $\log_2(6!)$  in the form  $a + \log_2 b$ , where  $a, b \in \mathbb{Z}$ .

$$\begin{aligned} \log_2(6!) &= \log_2 720 \\ &= \log_2(2 \times 360) \\ &= \log_2 2 + \log_2 360 \\ &= 1 + \log_2 360 \end{aligned}$$

$$\begin{aligned} \text{or} \quad \log_2(6!) &= \log_2 720 \\ &= \log_2(4 \times 180) \\ &= \log_2 4 + \log_2 180 \\ &= \log_2(2^2) + \log_2 180 \\ &= 2 + \log_2 180 \end{aligned}$$

$$\begin{aligned} \text{or} \quad \log_2(6!) &= \log_2 720 \\ &= \log_2(8 \times 90) \\ &= \log_2 8 + \log_2 90 \\ &= \log_2(2^3) + \log_2 90 \\ &= 3 + \log_2 90 \end{aligned}$$

$$\begin{aligned} \text{or} \quad \log_2(6!) &= 720 \\ &= \log_2(16 \times 45) \\ &= \log_2 16 + \log_2 45 \\ &= \log_2(2^4) + \log_2 45 \\ &= 4 + \log_2 45 \end{aligned}$$

$\log_2(6!) = 4 + \log_2 45$  where  $a = 4$  and  $b = 45$  gives the smallest possible value of  $b$ .

**12**  $\log(x^4) + \log\left(\frac{x^4}{y}\right) + \log\left(\frac{x^4}{y^2}\right) + \dots + \log\left(\frac{x^4}{y^9}\right)$

$$= \log(x^4) + \log(x^4) - \log y + \log(x^4) - \log(y^2) + \dots + \log(x^4) - \log(y^9)$$

$$= 10 \times \log(x^4) - (\log y + \log(y^2) + \dots + \log(y^9))$$

$$= \log((x^4)^{10}) - (\log y + 2 \log y + \dots + 9 \log y)$$

$$= \log(x^{40}) - (1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9) \log y$$

$$= \log(x^{40}) - 45 \log y$$

$$= \log(x^{40}) - \log(y^{45})$$

$$= \log\left(\frac{x^{40}}{y^{45}}\right)$$

$$\begin{aligned}
 \text{13 } \log \sqrt{3} - \log \sqrt[4]{3} + \log \sqrt[8]{3} - \log \sqrt[16]{3} + \dots &= \log(3^{\frac{1}{2}}) - \log(3^{\frac{1}{4}}) + \log(3^{\frac{1}{8}}) - \log(3^{\frac{1}{16}}) + \dots \\
 &= \frac{1}{2} \log 3 - \frac{1}{4} \log 3 + \frac{1}{8} \log 3 - \frac{1}{16} \log 3 + \dots \\
 &= \left(\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \dots\right) \log 3
 \end{aligned}$$

Now,  $\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \dots$  is an infinite geometric series with  $u_1 = \frac{1}{2}$  and  $r = -\frac{1}{2}$ .

Since  $|r| = \left|-\frac{1}{2}\right| = \frac{1}{2}$  which is  $< 1$ , the series will converge to  $\frac{u_1}{1-r}$ .

$$\frac{u_1}{1-r} = \frac{\frac{1}{2}}{1 - (-\frac{1}{2})} = \frac{\frac{1}{2}}{\frac{3}{2}} = \frac{1}{3}$$

$$\begin{aligned}
 \therefore \log \sqrt{3} - \log \sqrt[4]{3} + \log \sqrt[8]{3} - \log \sqrt[16]{3} + \dots &= \left(\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \dots\right) \log 3 \\
 &= \frac{1}{3} \log 3 \\
 &= \log(3^{\frac{1}{3}}) \\
 &= \log \sqrt[3]{3}
 \end{aligned}$$

$$\begin{aligned}
 \text{14 } \quad x^2 + y^2 &= 52xy, \quad 0 < y < x \\
 \therefore x^2 + y^2 - 2xy &= 52xy - 2xy \quad \{\text{subtracting } 2xy \text{ from both sides}\} \\
 \therefore (x - y)^2 &= 50xy \\
 \therefore \log[(x - y)^2] &= \log(50xy) \\
 \therefore 2\log(x - y) &= \log(25 \times x \times 2y) \\
 \therefore 2\log(x - y) &= \log 25 + \log x + \log(2y) \\
 \therefore \log(x - y) &= \frac{1}{2} \log 25 + \frac{1}{2}(\log x + \log(2y)) \\
 \therefore \log(x - y) &= \log(25^{\frac{1}{2}}) + \frac{1}{2}(\log x + \log(2y)) \\
 \therefore \log(x - y) &= \log 5 + \frac{1}{2}(\log x + \log(2y)) \\
 \therefore \log(x - y) - \log 5 &= \frac{1}{2}(\log x + \log(2y)) \\
 \therefore \log\left(\frac{x - y}{5}\right) &= \frac{1}{2}(\log x + \log(2y))
 \end{aligned}$$

### EXERCISE 3D

$$\begin{aligned}
 \text{1 a } \quad \ln(e^2) \\
 = 2
 \end{aligned}$$

$$\begin{aligned}
 \text{b } \quad \ln(e^4) \\
 = 4
 \end{aligned}$$

$$\begin{aligned}
 \text{c } \quad \ln((\sqrt{e})^3) \\
 = \ln((e^{\frac{1}{2}})^3) \\
 = \ln(e^{\frac{3}{2}}) \\
 = \frac{3}{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{d } \quad \ln 1 \\
 = \ln(e^0) \\
 = 0
 \end{aligned}$$

$$\begin{aligned}
 \text{e } \quad \ln\left(\frac{1}{e}\right) \\
 = \ln(e^{-1}) \\
 = -1
 \end{aligned}$$

$$\begin{aligned}
 \text{f } \quad \ln \sqrt[3]{e} \\
 = \ln(e^{\frac{1}{3}}) \\
 = \frac{1}{3}
 \end{aligned}$$

$$\begin{aligned}
 \text{g } \quad \ln\left(\frac{1}{e^2}\right) \\
 = \ln(e^{-2}) \\
 = -2
 \end{aligned}$$

$$\begin{aligned}
 \text{h } \quad \ln\left(\frac{1}{\sqrt{e}}\right) \\
 = \ln(e^{-\frac{1}{2}}) \\
 = -\frac{1}{2}
 \end{aligned}$$

$$\begin{aligned} \mathbf{2} \quad \mathbf{a} \quad & e^{\ln 3} \\ & = 3 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad & e^{2 \ln 3} \\ & = (e^{\ln 3})^2 \\ & = 3^2 \\ & = 9 \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad & e^{-\ln 5} \\ & = (e^{\ln 5})^{-1} \\ & = 5^{-1} \\ & = \frac{1}{5} \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad & e^{-2 \ln 2} \\ & = (e^{\ln 2})^{-2} \\ & = 2^{-2} \\ & = \frac{1}{2^2} \\ & = \frac{1}{4} \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad & \ln(e^a) \\ & = a \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad & \ln(e \times e^a) \\ & = \ln(e^{1+a}) \\ & = 1 + a \end{aligned}$$

$$\begin{aligned} \mathbf{g} \quad & \ln(e^a \times e^b) \\ & = \ln(e^{a+b}) \\ & = a + b \end{aligned}$$

$$\begin{aligned} \mathbf{h} \quad & \ln((e^a)^b) \\ & = \ln(e^{ab}) \\ & = ab \end{aligned}$$

$$\mathbf{3} \quad \mathbf{a} \quad \ln 12 \approx 2.485$$

$$\mathbf{b} \quad \ln 68 \approx 4.220$$

$$\mathbf{c} \quad \ln(1.4) \approx 0.336$$

$$\mathbf{d} \quad \ln(0.7) \approx -0.357$$

$$\mathbf{e} \quad \ln 500 \approx 6.215$$

$\mathbf{4}$   $x$  does not exist such that  $e^x = -2$  or  $0$  since  $e^x > 0$  for all  $x \in \mathbb{R}$ .  
 $\therefore \ln(-2)$  and  $\ln 0$  do not exist.

$$\begin{aligned} \mathbf{5} \quad \mathbf{a} \quad & 6 = e^{\ln 6} \\ & \approx e^{1.7918} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad & 60 = e^{\ln 60} \\ & \approx e^{4.0943} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad & 6000 = e^{\ln 6000} \\ & \approx e^{8.6995} \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad & 0.6 = e^{\ln(0.6)} \\ & \approx e^{-0.5108} \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad & 0.006 = e^{\ln(0.006)} \\ & \approx e^{-5.1160} \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad & 15 = e^{\ln 15} \\ & \approx e^{2.7081} \end{aligned}$$

$$\begin{aligned} \mathbf{g} \quad & 1500 = e^{\ln 1500} \\ & \approx e^{7.3132} \end{aligned}$$

$$\begin{aligned} \mathbf{h} \quad & 1.5 = e^{\ln(1.5)} \\ & \approx e^{0.4055} \end{aligned}$$

$$\begin{aligned} \mathbf{i} \quad & 0.15 = e^{\ln(0.15)} \\ & \approx e^{-1.8971} \end{aligned}$$

$$\begin{aligned} \mathbf{j} \quad & 0.00015 = e^{\ln(0.00015)} \\ & \approx e^{-8.8049} \end{aligned}$$

$$\begin{aligned} \mathbf{6} \quad \mathbf{a} \quad & \ln x = 3 \\ & \therefore x = e^3 \\ & \therefore x \approx 20.1 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad & \ln x = 1 \\ & \therefore x = e^1 \\ & \therefore x = e \approx 2.72 \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad & \ln x = 0 \\ & \therefore x = e^0 \\ & \therefore x = 1 \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad & \ln x = -1 \\ & \therefore x = e^{-1} \\ & \therefore x \approx 0.368 \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad & \ln x = -5 \\ & \therefore x = e^{-5} \\ & \therefore x \approx 0.00674 \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad & \ln x \approx 0.835 \\ & \therefore x \approx e^{0.835} \\ & \therefore x \approx 2.30 \end{aligned}$$

$$\begin{aligned} \mathbf{g} \quad & \ln x \approx 2.145 \\ & \therefore x \approx e^{2.145} \\ & \therefore x \approx 8.54 \end{aligned}$$

$$\begin{aligned} \mathbf{h} \quad & \ln x \approx -3.2971 \\ & \therefore x \approx e^{-3.2971} \\ & \therefore x \approx 0.0370 \end{aligned}$$

$$\mathbf{7} \quad \mathbf{a} \quad \mathbf{i} \quad \ln(e^x) = x$$

$$\mathbf{ii} \quad e^{\ln x} = x$$

$\mathbf{b}$   $y = e^x$  and  $y = \ln x$  are inverses of each other.

$$\begin{aligned} \mathbf{8} \quad \mathbf{a} \quad & \ln 15 + \ln 3 \\ &= \ln(15 \times 3) \\ &= \ln 45 \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad & \ln 4 + \ln 6 \\ &= \ln(4 \times 6) \\ &= \ln 24 \end{aligned}$$

$$\begin{aligned} \mathbf{g} \quad & 1 + \ln 4 \\ &= \ln(e^1) + \ln 4 \\ &= \ln(e \times 4) \\ &= \ln(4e) \end{aligned}$$

$$\begin{aligned} \mathbf{j} \quad & 2 + \ln 4 \\ &= \ln(e^2) + \ln 4 \\ &= \ln(e^2 \times 4) \\ &= \ln(4e^2) \end{aligned}$$

$$\begin{aligned} \mathbf{9} \quad \mathbf{a} \quad & 5 \ln 3 + \ln 4 \\ &= \ln(3^5) + \ln 4 \\ &= \ln(243 \times 4) \\ &= \ln 972 \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad & 3 \ln 4 - 2 \ln 2 \\ &= \ln(4^3) - \ln(2^2) \\ &= \ln\left(\frac{64}{4}\right) \\ &= \ln 16 \end{aligned}$$

$$\begin{aligned} \mathbf{g} \quad & -\ln 2 \\ &= \ln(2^{-1}) \\ &= \ln\left(\frac{1}{2}\right) \end{aligned}$$

$$\begin{aligned} \mathbf{j} \quad & 4 \ln 2 + 2 \\ &= \ln(2^4) + \ln(e^2) \\ &= \ln 16 + \ln(e^2) \\ &= \ln(16e^2) \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad & \ln 15 - \ln 3 \\ &= \ln\left(\frac{15}{3}\right) \\ &= \ln 5 \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad & \ln 5 + \ln(0.2) \\ &= \ln(5 \times 0.2) \\ &= \ln 1 \\ &= 0 \end{aligned}$$

$$\begin{aligned} \mathbf{h} \quad & \ln 6 - 1 \\ &= \ln 6 - \ln(e^1) \\ &= \ln\left(\frac{6}{e}\right) \end{aligned}$$

$$\begin{aligned} \mathbf{k} \quad & \ln 20 - 2 \\ &= \ln 20 - \ln(e^2) \\ &= \ln\left(\frac{20}{e^2}\right) \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad & 3 \ln 2 + 2 \ln 5 \\ &= \ln(2^3) + \ln(5^2) \\ &= \ln(8 \times 25) \\ &= \ln 200 \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad & \frac{1}{3} \ln 8 + \ln 3 \\ &= \ln(8^{\frac{1}{3}}) + \ln 3 \\ &= \ln(2 \times 3) \\ &= \ln 6 \end{aligned}$$

$$\begin{aligned} \mathbf{h} \quad & -\ln\left(\frac{1}{2}\right) \\ &= \ln\left(\left(\frac{1}{2}\right)^{-1}\right) \\ &= \ln 2 \end{aligned}$$

$$\begin{aligned} \mathbf{k} \quad & \frac{1}{2} \ln 9 - 1 \\ &= \ln(9^{\frac{1}{2}}) - \ln(e^1) \\ &= \ln 3 - \ln e \\ &= \ln\left(\frac{3}{e}\right) \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad & \ln 20 - \ln 5 \\ &= \ln\left(\frac{20}{5}\right) \\ &= \ln 4 \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad & \ln 2 + \ln 3 + \ln 5 \\ &= \ln(2 \times 3 \times 5) \\ &= \ln 30 \end{aligned}$$

$$\begin{aligned} \mathbf{i} \quad & \ln 5 + \ln 8 - \ln 2 \\ &= \ln(5 \times 8 \div 2) \\ &= \ln 20 \end{aligned}$$

$$\begin{aligned} \mathbf{l} \quad & \ln 12 - \ln 4 - \ln 3 \\ &= \ln(12 \div 4 \div 3) \\ &= \ln 1 \\ &= 0 \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad & 3 \ln 2 - \ln 8 \\ &= \ln(2^3) - \ln 8 \\ &= \ln\left(\frac{8}{8}\right) \\ &= \ln 1 \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad & \frac{1}{3} \ln\left(\frac{1}{27}\right) \\ &= \ln\left(\left(\frac{1}{27}\right)^{\frac{1}{3}}\right) \\ &= \ln\left(\frac{1}{27^{\frac{1}{3}}}\right) \\ &= \ln\left(\frac{1}{3}\right) \end{aligned}$$

$$\begin{aligned} \mathbf{i} \quad & -2 \ln\left(\frac{1}{4}\right) \\ &= \ln\left(\left(\frac{1}{4}\right)^{-2}\right) \\ &= \ln(4^2) \\ &= \ln 16 \end{aligned}$$

$$\begin{aligned} \mathbf{l} \quad & -3 \ln 2 + \frac{1}{2} \\ &= \ln(2^{-3}) + \ln(e^{\frac{1}{2}}) \\ &= \ln\left(\frac{1}{8}\right) + \ln \sqrt{e} \\ &= \ln\left(\frac{\sqrt{e}}{8}\right) \end{aligned}$$



$$\begin{aligned}
 10 \quad a \quad & \ln 27 \\
 &= \ln(3^3) \\
 &= 3 \ln 3
 \end{aligned}$$

$$\begin{aligned}
 b \quad & \ln \sqrt{3} \\
 &= \ln(3^{\frac{1}{2}}) \\
 &= \frac{1}{2} \ln 3
 \end{aligned}$$

$$\begin{aligned}
 c \quad & \ln\left(\frac{1}{16}\right) \\
 &= \ln\left(\frac{1}{2^4}\right) \\
 &= \ln(2^{-4}) \\
 &= -4 \ln 2
 \end{aligned}$$

$$\begin{aligned}
 d \quad & \ln\left(\frac{1}{6}\right) \\
 &= \ln(6^{-1}) \\
 &= -1 \times \ln 6 \\
 &= -\ln 6
 \end{aligned}$$

$$\begin{aligned}
 e \quad & \ln\left(\frac{1}{\sqrt{2}}\right) \\
 &= \ln(2^{-\frac{1}{2}}) \\
 &= -\frac{1}{2} \ln 2
 \end{aligned}$$

$$\begin{aligned}
 f \quad & \ln\left(\frac{e}{5}\right) \\
 &= \ln(e^1) - \ln 5 \\
 &= 1 - \ln 5
 \end{aligned}$$

$$\begin{aligned}
 g \quad & \ln(6e) \\
 &= \ln 6 + \ln e \\
 &= \ln 6 + 1
 \end{aligned}$$

$$\begin{aligned}
 h \quad & \ln \sqrt[3]{5} \\
 &= \ln(5^{\frac{1}{3}}) \\
 &= \frac{1}{3} \ln 5
 \end{aligned}$$

$$\begin{aligned}
 i \quad & \ln\left(\frac{1}{\sqrt[5]{2}}\right) \\
 &= \ln(2^{-\frac{1}{5}}) \\
 &= -\frac{1}{5} \ln 2
 \end{aligned}$$

$$\begin{aligned}
 j \quad & \ln\left(\frac{e^2}{8}\right) \\
 &= \ln(e^2) - \ln 8 \\
 &= \ln(e^2) - \ln(2^3) \\
 &= 2 - 3 \ln 2
 \end{aligned}$$

$$\begin{aligned}
 k \quad & \ln\left(\frac{\sqrt{3}}{e^4}\right) \\
 &= \ln(3^{\frac{1}{2}}) - \ln(e^4) \\
 &= \frac{1}{2} \ln 3 - 4
 \end{aligned}$$

$$\begin{aligned}
 l \quad & \ln\left(\frac{1}{16 \times \sqrt[3]{e}}\right) \\
 &= \ln 1 - \ln(16 \times \sqrt[3]{e}) \\
 &= 0 - (\ln 16 + \ln(e^{\frac{1}{3}})) \\
 &= -(\ln(2^4) + \frac{1}{3}) \\
 &= -(4 \ln 2 + \frac{1}{3}) \\
 &= -4 \ln 2 - \frac{1}{3}
 \end{aligned}$$

$$\begin{aligned}
 11 \quad & \ln\left(\frac{x}{y}\right) = 6 \\
 \therefore \ln x - \ln y &= 6 \\
 \therefore \ln x &= 6 + \ln y \quad \dots (1)
 \end{aligned}$$

$$\begin{aligned}
 & \text{Also, } \ln(x^3 y^4) = 4 \\
 \therefore \ln(x^3) + \ln(y^4) &= 4 \\
 \therefore 3 \ln x + 4 \ln y &= 4 \\
 \therefore 3(6 + \ln y) + 4 \ln y &= 4 \quad \{\text{using (1)}\} \\
 \therefore 18 + 3 \ln y + 4 \ln y &= 4 \\
 \therefore 7 \ln y &= -14 \\
 \therefore \ln y &= -2 \quad \dots (2) \\
 \therefore y &= e^{-2} \\
 \therefore y &= \frac{1}{e^2}
 \end{aligned}$$

$$\begin{aligned}
 \text{Substituting (2) into (1) gives } \ln x &= 6 + (-2) \\
 &= 4 \\
 \therefore x &= e^4
 \end{aligned}$$

## EXERCISE 3E

$$\begin{aligned} \mathbf{1} \quad \mathbf{a} \quad & y = 2^x \\ \therefore \log y &= \log(2^x) \\ \therefore \log y &= x \log 2 \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad & M = ad^4 \\ \therefore \log M &= \log(ad^4) \\ \therefore \log M &= \log a + \log(d^4) \\ \therefore \log M &= \log a + 4 \log d \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad & R = b\sqrt{l} \\ \therefore \log R &= \log(bl^{\frac{1}{2}}) \\ \therefore \log R &= \log b + \log(l^{\frac{1}{2}}) \\ \therefore \log R &= \log b + \frac{1}{2} \log l \end{aligned}$$

$$\begin{aligned} \mathbf{g} \quad & y = ab^x \\ \therefore \log y &= \log(ab^x) \\ \therefore \log y &= \log a + \log(b^x) \\ \therefore \log y &= \log a + x \log b \end{aligned}$$

$$\begin{aligned} \mathbf{i} \quad & L = \frac{ab}{c} \\ \therefore \log L &= \log\left(\frac{ab}{c}\right) \\ \therefore \log L &= \log(ab) - \log c \\ \therefore \log L &= \log a + \log b - \log c \end{aligned}$$

$$\begin{aligned} \mathbf{k} \quad & S = 200 \times 2^t \\ \therefore \log S &= \log(200 \times 2^t) \\ \therefore \log S &= \log 200 + \log(2^t) \\ \therefore \log S &= \log 200 + t \log 2 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad & y = 20b^3 \\ \therefore \log y &= \log(20b^3) \\ \therefore \log y &= \log 20 + \log(b^3) \\ \therefore \log y &= \log 20 + 3 \log b \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad & T = 5\sqrt{d} \\ \therefore \log T &= \log(5d^{\frac{1}{2}}) \\ \therefore \log T &= \log 5 + \log(d^{\frac{1}{2}}) \\ \therefore \log T &= \log 5 + \frac{1}{2} \log d \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad & Q = \frac{a}{b^n} \\ \therefore \log Q &= \log\left(\frac{a}{b^n}\right) \\ \therefore \log Q &= \log a - \log(b^n) \\ \therefore \log Q &= \log a - n \log b \end{aligned}$$

$$\begin{aligned} \mathbf{h} \quad & F = \frac{20}{\sqrt{n}} \\ \therefore \log F &= \log\left(\frac{20}{n^{\frac{1}{2}}}\right) \\ \therefore \log F &= \log 20 - \log(n^{\frac{1}{2}}) \\ \therefore \log F &= \log 20 - \frac{1}{2} \log n \end{aligned}$$

$$\begin{aligned} \mathbf{j} \quad & N = \sqrt{\frac{a}{b}} \\ \therefore N &= \left(\frac{a}{b}\right)^{\frac{1}{2}} \\ \therefore \log N &= \log\left(\left(\frac{a}{b}\right)^{\frac{1}{2}}\right) \\ \therefore \log N &= \frac{1}{2} \log\left(\frac{a}{b}\right) \\ \therefore \log N &= \frac{1}{2} \log a - \frac{1}{2} \log b \end{aligned}$$

$$\begin{aligned} \mathbf{l} \quad & y = \frac{a^m}{b^n} \\ \therefore \log y &= \log\left(\frac{a^m}{b^n}\right) \\ \therefore \log y &= \log(a^m) - \log(b^n) \\ \therefore \log y &= m \log a - n \log b \end{aligned}$$

$$\begin{aligned} \mathbf{2} \quad \mathbf{a} \quad & \log D = \log e + \log 2 \\ \therefore \log D &= \log(2e) \\ \therefore D &= 2e \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad & \log P = \frac{1}{2} \log x \\ \therefore \log P &= \log(x^{\frac{1}{2}}) \\ \therefore P &= \sqrt{x} \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad & \log B = 3 \log m - 2 \log n \\ \therefore \log B &= \log(m^3) - \log(n^2) \\ \therefore \log B &= \log\left(\frac{m^3}{n^2}\right) \\ \therefore B &= \frac{m^3}{n^2} \end{aligned}$$

$$\begin{aligned} \mathbf{g} \quad & \log P = 3 \log x + 1 \\ \therefore \log P &= \log(x^3) + \log(10^1) \\ \therefore \log P &= \log(10x^3) \\ \therefore P &= 10x^3 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad & \log_a F = \log_a 5 - \log_a t \\ \therefore \log_a F &= \log_a \left(\frac{5}{t}\right) \\ \therefore F &= \frac{5}{t} \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad & \log_n M = 2 \log_n b + \log_n c \\ \therefore \log_n M &= \log_n(b^2) + \log_n c \\ \therefore \log_n M &= \log_n(b^2 c) \\ \therefore M &= b^2 c \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad & \log N = -\frac{1}{3} \log p \\ \therefore \log N &= \log(p^{-\frac{1}{3}}) \\ \therefore \log N &= \log\left(\frac{1}{\sqrt[3]{p}}\right) \\ \therefore N &= \frac{1}{\sqrt[3]{p}} \end{aligned}$$

$$\begin{aligned} \mathbf{h} \quad & \log_a Q = 2 - \log_a x \\ \therefore \log_a Q &= \log_a(a^2) - \log_a x \\ \therefore \log_a Q &= \log_a\left(\frac{a^2}{x}\right) \\ \therefore Q &= \frac{a^2}{x} \end{aligned}$$

$$\begin{aligned} \mathbf{3} \quad \mathbf{a} \quad & \ln D = \ln x + 1 \\ \therefore \ln D - \ln x &= 1 \\ \therefore \ln\left(\frac{D}{x}\right) &= 1 \\ \therefore \frac{D}{x} &= e^1 \\ \therefore D &= ex \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad & \ln P = \frac{1}{2} \ln x \\ \therefore \ln P &= \ln(x^{\frac{1}{2}}) \\ \therefore P &= \sqrt{x} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad & \ln F = -\ln p + 2 \\ \therefore \ln F + \ln p &= 2 \\ \therefore \ln(Fp) &= 2 \\ \therefore Fp &= e^2 \\ \therefore F &= \frac{e^2}{p} \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad & \ln M = 2 \ln y + 3 \\ \therefore \ln M - 2 \ln y &= 3 \\ \therefore \ln M - \ln(y^2) &= 3 \\ \therefore \ln\left(\frac{M}{y^2}\right) &= 3 \\ \therefore \frac{M}{y^2} &= e^3 \\ \therefore M &= e^3 y^2 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & \ln B = 3 \ln t - 1 \\
 & \therefore \ln B - 3 \ln t = -1 \\
 & \therefore \ln B - \ln(t^3) = -1 \\
 & \therefore \ln\left(\frac{B}{t^3}\right) = -1 \\
 & \therefore \frac{B}{t^3} = e^{-1} \\
 & \therefore B = \frac{t^3}{e}
 \end{aligned}$$

$$\begin{aligned}
 \text{g} \quad & \ln Q \approx 3 \ln x + 2.159 \\
 & \therefore \ln Q - 3 \ln x \approx 2.159 \\
 & \therefore \ln Q - \ln(x^3) \approx 2.159 \\
 & \therefore \ln\left(\frac{Q}{x^3}\right) \approx 2.159 \\
 & \therefore \frac{Q}{x^3} \approx e^{2.159} \\
 & \therefore \frac{Q}{x^3} \approx 8.66 \\
 & \therefore Q \approx 8.66x^3
 \end{aligned}$$

$$\begin{aligned}
 \text{4 a} \quad & y = 3 \times 2^x \\
 & \therefore \log_2 y = \log_2(3 \times 2^x) \\
 & \therefore \log_2 y = \log_2 3 + \log_2(2^x) \\
 & \therefore \log_2 y = \log_2 3 + x
 \end{aligned}$$

$$\begin{aligned}
 \text{c i} \quad & \text{When } y = 3, \\
 & x = \log_2\left(\frac{3}{3}\right) \\
 & \therefore x = \log_2 1 \\
 & \therefore x = 0
 \end{aligned}$$

$$\begin{aligned}
 \text{ii} \quad & \text{When } y = 12, \\
 & x = \log_2\left(\frac{12}{3}\right) \\
 & \therefore x = \log_2 4 \\
 & \therefore x = \log_2(2^2) \\
 & \therefore x = 2
 \end{aligned}$$

$$\begin{aligned}
 \text{iii} \quad & \text{When } y = 30, \\
 & x = \log_2\left(\frac{30}{3}\right) \\
 & \therefore x = \log_2 10 \\
 & \therefore x \approx 3.32
 \end{aligned}$$

$$\begin{aligned}
 \text{5 a} \quad & \log_3 27 + \log_3\left(\frac{1}{3}\right) = \log_3 x \\
 & \therefore \log_3\left(27 \times \frac{1}{3}\right) = \log_3 x \\
 & \therefore \log_3 9 = \log_3 x \\
 & \therefore x = 9
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & \log_5 125 - \log_5 \sqrt{5} = \log_5 x \\
 & \therefore \log_5\left(\frac{125}{\sqrt{5}}\right) = \log_5 x \\
 & \therefore x = \frac{125}{\sqrt{5}} = 25\sqrt{5}
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & \ln N = -\frac{1}{3} \ln g \\
 & \therefore \ln N = \ln(g^{-\frac{1}{3}}) \\
 & \therefore N = g^{-\frac{1}{3}} \\
 & \therefore N = \frac{1}{\sqrt[3]{g}}
 \end{aligned}$$

$$\begin{aligned}
 \text{h} \quad & \ln D \approx 0.4 \ln n - 0.6582 \\
 & \therefore \ln D - 0.4 \ln n \approx -0.6582 \\
 & \therefore \ln D - \ln(n^{0.4}) \approx -0.6582 \\
 & \therefore \ln\left(\frac{D}{n^{0.4}}\right) \approx -0.6582 \\
 & \therefore \frac{D}{n^{0.4}} \approx e^{-0.6582} \\
 & \therefore \frac{D}{n^{0.4}} \approx 0.518 \\
 & \therefore D \approx 0.518n^{0.4}
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & \log_2 y = \log_2 3 + x \quad \{\text{from a}\} \\
 & \therefore x = \log_2 y - \log_2 3 \\
 & \therefore x = \log_2\left(\frac{y}{3}\right)
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & \log_5 x = \log_5 8 - \log_5(6 - x) \\
 & \therefore \log_5 x = \log_5\left(\frac{8}{6 - x}\right) \\
 & \therefore x = \frac{8}{6 - x} \quad \text{Note: } x > 0 \\
 & \therefore 6x - x^2 = 8 \quad \text{and } 6 - x > 0 \\
 & \therefore x^2 - 6x + 8 = 0 \quad \text{so } 0 < x < 6 \\
 & \therefore (x - 2)(x - 4) = 0 \\
 & \therefore x = 2 \text{ or } 4
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad & \log_{20} x = 1 + \log_{20} 10 \\
 & \therefore \log_{20} x = \log_{20}(20^1) + \log_{20} 10 \\
 & \quad = \log_{20} 200 \\
 & \therefore x = 200
 \end{aligned}$$



$$\begin{aligned}
 \text{e } \log x + \log(x+1) &= \log 30 \\
 \therefore \log[x(x+1)] &= \log 30 \\
 \therefore x^2 + x &= 30 \\
 \therefore x^2 + x - 30 &= 0 \\
 \therefore (x+6)(x-5) &= 0 \\
 \therefore x &= -6 \text{ or } 5 \\
 \therefore x &= 5 \quad \{x > 0\}
 \end{aligned}$$

$$\begin{aligned}
 \text{g } \log 24 &= \log 3 + x \log 2 \\
 \therefore \log 24 - \log 3 &= x \log 2 \\
 \therefore \log\left(\frac{24}{3}\right) &= x \log 2 \\
 \therefore \log 8 &= x \log 2 \\
 \therefore \log(2^3) &= \log(2^x) \\
 \therefore 2^3 &= 2^x \\
 \therefore x &= 3
 \end{aligned}$$

$$\begin{aligned}
 \text{f } \log(x+2) - \log(x-2) &= \log 5 \\
 \therefore \log\left(\frac{x+2}{x-2}\right) &= \log 5 \\
 \therefore \frac{x+2}{x-2} &= 5 \\
 \therefore x+2 &= 5x-10 \\
 \therefore -4x &= -12 \\
 \therefore x &= 3
 \end{aligned}$$

**Note:**  $x+2 > 0$  and  $x-2 > 0$

$$\therefore x > 2 \quad \checkmark$$

$$\begin{aligned}
 \text{h } x \log_2 3 + \log_2 36 &= 2 \\
 \therefore \log_2(3^x) + \log_2 36 &= 2 \\
 \therefore \log_2(3^x \times 36) &= \log_2(2^2) \\
 \therefore 3^x \times 36 &= 4 \\
 \therefore 3^x &= \frac{4}{36} \\
 \therefore 3^x &= \frac{1}{9} \\
 \therefore 3^x &= 3^{-2} \\
 \therefore x &= -2
 \end{aligned}$$

$$\begin{aligned}
 \text{6 a } \log_x y &= 2 \\
 \therefore \log_x y &= \log_x(x^2) \\
 \therefore y &= x^2 \quad \dots (*) \\
 \text{Also, } \log_{y-2} x &= 1 \\
 \therefore \log_{y-2} x &= \log_{y-2}[(y-2)^1] \\
 \therefore x &= y-2 \\
 \therefore x &= x^2 - 2 \quad \{\text{using } (*)\} \\
 \therefore x^2 - x - 2 &= 0 \\
 \therefore (x+1)(x-2) &= 0 \\
 \therefore x &= -1 \text{ or } 2 \\
 \therefore x &= 2 \quad \{\text{as } x > 0 \text{ for } \log_{y-2} x \text{ to exist}\}
 \end{aligned}$$

Substituting  $x = 2$  into  $(*)$  gives  $y = 2^2$

$$\therefore y = 4$$

So,  $x = 2$  and  $y = 4$ .

**b**  $\log_x y = 3$

$$\therefore \log_x y = \log_x (x^3)$$

$$\therefore y = x^3 \quad \dots (*)$$

Also,  $\log_{y+1}(x+1) = \frac{1}{2}$

$$\therefore \log_{y+1}(x+1) = \log_{y+1}(y+1)^{\frac{1}{2}}$$

$$\therefore x+1 = (y+1)^{\frac{1}{2}}$$

$$\therefore x+1 = \sqrt{y+1}$$

$$\therefore x+1 = \sqrt{x^3+1} \quad \{\text{using } (*)\}$$

$$\therefore (x+1)^2 = x^3 + 1$$

$$\therefore x^2 + 2x + 1 = x^3 + 1$$

$$\therefore x^3 - x^2 - 2x = 0$$

$$\therefore x(x^2 - x - 2) = 0$$

$$\therefore x^2 - x - 2 = 0 \quad \{\text{as } y > 0, \text{ hence } x > 0 \text{ from } (*), \text{ for } \log_x y \text{ to exist}\}$$

$$\therefore (x+1)(x-2) = 0$$

$$\therefore x = -1 \text{ or } 2$$

$$\therefore x = 2 \quad \{\text{as } x+1 > 0 \text{ for } \log_{y+1}(x+1) \text{ to exist}\}$$

Substituting  $x = 2$  into  $(*)$  gives  $y = 2^3$

$$\therefore y = 8$$

So,  $x = 2$  and  $y = 8$ .

**7 a**  $x = \log_2 7$

$$\therefore 2^x = 2^{\log_2 7}$$

$$\therefore 2^x = 7$$

**b**  $\log(2^x) = \log 7$

$$\therefore x \log 2 = \log 7$$

$$\therefore x = \frac{\log 7}{\log 2}$$

But  $x = \log_2 7 \quad \{\text{from a}\}$

$$\therefore \log_2 7 = \frac{\log 7}{\log 2} \approx 2.81$$

**8 a**  $a^x = b, \quad a, b > 0$

$$\therefore \log_a(a^x) = \log_a b$$

$$\therefore x = \log_a b$$

**c** Using **b**,  $x \log a = \log b$

$$\therefore x = \frac{\log b}{\log a}$$

and using part **a**,  $x = \log_a b = \frac{\log b}{\log a}$

**b**  $a^x = b$

$$\therefore \log(a^x) = \log b$$

## EXERCISE 3F

1 a  $\log_3 7$

$$= \frac{\log 7}{\log 3}$$

$$\approx 1.77$$

Check:  $\frac{\ln 7}{\ln 3} \approx 1.77 \quad \checkmark$

b  $\log_2 40$

$$= \frac{\log 40}{\log 2}$$

$$\approx 5.32$$

Check:  $\frac{\ln 40}{\ln 2} \approx 5.32 \quad \checkmark$

c  $\log_5 180$

$$= \frac{\log 180}{\log 5}$$

$$\approx 3.23$$

Check:  $\frac{\ln 180}{\ln 5} \approx 3.23 \quad \checkmark$

d  $\log_{\frac{1}{2}} 1250$

$$= \frac{\log 1250}{\log \frac{1}{2}}$$

$$\approx -10.3$$

Check:  $\frac{\ln 1250}{\ln \frac{1}{2}} \approx -10.3 \quad \checkmark$

e  $\log_3(0.067)$

$$= \frac{\log(0.067)}{\log 3}$$

$$\approx -2.46$$

Check:  $\frac{\ln(0.067)}{\ln 3} \approx -2.46 \quad \checkmark$

f  $\log_{0.4}(0.006\,984)$

$$= \frac{\log(0.006\,984)}{\log(0.4)}$$

$$\approx 5.42$$

Check:  $\frac{\ln(0.006\,984)}{\ln(0.4)} \approx 5.42 \quad \checkmark$

2  $\log_m n \times \log_n(m^2)$

$$= \frac{\log_n n}{\log_n m} \times \log_n(m^2) \quad \left\{ \log_b a = \frac{\log_c a}{\log_c b} \right\}$$

$$= \frac{1}{\log_n m} \times 2 \log_n m$$

$$= 1 \times 2$$

$$= 2$$

3  $\frac{4}{\log_5 4} + \frac{3}{\log_7 8} = \frac{4}{\frac{\log 4}{\log 5}} + \frac{3}{\frac{\log 8}{\log 7}} \quad \left\{ \log_b a = \frac{\log_c a}{\log_c b} \right\}$

$$= \frac{4 \log 5}{\log 4} + \frac{3 \log 7}{\log 8}$$

$$= \frac{4 \log 5}{2 \log 2} + \frac{3 \log 7}{3 \log 2}$$

$$= \frac{2 \log 5}{\log 2} + \frac{\log 7}{\log 2}$$

$$= \frac{\log(5^2)}{\log 2} + \frac{\log 7}{\log 2}$$

$$= \log_2 25 + \log_2 7 \quad \left\{ \frac{\log_c a}{\log_c b} = \log_b a \right\}$$

$$= \log_2(25 \times 7) \quad \{ \log_c(ab) = \log_c a + \log_c b \}$$

$$= \log_2 175$$

$$\therefore 2^{\frac{4}{\log_5 4} + \frac{3}{\log_7 8}} = 2^{\log_2 175} = 175$$

$$\begin{aligned}
 \text{4 a} \quad & \log_4(x^3) + \log_2 \sqrt{x} = 8 \\
 & \therefore \frac{\log_2(x^3)}{\log_2 4} + \log_2(x^{\frac{1}{2}}) = 8 \quad \left\{ \log_b a = \frac{\log_c a}{\log_c b} \right\} \\
 & \therefore \frac{1}{2} \times 3 \log_2 x + \frac{1}{2} \log_2 x = 8 \quad \{m \log_a b = \log_a(b^m)\} \\
 & \therefore 2 \log_2 x = 8 \\
 & \therefore \log_2 x = 4 \\
 & \therefore x = 2^4 \\
 & \therefore x = 16
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & \log_{\frac{1}{9}} x = \log_9 5 \\
 & \therefore \frac{\log x}{\log(\frac{1}{9})} = \frac{\log 5}{\log 9} \quad \left\{ \log_a b = \frac{\log b}{\log a} \right\} \\
 & \therefore \frac{\log x}{\log(9^{-1})} = \frac{\log 5}{\log 9} \\
 & \therefore \frac{\log x}{-\log 9} = \frac{\log 5}{\log 9} \\
 & \therefore \log x = -\log 5 \\
 & \therefore \log x = \log(5^{-1}) \\
 & \therefore x = \frac{1}{5}
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & \log_{16}(x^5) = \log_{64} 125 - \log_4 \sqrt{x} \\
 & \therefore \frac{\log_4(x^5)}{\log_4 16} = \frac{\log_4 125}{\log_4 64} - \log_4(x^{\frac{1}{2}}) \quad \left\{ \log_b a = \frac{\log_c a}{\log_c b} \right\} \\
 & \therefore \frac{1}{2} \times 5 \log_4 x = \frac{1}{3} \log_4 125 - \frac{1}{2} \log_4 x \\
 & \therefore 3 \log_4 x = \log_4(125^{\frac{1}{3}}) \\
 & \therefore \log_4 x = \frac{1}{3} \log_4 5 \\
 & \therefore \log_4 x = \log_4(5^{\frac{1}{3}}) \\
 & \therefore x = \sqrt[3]{5} \approx 1.71
 \end{aligned}$$



**d**

$$\log_3(x^3) - 4\log_9 x - 5\log_{27} \sqrt{x} = \log_9 4$$

$$\therefore 3\log_3 x - 4\log_9 x - 5\log_{27}(x^{\frac{1}{2}}) = \log_9 4$$

$$\therefore 3\log_3 x - 4\log_9 x - \frac{5}{2}\log_{27} x = \log_9 4$$

$$\therefore 3\log_3 x - 4\left(\frac{\log_3 x}{\log_3 9}\right) - \frac{5}{2}\left(\frac{\log_3 x}{\log_3 27}\right) = \frac{\log_3 4}{\log_3 9} \quad \left\{ \log_b a = \frac{\log_c a}{\log_c b} \right\}$$

$$\therefore 3\log_3 x - 4\left(\frac{\log_3 x}{\log_3(3^2)}\right) - \frac{5}{2}\left(\frac{\log_3 x}{\log_3(3^3)}\right) = \frac{\log_3 4}{\log_3(3^2)}$$

$$\therefore 3\log_3 x - 4\left(\frac{\log_3 x}{2}\right) - \frac{5}{2}\left(\frac{\log_3 x}{3}\right) = \frac{\log_3 4}{2}$$

$$\therefore 3\log_3 x - 2\log_3 x - \frac{5}{6}\log_3 x = \frac{1}{2}\log_3 4$$

$$\therefore \frac{1}{6}\log_3 x = \frac{1}{2}\log_3 4$$

$$\therefore \log_3 x = 3\log_3 4$$

$$\therefore \log_3 x = \log_3(4^3)$$

$$\therefore x = 4^3$$

$$\therefore x = 64$$

**e**

$$\log_x 4 + \log_2 x = 3$$

$$\therefore \frac{\log_2 4}{\log_2 x} + \log_2 x = 3 \quad \left\{ \log_b a = \frac{\log_c a}{\log_c b} \right\}$$

$$\therefore \frac{\log_2(2^2)}{\log_2 x} + \log_2 x = 3$$

$$\therefore \frac{2}{\log_2 x} + \log_2 x = 3$$

$$\therefore 2 + (\log_2 x)^2 = 3\log_2 x$$

$$\therefore (\log_2 x)^2 - 3\log_2 x + 2 = 0$$

$$\therefore (\log_2 x - 1)(\log_2 x - 2) = 0$$

$$\therefore \log_2 x = 1 \quad \text{or} \quad \log_2 x = 2$$

$$\therefore x = 2^1 \quad \text{or} \quad x = 2^2$$

$$\therefore x = 2 \text{ or } 4$$

$$\begin{aligned}
 5 \quad x &= \log_3(y^2) \\
 &= 2\log_3 y \\
 &= \frac{2\log_y y}{\log_y 3} \quad \left\{ \log_a b = \frac{\log_c a}{\log_c b} \right\}
 \end{aligned}$$

$$\therefore \log_y 3 = \frac{2}{x} \quad \dots (*)$$

$$\begin{aligned}
 \text{Thus } \log_y 81 &= \log_y(3^4) \\
 &= 4\log_y 3 \\
 &= 4\left(\frac{2}{x}\right) \quad \{\text{using } (*)\} \\
 &= \frac{8}{x}
 \end{aligned}$$

$$\begin{aligned}
 6 \quad m &= \log_4 3 \\
 &= \frac{\log_2 3}{\log_2 4} \quad \left\{ \log_b a = \frac{\log_c a}{\log_c b} \right\} \\
 &= \frac{\log_2 3}{\log_2(2^2)} \\
 &= \frac{\log_2 3}{2}
 \end{aligned}$$

$$\therefore \log_2 3 = 2m \quad \dots (*)$$

$$\begin{aligned}
 \text{Thus } \log_2 24 &= \log_2(8 \times 3) \\
 &= \log_2 8 + \log_2 3 \\
 &= \log_2(2^3) + 2m \quad \{\text{using } (*)\} \\
 &= 3\log_2 2 + 2m \\
 &= 3(1) + 2m \\
 &= 2m + 3
 \end{aligned}$$

$$\begin{aligned}
 7 \quad \text{a} \quad \text{When } x = 9, \quad \log_9(\log_3 x) &= \log_9(\log_3 9) \\
 &= \log_9(\log_3(3^2)) \\
 &= \log_9 2 \\
 \text{and } \log_3(\log_9 x) &= \log_3(\log_9 9) \\
 &= \log_3 1 \\
 &= 0
 \end{aligned}$$

Since  $\log_9 2 \neq 0$ ,  $\log_9(\log_3 x) \neq \log_3(\log_9 x)$  when  $x = 9$ .

So,  $\log_9(\log_3 x)$  and  $\log_3(\log_9 x)$  are not always equal.

$$\begin{aligned}
\text{b} \quad & \log_9(\log_3 x) = \log_3(\log_9 x) \\
\therefore & \frac{\log_3(\log_3 x)}{\log_3 9} = \log_3(\log_9 x) \quad \left\{ \log_b a = \frac{\log_c a}{\log_c b} \right\} \\
\therefore & \frac{\log_3(\log_3 x)}{\log_3(3^2)} = \log_3(\log_9 x) \\
\therefore & \frac{\log_3(\log_3 x)}{2} = \log_3(\log_9 x) \\
\therefore & \log_3(\log_3 x) = 2 \log_3(\log_9 x) \\
\therefore & \log_3(\log_3 x) = \log_3((\log_9 x)^2) \\
\therefore & \log_3 x = (\log_9 x)^2 \\
\therefore & \log_3 x = \left( \frac{\log_3 x}{\log_3 9} \right)^2 \quad \left\{ \log_b a = \frac{\log_c a}{\log_c b} \right\} \\
\therefore & \log_3 x = \left( \frac{\log_3 x}{\log_3(3^2)} \right)^2 \\
\therefore & \log_3 x = \left( \frac{\log_3 x}{2} \right)^2 \\
\therefore & \log_3 x = \frac{(\log_3 x)^2}{4} \\
\therefore & 4 \log_3 x = (\log_3 x)^2 \\
\therefore & \frac{4 \log_3 x}{\log_3 x} = \log_3 x \quad \{\text{as } \log_3 x > 0 \text{ for } \log_9(\log_3 x) \text{ to exist}\} \\
\therefore & 4 = \log_3 x \\
\therefore & x = 3^4 \\
\therefore & x = 81
\end{aligned}$$

$$\begin{aligned}
\text{c} \quad & \log_{a^2}(\log_a x) = \log_a(\log_{a^2} x) \\
\therefore & \frac{\log_a(\log_a x)}{\log_a(a^2)} = \log_a(\log_{a^2} x) \quad \left\{ \log_b a = \frac{\log_c a}{\log_c b} \right\} \\
\therefore & \frac{\log_a(\log_a x)}{2} = \log_a(\log_{a^2} x) \\
\therefore & \log_a(\log_a x) = 2 \log_a(\log_{a^2} x) \\
\therefore & \log_a(\log_a x) = \log_a((\log_{a^2} x)^2) \\
\therefore & \log_a x = (\log_{a^2} x)^2 \\
\therefore & \log_a x = \left( \frac{\log_a x}{\log_a(a^2)} \right)^2 \quad \left\{ \log_b a = \frac{\log_c a}{\log_c b} \right\} \\
\therefore & \log_a x = \left( \frac{\log_a x}{2} \right)^2 \\
\therefore & \log_a x = \frac{(\log_a x)^2}{4} \\
\therefore & 4 \log_a x = (\log_a x)^2 \\
\therefore & \frac{4 \log_a x}{\log_a x} = \log_a x \quad \{\text{as } \log_a x > 0 \text{ for } \log_{a^2}(\log_a x) \text{ to exist}\} \\
\therefore & 4 = \log_a x \\
\therefore & x = a^4
\end{aligned}$$

$$\mathbf{d} \quad \log_{a^k}(\log_a x) = \log_a(\log_{a^k} x), \quad k \neq 1$$

$$\therefore \frac{\log_a(\log_a x)}{\log_a(a^k)} = \log_a(\log_{a^k} x) \quad \left\{ \log_b a = \frac{\log_c a}{\log_c b} \right\}$$

$$\therefore \frac{\log_a(\log_a x)}{k} = \log_a(\log_{a^k} x)$$

$$\therefore \log_a(\log_a x) = k \log_a(\log_{a^k} x)$$

$$\therefore \log_a(\log_a x) = \log_a((\log_{a^k} x)^k)$$

$$\therefore \log_a x = (\log_{a^k} x)^k$$

$$\therefore \log_a x = \left( \frac{\log_a x}{\log_a(a^k)} \right)^k \quad \left\{ \log_b a = \frac{\log_c a}{\log_c b} \right\}$$

$$\therefore \log_a x = \left( \frac{\log_a x}{k} \right)^k$$

$$\therefore \log_a x = \frac{(\log_a x)^k}{k^k}$$

$$\therefore k^k \log_a x = (\log_a x)^k$$

$$\therefore \frac{k^k \log_a x}{\log_a x} = \frac{(\log_a x)^k}{\log_a x} \quad \{\text{as } \log_a x > 0 \text{ for } \log_{a^k}(\log_a x) \text{ to exist}\}$$

$$\therefore k^k = (\log_a x)^{k-1}$$

$$\therefore (k^k)^{\frac{1}{k-1}} = ((\log_a x)^{k-1})^{\frac{1}{k-1}}$$

$$\therefore k^{\frac{k}{k-1}} = (\log_a x)^{\frac{k-1}{k-1}}$$

$$\therefore \log_a x = k^{\frac{k}{k-1}} \quad \dots (*)$$

So, LHS

$$= \log_{a^k}(\log_a x)$$

$$= \log_{a^k} \left( k^{\frac{k}{k-1}} \right) \quad \{\text{using } (*)\}$$

$$= \frac{\log_a \left( k^{\frac{k}{k-1}} \right)}{\log_a(a^k)} \quad \left\{ \log_b a = \frac{\log_c a}{\log_c b} \right\}$$

$$= \frac{\log_a \left( k^{\frac{k}{k-1}} \right)}{k}$$

$$= \frac{1}{k} \log_a \left( k^{\frac{k}{k-1}} \right)$$

$$= \log_a \left( \left( k^{\frac{k}{k-1}} \right)^{\frac{1}{k}} \right)$$

$$= \log_a \left( k^{\frac{1}{k-1}} \right)$$

and RHS

$$= \log_a(\log_{a^k} x)$$

$$= \log_a \left( \frac{\log_a x}{\log_a(a^k)} \right) \quad \left\{ \log_b a = \frac{\log_c a}{\log_c b} \right\}$$

$$= \log_a \left( \frac{\log_a x}{k} \right)$$

$$= \log_a \left( \frac{k^{\frac{k}{k-1}}}{k} \right) \quad \{\text{using } (*)\}$$

$$= \log_a \left( k^{\frac{k}{k-1} - 1} \right)$$

$$= \log_a \left( k^{\frac{k-(k-1)}{k-1}} \right)$$

$$= \log_a \left( k^{\frac{1}{k-1}} \right)$$

$$= \text{LHS} \quad \checkmark$$



## EXERCISE 3G

**1 a**  $3^3 = 27$  and  $3^4 = 81$

Since  $27 < 40 < 81$ , then  $3^3 < 40 < 3^4$ , and the solution to  $3^x = 40$  lies between  $x = 3$  and  $x = 4$ .

**c**  $x = \frac{\log 40}{\log 3} \approx 3.36$

**b**  $3^x = 40$

$$\therefore \log(3^x) = \log 40$$

$$\therefore x \log 3 = \log 40$$

$$\therefore x = \frac{\log 40}{\log 3}$$

**2 a i**  $2^x = 10$

$$\therefore \log(2^x) = \log 10$$

$$\therefore x \log 2 = \log 10^1$$

$$\therefore x = \frac{1}{\log 2}$$

**ii**  $x = \frac{1}{\log 2} \approx 3.32$

**c i**  $4^x = 50$

$$\therefore \log(4^x) = \log 50$$

$$\therefore x \log 4 = \log 50$$

$$\therefore x = \frac{\log 50}{\log 4}$$

**ii**  $x = \frac{\log 50}{\log 4} \approx 2.82$

**e i**  $\left(\frac{3}{4}\right)^x = 0.1$

$$\therefore \log\left(\frac{3}{4}\right)^x = \log(10^{-1})$$

$$\therefore x \log\left(\frac{3}{4}\right) = -1$$

$$\therefore x = -\frac{1}{\log\left(\frac{3}{4}\right)}$$

**ii**  $x = -\frac{1}{\log\left(\frac{3}{4}\right)} \approx 8.00$

**b i**  $3^x = 20$

$$\therefore \log(3^x) = \log 20$$

$$\therefore x \log 3 = \log 20$$

$$\therefore x = \frac{\log 20}{\log 3}$$

**ii**  $x = \frac{\log 20}{\log 3} \approx 2.73$

**d i**  $\left(\frac{1}{2}\right)^x = 0.0625$

$$\therefore \log\left(\frac{1}{2}\right)^x = \log\left(\frac{1}{16}\right)$$

$$\therefore x \log(2^{-1}) = \log(2^{-4})$$

$$\therefore x = \frac{-4 \log 2}{-\log 2}$$

$$\therefore x = 4$$

**ii**  $x = 4$

**f i**  $10^x = 0.000\,015$

$$\therefore \log 10^x = \log(0.000\,015)$$

$$\therefore x \log 10 = \log(0.000\,015)$$

$$\therefore x = \log(0.000\,015)$$

**ii**  $x = \log(0.000\,015)$   
 $\approx -4.82$

**3 a**  $5^x = 40$

$$\therefore \log(5^x) = \log 40$$

$$\therefore x \log 5 = \log 40$$

$$\therefore x = \frac{\log 40}{\log 5} \approx 2.29$$

**b**  $3^x = 2^{x+3}$

$$\therefore \log(3^x) = \log(2^x \times 2^3)$$

$$\therefore \log(3^x) = \log(2^x) + \log(2^3)$$

$$\therefore x \log 3 = x \log 2 + 3 \log 2$$

$$\therefore x(\log 3 - \log 2) = 3 \log 2$$

$$\therefore x = \frac{3 \log 2}{\log 3 - \log 2} \approx 5.13$$

$$\begin{aligned}
 & \mathbf{c} \quad 2^{x+4} = 5^{2-x} \\
 & \quad \therefore \log(2^x \times 2^4) = \log(5^2 \times 5^{-x}) \\
 & \therefore \log(2^x) + \log(2^4) = \log(5^2) + \log(5^{-x}) \\
 & \therefore x \log 2 + 4 \log 2 = 2 \log 5 - x \log 5 \\
 & \therefore x(\log 2 + \log 5) = 2 \log 5 - 4 \log 2 \\
 & \quad \therefore x = \frac{2 \log 5 - 4 \log 2}{\log 2 + \log 5} \approx 0.194
 \end{aligned}$$

$$\begin{aligned}
 & \mathbf{4} \quad \mathbf{a} \quad e^x = 10 \\
 & \quad \therefore x = \ln 10
 \end{aligned}$$

$$\begin{aligned}
 & \mathbf{b} \quad e^x = 1000 \\
 & \quad \therefore x = \ln 1000
 \end{aligned}$$

$$\begin{aligned}
 & \mathbf{c} \quad 2e^x = 0.3 \\
 & \quad \therefore e^x = 0.15 \\
 & \quad \therefore x = \ln(0.15)
 \end{aligned}$$

$$\begin{aligned}
 & \mathbf{d} \quad e^{\frac{x}{2}} = 5 \\
 & \quad \therefore \frac{x}{2} = \ln 5 \\
 & \quad \therefore x = 2 \ln 5
 \end{aligned}$$

$$\begin{aligned}
 & \mathbf{e} \quad e^{2x} = 18 \\
 & \quad \therefore 2x = \ln 18 \\
 & \quad \therefore x = \frac{1}{2} \ln 18
 \end{aligned}$$

$$\begin{aligned}
 & \mathbf{f} \quad e^{-\frac{x}{2}} = 1 \\
 & \quad \therefore -\frac{x}{2} = \ln 1 \\
 & \quad \therefore -\frac{x}{2} = 0 \\
 & \quad \therefore x = 0
 \end{aligned}$$

$$\begin{aligned}
 & \mathbf{5} \quad \mathbf{a} \quad 3 \times 2^x = 75 \\
 & \quad \therefore 2^x = 25 \\
 & \therefore \log(2^x) = \log 25 \\
 & \therefore x \log 2 = \log 25 \\
 & \quad \therefore x = \frac{\log 25}{\log 2}
 \end{aligned}$$

$$\begin{aligned}
 & \mathbf{b} \quad 7 \times (1.5)^x = 20 \\
 & \quad \therefore (1.5)^x = \frac{20}{7} \\
 & \therefore \log((1.5)^x) = \log\left(\frac{20}{7}\right) \\
 & \therefore x \log(1.5) = \log\left(\frac{20}{7}\right) \\
 & \quad \therefore x = \frac{\log\left(\frac{20}{7}\right)}{\log(1.5)}
 \end{aligned}$$

$$\begin{aligned}
 & \mathbf{c} \quad 5 \times (0.8)^x = 3 \\
 & \quad \therefore (0.8)^x = 0.6 \\
 & \therefore \log((0.8)^x) = \log(0.6) \\
 & \therefore x \log(0.8) = \log(0.6) \\
 & \quad \therefore x = \frac{\log(0.6)}{\log(0.8)}
 \end{aligned}$$

$$\begin{aligned}
 & \mathbf{d} \quad 4 \times 2^{-x} = 0.12 \\
 & \quad \therefore 2^{-x} = 0.03 \\
 & \therefore \log(2^{-x}) = \log(0.03) \\
 & \therefore -x \log 2 = \log(0.03) \\
 & \quad \therefore x = -\frac{\log(0.03)}{\log 2}
 \end{aligned}$$

$$\begin{aligned}
 & \mathbf{e} \quad 300 \times 5^{0.1x} = 1000 \\
 & \quad \therefore 5^{0.1x} = \frac{10}{3} \\
 & \therefore \log(5^{0.1x}) = \log\left(\frac{10}{3}\right) \\
 & \therefore 0.1x \log 5 = \log\left(\frac{10}{3}\right) \\
 & \therefore x \log 5 = 10 \log\left(\frac{10}{3}\right) \\
 & \quad \therefore x = \frac{10 \log\left(\frac{10}{3}\right)}{\log 5}
 \end{aligned}$$

$$\begin{aligned}
 & \mathbf{f} \quad 32 \times e^{-0.25x} = 4 \\
 & \quad \therefore e^{-0.25x} = \frac{1}{8} \\
 & \therefore \ln(e^{-0.25x}) = \ln\left(\frac{1}{8}\right) \\
 & \therefore -0.25x \ln e = \ln 1 - \ln 8 \\
 & \therefore -0.25x = 0 - \ln 8 \\
 & \quad \therefore x = 4 \ln 8
 \end{aligned}$$

$$\begin{aligned}
 \text{6 a} \quad & 25^x - 3 \times 5^x = 0 \\
 & \therefore 5^x(5^x - 3) = 0 \\
 & \therefore 5^x = 0 \text{ or } 5^x - 3 = 0 \\
 & \quad \therefore 5^x = 3 \quad \{5^x \neq 0\} \\
 & \therefore \log(5^x) = \log 3 \\
 & \therefore x \log 5 = \log 3 \\
 & \therefore x = \frac{\log 3}{\log 5}
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & 8 \times 9^x - 3^x = 0 \\
 & \therefore 3^x(8 \times 3^x - 1) = 0 \\
 & \therefore 3^x = 0 \text{ or } 8 \times 3^x - 1 = 0 \\
 & \quad \therefore 8 \times 3^x = 1 \quad \{3^x \neq 0\} \\
 & \quad \therefore 3^x = \frac{1}{8} \\
 & \therefore \log(3^x) = \log\left(\frac{1}{8}\right) \\
 & \therefore x \log 3 = \log\left(\frac{1}{8}\right) \\
 & \quad \therefore x = \frac{\log(8^{-1})}{\log 3} \\
 & \quad \therefore x = -\frac{\log 8}{\log 3}
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & 2^x - 2 \times 4^x = 0 \\
 & \therefore 2^x(1 - 2 \times 2^x) = 0 \\
 & \therefore 2^x = 0 \text{ or } 1 - 2 \times 2^x = 0 \\
 & \quad \therefore 1 = 2 \times 2^x \quad \{2^x \neq 0\} \\
 & \quad \therefore 2^x = \frac{1}{2} \\
 & \quad \therefore 2^x = 2^{-1} \\
 & \quad \therefore x = -1
 \end{aligned}$$

$$\begin{aligned}
 \text{7} \quad & 3^{2x} = \frac{1}{16} \\
 & \therefore \ln(3^{2x}) = \ln\left(\frac{1}{16}\right) \\
 & \therefore 2x \ln 3 = \ln(2^{-4}) \\
 & \therefore 2x = \frac{-4 \ln 2}{\ln 3} \\
 & \therefore x = -\frac{2 \ln 2}{\ln 3}
 \end{aligned}$$

$$\begin{aligned}
 \text{8} \quad & 10^{2x} = 4^{x+3} \\
 & \therefore \ln(10^{2x}) = \ln(4^{x+3}) \\
 & \therefore 2x \ln 10 = (x+3) \ln 4 \\
 & \therefore 2x \ln 10 = x \ln 4 + 3 \ln 4 \\
 & \therefore 2x \ln 10 - x \ln 4 = 3 \ln 4 \\
 & \therefore x(2 \ln 10 - \ln 4) = 3 \ln 4 \\
 & \quad \therefore x = \frac{3 \ln 4}{2 \ln 10 - \ln 4} \\
 & \quad \therefore x = \frac{3 \ln(2^2)}{2 \ln(2 \times 5) - \ln(2^2)} \\
 & \quad \therefore x = \frac{6 \ln 2}{2 \ln 2 + 2 \ln 5 - 2 \ln 2} \\
 & \quad \therefore x = \frac{6 \ln 2}{2 \ln 5} \\
 & \quad \therefore x = \frac{3 \ln 2}{\ln 5}
 \end{aligned}$$

$$\begin{aligned}
 \text{9 a} \quad & e^{2x} = 2e^x \\
 & \therefore e^{2x} - 2e^x = 0 \\
 & \therefore e^x(e^x - 2) = 0 \\
 & \quad \therefore e^x = 0 \text{ or } 2 \\
 & \quad \therefore x = \ln 2 \quad \{e^x > 0\}
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & e^x = e^{-x} \\
 & \therefore e^{2x} = 1 \\
 & \therefore 2x = 0 \\
 & \therefore x = 0
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & e^{2x} - 5e^x + 6 = 0 \\
 \therefore & (e^x - 2)(e^x - 3) = 0 \\
 \therefore & e^x = 2 \text{ or } 3 \\
 \therefore & x = \ln 2 \text{ or } \ln 3
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad & e^x + 2 = 3e^{-x} \\
 \therefore & e^{2x} + 2e^x = 3 \\
 \therefore & e^{2x} + 2e^x - 3 = 0 \\
 \therefore & (e^x + 3)(e^x - 1) = 0 \\
 \therefore & e^x = -3 \text{ or } 1 \\
 \therefore & e^x = 1 \quad \{e^x > 0\} \\
 \therefore & x = 0
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & 1 + 12e^{-x} = e^x \\
 \therefore & e^x + 12 = e^{2x} \\
 \therefore & e^{2x} - e^x - 12 = 0 \\
 \therefore & (e^x - 4)(e^x + 3) = 0 \\
 \therefore & e^x = 4 \text{ or } -3 \\
 \therefore & e^x = 4 \quad \{e^x > 0\} \\
 \therefore & x = \ln 4
 \end{aligned}$$

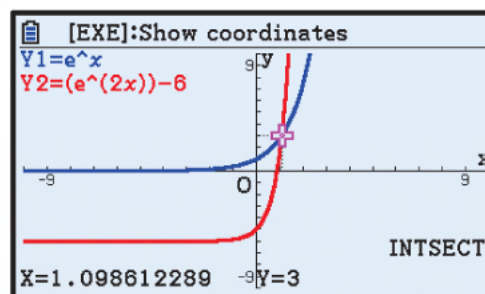
$$\begin{aligned}
 \text{f} \quad & e^x + e^{-x} = 3 \\
 \therefore & e^{2x} + 1 = 3e^x \\
 \therefore & e^{2x} - 3e^x + 1 = 0 \\
 \therefore & e^x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(1)}}{2(1)} \\
 & = \frac{3 \pm \sqrt{5}}{2} \\
 \therefore & x = \ln\left(\frac{3 + \sqrt{5}}{2}\right) \text{ or } \ln\left(\frac{3 - \sqrt{5}}{2}\right)
 \end{aligned}$$

10 a The functions  $y = e^x$  and  $y = e^{2x} - 6$  meet where

$$\begin{aligned}
 & e^x = e^{2x} - 6 \\
 \therefore & e^{2x} - e^x - 6 = 0 \\
 \therefore & (e^x + 2)(e^x - 3) = 0 \\
 \therefore & e^x = -2 \text{ or } 3 \\
 \therefore & e^x = 3 \quad \{e^x > 0\} \\
 \therefore & x = \ln 3
 \end{aligned}$$

When  $x = \ln 3$ ,  $y = e^{\ln 3} = 3$

$\therefore$  the functions meet at  $(\ln 3, 3)$ .

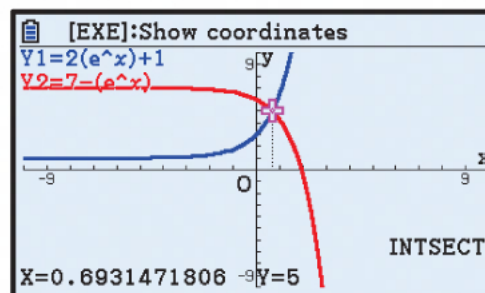


b The functions  $y = 2e^x + 1$  and  $y = 7 - e^x$  meet where

$$\begin{aligned}
 & 2e^x + 1 = 7 - e^x \\
 \therefore & 3e^x = 6 \\
 \therefore & e^x = 2 \\
 \therefore & x = \ln 2
 \end{aligned}$$

When  $x = \ln 2$ ,  $y = 2e^{\ln 2} + 1 = 5$

$\therefore$  the functions meet at  $(\ln 2, 5)$ .





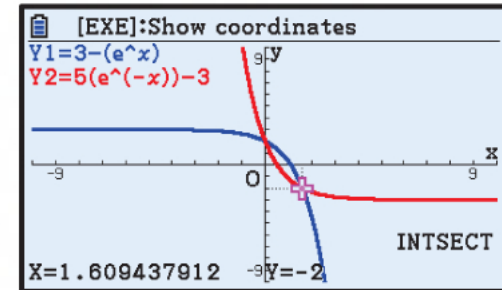
- The functions  $y = 3 - e^x$  and  $y = 5e^{-x} - 3$  meet where

$$\begin{aligned} 3 - e^x &= 5e^{-x} - 3 \\ \therefore e^x - 6 + 5e^{-x} &= 0 \\ \therefore e^{2x} - 6e^x + 5 &= 0 \\ \therefore (e^x - 1)(e^x - 5) &= 0 \\ \therefore e^x &= 1 \text{ or } 5 \\ \therefore x &= 0 \text{ or } \ln 5 \end{aligned}$$

When  $x = 0$ ,  $y = 3 - e^0 = 2$

When  $x = \ln 5$ ,  $y = 3 - e^{\ln 5} = -2$

$\therefore$  the functions meet at  $(0, 2)$  and at  $(\ln 5, -2)$ .



- 11 a  $y = 2e^x + 1$  meets  $y = 4e^{-x} - 6$  where

$$\begin{aligned} 2e^x + 1 &= 4e^{-x} - 6 \\ \therefore 2e^x + 7 - 4e^{-x} &= 0 \\ \therefore 2(e^x)^2 + 7e^x - 4 &= 0 \\ \therefore (2e^x - 1)(e^x + 4) &= 0 \\ \therefore 2e^x &= 1 \text{ or } e^x = -4 \\ \therefore 2e^x &= 1 \quad \{e^x > 0\} \\ \therefore e^x &= \frac{1}{2} \\ \therefore x &= \ln\left(\frac{1}{2}\right) \\ \therefore x &= \ln(2^{-1}) \\ \therefore x &= -\ln 2 \end{aligned}$$

$$\begin{aligned} \text{When } x = -\ln 2, \quad y &= 2e^{-\ln 2} + 1 \\ &= 2e^{\ln(2^{-1})} + 1 \\ &= 2(2^{-1}) + 1 \\ &= 2 \end{aligned}$$

$\therefore$  the intersection point of the graphs shown is  $(-\ln 2, 2)$ .

- b Let the  $x$ -coordinate of Q and R be  $a$ .

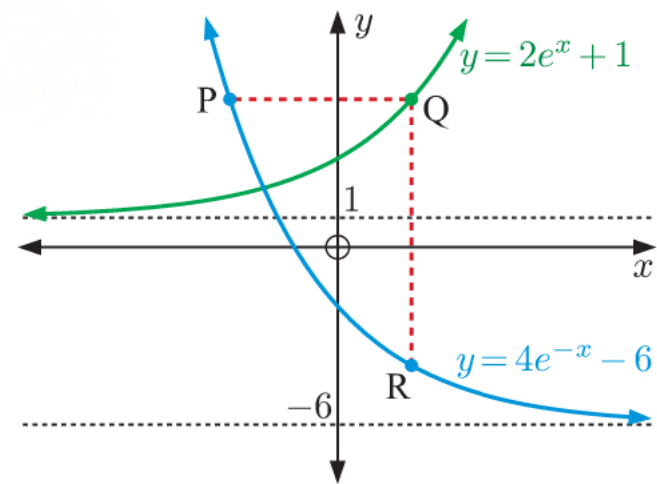
So, Q is  $(a, 2e^a + 1)$  and R is  $(a, 4e^{-a} - 6)$ .

Since  $QR = 9$ , the  $y$ -coordinate of R must be 9 units less than the  $y$ -coordinate of Q.

$$\begin{aligned} \therefore 4e^{-a} - 6 &= 2e^a + 1 - 9 \\ \therefore 2e^a - 2 - 4e^{-a} &= 0 \\ \therefore e^a - 1 - 2e^{-a} &= 0 \\ \therefore (e^a)^2 - e^a - 2 &= 0 \\ \therefore (e^a + 1)(e^a - 2) &= 0 \\ \therefore e^a &= -1 \text{ or } e^a = 2 \\ \therefore e^a &= 2 \quad \{e^a > 0\} \\ \therefore a &= \ln 2 \end{aligned}$$

$$\begin{aligned} \text{So, the } y\text{-coordinate of Q is } 2e^{\ln 2} + 1 &= 2(2) + 1 \\ &= 5 \end{aligned}$$

So, Q is  $(\ln 2, 5)$ .



Now, P has the same  $y$ -coordinate as Q, so P is  $(4e^{-x} - 6, 5)$ .

We need to find the  $x$ -coordinate of P.

When  $y = 5$ ,  $4e^{-x} - 6 = 5$

$$\therefore 4e^{-x} = 11$$

$$\therefore e^{-x} = \frac{11}{4}$$

$$\therefore \ln(e^{-x}) = \ln\left(\frac{11}{4}\right)$$

$$\therefore -x = \ln\left(\frac{11}{4}\right)$$

$$\therefore x = -\ln\left(\frac{11}{4}\right)$$

$$\therefore x = \ln\left(\left(\frac{11}{4}\right)^{-1}\right)$$

$$\therefore x = \ln\left(\frac{4}{11}\right)$$

So, P is  $(\ln(\frac{4}{11}), 5)$ .

Since P and Q have the same  $y$ -coordinate, PQ is the difference between the  $x$ -coordinates of P and Q.

$$\begin{aligned}\therefore PQ &= \ln 2 - \ln\left(\frac{4}{11}\right) \\ &= \ln 2 - (\ln 4 - \ln 11) \\ &= \ln 2 - \ln(2^2) + \ln 11 \\ &= \ln 2 - 2\ln 2 + \ln 11 \\ &= \ln 11 - \ln 2 \\ &= \ln\left(\frac{11}{2}\right) \text{ units}\end{aligned}$$

**12**  $P(t) = 852 \times (1.07)^t$  turtles

**a** When  $P(t) = 1000$ ,

$$852 \times (1.07)^t = 1000$$

$$\therefore (1.07)^t = \frac{1000}{852}$$

$$\therefore t \log(1.07) = \log\left(\frac{1000}{852}\right)$$

$$\begin{aligned}\therefore t &= \frac{\log\left(\frac{1000}{852}\right)}{\log(1.07)} \\ &\approx 2.37\end{aligned}$$

The population will reach 1000 turtles in about 2.37 years.

**b** When  $P(t) = 1500$ ,

$$852 \times (1.07)^t = 1500$$

$$\therefore (1.07)^t = \frac{1500}{852}$$

$$\therefore t \log(1.07) = \log\left(\frac{1500}{852}\right)$$

$$\begin{aligned}\therefore t &= \frac{\log\left(\frac{1500}{852}\right)}{\log(1.07)} \\ &\approx 8.36\end{aligned}$$

The population will reach 1500 turtles in about 8.36 years.

**13**  $W(t) = 20 \times 2^{0.15t}$  grams

**a** When  $W(t) = 30$ ,

$$20 \times 2^{0.15t} = 30$$

$$\therefore 2^{0.15t} = 1.5$$

$$\therefore \log(2^{0.15t}) = \log(1.5)$$

$$\therefore 0.15t \log 2 = \log(1.5)$$

$$\therefore t = \frac{\log(1.5)}{0.15 \times \log 2}$$

$$\therefore t \approx 3.90$$

It takes about 3.90 hours for the weight to reach 30 g.

**b** When  $W(t) = 100$ ,

$$20 \times 2^{0.15t} = 100$$

$$\therefore 2^{0.15t} = 5$$

$$\therefore \log(2^{0.15t}) = \log 5$$

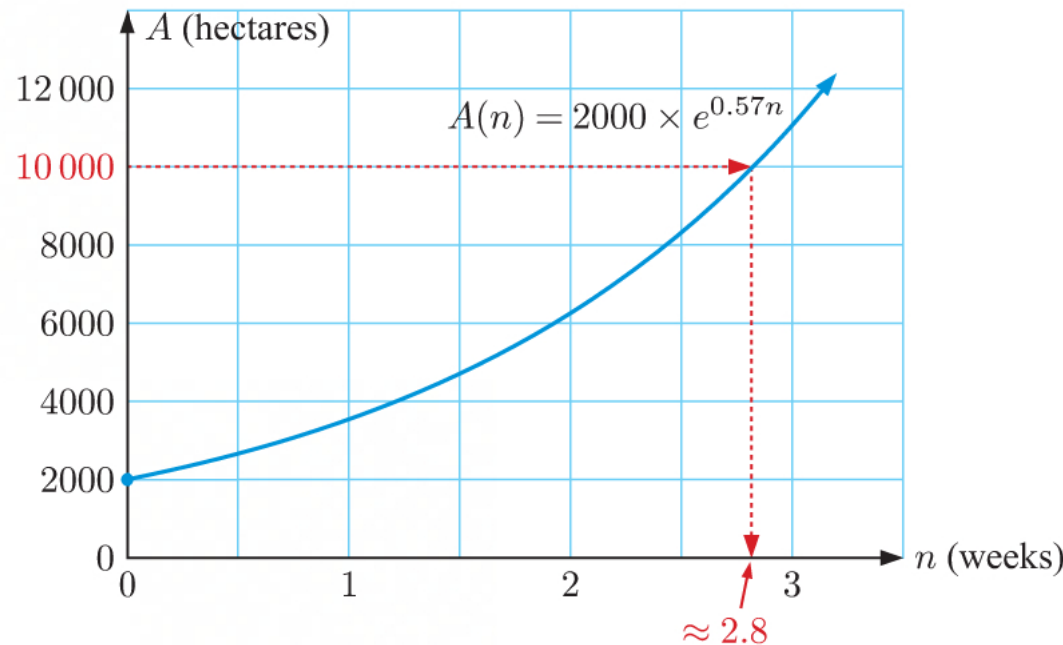
$$\therefore 0.15t \log 2 = \log 5$$

$$\therefore t = \frac{\log 5}{0.15 \times \log 2}$$

$$\therefore t \approx 15.5$$

It takes about 15.5 hours for the weight to reach 100 g.

**14 a**



**b** When  $A(n) = 10\,000$ ,  $t \approx 2.8$

$\therefore$  we estimate that it will take about 2.8 weeks for the infested area to reach 10 000 hectares.

**c** When  $A(n) = 10\,000$ ,  $2000 \times e^{0.57n} = 10\,000$

$$\therefore e^{0.57n} = 5$$

$$\therefore \ln(e^{0.57n}) = \ln 5$$

$$\therefore 0.57n = \ln 5$$

$$\therefore n = \frac{\ln 5}{0.57}$$

$$\therefore n \approx 2.82$$

$\therefore$  it takes about 2.82 weeks for the infested area to reach 10 000 hectares.

**15**  $u_0 = 360\,000$ ,  $u_n = 550\,000$ ,  $i = 7.5\% = 0.075$

$$u_n = u_0(1 + i)^n$$

$$\therefore 550\,000 = 360\,000 \times (1 + 0.075)^n$$

$$\therefore (1.075)^n = \frac{55}{36}$$

$$\therefore \log(1.075)^n = \log\left(\frac{55}{36}\right)$$

$$\therefore n \log(1.075) = \log\left(\frac{55}{36}\right)$$

$$\therefore n = \frac{\log\left(\frac{55}{36}\right)}{\log(1.075)} \approx 5.86$$

$\therefore$  we expect the house to be worth £550 000 in about 5.86 years or about 5 years and 10 months.

**16**  $u_0 = 10\,000$ ,  $u_n = 15\,000$ ,  $i = 4.8\% = 0.048$

$$u_n = u_0(1+i)^n$$

$$\therefore 15\,000 = 10\,000 \times (1+0.048)^n$$

$$\therefore (1.048)^n = 1.5$$

$$\therefore \log(1.048)^n = \log(1.5)$$

$$\therefore n \log(1.048) = \log(1.5)$$

$$\therefore n = \frac{\log(1.5)}{\log(1.048)}$$

$$\therefore n \approx 8.648$$

$\therefore$  it would take 9 years for Thabo's investment to grow to \$15 000.  
{interest compounded annually}

**17 a** 8.4% p.a. compounded monthly      **b**  $t_0 = 15\,000$ ,  $t_n = 25\,000$ , and  $r = 1.007$

is  $\frac{8.4\%}{12} = 0.7\%$  a month  
 $= 0.007$

So  $r = 1 + 0.007$

$$\therefore r = 1.007$$

$$t_n = t_0 \times r^n$$

$$\therefore 25\,000 = 15\,000 \times (1.007)^n$$

$$\therefore (1.007)^n = \frac{25}{15} = \frac{5}{3}$$

$$\therefore \log(1.007)^n = \log\left(\frac{5}{3}\right)$$

$$\therefore n \log(1.007) = \log\left(\frac{5}{3}\right)$$

$$\therefore n = \frac{\log\left(\frac{5}{3}\right)}{\log(1.007)} \approx 73.23$$

$\therefore$  Dien will withdraw the money after 74 months.

**18**  $M_t = 1000 \times e^{-0.04t}$

$$M_0 = 1000 \times e^0 = 1000$$

$\therefore$  the initial mass of the substance is 1000 grams.

**a** For the mass to halve,  $M_t = 500$

$$\therefore 1000e^{-0.04t} = 500$$

$$\therefore e^{-0.04t} = 0.5$$

$$\therefore -0.04t = \ln(0.5)$$

$$\therefore t = \frac{\ln(0.5)}{-0.04}$$

$$\therefore t \approx 17.3$$

$\therefore$  it will take about 17.3 years for the mass to halve.

**b** When  $M_t = 25$  g,

$$1000e^{-0.04t} = 25$$

$$\therefore e^{-0.04t} = 0.025$$

$$\therefore -0.04t = \ln(0.025)$$

$$\therefore t = \frac{\ln(0.025)}{-0.04}$$

$$\therefore t \approx 92.2$$

$\therefore$  it will take about 92.2 years for the mass to reach 25 grams.

**c** When  $M_t = 1\%$  of original value,

$$1000e^{-0.04t} = 0.01 \times 1000$$

$$\therefore e^{-0.04t} = 0.01$$

$$\therefore -0.04t = \ln(0.01)$$

$$\therefore t = \frac{\ln(0.01)}{-0.04}$$

$$\therefore t \approx 115$$

$\therefore$  it will take about 115 years for the mass to reach 1% of its original value.



**19**  $I = I_0 \times 2^{-0.02t}$  amps

When  $I$  is 10% of its original value,  $I = 10\%$  of  $I_0$

$$\therefore I_0 \times 2^{-0.02t} = 0.1 \times I_0$$

$$\therefore 2^{-0.02t} = 0.1$$

$$\therefore \log(2^{-0.02t}) = \log(0.1)$$

$$\therefore -0.02t \log 2 = \log(10^{-1})$$

$$\therefore -\frac{1}{50}t \log 2 = -1$$

$$\therefore t = \frac{50}{\log 2}$$

$\therefore$  it takes  $\frac{50}{\log 2}$  seconds for the current to drop to 10% of its original value.

**20**  $V(t) = 50(1 - e^{-0.2t})$   $\text{m s}^{-1}$

**a** When  $V(t) = 40$ ,  $50(1 - e^{-0.2t}) = 40$

$$\therefore 1 - e^{-0.2t} = \frac{4}{5}$$

$$\therefore e^{-0.2t} = \frac{1}{5}$$

$$\therefore \ln(e^{-0.2t}) = \ln\left(\frac{1}{5}\right)$$

$$\therefore -0.2t = \ln(5^{-1})$$

$$\therefore -\frac{1}{5}t = -\ln 5$$

$$\therefore t = 5 \ln 5$$

$\therefore$  it will take  $5 \ln 5$  seconds for the sky diver's speed to reach  $40 \text{ m s}^{-1}$ .

**b** When  $V(t) = v$ ,  $50(1 - e^{-0.2t}) = v$

$$\therefore 1 - e^{-0.2t} = \frac{v}{50}$$

$$\therefore e^{-0.2t} = 1 - \frac{v}{50}$$

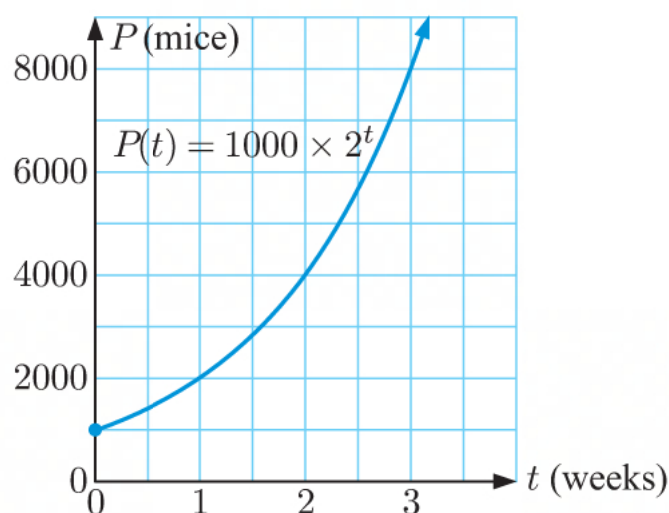
$$\therefore \ln(e^{-0.2t}) = \ln\left(1 - \frac{v}{50}\right)$$

$$\therefore -0.2t = \ln\left(1 - \frac{v}{50}\right)$$

$$\therefore t = -5 \ln\left(1 - \frac{v}{50}\right)$$

So, the time taken for the sky diver's speed to reach  $v \text{ m s}^{-1}$  is  $-5 \ln\left(1 - \frac{v}{50}\right)$  s.

**21 a**



**b** When  $P(t) = 20\,000$ ,

$$1000 \times 2^t = 20\,000$$

$$\therefore 2^t = 20$$

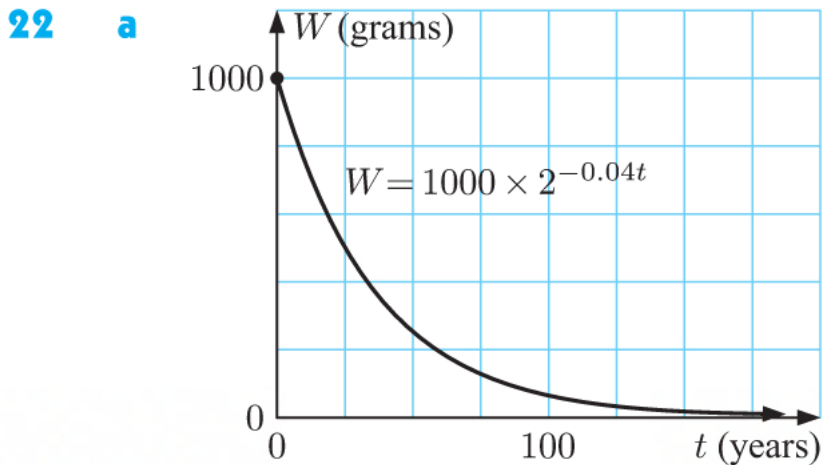
$$\therefore \log(2^t) = \log 20$$

$$\therefore t \log 2 = \log 20$$

$$\therefore t = \frac{\log 20}{\log 2} \approx 4.32$$

$\therefore$  it will take about 4.32 weeks for the population to reach 20 000 mice.

$$\begin{aligned}
 \text{c} \quad P &= 1000 \times 2^t \\
 \therefore 2^t &= \frac{P}{1000} \\
 \therefore \log(2^t) &= \log\left(\frac{P}{1000}\right) \\
 \therefore t \log 2 &= \log P - \log 1000 \\
 \therefore t &= \frac{\log P - \log(10^3)}{\log 2} \\
 \therefore t &= \frac{\log P - 3}{\log 2}
 \end{aligned}$$



**c i** When  $W = 20$ ,

$$\begin{aligned}
 t &= \frac{3 - \log 20}{0.04 \log 2} \\
 &\approx 141
 \end{aligned}$$

$\therefore$  it will take about 141 years for the weight to reach 20 grams.

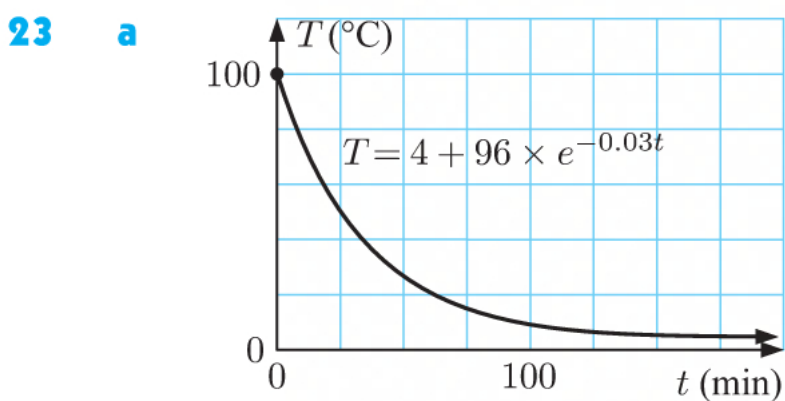
**b**

$$\begin{aligned}
 W &= 1000 \times 2^{-0.04t} \\
 \therefore 2^{-0.04t} &= \frac{W}{1000} \\
 \therefore \log(2^{-0.04t}) &= \log\left(\frac{W}{1000}\right) \\
 \therefore -0.04t \log 2 &= \log W - \log 1000 \\
 \therefore 0.04t \log 2 &= \log(10^3) - \log W \\
 \therefore t &= \frac{3 - \log W}{0.04 \log 2}
 \end{aligned}$$

**ii** When  $W = 0.001$ ,

$$\begin{aligned}
 t &= \frac{3 - \log(0.001)}{0.04 \log 2} \\
 &\approx 498
 \end{aligned}$$

$\therefore$  it will take about 498 years for the weight to reach 0.001 grams.



**c i** When  $T = 25$ ,

$$\begin{aligned}
 t &= \frac{\ln 96 - \ln 21}{0.03} \\
 &\approx 50.7
 \end{aligned}$$

$\therefore$  it will take about 50.7 minutes for the temperature to reach 25°C.

**b**

$$\begin{aligned}
 T &= 4 + 96 \times e^{-0.03t} \\
 \therefore 96 \times e^{-0.03t} &= T - 4 \\
 \therefore e^{-0.03t} &= \frac{T - 4}{96} \\
 \therefore -0.03t &= \ln\left(\frac{T - 4}{96}\right) \\
 \therefore -0.03t &= \ln(T - 4) - \ln 96 \\
 \therefore t &= \frac{\ln 96 - \ln(T - 4)}{0.03}
 \end{aligned}$$

**ii** When  $T = 5$ ,

$$\begin{aligned}
 t &= \frac{\ln 96 - \ln 1}{0.03} \\
 &\approx 152
 \end{aligned}$$

$\therefore$  it will take about 152 minutes for the temperature to reach 5°C.

**24**  $V(t) = 650(4 + 2 \times e^{-0.1t})$

**a** as  $t \rightarrow \infty$ ,  $e^{-0.1t} \rightarrow 0^+$   
 $\therefore$  the speed of the meteor is decreasing.

$$\begin{aligned}
 \text{b i } V(0) &= 650(4 + 2 \times e^{-0.1 \times 0}) \\
 &= 650(4 + 2 \times 1) \\
 &= 650(6) \\
 &= 3900
 \end{aligned}$$

The speed of the meteor when it was first sighted was  $3900 \text{ m s}^{-1}$ .

$$\begin{aligned}
 \text{ii } V(120) &= 650(4 + 2 \times e^{-0.1 \times 120}) \\
 &= 650(4 + 2 \times e^{-12}) \\
 &\approx 2600 \text{ m s}^{-1}
 \end{aligned}$$

The speed of the meteor after 2 minutes was about  $2600 \text{ m s}^{-1}$ .

$$\begin{aligned}
 \text{c } &\text{When } V(t) = 3000, \\
 &650(4 + 2 \times e^{-0.1t}) = 3000 \\
 \therefore &4 + 2 \times e^{-0.1t} = \frac{60}{13} \\
 \therefore &2 \times e^{-0.1t} = \frac{8}{13} \\
 \therefore &e^{-0.1t} = \frac{4}{13} \\
 \therefore &-0.1t = \ln\left(\frac{4}{13}\right) \\
 \therefore &t = \frac{\ln\left(\frac{4}{13}\right)}{-0.1} \\
 &\approx 11.8
 \end{aligned}$$

It will take about 11.8 seconds for the meteor's speed to reach  $3000 \text{ m s}^{-1}$ .

## INVESTIGATION 2

## THE “RULE OF 72”

- 1 **a** A population growing at 2% per year will double in approximately  $\frac{72}{2} = 36$  years.
- b** A population growing at 8% per year will double in approximately  $\frac{72}{8} = 9$  years.
- c** A population growing at 12% per year will double in approximately  $\frac{72}{12} = 6$  years.

- 2 Let  $T$  be the doubling time.

$$\begin{aligned}
 \text{a } &\text{For a population growing at 2\% per year, we require } (1.02)^T = 2, \\
 &\therefore \log((1.02)^T) = \log 2 \\
 &\therefore T \log(1.02) = \log 2 \\
 &\therefore T = \frac{\log 2}{\log(1.02)} \\
 &\approx 35.003
 \end{aligned}$$

$$\begin{aligned}
 &\text{For a population growing at 8\% per year, we require } (1.08)^T = 2 \\
 &\therefore \log((1.08)^T) = \log 2 \\
 &\therefore T \log(1.08) = \log 2 \\
 &\therefore T = \frac{\log 2}{\log(1.08)} \\
 &\approx 9.006
 \end{aligned}$$

$$\begin{aligned}
 &\text{For a population growing at 12\% per year, we require } (1.12)^T = 2 \\
 &\therefore \log((1.12)^T) = \log 2 \\
 &\therefore T \log(1.12) = \log 2 \\
 &\therefore T = \frac{\log 2}{\log(1.12)} \\
 &\approx 6.116
 \end{aligned}$$

**b** Our estimates in **1** using the “rule of 72” were very close to the values obtained in **2 a**.

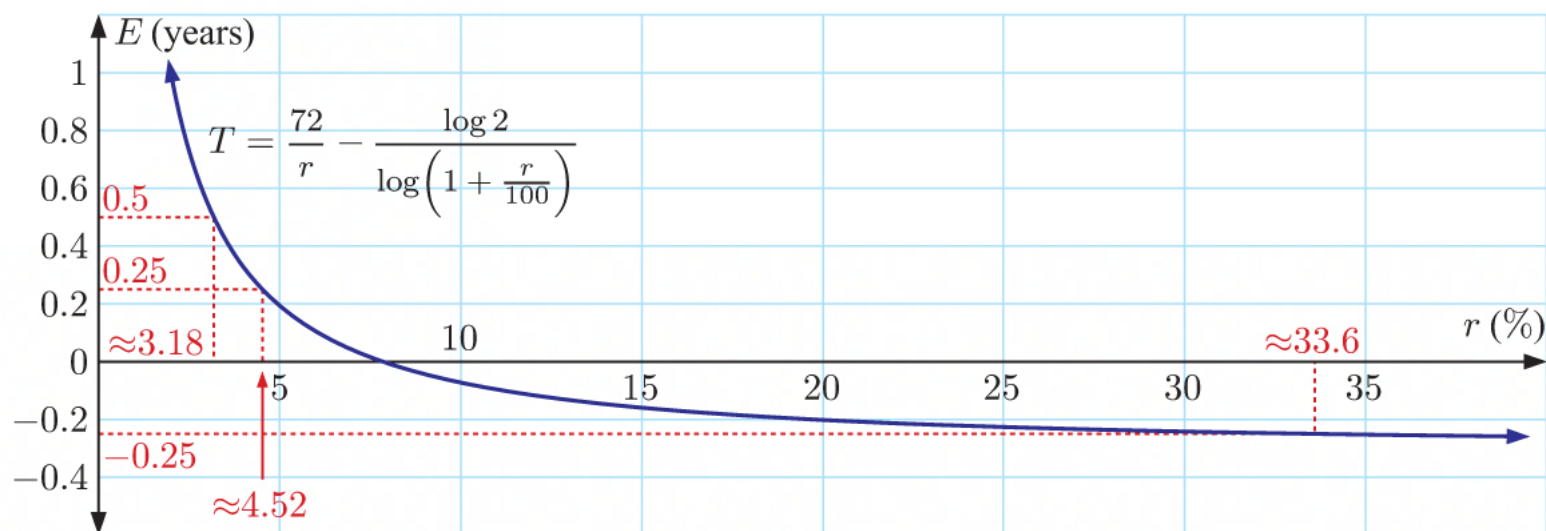
**3 a** Let the percentage increase each year be  $r\%$ , and the doubling time be  $T$ .

We require  $\left(1 + \frac{r}{100}\right)^T = 2$ , so  $\log\left(\left(1 + \frac{r}{100}\right)^T\right) = \log 2$

$$\therefore T \log\left(1 + \frac{r}{100}\right) = \log 2$$

$$\therefore T = \frac{\log 2}{\log\left(1 + \frac{r}{100}\right)}$$

**b** The error in estimating the exact doubling time is  $E = \frac{72}{r} - \frac{\log 2}{\log\left(1 + \frac{r}{100}\right)}$ .



**c i** 6 months is 0.5 years.

From the graph,  $E = 0.5$  when  $r \approx 3.18$ .

$\therefore$  the estimate is accurate to within 6 months for growth rates larger than about 3.18%.

**ii** 3 months is 0.25 years.

From the graph,  $E = 0.25$  when  $r \approx 4.52$   
and  $E = -0.25$  when  $r \approx 33.6$

$\therefore$  the estimate is accurate to within 3 months for growth rates between about 4.52% and 33.6%.

## EXERCISE 3H

**1 a**  $f : x \mapsto \log_2 x - 2$

**i**  $\log_2 x$  is defined when  $x > 0$

So, the domain is  $\{x \mid x > 0\}$  and the range is  $\{y \mid y \in \mathbb{R}\}$ .

**ii** As  $x \rightarrow 0^+$ ,  $y \rightarrow -\infty$ , so the vertical asymptote is  $x = 0$ .

As  $x \rightarrow \infty$ ,  $y \rightarrow \infty$ , so there is no horizontal asymptote.

When  $x = 0$ ,  $y$  is undefined, so there is no  $y$ -intercept.

When  $y = 0$ ,  $\log_2 x = 2$

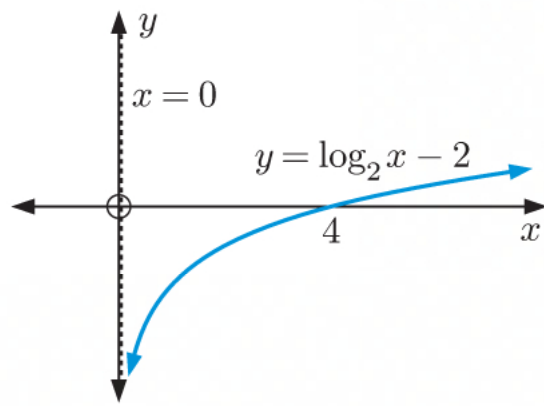
$$\therefore x = 2^2$$

$$\therefore x = 4$$

So, the  $x$ -intercept is 4.



iii



iv

$$\begin{aligned}
 f(x) &= -1 \\
 \therefore \log_2 x - 2 &= -1 \\
 \therefore \log_2 x &= 1 \\
 \therefore x &= 2^1 \\
 \therefore x &= 2
 \end{aligned}$$

$$\begin{aligned}
 \text{v } f \text{ is defined by } y &= \log_2 x - 2 \\
 \therefore f^{-1} \text{ is defined by } x &= \log_2 y - 2 \\
 \therefore x + 2 &= \log_2 y \\
 \therefore y &= 2^{x+2} \\
 \therefore f^{-1}(x) &= 2^{x+2}
 \end{aligned}$$

**b**  $f : x \mapsto \log_3(x + 1)$

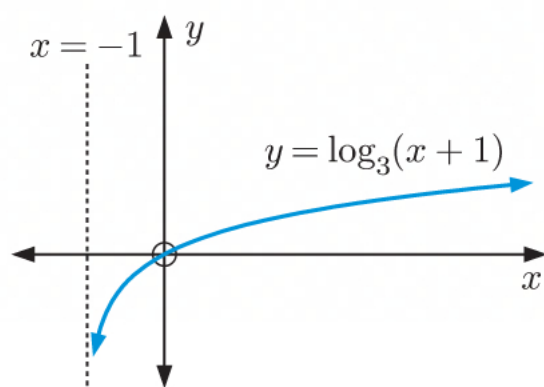
**i**  $\log_3(x + 1)$  is defined when  $x + 1 > 0$ , that is, when  $x > -1$ .  
So, the domain is  $\{x \mid x > -1\}$  and the range is  $\{y \mid y \in \mathbb{R}\}$ .

**ii** As  $x \rightarrow -1^+$ ,  $y \rightarrow -\infty$ , so the vertical asymptote is  $x = -1$ .  
As  $x \rightarrow \infty$ ,  $y \rightarrow \infty$ , so there is no horizontal asymptote.  
When  $x = 0$ ,  $\log_3 1 = 0$ , so the  $y$ -intercept is 0.

$$\begin{aligned}
 \text{When } y = 0, \quad \log_3(x + 1) &= 0 \\
 \therefore x + 1 &= 3^0 \\
 \therefore x &= 0
 \end{aligned}$$

So, the  $x$ -intercept is 0.

iii



iv

$$\begin{aligned}
 f(x) &= -1 \\
 \therefore \log_3(x + 1) &= -1 \\
 \therefore x + 1 &= 3^{-1} \\
 \therefore x &= -\frac{2}{3}
 \end{aligned}$$

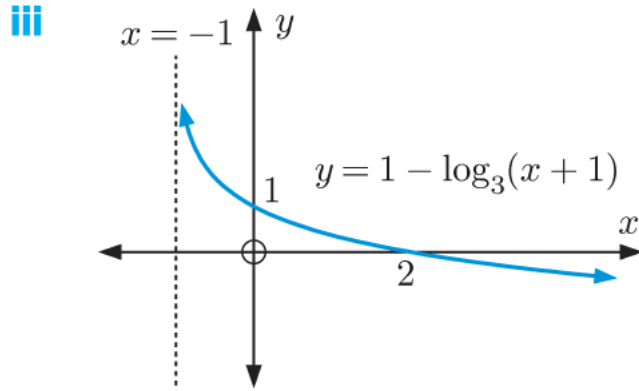
$$\begin{aligned}
 \text{v } f \text{ is defined by } y &= \log_3(x + 1) \\
 \therefore f^{-1} \text{ is defined by } x &= \log_3(y + 1) \\
 \therefore y + 1 &= 3^x \\
 \therefore y &= 3^x - 1 \\
 \therefore f^{-1}(x) &= 3^x - 1
 \end{aligned}$$

**c**  $f : x \mapsto 1 - \log_3(x + 1)$

**i**  $\log_3(x + 1)$  is defined when  $x + 1 > 0$ , that is, when  $x > -1$ .  
So, the domain is  $\{x \mid x > -1\}$  and the range is  $\{y \mid y \in \mathbb{R}\}$ .

- ii As  $x \rightarrow -1^+$ ,  $y \rightarrow \infty$ , so the vertical asymptote is  $x = -1$ .  
 As  $x \rightarrow \infty$ ,  $y \rightarrow -\infty$ , so there is no horizontal asymptote.  
 When  $x = 0$ ,  $1 - \log_3 1 = 1 - 0 = 1$ , so the  $y$ -intercept is 1.  
 When  $y = 0$ ,  $1 = \log_3(x + 1)$   
 $\therefore 3^1 = x + 1$   
 $\therefore x = 2$

So, the  $x$ -intercept is 2.



iv

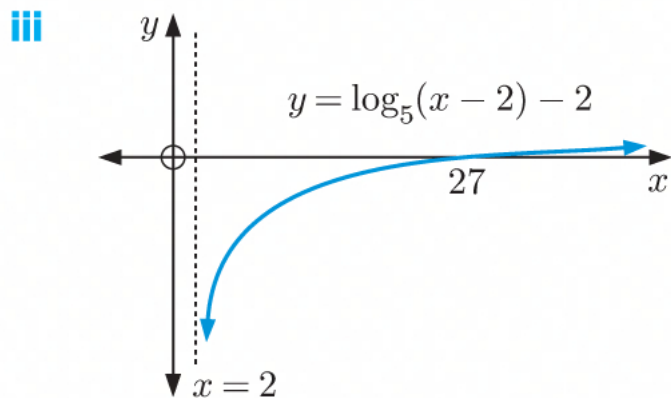
$$\begin{aligned} f(x) &= -1 \\ \therefore 1 - \log_3(x + 1) &= -1 \\ \therefore -\log_3(x + 1) &= -2 \\ \therefore \log_3(x + 1) &= 2 \\ \therefore x + 1 &= 3^2 \\ \therefore x &= 8 \end{aligned}$$

- v  $f$  is defined by  $y = 1 - \log_3(x + 1)$   
 $\therefore f^{-1}$  is defined by  $x = 1 - \log_3(y + 1)$   
 $\therefore \log_3(y + 1) = 1 - x$   
 $\therefore y + 1 = 3^{1-x}$   
 $\therefore y = 3^{1-x} - 1$   
 $\therefore f^{-1}(x) = 3^{1-x} - 1$

d  $f : x \mapsto \log_5(x - 2) - 2$

- i  $\log_5(x - 2)$  is defined when  $x - 2 > 0$ , that is, when  $x > 2$ .  
 So, the domain is  $\{x \mid x > 2\}$  and the range is  $\{y \mid y \in \mathbb{R}\}$ .  
 ii As  $x \rightarrow 2^+$ ,  $y \rightarrow -\infty$ , so the vertical asymptote is  $x = 2$ .  
 As  $x \rightarrow \infty$ ,  $y \rightarrow \infty$ , so there is no horizontal asymptote.  
 When  $x = 0$ ,  $y$  is undefined, so there is no  $y$ -intercept.  
 When  $y = 0$ ,  $\log_5(x - 2) = 2$   
 $\therefore x - 2 = 5^2$   
 $\therefore x = 27$

So, the  $x$ -intercept is 27.



iv

$$\begin{aligned} f(x) &= -1 \\ \therefore \log_5(x - 2) - 2 &= -1 \\ \therefore \log_5(x - 2) &= 1 \\ \therefore x - 2 &= 5^1 \\ \therefore x &= 7 \end{aligned}$$

$$\begin{aligned}
 \text{v} \quad & f \text{ is defined by } y = \log_5(x-2) - 2 \\
 \therefore f^{-1} \text{ is defined by } & x = \log_5(y-2) - 2 \\
 & \therefore x+2 = \log_5(y-2) \\
 & \therefore y-2 = 5^{x+2} \\
 & \therefore y = 5^{x+2} + 2 \\
 \therefore f^{-1}(x) &= 5^{x+2} + 2
 \end{aligned}$$

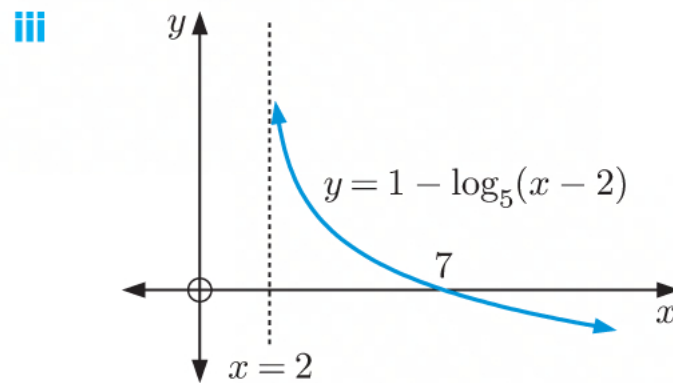
$$\text{e} \quad f : x \mapsto 1 - \log_5(x-2)$$

i  $\log_5(x-2)$  is defined when  $x-2 > 0$ , that is, when  $x > 2$ .  
So, the domain is  $\{x \mid x > 2\}$  and the range is  $\{y \mid y \in \mathbb{R}\}$ .

ii As  $x \rightarrow 2^+$ ,  $y \rightarrow \infty$ , so the vertical asymptote is  $x = 2$ .  
As  $x \rightarrow \infty$ ,  $y \rightarrow -\infty$ , so there is no horizontal asymptote.  
When  $x = 0$ ,  $y$  is undefined, so there is no  $y$ -intercept.

$$\begin{aligned}
 \text{When } y = 0, \quad & \log_5(x-2) = 1 \\
 & \therefore x-2 = 5^1 \\
 & \therefore x = 7
 \end{aligned}$$

So, the  $x$ -intercept is 7.



$$\begin{aligned}
 \text{iv} \quad & f(x) = -1 \\
 \therefore 1 - \log_5(x-2) &= -1 \\
 \therefore -\log_5(x-2) &= -2 \\
 \therefore \log_5(x-2) &= 2 \\
 \therefore x-2 &= 5^2 \\
 \therefore x &= 27
 \end{aligned}$$

$$\begin{aligned}
 \text{v} \quad & f \text{ is defined by } y = 1 - \log_5(x-2) \\
 \therefore f^{-1} \text{ is defined by } & x = 1 - \log_5(y-2) \\
 & \therefore \log_5(y-2) = 1-x \\
 & \therefore y-2 = 5^{1-x} \\
 & \therefore y = 5^{1-x} + 2 \\
 \therefore f^{-1}(x) &= 5^{1-x} + 2
 \end{aligned}$$

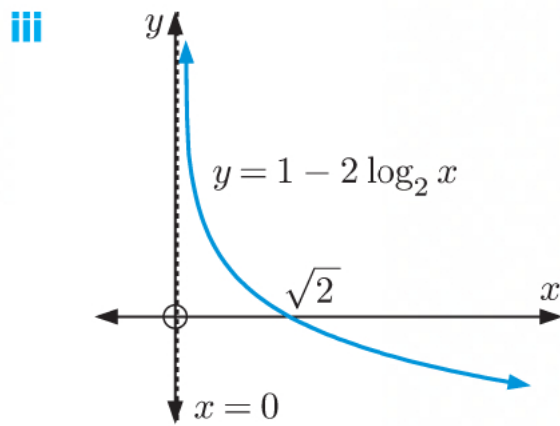
$$\text{f} \quad f : x \mapsto 1 - 2\log_2 x$$

i  $\log_2 x$  is defined when  $x > 0$ .  
So, the domain is  $\{x \mid x > 0\}$  and the range is  $\{y \mid y \in \mathbb{R}\}$ .

ii As  $x \rightarrow 0^+$ ,  $y \rightarrow \infty$ , so the vertical asymptote is  $x = 0$ .  
As  $x \rightarrow \infty$ ,  $y \rightarrow -\infty$ , so there is no horizontal asymptote.  
When  $x = 0$ ,  $y$  is undefined, so there is no  $y$ -intercept.

$$\begin{aligned}
 \text{When } y = 0, \quad & 2\log_2 x = 1 \\
 \therefore \log_2 x &= \frac{1}{2} \\
 \therefore x &= 2^{\frac{1}{2}} \\
 \therefore x &= \sqrt{2}
 \end{aligned}$$

So, the  $x$ -intercept is  $\sqrt{2}$ .



iv

$$f(x) = -1$$

$$\therefore 1 - 2 \log_2 x = -1$$

$$\therefore -2 \log_2 x = -2$$

$$\therefore 2 \log_2 x = 2$$

$$\therefore \log_2 x = 1$$

$$\therefore x = 2^1$$

$$\therefore x = 2$$

v

$f$  is defined by  $y = 1 - 2 \log_2 x$

$\therefore f^{-1}$  is defined by  $x = 1 - 2 \log_2 y$

$$\therefore 2 \log_2 y = 1 - x$$

$$\therefore \log_2 y = \frac{1-x}{2}$$

$$\therefore y = 2^{\frac{1-x}{2}}$$

$$\therefore f^{-1}(x) = 2^{\frac{1-x}{2}}$$

2 a  $f(x) = \ln x - 4$

i  $f(x)$  is a translation of  $y = \ln x$  through  $\begin{pmatrix} 0 \\ -4 \end{pmatrix}$ .

ii Domain =  $\{x \mid x > 0\}$ , Range =  $\{y \mid y \in \mathbb{R}\}$

iii As  $x \rightarrow 0^+$ ,  $y \rightarrow -\infty$ , so the vertical asymptote is  $x = 0$ .

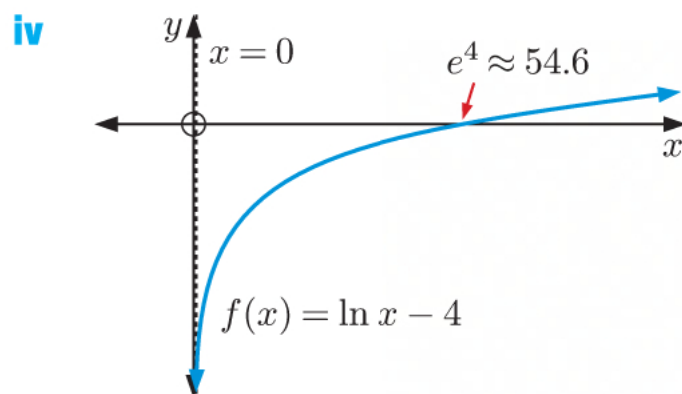
As  $x \rightarrow \infty$ ,  $y \rightarrow \infty$ , so there is no horizontal asymptote.

When  $x = 0$ ,  $y$  is undefined, so there is no  $y$ -intercept.

When  $y = 0$ ,  $\ln x = 4$

$$\therefore x = e^4 \approx 54.6$$

So, the  $x$ -intercept is  $e^4$ .



v

$f$  is defined by  $y = \ln x - 4$

$\therefore f^{-1}$  is defined by  $x = \ln y - 4$

$$\therefore x + 4 = \ln y$$

$$\therefore y = e^{x+4}$$

$$\therefore f^{-1}(x) = e^{x+4}$$

b  $f(x) = \ln(x-1) + 2$

i  $f(x)$  is a translation of  $y = \ln x$  through  $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ .

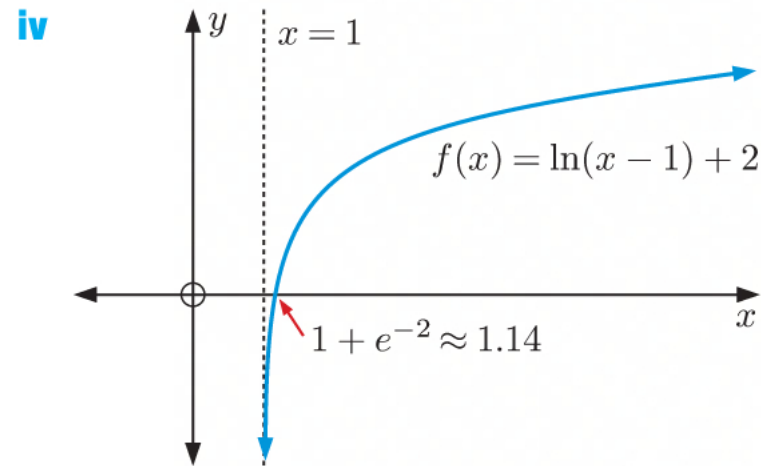
ii Domain =  $\{x \mid x > 1\}$ , Range =  $\{y \mid y \in \mathbb{R}\}$



- iii As  $x \rightarrow 1^+$ ,  $y \rightarrow -\infty$ , so the vertical asymptote is  $x = 1$ .  
 As  $x \rightarrow \infty$ ,  $y \rightarrow \infty$ , so there is no horizontal asymptote.  
 When  $x = 0$ ,  $y$  is undefined, so there is no  $y$ -intercept.

$$\begin{aligned}\text{When } y = 0, \quad \ln(x-1) &= -2 \\ \therefore x-1 &= e^{-2} \\ \therefore x &= 1 + e^{-2} \approx 1.14\end{aligned}$$

So, the  $x$ -intercept is  $1 + e^{-2}$ .



v  $f$  is defined by  $y = \ln(x-1) + 2$   
 $\therefore f^{-1}$  is defined by  $x = \ln(y-1) + 2$   
 $\therefore x-2 = \ln(y-1)$   
 $\therefore y-1 = e^{x-2}$   
 $\therefore y = e^{x-2} + 1$   
 $\therefore f^{-1}(x) = e^{x-2} + 1$

c  $f(x) = 3 \ln x - 1$

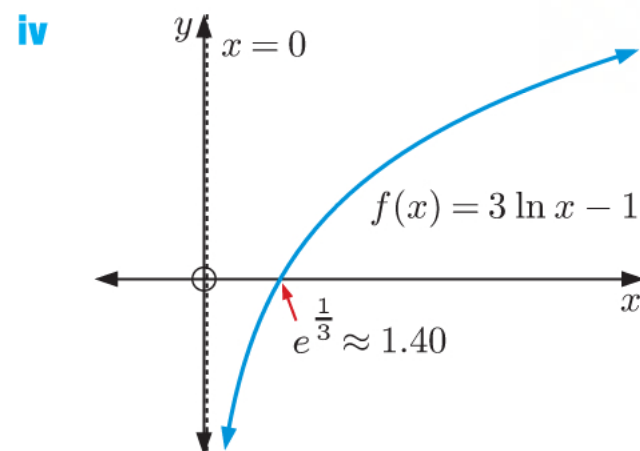
- i  $f(x)$  is a vertical stretch of  $y = \ln x$  with scale factor 3, then a translation through  $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$ .

ii Domain =  $\{x \mid x > 0\}$ , Range =  $\{y \mid y \in \mathbb{R}\}$

- iii As  $x \rightarrow 0^+$ ,  $y \rightarrow -\infty$ , so the vertical asymptote is  $x = 0$ .  
 As  $x \rightarrow \infty$ ,  $y \rightarrow \infty$ , so there is no horizontal asymptote.  
 When  $x = 0$ ,  $y$  is undefined, so there is no  $y$ -intercept.

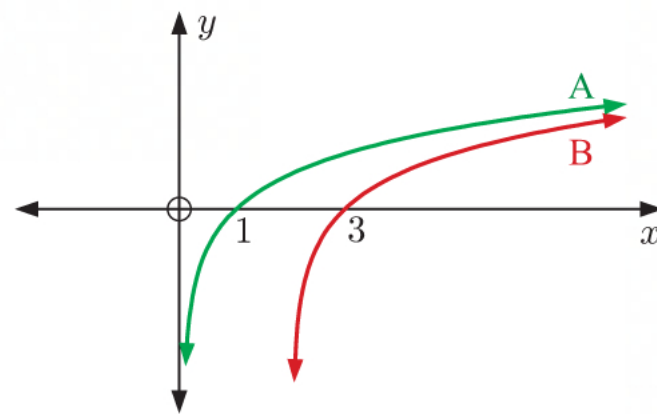
$$\begin{aligned}\text{When } y = 0, \quad 3 \ln x &= 1 \\ \therefore \ln x &= \frac{1}{3} \\ \therefore x &= e^{\frac{1}{3}} \approx 1.40\end{aligned}$$

So, the  $x$ -intercept is  $e^{\frac{1}{3}}$ .

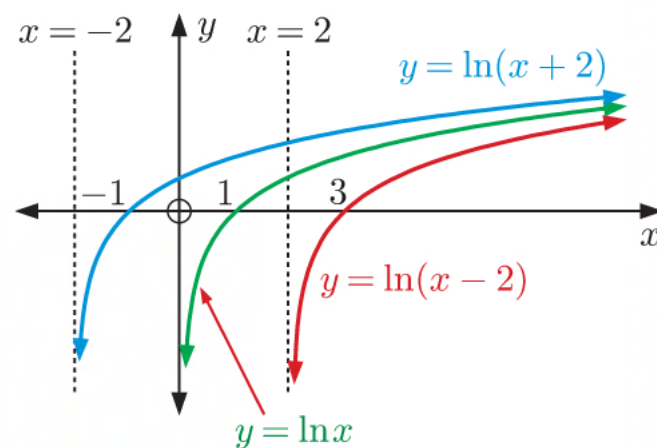


v  $f$  is defined by  $y = 3 \ln x - 1$   
 $\therefore f^{-1}$  is defined by  $x = 3 \ln y - 1$   
 $\therefore x+1 = 3 \ln y$   
 $\therefore \ln y = \frac{x+1}{3}$   
 $\therefore y = e^{\frac{x+1}{3}}$   
 $\therefore f^{-1}(x) = e^{\frac{x+1}{3}}$

- 3 a** For  $y = \ln x$ , when  $y = 0$ ,  $\ln x = 0$   
 $\therefore x = e^0$   
 $\therefore x = 1$   
 $\therefore$  A is  $y = \ln x$  as its  $x$ -intercept is 1,  
 so B must be  $y = \ln(x - 2)$ .



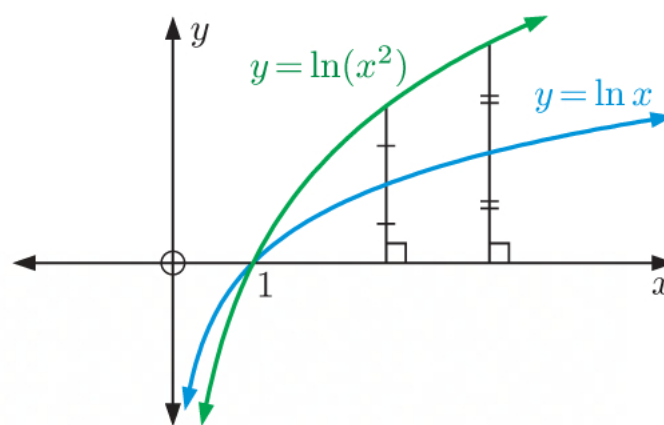
- b**  $y = \ln(x - 2)$  is a horizontal translation of  $y = \ln x$ , 2 units to the right.  
 $y = \ln(x + 2)$  is a horizontal translation of  $y = \ln x$ , 2 units to the left.



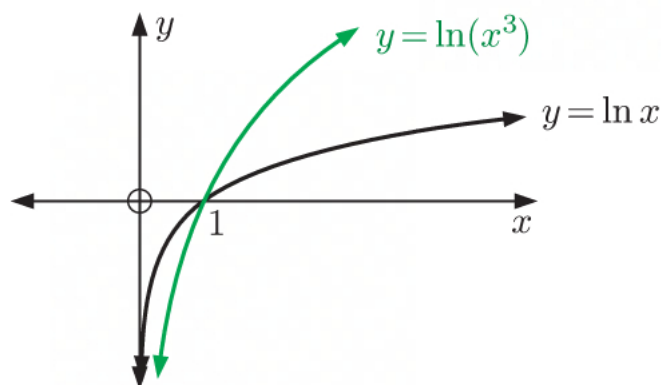
- c**  $y = \ln x$  has domain  $\{x \mid x > 0\}$ .  
 As  $x \rightarrow 0^+$ ,  $y \rightarrow -\infty$ , so  $y = \ln x$  has vertical asymptote  $x = 0$ .  
 $y = \ln(x - 2)$  has domain  $\{x \mid x > 2\}$ .  
 As  $x \rightarrow 2^+$ ,  $y \rightarrow -\infty$ , so  $y = \ln(x - 2)$  has vertical asymptote  $x = 2$ .  
 $y = \ln(x + 2)$  has domain  $\{x \mid x > -2\}$ .  
 As  $x \rightarrow -2^+$ ,  $y \rightarrow -\infty$ , so  $y = \ln(x + 2)$  has vertical asymptote  $x = -2$ .

- 4**  $y = \ln(x^2)$ ,  $x > 0$   
 $= 2 \ln x \quad \{m \ln b = \ln(b^m)\}$

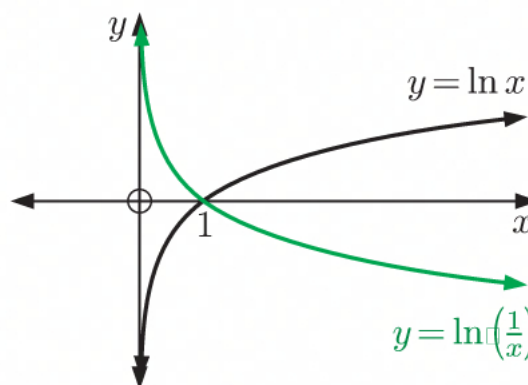
So, the  $y$ -values are twice as large for  $y = \ln(x^2)$  as they are for  $y = \ln x$ . Therefore, yes, Kelly is correct.



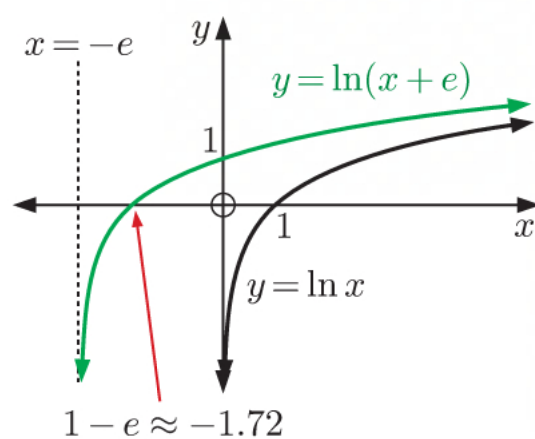
- 5 a**  $y = \ln(x^3) = 3 \ln x$  is a vertical stretch of  $y = \ln x$  with scale factor 3.



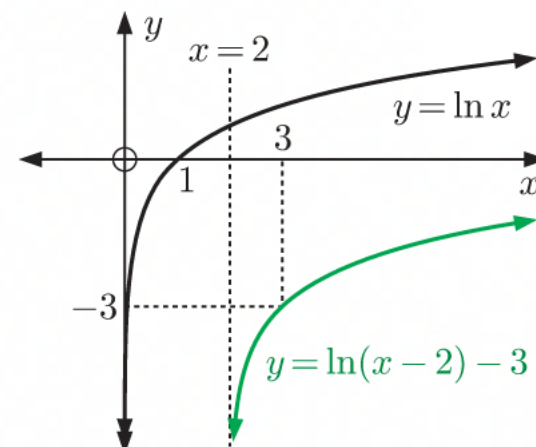
- b**  $y = \ln\left(\frac{1}{x}\right) = \ln(x^{-1}) = -\ln x$  is a reflection of  $y = \ln x$  in the  $x$ -axis.



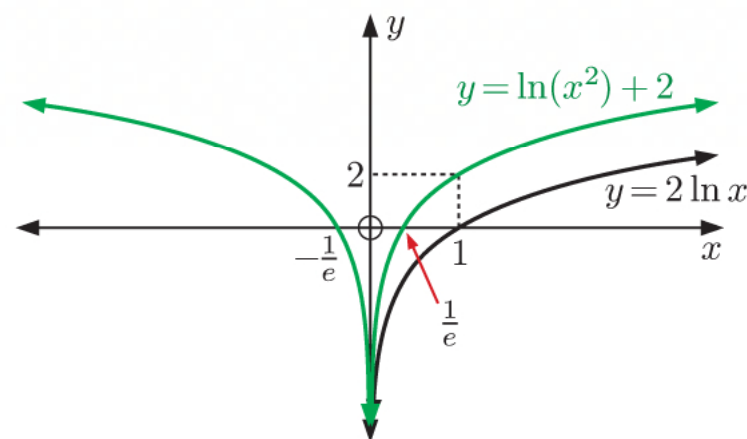
- c**  $y = \ln(x + e)$  is a horizontal translation of  $y = \ln x$ ,  $e$  units to the left.



- d**  $y = \ln(x - 2) - 3$  is a translation of  $y = \ln x$  through  $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$ .

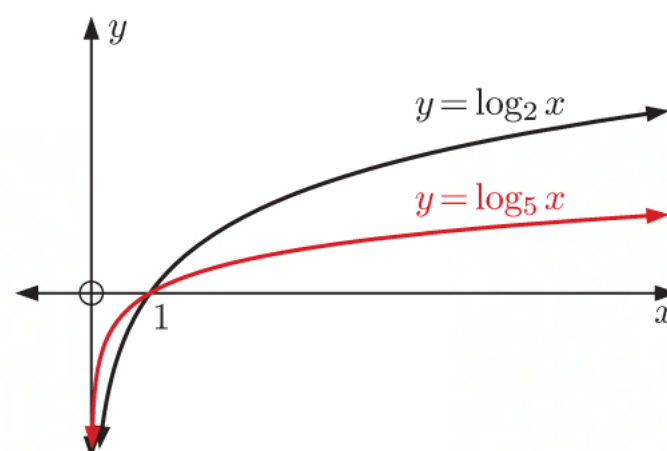


- e** For  $x > 0$ ,  $y = \ln(x^2) + 2 = 2 \ln x + 2$  is a vertical translation of  $y = 2 \ln x$ , 2 units upwards.  
For  $x < 0$ ,  $y = \ln(x^2) + 2 = \ln((-x)^2) + 2$  is a reflection of  $y = \ln(x^2) + 2$ ,  $x > 0$ , in the  $y$ -axis.

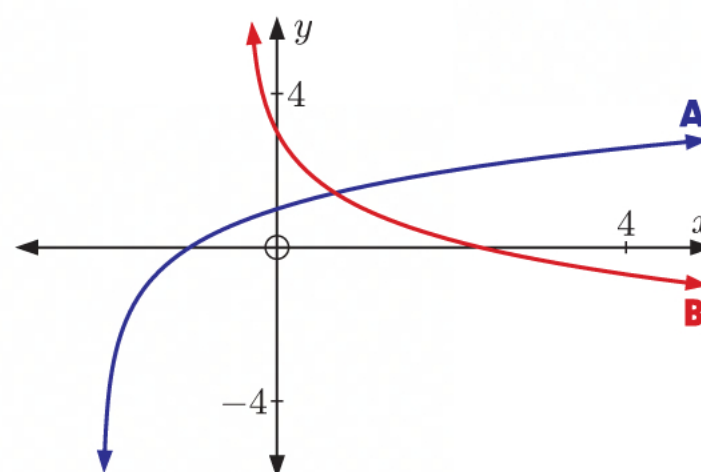


$$\begin{aligned} 6 \quad y &= \log_5 x \times \frac{\log_5 2}{\log_5 2} \\ &= \log_5 2 \times \frac{\log_5 x}{\log_5 2} \\ &= \log_5 2 \times \log_2 x \quad \left\{ \frac{\log_c a}{\log_c b} = \log_b a \right\} \end{aligned}$$

A vertical stretch with scale factor  $\log_5 2$  will map the graph of  $y = \log_2 x$  to the graph of  $y = \log_5 x$ .



- 7 a** For  $y = \log_2(x + 2)$ :  
as  $x \rightarrow \infty$ ,  $y \rightarrow \infty$   
 $\therefore$  **A** is  $y = \log_2(x + 2)$  since it is increasing.  
For  $y = 3 - \log_2(3x + 1)$ :  
as  $x \rightarrow \infty$ ,  $\log_2(3x + 1) \rightarrow \infty$   
 $\therefore 3 - \log_2(3x + 1) \rightarrow -\infty$   
 $\therefore$  **B** is  $y = 3 - \log_2(3x + 1)$  since it is decreasing.



- b** For  $y = \log_2(x + 2)$ :

$$\begin{aligned} \text{When } x = 0, \quad y &= \log_2(0 + 2) \\ &= \log_2 2 \\ &= 1 \end{aligned}$$

$$\begin{aligned} \text{When } y = 0, \quad \log_2(x + 2) &= 0 \\ \therefore x + 2 &= 1 \\ \therefore x &= -1 \end{aligned}$$

As  $x \rightarrow -2^+$ ,  $y \rightarrow -\infty$ , so the vertical asymptote is  $x = -2$ .

As  $x \rightarrow \infty$ ,  $y \rightarrow \infty$ , so there is no horizontal asymptote.

So,  $y = \log_2(x + 2)$  has  $x$ -intercept  $-1$ ,  $y$ -intercept  $1$ , and vertical asymptote  $x = -2$ .

For  $y = 3 - \log_2(3x + 1)$ :

$$\begin{aligned}\text{When } x = 0, \quad y &= 3 - \log_2(3(0) + 1) \\ &= 3 - \log_2 1 \\ &= 3 - 0 \\ &= 3\end{aligned}$$

$$\begin{aligned}\text{When } y = 0, \quad 3 - \log_2(3x + 1) &= 0 \\ \therefore \log_2(3x + 1) &= 3 \\ \therefore 3x + 1 &= 2^3 \\ \therefore 3x + 1 &= 8 \\ \therefore 3x &= 7 \\ \therefore x &= \frac{7}{3}\end{aligned}$$

As  $x \rightarrow -\frac{1}{3}^+$ ,  $y \rightarrow \infty$ , so the vertical asymptote is  $x = -\frac{1}{3}$ .

As  $x \rightarrow \infty$ ,  $y \rightarrow -\infty$ , so there is no horizontal asymptote.

So,  $y = 3 - \log_2(3x + 1)$  has  $x$ -intercept  $\frac{7}{3}$ ,  $y$ -intercept 3, and vertical asymptote  $x = -\frac{1}{3}$ .

**c**  $y = 3 - \log_2(3x + 1)$  meets  $y = \log_2(x + 2)$  where

$$\begin{aligned}3 - \log_2(3x + 1) &= \log_2(x + 2) \\ \therefore 3 &= \log_2(x + 2) + \log_2(3x + 1) \\ \therefore \log_2(2^3) &= \log_2((x + 2)(3x + 1)) \\ \therefore \log_2 8 &= \log_2(3x^2 + 7x + 2) \\ \therefore 8 &= 3x^2 + 7x + 2 \\ \therefore 3x^2 + 7x - 6 &= 0 \\ \therefore (3x - 2)(x + 3) &= 0 \\ \therefore x &= \frac{2}{3} \text{ or } -3 \\ \therefore x &= \frac{2}{3} \quad \{x > -2\}\end{aligned}$$

$$\begin{aligned}\text{Substituting } x = \frac{2}{3} \text{ into } y = 3 - \log_2(3x + 1) \text{ gives } y &= 3 - \log_2(3(\frac{2}{3}) + 1) \\ &= 3 - \log_2(2 + 1) \\ &= 3 - \log_2 3\end{aligned}$$

$\therefore$  the graphs intersect at  $(\frac{2}{3}, 3 - \log_2 3)$ .

**8 a**  $f$  is defined by  $y = 3^x$

$$\begin{aligned}\therefore f^{-1} \text{ is defined by } x &= 3^y \\ \therefore \log_3 x &= \log_3(3^y) \\ \therefore \log_3 x &= y \\ \therefore y &= \log_3 x \\ \therefore f^{-1}(x) &= \log_3 x\end{aligned}$$

The domain of  $f^{-1}$  is  $\{x \mid x > 0\}$ .

The range of  $f^{-1}$  is  $\{y \mid y \in \mathbb{R}\}$ .

**b**  $f$  is defined by  $y = 2^{x+1}$

$$\begin{aligned}\therefore f^{-1} \text{ is defined by } x &= 2^{y+1} \\ \therefore \log_2 x &= \log_2(2^{y+1}) \\ \therefore \log_2 x &= y + 1 \\ \therefore y &= \log_2 x - 1 \\ \therefore f^{-1}(x) &= \log_2 x - 1\end{aligned}$$

The domain of  $f^{-1}$  is  $\{x \mid x > 0\}$ .

The range of  $f^{-1}$  is  $\{y \mid y \in \mathbb{R}\}$ .



$$\begin{aligned}
 \text{c} \quad & f \text{ is defined by } y = e^{2x} \\
 \therefore f^{-1} \text{ is defined by } & x = e^{2y} \\
 \therefore \ln x = \ln(e^{2y}) & \\
 \therefore \ln x = 2y & \\
 \therefore y = \frac{1}{2} \ln x & \\
 \therefore f^{-1}(x) = \frac{1}{2} \ln x &
 \end{aligned}$$

The domain of  $f^{-1}$  is  $\{x \mid x > 0\}$ .

The range of  $f^{-1}$  is  $\{y \mid y \in \mathbb{R}\}$ .

$$\begin{aligned}
 \text{d} \quad & f \text{ is defined by } y = 5^x - 3 \\
 \therefore f^{-1} \text{ is defined by } & x = 5^y - 3 \\
 \therefore x + 3 = 5^y & \\
 \therefore \log_5(x + 3) = \log_5(5^y) & \\
 \therefore \log_5(x + 3) = y & \\
 \therefore y = \log_5(x + 3) & \\
 \therefore f^{-1}(x) = \log_5(x + 3) &
 \end{aligned}$$

The domain of  $f^{-1}$  is  $\{x \mid x > -3\}$ .

The range of  $f^{-1}$  is  $\{y \mid y \in \mathbb{R}\}$ .

9  $f(x) = be^x$ ,  $g(x) = \ln(bx)$

$$\begin{aligned}
 \text{a} \quad (f \circ g)(x) &= f(g(x)) \\
 &= f(\ln(bx)) \\
 &= be^{\ln(bx)} \\
 &= b(bx) \\
 &= b^2x
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad (g \circ f)(x) &= g(f(x)) \\
 &= g(be^x) \\
 &= \ln(b \times be^x) \\
 &= \ln(b^2e^x) \\
 &= \ln(b^2) + \ln(e^x) \\
 &= 2 \ln b + x
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad (f \circ g)(x) &= (g \circ f)(x) \\
 \therefore b^2x &= 2 \ln b + x \\
 \therefore b^2x - x &= 2 \ln b \\
 \therefore x(b^2 - 1) &= 2 \ln b \\
 \therefore x &= \frac{2 \ln b}{b^2 - 1}
 \end{aligned}$$

10  $f(x) = e^{x+2}$ ,  $g(x) = \ln x - 3$

$$\begin{aligned}
 \text{a} \quad (f \circ g)(x) &= f(g(x)) \\
 &= f(\ln x - 3), \quad x > 0 \\
 &= e^{\ln x - 3 + 2}, \quad x > 0 \\
 &= e^{\ln x - 1}, \quad x > 0 \\
 &= e^{\ln x} \times e^{-1}, \quad x > 0 \\
 &= \frac{x}{e}, \quad x > 0
 \end{aligned}$$

The domain is  $\{x \mid x > 0\}$ .

The range is  $\{y \mid y > 0\}$ .

$$\begin{aligned}
 \text{b} \quad (g \circ f)(x) &= g(f(x)) \\
 &= g(e^{x+2}) \\
 &= \ln(e^{x+2}) - 3 \\
 &= x + 2 - 3 \\
 &= x - 1
 \end{aligned}$$

The domain is  $\{x \mid x \in \mathbb{R}\}$ .

The range is  $\{y \mid y \in \mathbb{R}\}$ .

11  $f(x) = \log_2(1 - 3x)$ ,  $g(x) = 2x + 1$

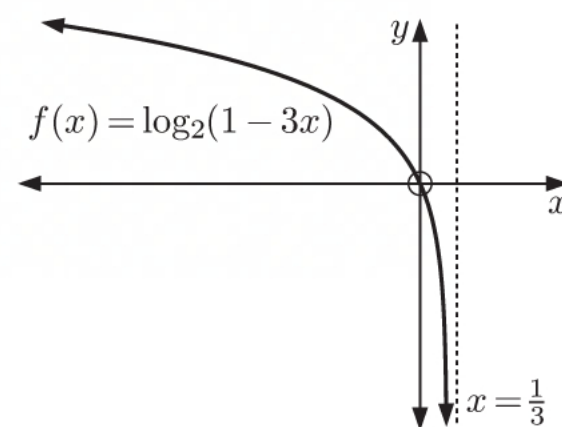
$$\begin{aligned}
 \text{a} \quad f(x) = \log_2(1 - 3x) \text{ is defined when } & 1 - 3x > 0 \\
 \therefore 1 > 3x & \\
 \therefore x < \frac{1}{3} &
 \end{aligned}$$

$\therefore$  the domain of  $f(x)$  is  $\{x \mid x < \frac{1}{3}\}$ .

As  $x \rightarrow \frac{1}{3}^-$ ,  $f(x) \rightarrow -\infty$ .

As  $x \rightarrow -\infty$ ,  $f(x) \rightarrow \infty$ .

$\therefore$  the range of  $f(x)$  is  $\{y \mid y \in \mathbb{R}\}$ .



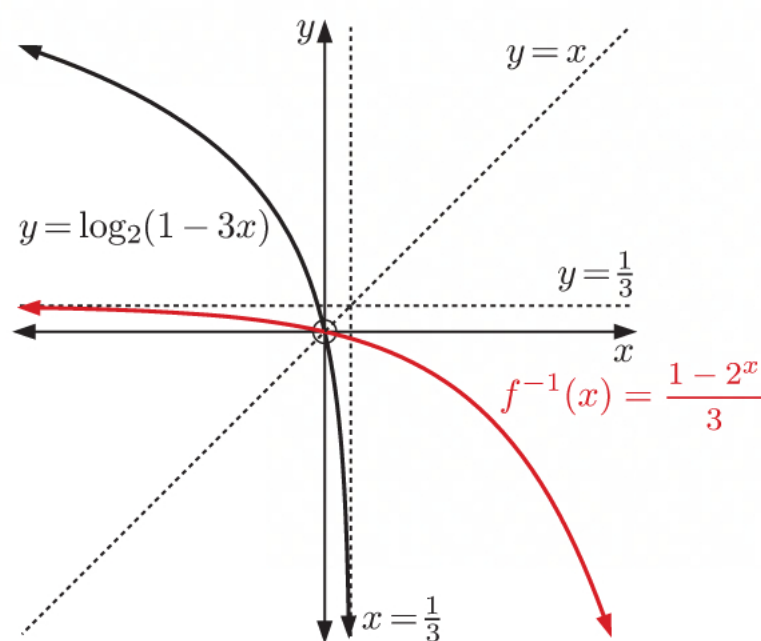
$$\begin{aligned}
 \text{b i} \quad & (f \circ g)(x) = 5 \\
 & \therefore f(g(x)) = 5 \\
 & \therefore f(2x+1) = 5 \\
 & \therefore \log_2(1-3(2x+1)) = 5 \\
 & \therefore \log_2(-2-6x) = 5 \\
 & \therefore \log_2(-2-6x) = \log_2(2^5) \\
 & \therefore -2-6x = 2^5 \\
 & \therefore -6x = 32+2 \\
 & \therefore x = -\frac{34}{6} \\
 & \therefore x = -\frac{17}{3}
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & f \text{ is defined by } y = \log_2(1-3x) \\
 & \therefore f^{-1} \text{ is defined by } x = \log_2(1-3y) \\
 & \therefore \log_2(2^x) = \log_2(1-3y) \\
 & \therefore 2^x = 1-3y \\
 & \therefore 3y = 1-2^x \\
 & \therefore y = \frac{1-2^x}{3} \\
 & \therefore f^{-1}(x) = \frac{1-2^x}{3}
 \end{aligned}$$

The domain of  $f^{-1}$  is  $\{x \mid x \in \mathbb{R}\}$ .

The range of  $f^{-1}$  is  $\{y \mid y < \frac{1}{3}\}$ .

$$\begin{aligned}
 \text{ii} \quad & (g \circ f)(x) = 0 \\
 & \therefore g(f(x)) = 0 \\
 & \therefore g(\log_2(1-3x)) = 0 \\
 & \therefore 2(\log_2(1-3x)) + 1 = 0 \\
 & \therefore \log_2(1-3x) = -\frac{1}{2} \\
 & \therefore \log_2(1-3x) = \log_2(2^{-\frac{1}{2}}) \\
 & \therefore 1-3x = 2^{-\frac{1}{2}} \\
 & \therefore 1-3x = \frac{1}{\sqrt{2}} \\
 & \therefore -3x = \frac{1}{\sqrt{2}} - 1 \\
 & \therefore x = \frac{1}{3} - \frac{1}{3\sqrt{2}}
 \end{aligned}$$



$$12 \quad f: x \mapsto e^{2x}, \quad g: x \mapsto 2x-1$$

$$\begin{aligned}
 \text{a} \quad & f \text{ is defined by } y = e^{2x} \\
 & \therefore f^{-1} \text{ is defined by } x = e^{2y} \\
 & \therefore 2y = \ln x \\
 & \therefore y = \frac{1}{2} \ln x \\
 & \therefore f^{-1}(x) = \frac{1}{2} \ln x
 \end{aligned}$$

$$\begin{aligned}
 (f^{-1} \circ g)(x) &= f^{-1}(g(x)) \\
 &= f^{-1}(2x-1) \\
 &= \frac{1}{2} \ln(2x-1)
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & (g \circ f)(x) = g(f(x)) \\
 &= g(e^{2x}) \\
 &= 2e^{2x} - 1
 \end{aligned}$$

$$\begin{aligned}
 & (g \circ f)(x) \text{ is defined by } y = 2e^{2x} - 1 \\
 & \therefore (g \circ f)^{-1}(x) \text{ is defined by } x = 2e^{2y} - 1
 \end{aligned}$$

$$\therefore x+1 = 2e^{2y}$$

$$\therefore e^{2y} = \frac{x+1}{2}$$

$$\therefore 2y = \ln\left(\frac{x+1}{2}\right)$$

$$\therefore y = \frac{1}{2} \ln\left(\frac{x+1}{2}\right)$$

$$\therefore (g \circ f)^{-1}(x) = \frac{1}{2} \ln\left(\frac{x+1}{2}\right)$$

**13 a**  $f(x) = \log_k(-x^2 + 2x + 8)$

From the graph,  $y = f(x)$  passes through  $(1, 2)$ .

$$\therefore 2 = \log_k(-1^2 + 2(1) + 8)$$

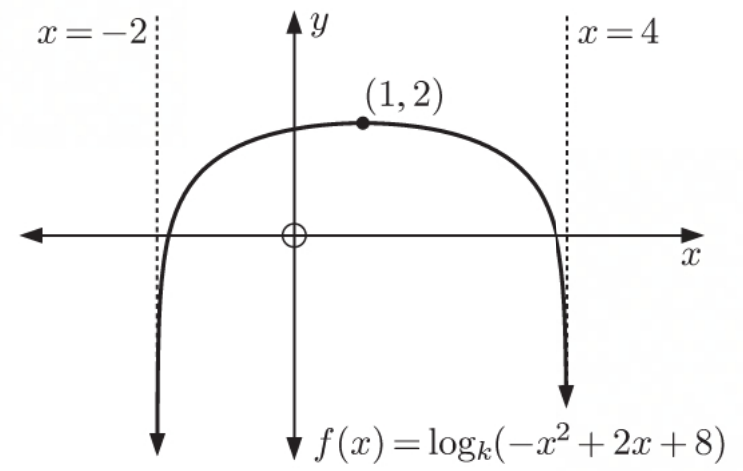
$$\therefore 2 = \log_k 9$$

$$\therefore 2 = \log_k(3^2)$$

$$\therefore 2 = 2\log_k 3$$

$$\therefore 1 = \log_k 3$$

$$\therefore k = 3$$



**b**  $f(x) = \log_3(-x^2 + 2x + 8)$

Now  $f(0) = \log_3 8$

and  $f(x) = 0$  when  $\log_3(-x^2 + 2x + 8) = 0$

$$\therefore -x^2 + 2x + 8 = 1$$

$$\therefore x^2 - 2x - 7 = 0$$

$$\therefore x^2 - 2x = 7$$

$$\therefore x^2 - 2x + (-1)^2 = 7 + (-1)^2$$

$$\therefore (x - 1)^2 = 8$$

$$\therefore x - 1 = \pm\sqrt{8}$$

$$\therefore x = 1 \pm 2\sqrt{2}$$

So, the  $y$ -intercept is  $\log_3 8$  and the  $x$ -intercepts are  $1 \pm 2\sqrt{2}$ .

**c**  $g(x) = f(x), -2 < x \leq 1$

$g$  is defined by  $y = \log_3(-x^2 + 2x + 8), -2 < x \leq 1$

$\therefore g^{-1}$  is defined by  $x = \log_3(-y^2 + 2y + 8), -2 < y \leq 1$

$$\therefore \log_3(3^x) = \log_3(-y^2 + 2y + 8)$$

$$\therefore 3^x = -y^2 + 2y + 8$$

$$\therefore y^2 - 2y = 8 - 3^x$$

$$\therefore y^2 - 2y + (-1)^2 = 8 - 3^x + (-1)^2$$

$$\therefore (y - 1)^2 = 9 - 3^x$$

$$\therefore y - 1 = -\sqrt{9 - 3^x} \quad \{\text{as } -3 < y - 1 \leq 0\}$$

$$\therefore y = 1 - \sqrt{9 - 3^x}$$

$$\therefore g^{-1}(x) = 1 - \sqrt{9 - 3^x}$$

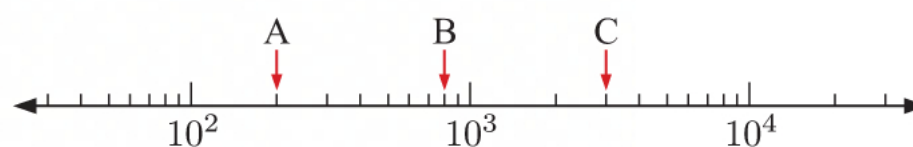
The domain of  $g^{-1}$  is  $\{x \mid x \leq 2\}$ .

The range of  $g^{-1}$  is  $\{y \mid -2 < y \leq 1\}$ .

## INVESTIGATION 3

## LOGARITHMIC SCALES

- 1 a** A - 200,  
B - 800,  
C - 3000



- b** The minor ticks correspond to integer multiples of each power of 10. For example, the minor ticks between  $10^1$  and  $10^2$  are:

$$20 = 2 \times 10^1 = 10^{\log 2} \times 10^1 = 10^{1+\log 2} \approx 10^{1.301}$$

$$30 = 3 \times 10^1 = 10^{\log 3} \times 10^1 = 10^{1+\log 3} \approx 10^{1.477}$$

$$40 = 4 \times 10^1 = 10^{\log 4} \times 10^1 = 10^{1+\log 4} \approx 10^{1.602}$$

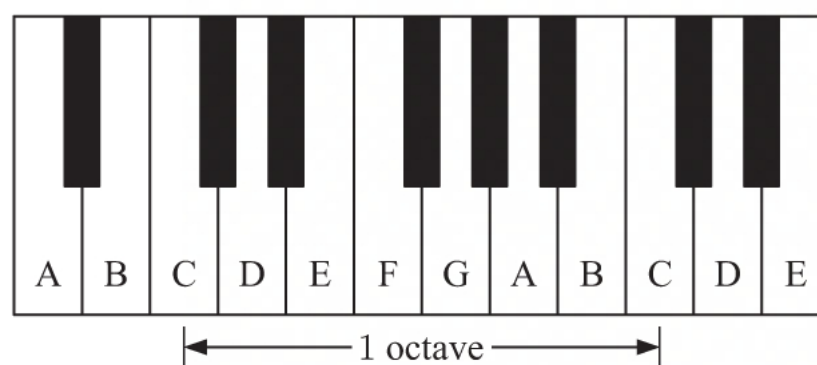
$$50 = 5 \times 10^1 = 10^{\log 5} \times 10^1 = 10^{1+\log 5} \approx 10^{1.699} \text{ and so on.}$$

The minor ticks are therefore placed between the major ticks in the positions  $\log 2$ ,  $\log 3$ ,  $\log 4$ ,  $\log 5$ , .... and these values are not evenly spaced.

- c** 0 is not on the logarithmic scale since  $\log 0$  is undefined.

- 2 a** Two notes separated by 3 octaves are 3 orders of magnitude apart.

**b**  $f = 261.6 \times 2^n \text{ Hz}$



- c** Let  $x$  be the difference in frequency between two adjacent notes.

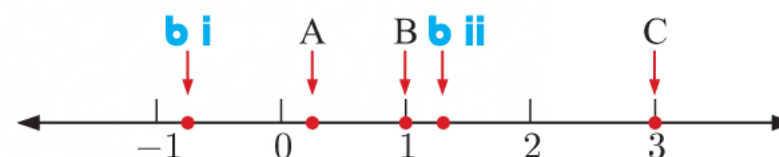
$$\therefore 261.6 + 12x = 2 \times 261.6 \quad \{12 \text{ notes in an octave}\}$$

$$\therefore 12x = 261.6$$

$$\therefore x = 21.8$$

$$\therefore \text{the ratio of frequencies between two adjacent notes} = 261.6 + 21.8 : 261.6 \\ \approx 1.08 : 1$$

- 3 a** C is  $10^{3-1} = 100$  times larger than B.



- 4 a i** If  $M = 0$ ,  $I = I_0$ , the earthquake intensity is equal to the reference intensity.  
**ii** If  $M = 1$ ,  $I = 10I_0$ , the earthquake intensity is 10 times greater than the reference intensity.

- b** The magnitude of an earthquake follows a logarithmic scale. A magnitude 6 earthquake is  $10^{6-3} = 1000$  times more intense than a magnitude 3 earthquake.

- c** The intensity  $I_4$  of an earthquake with magnitude 4 obeys  $4 = \log\left(\frac{I_4}{I_0}\right)$

$$\therefore \frac{I_4}{I_0} = 10^4$$

$$\therefore I_4 = 10^4 I_0$$

The magnitude of an earthquake with half this intensity is  $M = \log\left(\frac{\frac{1}{2} \times 10^4 I_0}{I_0}\right)$

$$= \log \frac{1}{2} + \log 10^4$$

$$= 4 - \log 2$$

$$\approx 3.70$$



- 5 a** The possible concentrations of  $\text{H}_3\text{O}^+$  take extremely small values that would otherwise be impossible to compare.

**b**  $\text{pH} = -\log C$

**i** If  $C = 0.000\,234$ ,  $\text{pH} = -\log(0.000\,234)$   
 $\approx 3.63$

**ii** If  $\text{pH} = 7$ ,  $7 = -\log C$   
 $\therefore \log C = -7$   
 $\therefore C = 10^{-7}$  units

**c** If  $C > 1$ , then  $\log C > 0$   
 $\therefore \text{pH} = -\log C < 0$

So, it is possible for a solution to have negative pH. In this case, the concentration of  $\text{H}_3\text{O}^+$  is greater than 1 unit.

## REVIEW SET 3A

**1 a**  $\log \sqrt{10}$   
 $= \log(10^{\frac{1}{2}})$   
 $= \frac{1}{2}$

**b**  $\log\left(\frac{1}{\sqrt[3]{10}}\right)$   
 $= \log(10^{-\frac{1}{3}})$   
 $= -\frac{1}{3}$

**c**  $\log(10^a \times 10^{b+1})$   
 $= \log(10^{a+b+1})$   
 $= a + b + 1$

**2 a**  $\log_4 64$   
 $= \log_4(4^3)$   
 $= 3$

**b**  $\log_2 256$   
 $= \log_2(2^8)$   
 $= 8$

**c**  $\log_2(0.25)$   
 $= \log_2\left(\frac{1}{4}\right)$   
 $= \log_2(2^{-2})$   
 $= -2$

**d**  $\log_{25} 5$   
 $= \log_{25}(25^{\frac{1}{2}})$   
 $= \frac{1}{2}$

**e**  $\log_8 1$   
 $= \log_8(8^0)$   
 $= 0$

**f**  $\log_{81} 3$   
 $= \log_{81}(81^{\frac{1}{4}})$   
 $= \frac{1}{4}$

**g**  $\log_9\left(\frac{1}{9}\right)$   
 $= \log_9(9^{-1})$   
 $= -1$

**h**  $\log_k \sqrt{k}$   
 $= \log_k(k^{\frac{1}{2}})$   
 $= \frac{1}{2}$   
 provided  $k > 0$ ,  
 $k \neq 1$

**3 a**  $\log 27$   
 $\approx 1.431$

**b**  $\log(0.58)$   
 $\approx -0.237$

**c**  $\log 400$   
 $\approx 2.602$

**d**  $\ln 40$   
 $\approx 3.689$

**4**  $y = \log_3 \sqrt{2-x}$   
 $\therefore y = \log_3((2-x)^{\frac{1}{2}})$   
 $\therefore y = \frac{1}{2} \log_3(2-x)$   
 $\therefore 2y = \log_3(2-x)$   
 $\therefore \log_3(3^{2y}) = \log_3(2-x)$   
 $\therefore 3^{2y} = 2-x$   
 $\therefore x = 2 - 3^{2y}$

$$\begin{array}{llll}
 \mathbf{5} \quad \mathbf{a} & 4 \ln 2 + 2 \ln 3 & \mathbf{b} & \frac{1}{2} \ln 9 - \ln 2 & \mathbf{c} & 2 \ln 5 - 1 & \mathbf{d} & \frac{1}{4} \ln 81 \\
 & = \ln(2^4) + \ln(3^2) & & = \ln(9^{\frac{1}{2}}) - \ln 2 & & = \ln(5^2) - \ln(e^1) & & = \ln(3^4)^{\frac{1}{4}} \\
 & = \ln(16 \times 9) & & = \ln 3 - \ln 2 & & = \ln\left(\frac{25}{e}\right) & & = \ln(3^1) \\
 & = \ln 144 & & = \ln\left(\frac{3}{2}\right) & & & & = \ln 3
 \end{array}$$

$$\begin{array}{lll}
 \mathbf{6} \quad \mathbf{a} & \log 16 + 2 \log 3 & \mathbf{b} \quad \log_2 16 - 2 \log_2 3 \\
 & = \log 16 + \log(3^2) & & = \log_2 16 - \log_2(3^2) \\
 & = \log(16 \times 9) & & = \log_2\left(\frac{16}{9}\right) \\
 & = \log 144 & & \\
 & & \mathbf{c} & 2 + \log_4 5 \\
 & & & = \log_4(4^2) + \log_4 5 \\
 & & & = \log_4(16 \times 5) \\
 & & & = \log_4 80
 \end{array}$$

$$\begin{array}{lll}
 \mathbf{7} \quad \mathbf{a} & \log_5 36 & \mathbf{b} \quad \log_5 54 \\
 & = \log_5(4 \times 9) & & = \log_5(2 \times 27) \\
 & = \log_5(2^2 \times 3^2) & & = \log_5(2 \times 3^3) \\
 & = \log_5(2^2) + \log_5(3^2) & & = \log_5 2 + \log_5(3^3) \\
 & = 2 \log_5 2 + 2 \log_5 3 & & = \log_5 2 + 3 \log_5 3 \\
 & = 2A + 2B & & = A + 3B \\
 & & \mathbf{c} & \log_5(8\sqrt{3}) \\
 & & & = \log_5(2^3 \times 3^{\frac{1}{2}}) \\
 & & & = \log_5(2^3) + \log_5(3^{\frac{1}{2}}) \\
 & & & = 3 \log_5 2 + \frac{1}{2} \log_5 3 \\
 & & & = 3A + \frac{1}{2}B \\
 & \mathbf{d} & \log_5(\sqrt{6}) & \mathbf{e} \quad \log_5(20.25) \\
 & & = \log_5(6^{\frac{1}{2}}) & & = \log_5\left(\frac{81}{4}\right) \\
 & & = \frac{1}{2} \log_5 6 & & = \log_5\left(\frac{3^4}{2^2}\right) \\
 & & = \frac{1}{2} \log_5(2 \times 3) & & = \log_5(3^4) - \log_5(2^2) \\
 & & = \frac{1}{2}(\log_5 2 + \log_5 3) & & = 4 \log_5 3 - 2 \log_5 2 \\
 & & = \frac{1}{2}(A + B) & & = 4B - 2A \\
 & & & \mathbf{f} & \log_5\left(\frac{8}{9}\right) \\
 & & & & = \log_5\left(\frac{2^3}{3^2}\right) \\
 & & & & = \log_5(2^3) - \log_5(3^2) \\
 & & & & = 3 \log_5 2 - 2 \log_5 3 \\
 & & & & = 3A - 2B
 \end{array}$$

$$\begin{array}{ll}
 \mathbf{8} \quad \mathbf{a} & M = ab^n \\
 & \therefore \log M = \log(ab^n) \\
 & \therefore \log M = \log a + \log(b^n) \\
 & \therefore \log M = \log a + n \log b
 \end{array}$$

$$\begin{array}{ll}
 \mathbf{b} & T = \frac{5}{\sqrt{l}} \\
 & \therefore \log T = \log\left(\frac{5}{l^{\frac{1}{2}}}\right) \\
 & \therefore \log T = \log 5 - \log(l^{\frac{1}{2}}) \\
 & \therefore \log T = \log 5 - \frac{1}{2} \log l
 \end{array}$$

$$\begin{array}{ll}
 \mathbf{c} & G = \frac{a^2b}{c} \\
 & \therefore \log G = \log\left(\frac{a^2b}{c}\right) \\
 & \therefore \log G = \log(a^2b) - \log c \\
 & \therefore \log G = \log(a^2) + \log b - \log c \\
 & \therefore \log G = 2 \log a + \log b - \log c
 \end{array}$$

$$\begin{aligned}
 9 \quad a \quad & 3^x = 300 \\
 & \therefore \log(3^x) = \log 300 \\
 & \therefore x \log 3 = \log 300 \\
 & \therefore x = \frac{\log 300}{\log 3} \\
 & \therefore x \approx 5.19
 \end{aligned}$$

$$\begin{aligned}
 c \quad & 3^{x+2} = 2^{1-x} \\
 & \therefore \log(3^{x+2}) = \log(2^{1-x}) \\
 & \therefore (x+2) \log 3 = (1-x) \log 2 \\
 & \therefore x \log 3 + 2 \log 3 = \log 2 - x \log 2 \\
 & \therefore x(\log 3 + \log 2) = \log 2 - 2 \log 3 \\
 & \therefore x \log(3 \times 2) = \log 2 - \log(3^2) \\
 & \therefore x \log 6 = \log\left(\frac{2}{9}\right) \\
 & \therefore x = \frac{\log\left(\frac{2}{9}\right)}{\log 6} \\
 & \therefore x \approx -0.839
 \end{aligned}$$

$$\begin{aligned}
 10 \quad a \quad & e^{2x} = 3e^x \\
 & \therefore e^{2x} - 3e^x = 0 \\
 & \therefore e^x(e^x - 3) = 0 \\
 & \therefore e^x - 3 = 0 \quad \{e^x > 0 \text{ for all } x\} \\
 & \therefore e^x = 3 \\
 & \therefore x = \ln 3
 \end{aligned}$$

$$\begin{aligned}
 11 \quad a \quad & \ln P = 1.5 \ln Q + \ln T \\
 & \therefore \ln P = \ln(Q^{1.5}) + \ln T \\
 & \quad = \ln(TQ^{1.5}) \\
 & \therefore P = TQ^{1.5}
 \end{aligned}$$

$$\begin{aligned}
 12 \quad a \quad & 3e^x - 5 = -2e^{-x} \\
 & \therefore 3e^{2x} - 5e^x = -2 \\
 & \therefore 3e^{2x} - 5e^x + 2 = 0 \\
 & \therefore (3e^x - 2)(e^x - 1) = 0 \\
 & \therefore 3e^x - 2 = 0 \quad \text{or} \quad e^x - 1 = 0 \\
 & \therefore e^x = \frac{2}{3} \quad \text{or} \quad e^x = 1 \\
 & \therefore x = \ln\left(\frac{2}{3}\right) \quad \text{or} \quad x = 0
 \end{aligned}$$

$$\begin{aligned}
 b \quad & 30 \times 5^{1-x} = 0.15 \\
 & \therefore 5^{1-x} = 0.005 \\
 & \therefore \log(5^{1-x}) = \log(0.005) \\
 & \therefore (1-x) \log 5 = \log(0.005) \\
 & \therefore 1-x = \frac{\log(0.005)}{\log 5} \\
 & \therefore x = 1 - \frac{\log(0.005)}{\log 5} \\
 & \therefore x \approx 4.29
 \end{aligned}$$

$$\begin{aligned}
 b \quad & e^{2x} - 7e^x + 12 = 0 \\
 & \therefore (e^x - 3)(e^x - 4) = 0 \\
 & \therefore e^x - 3 = 0 \quad \text{or} \quad e^x - 4 = 0 \\
 & \therefore e^x = 3 \quad \text{or} \quad e^x = 4 \\
 & \therefore x = \ln 3 \quad \text{or} \quad \ln 4
 \end{aligned}$$

$$\begin{aligned}
 b \quad & \ln M = 1.2 - 0.5 \ln N \\
 & \therefore \ln M + \ln(N^{\frac{1}{2}}) = 1.2 \\
 & \therefore \ln(M\sqrt{N}) = 1.2 \\
 & \therefore M\sqrt{N} = e^{1.2} \\
 & \therefore M = \frac{e^{1.2}}{\sqrt{N}}
 \end{aligned}$$

$$\begin{aligned}
 b \quad & 2 \ln x - 3 \ln\left(\frac{1}{x}\right) = 10 \\
 & \therefore \ln(x^2) - 3 \ln(x^{-1}) = 10 \\
 & \therefore \ln(x^2) + 3 \ln x = 10 \\
 & \therefore \ln(x^2) + \ln(x^3) = 10 \\
 & \therefore \ln(x^2 \times x^3) = 10 \\
 & \therefore \ln(x^5) = 10 \\
 & \therefore x^5 = e^{10} \\
 & \therefore x = (e^{10})^{\frac{1}{5}} \\
 & \therefore x = e^2
 \end{aligned}$$

$$\begin{aligned} \mathbf{13} \quad \mathbf{a} \quad \log_2 x &= -3 \\ \therefore x &= 2^{-3} \\ \therefore x &= \frac{1}{8} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \log_5 x &\approx 2.743 \\ \therefore x &\approx 5^{2.743} \\ \therefore x &\approx 82.7 \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad \log_3 x &\approx -3.145 \\ \therefore x &\approx 3^{-3.145} \\ \therefore x &\approx 0.0316 \end{aligned}$$

$$\begin{aligned} \mathbf{14} \quad \mathbf{a} \quad \mathbf{i} \quad 2^x &= 50 \\ \therefore \log(2^x) &= \log 50 \\ \therefore x \log 2 &= \log 50 \\ \therefore x &= \frac{\log 50}{\log 2} \end{aligned}$$

$$\mathbf{ii} \quad x = \frac{\log 50}{\log 2} \approx 5.64$$

$$\begin{aligned} \mathbf{c} \quad \mathbf{i} \quad (0.6)^x &= 0.01 \\ \therefore \log(0.6)^x &= \log(10^{-2}) \\ \therefore x \log(0.6) &= -2 \\ \therefore x &= \frac{-2}{\log(0.6)} \end{aligned}$$

$$\mathbf{ii} \quad x = \frac{-2}{\log(0.6)} \approx 9.02$$

$$\begin{aligned} \mathbf{b} \quad \mathbf{i} \quad 7^x &= 4 \\ \therefore \log(7^x) &= \log 4 \\ \therefore x \log 7 &= \log 4 \\ \therefore x &= \frac{\log 4}{\log 7} \end{aligned}$$

$$\mathbf{ii} \quad x = \frac{\log 4}{\log 7} \approx 0.71$$

$$\mathbf{15} \quad \log_a b = x$$

$$\begin{aligned} \text{Now, } \log_a \left( \frac{1}{b} \right) &= \log_a (b^{-1}) \\ &= -\log_a b \\ &= -x \end{aligned}$$

$$\begin{aligned} \mathbf{16} \quad \frac{8}{\log_5 9} &= \frac{8}{\left( \frac{\log_3 9}{\log_3 5} \right)} \quad \left\{ \log_a b = \frac{\log_c b}{\log_c a} \right\} \\ &= \frac{8 \log_3 5}{\log_3 9} \\ &= \frac{8 \log_3 5}{\log_3 (3^2)} \\ &= \frac{8 \log_3 5}{2} \\ &= 4 \log_3 5 \end{aligned}$$

$$\begin{aligned} \mathbf{17} \quad 16^x - 5 \times 8^x &= 0 \\ \therefore (2^4)^x - 5 \times (2^3)^x &= 0 \\ \therefore 2^{4x} - 5 \times 2^{3x} &= 0 \\ \therefore 2^{3x}(2^x - 5) &= 0 \\ \therefore 2^x - 5 &= 0 \quad \{2^{3x} \neq 0\} \\ \therefore 2^x &= 5 \\ \therefore x &= \log_2 5 \end{aligned}$$

$$\begin{aligned} \mathbf{18} \quad \mathbf{a} \quad \ln x &= 5 \\ \therefore x &= e^5 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad 3 \ln x + 2 &= 0 \\ \therefore 3 \ln x &= -2 \\ \therefore \ln x &= -\frac{2}{3} \\ \therefore x &= e^{-\frac{2}{3}} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad e^x &= 400 \\ \therefore x &= \ln 400 \end{aligned}$$



$$\begin{aligned}
 \text{d} \quad e^{2x+1} &= 11 \\
 \therefore 2x+1 &= \ln 11 \\
 \therefore 2x &= \ln 11 - 1 \\
 \therefore x &= \frac{\ln 11 - 1}{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad 25e^{\frac{x}{2}} &= 750 \\
 \therefore e^{\frac{x}{2}} &= 30 \\
 \therefore \frac{x}{2} &= \ln 30 \\
 \therefore x &= 2 \ln 30
 \end{aligned}$$

$$19 \quad f(x) = e^{3x-4} + 1$$

$$\begin{aligned}
 \text{a} \quad f \text{ is defined by } y &= e^{3x-4} + 1 \\
 \therefore f^{-1} \text{ is defined by } x &= e^{3y-4} + 1 \\
 \therefore e^{3y-4} &= x - 1 \\
 \therefore 3y - 4 &= \ln(x - 1) \\
 \therefore 3y &= \ln(x - 1) + 4 \\
 \therefore y &= \frac{\ln(x - 1) + 4}{3} \\
 \therefore f^{-1}(x) &= \frac{\ln(x - 1) + 4}{3}
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad f^{-1}(8) - f^{-1}(3) &= \frac{\ln 7 + 4}{3} - \frac{\ln 2 + 4}{3} \\
 &= \frac{\ln 7 + 4 - \ln 2 - 4}{3} \\
 &= \frac{1}{3} \ln\left(\frac{7}{2}\right)
 \end{aligned}$$

$$20 \quad x = 16y \quad \dots (*)$$

$$\begin{aligned}
 \text{Also, } \log_y x - \log_x y &= \frac{8}{3} \\
 \therefore \frac{\log x}{\log y} - \frac{\log y}{\log x} &= \frac{8}{3} \quad \left\{ \log_a b = \frac{\log b}{\log a} \right\} \\
 \therefore \frac{(\log x)^2 - (\log y)^2}{(\log x)(\log y)} &= \frac{8}{3} \\
 \therefore 3((\log x)^2 - (\log y)^2) &= 8(\log x)(\log y) \\
 \therefore 3(\log x)^2 - 3(\log y)^2 &= 8(\log x)(\log y) \\
 \therefore 3(\log x)^2 - 8(\log x)(\log y) - 3(\log y)^2 &= 0 \\
 \therefore (3\log x + \log y)(\log x - 3\log y) &= 0 \quad \{3a^2 - 8ab - 3b^2 = (3a + b)(a - 3b)\} \\
 \therefore 3\log x = -\log y \quad \text{or} \quad \log x &= 3\log y \\
 \therefore \log(x^3) = \log(y^{-1}) \quad \therefore \log x &= \log(y^3) \\
 \therefore x^3 = y^{-1} \quad \therefore x &= y^3 \\
 \therefore (16y)^3 = \frac{1}{y} \quad \{\text{using } (*)\} \quad \therefore 16y &= y^3 \quad \{\text{using } (*)\} \\
 \therefore 4096y^4 = 1 \quad \therefore 16y - y^3 &= 0 \\
 \therefore y^4 = \frac{1}{4096} \quad \therefore y(16 - y^2) &= 0 \\
 \therefore y = \pm \sqrt[4]{\frac{1}{4096}} \quad \therefore y(4 + y)(4 - y) &= 0 \\
 \therefore y = \frac{1}{8} \quad \{y > 0\} \quad \therefore y = 0 \quad \text{or} \quad \pm 4 \\
 &\therefore y = 4 \quad \{y > 0\}
 \end{aligned}$$

Using (\*), when  $y = \frac{1}{8}$ ,  $x = 16(\frac{1}{8}) = 2$

and when  $y = 4$ ,  $x = 16(4) = 64$

So, the solutions are  $x = 2$  and  $y = \frac{1}{8}$ , or  $x = 64$  and  $y = 4$ .

**21**  $g : x \mapsto \log_3(x + 2) - 2$

**a**  $g(x)$  is a translation of  $y = \log_3 x$  through  $\begin{pmatrix} -2 \\ -2 \end{pmatrix}$ .

**b** Domain =  $\{x \mid x > -2\}$ , Range =  $\{y \mid y \in \mathbb{R}\}$

**c** As  $x \rightarrow -2^+$ ,  $y \rightarrow -\infty$ , so the vertical asymptote is  $x = -2$ .

As  $x \rightarrow \infty$ ,  $y \rightarrow \infty$ , so there is no horizontal asymptote.

When  $x = 0$ ,  $y = \log_3 2 - 2 \approx -1.37$ .

So, the  $y$ -intercept is  $\approx -1.37$ .

$$\begin{aligned} \text{When } y = 0, \log_3(x + 2) &= 2 \\ \therefore x + 2 &= 3^2 \\ \therefore x &= 7 \end{aligned}$$

So, the  $x$ -intercept is 7.

**d**  $g$  is defined by  $y = \log_3(x + 2) - 2$

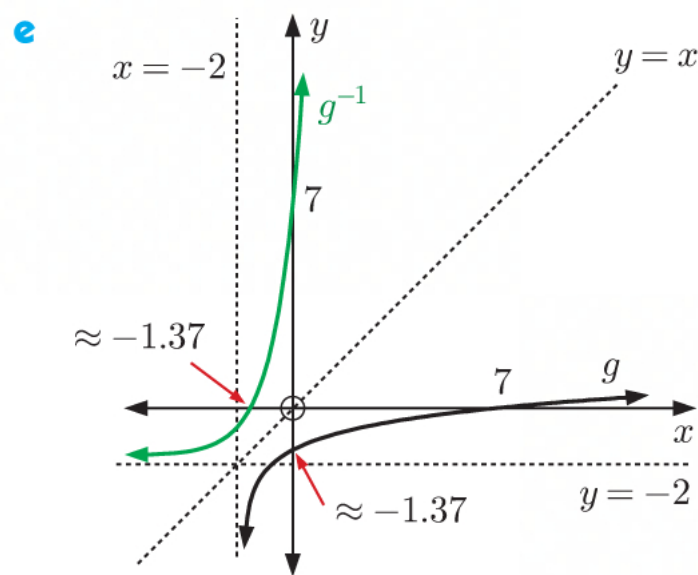
$\therefore g^{-1}$  is defined by  $x = \log_3(y + 2) - 2$

$$\therefore x + 2 = \log_3(y + 2)$$

$$\therefore y + 2 = 3^{x+2}$$

$$\therefore y = 3^{x+2} - 2$$

$$\therefore g^{-1}(x) = 3^{x+2} - 2$$



**22**  $W_t = 8000 \times e^{-\frac{t}{20}}$

$$W_0 = 8000 \times e^0 = 8000$$

$\therefore$  the initial weight is 8000 grams.

**a** For  $W_t$  to halve,  $W_t = 4000$

$$\therefore 8000 \times e^{-\frac{t}{20}} = 4000$$

$$\therefore e^{-\frac{t}{20}} = \frac{1}{2}$$

$$\therefore -\frac{t}{20} = \ln\left(\frac{1}{2}\right)$$

$$\therefore t = -20 \ln\left(\frac{1}{2}\right)$$

$$\therefore t \approx 13.9$$

$\therefore$  it will take about 13.9 weeks for the weight to halve.

**b** When  $W_t = 1000$ ,

$$8000 \times e^{-\frac{t}{20}} = 1000$$

$$\therefore e^{-\frac{t}{20}} = \frac{1}{8}$$

$$\therefore -\frac{t}{20} = \ln\left(\frac{1}{8}\right)$$

$$\therefore t = -20 \ln\left(\frac{1}{8}\right)$$

$$\therefore t \approx 41.6$$

$\therefore$  it will take about 41.6 weeks for the weight to reach 1000 grams.

- c** When  $W_t = 0.1\%$  of  $W_0$ ,

$$8000 \times e^{-\frac{t}{20}} = 0.001 \times 8000$$

$$\therefore e^{-\frac{t}{20}} = 0.001$$

$$\therefore -\frac{t}{20} = \ln(0.001)$$

$$\therefore t = -20 \ln(0.001)$$

$$\therefore t \approx 138$$

$\therefore$  it will take about 138 weeks for the weight to reach 0.1% of its original value.

**23**  $P(t) = 80 \times (1.15)^t$  seals

**a**  $P(0) = 80 \times (1.15)^0$   
 $= 80 \times 1$   
 $= 80$

So the initial population was 80 seals.

We need to find  $t$  when  $P(t) = 2 \times 80 = 160$

$$\therefore 80 \times (1.15)^t = 160$$

$$\therefore (1.15)^t = 2$$

$$\therefore \log(1.15)^t = \log 2$$

$$\therefore t \log(1.15) = \log 2$$

$$\therefore t = \frac{\log 2}{\log(1.15)}$$

$$\approx 4.96$$

$\therefore$  it took about 4.96 years or about 4 years and  $11\frac{1}{2}$  months for the population to double in size.

**b** Percentage increase in first 4 years  $= \left( \frac{P(4) - P(0)}{P(0)} \right) \times 100\%$

$$= \left( \frac{80 \times (1.15)^4 - 80}{80} \right) \times 100\%$$

$$\approx 74.9\%$$

**24 a**  $f(x) = \log_2(x + 4) - 1$

**i** Domain  $= \{x \mid x > -4\}$ , Range  $= \{y \mid y \in \mathbb{R}\}$

**ii** As  $x \rightarrow -4^+$ ,  $y \rightarrow -\infty$ , so the vertical asymptote is  $x = -4$ .

As  $x \rightarrow \infty$ ,  $y \rightarrow \infty$ , so there is no horizontal asymptote.

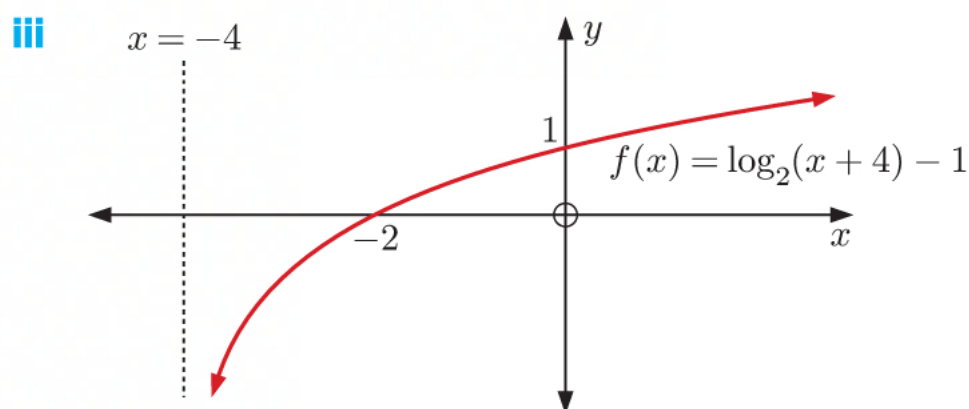
When  $x = 0$ ,  $\log_2 4 - 1 = 1$ , so the  $y$ -intercept is 1.

When  $y = 0$ ,  $\log_2(x + 4) = 1$

$$\therefore x + 4 = 2^1$$

$$\therefore x = -2$$

So, the  $x$ -intercept is  $-2$ .



b  $f(x) = \ln x + 2$

i Domain =  $\{x \mid x > 0\}$ , Range =  $\{y \mid y \in \mathbb{R}\}$

ii As  $x \rightarrow 0^+$ ,  $y \rightarrow -\infty$ , so the vertical asymptote is  $x = 0$ .

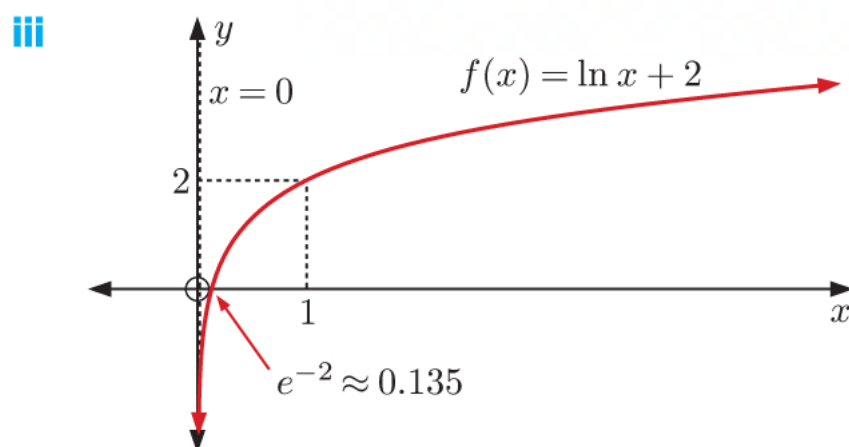
As  $x \rightarrow \infty$ ,  $y \rightarrow \infty$ , so there is no horizontal asymptote.

When  $x = 0$ ,  $y$  is undefined, so there is no  $y$ -intercept.

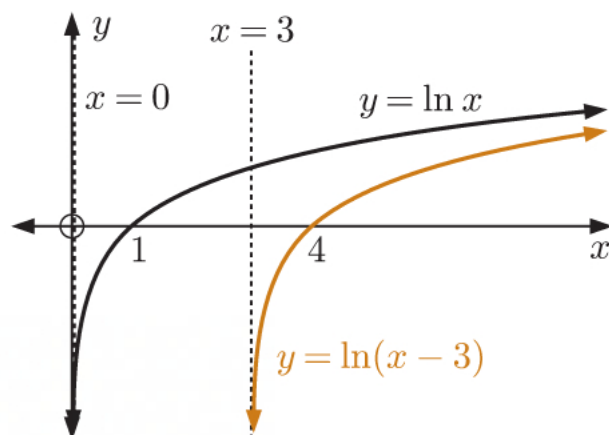
When  $y = 0$ ,  $\ln x = -2$

$$\therefore x = e^{-2} \approx 0.135$$

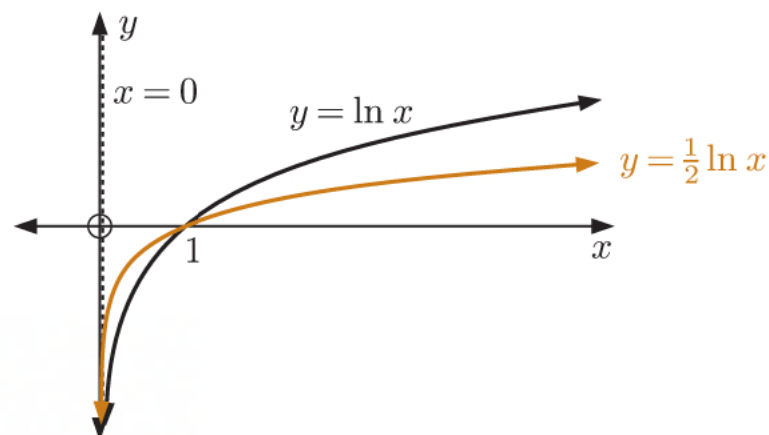
So, the  $x$ -intercept is  $e^{-2}$ .



25 a  $y = \ln(x - 3)$  is a horizontal translation of  $y = \ln x$ , 3 units to the right.



b  $y = \frac{1}{2} \ln x$  is a vertical stretch of  $y = \ln x$  with scale factor  $\frac{1}{2}$ .



26  $f(x) = e^x$ ,  $g(x) = \ln(x + 4)$ ,  $x > -4$

a  $(f \circ g)(5) = f(g(5))$   
 $= f(\ln 9)$   
 $= e^{\ln 9}$   
 $= 9$

b  $(g \circ f)(0) = g(f(0))$   
 $= g(e^0)$   
 $= g(1)$   
 $= \ln 5$



## REVIEW SET 3B

$$\begin{aligned}
 \text{1 a } \log \sqrt{1000} &= \log((10^3)^{\frac{1}{2}}) \\
 &= \log(10^{\frac{3}{2}}) \\
 &= \frac{3}{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{b } \log\left(\frac{10}{\sqrt[3]{10}}\right) &= \log\left(\frac{10^1}{10^{\frac{1}{3}}}\right) \\
 &= \log(10^{\frac{2}{3}}) \\
 &= \frac{2}{3}
 \end{aligned}$$

$$\begin{aligned}
 \text{c } \log\left(\frac{10^a}{10^{-b}}\right) &= \log(10^{a-(-b)}) \\
 &= \log(10^{a+b}) \\
 &= a + b
 \end{aligned}$$

$$\begin{aligned}
 \text{2 a } \log_2 128 &= \log_2(2^7) \\
 &= 7
 \end{aligned}$$

$$\begin{aligned}
 \text{b } \log_3\left(\frac{1}{27}\right) &= \log_3(3^{-3}) \\
 &= -3
 \end{aligned}$$

$$\begin{aligned}
 \text{c } \log_5\left(\frac{1}{\sqrt{5}}\right) &= \log_5(5^{-\frac{1}{2}}) \\
 &= -\frac{1}{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{3 a } 32 &= 10^{\log 32} \\
 &\approx 10^{1.5051}
 \end{aligned}$$

$$\begin{aligned}
 \text{b } 0.0013 &= 10^{\log(0.0013)} \\
 &\approx 10^{-2.8861}
 \end{aligned}$$

$$\begin{aligned}
 \text{c } 8.963 \times 10^{-5} &= 10^{\log(8.963)} \times 10^{-5} \\
 &= 10^{\log(8.963)-5} \\
 &\approx 10^{-4.0475}
 \end{aligned}$$

$$\begin{aligned}
 \text{4 a } \ln(e\sqrt{e}) &= \ln(e^1 e^{\frac{1}{2}}) \\
 &= \ln(e^{\frac{3}{2}}) \\
 &= \frac{3}{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{b } \ln\left(\frac{1}{e^3}\right) &= \ln(e^{-3}) \\
 &= -3
 \end{aligned}$$

$$\text{c } \ln(e^{2x}) = 2x$$

$$\begin{aligned}
 \text{d } \ln\left(\frac{e}{e^x}\right) &= \ln(e^{1-x}) \\
 &= 1 - x
 \end{aligned}$$

$$\begin{aligned}
 \text{5 a } \frac{\log_2 25}{\log_2 125} &= \frac{\log_2(5^2)}{\log_2(5^3)} \\
 &= \frac{2 \log_2 5}{3 \log_2 5} \\
 &= \frac{2}{3}
 \end{aligned}$$

$$\begin{aligned}
 \text{b } \frac{\log 64}{\log 32} &= \frac{\log(2^6)}{\log(2^5)} \\
 &= \frac{6 \log 2}{5 \log 2} \\
 &= \frac{6}{5}
 \end{aligned}$$

$$\begin{aligned}
 \text{c } \frac{\log_5 81}{\log_5 \sqrt{3}} &= \frac{\log_5(3^4)}{\log_5(3^{\frac{1}{2}})} \\
 &= \frac{4 \log_5 3}{\frac{1}{2} \log_5 3} \\
 &= \frac{4}{\frac{1}{2}} \\
 &= 8
 \end{aligned}$$

$$\begin{aligned}
 \text{6 a } e^{4 \ln x} &= (e^{\ln x})^4 \\
 &= x^4
 \end{aligned}$$

$$\text{b } \ln(e^5) = 5$$

$$\begin{aligned}
 \text{c } \ln(\sqrt{e}) &= \ln(e^{\frac{1}{2}}) \\
 &= \frac{1}{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad & 10^{\log x + \log 3} \\
 &= 10^{\log x} \times 10^{\log 3} \\
 &= x \times 3 \\
 &= 3x
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & \ln\left(\frac{1}{e^x}\right) = \ln(e^{-x}) \\
 &= -x
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & \frac{\log(x^2)}{\log_3 9} \\
 &= \frac{\log(x^2)}{\log_3(3^2)} \\
 &= \frac{\log(x^2)}{2} \\
 &= \frac{1}{2} \log(x^2) \\
 &= \log x
 \end{aligned}$$

$$\begin{aligned}
 7 \quad \text{a} \quad & 20 = e^{\ln 20} \\
 & \approx e^{2.9957}
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & 3000 = e^{\ln 3000} \\
 & \approx e^{8.0064}
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & 0.075 = e^{\ln(0.075)} \\
 & \approx e^{-2.5903}
 \end{aligned}$$

$$\begin{aligned}
 8 \quad \text{a} \quad \text{i} \quad & 5^x = 7 \\
 & \therefore \log(5^x) = \log 7 \\
 & \therefore x \log 5 = \log 7 \\
 & \therefore x = \frac{\log 7}{\log 5}
 \end{aligned}$$

$$\text{ii} \quad x = \frac{\log 7}{\log 5} \approx 1.21$$

$$\begin{aligned}
 \text{b} \quad \text{i} \quad & 2^x = 0.1 \\
 & \therefore \log(2^x) = \log(0.1) \\
 & \therefore x \log 2 = \log(10^{-1}) \\
 & \therefore x = -\frac{1}{\log 2}
 \end{aligned}$$

$$\text{ii} \quad x = -\frac{1}{\log 2} \approx -3.32$$

$$\begin{aligned}
 9 \quad \text{a} \quad & \ln 60 - \ln 20 \\
 &= \ln\left(\frac{60}{20}\right) \\
 &= \ln 3
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & \ln 4 + \ln 1 \\
 &= \ln 4 + 0 \\
 &= \ln 4
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & \ln 200 - \ln 8 + \ln 5 \\
 &= \ln\left(\frac{200}{8}\right) + \ln 5 \\
 &= \ln\left(\frac{200}{8} \times 5\right) \\
 &= \ln 125
 \end{aligned}$$

$$\begin{aligned}
 10 \quad \text{a} \quad & e^{2x} = 70 \\
 & \therefore \ln(e^{2x}) = \ln 70 \\
 & \therefore 2x = \ln 70 \\
 & \therefore x = \frac{\ln 70}{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & 3 \times (1.3)^x = 11 \\
 & \therefore (1.3)^x = \frac{11}{3} \\
 & \therefore \log(1.3)^x = \log\left(\frac{11}{3}\right) \\
 & \therefore x \log(1.3) = \log\left(\frac{11}{3}\right) \\
 & \therefore x = \frac{\log\left(\frac{11}{3}\right)}{\log(1.3)}
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & 5 \times 2^{0.3x} = 16 \\
 & \therefore 2^{0.3x} = \frac{16}{5} \\
 & \therefore \log(2^{0.3x}) = \log\left(\frac{16}{5}\right) \\
 & \therefore 0.3x \log 2 = \log\left(\frac{16}{5}\right) \\
 & \therefore x = \frac{\log\left(\frac{16}{5}\right)}{0.3 \log 2} \\
 & \therefore x = \frac{10 \log\left(\frac{16}{5}\right)}{3 \log 2}
 \end{aligned}$$

$$\begin{aligned}
 11 \quad & \log x = \ln x \\
 & \therefore \log x = \frac{\log x}{\log e} \quad \left\{ \log_a b = \frac{\log_c b}{\log_c a} \right\} \\
 & \therefore \log e \log x = \log x \\
 & \therefore \log x (\log e - 1) = 0 \\
 & \therefore \log x = 0 \quad \{\log e \neq 1\} \\
 & \therefore x = 10^0 \\
 & \therefore x = 1
 \end{aligned}$$

$\therefore x = 1$  is the only value of  $x$  for which  $\log x = \ln x$ .

$$\begin{aligned}
 \mathbf{12} \quad & \log_3(x-k) + \log_3(x+2) = 1 \\
 & \therefore \log_3((x-k)(x+2)) = \log_3(3^1) \\
 & \therefore (x-k)(x+2) = 3^1 \\
 & \therefore x^2 + 2x - kx - 2k = 3 \\
 & \therefore x^2 + (2-k)x - (2k+3) = 0 \quad \text{which has } a = 1, \quad b = 2-k, \quad \text{and } c = -(2k+3)
 \end{aligned}$$

$$\begin{aligned}
 \text{Now } \Delta &= b^2 - 4ac \\
 &= (2-k)^2 - 4(1)(-(2k+3)) \\
 &= 4 - 4k + k^2 + 4(2k+3) \\
 &= k^2 + 4k + 16 \\
 &= k^2 + 4k + 2^2 - 2^2 + 16 \\
 &= (k+2)^2 + 12
 \end{aligned}$$

$$\therefore \Delta > 0 \text{ for all } k \in \mathbb{R} \quad \{\text{as } (k+2)^2 \geq 0 \text{ for all } k \in \mathbb{R}\}$$

So,  $\log_3(x-k) + \log_3(x+2) = 1$  has a real solution for every  $k \in \mathbb{R}$ .

$$\begin{aligned}
 \mathbf{13} \quad \mathbf{a} \quad & P = 3 \times b^x \\
 & \therefore \log P = \log(3 \times b^x) \\
 & \therefore \log P = \log 3 + \log(b^x) \\
 & \therefore \log P = \log 3 + x \log b
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & m = \frac{n^3}{p^2} \\
 & \therefore \log m = \log\left(\frac{n^3}{p^2}\right) \\
 & \therefore \log m = \log(n^3) - \log(p^2) \\
 & \therefore \log m = 3 \log n - 2 \log p
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{14} \quad & \log_3 7 \times 2 \log_7 x = \log_3 7 \times 2 \times \frac{\log_3 x}{\log_3 7} \quad \left\{ \log_b a = \frac{\log_c a}{\log_c b} \right\} \\
 & = 2 \log_3 x
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{15} \quad \mathbf{a} \quad & \log_2(x^2) + \log_8 \sqrt{x} = 3 \\
 & \therefore 2 \log_2 x + \log_8(x^{\frac{1}{2}}) = 3 \\
 & \therefore 2 \log_2 x + \frac{\log_2(x^{\frac{1}{2}})}{\log_2 8} = 3 \quad \left\{ \log_b a = \frac{\log_c a}{\log_c b} \right\} \\
 & \therefore 2 \log_2 x + \frac{\frac{1}{2} \log_2 x}{\log_2(2^3)} = 3 \\
 & \therefore 2 \log_2 x + \frac{\frac{1}{2} \log_2 x}{3} = 3 \\
 & \therefore 2 \log_2 x + \frac{1}{6} \log_2 x = 3 \\
 & \therefore \frac{13}{6} \log_2 x = 3 \\
 & \therefore \log_2 x = \frac{18}{13} \\
 & \therefore \log_2 x = \log_2(2^{\frac{18}{13}}) \\
 & \therefore x = 2^{\frac{18}{13}}
 \end{aligned}$$

$$\text{b} \quad \log_{27} \left( \frac{1}{x} \right) + \log_3(x^4) = \log_3 10$$

$$\therefore \log_{27}(x^{-1}) + 4 \log_3 x = \log_3 10$$

$$\therefore -\log_{27} x + 4 \log_3 x = \log_3 10$$

$$\therefore -\frac{\log_3 x}{\log_3 27} + 4 \log_3 x = \log_3 10 \quad \left\{ \log_b a = \frac{\log_c a}{\log_c b} \right\}$$

$$\therefore -\frac{\log_3 x}{\log_3(3^3)} + 4 \log_3 x = \log_3 10$$

$$\therefore -\frac{\log_3 x}{3} + 4 \log_3 x = \log_3 10$$

$$\therefore \frac{11}{3} \log_3 x = \log_3 10$$

$$\therefore \log_3(x^{\frac{11}{3}}) = \log_3 10$$

$$\therefore x^{\frac{11}{3}} = 10$$

$$\therefore (x^{\frac{11}{3}})^{\frac{3}{11}} = 10^{\frac{3}{11}}$$

$$\therefore x = 10^{\frac{3}{11}}$$

$$\text{16 a} \quad \log T = 2 \log x - \log y$$

$$\therefore \log T = \log(x^2) - \log y$$

$$\therefore \log T = \log \left( \frac{x^2}{y} \right)$$

$$\therefore T = \frac{x^2}{y}$$

$$\text{b} \quad \log_2 K = \log_2 n + \frac{1}{2} \log_2 t$$

$$\therefore \log_2 K = \log_2 n + \log_2(t^{\frac{1}{2}})$$

$$\therefore \log_2 K = \log_2(n \times \sqrt{t})$$

$$\therefore K = n\sqrt{t}$$

$$\text{17 a} \quad \ln 32 = \ln(2^5) \\ = 5 \ln 2$$

$$\text{b} \quad \ln 125 = \ln(5^3) \\ = 3 \ln 5$$

$$\text{c} \quad \ln 729 = \ln(3^6) \\ = 6 \ln 3$$

$$\text{18} \quad \log_2 x \text{ is defined for all } x > 0$$

$$\therefore \text{the domain is } \{x \mid x > 0\}$$

$$\text{and the range is } \{y \mid y \in \mathbb{R}\}.$$

$$\ln(x+5) \text{ is defined for all } x > -5$$

$$\therefore \text{the domain is } \{x \mid x > -5\}$$

$$\text{and the range is } \{y \mid y \in \mathbb{R}\}.$$

So, the completed table is:

	$y = \log_2 x$	$y = \ln(x+5)$
Domain	$\{x \mid x > 0\}$	$\{x \mid x > -5\}$
Range	$\{y \mid y \in \mathbb{R}\}$	$\{y \mid y \in \mathbb{R}\}$

$$\text{19 a} \quad 4^x - 2^x - 20$$

$$= (2^x)^2 - 2^x - 20$$

$$= (2^x + 4)(2^x - 5) \quad \{a^2 - a - 20 = (a+4)(a-5)\}$$



**b**

$$\begin{aligned}
 2^x(2^x - 1) &= 20 \\
 \therefore (2^x)^2 - 2^x - 20 &= 0 \\
 \therefore (2^x + 4)(2^x - 5) &= 0 \quad \{\text{using a}\} \\
 \therefore 2^x &= -4 \text{ or } 5 \\
 \text{But } 2^x \text{ is never negative} \quad \therefore 2^x &= 5 \\
 \therefore \log(2^x) &= \log 5 \\
 \therefore x \log 2 &= \log 5 \\
 \therefore x &= \frac{\log 5}{\log 2} \text{ or } \log_2 5
 \end{aligned}$$

**c i** If  $p = \log_5 2$  then

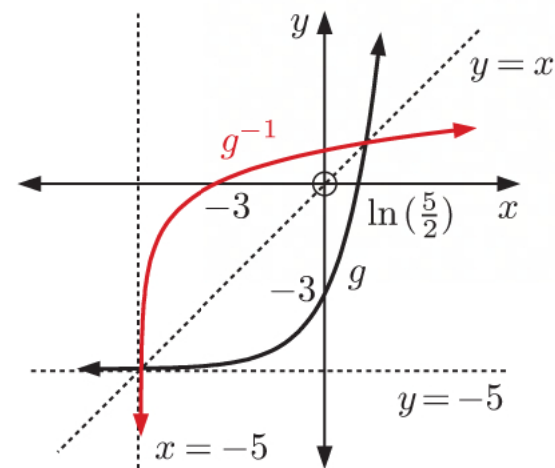
$$\begin{aligned}
 p &= \frac{\log 2}{\log 5} \\
 \therefore x &= \frac{1}{p}
 \end{aligned}$$

**ii**

$$\begin{aligned}
 8^x &= 5^{1-x} \\
 \therefore 2^{3x} &= 5^{1-x} \\
 \therefore \log(2^{3x}) &= \log(5^{1-x}) \\
 \therefore 3x \log 2 &= (1-x) \log 5 \\
 \therefore \frac{1-x}{3x} &= \frac{\log 2}{\log 5} = p \\
 \therefore 1-x &= 3px \\
 \therefore x(3p+1) &= 1 \\
 \therefore x &= \frac{1}{3p+1}
 \end{aligned}$$

**20**  $g: x \mapsto 2e^x - 5$ 

$$\begin{aligned}
 \text{a} \quad g \text{ is defined by } y &= 2e^x - 5 \\
 \therefore g^{-1} \text{ is defined by } x &= 2e^y - 5 \\
 \therefore x+5 &= 2e^y \\
 \therefore e^y &= \frac{x+5}{2} \\
 \therefore y &= \ln\left(\frac{x+5}{2}\right) \\
 \therefore g^{-1}(x) &= \ln\left(\frac{x+5}{2}\right)
 \end{aligned}$$

**b**

**c**  $g$ : Domain =  $\{x \mid x \in \mathbb{R}\}$ , Range =  $\{y \mid y > -5\}$   
 $g^{-1}$ : Domain =  $\{x \mid x > -5\}$ , Range =  $\{y \mid y \in \mathbb{R}\}$

**d** Consider  $g(x) = 2e^x - 5$ :As  $x \rightarrow -\infty$ ,  $y \rightarrow -5^+$ , so the horizontal asymptote is  $y = -5$ .When  $x = 0$ ,  $y = 2e^0 - 5$ 

$$\therefore y = -3$$

So, the  $y$ -intercept is  $-3$ .When  $y = 0$ ,  $2e^x - 5 = 0$ 

$$\therefore 2e^x = 5$$

$$\therefore e^x = \frac{5}{2}$$

$$\therefore x = \ln\left(\frac{5}{2}\right)$$

So, the  $x$ -intercept is  $\ln\left(\frac{5}{2}\right) \approx 0.916$ .

Since  $g^{-1}(x)$  is the reflection of  $g(x)$  in the line  $y = x$ , the  $x$  and  $y$ -intercepts of  $g(x)$  are the  $y$  and  $x$ -intercepts of  $g^{-1}(x)$  respectively.

The horizontal asymptote of  $g(x)$  corresponds to a vertical asymptote of  $g^{-1}(x)$ .

So,  $g^{-1}(x)$  has vertical asymptote  $x = -5$ ,  $x$ -intercept  $-3$ , and  $y$ -intercept  $\ln\left(\frac{5}{2}\right) \approx 0.916$ .

**21**  $T = 60e^{-0.1t} + 20$  °C

When  $T = 40$ ,  $60e^{-0.1t} + 20 = 40$

$$\therefore 60e^{-0.1t} = 20$$

$$\therefore e^{-0.1t} = \frac{1}{3}$$

$$\therefore -0.1t = \ln\left(\frac{1}{3}\right)$$

$$\therefore t = -10 \times -\ln 3$$

$$= 10 \ln 3 \text{ minutes as required}$$

**22**  $W(t) = 2500 \times 3^{-\frac{t}{3000}}$  grams

**a**  $W(0) = 2500 \times 3^0$   
 $= 2500 \times 1$   
 $= 2500$

$\therefore$  the initial weight was 2500 grams.

**b** When  $W(t) = 30\%$  of original weight,

$$2500 \times 3^{-\frac{t}{3000}} = 0.3 \times 2500$$

$$\therefore 3^{-\frac{t}{3000}} = 0.3$$

$$\therefore \log(3^{-\frac{t}{3000}}) = \log(0.3)$$

$$\therefore -\frac{t}{3000} \times \log 3 = \log(0.3)$$

$$\therefore t = \frac{-\log(0.3) \times 3000}{\log 3}$$

$$\therefore t \approx 3287.7$$

$\therefore$  it takes about 3288 years for the isotope to reduce to 30% of its original weight.

**c** Percentage change after 1500 years  $= \left( \frac{W(1500) - W(0)}{W(0)} \right) \times 100\%$   
 $= \left( \frac{2500 \times 3^{-\frac{1}{2}} - 2500}{2500} \right) \times 100\%$   
 $\approx -42.3\%$

$\therefore$  the percentage of weight lost after 1500 years is about 42.3%.

**23 a**  $5^{\frac{x}{2}} = 9$

$$\therefore \log(5^{\frac{x}{2}}) = \log 9$$

$$\therefore \frac{x}{2} \log 5 = \log 9$$

$$\therefore x = \frac{2 \log 9}{\log 5}$$

**b**  $e^x = 30$

$$\therefore x = \ln 30$$

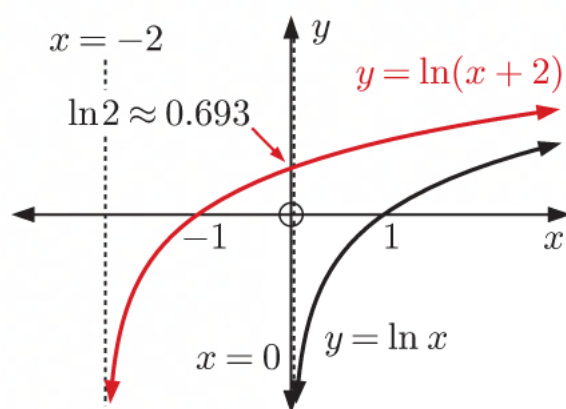
**c**  $e^{1-3x} = 2$

$$\therefore 1 - 3x = \ln 2$$

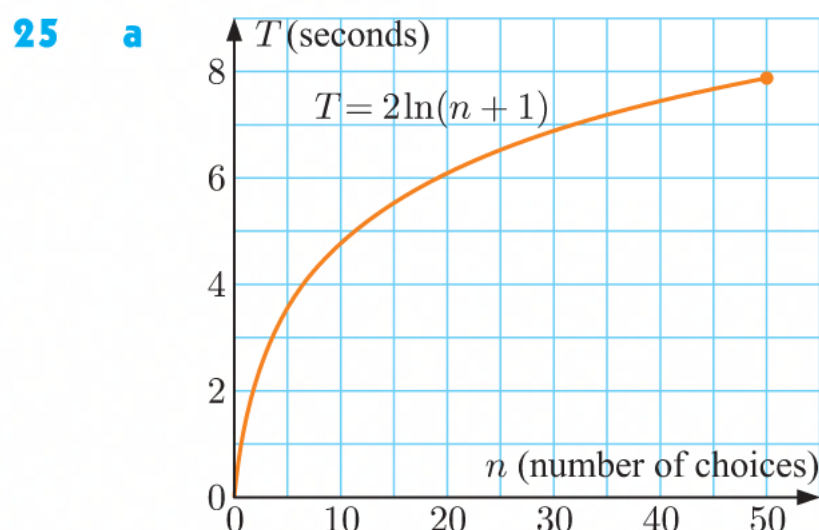
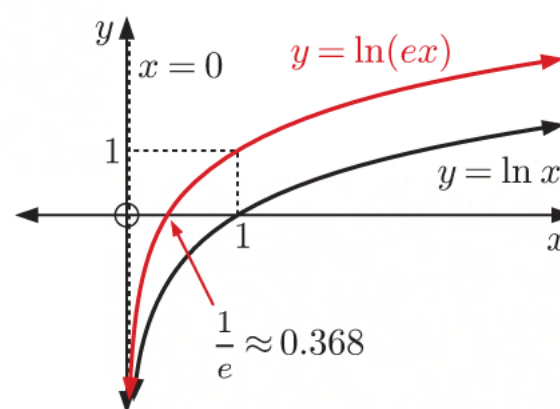
$$\therefore 3x = 1 - \ln 2$$

$$\therefore x = \frac{1 - \ln 2}{3}$$

- 24 a**  $y = \ln(x + 2)$  is a translation of  $y = \ln x$ , 2 units to the left.



- b**  $y = \ln(ex)$  is a horizontal stretch of  $y = \ln x$  with scale factor  $\frac{1}{e}$ .



- b**  $T = 2 \ln(n + 1)$  seconds
- i** When  $n = 5$ ,  $T = 2 \ln 6 \approx 3.58$  seconds
  - ii** When  $n = 15$ ,  $T = 2 \ln 16 \approx 5.55$  seconds
- c** When  $n = 20$ ,  $T = 2 \ln 21 \approx 6.09$  seconds  
 When  $n = 40$ ,  $T = 2 \ln 41 \approx 7.43$  seconds  
 $2 \ln 41 - 2 \ln 21 \approx 1.34$  seconds longer

- 26**  $f(x) = \log_3 x - 2$ ,  $g(x) = 3 - \sqrt{x}$

- a i**  $f$  is defined by  $y = \log_3 x - 2$   
 $\therefore f^{-1}$  is defined by  $x = \log_3 y - 2$   
 $\therefore x + 2 = \log_3 y$   
 $\therefore y = 3^{x+2}$   
 $\therefore f^{-1}(x) = 3^{x+2}$

The domain of  $f^{-1}$  is  $\{x \mid x \in \mathbb{R}\}$ . The range of  $f^{-1}$  is  $\{y \mid y > 0\}$ .

- ii**  $(f \circ g)(x) = f(g(x))$   
 $= f(3 - \sqrt{x})$ ,  $x \geq 0$   
 $= \log_3(3 - \sqrt{x}) - 2$ ,  $0 \leq x < 9$

The domain of  $(f \circ g)$  is  $\{x \mid 0 \leq x < 9\}$ . The range of  $(f \circ g)$  is  $\{y \mid y \leq -1\}$ .

- iii**  $(g \circ f)(x) = g(f(x))$   
 $= g(\log_3 x - 2)$ ,  $x > 0$   
 $= 3 - \sqrt{\log_3 x - 2}$ ,  $x \geq 9$

The domain of  $(g \circ f)$  is  $\{x \mid x \geq 9\}$ . The range of  $(g \circ f)$  is  $\{y \mid y \leq 3\}$ .

$$\begin{aligned}
 \text{b i} \quad & (f \circ g)(x) = -2 \\
 \therefore \log_3(3 - \sqrt{x}) - 2 &= -2 \\
 \therefore \log_3(3 - \sqrt{x}) &= 0 \\
 \therefore 3 - \sqrt{x} &= 1 \\
 \therefore 2 &= \sqrt{x} \\
 \therefore x &= 4
 \end{aligned}$$

$$\begin{aligned}
 \text{ii} \quad & (g \circ f)(x) = 0 \\
 \therefore 3 - \sqrt{\log_3 x - 2} &= 0 \\
 \therefore \sqrt{\log_3 x - 2} &= 3 \\
 \therefore \log_3 x - 2 &= 9 \\
 \therefore \log_3 x &= 11 \\
 x &= 3^{11} = 177\,147
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & (g \circ f) \text{ is defined by } y = 3 - \sqrt{\log_3 x - 2}, \quad x \geq 9 \\
 \therefore (g \circ f)^{-1} \text{ is defined by } x &= 3 - \sqrt{\log_3 y - 2}, \quad y \geq 9 \\
 \therefore \sqrt{\log_3 y - 2} &= 3 - x \\
 \therefore \log_3 y - 2 &= (3 - x)^2, \quad x \leq 3 \\
 \therefore \log_3 y - 2 &= 9 - 6x + x^2, \quad x \leq 3 \\
 \therefore \log_3 y &= x^2 - 6x + 11, \quad x \leq 3 \\
 \therefore y &= 3^{x^2 - 6x + 11}, \quad x \leq 3 \\
 \therefore (g \circ f)^{-1}(x) &= 3^{x^2 - 6x + 11}, \quad x \leq 3
 \end{aligned}$$

The domain of  $(g \circ f)^{-1}$  is  $\{x \mid x \leq 3\}$ . The range of  $(g \circ f)^{-1}$  is  $\{y \mid y \geq 9\}$ .



# Chapter 4

## INTRODUCTION TO COMPLEX NUMBERS

### EXERCISE 4A

1

$z$	$\operatorname{Re}(z)$	$\operatorname{Im}(z)$	$z^*$
$3 + 2i$	3	2	$3 - 2i$
$5 - i$	5	-1	$5 + i$
3	3	0	3
0	0	0	0
$-3 + 4i$	-3	4	$-3 - 4i$
$-7 - 2i$	-7	-2	$-7 + 2i$
$-11i$	0	-11	$11i$
$i\sqrt{3}$	0	$\sqrt{3}$	$-i\sqrt{3}$

2

$$\begin{aligned} \mathbf{a} \quad & \sqrt{-9} \\ &= \sqrt{9} \times \sqrt{-1} \\ &= 3i \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad & \sqrt{-5} \\ &= \sqrt{5} \times \sqrt{-1} \\ &= i\sqrt{5} \end{aligned}$$

3

$$\begin{aligned} \mathbf{a} \quad & x^2 = 25 \\ \therefore x &= \pm\sqrt{25} \\ \therefore x &= \pm 5 \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad & x^2 = -5 \\ \therefore x &= \pm\sqrt{-5} \\ \therefore x &= \pm\sqrt{5} \times \sqrt{-1} \\ \therefore x &= \pm i\sqrt{5} \end{aligned}$$

4

$$\begin{aligned} \mathbf{a} \quad & x^2 - 10x + 29 = 0 \\ \therefore x &= \frac{10 \pm \sqrt{100 - 4 \times 1 \times 29}}{2} \\ \therefore x &= \frac{10 \pm \sqrt{-16}}{2} \\ \therefore x &= 5 \pm \sqrt{-4} \\ \therefore x &= 5 \pm 2i \end{aligned}$$

b

$$\begin{aligned} & \sqrt{-64} \\ &= \sqrt{64} \times \sqrt{-1} \\ &= 8i \end{aligned}$$

e

$$\begin{aligned} & \sqrt{-8} \\ &= \sqrt{8} \times \sqrt{-1} \\ &= i\sqrt{8} \end{aligned}$$

b

$$\begin{aligned} & x^2 = -25 \\ \therefore x &= \pm\sqrt{-25} \\ \therefore x &= \pm\sqrt{25} \times \sqrt{-1} \\ \therefore x &= \pm 5i \end{aligned}$$

e

$$\begin{aligned} & 4x^2 = 9 \\ \therefore x^2 &= \frac{9}{4} \\ \therefore x &= \pm\sqrt{\frac{9}{4}} \\ \therefore x &= \pm\frac{3}{2} \end{aligned}$$

b

$$\begin{aligned} & x^2 + 6x + 25 = 0 \\ \therefore x &= \frac{-6 \pm \sqrt{36 - 4 \times 1 \times 25}}{2} \\ \therefore x &= \frac{-6 \pm \sqrt{-64}}{2} \\ \therefore x &= -3 \pm \sqrt{-16} \\ \therefore x &= -3 \pm 4i \end{aligned}$$

c

$$\begin{aligned} & \sqrt{-\frac{1}{4}} \\ &= \sqrt{\frac{1}{4}} \times \sqrt{-1} \\ &= \frac{1}{2}i \end{aligned}$$

c

$$\begin{aligned} & x^2 = 5 \\ \therefore x &= \pm\sqrt{5} \end{aligned}$$

f

$$\begin{aligned} & 4x^2 = -9 \\ \therefore x^2 &= -\frac{9}{4} \\ \therefore x &= \pm\sqrt{-\frac{9}{4}} \\ \therefore x &= \pm\sqrt{\frac{9}{4}} \times \sqrt{-1} \\ \therefore x &= \pm\frac{3}{2}i \end{aligned}$$

$$\begin{aligned}
 \text{c } x^2 + 14x + 50 &= 0, \\
 \therefore x &= \frac{-14 \pm \sqrt{14^2 - 4 \times 1 \times 50}}{2} \\
 \therefore x &= \frac{-14 \pm \sqrt{-4}}{2} \\
 \therefore x &= -7 \pm \sqrt{-1} \\
 \therefore x &= -7 \pm i
 \end{aligned}$$

$$\begin{aligned}
 \text{e } x^2 - 2\sqrt{3}x + 4 &= 0, \\
 \therefore x &= \frac{2\sqrt{3} \pm \sqrt{12 - 4 \times 1 \times 4}}{2} \\
 \therefore x &= \frac{2\sqrt{3} \pm \sqrt{-4}}{2} \\
 \therefore x &= \sqrt{3} \pm \sqrt{-1} \\
 \therefore x &= \sqrt{3} \pm i
 \end{aligned}$$

$$\begin{aligned}
 \text{d } 2x^2 + 5 &= 6x, \\
 \therefore 2x^2 - 6x + 5 &= 0 \\
 \therefore x &= \frac{6 \pm \sqrt{36 - 4 \times 2 \times 5}}{4} \\
 \therefore x &= \frac{6 \pm \sqrt{-4}}{4} \\
 \therefore x &= \frac{3 \pm \sqrt{-1}}{2} \\
 \therefore x &= \frac{3}{2} \pm \frac{1}{2}i
 \end{aligned}$$

$$\begin{aligned}
 \text{f } 2x + \frac{1}{x} &= 1, \\
 \therefore 2x^2 + 1 &= x \\
 \therefore 2x^2 - x + 1 &= 0 \\
 \therefore x &= \frac{1 \pm \sqrt{1 - 4 \times 2 \times 1}}{4} \\
 \therefore x &= \frac{1 \pm \sqrt{-7}}{4} \\
 \therefore x &= \frac{1}{4} \pm i\frac{\sqrt{7}}{4}
 \end{aligned}$$

## EXERCISE 4B

$$\begin{aligned}
 \text{1 a } x^2 - 9 \\
 &= (x + 3)(x - 3)
 \end{aligned}$$

$$\begin{aligned}
 \text{b } x^2 + 9 \\
 &= x^2 - 9i^2 \\
 &= (x + 3i)(x - 3i)
 \end{aligned}$$

$$\begin{aligned}
 \text{c } x^2 - 7 \\
 &= (x + \sqrt{7})(x - \sqrt{7})
 \end{aligned}$$

$$\begin{aligned}
 \text{d } x^2 + 7 \\
 &= x^2 - (i\sqrt{7})^2 \\
 &= (x + i\sqrt{7})(x - i\sqrt{7})
 \end{aligned}$$

$$\begin{aligned}
 \text{e } 4x^2 - 1 \\
 &= (2x + 1)(2x - 1)
 \end{aligned}$$

$$\begin{aligned}
 \text{f } 4x^2 + 1 \\
 &= 4x^2 - i^2 \\
 &= (2x + i)(2x - i)
 \end{aligned}$$

$$\begin{aligned}
 \text{g } 2x^2 - 9 \\
 &= (\sqrt{2}x)^2 - 9 \\
 &= (\sqrt{2}x + 3)(\sqrt{2}x - 3)
 \end{aligned}$$

$$\begin{aligned}
 \text{h } 2x^2 + 9 \\
 &= (\sqrt{2}x)^2 - 9i^2 \\
 &= (\sqrt{2}x + 3i)(\sqrt{2}x - 3i)
 \end{aligned}$$

$$\begin{aligned}
 \text{2 } (a + bi)(a - bi) &= a^2 - b^2i^2 \\
 &= a^2 - b^2(-1) \\
 &= a^2 + b^2 \quad \text{which is real}
 \end{aligned}$$

$$\begin{aligned}
 \text{3 a } x^2 + 16 &= 0 \\
 \therefore x^2 - 16i^2 &= 0 \\
 \therefore (x + 4i)(x - 4i) &= 0 \\
 \therefore x &= \pm 4i
 \end{aligned}$$

$$\begin{aligned}
 \text{c } x^2 + 5 &= 0 \\
 \therefore x^2 - 5i^2 &= 0 \\
 \therefore (x + i\sqrt{5})(x - i\sqrt{5}) &= 0 \\
 \therefore x &= \pm i\sqrt{5}
 \end{aligned}$$

$$\begin{aligned}
 \text{b } 36 + x^2 &= 0 \\
 \therefore x^2 - 36i^2 &= 0 \\
 \therefore (x + 6i)(x - 6i) &= 0 \\
 \therefore x &= \pm 6i
 \end{aligned}$$

$$\begin{aligned}
 \text{d } 9x^2 + 1 &= 0 \\
 \therefore 9x^2 - i^2 &= 0 \\
 \therefore (3x + i)(3x - i) &= 0 \\
 \therefore x &= \pm \frac{1}{3}i
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & 4x^2 + 25 = 0 \\
 & \therefore 4x^2 - 25i^2 = 0 \\
 & \therefore (2x + 5i)(2x - 5i) = 0 \\
 & \therefore x = \pm \frac{5}{2}i
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & 3x^2 + 2 = 0 \\
 & \therefore 3x^2 - 2i^2 = 0 \\
 & \therefore (\sqrt{3}x + i\sqrt{2})(\sqrt{3}x - i\sqrt{2}) = 0 \\
 & \therefore x = \pm i \frac{\sqrt{2}}{\sqrt{3}}
 \end{aligned}$$

$$\begin{aligned}
 4 \quad \text{a} \quad & x^3 - x \\
 & = x(x^2 - 1) \\
 & = x(x + 1)(x - 1)
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & x^3 + x \\
 & = x(x^2 + 1) \\
 & = x(x^2 - i^2) \\
 & = x(x + i)(x - i)
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & x^4 - 1 \\
 & = (x^2 + 1)(x^2 - 1) \\
 & = (x^2 - i^2)(x^2 - 1) \\
 & = (x + i)(x - i)(x + 1)(x - 1)
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad & x^4 - 16 \\
 & = (x^2 + 4)(x^2 - 4) \\
 & = (x^2 - 4i^2)(x^2 - 4) \\
 & = (x + 2i)(x - 2i)(x + 2)(x - 2)
 \end{aligned}$$

$$\begin{aligned}
 5 \quad \text{a} \quad & x^3 - 4x = 0 \\
 & \therefore x(x^2 - 4) = 0 \\
 & \therefore x(x + 2)(x - 2) = 0 \\
 & \therefore x = 0 \text{ or } \pm 2
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & x^3 + 4x = 0 \\
 & \therefore x(x^2 + 4) = 0 \\
 & \therefore x(x^2 - 4i^2) = 0 \\
 & \therefore x(x + 2i)(x - 2i) = 0 \\
 & \therefore x = 0 \text{ or } \pm 2i
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & x^3 - 3x = 0 \\
 & \therefore x(x^2 - 3) = 0 \\
 & \therefore x(x + \sqrt{3})(x - \sqrt{3}) = 0 \\
 & \therefore x = 0 \text{ or } \pm\sqrt{3}
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad & x^3 + 3x = 0 \\
 & \therefore x(x^2 + 3) = 0 \\
 & \therefore x(x^2 - 3i^2) = 0 \\
 & \therefore x(x + i\sqrt{3})(x - i\sqrt{3}) = 0 \\
 & \therefore x = 0 \text{ or } \pm i\sqrt{3}
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & x^4 - 1 = 0 \\
 & \therefore (x^2 + 1)(x^2 - 1) = 0 \\
 & \therefore (x^2 - i^2)(x^2 - 1) = 0 \\
 & \therefore (x + i)(x - i)(x + 1)(x - 1) = 0 \\
 & \therefore x = \pm i \text{ or } \pm 1
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & x^4 = 81 \\
 & \therefore x^4 - 81 = 0 \\
 & \therefore (x^2 + 9)(x^2 - 9) = 0 \\
 & \therefore (x^2 - 9i^2)(x^2 - 9) = 0 \\
 & \therefore (x + 3i)(x - 3i)(x + 3)(x - 3) = 0 \\
 & \therefore x = \pm 3i \text{ or } \pm 3
 \end{aligned}$$

$$\begin{aligned}
 6 \quad \text{a} \quad & x^4 + 2x^2 = 3 \\
 & \therefore x^4 + 2x^2 - 3 = 0 \\
 & \therefore (x^2 + 3)(x^2 - 1) = 0 \\
 & \therefore (x^2 - 3i^2)(x^2 - 1) = 0 \\
 & \therefore (x + i\sqrt{3})(x - i\sqrt{3})(x + 1)(x - 1) = 0 \\
 & \therefore x = \pm i\sqrt{3} \text{ or } \pm 1
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & x^4 = x^2 + 6 \\
 & \therefore x^4 - x^2 - 6 = 0 \\
 & \therefore (x^2 - 3)(x^2 + 2) = 0 \\
 & \therefore (x^2 - 3)(x^2 - 2i^2) = 0 \\
 & \therefore (x + \sqrt{3})(x - \sqrt{3})(x + i\sqrt{2})(x - i\sqrt{2}) = 0 \\
 & \therefore x = \pm\sqrt{3} \text{ or } \pm i\sqrt{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & x^4 + 5x^2 = 36 \\
 & \therefore x^4 + 5x^2 - 36 = 0 \\
 & \therefore (x^2 + 9)(x^2 - 4) = 0 \\
 & \therefore (x^2 - 9i^2)(x^2 - 4) = 0 \\
 & \therefore (x + 3i)(x - 3i)(x + 2)(x - 2) = 0 \\
 & \therefore x = \pm 3i \text{ or } \pm 2
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad & x^4 + 9x^2 + 14 = 0 \\
 & \therefore (x^2 + 7)(x^2 + 2) = 0 \\
 & \therefore (x^2 - 7i^2)(x^2 - 2i^2) = 0 \\
 & \therefore (x + i\sqrt{7})(x - i\sqrt{7})(x + i\sqrt{2})(x - i\sqrt{2}) = 0 \\
 & \therefore x = \pm i\sqrt{7} \text{ or } \pm i\sqrt{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & x^4 + 1 = 2x^2 \\
 & \therefore x^4 - 2x^2 + 1 = 0 \\
 & \therefore (x^2 - 1)^2 = 0 \\
 & \therefore (x + 1)^2(x - 1)^2 = 0 \\
 & \therefore x = \pm 1
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & x^4 + 2x^2 + 1 = 0 \\
 & \therefore (x^2 + 1)^2 = 0 \\
 & \therefore (x^2 - i^2)^2 = 0 \\
 & \therefore (x + i)^2(x - i)^2 = 0 \\
 & \therefore x = \pm i
 \end{aligned}$$

## EXERCISE 4C

$$1 \quad z = 5 - 2i, \quad w = 2 + i$$

$$\begin{aligned}
 \text{a} \quad & z + w \\
 & = (5 - 2i) + (2 + i) \\
 & = 7 - i
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & 2z \\
 & = 2(5 - 2i) \\
 & = 10 - 4i
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & iw = i(2 + i) \\
 & = 2i + i^2 \\
 & = -1 + 2i
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad & z - w \\
 & = (5 - 2i) - (2 + i) \\
 & = 5 - 2i - 2 - i \\
 & = 3 - 3i
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & 2z - 3w \\
 & = 2(5 - 2i) - 3(2 + i) \\
 & = 10 - 4i - 6 - 3i \\
 & = 4 - 7i
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & zw \\
 & = (5 - 2i)(2 + i) \\
 & = 10 + 5i - 4i - 2i^2 \\
 & = 12 + i
 \end{aligned}$$

$$\begin{aligned}
 \text{g} \quad & w^2 = (2 + i)^2 \\
 & = 4 + 4i + i^2 \\
 & = 3 + 4i
 \end{aligned}$$

$$\begin{aligned}
 \text{h} \quad & z^2 = (5 - 2i)^2 \\
 & = 25 - 20i + 4i^2 \\
 & = 21 - 20i
 \end{aligned}$$

$$2 \quad z = 1 + i, \quad w = -2 + 3i$$

$$\begin{aligned}
 \text{a} \quad & z + 2w \\
 & = (1 + i) + 2(-2 + 3i) \\
 & = 1 + i - 4 + 6i \\
 & = -3 + 7i
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & z^2 \\
 & = (1 + i)^2 \\
 & = 1 + 2i + i^2 \\
 & = 2i
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & z^3 = z^2 \times z \\
 & = 2i(1 + i) \quad \{\text{using b}\} \\
 & = 2i + 2i^2 \\
 & = -2 + 2i
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad & iz = i(1 + i) \\
 & = i + i^2 \\
 & = -1 + i
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & w^2 = (-2 + 3i)^2 \\
 & = 4 - 12i + 9i^2 \\
 & = -5 - 12i
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & zw \\
 & = (1 + i)(-2 + 3i) \\
 & = -2 + 3i - 2i + 3i^2 \\
 & = -5 + i
 \end{aligned}$$



$$\begin{aligned}
 \mathbf{g} \quad & z^2 w \\
 &= 2i(-2 + 3i) \quad \{\text{using } \mathbf{b}\} \\
 &= -4i + 6i^2 \\
 &= -6 - 4i
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{h} \quad & izw \\
 &= i(-5 + i) \quad \{\text{using } \mathbf{f}\} \\
 &= -5i + i^2 \\
 &= -1 - 5i
 \end{aligned}$$

$$\mathbf{3} \quad (a + \cancel{bi}) + (a - \cancel{bi}) = 2a \quad \text{which is real}$$

$$\begin{array}{llll}
 \mathbf{4} \quad \mathbf{a} \quad i^0 = 1 & i^4 = 1 & i^8 = 1 & i^{-1} = -i \\
 i^1 = i & i^5 = i & i^9 = i & i^{-2} = -1 \\
 i^2 = -1 & i^6 = -1 & & i^{-3} = i \\
 i^3 = -i & i^7 = -i & & i^{-4} = 1 \\
 & & & i^{-5} = -i
 \end{array}$$

$$\mathbf{b} \quad i^{4n+3} = -i \quad \text{where } n \text{ is any integer}$$

$$\begin{aligned}
 \mathbf{5} \quad (1+i)^4 &= [(1+i)^2]^2 & (1+i)^{101} &= (1+i)^{100} \times (1+i) \\
 &= (1+2i+i^2)^2 & &= [(1+i)^4]^{25} \times (1+i) \\
 &= (2i)^2 & &= [-4]^{25}(1+i) \\
 &= -4 & &= -2^{50}(1+i)
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{6} \quad \mathbf{a} \quad \sqrt{-4} \times \sqrt{-9} &= \sqrt{-1} \times \sqrt{4} \times \sqrt{-1} \times \sqrt{9} \\
 &= -1 \times \sqrt{36} \\
 &= -6
 \end{aligned}$$

$$\mathbf{b} \quad \text{No, } \sqrt{-4} \times \sqrt{-9} = -6 \neq \sqrt{36}$$

$$\mathbf{7} \quad z = 2 - i, \quad w = 1 + 3i$$

$$\begin{aligned}
 \mathbf{a} \quad \frac{z}{w} &= \left( \frac{2-i}{1+3i} \right) \times \left( \frac{1-3i}{1-3i} \right) \\
 &= \frac{2-6i-i+3i^2}{1-9i^2} \\
 &= \frac{-1-7i}{10} \\
 &= -\frac{1}{10} - \frac{7}{10}i
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad \frac{i}{z} &= \left( \frac{i}{2-i} \right) \times \left( \frac{2+i}{2+i} \right) \\
 &= \frac{2i+i^2}{4-i^2} \\
 &= \frac{-1+2i}{5} \\
 &= -\frac{1}{5} + \frac{2}{5}i
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad \frac{w}{iz} &= \frac{1+3i}{i(2-i)} \\
 &= \frac{1+3i}{2i-i^2} \\
 &= \left( \frac{1+3i}{1+2i} \right) \times \left( \frac{1-2i}{1-2i} \right) \\
 &= \frac{1-2i+3i-6i^2}{1-4i^2} \\
 &= \frac{7+i}{5} \\
 &= \frac{7}{5} + \frac{1}{5}i
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad z^{-2} &= \frac{1}{(2-i)^2} \\
 &= \frac{1}{4-4i+i^2} \\
 &= \left( \frac{1}{3-4i} \right) \times \left( \frac{3+4i}{3+4i} \right) \\
 &= \frac{3+4i}{9-16i^2} \\
 &= \frac{3+4i}{25} \\
 &= \frac{3}{25} + \frac{4}{25}i
 \end{aligned}$$

$$\begin{aligned}
 \text{8 a } \frac{i}{1-2i} &= \left( \frac{i}{1-2i} \right) \times \left( \frac{1+2i}{1+2i} \right) \\
 &= \frac{i+2i^2}{1-4i^2} \\
 &= \frac{-2+i}{5} \\
 &= -\frac{2}{5} + \frac{1}{5}i
 \end{aligned}$$

$$\begin{aligned}
 \text{b } \frac{i(2-i)}{3-2i} &= \frac{2i-i^2}{3-2i} \\
 &= \left( \frac{1+2i}{3-2i} \right) \times \left( \frac{3+2i}{3+2i} \right) \\
 &= \frac{3+2i+6i+4i^2}{9-4i^2} \\
 &= \frac{-1+8i}{13} \\
 &= -\frac{1}{13} + \frac{8}{13}i
 \end{aligned}$$

$$\begin{aligned}
 \text{c } \frac{1}{2-i} - \frac{2}{2+i} &= \left( \frac{1}{2-i} \right) \times \left( \frac{2+i}{2+i} \right) - \left( \frac{2}{2+i} \right) \times \left( \frac{2-i}{2-i} \right) \\
 &= \frac{2+i-2(2-i)}{(2-i)(2+i)} \\
 &= \frac{2+i-4+2i}{4-i^2} \\
 &= \frac{-2+3i}{5} \\
 &= -\frac{2}{5} + \frac{3}{5}i
 \end{aligned}$$

$$\begin{aligned}
 \text{d } \frac{i}{x+i} - \frac{i}{x-i} &= \left( \frac{i}{x+i} \right) \times \left( \frac{x-i}{x-i} \right) - \left( \frac{i}{x-i} \right) \times \left( \frac{x+i}{x+i} \right) \\
 &= \frac{i(x-i) - i(x+i)}{x^2 - i^2} \\
 &= \frac{xi - i^2 - xi - i^2}{x^2 - i^2} \\
 &= \frac{2}{x^2 + 1}
 \end{aligned}$$

$$\text{9 } z = 2 + i, \quad w = -1 + 2i$$

$$\begin{aligned}
 \text{a } 4z - 3w &= 4(2+i) - 3(-1+2i) \\
 &= 8+4i+3-6i \\
 &= 11-2i
 \end{aligned}$$

$$\therefore \operatorname{Im}(4z - 3w) = -2$$

$$\begin{aligned}
 \text{b } zw &= (2+i)(-1+2i) \\
 &= -2+4i-i+2i^2 \\
 &= -4+3i
 \end{aligned}$$

$$\therefore \operatorname{Re}(zw) = -4$$

$$\begin{aligned}
 \text{c } iz^2 &= i(2+i)^2 \\
 &= i(4+4i+i^2) \\
 &= i(3+4i) \\
 &= 3i+4i^2 \\
 &= -4+3i
 \end{aligned}$$

$$\therefore \operatorname{Im}(iz^2) = 3$$

$$\begin{aligned}
 \text{d } \frac{z}{w} &= \left( \frac{2+i}{-1+2i} \right) \times \left( \frac{-1-2i}{-1-2i} \right) \\
 &= \frac{-2-4i-i-2i^2}{1-4i^2} \\
 &= \frac{0-5i}{5} \\
 &= -i
 \end{aligned}$$

$$\therefore \operatorname{Re}\left(\frac{z}{w}\right) = 0$$

$$\begin{aligned}
 10 \quad \frac{4}{1+i} &= \left( \frac{4}{1+i} \right) \times \left( \frac{1-i}{1-i} \right) \\
 &= \frac{4-4i}{1-i^2} \\
 &= \frac{4-4i}{2} = 2-2i
 \end{aligned}$$

$$\begin{aligned}
 \text{So, } z &= \left( \frac{4}{1+i} + 7-2i \right)^2 \\
 &= ((2-2i) + 7-2i)^2 \\
 &= (9-4i)^2 \\
 &= 81-72i+16i^2 \\
 &= 65-72i
 \end{aligned}$$

$$\begin{aligned}
 11 \quad z &= \frac{3i}{\sqrt{2}-i} + 1 \\
 &= \left( \frac{3i}{\sqrt{2}-i} \right) \times \left( \frac{\sqrt{2}+i}{\sqrt{2}+i} \right) + 1 \\
 &= \frac{3i\sqrt{2}+3i^2}{2-i^2} + 1 \\
 &= \frac{3i\sqrt{2}-3}{3} + \frac{3}{3} \\
 &= \frac{3i\sqrt{2}}{3} \\
 &= 0+i\sqrt{2}
 \end{aligned}$$

$$\begin{aligned}
 12 \quad a \quad w &= \frac{z-1}{z^*+1} = \frac{a+bi-1}{a-bi+1} = \frac{(a-1)+bi}{(a+1)-bi} \\
 \therefore w &= \left( \frac{(a-1)+bi}{(a+1)-bi} \right) \times \left( \frac{(a+1)+bi}{(a+1)+bi} \right) \\
 &= \frac{(a^2-1)+(a-1)bi+(a+1)bi+b^2i^2}{(a+1)^2-b^2i^2} \\
 &= \frac{a^2-b^2-1}{(a+1)^2+b^2} + \frac{2ab}{(a+1)^2+b^2}i
 \end{aligned}$$

$$\begin{aligned}
 b \quad w \text{ is purely imaginary when } a^2-b^2-1 &= 0 \quad \text{and} \quad 2ab \neq 0 \\
 \therefore a^2-b^2 &= 1 \quad \text{and} \quad a \neq 0, b \neq 0
 \end{aligned}$$

$$\begin{aligned}
 13 \quad z^2 - (2+i)z + (3+i) &= 0 \\
 \therefore z &= \frac{2+i \pm \sqrt{(2+i)^2 - 4(1)(3+i)}}{2(1)} \\
 &= \frac{2+i \pm \sqrt{4+4i-1-12-4i}}{2} \\
 &= \frac{2+i \pm \sqrt{-9}}{2} \\
 &= \frac{2+i \pm 3i}{2} \\
 &= \frac{2+4i}{2} \quad \text{or} \quad \frac{2-2i}{2} \\
 &= 1+2i \quad \text{or} \quad 1-i
 \end{aligned}$$

## EXERCISE 4D

$$\begin{aligned}
 1 \quad a \quad 2x+3yi &= -x-6i \\
 \text{Equating real and imaginary parts,} \\
 2x &= -x \quad \text{and} \quad 3y = -6 \\
 \therefore 3x &= 0 \quad \text{and} \quad y = -2 \\
 \therefore x &= 0 \quad \text{and} \quad y = -2
 \end{aligned}$$

$$\begin{aligned}
 b \quad x^2+xi &= 4-2i \\
 \text{Equating real and imaginary parts,} \\
 x^2 &= 4 \quad \text{and} \quad x = -2 \\
 \therefore x &= \pm 2 \quad \text{and} \quad x = -2 \\
 \therefore x &= -2
 \end{aligned}$$

$$\text{c } (x + yi)(2 - i) = 8 + i$$

$$\therefore x + yi = \left(\frac{8+i}{2-i}\right) \times \left(\frac{2+i}{2+i}\right)$$

$$\therefore x + yi = \frac{16 + 8i + 2i + i^2}{4 - i^2}$$

$$\therefore x + yi = \frac{15 + 10i}{5}$$

$$\therefore x + yi = 3 + 2i$$

Equating real and imaginary parts,  
 $x = 3$  and  $y = 2$ .

$$\text{d } (3 + 2i)(x + yi) = -i$$

$$\therefore x + yi = \left(\frac{-i}{3+2i}\right) \times \left(\frac{3-2i}{3-2i}\right)$$

$$\therefore x + yi = \frac{-3i + 2i^2}{9 - 4i^2}$$

$$\therefore x + yi = \frac{-2 - 3i}{13}$$

$$\therefore x + yi = -\frac{2}{13} - \frac{3}{13}i$$

Equating real and imaginary parts,  
 $x = -\frac{2}{13}$  and  $y = -\frac{3}{13}$ .

$$\text{2 a } 2(x + yi) = x - yi$$

$$\therefore 2x + 2yi = x - yi$$

Equating real and imaginary parts,  $2x = x$  and  $2y = -y$   
 $\therefore x = 0$  and  $3y = 0$   
 $\therefore x = 0$  and  $y = 0$

$$\text{b } (x + 2i)(y - i) = -4 - 7i$$

$$\therefore xy - xi + 2yi - 2i^2 = -4 - 7i$$

$$\therefore (xy + 2) + (2y - x)i = -4 - 7i$$

Equating real and imaginary parts,

$$xy + 2 = -4 \quad \text{and} \quad 2y - x = -7$$

$$\therefore xy = -6 \quad \text{and} \quad x = 2y + 7$$

$$\therefore (2y + 7)y = -6$$

$$\therefore 2y^2 + 7y = -6$$

$$\therefore 2y^2 + 7y + 6 = 0$$

$$\therefore (2y + 3)(y + 2) = 0$$

$$\therefore y = -\frac{3}{2} \quad \text{or} \quad y = -2$$

$$\text{When } y = -\frac{3}{2}, \quad x = 2\left(-\frac{3}{2}\right) + 7 = 4$$

$$\text{and when } y = -2, \quad x = 2(-2) + 7 = 3$$

$$\therefore x = 4 \quad \text{and} \quad y = -\frac{3}{2}$$

$$\text{or } x = 3 \quad \text{and} \quad y = -2$$

$$\text{d } (x + yi)(2 + i) = 2x - (y + 1)i$$

$$\therefore 2x + xi + 2yi + yi^2 = 2x + (-y - 1)i$$

$$\therefore (2x - y) + (x + 2y)i = 2x + (-y - 1)i$$

Equating real and imaginary parts,  $2x - y = 2x$  and  $x + 2y = -y - 1$

$$\therefore -y = 0 \quad \text{and} \quad x = -3y - 1$$

$$\therefore y = 0 \quad \text{and} \quad x = -3y - 1$$

$$\therefore x = -3(0) - 1$$

$$\therefore x = -1$$

$$\therefore x = -1 \quad \text{and} \quad y = 0$$

$$\text{c } (x + i)(3 - yi) = 1 + 13i$$

$$\therefore 3x - xyi + 3i - yi^2 = 1 + 13i$$

$$\therefore (3x + y) + (3 - xy)i = 1 + 13i$$

Equating real and imaginary parts,

$$3x + y = 1 \quad \text{and} \quad 3 - xy = 13$$

$$\therefore y = 1 - 3x \quad \text{and} \quad xy = -10$$

$$\therefore x(1 - 3x) = -10$$

$$\therefore x - 3x^2 = -10$$

$$\therefore 0 = 3x^2 - x - 10$$

$$\therefore 0 = (3x + 5)(x - 2)$$

$$\therefore x = -\frac{5}{3} \quad \text{or} \quad x = 2$$

$$\text{When } x = -\frac{5}{3}, \quad y = 1 - 3\left(-\frac{5}{3}\right) = 6$$

$$\text{and when } x = 2, \quad y = 1 - 3 \times 2 = -5$$

$$\therefore x = -\frac{5}{3} \quad \text{and} \quad y = 6$$

$$\text{or } x = 2 \quad \text{and} \quad y = -5$$



$$3 \quad 3z + 17i = iz + 11$$

$$\therefore z(3 - i) = 11 - 17i$$

$$\begin{aligned}\therefore z &= \left( \frac{11 - 17i}{3 - i} \right) \times \left( \frac{3 + i}{3 + i} \right) \\ &= \frac{33 + 11i - 51i - 17i^2}{9 - i^2} \\ &= \frac{50 - 40i}{10} \\ &= 5 - 4i\end{aligned}$$

$$4 \quad 3(m + ni) = n - 2mi - (1 - 2i)$$

$$\therefore 3m + 3ni = (n - 1) + (2 - 2m)i$$

Equating real and imaginary parts,

$$3m = n - 1 \quad \text{and} \quad 3n = 2 - 2m$$

$$\therefore n = 3m + 1 \quad \text{and} \quad 3n = 2 - 2m$$

$$\therefore 3(3m + 1) = 2 - 2m$$

$$\therefore 9m + 3 = 2 - 2m$$

$$\therefore 11m = -1$$

$$\therefore m = -\frac{1}{11} \quad \text{and} \quad n = 3\left(-\frac{1}{11}\right) + 1 = \frac{8}{11}$$

$$5 \quad (a + bi)^2 = -16 - 30i$$

$$\therefore a^2 + 2abi + b^2i^2 = -16 - 30i$$

$$\therefore (a^2 - b^2) + 2abi = -16 - 30i$$

Equating real and imaginary parts,  $a^2 - b^2 = -16$  and  $2ab = -30$

$$\therefore a^2 - b^2 = -16 \quad \text{and} \quad b = -\frac{15}{a}$$

$$\therefore a^2 - \left(-\frac{15}{a}\right)^2 = -16$$

$$\therefore a^2 - \frac{225}{a^2} = -16$$

$$\therefore a^4 + 16a^2 - 225 = 0$$

$$\therefore (a^2 + 25)(a^2 - 9) = 0$$

$$\therefore a^2 = 9 \quad \{a^2 + 25 > 0\}$$

$$\therefore a = 3 \quad \text{and} \quad b = -\frac{15}{3} = -5$$

$$\text{or} \quad a = -3 \quad \text{and} \quad b = -\frac{15}{-3} = 5$$

$$6 \quad \text{Let } z = a + bi, \text{ then } (a + bi)^2 = 1 + i + \left( \frac{58}{9(3 - 7i)} \right) \times \left( \frac{3 + 7i}{3 + 7i} \right)$$

$$\therefore a^2 + 2abi + b^2i^2 = 1 + i + \frac{58(3 + 7i)}{9(9 - 49i^2)}$$

$$\therefore (a^2 - b^2) + 2abi = 1 + i + \frac{58(3 + 7i)}{9 \times 58}$$

$$\therefore (a^2 - b^2) + 2abi = 1 + i + \frac{3 + 7i}{9}$$

$$\therefore (a^2 - b^2) + 2abi = \frac{4}{3} + \frac{16}{9}i$$

Equating real and imaginary parts,  $a^2 - b^2 = \frac{4}{3}$  and  $2ab = \frac{16}{9}$

$$\therefore a^2 - b^2 = \frac{4}{3} \quad \text{and} \quad b = \frac{8}{9a}$$

$$\therefore a^2 - \left(\frac{8}{9a}\right)^2 = \frac{4}{3}$$

$$\therefore a^2 - \frac{64}{81a^2} = \frac{4}{3}$$

$$\therefore 81a^4 - 108a^2 - 64 = 0$$

$$\therefore (9a^2 - 16)(9a^2 + 4) = 0$$

$$\therefore a^2 = \frac{16}{9} \quad \{9a^2 + 4 > 0\}$$

$$\therefore a = \frac{4}{3} \quad \text{and} \quad b = \frac{8}{9(\frac{4}{3})} = \frac{2}{3}$$

$$\text{or } a = -\frac{4}{3} \quad \text{and} \quad b = \frac{8}{9(-\frac{4}{3})} = -\frac{2}{3}$$

So,  $z = \frac{4}{3} + \frac{2}{3}i$  or  $z = -\frac{4}{3} - \frac{2}{3}i$ .

## INVESTIGATION

## PROPERTIES OF CONJUGATES

**1**  $z_1 = 1 - i, \quad z_2 = 2 + i$

**a**  $z_1^* = 1 + i$

**b**  $z_2^* = 2 - i$

**c**  $(z_1^*)^* = (1 + i)^*$   
 $= 1 - i$   
 $= z_1$

**d**  $(z_2^*)^* = (2 - i)^*$   
 $= 2 + i$   
 $= z_2$

**e**  $z_1 + z_1^* = 1 - i + 1 + i$   
 $= 2$

**f**  $z_2 + z_2^* = 2 + i + 2 - i$   
 $= 4$

**g**  $z_1 z_1^* = (1 - i)(1 + i)$   
 $= 1 - i^2$   
 $= 1 + 1$   
 $= 2$

**h**  $z_2 z_2^* = (2 + i)(2 - i)$   
 $= 4 - i^2$   
 $= 4 + 1$   
 $= 5$

**i**  $(z_1 + z_2)^*$   
 $= (1 - i + 2 + i)^*$   
 $= (3)^*$   
 $= 3$

**j**  $z_1^* + z_2^*$   
 $= 1 + i + 2 - i$   
 $= 3$   
 $= (z_1 + z_2)^*$

**k**  $(z_1 - z_2)^*$   
 $= (1 - i - (2 + i))^*$   
 $= (1 - i - 2 - i)^*$   
 $= (-1 - 2i)^*$   
 $= -1 + 2i$

**l**  $z_1^* - z_2^*$   
 $= 1 + i - (2 - i)$   
 $= 1 + i - 2 + i$   
 $= -1 + 2i$   
 $= (z_1 - z_2)^*$

$$\begin{aligned}
 \text{m} \quad & (z_1 z_2)^* \\
 &= ((1-i)(2+i))^* \\
 &= (2+i-2i-i^2)^* \\
 &= (2+i-2i+1)^* \\
 &= (3-i)^* \\
 &= 3+i
 \end{aligned}$$

$$\begin{aligned}
 \text{n} \quad & z_1^* z_2^* \\
 &= (1+i)(2-i) \\
 &= 2-i+2i-i^2 \\
 &= 2-i+2i+1 \\
 &= 3+i \\
 &= (z_1 z_2)^*
 \end{aligned}$$

$$\begin{aligned}
 \text{o} \quad & \left(\frac{z_1}{z_2}\right)^* \\
 &= \left(\left(\frac{1-i}{2+i}\right) \times \left(\frac{2-i}{2-i}\right)\right)^* \\
 &= \left(\frac{2-i-2i+i^2}{4-i^2}\right)^* \\
 &= \left(\frac{1-3i}{5}\right)^* \\
 &= \frac{1}{5} + \frac{3}{5}i
 \end{aligned}$$

$$\begin{aligned}
 \text{p} \quad & \frac{z_1^*}{z_2^*} = \left(\frac{1+i}{2-i}\right) \times \left(\frac{2+i}{2+i}\right) \\
 &= \frac{2+i+2i+i^2}{4-i^2} \\
 &= \frac{1+3i}{5} \\
 &= \frac{1}{5} + \frac{3}{5}i \\
 &= \left(\frac{z_1}{z_2}\right)^*
 \end{aligned}$$

$$\begin{aligned}
 \text{q} \quad & (z_1^2)^* = ((1-i)^2)^* \\
 &= (1-2i+i^2)^* \\
 &= (-2i)^* \\
 &= 2i
 \end{aligned}$$

$$\begin{aligned}
 \text{r} \quad & (z_1^*)^2 = (1+i)^2 \\
 &= 1+2i+i^2 \\
 &= 2i \\
 &= (z_1^2)^*
 \end{aligned}$$

$$\begin{aligned}
 \text{s} \quad & (z_2^3)^* \\
 &= ((2+i)^3)^* \\
 &= (2^3 + 3(2)^2(i) + 3(2)(i)^2 + i^3)^* \\
 &= (8 + 12i + 6i^2 - i)^* \\
 &= (2 + 11i)^* \\
 &= 2 - 11i
 \end{aligned}$$

$$\begin{aligned}
 \text{t} \quad & (z_2^*)^3 \\
 &= (2-i)^3 \\
 &= 2^3 + 3(2)^2(-i) + 3(2)(-i)^2 + (-i)^3 \\
 &= 8 - 12i + 6i^2 - i^3 \\
 &= 2 - 12i + i \\
 &= 2 - 11i \\
 &= (z_2^3)^*
 \end{aligned}$$

- 3**
- $(z^*)^* = z$
  - $(z_1 + z_2)^* = z_1^* + z_2^*$  and  $(z_1 - z_2)^* = z_1^* - z_2^*$
  - $(z_1 z_2)^* = z_1^* \times z_2^*$  and  $\left(\frac{z_1}{z_2}\right)^* = \frac{z_1^*}{z_2^*}$ ,  $z_2 \neq 0$
  - $(z^n)^* = (z^*)^n$  for positive integers  $n$
  - $z + z^*$  and  $zz^*$  are real.

## EXERCISE 4E

**1** Let  $z_1 = a + bi$  and  $z_2 = c + di$ .

$$\begin{aligned}
 \therefore (z_1 - z_2)^* &= [(a + bi) - (c + di)]^* \\
 &= [(a - c) + (b - d)i]^* \\
 &= (a - c) - (b - d)i \\
 &= a - c - bi + di \\
 &= (a - bi) - (c - di) \\
 &= z_1^* - z_2^*
 \end{aligned}$$

**2**

$$\begin{aligned}
 & (w^* - z)^* - (w - 2z^*) \\
 &= (w^*)^* - z^* - w + 2z^* \\
 &= w - z^* - w + 2z^* \\
 &= -z^* + 2z^* \\
 &= z^*
 \end{aligned}$$

**3**  $z_1 = a + bi, \quad z_2 = c + di$

**a** 
$$\begin{aligned} \frac{z_1}{z_2} &= \left( \frac{a+bi}{c+di} \right) \times \left( \frac{c-di}{c-di} \right) \\ &= \frac{ac - adi + bci - bdi^2}{c^2 - d^2i^2} \\ &= \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2} \\ &= \left( \frac{ac + bd}{c^2 + d^2} \right) + \left( \frac{bc - ad}{c^2 + d^2} \right)i \end{aligned}$$

**b** 
$$\begin{aligned} \frac{z_1^*}{z_2^*} &= \left( \frac{a-bi}{c-di} \right) \times \left( \frac{c+di}{c+di} \right) \\ &= \frac{ac + adi - bci - bdi^2}{c^2 - d^2i^2} \\ &= \frac{(ac + bd) - (bc - ad)i}{c^2 + d^2} \\ &= \left( \frac{ac + bd}{c^2 + d^2} \right) - \left( \frac{bc - ad}{c^2 + d^2} \right)i \\ &= \left( \frac{z_1}{z_2} \right)^* \quad \text{for all } z_1 \text{ and } z_2 \neq 0 \quad \{\text{using a}\} \end{aligned}$$

**4** 
$$\begin{aligned} \left( \frac{z_1}{z_2} \right)^* \times z_2^* &= \left( \frac{z_1}{z_2} \times z_2 \right)^* \quad \{\text{from Example 8 b}\} \\ &= z_1^* \\ \therefore \left( \frac{z_1}{z_2} \right)^* &= \frac{z_1^*}{z_2^*} \quad \{\text{dividing both sides by } z_2^*\} \end{aligned}$$

**5**  $z = a + bi$

**a** If  $z = z^*$ , then  $a + bi = a - bi$   
 $\therefore 2bi = 0$   
 Equating imaginary parts,  $2b = 0$   
 $\therefore b = 0$

So,  $z = a + i(0) = a$  which is real.

**b** If  $z^* = -z$ , then  $a - bi = -(a + bi)$   
 $\therefore a - bi = -a - bi$   
 $\therefore 2a = 0$   
 $\therefore a = 0$

So,  $z = 0 + bi = bi$  which is purely imaginary or zero.

**6** Let  $z = a + bi$  and  $w = c + di$

**a** 
$$\begin{aligned} zw^* + z^*w &= (a + bi)(c - di) + (a - bi)(c + di) \\ &= ac - adi + bci - bdi^2 + ac + adi - bci - bdi^2 \\ &= ac + bd + ac + bd \\ &= 2ac + 2bd \quad \text{which is a real number} \end{aligned}$$

**b** 
$$\begin{aligned} zw^* - z^*w &= (a + bi)(c - di) - (a - bi)(c + di) \\ &= ac - adi + bci - bdi^2 - (ac + adi - bci - bdi^2) \\ &= ac - adi + bci + bd - ac - adi + bci - bd \\ &= 2bci - 2adi \\ &= (2bc - 2ad)i \quad \text{which is purely imaginary or zero} \end{aligned}$$

**c** 
$$\begin{aligned} \frac{z}{w} + \frac{z^*}{w^*} &= \left( \frac{a+bi}{c+di} \right) \times \left( \frac{c-di}{c-di} \right) + \left( \frac{a-bi}{c-di} \right) \times \left( \frac{c+di}{c+di} \right) \\ &= \frac{ac - adi + bci - bdi^2}{c^2 - d^2i^2} + \frac{ac + adi - bci - bdi^2}{c^2 - d^2i^2} \\ &= \frac{ac - adi + bci + bd + ac + adi - bci + bd}{c^2 + d^2} \\ &= \frac{2ac + 2bd}{c^2 + d^2} \quad \text{which is a real number} \end{aligned}$$



**7 a** If  $z = a + bi$   
 then  $z^2 = (a + bi)^2$   
 $= a^2 + 2abi + b^2i^2$   
 $= (a^2 - b^2) + (2ab)i$   
 $\therefore (z^2)^* = (a^2 - b^2) - 2abi$   
 $(z^*)^2 = (a - bi)^2$   
 $= a^2 - 2abi + b^2i^2$   
 $= (a^2 - b^2) - 2abi$   
 $\therefore (z^2)^* = (z^*)^2$  as required

**b**  $z^3 = z^2 \times z$   
 $= [(a^2 - b^2) + 2abi](a + bi)$  {using **a**}  
 $= a(a^2 - b^2) + b(a^2 - b^2)i + 2a^2bi + 2ab^2i^2$   
 $= a^3 - ab^2 + a^2bi - b^3i + 2a^2bi - 2ab^2$   
 $= (a^3 - 3ab^2) + (3a^2b - b^3)i$   
 $\therefore (z^3)^* = (a^3 - 3ab^2) - (3a^2b - b^3)i$   
 $(z^*)^3 = (z^*)^2 \times z^*$   
 $= [(a^2 - b^2) - 2abi](a - bi)$   
 $= a(a^2 - b^2) - b(a^2 - b^2)i - 2a^2bi + 2ab^2i^2$   
 $= a^3 - ab^2 - a^2bi + b^3i - 2a^2bi - 2ab^2$   
 $= (a^3 - 3ab^2) - (3a^2b - b^3)i$   
 $\therefore (z^3)^* = (z^*)^3$  as required

**8**  $w = \frac{z-1}{z^*+1}$  where  $z = a + bi$   
 $\therefore w = \left( \frac{(a-1) + bi}{(a+1) - bi} \right) \times \left( \frac{(a+1) + bi}{(a+1) + bi} \right)$   
 $= \frac{(a^2 - 1) + (a-1)bi + (a+1)bi + b^2i^2}{(a+1)^2 - b^2i^2}$   
 $= \frac{(a^2 - b^2 - 1) + 2abi}{(a+1)^2 + b^2}$   
 $= \frac{a^2 - b^2 - 1}{(a+1)^2 + b^2} + \frac{2ab}{(a+1)^2 + b^2} i$

**a**  $w$  is real if  $2ab = 0$ ,  
 that is, if  $a = 0$  or  $b = 0$ .  
 However, if  $b = 0$  and  $a = -1$ ,  
 $w$  is undefined and hence is not real.  
 $\therefore a = 0$  or  $(b = 0, a \neq -1)$ .

**b**  $w$  is purely imaginary if  
 $a^2 - b^2 - 1 = 0$  and  $2ab \neq 0$ ,  
 that is, if  $a^2 - b^2 = 1$   
 and  $a \neq 0, b \neq 0$ .

## REVIEW SET 4A

1 a  $x^2 - 6x + 11 = 0$

$$\therefore x = \frac{6 \pm \sqrt{36 - 4 \times 1 \times 11}}{2}$$

$$\therefore x = \frac{6 \pm \sqrt{-8}}{2}$$

$$\therefore x = 3 \pm \sqrt{-2}$$

$$\therefore x = 3 \pm i\sqrt{2}$$

b  $x^2 + 3 = 2\sqrt{2}x$

$$\therefore x^2 - 2\sqrt{2}x + 3 = 0$$

$$\therefore x = \frac{2\sqrt{2} \pm \sqrt{8 - 4 \times 1 \times 3}}{2}$$

$$\therefore x = \frac{2\sqrt{2} \pm \sqrt{-4}}{2}$$

$$\therefore x = \sqrt{2} \pm \sqrt{-1}$$

$$\therefore x = \sqrt{2} \pm i$$

2 a  $3x^2 + 16 = 0$

$$\therefore 3x^2 - 16i^2 = 0$$

$$\therefore (\sqrt{3}x + 4i)(\sqrt{3}x - 4i) = 0$$

$$\therefore x = \pm \frac{4}{\sqrt{3}}i$$

b  $x^4 + 2x^2 = 15$

$$\therefore x^4 + 2x^2 - 15 = 0$$

$$\therefore (x^2 + 5)(x^2 - 3) = 0$$

$$\therefore (x^2 - 5i^2)(x^2 - 3) = 0$$

$$\therefore (x + i\sqrt{5})(x - i\sqrt{5})(x + \sqrt{3})(x - \sqrt{3}) = 0$$

$$\therefore x = \pm i\sqrt{5} \text{ or } \pm\sqrt{3}$$

3  $z = 3 + i, w = -2 - i$

a  $2z - 3w = 2(3 + i) - 3(-2 - i)$   
 $= 6 + 2i + 6 + 3i$   
 $= 12 + 5i$

b  $\frac{z^*}{w} = \left( \frac{3-i}{-2-i} \right) \times \left( \frac{-2+i}{-2+i} \right)$   
 $= \frac{-6 + 3i + 2i - i^2}{4 - i^2}$   
 $= \frac{-5 + 5i}{5}$   
 $= -1 + i$

c  $z^3 = (3 + i)^3$   
 $= 3^3 + 3(3)^2(i) + 3(3)(i)^2 + i^3$   
 $= 27 + 27i + 9i^2 - i$   
 $= 18 + 26i$

4  $z = \frac{3}{i + \sqrt{3}} + \sqrt{3}$   
 $= \left( \frac{3}{\sqrt{3} + i} \right) \times \left( \frac{\sqrt{3} - i}{\sqrt{3} - i} \right) + \sqrt{3}$   
 $= \frac{3\sqrt{3} - 3i}{3 - i^2} + \sqrt{3}$   
 $= \frac{3\sqrt{3} - 3i}{4} + \frac{4\sqrt{3}}{4}$   
 $= \frac{7\sqrt{3}}{4} - \frac{3}{4}i$

$$\therefore \operatorname{Re}(z) = \frac{7\sqrt{3}}{4}, \operatorname{Im}(z) = -\frac{3}{4}$$

5  $\frac{5}{2-i} = \left( \frac{5}{2-i} \right) \times \left( \frac{2+i}{2+i} \right)$   
 $= \frac{10 + 5i}{4 - i^2}$   
 $= \frac{10 + 5i}{5} = 2 + i$

So,  $z = ((2 + i) - 3 - 2i)^3$   
 $= (-1 - i)^3$   
 $= -(i + 1)^3$   
 $= -(i^3 + 3i^2 + 3i + 1)$   
 $= -(-i - 3 + 3i + 1)$   
 $= -(2i - 2)$   
 $= 2 - 2i$

**6 a**  $a + bi = 4 = 4 + 0i$

Equating real and imaginary parts,  $a = 4$  and  $b = 0$ .

**b**  $(1 - 2i)(a + bi) = -5 - 10i$

$$\therefore a + bi = \left( \frac{-5 - 10i}{1 - 2i} \right) \times \left( \frac{1 + 2i}{1 + 2i} \right)$$

$$\therefore a + bi = \frac{-5 - 10i - 10i - 20i^2}{1 - 4i^2}$$

$$\therefore a + bi = \frac{15 - 20i}{5}$$

$$\therefore a + bi = 3 - 4i$$

Equating real and imaginary parts,  
 $a = 3$  and  $b = -4$ .

**c**  $(a + 2i)(1 + bi) = 17 - 19i$

$$\therefore a + abi + 2i + 2bi^2 = 17 - 19i$$

$$\therefore (a - 2b) + (ab + 2)i = 17 - 19i$$

Equating real and imaginary parts,

$$a - 2b = 17 \quad \text{and} \quad ab + 2 = -19$$

$$\therefore a = 2b + 17 \quad \text{and} \quad ab = -21$$

$$\therefore (2b + 17)b = -21$$

$$\therefore 2b^2 + 17b + 21 = 0$$

$$\therefore (2b + 3)(b + 7) = 0$$

$$\therefore b = -\frac{3}{2} \quad \text{or} \quad b = -7$$

$$\therefore b = -\frac{3}{2} \quad \text{and} \quad a = 2\left(-\frac{3}{2}\right) + 17 = 14$$

$$\text{or} \quad b = -7 \quad \text{and} \quad a = 2(-7) + 17 = 3$$

**7**  $2z - 1 = iz - i$

$$\therefore (2 - i)z = 1 - i$$

$$\therefore z = \left( \frac{1 - i}{2 - i} \right) \times \left( \frac{2 + i}{2 + i} \right)$$

$$= \frac{2 + i - 2i - i^2}{4 - i^2}$$

$$= \frac{3 - i}{5}$$

$$\therefore z = \frac{3}{5} - \frac{1}{5}i$$

**8** If  $\frac{2 - 3i}{2a + bi} = 3 + 2i$  then  $2a + bi = \left( \frac{2 - 3i}{3 + 2i} \right) \times \left( \frac{3 - 2i}{3 - 2i} \right)$

$$\therefore 2a + bi = \frac{6 - 4i - 9i + 6i^2}{9 - 4i^2}$$

$$\therefore 2a + bi = \frac{-13i}{13}$$

$$\therefore 2a + bi = 0 - i$$

Equating real and imaginary parts,  $2a = 0$  and  $b = -1$

$$\therefore a = 0 \quad \text{and} \quad b = -1$$

**9** If  $z = a + bi$   
then  $z + z^* = (a + bi) + (a - bi)$   
 $= 2a$  which is real

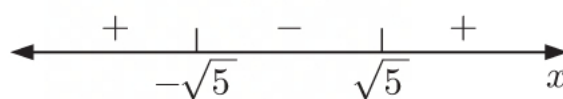
Also,  $zz^* = (a + bi)(a - bi)$   
 $= a^2 - abi + abi - b^2i^2$   
 $= a^2 + b^2$  which is real

$$\begin{aligned}
 10 \quad \frac{z+u}{z-u} &= \frac{x+2i+3+yi}{x+2i-(3+yi)} \\
 &= \left( \frac{(x+3)+(y+2)i}{(x-3)-(y-2)i} \right) \times \left( \frac{(x-3)+(y-2)i}{(x-3)+(y-2)i} \right) \\
 &= \frac{(x^2-9)+(x+3)(y-2)i+(y+2)(x-3)i+(y^2-4)i^2}{(x-3)^2-(y-2)^2i^2} \\
 &= \frac{(x^2-9)-(y^2-4)+((x+3)(y-2)+(y+2)(x-3))i}{(x-3)^2+(y-2)^2} \\
 &= \frac{(x^2-y^2-5)+(xy-2x+3y-6+xy-3y+2x-6)i}{(x-3)^2+(y-2)^2} \\
 &= \frac{x^2-y^2-5}{(x-3)^2+(y-2)^2} + \frac{2(xy-6)}{(x-3)^2+(y-2)^2} i
 \end{aligned}$$

$$\begin{aligned}
 \frac{z+u}{z-u} \text{ is purely imaginary if } x^2-y^2-5=0 \quad \text{and} \quad 2(xy-6) \neq 0 \\
 \therefore x^2=y^2+5 \quad \text{and} \quad xy \neq 6
 \end{aligned}$$

$$\text{Since } y^2 \geq 0, \quad x^2 \geq 5$$

$$\therefore x \leq -\sqrt{5} \text{ or } x \geq \sqrt{5}$$



$\therefore$  the smallest positive value of  $x$  for which  $\frac{z+u}{z-u}$  is imaginary is  $\sqrt{5}$  when  $y=0$ .

## REVIEW SET 4B

$$\begin{aligned}
 1 \quad a \quad x^2+121 &= x^2-121i^2 \\
 &= (x+11i)(x-11i)
 \end{aligned}$$

$$\begin{aligned}
 b \quad 3x^2+7 &= 3x^2-7i^2 \\
 &= (\sqrt{3}x+i\sqrt{7})(\sqrt{3}x-i\sqrt{7})
 \end{aligned}$$

$$\begin{aligned}
 2 \quad 4x^3+x &= 0 \\
 \therefore x(4x^2+1) &= 0 \\
 \therefore x(4x^2-i^2) &= 0 \\
 \therefore x(2x+i)(2x-i) &= 0 \\
 \therefore x=0 \text{ or } \pm \frac{1}{2}i
 \end{aligned}$$

$$\begin{aligned}
 3 \quad 2w^*-iz &= 2(3-2i)^*-i(4+i) \\
 &= 2(3+2i)-i(4+i) \\
 &= 6+4i-4i-i^2 \\
 &= 7
 \end{aligned}$$

$$\begin{aligned}
 4 \quad 2z+w &= i \\
 \therefore 6z+3w &= 0+3i \\
 \text{and } z-3w &= 7-10i \\
 \text{Adding, } 7z &= 7-7i \\
 \therefore z &= 1-i \\
 \text{Now, } 2z+w &= i \\
 \therefore z+w &= i-z \\
 &= i-(1-i) \\
 &= i-1+i \\
 &= -1+2i
 \end{aligned}$$

$$\begin{aligned}
 5 \quad \sqrt{z} &= \frac{2}{3-2i} + 2+5i \\
 &= \left( \frac{2}{3-2i} \right) \times \left( \frac{3+2i}{3+2i} \right) + 2+5i \\
 &= \frac{6+4i}{9-4i^2} + 2+5i \\
 &= \frac{6+4i}{13} + 2+5i \\
 &= \frac{32}{13} + \frac{69}{13}i \\
 \therefore z &= \left( \frac{32}{13} + \frac{69}{13}i \right)^2 \\
 &= \frac{32^2-69^2}{169} + \frac{2 \times 32 \times 69}{169}i \\
 &= -\frac{3737}{169} + \frac{4416}{169}i
 \end{aligned}$$



**6**  $w = \frac{z+1}{z^*+1}$  where  $z = a+bi$

$$\begin{aligned}\therefore w &= \left( \frac{(a+1)+bi}{(a+1)-bi} \right) \times \left( \frac{(a+1)+bi}{(a+1)+bi} \right) \\ &= \frac{(a+1)^2 + (a+1)bi + (a+1)bi + b^2i^2}{(a+1)^2 - b^2i^2} \\ &= \frac{(a+1)^2 - b^2}{(a+1)^2 + b^2} + \left( \frac{2(a+1)b}{(a+1)^2 + b^2} \right) i\end{aligned}$$

$w$  is purely imaginary if  $(a+1)^2 - b^2 = 0$  and  $2(a+1)b \neq 0$   
 $\therefore b^2 = (a+1)^2$  and  $a \neq -1, b \neq 0$   
 $\therefore b = \pm(a+1), a \neq -1, b \neq 0$

**7 a**  $x+yi=0$

Equating real and imaginary parts,  $x=0$  and  $y=0$ .

**b**  $(3-2i)(x+i) = 17+yi$   
 $3x+3i-2xi-2i^2 = 17+yi$   
 $(3x+2) + (3-2x)i = 17+yi$

Equating real and imaginary parts,  $3x+2=17$  and  $3-2x=y$   
 $\therefore 3x=15$  and  $y=3-2x$   
 $\therefore x=5$  and  $y=-7$

**c**  $(x+yi)^2 = x-yi$   
 $\therefore x^2 + 2ixy + y^2i^2 = x-yi$   
 $\therefore (x^2 - y^2) + 2xyi = x-yi$

Equating real and imaginary parts,  $x^2 - y^2 = x$  and  $2xy = -y$   
 $\therefore 2xy + y = 0$   
 $\therefore y(2x+1) = 0$   
 $\therefore y=0$  or  $x = -\frac{1}{2}$

When  $y=0$ ,  $x^2 = x$  and when  $x = -\frac{1}{2}$ ,  $(-\frac{1}{2})^2 - y^2 = -\frac{1}{2}$   
 $\therefore x^2 - x = 0$   $\therefore \frac{1}{4} - y^2 = -\frac{1}{2}$   
 $\therefore x(x-1) = 0$   $\therefore y^2 = \frac{3}{4}$   
 $\therefore x=0$  or  $1$   $\therefore y = \pm\frac{\sqrt{3}}{2}$

The possible solutions are:

$x$	0	1	$-\frac{1}{2}$	$-\frac{1}{2}$
$y$	0	0	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$

$$\mathbf{d} \quad (3x + 2yi)(1 - i) = (3y + 1)i - x$$

$$\therefore 3x - 3xi + 2yi - 2yi^2 = 3yi + i - x$$

$$\therefore (3x + 2y) + (2y - 3x)i = -x + (3y + 1)i$$

$$\text{Equating real and imaginary parts, } 3x + 2y = -x \quad \text{and} \quad 2y - 3x = 3y + 1$$

$$\therefore 4x = -2y \quad \text{and} \quad -1 - 3x = y$$

$$\therefore 4x = -2(-1 - 3x)$$

$$\therefore 4x = 2 + 6x$$

$$\therefore -2x = 2$$

$$\therefore x = -1 \quad \text{and} \quad y = -1 - 3(-1) = 2$$

$$\mathbf{8} \quad \text{Let } z = a + bi \quad \text{then} \quad (a + bi)^2 = 5 - 12i$$

$$\therefore a^2 + 2abi + b^2i^2 = 5 - 12i$$

$$\therefore (a^2 - b^2) + 2abi = 5 - 12i$$

$$\text{Equating real and imaginary parts, } a^2 - b^2 = 5 \quad \text{and} \quad 2ab = -12$$

$$\therefore a^2 - b^2 = 5 \quad \text{and} \quad b = -\frac{6}{a}$$

$$\text{So, } a^2 - \left(-\frac{6}{a}\right)^2 = 5$$

$$\therefore a^2 - \frac{36}{a^2} = 5$$

$$\therefore a^4 - 5a^2 - 36 = 0$$

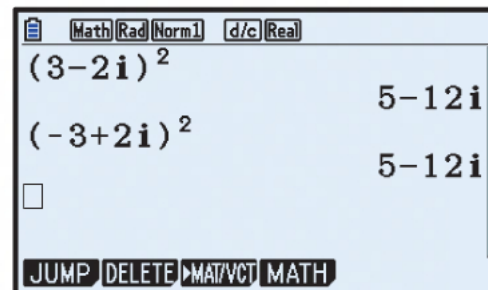
$$\therefore (a^2 - 9)(a^2 + 4) = 0$$

$$\therefore a^2 = 9 \quad \{a^2 + 4 > 0\}$$

$$\therefore a = 3 \quad \text{and} \quad b = -\frac{6}{3} = -2$$

$$\text{or } a = -3 \quad \text{and} \quad b = \frac{-6}{-3} = 2$$

$$\therefore z = 3 - 2i \quad \text{or} \quad z = -3 + 2i$$



The calculator screen shows the following calculations:

Expression	Result
$(3-2i)^2$	$5-12i$
$(-3+2i)^2$	$5-12i$

$$\begin{aligned} \mathbf{9} \quad (kz)^* &= (ka + kbi)^* \\ &= ka - kbi \\ &= k(a - bi) \\ &= kz^* \end{aligned}$$

$$\mathbf{10} \quad \text{Let } z = a + bi \quad \text{and} \quad w = c + di \quad \text{where } b \neq 0 \quad \text{and} \quad d \neq 0$$

$$\begin{aligned} \text{Now } z + w &= (a + c) + (b + d)i \quad \text{and} \quad zw = (a + bi)(c + di) \\ &= (ac - bd) + (bc + ad)i \end{aligned}$$

$$\text{As } z + w \text{ is real, } b + d = 0 \quad \text{and as } zw \text{ is real, } bc + ad = 0 \quad \dots (2)$$

$$\therefore b = -d \quad \dots (1)$$

$$\text{Substituting (1) into (2), } -dc + ad = 0$$

$$\therefore d(a - c) = 0$$

$$\text{But } d \neq 0, \therefore a = c \quad \text{and} \quad b = -d \quad \{\text{from (1)}\}$$

$$\therefore z^* = a - bi = c + di = w$$

# Chapter 5

## REAL POLYNOMIALS

### EXERCISE 5A

- 1 a** In  $P(x) = 2x^3 + x^2 - 2x + 5$ :
- i** The highest power of the variable  $x$  is 3.  
 $\therefore$  the polynomial has degree 3.
  - ii** The leading coefficient is 2.
  - iii** The constant term is 5.
- b** In  $f(x) = x^4 + 5x^2 + 3x - 2$ :
- i** The highest power of the variable  $x$  is 4.  
 $\therefore$  the polynomial has degree 4.
  - ii** The leading coefficient is 1.
  - iii** The constant term is  $-2$ .
- c** In  $Q(x) = -3x^5 - x^3 + 4x^2 + 1$ :
- i** The highest power of the variable  $x$  is 5.  
 $\therefore$  the polynomial has degree 5.
  - ii** The leading coefficient is  $-3$ .
  - iii** The constant term is 1.
- d** In  $g(x) = 5x^3 - 3x^2 + 4x$ :
- i** The highest power of the variable  $x$  is 3.  
 $\therefore$  the polynomial has degree 3.
  - ii** The leading coefficient is 5.
  - iii** The constant term is 0.
- e** In  $P(x) = 7 + 5x - x^4$ :
- i** The highest power of the variable  $x$  is 4.  
 $\therefore$  the polynomial has degree 4.
  - ii** The leading coefficient is  $-1$ .
  - iii** The constant term is 7.
- f** In  $h(x) = 2x^2 - 4x^3 + 7 - 3x$ :
- i** The highest power of the variable  $x$  is 3.  
 $\therefore$  the polynomial has degree 3.
  - ii** The leading coefficient is  $-4$ .
  - iii** The constant term is 7.
- 2 a** The coefficient of  $x^2$  in  $P(x) = 4x^3 + 7x^2 - 5x + 2$  is 7.
- b** The coefficient of  $x$  in  $f(x) = x^4 + 2x^3 - 4x + 1$  is  $-4$ .
- c** The coefficient of  $x^3$  in  $Q(x) = x^5 - 3x^4 + 2x^2 - 7$  is 0.
- d** The coefficient of  $x^2$  in  $g(x) = \frac{1}{3}x^3 - 2x + \frac{1}{4}x^2 + 3$  is  $\frac{1}{4}$ .

## EXERCISE 5B

1  $P(x) = x^2 + 2x + 3$  and  $Q(x) = 4x^2 + 5x + 6$

a 
$$\begin{aligned} 3P(x) &= 3(x^2 + 2x + 3) \\ &= 3x^2 + 6x + 9 \end{aligned}$$

b 
$$\begin{aligned} P(x) + Q(x) &= x^2 + 2x + 3 \\ &\quad + 4x^2 + 5x + 6 \\ &= 5x^2 + 7x + 9 \end{aligned}$$

c 
$$\begin{aligned} P(x) - 2Q(x) &= x^2 + 2x + 3 - 2(4x^2 + 5x + 6) \\ &= x^2 + 2x + 3 \\ &\quad - 8x^2 - 10x - 12 \\ &= -7x^2 - 8x - 9 \end{aligned}$$

d 
$$\begin{aligned} P(x)Q(x) &= (x^2 + 2x + 3)(4x^2 + 5x + 6) \\ &= x^2(4x^2 + 5x + 6) + 2x(4x^2 + 5x + 6) + 3(4x^2 + 5x + 6) \\ &= 4x^4 + 5x^3 + 6x^2 \\ &\quad + 8x^3 + 10x^2 + 12x \\ &\quad + 12x^2 + 15x + 18 \\ &= 4x^4 + 13x^3 + 28x^2 + 27x + 18 \end{aligned}$$

2 a 
$$\begin{aligned} f(x) + g(x) &= (x^2 - x + 2) + (x^3 - 3x + 5) \\ &= x^2 - x + 2 \\ &\quad + x^3 - 3x + 5 \\ &= x^3 + x^2 - 4x + 7 \end{aligned}$$

b 
$$\begin{aligned} g(x) - f(x) &= x^3 - 3x + 5 - (x^2 - x + 2) \\ &= x^3 - 3x + 5 \\ &\quad - x^2 + x - 2 \\ &= x^3 - x^2 - 2x + 3 \end{aligned}$$

c 
$$\begin{aligned} 2f(x) + 3g(x) &= 2(x^2 - x + 2) + 3(x^3 - 3x + 5) \\ &= 2x^2 - 2x + 4 \\ &\quad + 3x^3 - 9x + 15 \\ &= 3x^3 + 2x^2 - 11x + 19 \end{aligned}$$

d 
$$\begin{aligned} g(x) + xf(x) &= x^3 - 3x + 5 + x(x^2 - x + 2) \\ &= x^3 - 3x + 5 \\ &\quad + x^3 - x^2 + 2x \\ &= 2x^3 - x^2 - x + 5 \end{aligned}$$

e 
$$\begin{aligned} f(x)g(x) &= (x^2 - x + 2)(x^3 - 3x + 5) \\ &= x^2(x^3 - 3x + 5) - x(x^3 - 3x + 5) + 2(x^3 - 3x + 5) \\ &= x^5 - 3x^3 + 5x^2 \\ &\quad - x^4 + 3x^2 - 5x \\ &\quad + 2x^3 - 6x + 10 \\ &= x^5 - x^4 - x^3 + 8x^2 - 11x + 10 \end{aligned}$$



$$\begin{aligned}
& \mathbf{f} \quad [f(x)]^2 \\
&= (x^2 - x + 2)(x^2 - x + 2) \\
&= x^2(x^2 - x + 2) - x(x^2 - x + 2) + 2(x^2 - x + 2) \\
&= x^4 - x^3 + 2x^2 \\
&\quad - x^3 + x^2 - 2x \\
&\quad + 2x^2 - 2x + 4 \\
&= x^4 - 2x^3 + 5x^2 - 4x + 4
\end{aligned}$$

$$\begin{aligned}
& \mathbf{3} \quad \mathbf{a} \quad (x^2 - 2x + 3)(2x + 1) \\
&= 2x^3 + x^2 \\
&\quad - 4x^2 - 2x \\
&\quad + 6x + 3 \\
&= 2x^3 - 3x^2 + 4x + 3
\end{aligned}$$

$$\begin{aligned}
& \mathbf{c} \quad (x + 2)^3 \\
&= (x + 2)^2(x + 2) \\
&= (x^2 + 4x + 4)(x + 2) \\
&= x^3 + 2x^2 \\
&\quad + 4x^2 + 8x \\
&\quad + 4x + 8 \\
&= x^3 + 6x^2 + 12x + 8
\end{aligned}$$

$$\begin{aligned}
& \mathbf{e} \quad (2x - 1)^4 \\
&= (2x - 1)^2(2x - 1)^2 \\
&= (4x^2 - 4x + 1)(4x^2 - 4x + 1) \\
&= 16x^4 - 16x^3 + 4x^2 \\
&\quad - 16x^3 + 16x^2 - 4x \\
&\quad + 4x^2 - 4x + 1 \\
&= 16x^4 - 32x^3 + 24x^2 - 8x + 1
\end{aligned}$$

$$\begin{aligned}
& \mathbf{4} \quad \mathbf{a} \quad (2x^2 - 3x + 5)(3x - 1) \\
&= 6x^3 - 2x^2 \\
&\quad - 9x^2 + 3x \\
&\quad + 15x - 5 \\
&= 6x^3 - 11x^2 + 18x - 5
\end{aligned}$$

$$\begin{aligned}
& \mathbf{c} \quad (2x^2 + 3x + 2)(5 - x) \\
&= -2x^3 + 10x^2 \\
&\quad - 3x^2 + 15x \\
&\quad - 2x + 10 \\
&= -2x^3 + 7x^2 + 13x + 10
\end{aligned}$$

$$\begin{aligned}
& \mathbf{b} \quad (x - 1)^2(x^2 + 3x - 2) \\
&= (x^2 - 2x + 1)(x^2 + 3x - 2) \\
&= x^4 + 3x^3 - 2x^2 \\
&\quad - 2x^3 - 6x^2 + 4x \\
&\quad + x^2 + 3x - 2 \\
&= x^4 + x^3 - 7x^2 + 7x - 2
\end{aligned}$$

$$\begin{aligned}
& \mathbf{d} \quad (2x^2 - x + 3)^2 \\
&= (2x^2 - x + 3)(2x^2 - x + 3) \\
&= 4x^4 - 2x^3 + 6x^2 \\
&\quad - 2x^3 + x^2 - 3x \\
&\quad + 6x^2 - 3x + 9 \\
&= 4x^4 - 4x^3 + 13x^2 - 6x + 9
\end{aligned}$$

$$\begin{aligned}
& \mathbf{f} \quad (3x - 2)^2(2x + 1)(x - 4) \\
&= (9x^2 - 12x + 4)(2x^2 - 7x - 4) \\
&= 18x^4 - 63x^3 - 36x^2 \\
&\quad - 24x^3 + 84x^2 + 48x \\
&\quad + 8x^2 - 28x - 16 \\
&= 18x^4 - 87x^3 + 56x^2 + 20x - 16
\end{aligned}$$

$$\begin{aligned}
& \mathbf{b} \quad (4x^2 - x + 2)(2x + 5) \\
&= 8x^3 + 20x^2 \\
&\quad - 2x^2 - 5x \\
&\quad + 4x + 10 \\
&= 8x^3 + 18x^2 - x + 10
\end{aligned}$$

$$\begin{aligned}
& \mathbf{d} \quad (x - 2)^2(2x + 1) \\
&= (x^2 - 4x + 4)(2x + 1) \\
&= 2x^3 + x^2 \\
&\quad - 8x^2 - 4x \\
&\quad + 8x + 4 \\
&= 2x^3 - 7x^2 + 4x + 4
\end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & (x^2 - 3x + 2)(2x^2 + 4x - 1) \\
 &= 2x^4 + 4x^3 - x^2 \\
 &\quad - 6x^3 - 12x^2 + 3x \\
 &\quad + 4x^2 + 8x - 2 \\
 &= \underline{2x^4 - 2x^3 - 9x^2 + 11x - 2}
 \end{aligned}$$

$$\begin{aligned}
 \text{g} \quad & (x^2 - x + 3)^2 \\
 &= (x^2 - x + 3)(x^2 - x + 3) \\
 &= x^4 - x^3 + 3x^2 \\
 &\quad - x^3 + x^2 - 3x \\
 &\quad + 3x^2 - 3x + 9 \\
 &= \underline{x^4 - 2x^3 + 7x^2 - 6x + 9}
 \end{aligned}$$

$$\begin{aligned}
 \text{i} \quad & (2x + 5)^3 \\
 &= (2x + 5)^2(2x + 5) \\
 &= (4x^2 + 20x + 25)(2x + 5) \\
 &= 8x^3 + 20x^2 \\
 &\quad + 40x^2 + 100x \\
 &\quad + 50x + 125 \\
 &= \underline{8x^3 + 60x^2 + 150x + 125}
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & (3x^2 - x + 2)(5x^2 + 2x - 3) \\
 &= 15x^4 + 6x^3 - 9x^2 \\
 &\quad - 5x^3 - 2x^2 + 3x \\
 &\quad + 10x^2 + 4x - 6 \\
 &= \underline{15x^4 + x^3 - x^2 + 7x - 6}
 \end{aligned}$$

$$\begin{aligned}
 \text{h} \quad & (2x^2 + x - 4)^2 \\
 &= (2x^2 + x - 4)(2x^2 + x - 4) \\
 &= 4x^4 + 2x^3 - 8x^2 \\
 &\quad + 2x^3 + x^2 - 4x \\
 &\quad - 8x^2 - 4x + 16 \\
 &= \underline{4x^4 + 4x^3 - 15x^2 - 8x + 16}
 \end{aligned}$$

$$\begin{aligned}
 \text{j} \quad & (x^3 + x^2 - 2)^2 \\
 &= (x^3 + x^2 - 2)(x^3 + x^2 - 2) \\
 &= x^6 + x^5 - 2x^3 \\
 &\quad + x^5 + x^4 - 2x^2 \\
 &\quad - 2x^3 - 2x^2 + 4 \\
 &= \underline{x^6 + 2x^5 + x^4 - 4x^3 - 4x^2 + 4}
 \end{aligned}$$

**5**  $Q(x) = (x - 1)P(x)$

If we let  $P(x) = ax^3 + bx^2 + cx + d$ , where  $a \neq 0$ , then

$$\begin{aligned}
 Q(x) &= (x - 1)(ax^3 + bx^2 + cx + d) \\
 &= x(ax^3 + bx^2 + cx + d) - (ax^3 + bx^2 + cx + d) \\
 &= ax^4 + bx^3 + cx^2 + dx - ax^3 - bx^2 - cx - d \\
 &= ax^4 + (b - a)x^3 + (c - b)x^2 + (d - c)x - d
 \end{aligned}$$

The degree of  $Q(x)$  is 4.

**6**  $P(x) = (x^2 - 3x + 4)Q(x)$

If  $P(x)$  has degree 5, then  $Q(x)$  must have degree 3, since  $x^2 \times x^3 = x^5$ .

**7** Let  $P(x) = 3x^2 + ax + 5$  and  $Q(x) = -5x^4 + bx^3 + cx^2 + dx - 2$ .

$$\begin{aligned}
 \text{Now } P(x)Q(x) &= (3x^2 + ax + 5)(-5x^4 + bx^3 + cx^2 + dx - 2) \\
 &= 3x^2(-5x^4 + bx^3 + cx^2 + dx - 2) + ax(-5x^4 + bx^3 + cx^2 + dx - 2) \\
 &\quad + 5(-5x^4 + bx^3 + cx^2 + dx - 2) \\
 &= -15x^6 + 3bx^5 + \dots + 5dx - 10
 \end{aligned}$$

$P(x)Q(x)$  has degree 6, leading coefficient  $-15$ , and constant term  $-10$ .

**8 a** If  $p \neq q$ , the degree of  $P(x) + Q(x)$  is the maximum of  $p$  and  $q$  since the highest power of  $x$  is the maximum of  $p$  and  $q$ .

If  $p = q$ , the degree of  $P(x) + Q(x) \leq p$ , since the coefficients of the higher powers of  $x$  may cancel.

**b**  $kP(x)$  has degree  $p$  since any non-zero scalar multiple of a polynomial results in a polynomial of the same degree.

- c  $P(x)Q(x)$  has degree  $p + q$  since the highest power of  $x$  is  $x^p \times x^q = x^{p+q}$ .
- d  $(P(x))^2$  has degree  $2p$  since the highest power of  $x$  is  $x^p \times x^p = x^{2p}$ .

**ACTIVITY 1****POLYNOMIALS IN POWERS OF  $x \pm k$** 

- 1 a Let  $P(x) = x^3 - 2x^2 + 3x + 5$   
 and  $Q(x) = P(x + 2)$   

$$= (x + 2)^3 - 2(x + 2)^2 + 3(x + 2) + 5$$

$$= x^3 + 6x^2 + 12x + 8$$

$$\quad - 2x^2 - 8x - 8$$

$$\quad \quad + 3x + 6$$

$$\quad \quad \quad + 5$$

$$= x^3 + 4x^2 + 7x + 11$$

$$\therefore P(x) = Q(x - 2)$$

$$= (x - 2)^3 + 4(x - 2)^2 + 7(x - 2) + 11$$
- b Let  $P(x) = x^4 + 2x^2 - 6$   
 and  $Q(x) = P(x - 1)$   

$$= (x - 1)^4 + 2(x - 1)^2 - 6$$

$$= x^4 - 4x^3 + 6x^2 - 4x + 1$$

$$\quad + 2x^2 - 4x + 2$$

$$\quad \quad - 6$$

$$= x^4 - 4x^3 + 8x^2 - 8x - 3$$

$$\therefore P(x) = Q(x + 1)$$

$$= (x + 1)^4 - 4(x + 1)^3 + 8(x + 1)^2 - 8(x + 1) - 3$$
- c Let  $P(z) = z^4$   
 and  $Q(z) = P(z + 1)$   

$$= (z + 1)^4$$

$$= z^4 + 4z^3 + 6z^2 + 4z + 1$$

$$\therefore P(z) = Q(z - 1)$$

$$= (z - 1)^4 + 4(z - 1)^3 + 6(z - 1)^2 + 4(z - 1) + 1$$
- d Let  $P(z) = z^4 + z^2$   
 and  $Q(z) = P(z - 3)$   

$$= (z - 3)^4 + (z - 3)^2$$

$$= z^4 - 12z^3 + 54z^2 - 108z + 81$$

$$\quad + z^2 - 6z + 9$$

$$= z^4 - 12z^3 + 55z^2 - 114z + 90$$

$$\therefore P(z) = Q(z + 3)$$

$$= (z + 3)^4 - 12(z + 3)^3 + 55(z + 3)^2 - 114(z + 3) + 90$$

**2 a** Let  $P(z) = z^3 - 8z^2 + 28z - 20$   
 and  $Q(z) = P(z + 3)$

$$\begin{aligned}
 &= (z + 3)^3 - 8(z + 3)^2 + 28(z + 3) - 20 \\
 &= z^3 + 9z^2 + 27z + 27 \\
 &\quad - 8z^2 - 48z - 72 \\
 &\quad + 28z + 84 \\
 &\quad - 20 \\
 &= z^3 + z^2 + 7z + 19 \\
 \therefore P(z) &= Q(z - 3) \\
 &= (z - 3)^3 + (z - 3)^2 + 7(z - 3) + 19
 \end{aligned}$$

All terms of which are positive for  $z > 3$

$$\therefore z^3 - 8z^2 + 28z - 20 > 0 \text{ for all } z > 3.$$

**b** Let  $P(x) = x^4 - 4x^3 + 8x^2 - 5x + 3$   
 and  $Q(x) = P(x + 1)$

$$\begin{aligned}
 &= (x + 1)^4 - 4(x + 1)^3 + 8(x + 1)^2 - 5(x + 1) + 3 \\
 &= x^4 + 4x^3 + 6x^2 + 4x + 1 \\
 &\quad - 4x^3 - 12x^2 - 12x - 4 \\
 &\quad + 8x^2 + 16x + 8 \\
 &\quad - 5x - 5 \\
 &\quad + 3 \\
 &= x^4 + 2x^2 + 3x + 3 \\
 \therefore P(x) &= Q(x - 1) \\
 &= (x - 1)^4 + 2(x - 1)^2 + 3(x - 1) + 3
 \end{aligned}$$

All terms of which are positive for  $x > 1$

$$\therefore x^4 - 4x^3 + 8x^2 - 5x + 3 > 0 \text{ for all } x > 1.$$

**c** Let  $P(x) = x^3 + 8x^2 + 24x + 25$   
 and  $Q(x) = P(x - 2)$

$$\begin{aligned}
 &= (x - 2)^3 + 8(x - 2)^2 + 24(x - 2) + 25 \\
 &= x^3 - 6x^2 + 12x - 8 \\
 &\quad + 8x^2 - 32x + 32 \\
 &\quad + 24x - 48 \\
 &\quad + 25 \\
 &= x^3 + 2x^2 + 4x + 1 \\
 \therefore P(x) &= Q(x + 2) \\
 &= (x + 2)^3 + 2(x + 2)^2 + 4(x + 2) + 1
 \end{aligned}$$

All terms of which are positive for  $x > -2$

$$\therefore x^3 + 8x^2 + 24x + 25 > 0 \text{ for all } x > -2.$$



**EXERCISE 5C**

**1 a** When  $x = 3$ ,  $x^2 - 5x + 6 = 3^2 - 5(3) + 6$   
 $= 9 - 15 + 6$   
 $= 0$

$\therefore 3$  is a zero of  $x^2 - 5x + 6$ .

**b** When  $x = -1$ ,  $x^3 + 2x^2 - 4x + 3 = (-1)^3 + 2(-1)^2 - 4(-1) + 3$   
 $= -1 + 2 + 4 + 3$   
 $= 8 \neq 0$

$\therefore -1$  is not a zero of  $x^3 + 2x^2 - 4x + 3$ .

**c** When  $x = 2$ ,  $3x^2 - 5x - 4 = 3(2)^2 - 5(2) - 4$   
 $= 12 - 10 - 4$   
 $= -2 \neq 0$

$\therefore 2$  is not a root of  $3x^2 - 5x - 4 = 0$ .

**d** When  $x = -2$ ,  $x^3 + 3x^2 + 5x + 6 = (-2)^3 + 3(-2)^2 + 5(-2) + 6$   
 $= -8 + 12 - 10 + 6$   
 $= 0$

$\therefore -2$  is a root of  $x^3 + 3x^2 + 5x + 6 = 0$ .

**2 a** We wish to find  $x$  such that  
 $3x^2 - 2x - 5 = 0$

$$\begin{aligned}\therefore x &= \frac{2 \pm \sqrt{4 - 4(3)(-5)}}{6} \\ &= \frac{2 \pm \sqrt{64}}{6} \\ &= \frac{2 \pm 8}{6} \\ &= -1 \text{ or } \frac{5}{3}\end{aligned}$$

The zeros are  $-1$  and  $\frac{5}{3}$ .

**c** We wish to find  $z$  such that  
 $z^2 - 6z + 6 = 0$

$$\begin{aligned}\therefore z &= \frac{6 \pm \sqrt{36 - 4(1)(6)}}{2} \\ &= \frac{6 \pm \sqrt{12}}{2} \\ &= 3 \pm \sqrt{3}\end{aligned}$$

The zeros are  $3 - \sqrt{3}$  and  $3 + \sqrt{3}$ .

**b** We wish to find  $x$  such that  
 $x^2 + 6x + 10 = 0$

$$\begin{aligned}\therefore x &= \frac{-6 \pm \sqrt{36 - 4(1)(10)}}{2} \\ &= \frac{-6 \pm \sqrt{-4}}{2} \\ &= \frac{-6 \pm 2i}{2} \\ &= -3 \pm i\end{aligned}$$

The zeros are  $-3 - i$  and  $-3 + i$ .

**d** We wish to find  $x$  such that  
 $x^3 - 4x = 0$

$$\begin{aligned}\therefore x(x^2 - 4) &= 0 \\ \therefore x(x + 2)(x - 2) &= 0 \\ \therefore x &= 0, -2, \text{ or } 2\end{aligned}$$

The zeros are  $0$ ,  $-2$ , and  $2$ .

**e** We wish to find  $z$  such that  $z^3 + 2z = 0$   
 $\therefore z(z^2 + 2) = 0$   
 $\therefore z(z^2 - 2i^2) = 0$   
 $\therefore z(z + i\sqrt{2})(z - i\sqrt{2}) = 0$   
 $\therefore z = 0 \text{ or } \pm i\sqrt{2}$

The zeros are  $0$ ,  $-i\sqrt{2}$ , and  $i\sqrt{2}$ .

**f** We wish to find  $z$  such that  $z^4 + 4z^2 - 5 = 0$   
 $\therefore (z^2 + 5)(z^2 - 1) = 0$   
 $\therefore (z^2 - 5i^2)(z^2 - 1) = 0$   
 $\therefore (z + i\sqrt{5})(z - i\sqrt{5})(z + 1)(z - 1) = 0$   
 $\therefore z = \pm i\sqrt{5} \text{ or } \pm 1$

The zeros are  $-i\sqrt{5}$ ,  $i\sqrt{5}$ ,  $-1$ , and  $1$ .

**3 a**  $5x^2 = 3x + 2$   
 $\therefore 5x^2 - 3x - 2 = 0$   
 $\therefore (5x + 2)(x - 1) = 0$   
The roots are  $1$  and  $-\frac{2}{5}$ .

**c**  $-2z(z^2 - 2z + 2) = 0$   
 $\therefore z = 0 \text{ or } z = \frac{2 \pm \sqrt{4 - 4(1)(2)}}{2}$   
 $\therefore z = 0 \text{ or } z = \frac{2 \pm \sqrt{-4}}{2}$   
 $\therefore z = 0 \text{ or } z = 1 \pm i$   
The roots are  $0$ ,  $1 - i$ , and  $1 + i$ .

**e**  $z^3 + 5z = 0$   
 $\therefore z(z^2 + 5) = 0$   
 $\therefore z(z^2 - 5i^2) = 0$   
 $\therefore z(z + i\sqrt{5})(z - i\sqrt{5}) = 0$   
The roots are  $0$ ,  $-i\sqrt{5}$ , and  $i\sqrt{5}$ .

**f**  $z^4 = 3z^2 + 10$   
 $\therefore z^4 - 3z^2 - 10 = 0$   
 $\therefore (z^2 - 5)(z^2 + 2) = 0$   
 $\therefore (z^2 - 5)(z^2 - 2i^2) = 0$   
 $\therefore (z + \sqrt{5})(z - \sqrt{5})(z + i\sqrt{2})(z - i\sqrt{2}) = 0$   
The roots are  $-\sqrt{5}$ ,  $\sqrt{5}$ ,  $-i\sqrt{2}$ , and  $i\sqrt{2}$ .

**4 a**  $(x - 4)(x - 2)(x + 3)$   
 $= (x^2 - 6x + 8)(x + 3)$   
 $= x^3 - 3x^2 - 10x + 24 \quad \checkmark$

**c**  $(2x + 1)(2x - 3)(x + 2)$   
 $= (4x^2 - 4x - 3)(x + 2)$   
 $= 4x^3 + 4x^2 - 11x - 6 \quad \checkmark$

**b**  $(2x + 1)(x^2 + 3) = 0$   
 $\therefore (2x + 1)(x^2 - 3i^2) = 0$   
 $\therefore (2x + 1)(x + i\sqrt{3})(x - i\sqrt{3}) = 0$   
The roots are  $-\frac{1}{2}$ ,  $-i\sqrt{3}$ , and  $i\sqrt{3}$ .

**d**  $x^3 = 5x$   
 $\therefore x^3 - 5x = 0$   
 $\therefore x(x^2 - 5) = 0$   
 $\therefore x(x + \sqrt{5})(x - \sqrt{5}) = 0$   
 $\therefore x = 0 \text{ or } \pm \sqrt{5}$   
The roots are  $0$ ,  $-\sqrt{5}$ , and  $\sqrt{5}$ .

**b**  $(3x + 1)(x + 3)(x - 4)$   
 $= (3x^2 + 10x + 3)(x - 4)$   
 $= 3x^3 - 2x^2 - 37x - 12 \quad \checkmark$

$$\begin{aligned} 5 \quad a \quad & 2x^2 - 7x - 15 \\ &= (2x + 3)(x - 5) \end{aligned}$$

$$\begin{aligned} c \quad & x^3 + 2x^2 - 4x \\ &= x(x^2 + 2x - 4) \\ & x^2 + 2x - 4 \text{ is zero when} \\ x &= \frac{-2 \pm \sqrt{4 - 4(1)(-4)}}{2} \\ &= -1 \pm \sqrt{5} \end{aligned}$$

$$\begin{aligned} \therefore \quad & x^3 + 2x^2 - 4x \\ &= x(x - [-1 - \sqrt{5}]) (x - [-1 + \sqrt{5}]) \\ &= x(x + 1 + \sqrt{5})(x + 1 - \sqrt{5}) \end{aligned}$$

$$\begin{aligned} e \quad & z^4 - 6z^2 + 5 \\ &= (z^2 - 1)(z^2 - 5) \\ &= (z + 1)(z - 1)(z + \sqrt{5})(z - \sqrt{5}) \end{aligned}$$

$$\begin{aligned} 6 \quad a \quad & x^2 - 6x - 16 = (x - 8)(x + 2) \\ & \text{The zeros are 8 and } -2. \end{aligned}$$

$$\begin{aligned} c \quad & x^2 + 3x + 4 \text{ has } \Delta = (3)^2 - 4(1)(4) \\ &= -7 \\ & \text{which is } < 0 \\ \therefore & \text{there are no real linear factors, no} \\ & \text{real zeros.} \end{aligned}$$

$$\begin{aligned} e \quad & 6z^3 - z^2 - 2z = z(6z^2 - z - 2) \\ &= z(3z - 2)(2z + 1) \\ & \text{The zeros are } 0, \frac{2}{3}, \text{ and } -\frac{1}{2}. \end{aligned}$$

$$\begin{aligned} 7 \quad & P(x) = a(x - \alpha)(x - \beta)(x - \gamma) \\ P(\alpha) &= a(\alpha - \alpha)(\alpha - \beta)(\alpha - \gamma) \\ &= a(0)(\alpha - \beta)(\alpha - \gamma) \\ &= 0 \end{aligned}$$

$\therefore \alpha$  is a zero of  $P(x)$ .

$$\begin{aligned} P(\gamma) &= a(\gamma - \alpha)(\gamma - \beta)(\gamma - \gamma) \\ &= a(\gamma - \alpha)(\gamma - \beta)(0) \\ &= 0 \end{aligned}$$

$\therefore \gamma$  is a zero of  $P(x)$ .

$$b \quad z^2 - 6z + 16 \text{ is zero when}$$

$$\begin{aligned} z &= \frac{6 \pm \sqrt{36 - 4(1)(16)}}{2} \\ &= 3 \pm i\sqrt{7} \end{aligned}$$

$$\begin{aligned} \therefore \quad & z^2 - 6z + 16 \\ &= (z - [3 + i\sqrt{7}]) (z - [3 - i\sqrt{7}]) \\ &= (z - 3 + i\sqrt{7})(z - 3 - i\sqrt{7}) \end{aligned}$$

$$\begin{aligned} d \quad & 6z^3 - z^2 - 2z \\ &= z(6z^2 - z - 2) \\ &= z(2z + 1)(3z - 2) \end{aligned}$$

$$\begin{aligned} f \quad & z^4 - z^2 - 2 \\ &= (z^2 - 2)(z^2 + 1) \\ &= (z + \sqrt{2})(z - \sqrt{2})(z + i)(z - i) \end{aligned}$$

$$\begin{aligned} b \quad & 2x^2 - 7x - 15 = (2x + 3)(x - 5) \\ & \text{The zeros are } -\frac{3}{2} \text{ and } 5. \end{aligned}$$

$$\begin{aligned} d \quad & x^3 + 2x^2 - 4x = x(x^2 + 2x - 4) \\ & x^2 + 2x - 4 \text{ is zero when} \\ x &= \frac{-2 \pm \sqrt{4 - 4(1)(-4)}}{2} \\ &= -1 \pm \sqrt{5} \end{aligned}$$

$$\begin{aligned} \therefore \quad & x^3 + 2x^2 - 4x \\ &= x(x + 1 + \sqrt{5})(x + 1 - \sqrt{5}) \end{aligned}$$

The zeros are 0,  $-1 \pm \sqrt{5}$ .

$$\begin{aligned} f \quad & z^4 + 3z^2 - 18 \\ &= (z^2 + 6)(z^2 - 3) \\ &= (z^2 + i\sqrt{6})(z - i\sqrt{6})(z + \sqrt{3})(z - \sqrt{3}) \end{aligned}$$

The real linear factors are  $(z + \sqrt{3})(z - \sqrt{3})$ , with corresponding zeros  $\pm\sqrt{3}$ .

$$\begin{aligned} P(\beta) &= a(\beta - \alpha)(\beta - \beta)(\beta - \gamma) \\ &= a(\beta - \alpha)(0)(\beta - \gamma) \\ &= 0 \end{aligned}$$

$\therefore \beta$  is a zero of  $P(x)$ .



- 8 a**  $P(\alpha) = 0$ ,  $P(\beta) = 0$ ,  $P(\gamma) = 0$
- b** If  $P(x)$  has a factor of  $(x - a)$ , then  $P(a) = 0$  which implies that  $P(x)$  has  $x$ -intercept  $a$  which is distinct from the  $x$ -intercepts of  $\alpha$ ,  $\beta$ , and  $\gamma$ .  
But  $P(x)$  has only three  $x$ -intercepts and since  $a \neq \alpha$ ,  $\beta$ , or  $\gamma$ ,  $a$  cannot be one of them.  
 $\therefore P(x)$  cannot have a factor of  $(x - a)$ .
- 9 a** The zeros 0 and  $-\frac{2}{3}$  come from the linear factors  $x$  and  $(3x + 2)$ .  
 $\therefore P(x) = ax(3x + 2)$ ,  $a \neq 0$ .
- b** The zeros  $2 \pm \sqrt{3}$  have  $\text{sum} = 2 + \sqrt{3} + 2 - \sqrt{3} = 4$   
and  $\text{product} = (2 + \sqrt{3})(2 - \sqrt{3}) = 1$   
 $\therefore$  they come from the quadratic factor  $x^2 - 4x + 1$ .  
 $\therefore P(x) = a(x^2 - 4x + 1)$ ,  $a \neq 0$ .
- c** The zeros  $-1$  and  $1 - \sqrt{2}$  come from the linear factors  $(x + 1)$  and  $(x - 1 + \sqrt{2})$ .  
 $\therefore P(x) = a(x + 1)(x - 1 + \sqrt{2})$ ,  $a \neq 0$ .
- d** The zeros  $3 \pm i$  have  $\text{sum} = 3 + i + 3 - i = 6$   
and  $\text{product} = (3 + i)(3 - i) = 10$   
 $\therefore$  they come from the quadratic factor  $x^2 - 6x + 10$ .  
 $\therefore P(x) = a(x^2 - 6x + 10)$ ,  $a \neq 0$ .
- e** The zeros  $1 \pm 3i$  have  $\text{sum} = 1 + 3i + 1 - 3i = 2$   
and  $\text{product} = (1 + 3i)(1 - 3i) = 10$   
 $\therefore$  they come from the quadratic factor  $x^2 - 2x + 10$ .  
 $\therefore P(x) = a(x^2 - 2x + 10)$ ,  $a \neq 0$ .
- f** The zeros  $-2 \pm 5i$  have  $\text{sum} = -2 + 5i - 2 - 5i = -4$   
and  $\text{product} = (-2 + 5i)(-2 - 5i) = 29$   
 $\therefore$  they come from the quadratic factor  $x^2 + 4x + 29$ .  
 $\therefore P(x) = a(x^2 + 4x + 29)$ ,  $a \neq 0$ .
- g** The zeros  $-3 \pm \sqrt{2}$  have  $\text{sum} = -3 + \sqrt{2} - 3 - \sqrt{2} = -6$   
and  $\text{product} = (-3 + \sqrt{2})(-3 - \sqrt{2}) = 7$   
 $\therefore$  they come from the quadratic factor  $x^2 + 6x + 7$ .  
 $\therefore P(x) = a(x^2 + 6x + 7)$ ,  $a \neq 0$ .
- h** The zeros  $\pm i\sqrt{2}$  have  $\text{sum} = 0$   
and  $\text{product} = 2$   
 $\therefore$  they come from the quadratic factor  $x^2 + 2$ .  
 $\therefore P(x) = a(x^2 + 2)$ ,  $a \neq 0$ .
- i** The zeros  $\sqrt{2} \pm i$  have  $\text{sum} = \sqrt{2} + i + \sqrt{2} - i = 2\sqrt{2}$   
and  $\text{product} = (\sqrt{2} + i)(\sqrt{2} - i) = 3$   
 $\therefore$  they come from the quadratic factor  $x^2 - 2\sqrt{2}x + 3$ .  
 $\therefore P(x) = a(x^2 - 2\sqrt{2}x + 3)$ ,  $a \neq 0$ .



- 10** The zeros  $p \pm qi$  have  $\text{sum} = p + qi + p - qi = 2p$   
 and  $\text{product} = (p + qi)(p - qi) = p^2 + q^2$   
 $\therefore$  they come from the quadratic factor  $x^2 - 2px + (p^2 + q^2)$ .  
 $\therefore P(x) = a(x^2 - 2px + (p^2 + q^2))$ ,  $a \neq 0$ .
- 11** **a** The zeros  $\pm 2$  have  $\text{sum} = 0$   
 and  $\text{product} = -4$   
 $\therefore$  they come from the quadratic factor  $x^2 - 4$ .  
 3 comes from the linear factor  $x - 3$ .  
 $\therefore P(x) = a(x^2 - 4)(x - 3)$ ,  $a \neq 0$ .
- b** The zeros  $\pm 4i$  have  $\text{sum} = 0$   
 and  $\text{product} = 16$   
 $\therefore$  they come from the quadratic factor  $x^2 + 16$ .  
 $-2$  comes from the linear factor  $x + 2$ .  
 $\therefore P(x) = a(x + 2)(x^2 + 16)$ ,  $a \neq 0$ .
- c** The zeros  $-1 \pm i$  have  $\text{sum} = -1 + i - 1 - i = -2$   
 and  $\text{product} = (-1 + i)(-1 - i) = 2$   
 $\therefore$  they come from the quadratic factor  $x^2 + 2x + 2$ .  
 3 comes from the linear factor  $x - 3$ .  
 $\therefore P(x) = a(x - 3)(x^2 + 2x + 2)$ ,  $a \neq 0$ .
- d** The zeros  $-3 \pm 2i$  have  $\text{sum} = -3 + 2i - 3 - 2i = -6$   
 and  $\text{product} = (-3 + 2i)(-3 - 2i) = 13$   
 $\therefore$  they come from the quadratic factor  $x^2 + 6x + 13$ .  
 $-5$  comes from the linear factor  $x + 5$ .  
 $\therefore P(x) = a(x + 5)(x^2 + 6x + 13)$ ,  $a \neq 0$ .
- e** The zeros  $3 \pm \sqrt{2}$  have  $\text{sum} = 3 + \sqrt{2} + 3 - \sqrt{2} = 6$   
 and  $\text{product} = (3 + \sqrt{2})(3 - \sqrt{2}) = 7$   
 $\therefore$  they come from the quadratic factor  $x^2 - 6x + 7$ .  
 $-1$  comes from the linear factor  $x + 1$ .  
 $\therefore P(x) = a(x + 1)(x^2 - 6x + 7)$ ,  $a \neq 0$ .
- f** The zeros  $-2 \pm i\sqrt{2}$  have  $\text{sum} = -2 + i\sqrt{2} - 2 - i\sqrt{2} = -4$   
 and  $\text{product} = (-2 + i\sqrt{2})(-2 - i\sqrt{2}) = 6$   
 $\therefore$  they come from the quadratic factor  $x^2 + 4x + 6$ .  
 $\sqrt{2}$  comes from the linear factor  $x - \sqrt{2}$ .  
 $\therefore P(x) = a(x - \sqrt{2})(x^2 + 4x + 6)$ ,  $a \neq 0$ .
- 12** **a** The zeros  $-2, 0, 1$ , and  $5$  come from the linear factors  $(x + 2)$ ,  $x$ ,  $(x - 1)$ , and  $(x - 5)$ .  
 $\therefore P(x) = ax(x + 2)(x - 1)(x - 5)$ ,  $a \neq 0$ .
- b** The zeros  $\pm 1$  have  $\text{sum} = 0$   
 and  $\text{product} = -1$   
 $\therefore$  they come from the quadratic factor  $x^2 - 1$ .  
 The zeros  $\pm\sqrt{2}$  have  $\text{sum} = 0$   
 and  $\text{product} = -2$   
 $\therefore$  they come from the quadratic factor  $x^2 - 2$ .  
 $\therefore P(x) = a(x^2 - 1)(x^2 - 2)$ ,  $a \neq 0$ .

- c** The zeros 2 and  $-1$  come from the linear factors  $(x - 2)$  and  $(x + 1)$ .

The zeros  $\pm i\sqrt{3}$  have  $\text{sum} = 0$   
and  $\text{product} = 3$

$\therefore$  they come from the quadratic factor  $x^2 + 3$ .

$\therefore P(x) = a(x - 2)(x + 1)(x^2 + 3), a \neq 0$ .

- d** The zeros  $2 \pm \sqrt{3}$  have  $\text{sum} = 2 + \sqrt{3} + 2 - \sqrt{3} = 4$   
and  $\text{product} = (2 + \sqrt{3})(2 - \sqrt{3}) = 1$

$\therefore$  they come from the quadratic factor  $x^2 - 4x + 1$ .

The zeros 4 and 0 come from the linear factors  $(x - 4)$  and  $x$ .

$\therefore P(x) = ax(x - 4)(x^2 - 4x + 1), a \neq 0$ .

- e** The zeros  $\pm\sqrt{3}$  have  $\text{sum} = 0$   
and  $\text{product} = -3$

$\therefore$  they come from the quadratic factor  $x^2 - 3$ .

The zeros  $1 \pm i$  have  $\text{sum} = 1 + i + 1 - i = 2$   
and  $\text{product} = (1 + i)(1 - i) = 2$

$\therefore$  they come from the quadratic factor  $x^2 - 2x + 2$ .

$\therefore P(x) = a(x^2 - 3)(x^2 - 2x + 2), a \neq 0$ .

- f** The zeros  $2 \pm \sqrt{5}$  have  $\text{sum} = 2 + \sqrt{5} + 2 - \sqrt{5} = 4$   
and  $\text{product} = (2 + \sqrt{5})(2 - \sqrt{5}) = -1$

$\therefore$  they come from the quadratic factor  $x^2 - 4x - 1$ .

The zeros  $-2 \pm 3i$  have  $\text{sum} = -2 + 3i - 2 - 3i = -4$   
and  $\text{product} = (-2 + 3i)(-2 - 3i) = 13$

$\therefore$  they come from the quadratic factor  $x^2 + 4x + 13$ .

$\therefore P(x) = a(x^2 - 4x - 1)(x^2 + 4x + 13), a \neq 0$ .

- 13 a** Suppose the quadratic function is  $f(x) = ax^2 + bx + c$ , where  $a, b, c \in \mathbb{Q}$ .

The sum of the zeros is  $-\frac{b}{a}$ , which is also rational, and the product of the zeros is  $\frac{c}{a}$ , which is also rational.

The quadratic has one zero  $p + q\sqrt{n}$ , where  $p, q, n \in \mathbb{Q}$ ,  $\sqrt{n} \notin \mathbb{Q}$ .

Let the other zero have the form  $r + s\sqrt{m}$ , where  $r, s, m \in \mathbb{Q}$ ,  $\sqrt{m} \notin \mathbb{Q}$ .

$\therefore$  the sum of the zeros  $p + q\sqrt{n} + r + s\sqrt{m}$  is rational

$\therefore q\sqrt{n} + s\sqrt{m}$  is rational

$\therefore q\sqrt{n} + s\sqrt{m} = 0 \quad \{\sqrt{n}, \sqrt{m} \notin \mathbb{Q}\}$

$\therefore s\sqrt{m} = -q\sqrt{n}$

Now the product of the zeros  $(p + q\sqrt{n})(r + s\sqrt{m})$  is also rational

$\therefore pr - pq\sqrt{n} + rq\sqrt{n} - q^2n$  is rational

$\therefore q\sqrt{n}(r - p)$  is rational

$\therefore r = p \quad \{\sqrt{n} \notin \mathbb{Q}\}$

$\therefore$  the other zero has the form  $p - q\sqrt{n}$ .

- b** If  $1 - \sqrt{2}$  is a zero of  $x^2 + ax + b$ ,  $a, b \in \mathbb{Q}$ , then the other zero is  $1 + \sqrt{2}$ . {from **a**}

The zeros  $1 \pm \sqrt{2}$  have  $\text{sum} = 1 + \sqrt{2} + 1 - \sqrt{2} = 2$   
and  $\text{product} = (1 + \sqrt{2})(1 - \sqrt{2}) = -1$

$\therefore$  they come from the quadratic factor  $x^2 - 2x - 1$ .

$\therefore a = -2, b = -1$

- c** If  $2 + \sqrt{5}$  is a zero of a quadratic with rational coefficients, the other zero is  $2 - \sqrt{5}$ . {from **a**}

The zeros  $2 \pm \sqrt{5}$  have  $\text{sum} = 2 + \sqrt{5} + 2 - \sqrt{5} = 4$   
and  $\text{product} = (2 + \sqrt{5})(2 - \sqrt{5}) = -1$

$\therefore$  they come from the quadratic factor  $x^2 - 4x - 1$ .

$\therefore P(x) = a(x^2 - 4x - 1), a \neq 0$ .

- 14 a** Let the real quadratic factor be  $P(x) = \alpha x^2 + \beta x + \gamma$ ,  $\alpha, \beta, \gamma \in \mathbb{R}$ .

Let  $z = p + qi$  be a zero of  $P(x)$ .

$$\therefore P(z) = 0$$

$$\therefore \alpha z^2 + \beta z + \gamma = 0$$

Taking the complex conjugate of both sides we obtain

$$(\alpha z^2 + \beta z + \gamma)^* = 0^*$$

$$\therefore (\alpha z^2)^* + (\beta z)^* + \gamma^* = 0 \quad \{(z_1 + z_2)^* = z_1^* + z_2^*\}$$

$$\therefore \alpha^* (z^2)^* + \beta^* z^* + \gamma^* = 0 \quad \{(z_1 z_2)^* = z_1^* z_2^*\}$$

$$\therefore \alpha (z^2)^* + \beta z^* + \gamma = 0 \quad \{\alpha, \beta, \gamma \in \mathbb{R}\}$$

$$\therefore \alpha (z^*)^2 + \beta z^* + \gamma = 0 \quad \{(z^n)^* = (z^*)^n\}$$

$$\therefore P(z^*) = 0$$

So  $z^* = p - qi$  is also a zero of  $P(x)$ .

- b** If  $3 + i$  is a zero of  $x^2 + ax + b$ ,  $a, b \in \mathbb{R}$ , then  $3 - i$  is also a zero. {from **a**}

The zeros  $3 \pm i$  have  $\text{sum} = 3 + i + 3 - i = 6$   
and  $\text{product} = (3 + i)(3 - i) = 10$

$\therefore$  they come from the quadratic factor  $x^2 - 6x + 10$ .

$\therefore a = -6, b = 10$

- c** If a real quadratic has zero  $\sqrt{2} + i$ , then the other zero is  $\sqrt{2} - i$ . {from **a**}

The zeros  $\sqrt{2} \pm i$  have  $\text{sum} = \sqrt{2} + i + \sqrt{2} - i = 2\sqrt{2}$   
and  $\text{product} = (\sqrt{2} + i)(\sqrt{2} - i) = 3$

$\therefore$  they come from the quadratic  $x^2 - 2\sqrt{2}x + 3$ .

$\therefore P(x) = a(x^2 - 2\sqrt{2}x + 3), a \neq 0$ .



**d** If  $a + ai$  is a zero of  $x^2 + 4x + b$ ,  $a, b \in \mathbb{R}$ , the other zero is  $a - ai$ .

$$\begin{aligned} \therefore (a + ai)^2 + 4(a + ai) + b &= 0 \quad \text{and} \quad (a - ai)^2 + 4(a - ai) + b = 0 \\ \therefore a^2 + 2a^2i - a^2 + 4a + 4ai + b &= 0 \quad \text{and} \quad a^2 - 2a^2i - a^2 + 4a - 4ai + b = 0 \\ \therefore 2a^2i + 4a + 4ai + b &= 0 \quad \text{and} \quad -2a^2i + 4a - 4ai + b = 0 \quad \dots (2) \\ \therefore b &= -2a^2i - 4a - 4ai \quad \dots (1) \end{aligned}$$

$$\begin{aligned} \text{Substituting (1) into (2) gives } -2a^2i + 4a - 4ai - 2a^2i - 4a - 4ai &= 0 \\ \therefore 4a^2i + 8ai &= 0 \\ \therefore 4ai(a + 2) &= 0 \\ \therefore a &= 0 \text{ or } -2 \end{aligned}$$

$$\begin{aligned} \text{If } a = 0, \quad b = 0, \quad \text{and if } a = -2, \quad b &= -2(-2)^2i - 4(-2) - 4(-2)i \\ &= -8i + 8 + 8i \\ &= 8 \end{aligned}$$

$$\therefore a = -2, \quad b = 8, \quad \text{or} \quad a = 0, \quad b = 0$$

## EXERCISE 5D

**1 a**  $2x^2 + 4x + 5 = ax^2 + (2b - 6)x + c$

Since this is true for all  $x$ , we equate coefficients:

$$\therefore \underbrace{a = 2}_{x^2 \text{ s}} \quad \underbrace{2b - 6 = 4}_{x \text{ s}} \quad \text{and} \quad \underbrace{c = 5}_{\text{constant terms}}$$

$$\begin{aligned} \therefore a = 2, \quad c = 5, \quad \text{and} \quad 2b &= 10 \\ \therefore b &= 5 \end{aligned}$$

So,  $a = 2$ ,  $b = 5$ ,  $c = 5$ .

**b** 
$$\begin{aligned} x^3 + 2x^2 - 3x + 4 &= (x - 2)(x^2 + ax + b) + c \\ &= x^3 + ax^2 + bx - 2x^2 - 2ax - 2b + c \\ &= x^3 + (a - 2)x^2 + (b - 2a)x - 2b + c \end{aligned}$$

Since this is true for all  $x$ , we equate coefficients:

$$\therefore \underbrace{a - 2 = 2}_{x^2 \text{ s}} \quad \underbrace{b - 2a = -3}_{x \text{ s}} \quad \text{and} \quad \underbrace{-2b + c = 4}_{\text{constant terms}}$$

$$\begin{aligned} \therefore a &= 4 \\ \therefore b - 2(4) &= -3 \quad \Rightarrow \quad b = 5 \\ \therefore -2(5) + c &= 4 \quad \Rightarrow \quad c = 14 \end{aligned}$$

So,  $a = 4$ ,  $b = 5$ ,  $c = 14$ .



$$\begin{aligned}
 \text{c } 2x^3 - x^2 + 6 &= (x-1)^2(2x+a) + bx + c \\
 &= (x^2 - 2x + 1)(2x+a) + bx + c \\
 &= 2x^3 + ax^2 - 4x^2 - 2ax + 2x + a + bx + c \\
 &= 2x^3 + (a-4)x^2 + (-2a+2+b)x + a+c
 \end{aligned}$$

Since this is true for all  $x$ , we equate coefficients:

$$\therefore \underbrace{a-4=-1}_{x^2 \text{ s}} \quad \underbrace{-2a+2+b=0}_{x \text{ s}} \quad \text{and} \quad \underbrace{a+c=6}_{\text{constant terms}}$$

$$\therefore a = 3$$

$$\begin{aligned}
 \text{Now } -2(3) + 2 + b &= 0 \Rightarrow b = 4 \\
 \text{and } 3 + c &= 6 \Rightarrow c = 3
 \end{aligned}$$

So,  $a = 3$ ,  $b = 4$ ,  $c = 3$ .

$$\begin{aligned}
 \text{2 a } z^4 + 4 &= (z^2 + az + 2)(z^2 + bz + 2) \\
 &= z^4 + bz^3 + 2z^2 \\
 &\quad + az^3 + abz^2 + 2az \\
 &\quad + 2z^2 + 2bz + 4 \\
 &= z^4 + (a+b)z^3 + (4+ab)z^2 + (2a+2b)z + 4 \quad \text{for all } z
 \end{aligned}$$

$$\text{Equating coefficients gives } \begin{cases} a+b=0 & \dots (1) & \{z^3 \text{ s}\} \\ 4+ab=0 & \dots (2) & \{z^2 \text{ s}\} \\ 2a+2b=0 & \dots (3) & \{z \text{ s}\} \end{cases}$$

From (1) and (3),  $b = -a$

$$\begin{aligned}
 \therefore \text{ in (2), } 4 + a(-a) &= 0 \quad \therefore a^2 = 4 \\
 \therefore a &= \pm 2 \quad \text{and so } b = \mp 2
 \end{aligned}$$

$$\therefore a = 2, b = -2 \quad \text{or} \quad a = -2, b = 2$$

$$\begin{aligned}
 \text{b } z^4 + z^3 + z^2 - 9z - 10 &= (z^2 + az + 5)(z^2 + bz - 2) \\
 &= z^4 + bz^3 - 2z^2 \\
 &\quad + az^3 + abz^2 - 2az \\
 &\quad + 5z^2 + 5bz - 10 \\
 &= z^4 + (a+b)z^3 + (3+ab)z^2 + (-2a+5b)z - 10 \quad \text{for all } z
 \end{aligned}$$

$$\text{Equating coefficients gives } \begin{cases} a+b=1 & \dots (1) & \{z^3 \text{ s}\} \\ 3+ab=1 & \dots (2) & \{z^2 \text{ s}\} \\ -2a+5b=-9 & \dots (3) & \{z \text{ s}\} \end{cases}$$

From (1),  $b = 1 - a$

$$\begin{aligned}
 \therefore \text{ in (2), } 3 + a(1-a) &= 1 \quad \therefore -a^2 + a + 2 = 0 \\
 \therefore -(a-2)(a+1) &= 0 \\
 \therefore a &= 2 \text{ or } -1 \quad \text{and so } b = -1 \text{ or } 2
 \end{aligned}$$

But only  $a = 2$ ,  $b = -1$  holds in (3)

$$\therefore a = 2, b = -1$$

$$\begin{aligned}
2z^4 + 5z^3 + 4z^2 + 7z + 6 &= (z^2 + az + 2)(2z^2 + bz + 3) \\
&= 2z^4 + bz^3 + 3z^2 \\
&\quad + 2az^3 + abz^2 + 3az \\
&\quad + 4z^2 + 2bz + 6 \\
&= 2z^4 + (b + 2a)z^3 + (7 + ab)z^2 + (3a + 2b)z + 6 \quad \text{for all } z
\end{aligned}$$

$$\text{Equating coefficients gives } \begin{cases} b + 2a = 5 & \dots (1) & \{z^3 \text{ s}\} \\ 7 + ab = 4 & \dots (2) & \{z^2 \text{ s}\} \\ 3a + 2b = 7 & \dots (3) & \{z \text{ s}\} \end{cases}$$

$$\text{From (1), } b = 5 - 2a$$

$$\therefore \text{ in (2), } 7 + a(5 - 2a) = 4 \quad \therefore 2a^2 - 5a - 3 = 0$$

$$\therefore (2a + 1)(a - 3) = 0$$

$$\therefore a = 3 \text{ or } -\frac{1}{2} \quad \text{and so } b = -1 \text{ or } 6$$

$$\text{But only } a = 3, b = -1 \text{ holds in (3)}$$

$$\therefore a = 3, b = -1$$

$$\begin{aligned}
3 \text{ Consider } z^4 + 64 &= (z^2 + az + 8)(z^2 + bz + 8) \\
&= z^4 + bz^3 + 8z^2 \\
&\quad + az^3 + abz^2 + 8az \\
&\quad + 8z^2 + 8bz + 64 \\
&= z^4 + (a + b)z^3 + (16 + ab)z^2 + (8a + 8b)z + 64 \quad \text{for all } z
\end{aligned}$$

$$\text{Equating coefficients gives } \begin{cases} a + b = 0 & \dots (1) \\ 16 + ab = 0 & \dots (2) \\ 8a + 8b = 0 & \dots (3) \end{cases}$$

$$\text{From (1) and (3), } b = -a$$

$$\therefore \text{ in (2), } 16 + a(-a) = 0 \quad \therefore a^2 = 16$$

$$\therefore a = \pm 4 \quad \text{and so } b = \mp 4$$

$$\therefore a = 4, b = -4 \text{ or } a = -4, b = 4$$

$$\therefore z^4 + 64 \text{ can be factorised into } (z^2 + 4z + 8)(z^2 - 4z + 8)$$

$$\begin{aligned}
\text{Now consider } z^4 + 64 &= (z^2 + az + 16)(z^2 + bz + 4) \\
&= z^4 + bz^3 + 4z^2 \\
&\quad + az^3 + abz^2 + 4az \\
&\quad + 16z^2 + 16bz + 64 \\
&= z^4 + (a + b)z^3 + (ab + 20)z^2 + (4a + 16b)z + 64 \quad \text{for all } z
\end{aligned}$$

$$\text{Equating coefficients gives } \begin{cases} a + b = 0 & \dots (1) \\ ab + 20 = 0 & \dots (2) \\ 4a + 16b = 0 & \dots (3) \end{cases}$$

$$\text{From (1), } b = -a$$

$$\therefore \text{ in (2), } a(-a) + 20 = 0 \quad \therefore a^2 = 20$$

$$\therefore a = \pm\sqrt{20} \quad \text{and so } b = \mp\sqrt{20}$$

However, neither of these solutions satisfy (3).

$\therefore z^4 + 64$  cannot be factorised in this way.

$$\begin{aligned}
4 \quad x^4 - 4x^2 + 8x - 4 &= (x^2 + ax + 2)(x^2 + bx - 2) \\
&= x^4 + bx^3 - 2x^2 \\
&\quad + ax^3 + abx^2 - 2ax \\
&\quad + 2x^2 + 2bx - 4 \\
&= x^4 + (a+b)x^3 + abx^2 + (2b-2a)x - 4 \quad \text{for all } x
\end{aligned}$$

$$\text{Equating coefficients gives } \begin{cases} a+b=0 & \dots (1) \\ ab=-4 & \dots (2) \\ 2a-2b=8 & \dots (3) \end{cases}$$

From (1),  $b = -a$

$$\therefore \text{ in (2), } a(-a) = -4$$

$$\therefore -a^2 = -4$$

$$\therefore a = \pm 2 \quad \text{and so } b = \mp 2$$

But only  $a = -2, b = 2$  holds in (3)

$$\therefore x^4 - 4x^2 + 8x - 4 = (x^2 - 2x + 2)(x^2 + 2x - 2)$$

$$\text{Now if } x^4 + 8x = 4x^2 + 4$$

$$\text{then } x^4 - 4x^2 + 8x - 4 = 0$$

$$\therefore (x^2 - 2x + 2)(x^2 + 2x - 2) = 0$$

$$\therefore x^2 - 2x + 2 = 0 \quad \text{or} \quad x^2 + 2x - 2 = 0$$

$$\therefore x = \frac{2 \pm \sqrt{4 - 4(1)(2)}}{2} \quad \text{or} \quad x = \frac{-2 \pm \sqrt{4 - 4(1)(-2)}}{2}$$

$$\therefore x = \frac{2 \pm 2i}{2} \quad \text{or} \quad x = \frac{-2 \pm 2\sqrt{3}}{2}$$

$$\therefore x = 1 \pm i \quad \text{or} \quad -1 \pm \sqrt{3}$$

$$\therefore x = -1 \pm \sqrt{3} \quad \text{are the only } x \in \mathbb{R} \text{ such that } x^4 + 8x = 4x^2 + 4.$$

$$\begin{aligned}
5 \quad x^4 - 10x^2 + 1 &= (x^2 + cx + 1)(x^2 + dx + 1) \\
&= x^4 + dx^3 + x^2 \\
&\quad + cx^3 + cdx^2 + cx \\
&\quad + x^2 + dx + 1 \\
&= x^4 + (c+d)x^3 + (cd+2)x^2 + (c+d)x + 1 \quad \text{for all } x
\end{aligned}$$

$$\text{Equating coefficients gives } \begin{cases} c+d=0 & \dots (1) \\ cd+2=-10 & \dots (2) \\ c+d=0 & \dots (3) \end{cases}$$

From (1) and (3),  $d = -c$

$$\therefore \text{ in (2), } c(-c) + 2 = -10$$

$$\therefore c^2 = 12$$

$$\therefore c = \pm\sqrt{12}, \quad d = \mp\sqrt{12}$$

$$\therefore c = \pm 2\sqrt{3}, \quad d = \mp 2\sqrt{3}$$

$$\therefore x^4 - 10x^2 + 1 = (x^2 + 2\sqrt{3}x + 1)(x^2 - 2\sqrt{3}x + 1)$$

Now if  $x^4 + 1 = 10x^2$

then  $x^4 - 10x^2 + 1 = 0$

$$\therefore (x^2 + 2\sqrt{3}x + 1)(x^2 - 2\sqrt{3}x + 1) = 0$$

$$\therefore x^2 + 2\sqrt{3}x + 1 = 0 \quad \text{or} \quad x^2 - 2\sqrt{3}x + 1 = 0$$

$$\therefore x = \frac{-2\sqrt{3} \pm \sqrt{12 - 4(1)(1)}}{2} \quad \text{or} \quad x = \frac{2\sqrt{3} \pm \sqrt{12 - 4(1)(1)}}{2}$$

$$\therefore x = \frac{-2\sqrt{3} \pm 2\sqrt{2}}{2} \quad \text{or} \quad x = \frac{2\sqrt{3} \pm 2\sqrt{2}}{2}$$

$$\therefore x = -\sqrt{3} \pm \sqrt{2} \quad \text{or} \quad \sqrt{3} \pm \sqrt{2}$$

**6 a** Since  $(2z - 3)$  is a factor,

$$\begin{aligned} 2z^3 + az^2 - 17z + 12 &= (2z - 3)(z^2 + bz - 4) \quad \text{for some constant } b \\ &= 2z^3 + 2bz^2 - 8z - 3z^2 - 3bz + 12 \\ &= 2z^3 + (2b - 3)z^2 - (8 + 3b)z + 12 \end{aligned}$$

Equating coefficients gives  $2b - 3 = a$  and  $8 + 3b = 17$

$$\therefore 3b = 9$$

$$\therefore b = 3$$

$$\therefore a = 3$$

$$\begin{aligned} \text{Now } z^2 + 3z - 4 \text{ has zeros } z &= \frac{-3 \pm \sqrt{9 - 4(1)(-4)}}{2} \\ &= \frac{-3 \pm 5}{2} \\ &= 1 \text{ or } -4 \end{aligned}$$

$\therefore a = 3$ , and the zeros of the cubic are  $-4$ ,  $1$ , and  $\frac{3}{2}$ .

**b** Since  $(3z + 2)$  is a factor,

$$\begin{aligned} 3z^3 - z^2 + (a - 4)z + a &= (3z + 2)(z^2 + bz + c) \quad \text{for some constants } b \text{ and } c \\ &= 3z^3 + 3bz^2 + 3cz + 2z^2 + 2bz + 2c \\ &= 3z^3 + (3b + 2)z^2 + (2b + 3c)z + 2c \end{aligned}$$

$$\text{Equating coefficients gives } \begin{cases} 3b + 2 = -1 & \dots (1) \\ 2b + 3c = a - 4 & \dots (2) \\ 2c = a & \dots (3) \end{cases}$$

From (1),  $b = -1$

Substituting  $b = -1$  and (3) into (2) gives:

$$2(-1) + 3c = 2c - 4$$

$$\therefore -2 + 3c = 2c - 4$$

$$\therefore c = -2 \text{ and so } a = -4$$

$$\begin{aligned} \therefore 3z^3 - z^2 + (a - 4)z + a &= 3z^3 - z^2 - 8z - 4 \\ &= (3z + 2)(z^2 - z - 2) \\ &= (3z + 2)(z + 1)(z - 2) \end{aligned}$$

$\therefore a = -4$ , and the zeros of the cubic are  $-\frac{2}{3}$ ,  $-1$ , and  $2$ .



- 7 a** Since  $(2x + 1)$  and  $(x - 2)$  are factors,

$$\begin{aligned}
 P(x) &= 2x^4 + ax^3 + bx^2 + 18x + 8 \\
 &= (2x + 1)(x - 2)(x^2 + cx - 4) \quad \text{for some constant } c \\
 &= (2x^2 - 3x - 2)(x^2 + cx - 4) \\
 &= 2x^4 + 2cx^3 - 8x^2 \\
 &\quad - 3x^3 - 3cx^2 + 12x \\
 &\quad - 2x^2 - 2cx + 8 \\
 &= 2x^4 + (2c - 3)x^3 + (-10 - 3c)x^2 + (12 - 2c)x + 8
 \end{aligned}$$

Equating coefficients gives  $2c - 3 = a$ ,  $-10 - 3c = b$ , and  $12 - 2c = 18$

$$\therefore -2c = 6$$

$$\therefore c = -3$$

$$\therefore a = -9 \text{ and } b = -1$$

$$\begin{aligned}
 \therefore P(x) &= 2x^4 - 9x^3 - x^2 + 12x + 8 \\
 &= (2x + 1)(x - 2)(x^2 - 3x - 4) \\
 &= (2x + 1)(x - 2)(x + 1)(x - 4)
 \end{aligned}$$

$\therefore P(x)$  has zeros  $-1$ ,  $-\frac{1}{2}$ ,  $2$ , and  $4$ .

- b** Since  $(x + 3)$  and  $(2x - 1)$  are factors of the quartic,

$$\begin{aligned}
 2x^4 + ax^3 + bx^2 + ax + 3 &= (x + 3)(2x - 1)(x^2 + cx - 1) \quad \text{for some constant } c \\
 &= (2x^2 + 5x - 3)(x^2 + cx - 1) \\
 &= 2x^4 + 2cx^3 - 2x^2 \\
 &\quad + 5x^3 + 5cx^2 - 5x \\
 &\quad - 3x^2 - 3cx + 3 \\
 &= 2x^4 + (2c + 5)x^3 + (5c - 5)x^2 + (-5 - 3c)x + 3
 \end{aligned}$$

$$\text{Equating coefficients gives } \begin{cases} 2c + 5 = a & \dots (1) \\ 5c - 5 = b & \dots (2) \\ -5 - 3c = a & \dots (3) \end{cases}$$

Equating (1) and (3) gives  $2c + 5 = -5 - 3c$

$$\therefore c = -2 \text{ and so } a = 1 \text{ and } b = -15$$

$$\begin{aligned}
 \therefore 2x^4 + ax^3 + bx^2 + ax + 3 &= 2x^4 + x^3 - 15x^2 + x + 3 \\
 &= (x + 3)(2x - 1)(x^2 - 2x - 1)
 \end{aligned}$$

Now  $x^2 - 2x - 1$  has zeros  $\frac{2 \pm \sqrt{4 + 4}}{2} = 1 \pm \sqrt{2}$

$\therefore$  the zeros of the quartic are  $-3$ ,  $\frac{1}{2}$ , and  $1 \pm \sqrt{2}$ .

- 8 If  $4x^4 - 4x^3 + kx^2 - 6x + 9$  is a perfect square, then

$$\begin{aligned}
 4x^4 - 4x^3 + kx^2 - 6x + 9 &= (ax^2 + bx + c)^2 \quad \text{for some constants } a, b, \text{ and } c \\
 &= (ax^2 + bx + c)(ax^2 + bx + c) \\
 &= a^2x^4 + abx^3 + acx^2 \\
 &\quad + abx^3 + b^2x^2 + bcx \\
 &\quad + acx^2 + bcx + c^2 \\
 &= a^2x^4 + 2abx^3 + (2ac + b^2)x^2 + 2bcx + c^2
 \end{aligned}$$

Equating coefficients gives  $a^2 = 4$ ,  $2ab = -4$ ,  $2ac + b^2 = k$ ,  $2bc = -6$ ,  $c^2 = 9$

$$\therefore a = \pm 2, \quad c = \pm 3 \Rightarrow b = \mp 1$$

$$\begin{aligned}
 \therefore k &= 2ac + b^2 \\
 &= 2(2)(3) + (-1)^2 \\
 &= 13
 \end{aligned}$$

9  $x^4 - 2x^3 - 3x^2 + 4x + 3 = a(x^2 - x)^2 + b(x^2 - x) + c$

$$\begin{aligned}
 &= a(x^4 - 2x^3 + x^2) + bx^2 - bx + c \\
 &= ax^4 - 2ax^3 + ax^2 + bx^2 - bx + c \\
 &= ax^4 - 2ax^3 + (a + b)x^2 - bx + c
 \end{aligned}$$

Equating coefficients gives  $a = 1$ ,  $-2a = -2$ ,  $a + b = -3$ ,  $-b = 4$ ,  $c = 3$

$$\therefore a = 1, \quad b = -4, \quad c = 3$$

$$\therefore x^4 - 2x^3 - 3x^2 + 4x + 3 = (x^2 - x)^2 - 4(x^2 - x) + 3$$

Let  $X = x^2 - x$

$$\therefore X^2 - 4X + 3 = 0$$

$$\therefore (X - 1)(X - 3) = 0$$

$$\therefore X = 1 \text{ or } 3$$

$$\therefore x^2 - x = 1$$

$$\text{or } x^2 - x = 3$$

$$\therefore x^2 - x - 1 = 0$$

$$\therefore x^2 - x - 3 = 0$$

$$\begin{aligned}
 \therefore x &= \frac{1 \pm \sqrt{1 - 4(1)(-1)}}{2} \\
 &= \frac{1 \pm \sqrt{5}}{2}
 \end{aligned}$$

$$\begin{aligned}
 \therefore x &= \frac{1 \pm \sqrt{1 - 4(1)(-3)}}{2} \\
 &= \frac{1 \pm \sqrt{13}}{2}
 \end{aligned}$$

$\therefore$  the zeros of the polynomial are  $\frac{1 \pm \sqrt{5}}{2}$  and  $\frac{1 \pm \sqrt{13}}{2}$ .

- 10 a Let  $P(x) = x^3 + x^2 - 16x + k$ ,  $k \in \mathbb{R}$ .

If  $P(x)$  has two identical linear factors, then

$$\begin{aligned}
 x^3 + x^2 - 16x + k &= (x - a)^2(x - b) \quad \text{for some constants } a \text{ and } b \\
 &= (x^2 - 2ax + a^2)(x - b) \\
 &= x^3 - bx^2 - 2ax^2 + 2abx + a^2x - a^2b \\
 &= x^3 + (-2a - b)x^2 + (2ab + a^2)x - a^2b
 \end{aligned}$$

$$\text{Equating coefficients gives } \begin{cases} -2a - b = 1 & \dots (1) \\ 2ab + a^2 = -16 & \dots (2) \\ -a^2b = k & \dots (3) \end{cases}$$

From (1),  $b = -2a - 1$ , and substituting this into (2) gives

$$\begin{aligned} 2a(-2a - 1) + a^2 &= -16 \\ \therefore -4a^2 - 2a + a^2 &= -16 \\ \therefore -3a^2 - 2a + 16 &= 0 \\ \therefore -(3a + 8)(a - 2) &= 0 \\ \therefore a &= -\frac{8}{3} \text{ or } 2 \quad \text{and so} \quad b = \frac{13}{3} \text{ or } -5 \end{aligned}$$

$$\begin{aligned} \text{Now, in (3), } k &= -\left(-\frac{8}{3}\right)^2\left(\frac{13}{3}\right) \quad \text{or} \quad -(2)^2(-5) \\ &= -\frac{832}{27} \text{ or } 20 \end{aligned}$$

$$\text{If } k = 20, \quad P(x) = (x - 2)^2(x + 5)$$

$$\begin{aligned} \text{If } k = -\frac{832}{27}, \quad P(x) &= \left(x + \frac{8}{3}\right)^2\left(x - \frac{13}{3}\right) \\ &= \frac{1}{27}(3x + 8)^2(3x - 13) \end{aligned}$$

**b** Let  $P(x) = x^3 + 3x^2 - 9x + c$ ,  $c \in \mathbb{R}$ .

If  $P(x)$  has two identical linear factors, then

$$\begin{aligned} x^3 + 3x^2 - 9x + c &= (x - a)^2(x - b) \quad \text{for some constants } a \text{ and } b \\ &= (x^2 - 2ax + a^2)(x - b) \\ &= x^3 - bx^2 - 2ax^2 + 2abx + a^2x - a^2b \\ &= x^3 + (-2a - b)x^2 + (2ab + a^2)x - a^2b \end{aligned}$$

$$\text{Equating coefficients gives} \quad \begin{cases} -2a - b = 3 & \dots (1) \\ 2ab + a^2 = -9 & \dots (2) \\ -a^2b = c & \dots (3) \end{cases}$$

From (1),  $b = -2a - 3$ , and substituting this into (2) gives

$$\begin{aligned} 2a(-2a - 3) + a^2 &= -9 \\ \therefore -4a^2 - 6a + a^2 &= -9 \\ \therefore 3a^2 + 6a - 9 &= 0 \\ \therefore 3(a^2 + 2a - 3) &= 0 \\ \therefore 3(a - 1)(a + 3) &= 0 \\ \therefore a &= 1 \text{ or } -3 \quad \text{and so} \quad b = -5 \text{ or } 3 \end{aligned}$$

$$\begin{aligned} \text{Now, in (3), } c &= -(1)^2(-5) \quad \text{or} \quad -(-3)^2(3) \\ &= 5 \text{ or } -27 \end{aligned}$$

$$\text{If } c = 5, \quad P(x) = (x - 1)^2(x + 5)$$

$$\text{If } c = -27, \quad P(x) = (x + 3)^2(x - 3)$$

**c** Let  $P(x) = 3x^3 + 4x^2 - x + m$ ,  $m \in \mathbb{R}$ .

If  $P(x)$  has two identical linear factors, then

$$\begin{aligned} 3x^3 + 4x^2 - x + m &= (x - a)^2(3x + b) \quad \text{for some constants } a \text{ and } b \\ &= (x^2 - 2ax + a^2)(3x + b) \\ &= 3x^3 + bx^2 - 6ax^2 - 2abx + 3a^2x + a^2b \\ &= 3x^3 + (b - 6a)x^2 + (3a^2 - 2ab)x + a^2b \end{aligned}$$

$$\text{Equating coefficients gives} \quad \begin{cases} b - 6a = 4 & \dots (1) \\ 3a^2 - 2ab = -1 & \dots (2) \\ a^2b = m & \dots (3) \end{cases}$$

From (1),  $b = 4 + 6a$ , and substituting this into (2) gives

$$\begin{aligned} 3a^2 - 2a(4 + 6a) &= -1 \\ \therefore 3a^2 - 8a - 12a^2 &= -1 \\ \therefore 9a^2 + 8a - 1 &= 0 \\ \therefore (9a - 1)(a + 1) &= 0 \\ \therefore a &= -1 \text{ or } \frac{1}{9} \end{aligned}$$

When  $a = -1$ ,  $b = -2$ , and when  $a = \frac{1}{9}$ ,  $b = \frac{14}{3}$ .

Now, from (3), when  $a = -1$  and  $b = -2$ ,  $m = -2$ ,

$$\text{and when } a = \frac{1}{9} \text{ and } b = \frac{14}{3}, m = \frac{14}{243}.$$

If  $m = -2$ ,  $P(x) = (x + 1)^2(3x - 2)$  which has zeros  $-1$  (repeated) and  $\frac{2}{3}$ .

If  $m = \frac{14}{243}$ ,  $P(x) = (x - \frac{1}{9})^2(3x + \frac{14}{3})$  which has zeros  $\frac{1}{9}$  (repeated) and  $-\frac{14}{9}$ .

## EXERCISE 5E.1

1 a

$$\begin{array}{r} x \\ x+2 \overline{) x^2 + 2x - 3} \\ \underline{-(x^2 + 2x)} \phantom{-3} \\ -3 \end{array}$$

$$\therefore Q(x) = x, \quad R = -3$$

$$\therefore x^2 + 2x - 3 = x(x + 2) - 3$$

b

$$\begin{array}{r} x-4 \\ x-1 \overline{) x^2 - 5x + 1} \\ \underline{-(x^2 - x)} \phantom{+1} \\ -4x + 1 \\ \underline{-(-4x + 4)} \\ -3 \end{array}$$

$$\therefore Q(x) = x - 4, \quad R = -3$$

$$\therefore x^2 - 5x + 1 = (x - 4)(x - 1) - 3$$

c

$$\begin{array}{r} 2x^2 + 10x + 16 \\ x-2 \overline{) 2x^3 + 6x^2 - 4x + 3} \\ \underline{-(2x^3 - 4x^2)} \phantom{+3} \\ 10x^2 - 4x \phantom{+3} \\ \underline{-(10x^2 - 20x)} \phantom{+3} \\ 16x + 3 \\ \underline{-(16x - 32)} \\ 35 \end{array}$$

$$\therefore Q(x) = 2x^2 + 10x + 16, \quad R = 35$$

$$\therefore 2x^3 + 6x^2 - 4x + 3 = (2x^2 + 10x + 16)(x - 2) + 35$$

2 a

$$\begin{array}{r} x+1 \\ x-4 \overline{) x^2 - 3x + 6} \\ \underline{-(x^2 - 4x)} \phantom{+6} \\ x + 6 \\ \underline{-(x - 4)} \\ 10 \end{array}$$

$$\therefore x^2 - 3x + 6 = (x + 1)(x - 4) + 10$$

b

$$\begin{array}{r} x+1 \\ x+3 \overline{) x^2 + 4x - 11} \\ \underline{-(x^2 + 3x)} \phantom{-11} \\ x - 11 \\ \underline{-(x + 3)} \\ -14 \end{array}$$

$$\therefore x^2 + 4x - 11 = (x + 1)(x + 3) - 14$$



$$\begin{array}{r}
 \text{c} \qquad \qquad \qquad 2x - 3 \\
 x - 2 \overline{) 2x^2 - 7x + 2} \\
 \underline{-(2x^2 - 4x)} \quad \downarrow \\
 -3x + 2 \\
 \underline{-(-3x + 6)} \\
 -4
 \end{array}$$

$$\therefore 2x^2 - 7x + 2 = (2x - 3)(x - 2) - 4$$

$$\begin{array}{r}
 \text{e} \qquad \qquad \qquad x^2 + 4x + 4 \\
 3x - 1 \overline{) 3x^3 + 11x^2 + 8x + 7} \\
 \underline{-(3x^3 - x^2)} \quad \downarrow \quad \downarrow \\
 12x^2 + 8x \\
 \underline{-(12x^2 - 4x)} \quad \downarrow \\
 12x + 7 \\
 \underline{-(12x - 4)} \\
 11
 \end{array}$$

$$\begin{aligned}
 \therefore 3x^3 + 11x^2 + 8x + 7 \\
 = (x^2 + 4x + 4)(3x - 1) + 11
 \end{aligned}$$

$$\begin{array}{r}
 \text{3 a} \qquad \qquad \qquad x + 2 \\
 x - 2 \overline{) x^2 + 0x + 5} \\
 \underline{-(x^2 - 2x)} \quad \downarrow \\
 2x + 5 \\
 \underline{-(2x - 4)} \\
 9
 \end{array}$$

$$\therefore \frac{x^2 + 5}{x - 2} = x + 2 + \frac{9}{x - 2}$$

$$\begin{array}{r}
 \text{c} \qquad \qquad \qquad 3x - 4 \\
 x + 2 \overline{) 3x^2 + 2x - 5} \\
 \underline{-(3x^2 + 6x)} \quad \downarrow \\
 -4x - 5 \\
 \underline{-(-4x - 8)} \\
 3
 \end{array}$$

$$\therefore \frac{3x^2 + 2x - 5}{x + 2} = 3x - 4 + \frac{3}{x + 2}$$

$$\begin{array}{r}
 \text{d} \qquad \qquad \qquad x^2 + x - 2 \\
 2x + 1 \overline{) 2x^3 + 3x^2 - 3x - 2} \\
 \underline{-(2x^3 + x^2)} \quad \downarrow \quad \downarrow \\
 2x^2 - 3x \\
 \underline{-(2x^2 + x)} \quad \downarrow \\
 -4x - 2 \\
 \underline{-(-4x - 2)} \\
 0
 \end{array}$$

$$\begin{aligned}
 \therefore 2x^3 + 3x^2 - 3x - 2 \\
 = (x^2 + x - 2)(2x + 1)
 \end{aligned}$$

$$\begin{array}{r}
 \text{f} \qquad \qquad \qquad x^3 - 2x^2 + \frac{5}{2}x - \frac{1}{4} \\
 2x + 3 \overline{) 2x^4 - x^3 - x^2 + 7x + 4} \\
 \underline{-(2x^4 + 3x^3)} \quad \downarrow \quad \downarrow \quad \downarrow \\
 -4x^3 - x^2 \\
 \underline{-(-4x^3 - 6x^2)} \quad \downarrow \quad \downarrow \\
 5x^2 + 7x \\
 \underline{-(5x^2 + \frac{15}{2}x)} \quad \downarrow \\
 -\frac{1}{2}x + 4 \\
 \underline{-(-\frac{1}{2}x - \frac{3}{4})} \\
 \frac{19}{4}
 \end{array}$$

$$\begin{aligned}
 \therefore 2x^4 - x^3 - x^2 + 7x + 4 \\
 = (x^3 - 2x^2 + \frac{5}{2}x - \frac{1}{4})(2x + 3) + \frac{19}{4}
 \end{aligned}$$

$$\begin{array}{r}
 \text{b} \qquad \qquad \qquad 2x + 1 \\
 x + 1 \overline{) 2x^2 + 3x + 0} \\
 \underline{-(2x^2 + 2x)} \quad \downarrow \\
 x + 0 \\
 \underline{-(x + 1)} \\
 -1
 \end{array}$$

$$\therefore \frac{2x^2 + 3x}{x + 1} = 2x + 1 - \frac{1}{x + 1}$$

$$\begin{array}{r}
 \text{d} \qquad \qquad \qquad x^2 + 3x - 2 \\
 x - 1 \overline{) x^3 + 2x^2 - 5x + 2} \\
 \underline{-(x^3 - x^2)} \quad \downarrow \quad \downarrow \\
 3x^2 - 5x \\
 \underline{-(3x^2 - 3x)} \quad \downarrow \\
 -2x + 2 \\
 \underline{-(-2x + 2)} \\
 0
 \end{array}$$

$$\therefore \frac{x^3 + 2x^2 - 5x + 2}{x - 1} = x^2 + 3x - 2$$

$$\begin{array}{r}
 \text{e} \qquad \qquad \qquad 2x^2 - 8x + 31 \\
 x + 4 \overline{) \begin{array}{r} 2x^3 + 0x^2 - x + 0 \\ -(2x^3 + 8x^2) \phantom{-x + 0} \\ \hline -8x^2 - x \phantom{+ 0} \\ -(-8x^2 - 32x) \phantom{+ 0} \\ \hline 31x + 0 \\ -(31x + 124) \\ \hline -124 \end{array}} \\
 \hline
 \therefore \frac{2x^3 - x}{x + 4} = 2x^2 - 8x + 31 - \frac{124}{x + 4}
 \end{array}$$

$$\begin{array}{r}
 \text{f} \qquad \qquad \qquad x^2 + 3x + 6 \\
 x - 2 \overline{) \begin{array}{r} x^3 + x^2 + 0x - 5 \\ -(x^3 - 2x^2) \phantom{- 5} \\ \hline 3x^2 + 0x \phantom{- 5} \\ -(3x^2 - 6x) \phantom{- 5} \\ \hline 6x - 5 \\ -(6x - 12) \\ \hline 7 \end{array}} \\
 \hline
 \therefore \frac{x^3 + x^2 - 5}{x - 2} = x^2 + 3x + 6 + \frac{7}{x - 2}
 \end{array}$$

$$\begin{array}{r}
 \text{4 a} \qquad \qquad \qquad 3x^2 - 8x + 4 \\
 x + 1 \overline{) \begin{array}{r} 3x^3 - 5x^2 - 4x + 4 \\ -(3x^3 + 3x^2) \phantom{- 4x + 4} \\ \hline -8x^2 - 4x \phantom{+ 4} \\ -(-8x^2 - 8x) \phantom{+ 4} \\ \hline 4x + 4 \\ -(4x + 4) \\ \hline 0 \end{array}} \\
 \hline
 \therefore \frac{3x^3 - 5x^2 - 4x + 4}{x + 1} = 3x^2 - 8x + 4 \quad \{\text{the remainder is 0}\} \\
 \therefore 3x^3 - 5x^2 - 4x + 4 = (x + 1)(3x^2 - 8x + 4) \\
 \qquad \qquad \qquad = (x + 1)(3x - 2)(x - 2)
 \end{array}$$

$$\begin{array}{r}
 \text{b} \qquad \qquad \qquad 2x^2 + 11x + 12 \\
 x - 2 \overline{) \begin{array}{r} 2x^3 + 7x^2 - 10x - 24 \\ -(2x^3 - 4x^2) \phantom{- 10x - 24} \\ \hline 11x^2 - 10x \phantom{- 24} \\ -(11x^2 - 22x) \phantom{- 24} \\ \hline 12x - 24 \\ -(12x - 24) \\ \hline 0 \end{array}} \\
 \hline
 \therefore \frac{2x^3 + 7x^2 - 10x - 24}{x - 2} = 2x^2 + 11x + 12 \quad \{\text{the remainder is 0}\} \\
 \therefore 2x^3 + 7x^2 - 10x - 24 = (x - 2)(2x^2 + 11x + 12) \\
 \qquad \qquad \qquad = (x - 2)(2x + 3)(x + 4)
 \end{array}$$

$$\begin{array}{r}
 \text{c} \quad \begin{array}{r} 4x^2 + x - 3 \\ x + 5 \overline{) 4x^3 + 21x^2 + 2x - 15} \\ \underline{-(4x^3 + 20x^2)} \phantom{-15} \downarrow \\ x^2 + 2x \phantom{-15} \downarrow \\ \underline{-(x^2 + 5x)} \phantom{-15} \downarrow \\ -3x - 15 \phantom{-15} \downarrow \\ \underline{-(-3x - 15)} \\ 0 \end{array} \\
 \therefore \frac{4x^3 + 21x^2 + 2x - 15}{x + 5} = 4x^2 + x - 3 \quad \{\text{the remainder is } 0\} \\
 \therefore 4x^3 + 21x^2 + 2x - 15 = (x + 5)(4x^2 + x - 3) \\
 = (x + 5)(4x - 3)(x + 1)
 \end{array}$$

$$\begin{array}{r}
 \text{5 a} \quad \begin{array}{r} x^2 - ax + a^2 \\ x + a \overline{) x^3 + 0x^2 + 0x + a^3} \\ \underline{-(x^3 + ax^2)} \phantom{+ 0x + a^3} \downarrow \\ -ax^2 + 0x \phantom{+ a^3} \downarrow \\ \underline{-(-ax^2 - a^2x)} \phantom{+ a^3} \downarrow \\ a^2x + a^3 \phantom{+ a^3} \downarrow \\ \underline{-(a^2x + a^3)} \\ 0 \end{array}
 \end{array}$$

$$\therefore \frac{x^3 + a^3}{x + a} = x^2 - ax + a^2$$

$$\begin{aligned}
 \text{Check: } (x^2 - ax + a^2)(x + a) &= x^3 + ax^2 \\
 &\quad - ax^2 - a^2x \\
 &\quad + a^2x + a^3 \\
 &= x^3 + a^3 \quad \checkmark
 \end{aligned}$$

$$\begin{aligned}
 \text{b i} \quad x^3 + 1 &= x^3 + 1^3 \\
 &= (x^2 - x + 1)(x + 1)
 \end{aligned}$$

$$\begin{aligned}
 \text{ii} \quad x^3 + 8 &= x^3 + 2^3 \\
 &= (x^2 - 2x + 4)(x + 2)
 \end{aligned}$$

$$\begin{aligned}
 \text{iii} \quad x^3 + 125 &= x^3 + 5^3 \\
 &= (x^2 - 5x + 25)(x + 5)
 \end{aligned}$$

$$\begin{array}{r}
 \text{6 a i} \quad \begin{array}{r} x + a \\ x - a \overline{) x^2 + 0x + a^2} \\ \underline{-(x^2 - ax)} \phantom{+ a^2} \downarrow \\ ax - a^2 \phantom{+ a^2} \downarrow \\ \underline{-(ax - a^2)} \\ 0 \end{array}
 \end{array}$$

$$\therefore \frac{x^2 - a^2}{x - a} = x + a$$

$$\begin{array}{r}
 \text{ii} \quad \begin{array}{r} x^2 + ax + a^2 \\ x - a \overline{) x^3 + 0x^2 + 0x - a^3} \\ \underline{-(x^3 - ax^2)} \phantom{+ 0x - a^3} \downarrow \\ ax^2 + 0x \phantom{- a^3} \downarrow \\ \underline{-(ax^2 - a^2x)} \phantom{- a^3} \downarrow \\ a^2x - a^3 \phantom{- a^3} \downarrow \\ \underline{-(a^2x - a^3)} \\ 0 \end{array}
 \end{array}$$

$$\therefore \frac{x^3 - a^3}{x - a} = x^2 + ax + a^2$$

iii

$$\begin{array}{r}
 x^3 + ax^2 + a^2x + a^3 \\
 x - a \overline{) \begin{array}{l} x^4 + 0x^3 + 0x^2 + 0x - a^4 \\ -(x^4 - ax^3) \\ \hline ax^3 + 0x^2 \\ -(ax^3 - a^2x^2) \\ \hline a^2x^2 + 0x \\ -(a^2x^2 - a^3x) \\ \hline a^3x - a^4 \\ -(a^3x - a^4) \\ \hline 0 \end{array} }
 \end{array}$$

$$\therefore \frac{x^4 - a^4}{x - a} = x^3 + ax^2 + a^2x + a^3$$

iv

$$\begin{array}{r}
 x^4 + ax^3 + a^2x^2 + a^3x + a^4 \\
 x - a \overline{) \begin{array}{l} x^5 + 0x^4 + 0x^3 + 0x^2 + 0x - a^5 \\ -(x^5 - ax^4) \\ \hline ax^4 + 0x^3 \\ -(ax^4 - a^2x^3) \\ \hline a^2x^3 + 0x^2 \\ -(a^2x^3 - a^3x^2) \\ \hline a^3x^2 + 0x \\ -(a^3x^2 - a^4x) \\ \hline a^4x - a^5 \\ -(a^4x - a^5) \\ \hline 0 \end{array} }
 \end{array}$$

$$\therefore \frac{x^5 - a^5}{x - a} = x^4 + ax^3 + a^2x^2 + a^3x + a^4$$

**b**  $x^n - a^n = (x - a)(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \dots + a^{n-2}x + a^{n-1})$

**c i**  $x^3 - 8 = x^3 - 2^3 = (x - 2)(x^2 + 2x + 4)$

**ii**  $x^4 - 1 = x^4 - 1^4 = (x - 1)(x^3 + x^2 + x + 1)$

**iii**  $x^4 - 16 = x^4 - 2^4 = (x - 2)(x^3 + 2x^2 + 4x + 8)$

**iv**  $x^5 - 243 = x^5 - 3^5 = (x - 3)(x^4 + 3x^3 + 9x^2 + 27x + 81)$



## ACTIVITY 2

## SYNTHETIC DIVISION

$$1 \quad a \quad \begin{array}{r|rrr} 1 & 3 & -2 & -3 \\ & 0 & 3 & 1 \\ \hline & 3 & 1 & -2 \end{array}$$

$$\begin{aligned} \therefore \frac{3x^2 - 2x - 3}{x - 1} &= 3x + 1 + \frac{-2}{x - 1} \\ &= 3x + 1 - \frac{2}{x - 1} \end{aligned}$$

$$c \quad \begin{array}{r|rrr} -1 & 3 & -1 & 2 \\ & 0 & -3 & 4 \\ \hline & 3 & -4 & 6 \end{array}$$

$$\therefore \frac{3z^2 - z + 2}{z + 1} = 3z - 4 + \frac{6}{z + 1}$$

$$e \quad \begin{array}{r|rrrrr} 3 & 1 & -2 & 1 & 0 & -4 \\ & 0 & 3 & 3 & 12 & 36 \\ \hline & 1 & 1 & 4 & 12 & 32 \end{array}$$

$$\begin{aligned} \therefore \frac{z^4 - 2z^3 + z^2 - 4}{z - 3} \\ = z^3 + z^2 + 4z + 12 + \frac{32}{z - 3} \end{aligned}$$

$$b \quad \begin{array}{r|rrrr} -3 & 1 & 5 & 6 & 5 \\ & 0 & -3 & -6 & 0 \\ \hline & 1 & 2 & 0 & 5 \end{array}$$

$$\therefore \frac{x^3 + 5x^2 + 6x + 5}{x + 3} = x^2 + 2x + \frac{5}{x + 3}$$

$$d \quad \begin{array}{r|rrrr} -3 & 1 & 0 & 0 & 27 \\ & 0 & -3 & 9 & -27 \\ \hline & 1 & -3 & 9 & 0 \end{array}$$

$$\therefore \frac{x^3 + 27}{x + 3} = x^2 - 3x + 9$$

$$f \quad \begin{array}{r|rrrrr} -1 & 1 & 0 & 1 & -1 & 0 \\ & 0 & -1 & 1 & -2 & 3 \\ \hline & 1 & -1 & 2 & -3 & 3 \end{array}$$

$$\begin{aligned} \therefore \frac{z^4 + z^2 - z}{z + 1} \\ = z^3 - z^2 + 2z - 3 + \frac{3}{z + 1} \end{aligned}$$

$$2 \quad a \quad \begin{array}{r|rrrr} 3 & 1 & -1 & -3 & -5 \\ & 0 & 3 & 6 & 9 \\ \hline & 1 & 2 & 3 & 4 \end{array}$$

The quotient is  $x^2 + 2x + 3$ , and the remainder is 4.

$$\begin{aligned} \therefore \frac{x^3 - x^2 - 3x - 5}{x - 3} &= x^2 + 2x + 3 + \frac{4}{x - 3} \\ \therefore x^3 - x^2 - 3x - 5 &= \underbrace{(x^2 + 2x + 3)}_{Q(x)} \times (x - 3) + \underbrace{4}_R \end{aligned}$$

$$b \quad \frac{Q(x)}{x + 1} = \frac{x^2 + 2x + 3}{x + 1} \quad \begin{array}{r|rrr} -1 & 1 & 2 & 3 \\ & 0 & -1 & -1 \\ \hline & 1 & 1 & 2 \end{array}$$

The quotient is  $x + 1$ , and the remainder is 2.

$$\begin{aligned} \therefore \frac{x^2 + 2x + 3}{x + 1} &= x + 1 + \frac{2}{x + 1} \\ \therefore x^2 + 2x + 3 &= (x + 1)^2 + 2 \\ \therefore x^3 - x^2 - 3x - 5 &= (x - 3)[(x + 1)^2 + 2] + 4 \\ &= (x - 3)(x + 1)^2 + 2x - 6 + 4 \\ &= \underbrace{(x + 1)(x + 1)(x - 3)}_{A(x)} + \underbrace{2(x - 1)}_{B(x)} \end{aligned}$$

## EXERCISE 5E.2

1 a

$$\begin{array}{r}
 x+1 \\
 x^2+x+1 \overline{) \begin{array}{l} x^3+2x^2+x-3 \\ -(x^3+x^2+x) \phantom{-3} \\ \hline x^2+0x-3 \\ -(x^2+x+1) \\ \hline -x-4 \end{array}}
 \end{array}$$

$\therefore$  the quotient is  $x+1$  and  
the remainder is  $-x-4$ .

c

$$\begin{array}{r}
 3x \\
 x^2+1 \overline{) \begin{array}{l} 3x^3+0x^2+x-1 \\ -(3x^3 \phantom{+0x^2} + 3x) \phantom{-1} \\ \hline -2x-1 \end{array}}
 \end{array}$$

$\therefore$  the quotient is  $3x$  and  
the remainder is  $-2x-1$ .

2 a

$$\begin{array}{r}
 1 \\
 x^2+x+1 \overline{) \begin{array}{l} x^2-x+1 \\ -(x^2+x+1) \\ \hline -2x \end{array}}
 \end{array}$$

$$\therefore \frac{x^2-x+1}{x^2+x+1} = 1 - \frac{2x}{x^2+x+1}$$

$$\therefore x^2-x+1 = (x^2+x+1) - 2x$$

c

$$\begin{array}{r}
 x^2+x+3 \\
 x^2-x+1 \overline{) \begin{array}{l} x^4+0x^3+3x^2+x-1 \\ -(x^4-x^3+x^2) \phantom{-1} \\ \hline x^3+2x^2+x \\ -(x^3-x^2+x) \phantom{-1} \\ \hline 3x^2+0x-1 \\ -(3x^2-3x+3) \\ \hline 3x-4 \end{array}}
 \end{array}$$

$$\therefore \frac{x^4+3x^2+x-1}{x^2-x+1} = x^2+x+3 + \frac{3x-4}{x^2-x+1}$$

$$\therefore x^4+3x^2+x-1 = (x^2+x+3)(x^2-x+1) + 3x-4$$

b

$$\begin{array}{r}
 3 \\
 x^2-1 \overline{) \begin{array}{l} 3x^2-x+0 \\ -(3x^2 \phantom{-x} - 3) \\ \hline -x+3 \end{array}}
 \end{array}$$

$\therefore$  the quotient is 3 and  
the remainder is  $-x+3$ .

d

For  $\frac{x-4}{x^2+2x-1}$ , the quotient is 0 and  
the remainder is  $x-4$ .

b

$$\begin{array}{r}
 x \\
 x^2+2 \overline{) \begin{array}{l} x^3+0x^2+0x+0 \\ -(x^3 \phantom{+0x^2} + 2x) \phantom{+0} \\ \hline -2x+0 \end{array}}
 \end{array}$$

$$\therefore \frac{x^3}{x^2+2} = x - \frac{2x}{x^2+2}$$

$$\therefore x^3 = x(x^2+2) - 2x$$

$$\text{d} \quad \frac{2x^3 - x + 6}{(x-1)^2} = \frac{2x^3 - x + 6}{x^2 - 2x + 1}$$

$$\begin{array}{r} x^2 - 2x + 1 \overline{) \begin{array}{r} 2x^3 + 0x^2 - x + 6 \\ -(2x^3 - 4x^2 + 2x) \phantom{+ 6} \\ \hline 4x^2 - 3x + 6 \\ -(4x^2 - 8x + 4) \\ \hline 5x + 2 \end{array}} \end{array}$$

$$\therefore \frac{2x^3 - x + 6}{(x-1)^2} = 2x + 4 + \frac{5x + 2}{(x-1)^2}$$

$$\therefore 2x^3 - x + 6 = (2x + 4)(x-1)^2 + 5x + 2$$

$$\text{e} \quad \frac{x^4}{(x+1)^2} = \frac{x^4}{x^2 + 2x + 1}$$

$$\begin{array}{r} x^2 + 2x + 1 \overline{) \begin{array}{r} x^4 + 0x^3 + 0x^2 + 0x + 0 \\ -(x^4 + 2x^3 + x^2) \phantom{+ 0x + 0} \\ \hline -2x^3 - x^2 + 0x \\ -(-2x^3 - 4x^2 - 2x) \phantom{+ 0} \\ \hline 3x^2 + 2x + 0 \\ -(3x^2 + 6x + 3) \\ \hline -4x - 3 \end{array}} \end{array}$$

$$\therefore \frac{x^4}{(x+1)^2} = x^2 - 2x + 3 - \frac{4x + 3}{(x+1)^2}$$

$$\therefore x^4 = (x^2 - 2x + 3)(x+1)^2 - 4x - 3$$

$$\text{f} \quad \frac{x^4 - 2x^3 + x + 5}{(x-1)(x+2)} = \frac{x^4 - 2x^3 + x + 5}{x^2 + x - 2}$$

$$\begin{array}{r} x^2 + x - 2 \overline{) \begin{array}{r} x^4 - 2x^3 + 0x^2 + x + 5 \\ -(x^4 + x^3 - 2x^2) \phantom{+ x + 5} \\ \hline -3x^3 + 2x^2 + x \\ -(-3x^3 - 3x^2 + 6x) \phantom{+ 5} \\ \hline 5x^2 - 5x + 5 \\ -(5x^2 + 5x - 10) \\ \hline -10x + 15 \end{array}} \end{array}$$

$$\therefore \frac{x^4 - 2x^3 + x + 5}{(x-1)(x+2)} = x^2 - 3x + 5 + \frac{15 - 10x}{(x-1)(x+2)}$$

$$\therefore x^4 - 2x^3 + x + 5 = (x^2 - 3x + 5)(x-1)(x+2) + 15 - 10x$$

$$\begin{aligned} 3 \quad \frac{P(x)}{x-2} &= \frac{(x-2)(x^2+2x+3)+7}{x-2} \\ &= x^2+2x+3+\frac{7}{x-2} \end{aligned}$$

$\therefore$  the quotient is  $x^2+2x+3$  and the remainder is 7.

$$\begin{aligned} 4 \quad \frac{f(x)}{x^2+x-2} &= \frac{(x-1)(x+2)(x^2-3x+5)+15-10x}{(x-1)(x+2)} \\ &= x^2-3x+5+\frac{15-10x}{(x-1)(x+2)} \end{aligned}$$

$\therefore$  the quotient is  $x^2-3x+5$  and the remainder is  $15-10x$ .

### ACTIVITY 3

### POLYNOMIAL DIVISION USING A GRID

1 a

	$x$	1		
$x^2$	$x^3$	$x^2$	$-x$	$-4$
$x$	$x^2$	$x$		
1	$x$	1		

$$\therefore \frac{x^3+2x^2+x-3}{x^2+x+1} = x+1+\frac{-x-4}{x^2+x+1}$$

$$\therefore x^3+2x^2+x-3 = (x+1)(x^2+x+1) - x - 4$$

b

	3		
$x^2$	$3x^2$	$-x$	3
$0x$	0		
$-1$	$-3$		

$$\therefore \frac{3x^2-x}{x^2-1} = 3+\frac{-x+3}{x^2-1}$$

$$\therefore 3x^2-x = 3(x^2-1) - x + 3$$

d The quotient is 0 and the remainder is  $x-4$ .

c

	$3x$	0		
$x^2$	$3x^3$	$0x^2$	$-2x$	$-1$
$0x$	$0x^2$	$0x$		
1	$3x$	0		

$$\therefore \frac{3x^3+x-1}{x^2+1} = 3x+\frac{-2x-1}{x^2+1}$$

$$\therefore 3x^3+x-1 = 3x(x^2+1) - 2x - 1$$

2 a

	1		
$x^2$	$x^2$	$-2x$	
$x$	$x$		
1	1		

$$\therefore \frac{x^2-x+1}{x^2+x+1} = 1+\frac{-2x}{x^2+x+1}$$

$$\therefore x^2-x+1 = (x^2+x+1) - 2x$$

b

	$x$		
$x^2$	$x^3$		$-2x$
$0x$	$0x^2$		
2	$2x$		

$$\therefore \frac{x^3}{x^2+2} = x+\frac{-2x}{x^2+2}$$

$$\therefore x^3 = x(x^2+2) - 2x$$



**c**

	$x^2$	$x$	3		
$x^2$	$x^4$	$x^3$	$3x^2$	$3x$	$-4$
$-x$	$-x^3$	$-x^2$	$-3x$		
1	$x^2$	$x$	3		

$$\therefore \frac{x^4 + 3x^2 + x - 1}{x^2 - x + 1} = x^2 + x + 3 + \frac{3x - 4}{x^2 - x + 1}$$

$$\therefore x^4 + 3x^2 + x - 1 = (x^2 + x + 3)(x^2 - x + 1) + 3x - 4$$

**d**

$$\frac{2x^3 - x + 6}{(x - 1)^2} = \frac{2x^3 - x + 6}{x^2 - 2x + 1}$$

	$2x$	4		
$x^2$	$2x^3$	$4x^2$	$5x$	2
$-2x$	$-4x^2$	$-8x$		
1	$2x$	4		

$$\therefore \frac{2x^3 - x + 6}{(x - 1)^2} = 2x + 4 + \frac{5x + 2}{(x - 1)^2}$$

$$\therefore 2x^3 - x + 6 = (2x + 4)(x - 1)^2 + 5x + 2$$

**e**

$$\frac{x^4}{(x + 1)^2} = \frac{x^4}{x^2 + 2x + 1}$$

	$x^2$	$-2x$	3		
$x^2$	$x^4$	$-2x^3$	$3x^2$	$-4x$	3
$2x$	$2x^3$	$-4x^2$	$6x$		
1	$x^2$	$-2x$	3		

$$\therefore \frac{x^4}{(x + 1)^2} = x^2 - 2x + 3 + \frac{-4x - 3}{(x + 1)^2}$$

$$\therefore x^4 = (x^2 - 2x + 3)(x + 1)^2 - 4x - 3$$

**f**

$$\frac{x^4 - 2x^3 + x + 5}{(x - 1)(x + 2)} = \frac{x^4 - 2x^3 + x + 5}{x^2 + x - 2}$$

	$x^2$	$-3x$	5		
$x^2$	$x^4$	$-3x^3$	$5x^2$	$-10x$	15
$x$	$x^3$	$-3x^2$	$5x$		
$-2$	$-2x^2$	$6x$	$-10$		

$$\therefore \frac{x^4 - 2x^3 + x + 5}{(x - 1)(x + 2)} = x^2 - 3x + 5 + \frac{15 - 10x}{(x - 1)(x + 2)}$$

$$\therefore x^4 - 2x^3 + x + 5 = (x^2 - 3x + 5)(x - 1)(x + 2) + 15 - 10x$$

## EXERCISE 5F

- 1 a If  $P(2) = 7$ , then  $P(x) = Q(x)(x-2) + 7$  and  $P(x)$  divided by  $x-2$  leaves a remainder of 7.
- b If  $P(x) = Q(x)(x+3) - 8$ , then  $P(-3) = -8$  and  $P(x)$  divided by  $x+3$  leaves a remainder of  $-8$ .
- c If  $P(x)$  divided by  $x-5$  has a remainder of 11, then  $P(x) = Q(x)(x-5) + 11$  and  $P(5) = 11$ .

- 2 a If  $P(x) = x^3 + 2x^2 - 7x + 5$ , then

$$\begin{aligned} P(1) &= (1)^3 + 2(1)^2 - 7(1) + 5 \\ &= 1 + 2 - 7 + 5 \\ &= 1 \end{aligned}$$

$\therefore$  when  $x^3 + 2x^2 - 7x + 5$  is divided by  $x-1$ , the remainder is 1.

{Remainder theorem}

- b If  $P(x) = 2x^3 - 8x + 11$ , then

$$\begin{aligned} P(-3) &= 2(-3)^3 - 8(-3) + 11 \\ &= -54 + 24 + 11 \\ &= -19 \end{aligned}$$

$\therefore$  when  $2x^3 - 8x + 11$  is divided by  $x+3$ , the remainder is  $-19$ .

{Remainder theorem}

- c If  $P(x) = x^4 - 2x^2 + 3x - 1$

$$\begin{aligned} P(-2) &= (-2)^4 - 2(-2)^2 + 3(-2) - 1 \\ &= 16 - 8 - 6 - 1 \\ &= 1 \end{aligned}$$

$\therefore$  when  $x^4 - 2x^2 + 3x - 1$  is divided by  $x+2$ , the remainder is 1.

{Remainder theorem}

- 3 a  $P(x) = x^3 - 2x + a$

Now  $P(2) = 7$  {Remainder theorem}

$$\therefore 2^3 - 2(2) + a = 7$$

$$\therefore 4 + a = 7$$

$$\therefore a = 3$$

- b  $P(x) = 2x^3 + x^2 + ax - 5$

Now  $P(-1) = -8$  {Remainder theorem}

$$\therefore 2(-1)^3 + (-1)^2 + a(-1) - 5 = -8$$

$$\therefore -2 + 1 - a - 5 = -8$$

$$\therefore -a - 6 = -8$$

$$\therefore -a = -2$$

$$\therefore a = 2$$

**4**  $P(x) = x^3 + 2x^2 + ax + b$

Now  $P(1) = 4$  and  $P(-2) = 16$  {Remainder theorem}

If  $P(1) = 4$  then  $1 + 2 + a + b = 4$  and so  $a + b = 1$  .... (1)

If  $P(-2) = 16$  then  $(-2)^3 + 2(-2)^2 + a(-2) + b = 16$   
 $\therefore -8 + 8 - 2a + b = 16$   
 $\therefore -2a + b = 16$  .... (2)

Solving simultaneously:  $-a - b = -1$  {-(1)}  
 $-2a + b = 16$  {(2)}

Adding,  $-3a = 15$   
 $\therefore a = -5$  and so  $b = 6$   
 $\therefore a = -5$  and  $b = 6$

**5**  $P(x) = 2x^n + ax^2 - 6$

Now  $P(1) = -7$  {Remainder theorem}

$\therefore 2(1)^n + a(1)^2 - 6 = -7$

$\therefore 2 + a - 6 = -7$

$\therefore a = -3$

So,  $P(x) = 2x^n - 3x^2 - 6$ .

Also,  $P(-3) = 129$  {Remainder theorem}

$\therefore 2(-3)^n - 3(-3)^2 - 6 = 129$

$2(-3)^n - 27 - 6 = 129$

$2(-3)^n = 162$

$(-3)^n = 81$

$\therefore n = 4$

So  $a = -3$  and  $n = 4$ .

**6**  $f(x) = 2x^3 + ax^2 - 3x + b$

Now  $f(-1) = 7$  and  $f(2) = 28$  {Remainder theorem}

So,  $2(-1)^3 + a(-1)^2 - 3(-1) + b = 7$  and  $2(2)^3 + a(2)^2 - 3(2) + b = 28$

$\therefore -2 + a + 3 + b = 7$

$\therefore 16 + 4a - 6 + b = 28$

$\therefore a + b = 6$  .... (1)

$\therefore 4a + b = 18$  .... (2)

Solving simultaneously:  $4a + b = 18$  {(2)}  
 $a + b = 6$  {(1)}

Subtracting,  $3a = 12$   
 $\therefore a = 4$

Substituting  $a = 4$  into (1),  $4 + b = 6$

$\therefore b = 2$

$\therefore f(x) = 2x^3 + 4x^2 - 3x + 2$

$f(-3) = 2(-3)^3 + 4(-3)^2 - 3(-3) + 2$

$= -54 + 36 + 9 + 2$

$= -7$

$\therefore$  when  $f(x)$  is divided by  $x + 3$ , the remainder is  $-7$ .

**7 a**  $P(x) = Q(x)(2x - 1) + R$

$$\begin{aligned} P\left(\frac{1}{2}\right) &= Q\left(\frac{1}{2}\right)\left(2 \times \frac{1}{2} - 1\right) + R \\ &= Q\left(\frac{1}{2}\right) \times 0 + R \\ &= R \end{aligned}$$

**b i** If  $P(x) = 4x^2 - 10x + 1$ , then

$$\begin{aligned} R &= P\left(\frac{1}{2}\right) \quad \{\text{Remainder theorem}\} \\ &= 4\left(\frac{1}{2}\right)^2 - 10\left(\frac{1}{2}\right) + 1 \\ &= 1 - 5 + 1 \\ &= -3 \end{aligned}$$

**ii** If  $P(x) = 2x^3 - 5x^2 + 8$ , then

$$\begin{aligned} R &= P\left(\frac{1}{2}\right) \quad \{\text{Remainder theorem}\} \\ &= 2\left(\frac{1}{2}\right)^3 - 5\left(\frac{1}{2}\right)^2 + 8 \\ &= \frac{1}{4} - \frac{5}{4} + 8 \\ &= 7 \end{aligned}$$

**8** If  $P(x) = 4x^3 + 7x - 3$ , then

$$\begin{aligned} R &= P\left(-\frac{1}{2}\right) \quad \{\text{Remainder theorem}\} \\ &= 4\left(-\frac{1}{2}\right)^3 + 7\left(-\frac{1}{2}\right) - 3 \\ &= -\frac{1}{2} - \frac{7}{2} - 3 \\ &= -7 \end{aligned}$$

**9**  $P(x) = 2x^3 + ax^2 + bx + 4$

Now  $P(-1) = -5$  and  $P\left(\frac{1}{2}\right) = 10$  {Remainder theorem}

$$\begin{aligned} \text{So, } 2(-1)^3 + a(-1)^2 + b(-1) + 4 &= -5 \\ \therefore -2 + a - b + 4 &= -5 \\ \therefore a - b &= -7 \quad \dots (1) \end{aligned}$$

$$\begin{aligned} \text{and } 2\left(\frac{1}{2}\right)^3 + a\left(\frac{1}{2}\right)^2 + b\left(\frac{1}{2}\right) + 4 &= 10 \\ \therefore \frac{1}{4} + \frac{1}{4}a + \frac{1}{2}b + 4 &= 10 \\ \therefore \frac{1}{4}a + \frac{1}{2}b &= \frac{23}{4} \\ \therefore a + 2b &= 23 \quad \dots (2) \end{aligned}$$

Solving simultaneously:  $a + 2b = 23$  {(2)}

$a - b = -7$  {(1)}

$$\begin{array}{r} \text{Subtracting,} \\ \hline 3b = 30 \\ \therefore b = 10 \end{array}$$

Substituting  $b = 10$  into (1),  $a - 10 = -7$

$$\therefore a = 3$$

So,  $a = 3$ ,  $b = 10$ .

**10**  $P(z) = Q(z)(z^2 - 3z + 2) + (4z - 7) = Q(z)(z - 2)(z - 1) + 4z - 7$

**a** remainder  $= P(1)$  {Remainder theorem}

$$\begin{aligned} &= Q(1) \times (1 - 2) \times 0 + 4 - 7 \\ &= -3 \end{aligned}$$

**b** remainder  $= P(2)$  {Remainder theorem}

$$\begin{aligned} &= Q(2) \times 0 \times (2 - 1) + 4(2) - 7 \\ &= 1 \end{aligned}$$



$$11 \quad \frac{P(x)}{x^2 - x + 3} = x^2 + 4x - 1 + \frac{R(x)}{x^2 - x + 3}$$

$$\therefore P(x) = (x^2 + 4x - 1)(x^2 - x + 3) + R(x)$$

$$\text{Let } R(x) = ax + b$$

$$\text{Now } P(1) = 10 \quad \{\text{Remainder theorem}\}$$

$$\therefore (1^2 + 4(1) - 1)(1^2 - 1 + 3) + R(1) = 10$$

$$\therefore 12 + a + b = 10$$

$$\therefore a + b = -2 \quad \dots (1)$$

$$\text{and } P(-3) = -42 \quad \{\text{Remainder theorem}\}$$

$$\therefore ((-3)^2 + 4(-3) - 1)((-3)^2 - (-3) + 3) + R(-3) = -42$$

$$\therefore -60 - 3a + b = -42$$

$$\therefore -3a + b = 18 \quad \dots (2)$$

$$\text{Solving (1) and (2), } a + b = -2$$

$$-3a + b = 18$$

$$\text{Subtracting, } \begin{array}{r} a + b = -2 \\ -3a + b = 18 \\ \hline 4a = -20 \end{array}$$

$$\therefore a = -5 \text{ and so } b = 3$$

$$\therefore R(x) = -5x + 3$$

$$12 \quad \text{Suppose } P(z) \text{ is divided by } (z - 3)(z + 1).$$

$$\therefore P(z) = Q(z) \times (z - 3)(z + 1) + (Az + B) \quad \{\text{the remainder must be linear}\}$$

$$\text{Now } P(-1) = -8 \quad \{\text{Remainder theorem}\}$$

$$\therefore Q(-1) \times 0 + (-A + B) = -8$$

$$\therefore -A + B = -8 \quad \dots (1)$$

$$\text{and } P(3) = 4 \quad \{\text{Remainder theorem}\}$$

$$\therefore Q(3) \times 0 + (3A + B) = 4$$

$$\therefore 3A + B = 4 \quad \dots (2)$$

$$\text{Solving simultaneously, } \begin{array}{r} -A + B = -8 \quad \{(1)\} \\ -3A + B = -4 \quad \{- (2)\} \\ \hline -4A = -12 \end{array}$$

$$\therefore A = 3 \text{ and so } B = -5$$

$$\therefore R(z) = 3z - 5$$

$$13 \quad \text{Suppose } P(x) \text{ is divided by } (x - a)(x - b) \text{ and has remainder } Cx + D.$$

$$\therefore P(x) = Q(x) \times (x - a)(x - b) + Cx + D$$

$$\text{Now by the Remainder theorem, } P(a) = Ca + D \quad \dots (1) \text{ and } P(b) = Cb + D \quad \dots (2)$$

$$\text{Subtracting (1) from (2), } P(b) - P(a) = Cb - Ca = C(b - a)$$

$$\therefore C = \frac{P(b) - P(a)}{b - a}$$

$$\text{Substituting into (1) gives } D = P(a) - Ca = P(a) - \left( \frac{P(b) - P(a)}{b - a} \right) a$$

$$\begin{aligned} \therefore \text{remainder} &= \left( \frac{P(b) - P(a)}{b - a} \right) x + P(a) - \left( \frac{P(b) - P(a)}{b - a} \right) a \\ &= \left( \frac{P(b) - P(a)}{b - a} \right) (x - a) + P(a) \end{aligned}$$

**EXERCISE 5G**

- 1 a** Let  $P(x) = x^3 - 3x^2 + 3x - 2$   
 $P(2) = 2^3 - 3(2)^2 + 3(2) - 2$   
 $= 8 - 12 + 6 - 2$   
 $= 0$   
 $\therefore (x - 2)$  is a factor of  $x^3 - 3x^2 + 3x - 2$  {Factor theorem}
- b** Let  $P(x) = x^4 - 2x^3 + 3x^2 - 4$   
 $P(-1) = (-1)^4 - 2(-1)^3 + 3(-1)^2 - 4$   
 $= 1 + 2 + 3 - 4$   
 $= 2 \neq 0$   
 $\therefore (x + 1)$  is not a factor of  $x^4 - 2x^3 + 3x^2 - 4$  {Factor theorem}
- c** Let  $P(x) = 2x^3 + 5x^2 + x + 12$   
 $P(-3) = 2(-3)^3 + 5(-3)^2 - 3 + 12$   
 $= -54 + 45 - 3 + 12$   
 $= 0$   
 $\therefore (x + 3)$  is a factor of  $2x^3 + 5x^2 + x + 12$  {Factor theorem}
- 2 a**  $(x - 1)$  is a factor of  $2x^3 - 3x^2 + ax - 4$   
 $\therefore 2(1)^3 - 3(1)^2 + a(1) - 4 = 0$  {Factor theorem}  
 $\therefore 2 - 3 + a - 4 = 0$   
 $\therefore a = 5$
- b**  $(x + 2)$  is a factor of  $3x^5 + ax^3 - 7x + 3$   
 $\therefore 3(-2)^5 + a(-2)^3 - 7(-2) + 3 = 0$  {Factor theorem}  
 $\therefore -96 - 8a + 14 + 3 = 0$   
 $\therefore 8a = -79$   
 $\therefore a = -\frac{79}{8}$
- 3 a** Let  $P(x) = 2x^3 + x^2 + kx - 4$ .  
 Since  $(x + 2)$  is a factor,  $P(-2) = 0$  {Factor theorem}  
 $\therefore 2(-2)^3 + (-2)^2 + k(-2) - 4 = 0$   
 $\therefore -16 + 4 - 2k - 4 = 0$   
 $\therefore -2k = 16$   
 $\therefore k = -8$

We now use division to find the other factors of  $P(x)$ :

$$\begin{array}{r|rrrr} -2 & 2 & 1 & -8 & -4 \\ & 0 & -4 & 6 & 4 \\ \hline & 2 & -3 & -2 & 0 \end{array}$$

$$\begin{aligned} \therefore P(x) &= (x + 2)(2x^2 - 3x - 2) \\ &= (x + 2)(2x + 1)(x - 2) \end{aligned}$$

**b** Let  $P(x) = x^4 - 3x^3 - kx^2 + 6x$ .

Since  $(x - 3)$  is a factor,  $P(3) = 0$  {Factor theorem}

$$\begin{aligned}\therefore (3)^4 - 3(3)^3 - k(3)^2 + 6(3) &= 0 \\ \therefore -9k + 18 &= 0 \\ \therefore 9k &= 18 \\ \therefore k &= 2\end{aligned}$$

We now use synthetic division to find the other factors of  $P(x)$ :

$$\begin{array}{r|rrrrr} 3 & 1 & -3 & -2 & 6 & 0 \\ & & 3 & 0 & -6 & 0 \\ \hline & 1 & 0 & -2 & 0 & 0 \end{array}$$

$$\begin{aligned}\therefore P(x) &= (x - 3)(x^3 - 2x) \\ &= x(x - 3)(x^2 - 2) \\ &= x(x - 3)(x + \sqrt{2})(x - \sqrt{2})\end{aligned}$$

**4 a** Since  $(x - 4)$  is a factor,  $P(4) = 0$  {Factor theorem}

$$\begin{aligned}\therefore 3(4)^3 - 17(4)^2 + k(4) + 8 &= 0 \\ \therefore 192 - 272 + 4k + 8 &= 0 \\ \therefore 4k &= 72 \\ \therefore k &= 18\end{aligned}$$

**b**  $P(x) = 3x^3 - 17x^2 + 18x + 8$   
 $= (x - 4)(ax^2 + bx + c)$

The  $x^3$  term is  $3x^3$ , so we require  $a = 3$ .

The constant term is 8, so we require  $c = -2$ .

$$\begin{aligned}\therefore P(x) &= (x - 4)(3x^2 + bx - 2) \\ &= 3x^3 + bx^2 - 2x \\ &\quad - 12x^2 - 4bx + 8 \\ &= 3x^3 + (b - 12)x^2 + (-2 - 4b)x + 8\end{aligned}$$

Equating coefficients,  $b - 12 = -17$   
 $\therefore b = -5$

and  $-2b - 4b = -2 - 4(-5)$   
 $= -2 + 20$   
 $= 18$  ✓

$$\therefore P(x) = (x - 4)(3x^2 - 5x - 2)$$

**c**  $P(x) = (x - 4)(3x^2 - 5x - 2)$   
 $= (x - 4)(3x + 1)(x - 2)$

$\therefore$  the roots of  $P(x) = 0$  are  $x = -\frac{1}{3}$ , 2, or 4.

**5** Let  $P(x) = 2x^3 + ax^2 + bx + 5$ .

Since  $(x - 1)$  is a factor,  $P(1) = 0$  {Factor theorem}

$$\begin{aligned}\therefore 2(1)^3 + a(1)^2 + b(1) + 5 &= 0 \\ \therefore 2 + a + b + 5 &= 0 \\ \therefore a + b &= -7 \quad \dots (1)\end{aligned}$$

Since  $(x + 5)$  is a factor,  $P(-5) = 0$  {Factor theorem}

$$\therefore 2(-5)^3 + a(-5)^2 + b(-5) + 5 = 0$$

$$\therefore -250 + 25a - 5b + 5 = 0$$

$$\therefore 25a - 5b = 245$$

$$\therefore 5a - b = 49 \quad \dots (2)$$

Adding (1) and (2) gives:  $6a = 42$

$$\therefore a = 7 \text{ and } b = -14$$

**6**  $P(z) = z^3 - z^2 + (k - 5)z + (k^2 - 1)$

Using synthetic division to divide  $P(z)$  by  $(z - 3)$ :

$$\begin{array}{r|rrrr} 3 & 1 & -1 & k-5 & k^2-1 \\ & 0 & 3 & 6 & 3k+3 \\ \hline & 1 & 2 & k+1 & k^2+3k+2 \end{array}$$

So  $P(z) = (z - 3)(z^2 + 2z + (k + 1)) + k^2 + 3k + 2$

Since 3 is a zero of  $P(z)$ ,  $P(3) = 0$  {Factor theorem}

$$\therefore (3 - 3)(z^2 + 2z + k + 1) + k^2 + 3k + 2 = 0$$

$$\therefore k^2 + 3k + 2 = 0$$

$$\therefore (k + 1)(k + 2) = 0$$

$$\therefore k = -1 \text{ or } -2$$

If  $k = -1$ ,  $P(z) = (z - 3)(z^2 + 2z)$

$$= z(z - 3)(z + 2)$$

$\therefore$  the zeros of  $P(z)$  are 0, 3 and  $-2$ .

If  $k = -2$ ,  $P(z) = (z - 3)(z^2 + 2z - 1)$

The quadratic factor has zeros  $\frac{-2 \pm \sqrt{4 + 4}}{2} = -1 \pm \sqrt{2}$

$\therefore$  the zeros of  $P(z)$  are 3 and  $-1 \pm \sqrt{2}$ .

**7**  $P(z) = z^3 + mz^2 + (3m - 2)z - 10m - 4$

Now  $P(2) = 8 + 4m + 6m - 4 - 10m - 4$   
 $= 0$

$\therefore (z - 2)$  is always a factor {Factor theorem}

Now consider  $P(z)$  divided by  $(z - 2)^2$ :

$$\begin{array}{r|rrrr} 2 & 1 & m & 3m-2 & -10m-4 \\ & 0 & 2 & 2m+4 & 10m+4 \\ \hline 2 & 1 & m+2 & 5m+2 & 0 \\ & 0 & 2 & 2m+8 & \\ \hline & 1 & m+4 & 7m+10 & \end{array}$$

For  $(z - 2)^2$  to be a factor,  $7m + 10 = 0$

$$\therefore m = -\frac{10}{7}$$



**8 a**  $(x + 3)$  is a factor of  $P(x) = 2x^3 + 9x^2 + ax + b$

$$\therefore P(-3) = 0 \quad \{\text{Factor theorem}\}$$

$$\therefore 2(-3)^3 + 9(-3)^2 + a(-3) + b = 0$$

$$\therefore -54 + 81 - 3a + b = 0$$

$$\therefore 3a - b = 27 \quad \dots (1)$$

When  $P(x)$  is divided by  $x + 4$ , the remainder is  $-18$ .

$$\therefore P(-4) = -18 \quad \{\text{Remainder theorem}\}$$

$$\therefore 2(-4)^3 + 9(-4)^2 + a(-4) + b = -18$$

$$\therefore -128 + 144 - 4a + b = -18$$

$$\therefore 4a - b = 34 \quad \dots (2)$$

Solving simultaneously,  $4a - b = 34 \quad \{(2)\}$

$$3a - b = 27 \quad \{(1)\}$$

$$\text{Subtracting,} \quad \underline{\quad\quad\quad} a = 7 \text{ and so } b = -6$$

**b**  $P(x) = 2x^3 + 9x^2 + 7x - 6$

$$\therefore P(2) = 2(2)^3 + 9(2)^2 + 7(2) - 6$$

$$= 16 + 36 + 14 - 6$$

$$= 60$$

$\therefore$  when  $P(x)$  is divided by  $x - 2$ , the remainder is 60.  $\{\text{Remainder theorem}\}$

$$\begin{aligned} \text{c } P(x) &= 2x^3 + 9x^2 + 7x - 6 \\ &= (x + 3)(px^2 + qx + r) \end{aligned}$$

The  $x^3$  term is  $2x^3$ , so we require  $p = 2$ .

The constant term is  $-6$ , so we require  $r = -2$ .

$$\begin{aligned} \therefore P(x) &= (x + 3)(2x^2 + qx - 2) \\ &= 2x^3 + qx^2 - 2x \\ &\quad + 6x^2 + 3qx - 6 \\ &= 2x^3 + (q + 6)x^2 + (3q - 2)x - 6 \end{aligned}$$

Equating coefficients,  $q + 6 = 9$

$$\therefore q = 3$$

$$\text{and } 3(3) - 2 = 9 - 2$$

$$= 7 \quad \checkmark$$

$$\therefore P(x) = (x + 3)(2x^2 + 3x - 2)$$

$$\begin{aligned} \text{d } P(x) &= (x + 3)(2x^2 + 3x - 2) \\ &= (x + 3)(2x - 1)(x + 2) \end{aligned}$$

$\therefore$  the zeros of  $P(x)$  are  $-3$ ,  $-2$ , and  $\frac{1}{2}$ .

**9 a**  $(2x - 1)$  is a factor of  $P(x) = 2x^3 + ax^2 - 8x + b$

$$\therefore P\left(\frac{1}{2}\right) = 0 \quad \{\text{Factor theorem}\}$$

$$\therefore 2\left(\frac{1}{2}\right)^3 + a\left(\frac{1}{2}\right)^2 - 8\left(\frac{1}{2}\right) + b = 0$$

$$\therefore \frac{1}{4} + \frac{1}{4}a - 4 + b = 0$$

$$\therefore \frac{1}{4}a + b = \frac{15}{4}$$

$$\therefore a + 4b = 15 \quad \dots (1)$$

When  $P(x)$  is divided by  $x - 1$ , the remainder is 3.

$$\therefore P(1) = 3 \quad \{\text{Remainder theorem}\}$$

$$\therefore 2(1)^3 + a(1)^2 - 8(1) + b = 3$$

$$\therefore 2 + a - 8 + b = 3$$

$$\therefore a + b = 9 \quad \dots (2)$$

Solving simultaneously,  $a + 4b = 15 \quad \{(1)\}$

$$a + b = 9 \quad \{(2)\}$$

Subtracting,

$$3b = 6$$

$$\therefore b = 2 \text{ and so } a = 7$$

$$\begin{aligned} \mathbf{b} \quad P(x) &= 2x^3 + 7x^2 - 8x + 2 \\ &= (2x - 1)(px^2 + qx + r) \end{aligned}$$

The  $x^3$  term is  $2x^3$ , so we require  $p = 1$ .

The constant term is 2, so we require  $r = -2$ .

$$\begin{aligned} \therefore P(x) &= (2x - 1)(x^2 + qx - 2) \\ &= 2x^3 + 2qx^2 - 4x \\ &\quad - x^2 - qx + 2 \\ &= 2x^3 + (2q - 1)x^2 - (4 + q)x + 2 \end{aligned}$$

Equating coefficients,  $2q - 1 = 7$

$$\therefore 2q = 8$$

$$\therefore q = 4$$

and  $-(4 + 4) = -8 \quad \checkmark$

$$\begin{aligned} \therefore P(x) &= (2x - 1)(x^2 + 4x - 2) \text{ which has roots } x = \frac{1}{2}, \frac{-4 \pm \sqrt{16 + 8}}{2} \\ &= \frac{1}{2}, \frac{-4 \pm 2\sqrt{6}}{2} \\ &= \frac{1}{2}, -2 \pm \sqrt{6} \end{aligned}$$

$\therefore$  the irrational roots of  $P(x)$  are  $x = -2 \pm \sqrt{6}$ .

**10** Consider  $P(x) = x^n + 1$ ,  $n \in \mathbb{Z}$ .

Now  $(x + 1)$  is a factor of  $P(x) \Leftrightarrow P(-1) = 0 \quad \{\text{Factor theorem}\}$

$$\Leftrightarrow (-1)^n + 1 = 0$$

$$\Leftrightarrow (-1)^n = -1$$

$$\Leftrightarrow n \text{ is odd}$$

$\therefore (x + 1)$  is a factor of  $x^n + 1 \Leftrightarrow n$  is odd.

- 11**  $P(x) = x^3 - 3ax - 9$  and if  $(x - 1 - a)$  is a factor then  $P(1 + a) = 0$  {Factor theorem}

Dividing  $P(x)$  by  $(x - 1 - a)$ :

$$\begin{array}{r|rrrr}
 1+a & 1 & 0 & -3a & -9 \\
 & 0 & 1+a & 1+2a+a^2 & a^3+1 \\
 \hline
 & 1 & 1+a & a^2-a+1 & a^3-8
 \end{array}$$

$$\therefore a^3 - 8 = 0$$

$$\therefore a = 2 \quad \{a \in \mathbb{R}\}$$

## EXERCISE 5H

- 1** Since  $P(x) = x^2 + ax + b$  is real, both  $2 - i\sqrt{3}$  and  $2 + i\sqrt{3}$  are zeros.  
 These have sum = 4 and product =  $4 - 3i^2 = 7$ .  
 $\therefore$  the zeros  $2 \pm i\sqrt{3}$  come from the quadratic  $x^2 - 4x + 7$ .  
 $\therefore a = -4, b = 7$ .
- 2** Since it is a real polynomial of degree 3, the zeros must be  $-\frac{1}{2}, 1 - 3i$ , and  $1 + 3i$ .  
 The zeros  $1 \pm 3i$  have sum = 2 and product =  $1 - 9i^2 = 10$   
 So the factors of the cubic polynomial  $P(x)$  are  $(2x + 1)$  and  $(x^2 - 2x + 10)$   
 $\therefore P(x) = a(2x + 1)(x^2 - 2x + 10), a \neq 0$
- 3**  $p(1) = p(2 + i) = 0$   
 Hence the zeros of  $p(x)$  must be  $1, 2 \pm i$  {as  $p(x)$  is real and cubic}  
 The zeros  $2 \pm i$  have sum = 4 and product =  $4 - i^2 = 5$   
 So the factors of  $p(x)$  must be  $(x - 1)$  and  $(x^2 - 4x + 5)$   
 $\therefore p(x) = a(x - 1)(x^2 - 4x + 5), a \neq 0$   
 Since  $p(0) = -20, -20 = a(-1)(5)$   
 $\therefore a = 4$   
 $\therefore p(x) = 4(x - 1)(x^2 - 4x + 5)$   
 $= 4(x^3 - 4x^2 + 5x - x^2 + 4x - 5)$   
 $= 4(x^3 - 5x^2 + 9x - 5)$   
 $= 4x^3 - 20x^2 + 36x - 20$
- 4 a** Since  $P(z) = z^3 + pz + q$  is real, both  $2 - 3i$  and  $2 + 3i$  are zeros.  
 These have sum = 4 and product =  $4 - 9i^2 = 13$   
 $\therefore (z^2 - 4z + 13)$  is a factor of  $P(z)$   
 $\therefore z^3 + pz + q = (z^2 - 4z + 13)(z + a)$  for some constant  $a$   
 $= z^3 + (a - 4)z^2 + (13 - 4a)z + 13a$   
 Equating coefficients:  $a - 4 = 0, 13 - 4a = p, \text{ and } 13a = q$   
 $\therefore a = 4, p = -3, q = 52$   
 $\therefore$  the other zeros are  $-4$  and  $2 + 3i$ .

**b** Check: Since  $P(2 - 3i) = 0$ ,  $(2 - 3i)^3 + p(2 - 3i) + q = 0$

$$\text{Expanding, } (-46 - 9i) + p(2 - 3i) + q = 0$$

$$\therefore (-46 + 2p + q) + (-9 - 3p)i = 0$$

$$\text{Equating real and imaginary parts, } -46 + 2p + q = 0 \quad \dots (1)$$

$$\text{and } -9 - 3p = 0 \quad \dots (2)$$

$$\text{From (2), } p = -3, \text{ so in (1), } -46 - 6 + q = 0 \therefore p = -3, q = 52 \quad \checkmark$$

**5**  $3 + i$  is a root of  $z^4 - 2z^3 + az^2 + bz + 10 = 0$  where the coefficients are real.

$\therefore 3 - i$  is also a root.

The roots  $3 \pm i$  have sum = 6 and product =  $9 - i^2 = 10$

$\therefore z^2 - 6z + 10$  is a factor of  $z^4 - 2z^3 + az^2 + bz + 10$

$$\begin{aligned} \therefore z^4 - 2z^3 + az^2 + bz + 10 &= (z^2 - 6z + 10)(z^2 + sz + 1) \quad \text{for some constant } s \\ &= z^4 + sz^3 + z^2 \\ &\quad - 6z^3 - 6sz^2 - 6z \\ &\quad + 10z^2 + 10sz + 10 \\ &= z^4 + (s - 6)z^3 + (11 - 6s)z^2 + (10s - 6)z + 10 \end{aligned}$$

Equating coefficients:

$$\begin{aligned} s - 6 &= -2, & 11 - 6s &= a & \text{and } 10s - 6 &= b \\ \therefore s &= 4 & a &= 11 - 6(4) = -13 & b &= 10(4) - 6 = 34 \end{aligned}$$

$\therefore$  the other factor is  $z^2 + 4z + 1$  which has zeros  $\frac{-4 \pm \sqrt{16 - 4}}{2} = -2 \pm \sqrt{3}$

$\therefore a = -13, b = 34$  and the other roots are  $z = 3 - i$  and  $z = -2 \pm \sqrt{3}$ .

**6**  $2 + i$  is a root of  $x^4 + ax^3 + 8x^2 + 3ax + b = 0$  where the coefficients are real.

$\therefore 2 - i$  is also a root.

The roots  $2 \pm i$  have sum = 4 and product =  $4 - i^2 = 5$

$\therefore x^2 - 4x + 5$  is a factor

$$\begin{aligned} \therefore x^4 + ax^3 + 8x^2 + 3ax + b &= (x^2 - 4x + 5)\left(x^2 + sx + \frac{b}{5}\right) \quad \text{for some constant } s \\ &= x^4 + sx^3 + \frac{b}{5}x^2 \\ &\quad - 4x^3 - 4sx^2 - \frac{4}{5}xb \\ &\quad + 5x^2 + 5sx + b \\ &= x^4 + (s - 4)x^3 + \left(\frac{b}{5} - 4s + 5\right)x^2 + \left(5s - \frac{4}{5}b\right)x + b \end{aligned}$$

$$\text{Equating coefficients gives } \begin{cases} s - 4 = a & \dots (1) \\ 5 + \frac{b}{5} - 4s = 8 & \dots (2) \\ 5s - \frac{4}{5}b = 3a & \dots (3) \end{cases}$$

$$\text{From (2): } \frac{b}{5} - 4s = 3$$

$$\therefore b - 20s = 15$$

$$\therefore b = 15 + 20s$$

$$\therefore \text{ in (3): } 5s - \frac{4}{5}(15 + 20s) = 3a$$

$$\therefore 5s - 12 - 16s = 3a$$

$$\therefore -11s - 12 = 3a$$

$$\therefore a = -\frac{11}{3}s - 4$$



$$\therefore \text{ in (1): } s - 4 = -\frac{11}{3}s - 4$$

$$s = -\frac{11}{3}s$$

$$\therefore s = 0$$

$$\therefore a = -4 \text{ and } b = 15$$

$\therefore$  the other factor is  $x^2 + 3$  which has zeros  $\pm i\sqrt{3}$

$\therefore a = -4, b = 15$  and the other roots are  $x = 2 - i$  and  $x = \pm i\sqrt{3}$ .

**7** Let the purely imaginary zero be  $bi$ ,  $b \in \mathbb{R}$ .

Since  $P(z)$  is real, another zero is  $-bi$ .

$\therefore z^2 + b^2$  is a factor of  $P(z)$

$$\begin{aligned} \therefore z^3 + az^2 + 3z + 9 &= (z^2 + b^2)(z + c) \text{ for some constant } c \\ &= z^3 + cz^2 + b^2z + b^2c \end{aligned}$$

Equating coefficients,  $b^2 = 3$ ,  $b^2c = 9$ , and  $a = c$

$$\therefore b = \pm\sqrt{3} \text{ and so } a = c = 3$$

$$\therefore P(z) = (z + 3)(z^2 + 3)$$

$$\therefore P(z) = (z + 3)(z + i\sqrt{3})(z - i\sqrt{3})$$

**8** Let  $ai$ ,  $a \in \mathbb{R}$  be the purely imaginary zero of  $3x^3 + kx^2 + 15x + 10$ .

$\therefore$  as  $P(x)$  is real,  $-ai$  is also a zero.

The zeros  $\pm ai$  have sum  $= 0$  and product  $= -a^2i^2 = a^2$

$\therefore x^2 + a^2$  is a factor of  $P(x)$

$$\begin{aligned} \therefore 3x^3 + kx^2 + 15x + 10 &= (x^2 + a^2)(3x + b) \text{ for some constant } b \\ &= 3x^3 + bx^2 + 3a^2x + a^2b \end{aligned}$$

Equating coefficients,  $k = b$ ,  $3a^2 = 15$ , and  $a^2b = 10$

$$\therefore a^2 = 5 \text{ and so } b = k = 2$$

$$\therefore P(x) = (x^2 + 5)(3x + 2)$$

$$\therefore P(x) = (x - i\sqrt{5})(x + i\sqrt{5})(3x + 2)$$

**9 a** Since  $P(x)$  is real, both  $1 + ki$  and  $1 - ki$  are zeros.

These have sum  $= 2$  and product  $= 1 - k^2i^2 = 1 + k^2$

$\therefore x^2 - 2x + (1 + k^2)$  is a factor of  $P(x)$ .

**b** By comparison with  $P(x)$ ,  $1 + k^2$  is a factor of  $-10$ .

$$\therefore 1 + k^2 = \pm 1, \pm 2, \pm 5, \pm 10 \text{ where } k \in \mathbb{Z}$$

$$\therefore k^2 = 0, 1, 4, 9$$

$$\therefore k = 0, \pm 1, \pm 2, \pm 3$$

**c** As  $p$  and  $q$  are integer zeros, they come from

$$(x - p)(x - q) = x^2 - (p + q)x + pq$$

$\therefore pq$  is a factor of  $-10$

The possibilities are as shown in the table:

$1 + k^2$	$pq$
1	-10
2	-5
5	-2
10	-1

**d** Without loss of generality, we assume  $p > q$ .

Since  $p + q = -1$ , the only possibility is  $p = 1$ ,  $q = -2$ ,

and  $1 + k^2 = 5$

$$\therefore k = \pm 2$$

$$\therefore P(x) = (x - 1)(x + 2)(x - 1 - 2i)(x - 1 + 2i)$$

So, the zeros of  $P(x)$  are  $1$ ,  $-2$ , and  $1 \pm 2i$ .

## INVESTIGATION 1

## SUM AND PRODUCT OF ROOTS

**1 a**  $P_2(x) = a(x - \alpha)(x - \beta)$

$$= a(x^2 - \beta x - \alpha x + \alpha\beta)$$

$$= a(x^2 - [\alpha + \beta]x + \alpha\beta)$$

**b**  $P_3(x) = a(x - \alpha)(x - \beta)(x - \gamma)$

$$= P_2(x)(x - \gamma)$$

$$= a(x^2 - [\alpha + \beta]x + \alpha\beta)(x - \gamma)$$

$$= a(x^3 - [\alpha + \beta]x^2 + \alpha\beta x - \gamma x^2 + \gamma[\alpha + \beta]x - \alpha\beta\gamma)$$

$$= a(x^3 - [\alpha + \beta + \gamma]x^2 + [\alpha\beta + \alpha\gamma + \beta\gamma]x - \alpha\beta\gamma)$$

**c**  $P_4(x) = a(x - \alpha)(x - \beta)(x - \gamma)(x - \delta)$

$$= P_3(x)(x - \delta)$$

$$= a(x^3 - [\alpha + \beta + \gamma]x^2 + [\alpha\beta + \alpha\gamma + \beta\gamma]x - \alpha\beta\gamma)(x - \delta)$$

$$= a(x^4 - [\alpha + \beta + \gamma]x^3 + [\alpha\beta + \alpha\gamma + \beta\gamma]x^2 - \alpha\beta\gamma x - \delta x^3 + \delta[\alpha + \beta + \gamma]x^2 - \delta[\alpha\beta + \alpha\gamma + \beta\gamma]x + \alpha\beta\gamma\delta)$$

$$= a(x^4 - [\alpha + \beta + \gamma + \delta]x^3 + [\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta]x^2 - [\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta]x + \alpha\beta\gamma\delta)$$

**2 a, b** For  $P_2(x) = a(x^2 - [\alpha + \beta]x + \alpha\beta)$ :

The constant term is  $a\alpha\beta$  and the product of the roots of  $P_2(x) = 0$  is  $\alpha\beta$ .

The coefficient of  $x$  is  $-a[\alpha + \beta]$  and the sum of the roots is  $\alpha + \beta$ .

For  $P_3(x) = a(x^3 - [\alpha + \beta + \gamma]x^2 + [\alpha\beta + \alpha\gamma + \beta\gamma]x - \alpha\beta\gamma)$ :

The constant term is  $-a\alpha\beta\gamma$  and the product of the roots of  $P_3(x) = 0$  is  $-\alpha\beta\gamma$ .

The coefficient of  $x^2$  is  $-a[\alpha + \beta + \gamma]$  and the sum of the roots is  $\alpha + \beta + \gamma$ .

For  $P_4(x) = a(x^4 - [\alpha + \beta + \gamma + \delta]x^3 + [\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta]x^2 - [\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta]x + \alpha\beta\gamma\delta)$ :

The constant term is  $a\alpha\beta\gamma\delta$  and the product of the roots of  $P_4(x) = 0$  is  $\alpha\beta\gamma\delta$ .

The coefficient of  $x^3$  is  $-a[\alpha + \beta + \gamma + \delta]$  and the sum of the roots is  $\alpha + \beta + \gamma + \delta$ .

**3** For the polynomial equation  $\sum_{r=0}^n a_r x^r = 0$ ,  $a_n \neq 0$

the sum of the roots is  $-\frac{a_{n-1}}{a_n}$ , and the product of the roots is  $\frac{(-1)^n a_0}{a_n}$ .

## EXERCISE 5I

1 a  $2x^2 - 3x + 4 = 0$

$$\therefore \text{the sum of the roots} = -\frac{(-3)}{2} = \frac{3}{2}$$

The polynomial equation has degree 2.

$$\therefore \text{the product of the roots} = \frac{(-1)^2 4}{2} = 2$$

c  $x^4 - x^3 + 2x^2 + 3x - 4 = 0$

$$\therefore \text{the sum of the roots} = -\frac{(-1)}{1} = 1$$

The polynomial equation has degree 4.

$$\therefore \text{the product of the roots} = \frac{(-1)^4(-4)}{1} = -4$$

e  $x^7 - x^5 + 2x - 9 = 0$

$$\therefore x^7 + (0)x^6 - x^5 + 2x - 9 = 0$$

$$\therefore \text{the sum of the roots} = -\frac{0}{1} = 0$$

The polynomial equation has degree 7.

$$\therefore \text{the product of the roots} = \frac{(-1)^7(-9)}{1} = 9$$

b  $3x^3 - 4x^2 + 8x - 5 = 0$

$$\therefore \text{the sum of the roots} = -\frac{(-4)}{3} = \frac{4}{3}$$

The polynomial equation has degree 3.

$$\therefore \text{the product of the roots} = \frac{(-1)^3(-5)}{3} = \frac{5}{3}$$

d  $2x^5 - 3x^4 + x^2 - 8 = 0$

$$\therefore \text{the sum of the roots} = -\frac{(-3)}{2} = \frac{3}{2}$$

The polynomial equation has degree 5.

$$\therefore \text{the product of the roots} = \frac{(-1)^5(-8)}{2} = 4$$

f  $x^6 - 1 = 0$

$$\therefore x^6 + (0)x^5 - 1 = 0$$

$$\therefore \text{the sum of the roots} = -\frac{0}{1} = 0$$

The polynomial equation has degree 6.

$$\therefore \text{the product of the roots} = \frac{(-1)^6(-1)}{1} = -1$$

2 a The zeros of the cubic polynomial  $P(x)$  are  $3 \pm \sqrt{2}$  and  $\frac{2}{3}$ .

$$\begin{aligned} \therefore \text{the sum of the zeros} &= (3 + \sqrt{2}) + (3 - \sqrt{2}) + \frac{2}{3} \\ &= 6 + \frac{2}{3} \\ &= \frac{20}{3} \end{aligned}$$

$$\begin{aligned} \text{The product of the zeros} &= \frac{2}{3} \times (3 + \sqrt{2})(3 - \sqrt{2}) \\ &= \frac{2}{3}(9 - 2) \\ &= \frac{14}{3} \end{aligned}$$

b Let  $P(x) = 6x^3 + ax^2 + bx + c$  {leading coefficient is 6}.

$$\therefore \text{the sum of the zeros is } -\frac{a}{6} = \frac{20}{3}$$

$$\therefore -a = 2 \times 20$$

$$\therefore a = -40$$

So, the coefficient of  $x^2$  is  $-40$ .

c The product of the zeros is  $\frac{(-1)^3 c}{6} = \frac{14}{3}$

$$\therefore -c = 2 \times 14$$

$$\therefore c = -28$$

So, the constant term is  $-28$ .



- 3 a** The zeros of the cubic polynomial  $P(x)$  are  $1 \pm i\sqrt{2}$  and 1.

$$\begin{aligned}\therefore \text{ the sum of the zeros} &= (1 + i\sqrt{2}) + (1 - i\sqrt{2}) + 1 \\ &= 2 + 1 \\ &= 3\end{aligned}$$

$$\begin{aligned}\text{The product of the zeros} &= (1 + i\sqrt{2})(1 - i\sqrt{2}) \\ &= 1 - 2i^2 \\ &= 3\end{aligned}$$

- b** Let  $P(x) = \frac{2}{3}x^3 + ax^2 + bx + c$  {leading coefficient is  $\frac{2}{3}$ }

$$\therefore \text{ the sum of the zeros is } -\frac{a}{(\frac{2}{3})} = 3$$

$$\begin{aligned}\therefore -\frac{3}{2}a &= 3 \\ \therefore a &= -2\end{aligned}$$

So, the coefficient of  $x^2$  is  $-2$ .

**c** The product of the zeros is  $\frac{(-1)^3 c}{(\frac{2}{3})} = 3$

$$\begin{aligned}\therefore -\frac{3}{2}c &= 3 \\ \therefore c &= -2\end{aligned}$$

So, the constant term is  $-2$ .

- 4 a** A polynomial of degree 4 has zeros  $-2$ ,  $3$ , and  $\sqrt{k} \pm 1$ .

The constant term is the  $y$ -intercept.

$\therefore$  the constant term is 18.

$$\therefore \text{ the product of the zeros is } -2(3)(\sqrt{k} + 1)(\sqrt{k} - 1) = \frac{(-1)^4(18)}{(-1)}$$

$$\therefore -6(k - 1) = -18$$

$$\therefore -6k + 6 = -18$$

$$\therefore -6k = -24$$

$$\therefore k = 4$$

- b** Let  $a$  be the coefficient of  $x^3$ .

$$\begin{aligned}\therefore \text{ the sum of the zeros is } -2 + 3 + (\sqrt{4} + 1) + (\sqrt{4} - 1) &= -\frac{a}{(-1)} \\ \therefore a &= -2 + 3 + 3 + 1 \\ \therefore a &= 5\end{aligned}$$

So, the coefficient of  $x^3$  is 5.

- 5** The polynomial of degree 5 has zeros  $\frac{1}{2}$ ,  $1 \pm \sqrt{2}$ , and  $m \pm ni$ .

$$\begin{aligned}\therefore \text{ the sum of the zeros is } \frac{1}{2} + (1 + \sqrt{2}) + (1 - \sqrt{2}) + (m + ni) + (m - ni) &= -\frac{3}{2} \\ \therefore \frac{1}{2} + 2 + 2m &= -\frac{3}{2} \\ \therefore \frac{5}{2} + 2m &= -\frac{3}{2} \\ \therefore 2m &= -4 \\ \therefore m &= -2\end{aligned}$$



The constant term is the  $y$ -intercept.

$\therefore$  the constant term is 5.

$\therefore$  the product of the zeros is  $\frac{1}{2} \times (1 + \sqrt{2})(1 - \sqrt{2})(-2 + ni)(-2 - ni) = \frac{(-1)^5 5}{2}$

$$\therefore \frac{1}{2}(1 - 2)(4 + n^2) = -\frac{5}{2}$$

$$\therefore -\frac{1}{2}(4 + n^2) = -\frac{5}{2}$$

$$\therefore 4 + n^2 = 5$$

$$\therefore n^2 = 1$$

$$\therefore n = 1 \quad \{n > 0\}$$

So,  $m = -2$  and  $n = 1$ .

**6** The quartic polynomial has zeros  $a \pm i$  and  $3 \pm a$ .

$\therefore$  the product of the zeros is  $(a + i)(a - i)(3 + a)(3 - a) = \frac{(-1)^4 25}{1}$

$$\therefore (a^2 + 1)(9 - a^2) = 25$$

$$\therefore -a^4 + 8a^2 + 9 = 25$$

$$\therefore a^4 - 8a^2 + 16 = 0$$

$$\therefore (a^2)^2 - 8(a^2) + 16 = 0$$

$$\therefore (a^2 - 4)^2 = 0$$

$$\therefore a^2 = 4$$

$$\therefore a = \pm 2$$

**7 a**  $x^3 - px^2 + qx - r = 0$  has non-zero roots  $p$ ,  $q$ , and  $r$ .

$\therefore$  the sum of the roots is  $p + q + r = -\frac{(-p)}{1}$

$$\therefore p + q + r = p$$

$$\therefore q + r = 0$$

$$\therefore q = -r$$

$\therefore$  the product of the roots is  $pqr = \frac{(-1)^3(-r)}{1}$

$$\therefore pqr = r$$

$$\therefore pq = 1$$

$$\therefore p = \frac{1}{q}$$

$$\therefore p = -\frac{1}{r} \quad \{\text{as } q = -r\}$$

So,  $q = -r$  and  $p = -\frac{1}{r}$ , as required.

**b** By the Factor theorem, if  $p$ ,  $q$ , and  $r$  are zeros of the polynomial, then  $(x - p)$ ,  $(x - q)$ , and  $(x - r)$  are factors.

$$\begin{aligned} \therefore x^3 - px^2 + qx - r &= (x - p)(x - q)(x - r) \\ &= (x^2 - (p + q)x + pq)(x - r) \\ &= x^3 - (p + q + r)x^2 + (pq + pr + qr)x - pqr \end{aligned}$$

Equating coefficients gives  $pq + pr + qr = q$

$$\therefore \left(-\frac{1}{r}\right)(-r) + \left(-\frac{1}{r}\right)r + (-r)r = -r \quad \{\text{using a}\}$$

$$\therefore 1 - 1 - r^2 = -r$$

$$\therefore r^2 - r = 0$$

$$\therefore r(r - 1) = 0$$

$$\therefore r = 0 \text{ or } r = 1$$

$$\therefore r = 1 \quad \{r \neq 0\}$$

$$\therefore p = -\frac{1}{r} = -1 \text{ and } q = -r = -1$$

So,  $p = -1$ ,  $q = -1$ , and  $r = 1$ .

**8** Suppose the common root is  $\alpha$  and the other roots are  $\beta$  and  $\gamma$ .

$$\therefore x^2 + ax + bc = (x - \alpha)(x - \beta) \text{ and } x^2 + bx + ca = (x - \alpha)(x - \gamma)$$

$$\text{Thus } \alpha + \beta = -a \text{ and } \alpha + \gamma = -b \text{ so } (\alpha + \beta)(\alpha + \gamma) = ab \quad \dots (1)$$

$$\text{Also, } \alpha\beta = bc \text{ and } \alpha\gamma = ca \text{ so } \alpha^2\beta\gamma = abc^2 \quad \dots (2)$$

$$\text{Now } \alpha \text{ is a common root of both equations } \therefore \alpha^2 + a\alpha + bc = 0 \quad \dots (3) \text{ and}$$

$$\alpha^2 + b\alpha + ca = 0 \quad \dots (4)$$

$$\text{Subtracting (4) from (3), } (a - b)\alpha - (a - b)c = 0$$

$$\therefore (a - b)(\alpha - c) = 0$$

$$\therefore a = b \text{ or } \alpha = c$$

But if  $a = b$ , both equations are the same, and so the equations would have two common roots.

$$\therefore \alpha = c \quad \dots (5)$$

$$\text{Using (2), } \alpha^2\beta\gamma = abc^2$$

$$\therefore c^2\beta\gamma = abc^2$$

$$\therefore \beta\gamma = ab \quad \dots (6)$$

$$\text{Using (1), } \alpha^2 + \alpha(\beta + \gamma) + \beta\gamma = ab$$

$$\therefore c^2 + c(\beta + \gamma) = 0 \quad \{\text{using (5), (6)}\}$$

$$\therefore \beta + \gamma = -c \quad \dots (7)$$

From (6) and (7),  $\beta$  and  $\gamma$  are the roots of  $x^2 + cx + ab = 0$ , as required.

**9 a**  $f(x)$  has degree  $n$ .

$$\therefore g(x) = [f(x)]^2 \text{ has degree } 2n.$$

**b** The sum of the roots of  $f(x)$  is  $-\frac{a_{n-1}}{a_n} = 5$

$$\therefore -a_{n-1} = 5 \quad \{a_n = 1\}$$

$$\therefore a_{n-1} = -5$$

$$\text{So, } f(x) = x^n - 5x^{n-1} + \dots$$

$$\therefore g(x) = (x^n - 5x^{n-1} + \dots)(x^n - 5x^{n-1} + \dots)$$

$$= x^{2n} - 5x^{2n-1} - 5x^{2n-1} + \dots$$

$$= x^{2n} - 10x^{2n-1} + \dots$$

$$\therefore \text{the sum of the roots of } g(x) \text{ is } \frac{-(-10)}{1} = 10.$$

- c The product of the roots of  $f(x)$  is  $\frac{(-1)^n a_0}{a_n} = -3$   
 $\therefore (-1)^n a_0 = -3 \quad \{a_n = 1\}$   
 $\therefore a_0 = 3 \text{ or } -3$

If  $a_0 = 3$ ,  $g(x) = (x^n - 5x^{n-1} + \dots + 3)(x^n - 5x^{n-1} + \dots + 3)$   
 $= x^{2n} - 10x^{2n-1} + \dots + 9$

If  $a_0 = -3$ ,  $g(x) = (x^n - 5x^{n-1} + \dots - 3)(x^n - 5x^{n-1} + \dots - 3)$   
 $= x^{2n} - 10x^{2n-1} + \dots + 9$

In either case, the constant term of  $g(x)$  is 9.

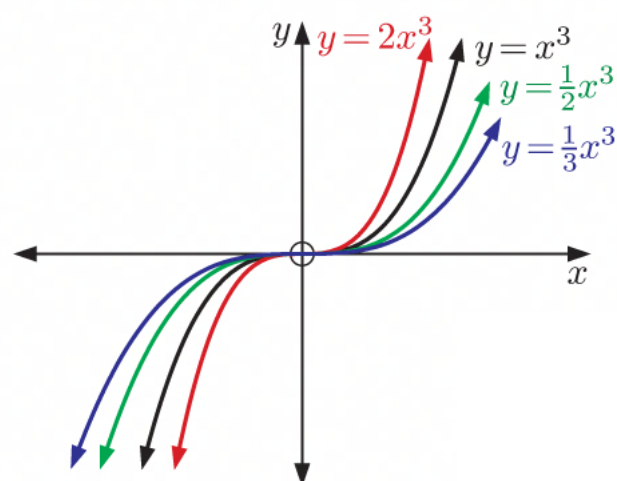
$\therefore$  the product of the roots of  $g(x)$  is  $\frac{(-1)^{2n}(9)}{1} = 9 \quad \{(-1)^{2n} = 1 \text{ for all } n \in \mathbb{Z}\}$

- d The constant term of  $g(x)$  is the  $y$ -intercept.  
 $\therefore$  the  $y$ -intercept of  $y = g(x)$  is 9.

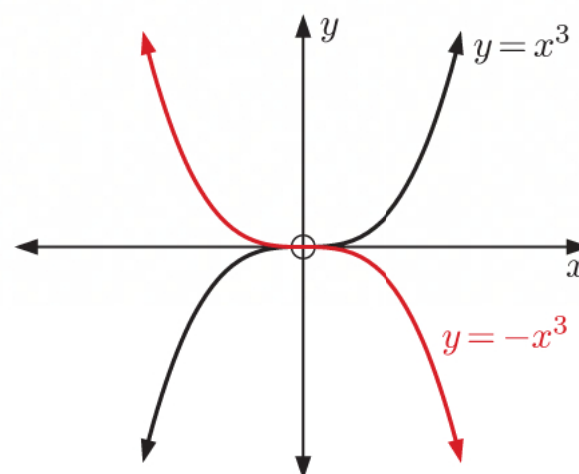
## INVESTIGATION 2

## FAMILIES OF CUBICS

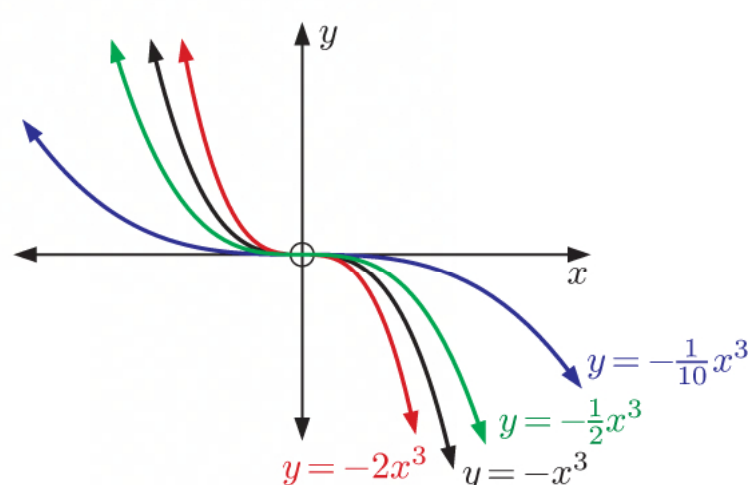
1 a i



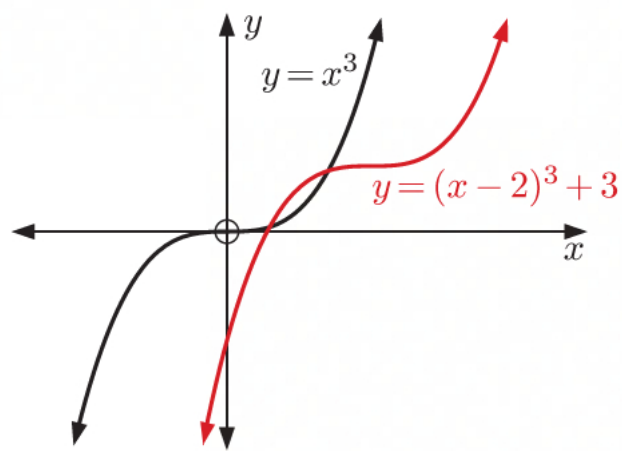
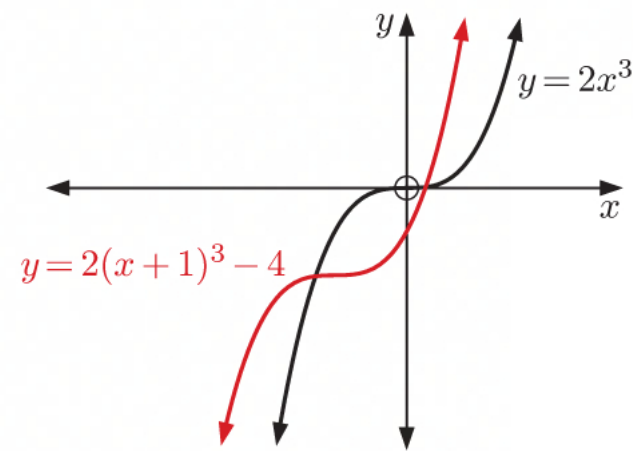
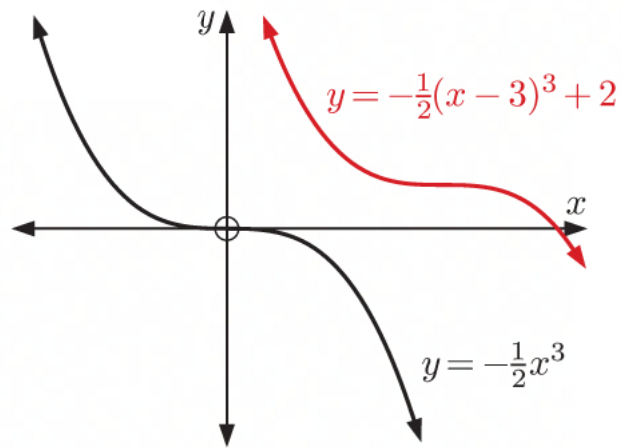
ii



iii



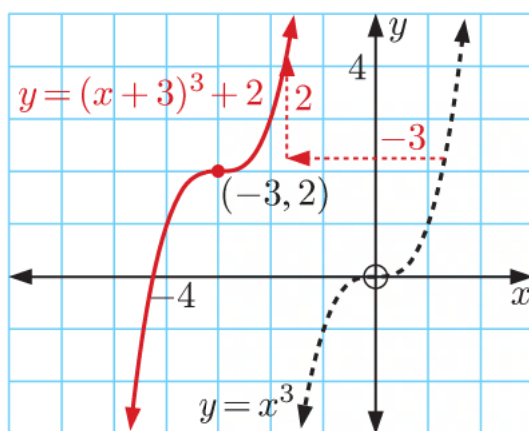
- b
- The graph of  $y = ax^3$  has shape:
    - if  $a > 0$
    - if  $a < 0$ .
  - As the size of  $a$  increases, the graph of  $y = ax^3$  becomes steeper.

**2 a i****ii****iii**

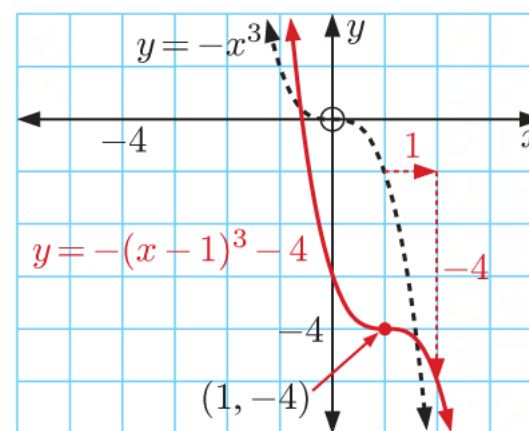
- b** • The graph of  $y = a(x-b)^3 + c$  is obtained by translating the graph of  $y = ax^3$  by the vector  $\begin{pmatrix} b \\ c \end{pmatrix}$ .

**EXERCISE 5J**

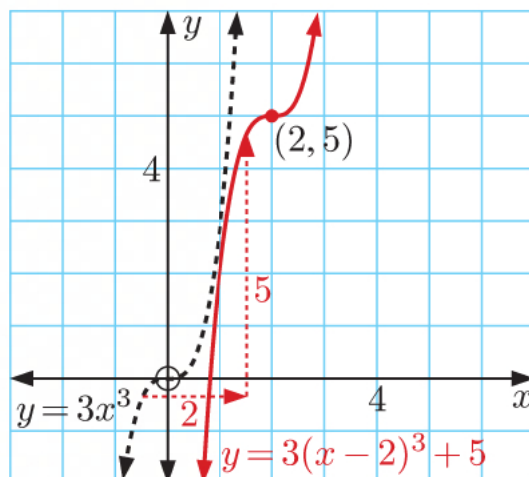
- 1 a** We translate  $y = x^3$  3 units left and 2 units up.



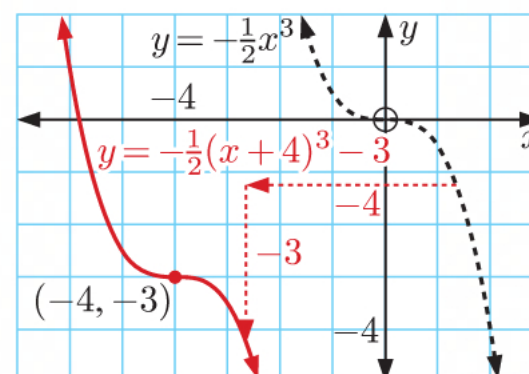
- b** We translate  $y = -x^3$  1 unit right and 4 units down.



- c** We translate  $y = 3x^3$  2 units right and 5 units up.



- d** We translate  $y = -\frac{1}{2}x^3$  4 units to the left and 3 units down.

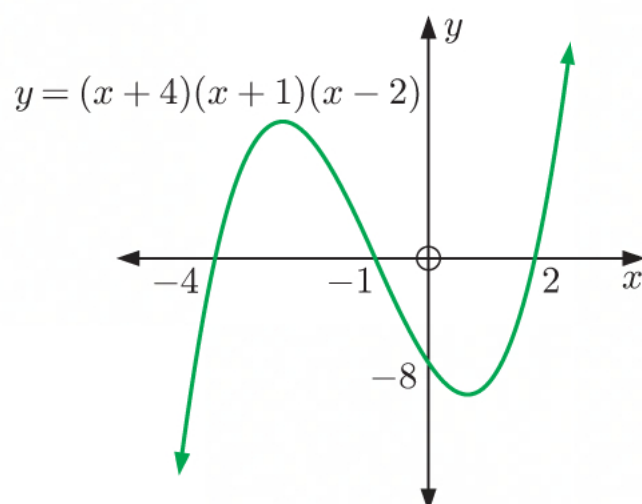




- 2 a** The graph cuts the  $x$ -axis at  $-4$ ,  $-1$ , and  $2$ .

When  $x = 0$ ,  $y = (4)(1)(-2) = -8$

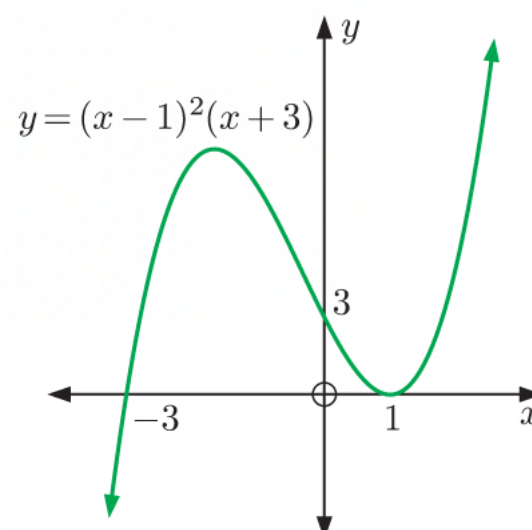
$\therefore$  the  $y$ -intercept is  $-8$ .



- b** The graph touches the  $x$ -axis at  $1$ , and cuts the  $x$ -axis at  $-3$ .

When  $x = 0$ ,  $y = (-1)^2(3) = 3$

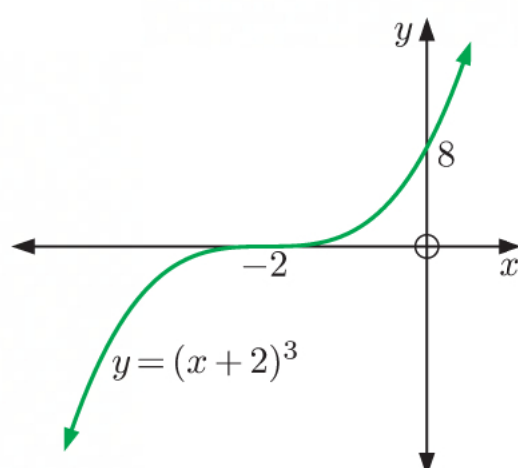
$\therefore$  the  $y$ -intercept is  $3$ .



- c** The graph is horizontal at  $x = -2$ .

When  $x = 0$ ,  $y = 2^3 = 8$

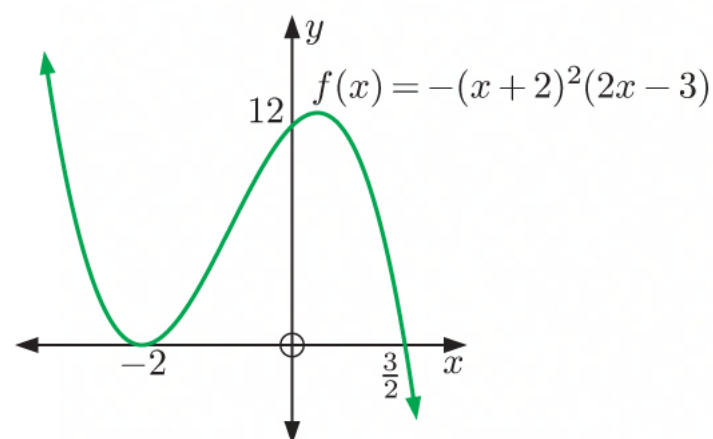
$\therefore$  the  $y$ -intercept is  $8$ .



- d** The graph touches the  $x$ -axis at  $-2$ , and cuts the  $x$ -axis at  $\frac{3}{2}$ .

When  $x = 0$ ,  $f(x) = -(2)^2(-3) = 12$

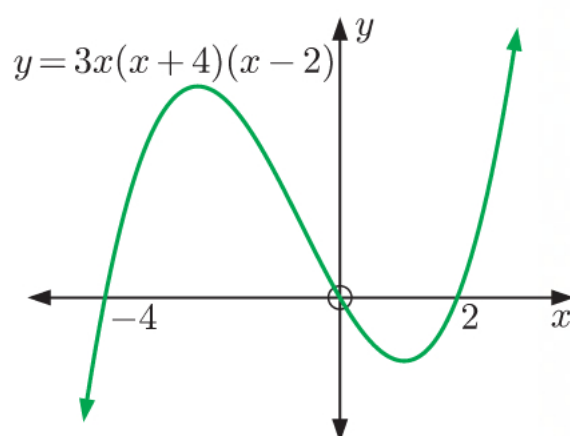
$\therefore$  the  $y$ -intercept is  $12$ .



- e** The graph cuts the  $x$ -axis at  $0$ ,  $-4$ , and  $2$ .

When  $x = 0$ ,  $y = 0(4)(-2) = 0$

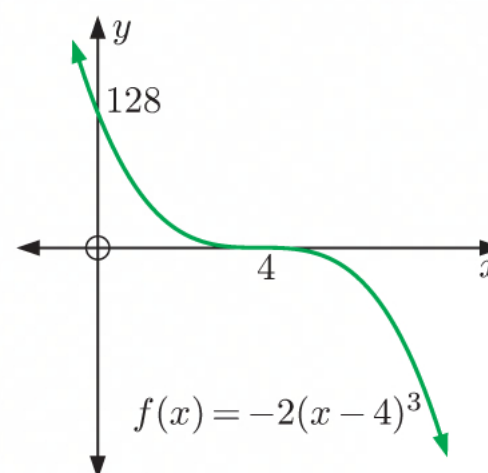
$\therefore$  the  $y$ -intercept is  $0$ .



- f** The graph is horizontal at  $x = 4$ .

When  $x = 0$ ,  $f(x) = -2(-4)^3 = 128$

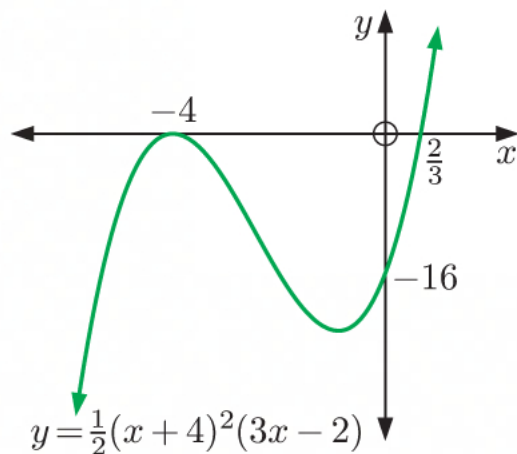
$\therefore$  the  $y$ -intercept is  $128$ .



- g** The graph touches the  $x$ -axis at  $-4$ , and cuts the  $x$ -axis at  $\frac{2}{3}$ .

$$\text{When } x = 0, \quad y = \frac{1}{2}(4)^2(-2) = -16$$

$\therefore$  the  $y$ -intercept is  $-16$ .

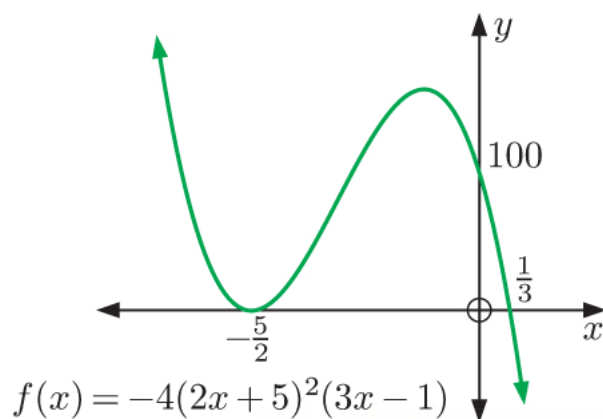


- i** The graph touches the  $x$ -axis at  $-\frac{5}{2}$ , and cuts the  $x$ -axis at  $\frac{1}{3}$ .

When  $x = 0$ ,

$$f(x) = -4(5)^2(-1) = 100$$

$\therefore$  the  $y$ -intercept is  $100$ .



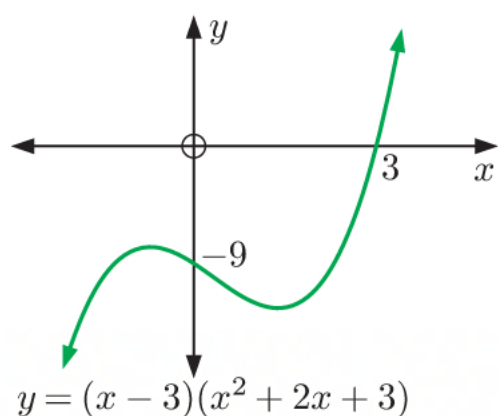
- k**  $y = (x-3)(x^2+2x+3)$   
 $x^2+2x+3$  has  $\Delta = (2)^2 - 4(1)(3)$   
 $= 4 - 12$   
 $= -8$

$\therefore$  there is only one  $x$ -intercept.

The graph cuts the  $x$ -axis at  $3$ .

$$\text{When } x = 0, \quad y = (-3)(3) = -9$$

$\therefore$  the  $y$ -intercept is  $-9$ .

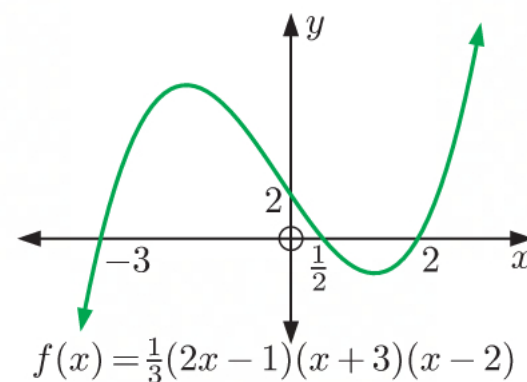


- h** The graph cuts the  $x$ -axis at  $\frac{1}{2}$ ,  $-3$ , and  $2$ .

When  $x = 0$ ,

$$f(x) = \frac{1}{3}(-1)(3)(-2) = 2$$

$\therefore$  the  $y$ -intercept is  $2$ .



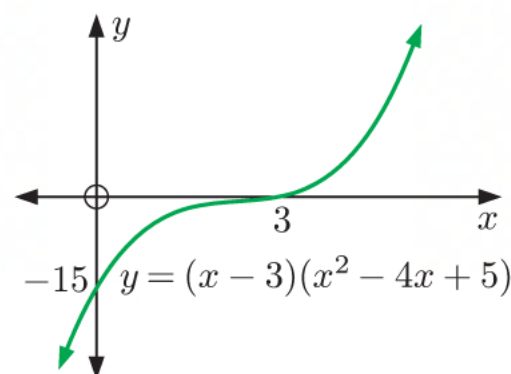
- j**  $y = (x-3)(x^2-4x+5)$   
 $x^2-4x+5$  has  $\Delta = (-4)^2 - 4(1)(5)$   
 $= 16 - 20$   
 $= -4$

$\therefore$  there is only one  $x$ -intercept.

The graph cuts the  $x$ -axis at  $3$ .

$$\text{When } x = 0, \quad y = (-3)(5) = -15$$

$\therefore$  the  $y$ -intercept is  $-15$ .

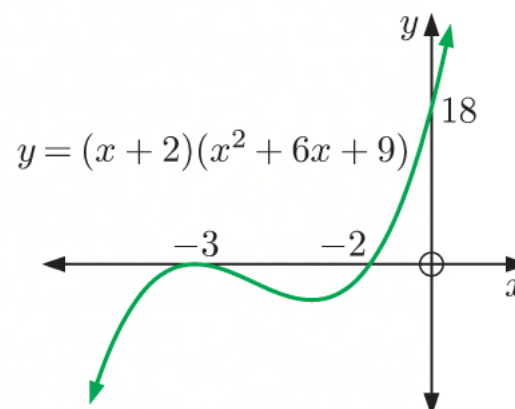


- l**  $y = (x+2)(x^2+6x+9)$   
 $= (x+2)(x+3)^2$

The graph touches the  $x$ -axis at  $-3$ , and cuts the  $x$ -axis at  $-2$ .

$$\text{When } x = 0, \quad y = (2)(3)^2 = 18$$

$\therefore$  the  $y$ -intercept is  $18$ .



- 3 a** The  $x$ -intercepts are  $-1$ ,  $2$ , and  $3$ .

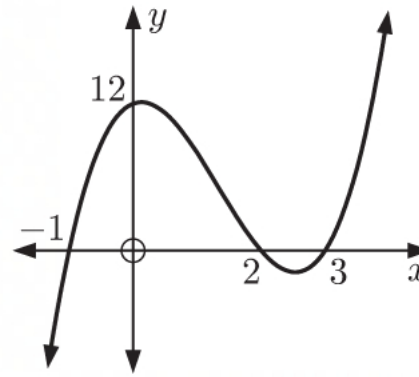
$$\therefore y = a(x+1)(x-2)(x-3)$$

But when  $x = 0$ ,  $y = 12$

$$\therefore a(1)(-2)(-3) = 12$$

$$\therefore a = 2$$

So,  $y = 2(x+1)(x-2)(x-3)$



- b** The graph touches the  $x$ -axis at  $3$ , indicating a squared factor  $(x-3)^2$ .

The other  $x$ -intercept is  $-1$ ,

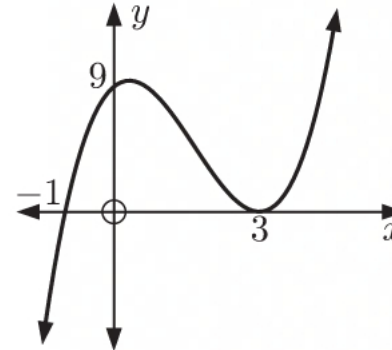
so  $y = a(x+1)(x-3)^2$ .

But when  $x = 0$ ,  $y = 9$

$$\therefore a(1)(-3)^2 = 9$$

$$\therefore a = 1$$

So,  $y = (x+1)(x-3)^2$



- c** The  $x$ -intercepts are  $-4$ ,  $-2$ , and  $1$ .

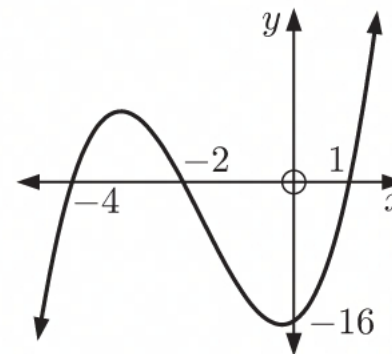
$$\therefore y = a(x+4)(x+2)(x-1)$$

But when  $x = 0$ ,  $y = -16$

$$\therefore a(4)(2)(-1) = -16$$

$$\therefore a = 2$$

So,  $y = 2(x+4)(x+2)(x-1)$



- d** The  $x$ -intercepts are  $-3$ ,  $-\frac{1}{2}$ , and  $\frac{1}{2}$ .

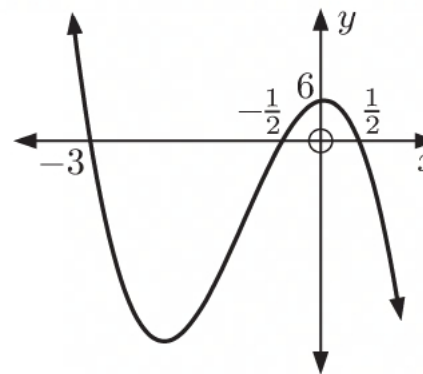
$$\therefore y = a(x+3)(2x+1)(2x-1)$$

But when  $x = 0$ ,  $y = 6$

$$\therefore a(3)(1)(-1) = 6$$

$$\therefore a = -2$$

So,  $y = -2(x+3)(2x+1)(2x-1)$



- e** The graph touches the  $x$ -axis at  $-4$ , indicating a squared factor  $(x+4)^2$ .

The other  $x$ -intercept is  $3$ ,

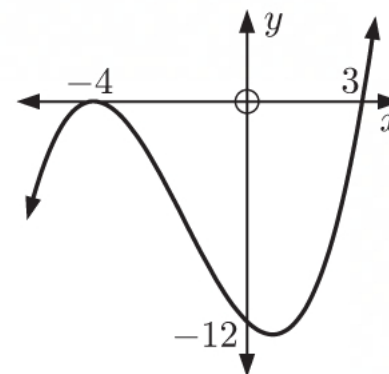
so  $y = a(x+4)^2(x-3)$ .

But when  $x = 0$ ,  $y = -12$

$$\therefore a(4)^2(-3) = -12$$

$$\therefore a = \frac{1}{4}$$

So,  $y = \frac{1}{4}(x+4)^2(x-3)$



**f** The  $x$ -intercepts are  $-5$ ,  $-2$ , and  $5$ .

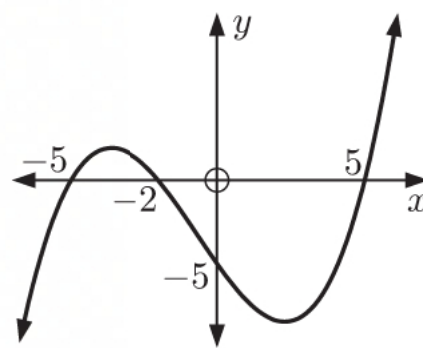
$$\therefore y = a(x + 5)(x + 2)(x - 5)$$

But when  $x = 0$ ,  $y = -5$

$$\therefore a(5)(2)(-5) = -5$$

$$\therefore a = \frac{1}{10}$$

$$\text{So, } y = \frac{1}{10}(x + 5)(x + 2)(x - 5)$$



**g** The graph is horizontal on the  $x$ -axis at 2, indicating a cubed factor  $(x - 2)^3$ .

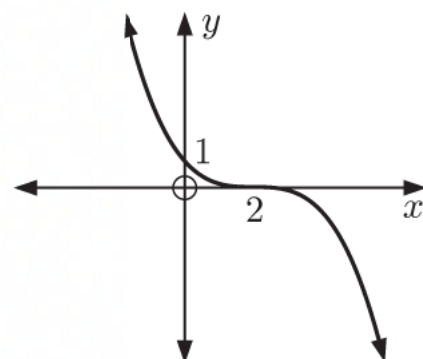
$$\therefore y = a(x - 2)^3$$

But when  $x = 0$ ,  $y = 1$

$$\therefore a(-2)^3 = 1$$

$$\therefore a = -\frac{1}{8}$$

$$\text{So, } y = -\frac{1}{8}(x - 2)^3$$



**h** The graph touches the  $x$ -axis at 3, indicating a squared factor  $(x - 3)^2$ .

The other  $x$ -intercept is  $-4$ ,

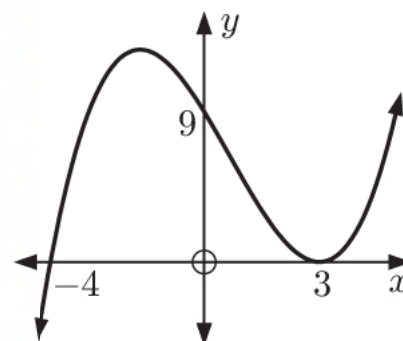
$$\text{so } y = a(x - 3)^2(x + 4).$$

But when  $x = 0$ ,  $y = 9$

$$\therefore a(-3)^2(4) = 9$$

$$\therefore a = \frac{1}{4}$$

$$\text{So, } y = \frac{1}{4}(x + 4)(x - 3)^2$$



**i** The  $x$ -intercepts are  $-3$ ,  $-2$ , and  $-\frac{1}{2}$ .

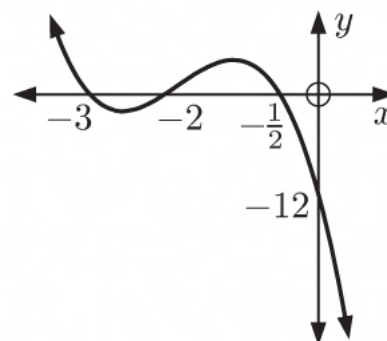
$$\therefore y = a(x + 3)(x + 2)(2x + 1)$$

But when  $x = 0$ ,  $y = -12$

$$\therefore a(3)(2)(1) = -12$$

$$\therefore a = -2$$

$$\text{So, } y = -2(x + 3)(x + 2)(2x + 1)$$



**4 a**  $-2$  is an  $x$ -intercept of the graph of  $f(x) = 2x^3 - 3x^2 - 11x + 6$

$\Leftrightarrow x = -2$  is a solution of  $f(x) = 0$ .

$$\begin{aligned} \text{Now, } f(-2) &= 2(-2)^3 - 3(-2)^2 - 11(-2) + 6 \\ &= -16 - 12 + 22 + 6 = 0 \quad \checkmark \end{aligned}$$

So,  $-2$  is an  $x$ -intercept of the graph of  $f(x)$ .

**b** Since  $x = -2$  is a solution to  $f(x) = 0$ , then  $x + 2$  is a factor of  $f(x)$ .

$$\begin{aligned} \therefore 2x^3 - 3x^2 - 11x + 6 &= (x + 2)(2x^2 + bx + 3) \\ &= 2x^3 + bx^2 + 3x \\ &\quad + 4x^2 + 2bx + 6 \\ &= 2x^3 + (b + 4)x^2 + (3 + 2b)x + 6 \end{aligned}$$

Equating coefficients:  $b + 4 = -3$  and  $3 + 2b = -11$

$$\therefore b = -7$$

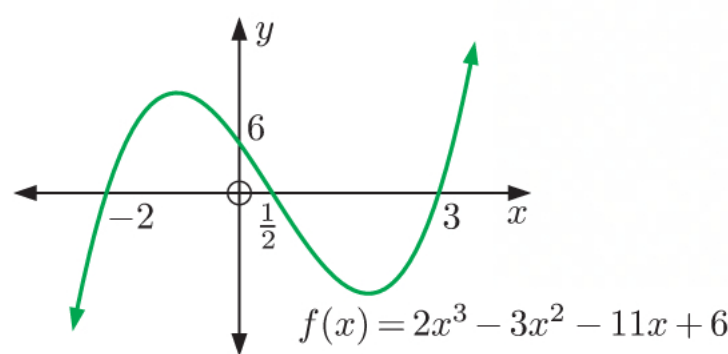
$$\begin{aligned} \therefore 2x^3 - 3x^2 - 11x + 6 &= (x + 2)(2x^2 - 7x + 3) \\ &= (x + 2)(2x - 1)(x - 3) \end{aligned}$$



- c** The  $x$ -intercepts are  $-2$ ,  $\frac{1}{2}$ , and  $3$ .

When  $x = 0$ ,  $f(x) = 6$

$\therefore$  the  $y$ -intercept is  $6$ .



- d** The domain is  $\{x \mid x \in \mathbb{R}\}$ .

The range is  $\{y \mid y \in \mathbb{R}\}$ .

**5 a**  $y = a(x - 3)(x - 1)(x + 2)$

But when  $x = 2$ ,  $y = -4$

$$\therefore a(-1)(1)(4) = -4$$

$$\therefore -4a = -4$$

$$\therefore a = 1$$

$$\therefore y = (x - 3)(x - 1)(x + 2)$$

**c**  $y = a(x - 1)^2(x + 2)$

But when  $x = 4$ ,  $y = 54$

$$\therefore a(3)^2(6) = 54$$

$$\therefore 54a = 54$$

$$\therefore a = 1$$

$$\therefore y = (x - 1)^2(x + 2)$$

**e** as  $x \rightarrow \infty$ ,  $y \rightarrow \infty$

as  $x \rightarrow -\infty$ ,  $y \rightarrow -\infty$

**b**  $y = ax(x + 2)(2x - 1)$

But when  $x = -1$ ,  $y = 9$

$$\therefore a(-1)(1)(-3) = 9$$

$$\therefore 3a = 9$$

$$\therefore a = 3$$

$$\therefore y = 3x(x + 2)(2x - 1)$$

**d**  $y = a(3x + 2)^2(x - 4)$

But when  $x = 1$ ,  $y = 25$

$$\therefore a(5)^2(-3) = 25$$

$$\therefore -75a = 25$$

$$\therefore a = -\frac{1}{3}$$

$$\therefore y = -\frac{1}{3}(3x + 2)^2(x - 4)$$

**6 a**  $1$  is a zero of  $-2x^3 - 2x^2 + 10x - 6$

$\therefore x - 1$  is a factor of  $-2x^3 - 2x^2 + 10x - 6$

$$\therefore -2x^3 - 2x^2 + 10x - 6 = (x - 1)(-2x^2 + bx + 6)$$

$$= -2x^3 + bx^2 + 6x$$

$$+ 2x^2 - bx - 6$$

$$= -2x^3 + (b + 2)x^2 + (6 - b)x - 6$$

Equating coefficients:  $b + 2 = -2$  and  $6 - b = 10$

$$\therefore b = -4$$

$$\therefore -2x^3 - 2x^2 + 10x - 6 = (x - 1)(-2x^2 - 4x + 6)$$

$$= -2(x - 1)(x^2 + 2x - 3)$$

$$= -2(x - 1)(x + 3)(x - 1)$$

$$= -2(x + 3)(x - 1)^2$$

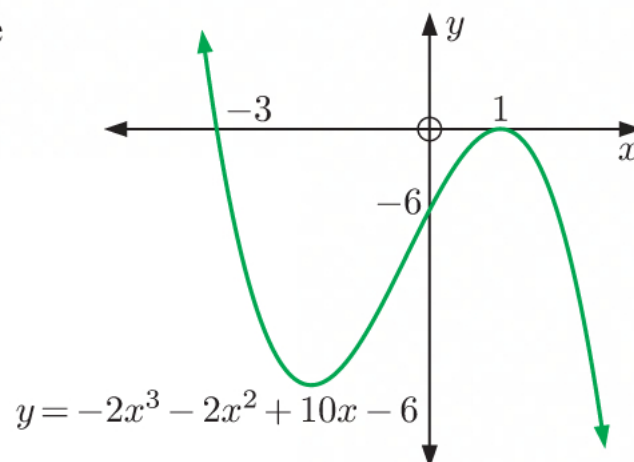
- b** The graph touches the  $x$ -axis at  $1$ , and cuts the  $x$ -axis at  $-3$ .

When  $x = 0$ ,  $y = -2(-1)^2(3) = -6$

$\therefore$  the  $y$ -intercept is  $-6$ .

**c** as  $x \rightarrow \infty$ ,  $y \rightarrow -\infty$

as  $x \rightarrow -\infty$ ,  $y \rightarrow \infty$



- 7 a**  $\frac{1}{2}$  and  $-3$  are zeros of a cubic polynomial, so  $y = (2x - 1)(x + 3)(ax + b)$ ,  $a \neq 0$ .

When  $x = 0$ ,  $y = 30$

$$\therefore (-1)(3)b = 30$$

$$\therefore b = -10$$

When  $x = 1$ ,  $y = -20$

$$\therefore (1)(4)(a - 10) = -20$$

$$\therefore a - 10 = -5$$

$$\therefore a = 5$$

So, the cubic has equation  $y = (2x - 1)(x + 3)(5x - 10)$

which we can write as  $y = 5(2x - 1)(x + 3)(x - 2)$

- b** The zero at  $-2$  touches the  $x$ -axis, so  $y = (x + 2)^2(ax + b)$ ,  $a \neq 0$ .

When  $x = 0$ ,  $y = 8$

$$\therefore (2)^2b = 8$$

$$\therefore b = 2$$

When  $x = -1$ ,  $y = 4$

$$\therefore (1)^2(-a + 2) = 4$$

$$\therefore 2 - a = 4$$

$$\therefore a = -2$$

So, the cubic has equation  $y = (x + 2)^2(-2x + 2)$

which we can write as  $y = -2(x + 2)^2(x - 1)$

- c** There is a zero at  $2$ , so  $y = (x - 2)(ax^2 + bx + c)$ ,  $a \neq 0$ .

When  $x = 0$ ,  $y = -4$

$$\therefore (-2)c = -4$$

$$\therefore c = 2$$

When  $x = 1$ ,  $y = -1$

$$\therefore (-1)(a + b + 2) = -1$$

$$\therefore a + b + 2 = 1$$

$$\therefore a + b = -1 \quad \dots (1)$$

When  $x = -1$ ,  $y = -21$

$$\therefore (-3)(a - b + 2) = -21$$

$$\therefore a - b + 2 = 7$$

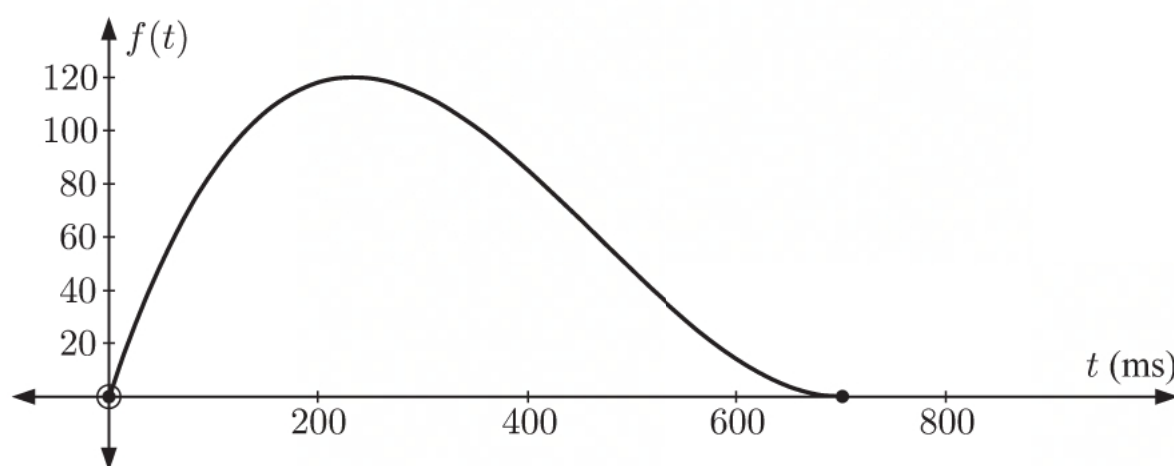
$$\therefore a - b = 5 \quad \dots (2)$$

Adding (1) and (2) gives  $2a = 4$

$$\therefore a = 2 \text{ and so } b = -3$$

So, the cubic has equation  $y = (x - 2)(2x^2 - 3x + 2)$

- 8 a**  $f(t) = kt(t - a)^2$



From the graph,  $a$  is the  $t$ -value at the point where the graph touches the  $t$ -axis.

$\therefore a = 700$  milliseconds

This represents the time when the barrier has returned to its original position.

**b** When  $t = 100$  ms,  $f(t) = 85$  mm

$$\therefore 85 = k \times 100(100 - 700)^2$$

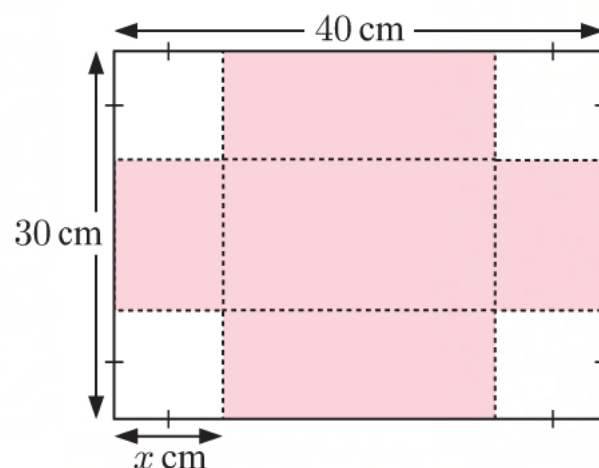
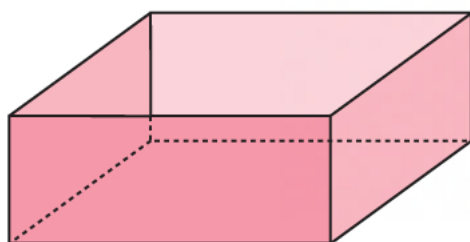
$$\therefore 85 = 100 \times k \times 360\,000$$

$$\therefore k = \frac{85}{36\,000\,000} = \frac{17}{7\,200\,000}$$

$$\therefore f(t) = \frac{17}{7\,200\,000} t(t - 700)^2$$

**9 a** volume of container = area of base  $\times$  height

$$\begin{aligned}\therefore V &= (40 - 2x) \times (30 - 2x) \times x \\ &= x(40 - 2x)(30 - 2x) \text{ cm}^3\end{aligned}$$



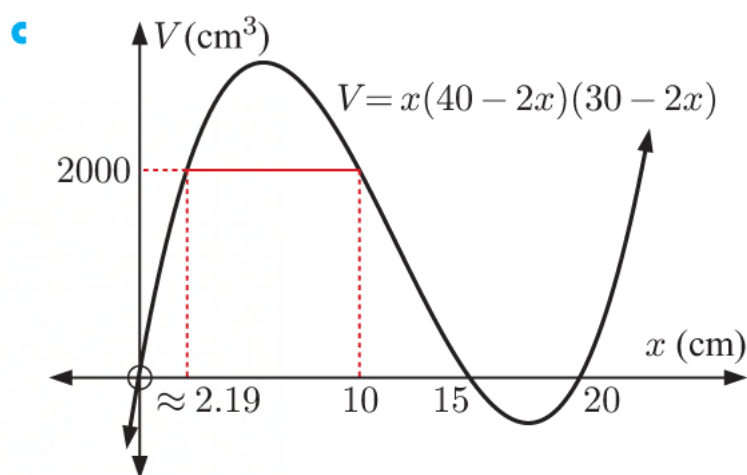
**b** Each of the dimensions of the container must be  $> 0$  in order for the container to exist.

$$\therefore x > 0 \quad 40 - 2x > 0 \quad \text{and} \quad 30 - 2x > 0$$

$$\therefore 2x < 40 \quad \text{and} \quad 2x < 30$$

$$\therefore x < 20 \quad \text{and} \quad x < 15$$

$$\therefore 0 < x < 15$$



**d** Using technology, when  $V = 2000$ ,  
 $x = 10$  or  $x \approx 2.19$

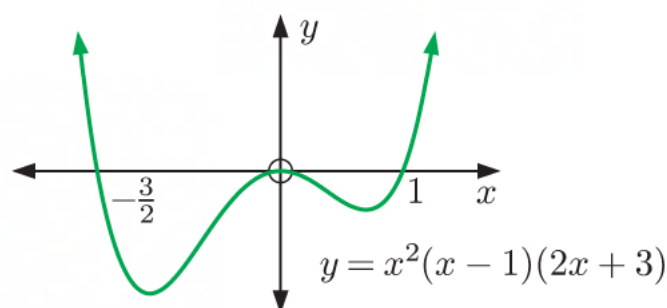
$\therefore$  the squares should be either  
10 cm  $\times$  10 cm, or  
 $\approx 2.19$  cm  $\times$   $\approx 2.19$  cm.

## EXERCISE 5K

**1 a**  $a > 0$ , so the graph opens upwards.  
The graph touches the  $x$ -axis at 0, and cuts the  $x$ -axis at 1 and  $-\frac{3}{2}$ .

$$\text{When } x = 0, \quad y = 0^2(-1)(3) = 0$$

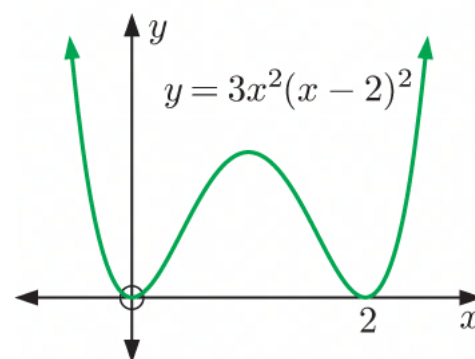
$\therefore$  the  $y$ -intercept is 0.



**b**  $a > 0$ , so the graph opens upwards.  
The graph touches the  $x$ -axis at 0 and 2.

$$\text{When } x = 0, \quad y = 3(0)^2(-2)^2 = 0$$

$\therefore$  the  $y$ -intercept is 0.

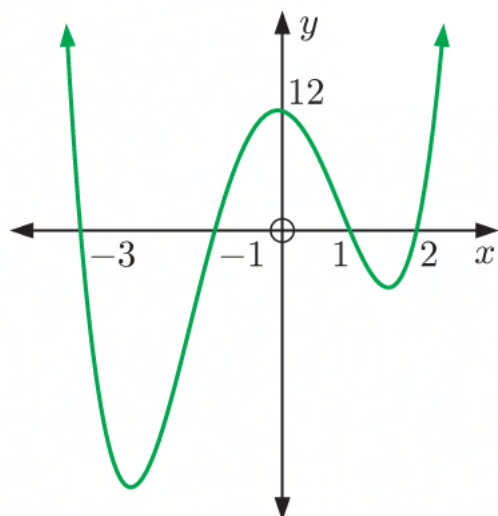


- c**  $a > 0$ , so the graph opens upwards.  
The graph cuts the  $x$ -axis at  $-3$ ,  $-1$ ,  $1$ , and  $2$ .

When  $x = 0$ ,

$$f(x) = 2(3)(1)(-1)(-2) = 12$$

$\therefore$  the  $y$ -intercept is  $12$ .

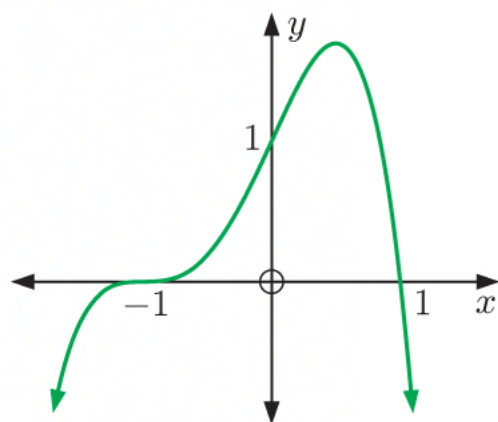


$$f(x) = 2(x+3)(x+1)(x-1)(x-2)$$

- e**  $a < 0$ , so the graph opens downwards.  
The graph cuts the  $x$ -axis at  $1$ , and is “flat” at  $x = -1$ .

When  $x = 0$ ,  $f(x) = -(-1)(1)^3 = 1$

$\therefore$  the  $y$ -intercept is  $1$ .



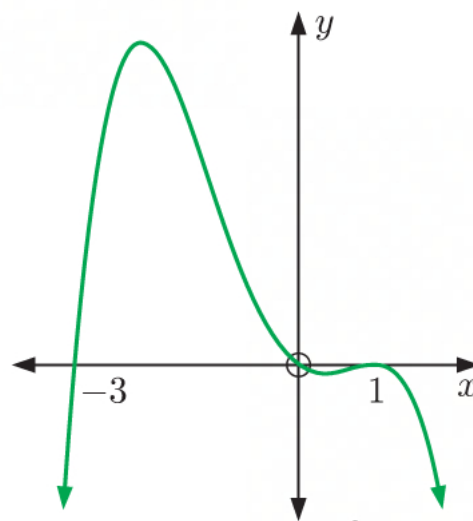
$$f(x) = -(x-1)(x+1)^3$$

- d**  $a < 0$ , so the graph opens downwards.  
The graph cuts the  $x$ -axis at  $0$  and  $-3$ , and touches the  $x$ -axis at  $1$ .

When  $x = 0$ ,

$$y = -2(0)(-1)^2(3) = 0$$

$\therefore$  the  $y$ -intercept is  $0$ .



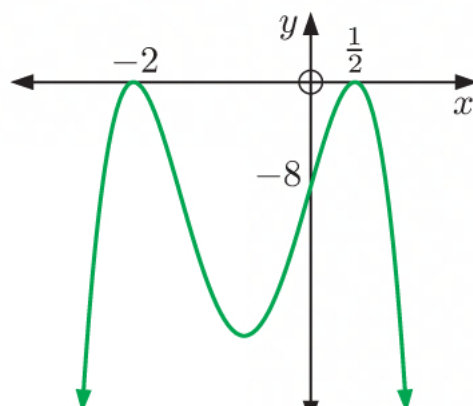
$$y = -2x(x-1)^2(x+3)$$

- f**  $a < 0$ , so the graph opens downwards.  
The graph touches the  $x$ -axis at  $-2$  and  $\frac{1}{2}$ .

When  $x = 0$ ,

$$f(x) = -2(2)^2(-1)^2 = -8$$

$\therefore$  the  $y$ -intercept is  $-8$ .



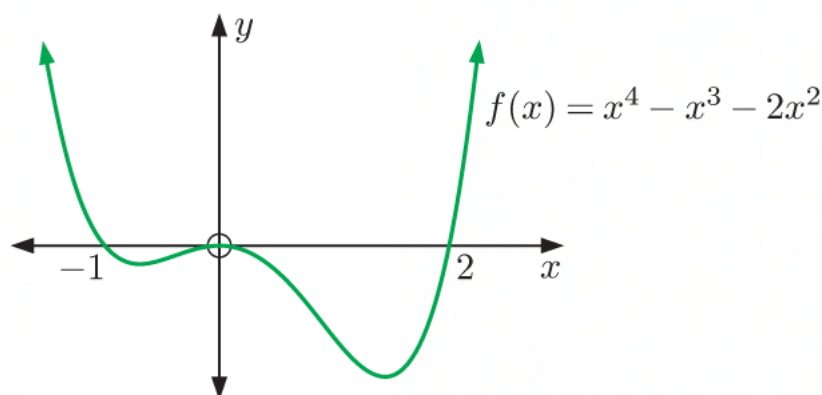
$$f(x) = -2(x+2)^2(2x-1)^2$$

**2 a**  $x^4 - x^3 - 2x^2 = x^2(x^2 - x - 2)$   
 $= x^2(x-2)(x+1)$

- b**  $a > 0$ , so the graph opens upwards.  
The graph touches the  $x$ -axis at  $0$ , and cuts the  $x$ -axis at  $2$  and  $-1$ .

When  $x = 0$ ,  $f(x) = 0^2(-2)(1) = 0$

$\therefore$  the  $y$ -intercept is  $0$ .





- 3 a** The graph touches the  $x$ -axis at  $-1$  and  $1$ .

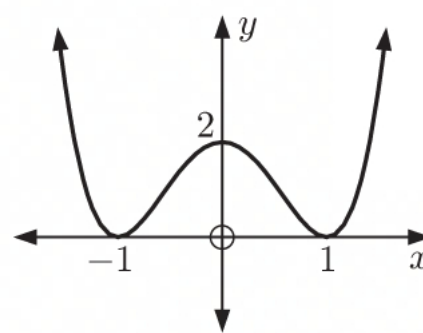
$$\therefore y = a(x+1)^2(x-1)^2, \quad a \neq 0$$

But when  $x = 0$ ,  $y = 2$

$$\therefore 2 = a(1)^2(-1)^2$$

$$\therefore 2 = a$$

$$\therefore y = 2(x+1)^2(x-1)^2$$



- b** The graph touches the  $x$ -axis at  $-1$ , and cuts the  $x$ -axis at  $-3$  and  $\frac{2}{3}$ .

$$\therefore y = a(x+3)(x+1)^2(3x-2), \quad a \neq 0$$

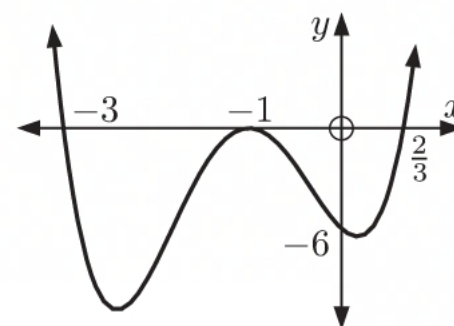
But when  $x = 0$ ,  $y = -6$

$$\therefore -6 = a(3)(1)^2(-2)$$

$$\therefore -6 = -6a$$

$$\therefore a = 1$$

$$\therefore y = (x+3)(x+1)^2(3x-2)$$



- c** The graph touches the  $x$ -axis at  $2$ , and cuts the  $x$ -axis at  $-2$  and  $-1$ .

$$\therefore y = a(x+2)(x+1)(x-2)^2, \quad a \neq 0$$

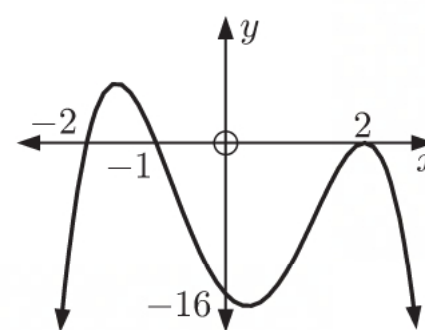
But when  $x = 0$ ,  $y = -16$

$$\therefore -16 = a(-2)^2(2)(1)$$

$$\therefore -16 = 8a$$

$$\therefore a = -2$$

$$\therefore y = -2(x+2)(x+1)(x-2)^2$$



- d** The graph cuts the  $x$ -axis at  $-3$ ,  $-1$ ,  $\frac{3}{2}$ , and  $3$ .

$$\therefore y = a(x+3)(x+1)(2x-3)(x-3), \quad a \neq 0$$

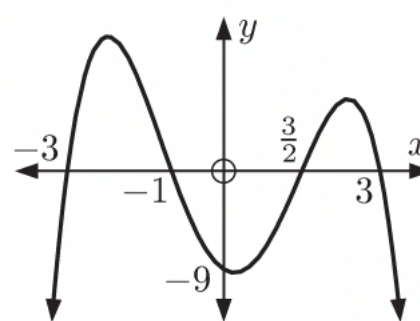
But when  $x = 0$ ,  $y = -9$

$$\therefore -9 = a(3)(1)(-3)(-3)$$

$$\therefore -9 = 27a$$

$$\therefore a = -\frac{1}{3}$$

$$\therefore y = -\frac{1}{3}(x+3)(x+1)(2x-3)(x-3)$$



- e** The graph cuts the  $x$ -axis at  $-1$ , and is horizontal at  $x = 4$ .

$$\therefore y = a(x+1)(x-4)^3, \quad a \neq 0$$

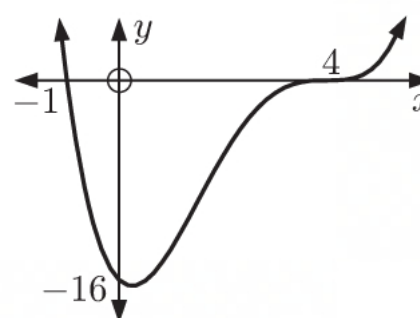
But when  $x = 0$ ,  $y = -16$

$$\therefore -16 = a(1)(-4)^3$$

$$\therefore -16 = -64a$$

$$\therefore a = \frac{1}{4}$$

$$\therefore y = \frac{1}{4}(x+1)(x-4)^3$$



- f** The graph touches the  $x$ -axis at 0, and cuts the  $x$ -axis at  $-2$  and  $3$ .

$$\therefore y = ax^2(x+2)(x-3), \quad a \neq 0$$

But when  $x = -3$ ,  $y = 54$

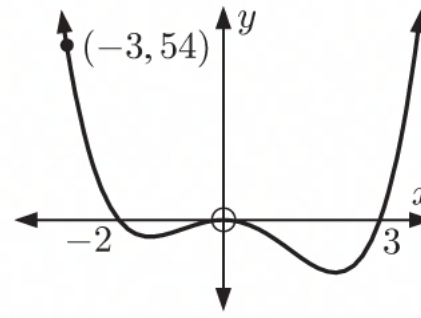
$$\therefore 54 = a(-3)^2(-3+2)(-3-3)$$

$$\therefore 54 = a(9)(-1)(-6)$$

$$\therefore 54 = 54a$$

$$\therefore a = 1$$

$$\therefore y = x^2(x+2)(x-3)$$



- 4 a** The graph cuts the  $x$ -axis at  $-4$  and  $\frac{1}{2}$ , and touches the  $x$ -axis at  $2$ .

$$\therefore y = a(x+4)(2x-1)(x-2)^2, \quad a \neq 0$$

When  $x = 1$ ,  $y = 5$

$$\therefore a(5)(1)(-1)^2 = 5$$

$$\therefore 5a = 5$$

$$\therefore a = 1$$

So,  $y = (x+4)(2x-1)(x-2)^2$

- b** The graph touches the  $x$ -axis at  $\frac{2}{3}$  and  $-3$ .

$$\therefore y = a(3x-2)^2(x+3)^2, \quad a \neq 0$$

When  $x = -4$ ,  $y = 49$

$$\therefore a(-14)^2(-1)^2 = 49$$

$$\therefore 196a = 49$$

$$\therefore a = \frac{1}{4}$$

So,  $y = \frac{1}{4}(3x-2)^2(x+3)^2$

- c** The graph cuts the  $x$ -axis at  $\pm\frac{1}{2}$  and  $\pm 2$ .

$$\therefore y = a(2x+1)(2x-1)(x+2)(x-2), \quad a \neq 0$$

When  $x = 1$ ,  $y = -18$

$$\therefore a(3)(1)(3)(-1) = -18$$

$$\therefore -9a = -18$$

$$\therefore a = 2$$

So,  $y = 2(2x+1)(2x-1)(x+2)(x-2)$

- d** The graph touches the  $x$ -axis at  $1$ , so  $(x-1)^2$  is a factor.

$$\therefore y = (x-1)^2(ax^2+bx+c), \quad a \neq 0$$

When  $x = 0$ ,  $y = -1$

$$\therefore (-1)^2(c) = -1$$

$$\therefore c = -1$$

When  $x = -1$ ,  $y = -4$

$$\therefore (-2)^2(a-b-1) = -4$$

$$\therefore 4(a-b-1) = -4$$

$$\therefore a-b-1 = -1$$

$$\therefore a = b$$

When  $x = 2$ ,  $y = 15$

$$\therefore (1)^2(4b+2b-1) = 15$$

$$\therefore 6b-1 = 15$$

$$\therefore 6b = 16$$

$$\therefore b = \frac{8}{3} \text{ and so } a = \frac{8}{3}$$

So,  $y = (x-1)^2\left(\frac{8}{3}x^2 + \frac{8}{3}x - 1\right)$

- 5 a** The graph touches the  $x$ -axis at  $-2$ , and cuts the  $x$ -axis at  $-4$  and  $1$ .

$$\therefore y = a(x+2)^2(x+4)(x-1)$$

But when  $x = 0$ ,  $y = -8$

$$\therefore -8 = a(2)^2(4)(-1)$$

$$\therefore -8 = -16a$$

$$\therefore a = \frac{1}{2}$$

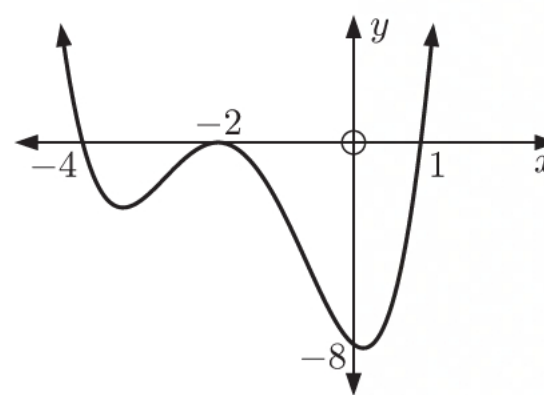
$$\therefore y = \frac{1}{2}(x+2)^2(x+4)(x-1)$$

$$= \frac{1}{2}(x^2 + 4x + 4)(x^2 + 3x - 4)$$

$$= \frac{1}{2}(x^4 + 3x^3 - 4x^2 + 4x^3 + 12x^2 - 16x + 4x^2 + 12x - 16)$$

$$= \frac{1}{2}(x^4 + 7x^3 + 12x^2 - 4x - 16)$$

$$\therefore y = \frac{1}{2}x^4 + \frac{7}{2}x^3 + 6x^2 - 2x - 8$$



- b** The graph touches the  $x$ -axis at  $1$ , and cuts the  $x$ -axis at  $-3$  and  $-1$ .

$$\therefore y = a(x-1)^2(x+3)(x+1)$$

But when  $x = 0$ ,  $y = -6$

$$\therefore -6 = a(-1)^2(3)(1)$$

$$\therefore -6 = 3a$$

$$\therefore a = -2$$

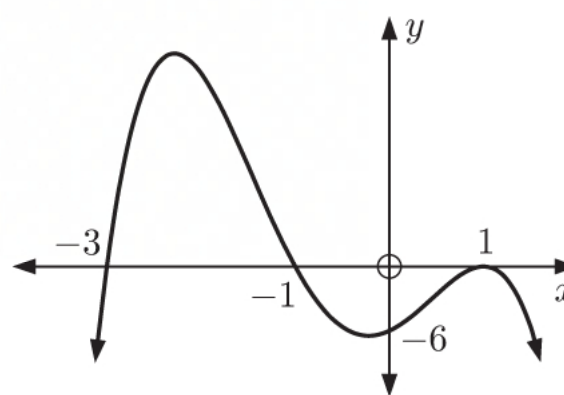
$$\therefore y = -2(x-1)^2(x+3)(x+1)$$

$$= -2(x^2 - 2x + 1)(x^2 + 4x + 3)$$

$$= -2(x^4 + 4x^3 + 3x^2 - 2x^3 - 8x^2 - 6x + x^2 + 4x + 3)$$

$$= -2(x^4 + 2x^3 - 4x^2 - 2x + 3)$$

$$\therefore y = -2x^4 - 4x^3 + 8x^2 + 4x - 6$$



- 6** Since the graph of  $f(x) = x^4 - x^3 - 5x^2 + 12$  touches the  $x$ -axis at  $2$ , then  $f(x) = (x-2)^2(x^2 + bx + 3)$ .

$$\mathbf{a} \quad x^4 - x^3 - 5x^2 + 12 = (x-2)^2(x^2 + bx + 3)$$

$$= (x^2 - 4x + 4)(x^2 + bx + 3)$$

$$= x^4 + bx^3 + 3x^2$$

$$- 4x^3 - 4bx^2 - 12x$$

$$+ 4x^2 + 4bx + 12$$

$$= x^4 + (b-4)x^3 + (7-4b)x^2 + (4b-12)x + 12$$

$$\text{Equating coefficients:} \quad b-4 = -1 \quad 7-4b = -5 \quad \text{and} \quad 4b-12 = 0$$

$$\therefore b = 3$$

$$\therefore x^4 - x^3 - 5x^2 + 12 = (x-2)^2(x^2 + 3x + 3)$$

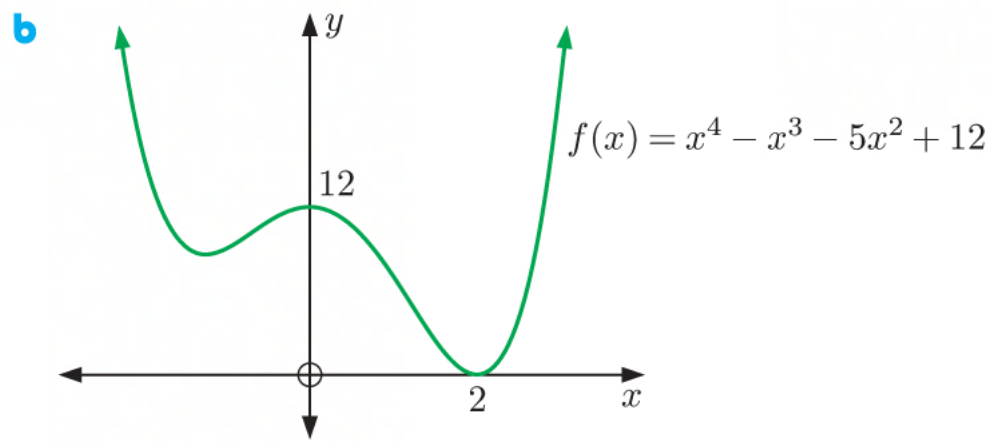
$$\text{For } x^2 + 3x + 3, \quad \Delta = b^2 - 4ac$$

$$= 3^2 - 4(1)(3)$$

$$= -3 \quad \text{which is } < 0$$

$$\therefore x^2 + 3x + 3 \text{ has no real zeros.}$$

$$\therefore \text{there are no other } x\text{-intercepts.}$$



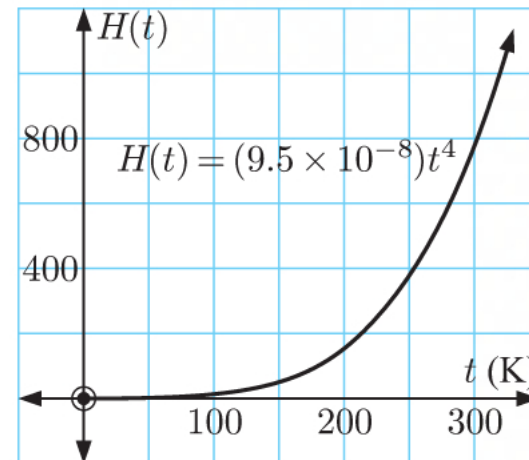
**7**  $H(t) \approx (9.5 \times 10^{-8})t^4$  joules per second

**a** When  $t = 310$ ,  $H(t) \approx (9.5 \times 10^{-8}) \times 310^4$   
 $\approx 877$

The heat radiated is about 877 joules per second.

**b** When  $t = 312$ ,  $H(t) \approx (9.5 \times 10^{-8}) \times 312^4$   
 $\approx 900$

The heat radiated is about 900 joules per second.



**8 a**  $x^2 + x + 1$  is a factor of  $x^4 - 4x^3 + 3x^2 + 2x + 7$

$$\begin{aligned} \therefore x^4 - 4x^3 + 3x^2 + 2x + 7 &= (x^2 + x + 1)(x^2 + bx + 7) \\ &= x^4 + bx^3 + 7x^2 \\ &\quad + x^3 + bx^2 + 7x \\ &\quad + x^2 + bx + 7 \\ &= x^4 + (b+1)x^3 + (8+b)x^2 + (7+b)x + 7 \end{aligned}$$

Equating coefficients:  $b+1 = -4$      $8+b = 3$     and     $7+b = 2$   
 $\therefore b = -5$

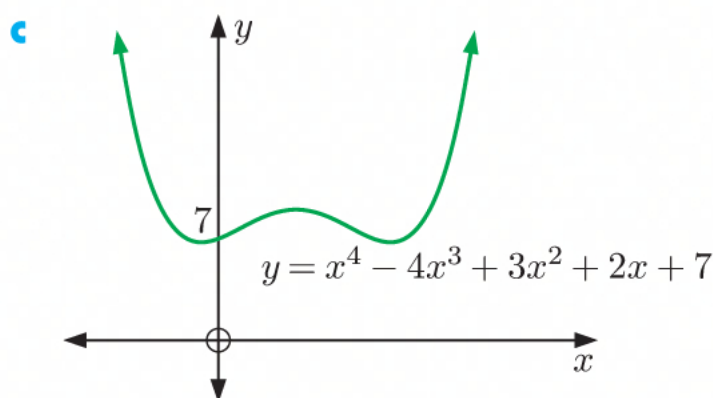
$$\therefore x^4 - 4x^3 + 3x^2 + 2x + 7 = (x^2 + x + 1)(x^2 - 5x + 7)$$

**b** For  $x^2 + x + 1$ ,  $a > 0$ ,  
 and  $\Delta = 1^2 - 4(1)(1) = -3$  which is  $< 0$   
 $\therefore x^2 + x + 1$  has no real zeros.  
 $\therefore$  there are no  $x$ -intercepts.

For  $x^2 - 5x + 7$ ,  $a > 0$ ,  
 and  $\Delta = (-5)^2 - 4(1)(7) = -3$  which is  $< 0$   
 $\therefore x^2 - 5x + 7$  has no real zeros.  
 $\therefore$  there are no  $x$ -intercepts.

$\therefore$  the graph of  $f(x) = x^4 - 4x^3 + 3x^2 + 2x + 7$  does not meet the  $x$ -axis at all.  
 It lies entirely above the  $x$ -axis.

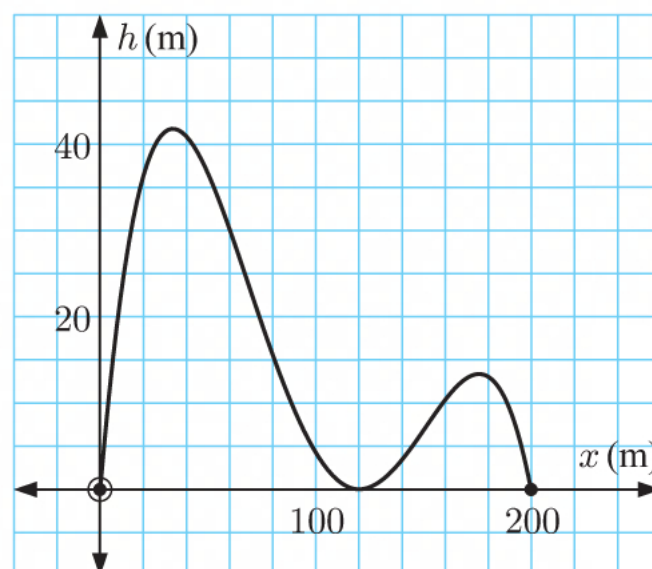




- 9 a The graph touches the  $x$ -axis at 120, and cuts the  $x$ -axis at 0 and 200.

$$\therefore h(x) = \frac{-x(x-120)^2(x-200)}{k}, \quad k \neq 0$$

So,  $a = 120$  and  $b = 200$ .



- b When  $x = 100$ ,  $h(x) = 4$

$$\begin{aligned} \therefore 4 &= \frac{(-100)(100-120)^2(100-200)}{k} \\ &= \frac{4\,000\,000}{k} \end{aligned}$$

$$\therefore 4k = 4\,000\,000$$

$$\therefore k = 1\,000\,000$$

- c When  $x = 150$ ,  $h(x) = \frac{-150(150-120)^2(150-200)}{1\,000\,000}$   
 $= 6.75$

The roller coaster is 6.75 m above the ground.

## EXERCISE 5L

- 1 a Using technology,  $-1$  is a zero.

$\therefore (x+1)$  is a factor.

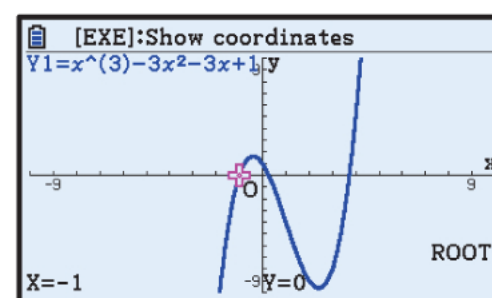
$$\begin{aligned} \therefore x^3 - 3x^2 - 3x + 1 &= (x+1)(x^2 + ax + 1) \quad \text{for some constant } a \\ &= x^3 + ax^2 + x + x^2 + ax + 1 \\ &= x^3 + (a+1)x^2 + (a+1)x + 1 \end{aligned}$$

$$\begin{aligned} \text{Equating coefficients of } x^2: \quad a+1 &= -3 \\ \therefore a &= -4 \end{aligned}$$

$$\begin{aligned} \text{Equating coefficients of } x: \quad a+1 &= -4+1 \\ &= -3 \quad \checkmark \end{aligned}$$

So the quadratic factor is  $(x^2 - 4x + 1)$  which has zeros  $\frac{4 \pm \sqrt{16 - 4(1)(1)}}{2} = \frac{4 \pm \sqrt{12}}{2}$   
 $= 2 \pm \sqrt{3}$

$\therefore$  the zeros are  $-1$  and  $2 \pm \sqrt{3}$ .



**b** Using technology, 1 is a zero.

$\therefore (x - 1)$  is a factor.

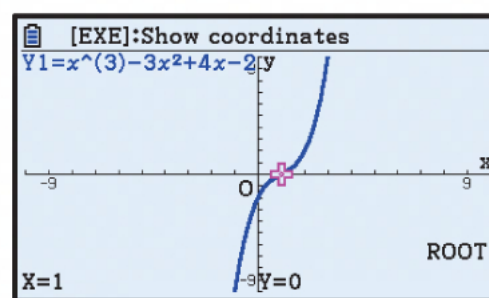
$$\begin{aligned} \therefore x^3 - 3x^2 + 4x - 2 &= (x - 1)(x^2 + ax + 2) \quad \text{for some constant } a \\ &= x^3 + ax^2 + 2x - x^2 - ax - 2 \\ &= x^3 + (a - 1)x^2 + (2 - a)x - 2 \end{aligned}$$

Equating coefficients of  $x^2$ :  $a - 1 = -3$

$$\therefore a = -2$$

Equating coefficients of  $x$ :  $2 - a = 2 - (-2) = 4$  ✓

So the quadratic factor is  $(x^2 - 2x + 2)$  which has zeros



$$\begin{aligned} \frac{2 \pm \sqrt{4 - 4(1)(2)}}{2} &= \frac{2 \pm \sqrt{-4}}{2} \\ &= 1 \pm i \end{aligned}$$

$\therefore$  the zeros are 1 and  $1 \pm i$ .

**c** Using technology,  $3.5 = \frac{7}{2}$  is a zero.

$\therefore (2x - 7)$  is a factor.

$$\begin{aligned} \therefore 2x^3 - 3x^2 - 4x - 35 &= (2x - 7)(x^2 + ax + 5) \quad \text{for some constant } a \\ &= 2x^3 + 2ax^2 + 10x - 7x^2 - 7ax - 35 \\ &= 2x^3 + (2a - 7)x^2 + (10 - 7a)x - 35 \end{aligned}$$

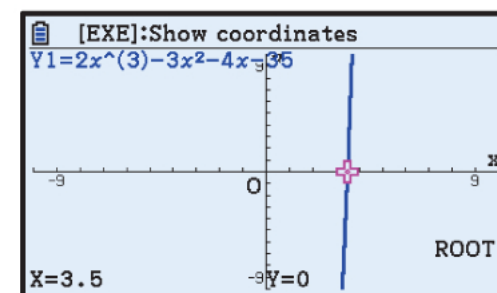
Equating coefficients of  $x^2$ :  $2a - 7 = -3$

$$\therefore 2a = 4$$

$$\therefore a = 2$$

Equating coefficients of  $x$ :  $10 - 7a = 10 - 7(2) = -4$  ✓

So the quadratic factor is  $(x^2 + 2x + 5)$  which has zeros



$$\begin{aligned} \frac{-2 \pm \sqrt{4 - 4(1)(5)}}{2} &= \frac{-2 \pm \sqrt{-16}}{2} \\ &= -1 \pm 2i \end{aligned}$$

$\therefore$  the zeros are  $\frac{7}{2}$  and  $-1 \pm 2i$ .

**d** Using technology,  $0.5 = \frac{1}{2}$  is a zero.

$\therefore (2x - 1)$  is a factor.

$$\begin{aligned} \therefore 2x^3 - x^2 + 20x - 10 &= (2x - 1)(x^2 + ax + 10) \quad \text{for some constant } a \\ &= 2x^3 + 2ax^2 + 20x - x^2 - ax - 10 \\ &= 2x^3 + (2a - 1)x^2 + (20 - a)x - 10 \end{aligned}$$

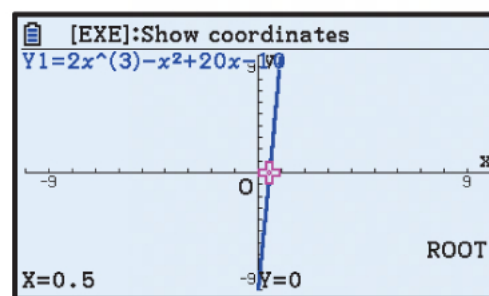
Equating coefficients of  $x^2$ :  $2a - 1 = -1$

$$\therefore a = 0$$

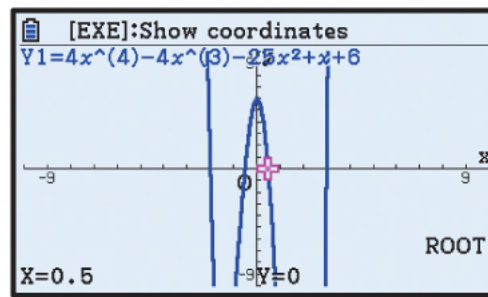
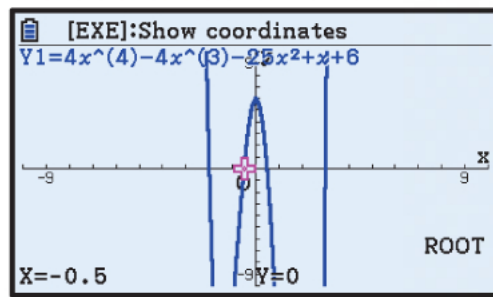
Equating coefficients of  $x$ :  $20 - a = 20$  ✓

So the quadratic factor is  $(x^2 + 10)$  which has zeros  $\pm i\sqrt{10}$ .

$\therefore$  the zeros are  $\frac{1}{2}$  and  $\pm i\sqrt{10}$ .



e



Using technology,  $-\frac{1}{2}$  and  $\frac{1}{2}$  are zeros.

$\therefore (2x + 1)$  and  $(2x - 1)$  are factors.

$$\begin{aligned}
 \therefore 4x^4 - 4x^3 - 25x^2 + x + 6 &= (2x + 1)(2x - 1)(x^2 + ax - 6) \quad \text{for some constant } a \\
 &= (4x^2 - 1)(x^2 + ax - 6) \\
 &= 4x^4 + 4ax^3 - 24x^2 \\
 &\quad - x^2 - ax + 6 \\
 &= 4x^4 + 4ax^3 - 25x^2 - ax + 6
 \end{aligned}$$

Equating coefficients of  $x^3$ :  $4a = -4$

$$\therefore a = -1$$

Equating coefficients of  $x$ :  $-a = 1$  ✓

So the quadratic factor is  $(x^2 - x + 6) = (x - 3)(x + 2)$

$\therefore$  the zeros are  $\pm\frac{1}{2}$ , 3 and  $-2$ .

f Using technology, 2 is a repeated zero.

{graph touches the  $x$ -axis at 2}

$\therefore (x - 2)^2$  is a factor.

$$\begin{aligned}
 \therefore x^4 - 6x^3 + 22x^2 - 48x + 40 &= (x - 2)^2(x^2 + ax + 10) \quad \text{for some constant } a \\
 &= (x^2 - 4x + 4)(x^2 + ax + 10) \\
 &= x^4 + ax^3 + 10x^2 \\
 &\quad - 4x^3 - 4ax^2 - 40x \\
 &\quad + 4x^2 + 4ax + 40 \\
 &= x^4 + (a - 4)x^3 + (-4a + 14)x^2 + (4a - 40)x + 40
 \end{aligned}$$

Equating coefficients of  $x^3$ :  $a - 4 = -6$

$$\therefore a = -2$$

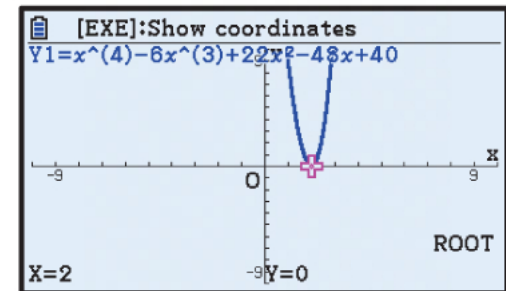
Equating coefficients of  $x^2$ :  $-4a + 14 = -4(-2) + 14 = 22$  ✓

Equating coefficients of  $x$ :  $4a - 40 = 4(-2) - 40 = -48$  ✓

So the other quadratic factor is  $(x^2 - 2x + 10)$  which has zeros

$$\begin{aligned}
 \frac{2 \pm \sqrt{4 - 4(1)(10)}}{2} &= \frac{2 \pm \sqrt{-36}}{2} \\
 &= 1 \pm 3i
 \end{aligned}$$

$\therefore$  the zeros are 2 (repeated) and  $1 \pm 3i$ .



**2 a** Using technology,  $-2$  is a root.

$\therefore (x + 2)$  is a factor.

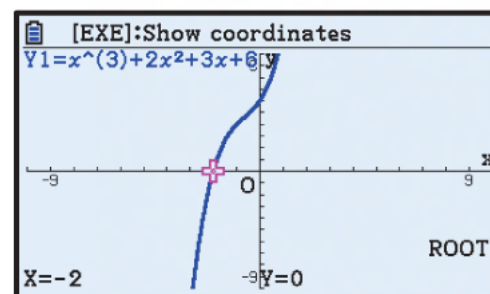
$$\begin{aligned} \therefore x^3 + 2x^2 + 3x + 6 &= (x + 2)(x^2 + ax + 3) \quad \text{for some constant } a \\ &= x^3 + ax^2 + 3x + 2x^2 + 2ax + 6 \\ &= x^3 + (a + 2)x^2 + (2a + 3)x + 6 \end{aligned}$$

Equating coefficients of  $x^2$ :  $a + 2 = 2$   
 $\therefore a = 0$

Equating coefficients of  $x$ :  $2a + 3 = 3$  ✓

$$\begin{aligned} \therefore (x + 2)(x^2 + 3) &= 0 \\ \therefore x &= -2 \text{ or } \pm i\sqrt{3} \end{aligned}$$

So the roots are  $-2$  and  $\pm i\sqrt{3}$ .



**b** Using technology,  $1$  is a root.

$\therefore (x - 1)$  is a factor.

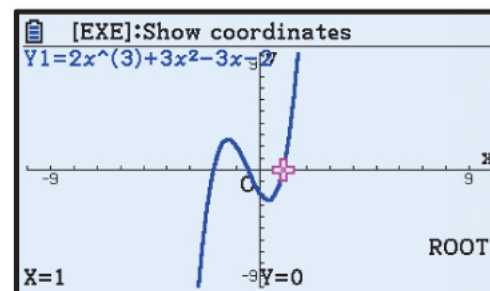
$$\begin{aligned} \therefore 2x^3 + 3x^2 - 3x - 2 &= (x - 1)(2x^2 + ax + 2) \quad \text{for some constant } a \\ &= 2x^3 + ax^2 + 2x - 2x^2 - ax - 2 \\ &= 2x^3 + (a - 2)x^2 + (-a + 2)x - 2 \end{aligned}$$

Equating coefficients of  $x^2$ :  $a - 2 = 3$   
 $\therefore a = 5$

Equating coefficients of  $x$ :  $-a + 2 = -5 + 2 = -3$  ✓

$$\begin{aligned} \therefore (x - 1)(2x^2 + 5x + 2) &= 0 \\ \therefore (x - 1)(2x + 1)(x + 2) &= 0 \\ \therefore x &= 1, -\frac{1}{2}, \text{ or } -2 \end{aligned}$$

So the roots are  $1$ ,  $-\frac{1}{2}$ , and  $-2$ .



**c** Using technology,  $2$  is a root.

$\therefore (x - 2)$  is a factor.

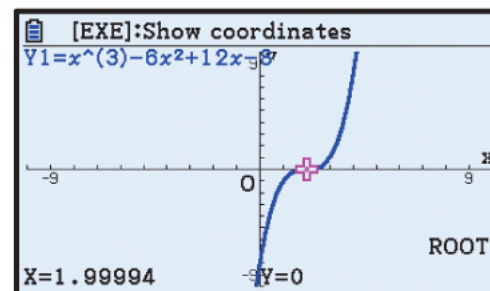
$$\begin{aligned} \therefore x^3 - 6x^2 + 12x - 8 &= (x - 2)(x^2 + ax + 4) \quad \text{for some constant } a \\ &= x^3 + ax^2 + 4x - 2x^2 - 2ax - 8 \\ &= x^3 + (a - 2)x^2 + (-2a + 4)x - 8 \end{aligned}$$

Equating coefficients of  $x^2$ :  $a - 2 = -6$   
 $\therefore a = -4$

Equating coefficients of  $x$ :  $-2a + 4 = -2(-4) + 4 = 12$  ✓

$$\begin{aligned} \therefore (x - 2)(x^2 - 4x + 4) &= 0 \\ \therefore (x - 2)(x - 2)^2 &= 0 \\ \therefore (x - 2)^3 &= 0 \\ \therefore x &= 2 \end{aligned}$$

So the only root is  $2$  (repeated 3 times).





**d** Using technology, 3 is a root.

$\therefore (x - 3)$  is a factor.

$$\begin{aligned} \therefore 2x^3 - 5x^2 - 9x + 18 &= (x - 3)(2x^2 + ax - 6) \quad \text{for some constant } a \\ &= 2x^3 + ax^2 - 6x - 6x^2 - 3ax + 18 \\ &= 2x^3 + (a - 6)x^2 + (-3a - 6)x + 18 \end{aligned}$$

Equating coefficients of  $x^2$ :  $a - 6 = -5$   
 $\therefore a = 1$

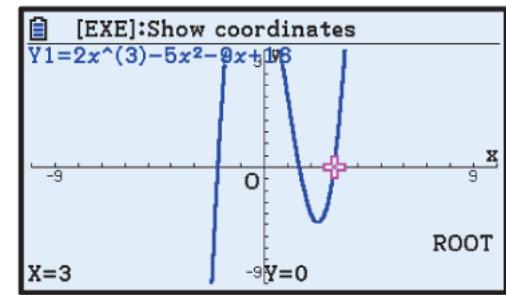
Equating coefficients of  $x$ :  $-3a - 6 = -3(1) - 6 = -9$  ✓

$$\therefore (x - 3)(2x^2 + x - 6) = 0$$

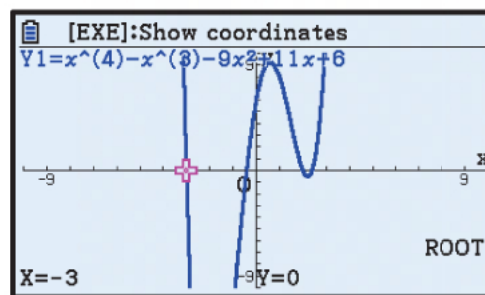
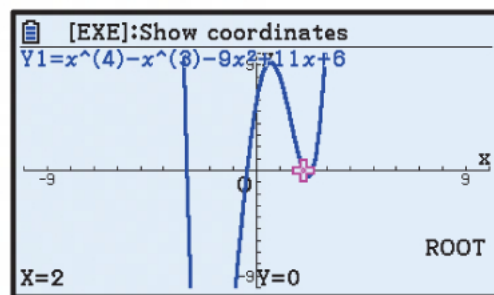
$$\therefore (x - 3)(2x - 3)(x + 2) = 0$$

$$\therefore x = 3, \frac{3}{2}, \text{ or } -2$$

So the roots are 3,  $\frac{3}{2}$ , and  $-2$ .



**e**



Using technology, 2 and  $-3$  are roots.

$\therefore (x - 2)$  and  $(x + 3)$  are factors.

$$\begin{aligned} \therefore x^4 - x^3 - 9x^2 + 11x + 6 &= (x - 2)(x + 3)(x^2 + ax - 1) \quad \text{for some constant } a \\ &= (x^2 + x - 6)(x^2 + ax - 1) \\ &= x^4 + ax^3 - x^2 + x^3 + ax^2 - x - 6x^2 - 6ax + 6 \\ &= x^4 + (a + 1)x^3 + (a - 7)x^2 + (-6a - 1)x + 6 \end{aligned}$$

Equating coefficients of  $x^3$ :  $a + 1 = -1$   
 $\therefore a = -2$

Equating coefficients of  $x^2$ :  $a - 7 = -2 - 7 = -9$  ✓

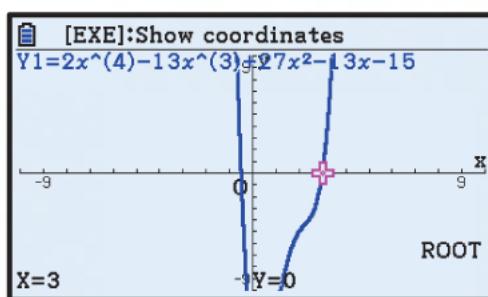
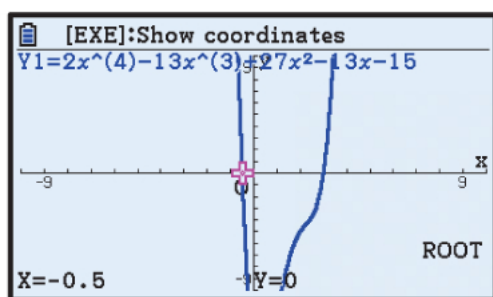
Equating coefficients of  $x$ :  $-6a - 1 = -6(-2) - 1 = 11$  ✓

$$\therefore (x - 2)(x + 3)(x^2 - 2x - 1) = 0$$

$$\therefore x = 2, -3, \text{ or } \frac{2 \pm \sqrt{4 - 4(1)(-1)}}{2} = 1 \pm \sqrt{2}$$

So the roots are 2,  $-3$ , and  $1 \pm \sqrt{2}$ .

f



Using technology,  $-0.5 = -\frac{1}{2}$  and 3 are roots.

$\therefore (2x + 1)$  and  $(x - 3)$  are factors.

$$\begin{aligned}
 \therefore & 2x^4 - 13x^3 + 27x^2 - 13x - 15 \\
 &= (2x + 1)(x - 3)(x^2 + ax + 5) \quad \text{for some constant } a \\
 &= (2x^2 - 5x - 3)(x^2 + ax + 5) \\
 &= 2x^4 + 2ax^3 + 10x^2 \\
 &\quad - 5x^3 - 5ax^2 - 25x \\
 &\quad - 3x^2 - 3ax - 15 \\
 &= 2x^4 + (2a - 5)x^3 + (-5a + 7)x^2 + (-3a - 25)x - 15
 \end{aligned}$$

Equating coefficients of  $x^3$ :  $2a - 5 = -13$

$$\therefore 2a = -8$$

$$\therefore a = -4$$

Equating coefficients of  $x^2$ :  $-5a + 7 = -5(-4) + 7 = 27$  ✓

Equating coefficients of  $x$ :  $-3a - 25 = -3(-4) - 25 = -13$  ✓

$$\therefore (2x + 1)(x - 3)(x^2 - 4x + 5) = 0$$

$$\therefore x = -\frac{1}{2}, 3, \text{ or } \frac{4 \pm \sqrt{16 - 4(1)(5)}}{2} = 2 \pm i$$

So the roots are  $-\frac{1}{2}$ , 3, and  $2 \pm i$ .

**3 a** Using technology,  $x = 2$  is a solution.

$\therefore (x - 2)$  is a factor.

$$\begin{aligned}
 \therefore & x^3 - 3x^2 + 4x - 4 \\
 &= (x - 2)(x^2 + ax + 2) \quad \text{for some constant } a \\
 &= x^3 + ax^2 + 2x \\
 &\quad - 2x^2 - 2ax - 4 \\
 &= x^3 + (a - 2)x^2 + (-2a + 2)x - 4
 \end{aligned}$$

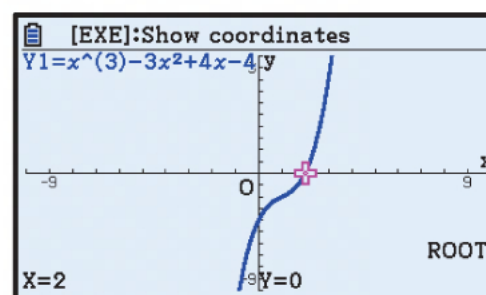
Equating coefficients of  $x^2$ :  $a - 2 = -3$

$$\therefore a = -1$$

Equating coefficients of  $x$ :  $-2a + 2 = -2(-1) + 2 = 4$  ✓

$$\therefore (x - 2)(x^2 - x + 2) = 0$$

$$\therefore x = 2 \text{ or } \frac{1 \pm \sqrt{1 - 4(1)(2)}}{2} = \frac{1 \pm i\sqrt{7}}{2}$$



**b** Using technology,  $x = -3$  is a solution.

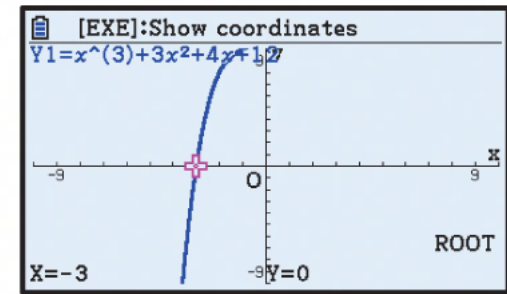
$\therefore (x + 3)$  is a factor.

$$\begin{aligned} \therefore x^3 + 3x^2 + 4x + 12 &= (x + 3)(x^2 + ax + 4) \quad \text{for some constant } a \\ &= x^3 + ax^2 + 4x + 3x^2 + 3ax + 12 \\ &= x^3 + (a + 3)x^2 + (3a + 4)x + 12 \end{aligned}$$

$$\begin{aligned} \text{Equating coefficients of } x^2: \quad a + 3 &= 3 \\ \therefore a &= 0 \end{aligned}$$

$$\text{Equating coefficients of } x: \quad 3a + 4 = 4 \quad \checkmark$$

$$\begin{aligned} \therefore (x + 3)(x^2 + 4) &= 0 \\ \therefore x &= -3 \quad \text{or} \quad x = \pm 2i \end{aligned}$$



**c** Using technology,  $x = 0.5 = \frac{1}{2}$  is a solution.

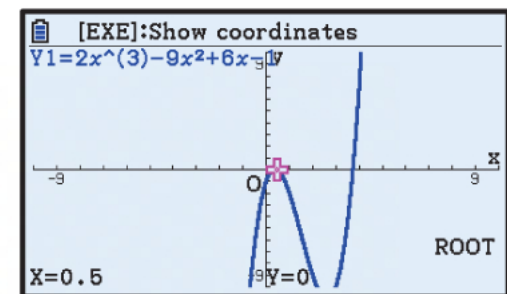
$\therefore (2x - 1)$  is a factor.

$$\begin{aligned} \therefore 2x^3 - 9x^2 + 6x - 1 &= (2x - 1)(x^2 + ax + 1) \quad \text{for some constant } a \\ &= 2x^3 + 2ax^2 + 2x - x^2 - ax - 1 \\ &= 2x^3 + (2a - 1)x^2 + (-a + 2)x - 1 \end{aligned}$$

$$\begin{aligned} \text{Equating coefficients of } x^2: \quad 2a - 1 &= -9 \\ \therefore a &= -4 \end{aligned}$$

$$\text{Equating coefficients of } x: \quad -a + 2 = 4 + 2 = 6 \quad \checkmark$$

$$\begin{aligned} \therefore (2x - 1)(x^2 - 4x + 1) &= 0 \\ \therefore x &= \frac{1}{2} \quad \text{or} \quad \frac{4 \pm \sqrt{16 - 4(1)(1)}}{2} = 2 \pm \sqrt{3} \end{aligned}$$



**d** Using technology,  $x = 2$  is a solution.

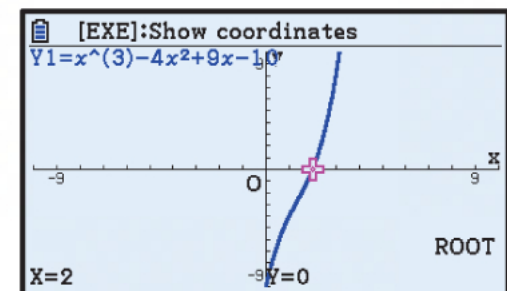
$\therefore (x - 2)$  is a factor.

$$\begin{aligned} \therefore x^3 - 4x^2 + 9x - 10 &= (x - 2)(x^2 + ax + 5) \quad \text{for some constant } a \\ &= x^3 + ax^2 + 5x - 2x^2 - 2ax - 10 \\ &= x^3 + (a - 2)x^2 + (-2a + 5)x - 10 \end{aligned}$$

$$\begin{aligned} \text{Equating coefficients of } x^2: \quad a - 2 &= -4 \\ \therefore a &= -2 \end{aligned}$$

$$\text{Equating coefficients of } x: \quad -2a + 5 = -2(-2) + 5 = 9 \quad \checkmark$$

$$\begin{aligned} \therefore (x - 2)(x^2 - 2x + 5) &= 0 \\ \therefore x &= 2 \quad \text{or} \quad \frac{2 \pm \sqrt{4 - 4(1)(5)}}{2} = 1 \pm 2i \end{aligned}$$



e Using technology,  $x = 1$  is a solution.

$\therefore (x - 1)$  is a factor.

$$\begin{aligned} &\therefore 4x^3 - 8x^2 + x + 3 \\ &= (x - 1)(4x^2 + ax - 3) \quad \text{for some constant } a \\ &= 4x^3 + ax^2 - 3x \\ &\quad - 4x^2 - ax + 3 \\ &= 4x^3 + (a - 4)x^2 + (-a - 3)x + 3 \end{aligned}$$

Equating coefficients of  $x^2$ :  $a - 4 = -8$

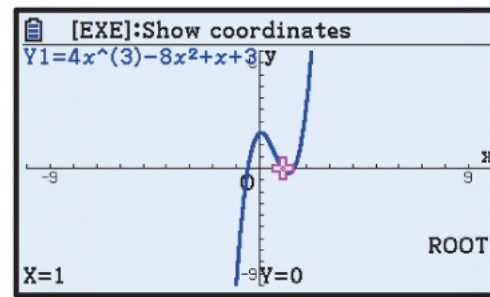
$$\therefore a = -4$$

Equating coefficients of  $x$ :  $-a - 3 = 4 - 3 = 1$  ✓

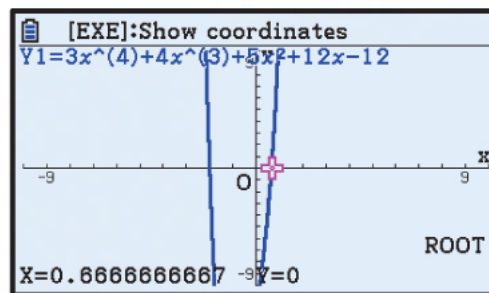
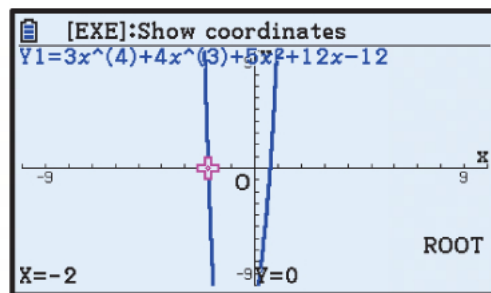
$$\therefore (x - 1)(4x^2 - 4x - 3) = 0$$

$$\therefore (x - 1)(2x - 3)(2x + 1) = 0$$

$$\therefore x = 1, \frac{3}{2}, \text{ or } -\frac{1}{2}$$



f



Using technology,  $x = -2$  and  $x = \frac{2}{3}$  are solutions.

$\therefore (x + 2)$  and  $(3x - 2)$  are factors.

$$\begin{aligned} \therefore 3x^4 + 4x^3 + 5x^2 + 12x - 12 &= (x + 2)(3x - 2)(x^2 + ax + 3) \quad \text{for some constant } a \\ &= (3x^2 + 4x - 4)(x^2 + ax + 3) \\ &= 3x^4 + 3ax^3 + 9x^2 \\ &\quad + 4x^3 + 4ax^2 + 12x \\ &\quad - 4x^2 - 4ax - 12 \\ &= 3x^4 + (3a + 4)x^3 + (4a + 5)x^2 + (-4a + 12)x - 12 \end{aligned}$$

Equating coefficients of  $x^3$ :  $3a + 4 = 4$

$$\therefore a = 0$$

Equating coefficients of  $x^2$ :  $4a + 5 = 5$  ✓

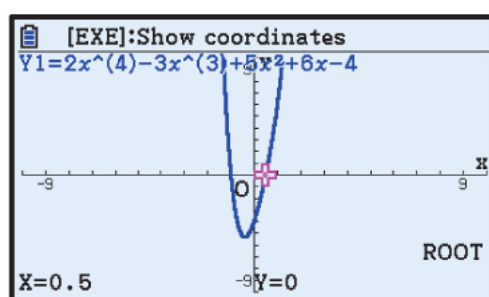
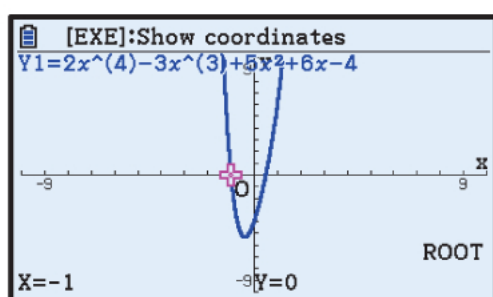
Equating coefficients of  $x$ :  $-4a + 12 = 12$  ✓

$$\therefore (x + 2)(3x - 2)(x^2 + 3) = 0$$

$$\therefore x = -2, \frac{2}{3}, \text{ or } \pm i\sqrt{3}$$



9



Using technology,  $x = -1$  and  $x = 0.5 = \frac{1}{2}$  are solutions.

$\therefore (x + 1)$  and  $(2x - 1)$  are factors.

$$\begin{aligned}
 \therefore 2x^4 - 3x^3 + 5x^2 + 6x - 4 &= (x + 1)(2x - 1)(x^2 + ax + 4) \text{ for some constant } a \\
 &= (2x^2 + x - 1)(x^2 + ax + 4) \\
 &= 2x^4 + 2ax^3 + 8x^2 \\
 &\quad + x^3 + ax^2 + 4x \\
 &\quad - x^2 - ax - 4 \\
 &= 2x^4 + (2a + 1)x^3 + (a + 7)x^2 + (-a + 4)x - 4
 \end{aligned}$$

Equating coefficients of  $x^3$ :  $2a + 1 = -3$

$$\therefore a = -2$$

Equating coefficients of  $x^2$ :  $a + 7 = -2 + 7 = 5$  ✓

Equating coefficients of  $x$ :  $-a + 4 = 2 + 4 = 6$  ✓

$$\therefore (x + 1)(2x - 1)(x^2 - 2x + 4) = 0$$

$$\therefore x = -1, \frac{1}{2}, \text{ or } \frac{2 \pm \sqrt{4 - 4(1)(4)}}{2} = 1 \pm i\sqrt{3}$$

**h** Using technology,  $x = -2.5 = -\frac{5}{2}$  is a solution.

$\therefore (2x + 5)$  is a factor.

$$\begin{aligned}
 \therefore 2x^3 + 5x^2 + 8x + 20 &= (2x + 5)(x^2 + ax + 4) \text{ for some constant } a \\
 &= 2x^3 + 2ax^2 + 8x \\
 &\quad + 5x^2 + 5ax + 20 \\
 &= 2x^3 + (2a + 5)x^2 + (5a + 8)x + 20
 \end{aligned}$$

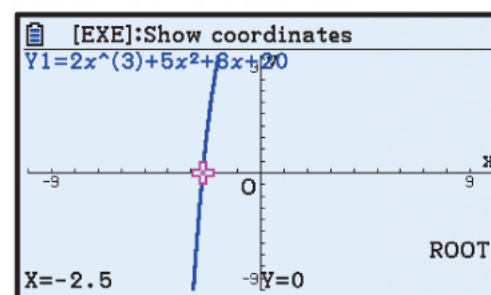
Equating coefficients of  $x^2$ :  $2a + 5 = 5$

$$\therefore a = 0$$

Equating coefficients of  $x$ :  $5a + 8 = 8$  ✓

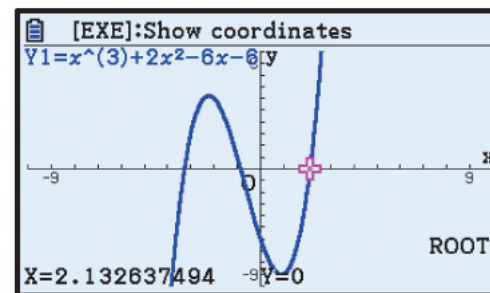
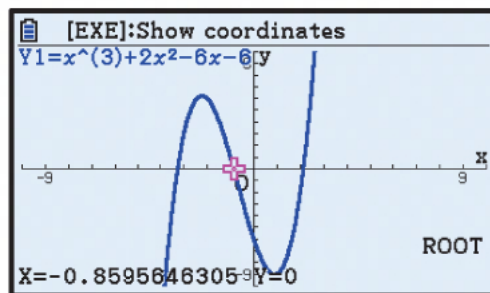
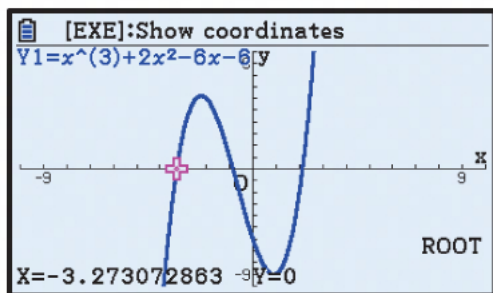
$$\therefore (2x + 5)(x^2 + 4) = 0$$

$$\therefore x = -\frac{5}{2} \text{ or } \pm 2i$$



4

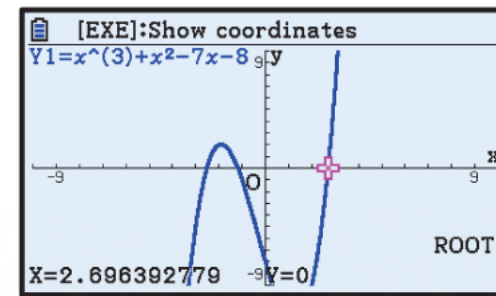
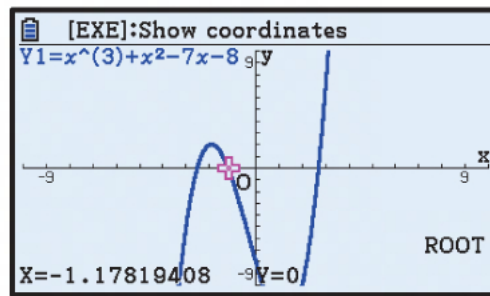
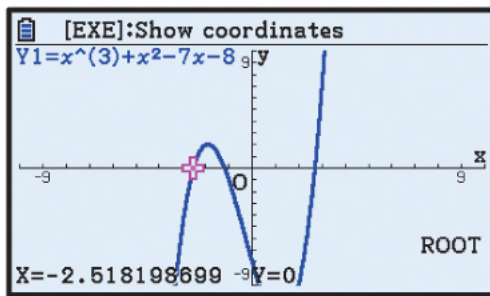
a



Using technology,  $x^3 + 2x^2 - 6x - 6$  has zeros of  $-3.27$ ,  $-0.860$ , and  $2.13$

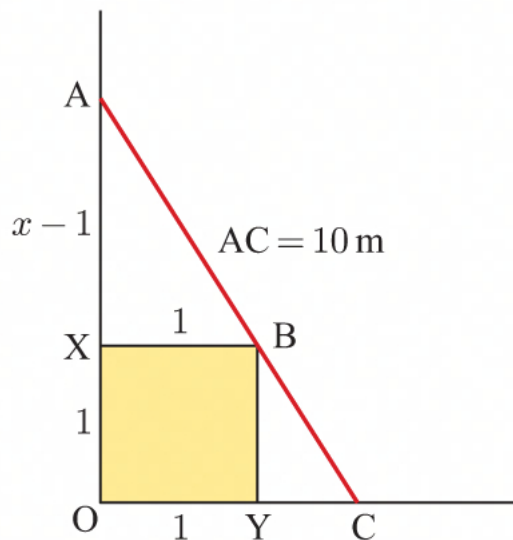
$\therefore x \approx -3.27, -0.860, \text{ or } 2.13$

b



Using technology,  $x^3 + x^2 - 7x - 8$  has zeros of  $-2.52$ ,  $-1.18$ , and  $2.70$   
 $\therefore x \approx -2.52$ ,  $-1.18$ , or  $2.70$

5



Let the height of the wall where the ladder touches be  $x$  m.

$$\frac{x-1}{1} = \frac{x}{OC} \quad \{\text{triangles AXB and AOC are similar}\}$$

$$\therefore OC = \frac{x}{x-1}$$

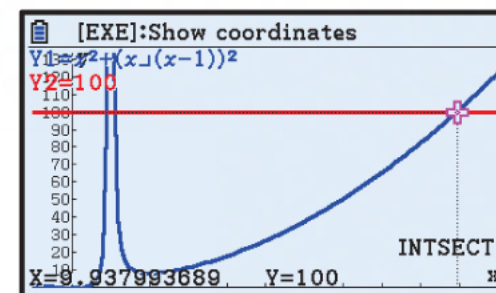
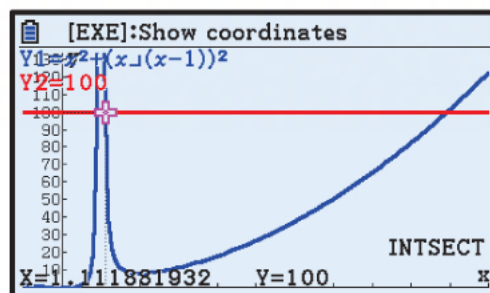
$$\text{but } x^2 + OC^2 = 10^2 \quad \{\text{Pythagoras}\}$$

$$x^2 + \left(\frac{x}{x-1}\right)^2 = 100$$

Using technology to find the intersection of

$$y = x^2 + \left(\frac{x}{x-1}\right)^2 \quad \text{and}$$

$$y = 100, \quad x \approx 1.11 \quad \text{or} \quad 9.94$$



So, height  $\approx 1.11$  m or  $9.94$  m up the wall.

## EXERCISE 5M

- 1 a 2 is a zero of  $x^3 + 3x^2 - 4x - 12$ , so  $(x - 2)$  is a factor.  
 $\therefore x^3 + 3x^2 - 4x - 12 = (x - 2)(x^2 + ax + 6)$  for some constant  $a$

$$\begin{aligned} &= x^3 + ax^2 + 6x \\ &\quad - 2x^2 - 2ax - 12 \\ &= x^3 + (a - 2)x^2 + (6 - 2a)x - 12 \end{aligned}$$

$$\text{Equating coefficients of } x^2: \quad a - 2 = 3$$

$$\therefore a = 5$$

$$\text{Equating coefficients of } x: \quad 6 - 2a = -4 \quad \checkmark$$

$$\begin{aligned} \therefore x^3 + 3x^2 - 4x - 12 &= (x - 2)(x^2 + 5x + 6) \\ &= (x - 2)(x + 2)(x + 3) \end{aligned}$$

b

$$x^3 + 3x^2 - 4x - 12 \geq 0$$

$$\therefore (x - 2)(x + 2)(x + 3) \geq 0$$

Sign diagram of LHS:  $\leftarrow \begin{array}{c} - & + & - & + \\ -3 & -2 & 2 & x \end{array} \rightarrow$

$$\therefore -3 \leq x \leq -2 \quad \text{or} \quad x \geq 2$$

$$\begin{aligned}
 \text{2 a} \quad & 3\left(-\frac{2}{3}\right)^3 - 10\left(-\frac{2}{3}\right)^2 - 17\left(-\frac{2}{3}\right) - 6 \\
 &= 3\left(-\frac{8}{27}\right) - 10\left(\frac{4}{9}\right) + \frac{34}{3} - 6 \\
 &= -\frac{8}{9} - \frac{40}{9} + \frac{34}{3} - 6 \\
 &= 0
 \end{aligned}$$

$\therefore -\frac{2}{3}$  is a zero of  $3x^3 - 10x^2 - 17x - 6$ .

$$\begin{aligned}
 \text{b} \quad 3x^3 - 10x^2 - 17x - 6 &= (3x + 2)(x^2 + ax - 3) \quad \text{for some constant } a \\
 &= 3x^3 + 3ax^2 - 9x \\
 &\quad + 2x^2 + 2ax - 6 \\
 &= 3x^3 + (3a + 2)x^2 + (2a - 9)x - 6
 \end{aligned}$$

Equating coefficients of  $x^2$ :  $3a + 2 = -10$

$$\therefore 3a = -12$$

$$\therefore a = -4$$

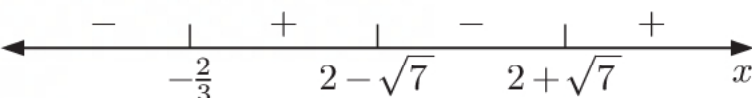
Equating coefficients of  $x$ :  $2a - 9 = -17$  ✓

$$\therefore 3x^3 - 10x^2 - 17x - 6 = (3x + 2)(x^2 - 4x - 3)$$

$$\begin{aligned}
 x^2 - 4x - 3 = 0 \quad \text{when} \quad x &= \frac{4 \pm \sqrt{16 - 4(1)(-3)}}{2} \\
 &= \frac{4 \pm \sqrt{28}}{2} \\
 &= 2 \pm \sqrt{7}
 \end{aligned}$$

So, the remaining zeros of  $3x^3 - 10x^2 - 17x - 6$  are  $2 \pm \sqrt{7}$ .

$$\begin{aligned}
 \text{c} \quad & 3x^3 - 6 < 10x^2 + 17x \\
 \therefore 3x^3 - 10x^2 - 17x - 6 &< 0 \\
 \therefore (3x + 2)(x^2 - 4x - 3) &< 0
 \end{aligned}$$

Sign diagram of LHS: 

$$\therefore x < -\frac{2}{3} \quad \text{or} \quad 2 - \sqrt{7} < x < 2 + \sqrt{7}$$

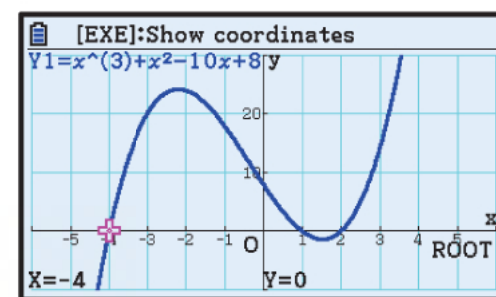
$$\text{3 a} \quad x^3 + x^2 - 10x + 8 < 0$$

Using technology, the roots are  $x = -4$ ,  $1$ , and  $2$ .

$$\therefore x^3 + x^2 - 10x + 8 = (x + 4)(x - 1)(x - 2)$$

which has sign diagram: 

$$\therefore x < -4 \quad \text{or} \quad 1 < x < 2$$



**b**  $x^3 + x^2 - 11x - 15 \geq 0$

Using technology,  $x = -3$  is a root, so  $(x + 3)$  is a factor of the cubic.

$$\begin{aligned} \therefore x^3 + x^2 - 11x - 15 &= (x + 3)(x^2 + ax - 5) \quad \text{for some } a \\ &= x^3 + ax^2 - 5x + 3x^2 + 3ax - 15 \\ &= x^3 + (a + 3)x^2 + (3a - 5)x - 15 \end{aligned}$$

Equating coefficients of  $x^2$ :  $a + 3 = 1$

$$\therefore a = -2$$

Equating coefficients of  $x$ :  $3a - 5 = -11$  ✓

$$\therefore x^3 + x^2 - 11x - 15 = (x + 3)(x^2 - 2x - 5)$$

$$\begin{aligned} x^2 - 2x - 5 = 0 \quad \text{when} \quad x &= \frac{2 \pm \sqrt{4 - 4(1)(-5)}}{2} \\ &= \frac{2 \pm \sqrt{24}}{2} \\ &= 1 \pm \sqrt{6} \end{aligned}$$

Sign diagram of LHS is:  $\begin{array}{ccccccc} & - & & + & & - & & + \\ & & -3 & & 1 - \sqrt{6} & & 1 + \sqrt{6} & & x \end{array}$

$$\therefore -3 \leq x \leq 1 - \sqrt{6} \quad \text{or} \quad x \geq 1 + \sqrt{6}$$

**c**  $2x^3 - 9x^2 + 2x + 1 \leq 0$

Using technology,  $x = \frac{1}{2}$  is a root, so  $(2x - 1)$  is a factor of the cubic.

$$\begin{aligned} \therefore 2x^3 - 9x^2 + 2x + 1 &= (2x - 1)(x^2 + ax - 1) \quad \text{for some constant } a \\ &= 2x^3 + 2ax^2 - 2x - x^2 - ax + 1 \\ &= 2x^3 + (2a - 1)x^2 + (-2 - a)x + 1 \end{aligned}$$

Equating coefficients of  $x^2$ :  $2a - 1 = -9$

$$\therefore 2a = -8$$

$$\therefore a = -4$$

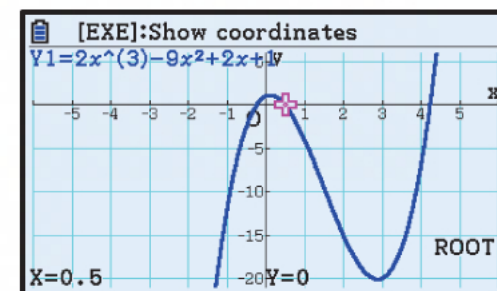
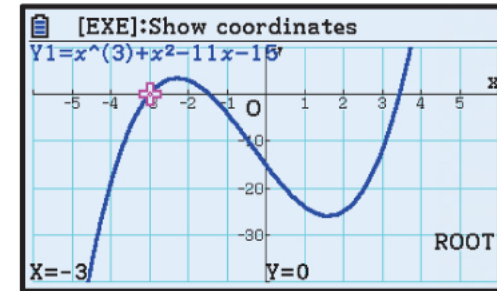
Equating coefficients of  $x$ :  $-2 - a = 2$  ✓

$$\therefore 2x^3 - 9x^2 + 2x + 1 = (2x - 1)(x^2 - 4x - 1)$$

$$\begin{aligned} x^2 - 4x - 1 = 0 \quad \text{when} \quad x &= \frac{4 \pm \sqrt{16 - 4(1)(-1)}}{2} \\ &= \frac{4 \pm \sqrt{20}}{2} \\ &= 2 \pm \sqrt{5} \end{aligned}$$

Sign diagram of LHS is:  $\begin{array}{ccccccc} & - & & + & & - & & + \\ & & 2 - \sqrt{5} & & \frac{1}{2} & & 2 + \sqrt{5} & & x \end{array}$

$$\therefore x \leq 2 - \sqrt{5} \quad \text{or} \quad \frac{1}{2} \leq x \leq 2 + \sqrt{5}$$





**d**  $3x^3 + 10x^2 > 22x + 40$

$$\therefore 3x^3 + 10x^2 - 22x - 40 > 0$$

Using technology,  $x = -\frac{4}{3}$  is a root, so  $(3x + 4)$  is a factor of the cubic.

$$\begin{aligned} \therefore 3x^3 + 10x^2 - 22x - 40 &= (3x + 4)(x^2 + ax - 10) \quad \text{for some constant } a \\ &= 3x^3 + 3ax^2 - 30x \\ &\quad + 4x^2 + 4ax - 40 \\ &= 3x^3 + (3a + 4)x^2 + (4a - 30)x - 40 \end{aligned}$$

Equating coefficients of  $x^2$ :  $3a + 4 = 10$

$$\therefore 3a = 6$$

$$\therefore a = 2$$

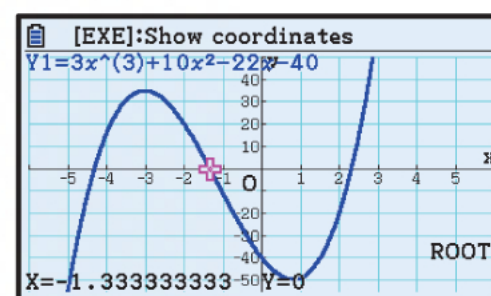
Equating coefficients of  $x$ :  $4a - 30 = -22$  ✓

$$\therefore 3x^3 + 10x^2 - 22x - 40 = (3x + 4)(x^2 + 2x - 10)$$

$$\begin{aligned} x^2 + 2x - 10 = 0 \quad \text{when} \quad x &= \frac{-2 \pm \sqrt{4 - 4(1)(-10)}}{2} \\ &= \frac{-2 \pm \sqrt{44}}{2} \\ &= -1 \pm \sqrt{11} \end{aligned}$$

Sign diagram of LHS is:  $\leftarrow \begin{array}{ccccccc} & - & & + & & - & & + \\ & & | & & | & & | & \\ & & -1 - \sqrt{11} & & -\frac{4}{3} & & -1 + \sqrt{11} & \\ & & & & & & & x \end{array} \rightarrow$

$$\therefore -1 - \sqrt{11} < x < -\frac{4}{3} \quad \text{or} \quad x > -1 + \sqrt{11}$$



**e**  $4x^3 - 7 \leq 28x - x^2$

$$\therefore 4x^3 + x^2 - 28x - 7 \leq 0$$

Using technology,  $x = -\frac{1}{4}$  is a root, so  $(4x + 1)$  is a factor of the cubic.

$$\begin{aligned} \therefore 4x^3 + x^2 - 28x - 7 &= (4x + 1)(x^2 + ax - 7) \quad \text{for some constant } a \\ &= 4x^3 + 4ax^2 - 28x \\ &\quad + x^2 + ax - 7 \\ &= 4x^3 + (4a + 1)x^2 + (a - 28)x - 7 \end{aligned}$$

Equating coefficients of  $x^2$ :  $4a + 1 = 1$

$$\therefore 4a = 0$$

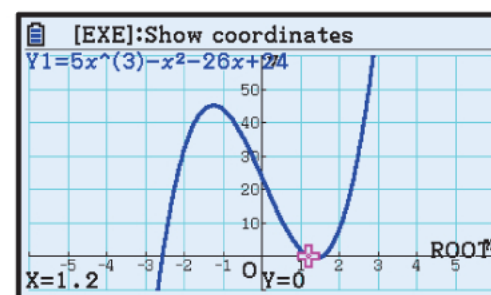
$$\therefore a = 0$$

Equating coefficients of  $x$ :  $a - 28 = -28$  ✓

$$\begin{aligned} \therefore 4x^3 + x^2 - 28x - 7 &= (4x + 1)(x^2 - 7) \\ &= (4x + 1)(x + \sqrt{7})(x - \sqrt{7}) \end{aligned}$$

Sign diagram of LHS is:  $\leftarrow \begin{array}{ccccccc} & - & & + & & - & & + \\ & & | & & | & & | & \\ & & -\sqrt{7} & & -\frac{1}{4} & & \sqrt{7} & \\ & & & & & & & x \end{array} \rightarrow$

$$\therefore x \leq -\sqrt{7} \quad \text{or} \quad -\frac{1}{4} \leq x \leq \sqrt{7}$$



**f**  $5x^3 - 26x > x^2 - 24$

$$\therefore 5x^3 - x^2 - 26x + 24 > 0$$

Using technology,  $x = \frac{6}{5}$  is a root, so  $(5x - 6)$  is a factor of the cubic.

$$\begin{aligned} \therefore 5x^3 - x^2 - 26x + 24 &= (5x - 6)(x^2 + ax - 4) \quad \text{for some constant } a \\ &= 5x^3 + 5ax^2 - 20x \\ &\quad - 6x^2 - 6ax + 24 \\ &= 5x^3 - (5a - 6)x^2 + (-20 - 6a)x + 24 \end{aligned}$$

Equating coefficients of  $x^2$ :  $5a - 6 = -1$

$$\therefore 5a = 5$$

$$\therefore a = 1$$

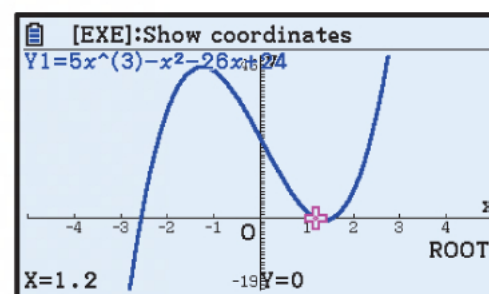
Equating coefficients of  $x$ :  $-20 - 6a = -26$  ✓

$$\therefore 5x^3 - x^2 - 26x + 24 = (5x - 6)(x^2 + x - 4)$$

$$\begin{aligned} x^2 + x - 4 = 0 \quad \text{when} \quad x &= \frac{-1 \pm \sqrt{1 - 4(1)(-4)}}{2} \\ &= \frac{-1 \pm \sqrt{17}}{2} \end{aligned}$$

Sign diagram of LHS is:  $\leftarrow \begin{array}{c} - \quad + \quad - \quad + \\ \frac{-1 - \sqrt{17}}{2} \quad -\frac{6}{5} \quad \frac{-1 + \sqrt{17}}{2} \end{array} \rightarrow x$

$$\therefore \frac{-1 - \sqrt{17}}{2} < x < \frac{6}{5} \quad \text{or} \quad x > \frac{-1 + \sqrt{17}}{2}$$

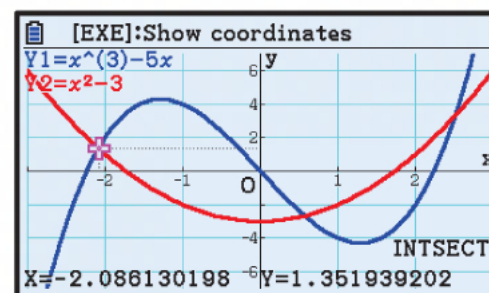


- 4 a** We graph  $y = x^3 - 5x$  and  $y = x^2 - 3$  on the same set of axes.

Using technology, the graphs intersect at  $x \approx -2.09$ ,  $x \approx 0.572$ , and  $x \approx 2.51$ .

$x^3 - 5x > x^2 - 3$  whenever the graph of  $y = x^3 - 5x$  is above the graph of  $y = x^2 - 3$ .

This occurs when  $-2.09 < x < 0.572$  or  $x > 2.51$ .

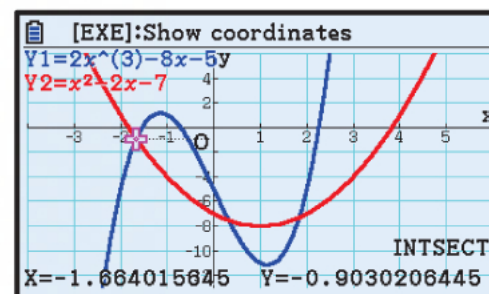


- b** We graph  $y = 2x^3 - 8x - 5$  and  $y = x^2 - 2x - 7$  on the same set of axes.

Using technology, the graphs intersect at  $x \approx -1.66$ ,  $x \approx 0.327$ , and  $x \approx 1.84$ .

$2x^3 - 8x - 5 < x^2 - 2x - 7$  whenever the graph of  $y = 2x^3 - 8x - 5$  is below the graph of  $y = x^2 - 2x - 7$ .

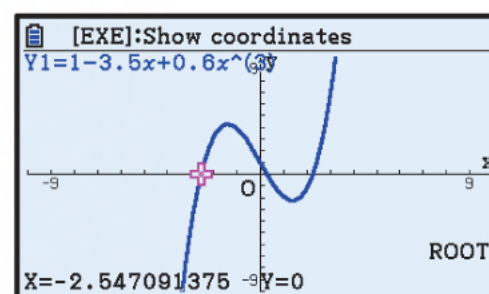
This occurs when  $x < -1.66$  or  $0.327 < x < 1.84$ .



- c** We graph  $y = 1 - 3.5x + 0.6x^3$ .

$1 - 3.5x + 0.6x^3 \geq 0$  when the graph of  $y = 1 - 3.5x + 0.6x^3$  is on or above the  $x$ -axis.

Using technology, this occurs when  $-2.55 < x < 0.290$  or  $x \geq 2.26$ .

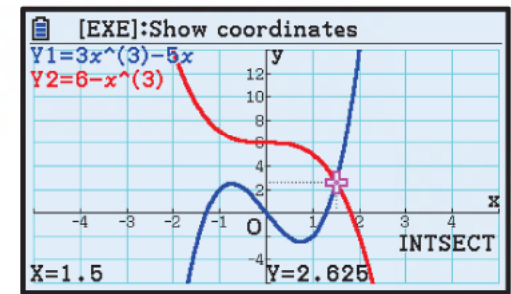


- d** We graph  $y = 3x^3 - 5x$  and  $y = 6 - x^3$  on the same set of axes.

Using technology, the graphs intersect at  $x = \frac{3}{2}$ .

$3x^3 - 5x \leq 6 - x^3$  when the graph of  $y = 3x^3 - 5x$  is on or below the graph of  $y = 6 - x^3$ .

This occurs when  $x \leq \frac{3}{2}$ .



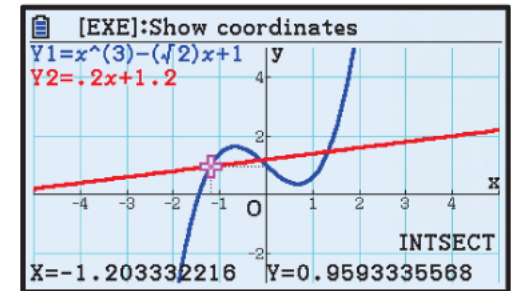
- e** We graph  $y = x^3 - \sqrt{2}x + 1$  and  $y = 0.2x + 1.2$  on the same set of axes.

Using technology, the graphs intersect at  $x \approx -1.20$ ,

$x \approx -0.125$ , and  $x \approx 1.33$ .

$x^3 - \sqrt{2}x + 1 > 0.2x + 1.2$  when the graph of  $y = x^3 - \sqrt{2}x + 1$  is above the graph of  $y = 0.2x + 1.2$ .

This occurs when  $-1.20 < x < -0.125$  or  $x > 1.33$ .

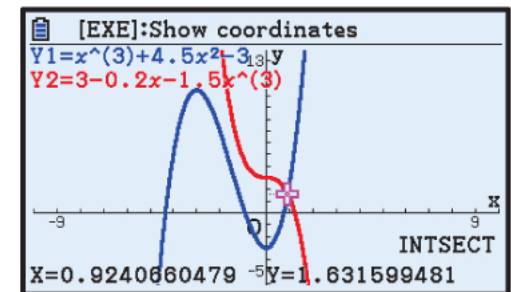


- f** We graph  $y = x^3 + 4.5x^2 - 3$  and  $y = 3 - 0.2x - 1.5x^3$  on the same set of axes.

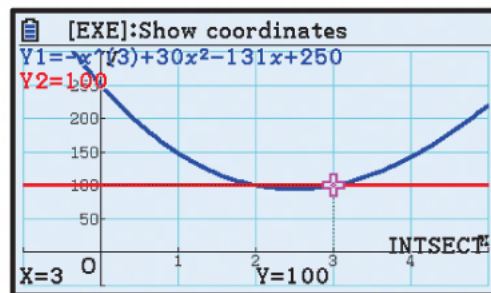
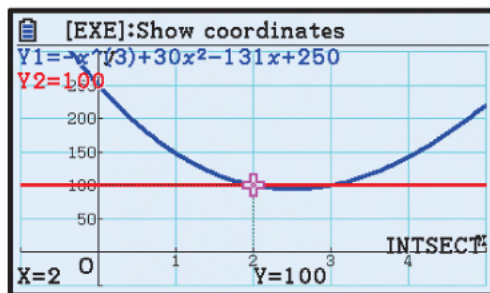
Using technology, the graphs intersect at  $x \approx 0.924$ .

$x^3 + 4.5x^2 - 3 < 3 - 0.2x - 1.5x^3$  when the graph of  $y = x^3 + 4.5x^2 - 3$  is below the graph of  $y = 3 - 0.2x - 1.5x^3$ .

This occurs when  $x < 0.924$ .



5

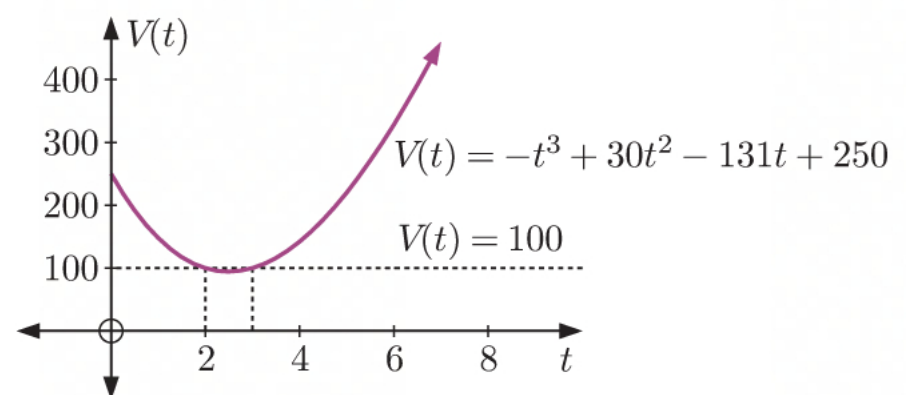


We graph  $y = -x^3 + 30x^2 - 131x + 250$  and  $y = 100$  on the same set of axes.

Using technology, the graphs intersect at  $x = 2$  and  $x = 3$ .

$\therefore -t^3 + 30t^2 - 131t + 250 < 100$  for  $2 < t < 3$ .

$\therefore$  last year, irrigation was prohibited from the end of the 2nd month to the end of the 3rd month, which corresponds to March.





## REVIEW SET 5A

1  $P(z) = 2z^3 - z^2 + 4z$  and  $Q(z) = 3 - z$

a 
$$\begin{aligned} P(z) + 2Q(z) &= 2z^3 - z^2 + 4z + 2(3 - z) \\ &= 2z^3 - z^2 + 4z \\ &\quad - 2z + 6 \\ &= 2z^3 - z^2 + 2z + 6 \end{aligned}$$

b 
$$\begin{aligned} P(z)Q(z) &= (2z^3 - z^2 + 4z)(3 - z) \\ &= 2z^3(3 - z) - z^2(3 - z) + 4z(3 - z) \\ &= -2z^4 + 6z^3 \\ &\quad + z^3 - 3z^2 \\ &\quad - 4z^2 + 12z \\ &= -2z^4 + 7z^3 - 7z^2 + 12z \end{aligned}$$

c 
$$\begin{aligned} \frac{P(z)}{Q(z)} &= \frac{2z^3 - z^2 + 4z}{3 - z} \\ &= \frac{-2z^3 + z^2 - 4z}{z - 3} \end{aligned}$$

$$\begin{array}{r} -2z^2 - 5z - 19 \\ z - 3 \overline{) \begin{array}{r} -2z^3 + z^2 - 4z + 0 \\ -(-2z^3 + 6z^2) \phantom{- 4z + 0} \\ \hline -5z^2 - 4z \phantom{+ 0} \\ -(-5z^2 + 15z) \phantom{+ 0} \\ \hline -19z + 0 \\ -(-19z + 57) \\ \hline -57 \end{array}} \end{array}$$

The quotient is  $-2z^2 - 5z - 19$  and the remainder is  $-57$ .

$$\begin{aligned} \therefore \frac{P(z)}{Q(z)} &= -2z^2 - 5z - 19 - \frac{57}{z - 3} \\ &= -2z^2 - 5z - 19 + \frac{57}{3 - z} \end{aligned}$$

2 a 
$$\begin{aligned} (3x^3 + 2x - 5)(4x - 3) &= 3x^3(4x - 3) + 2x(4x - 3) - 5(4x - 3) \\ &= 12x^4 - 9x^3 + 8x^2 - 6x \\ &\quad - 20x + 15 \\ &= 12x^4 - 9x^3 + 8x^2 - 26x + 15 \end{aligned}$$

b 
$$\begin{aligned} (2x^2 - x + 3)^2 &= (2x^2 - x + 3)(2x^2 - x + 3) \\ &= 2x^2(2x^2 - x + 3) - x(2x^2 - x + 3) + 3(2x^2 - x + 3) \\ &= 4x^4 - 2x^3 + 6x^2 \\ &\quad - 2x^3 + x^2 - 3x \\ &\quad + 6x^2 - 3x + 9 \\ &= 4x^4 - 4x^3 + 13x^2 - 6x + 9 \end{aligned}$$



**3 a**

$$\begin{array}{r}
 x^2 - 2x + 4 \\
 x + 2 \overline{) \begin{array}{l} x^3 + 0x^2 + 0x + 0 \\ -(x^3 + 2x^2) \phantom{+ 0x + 0} \\ \hline -2x^2 + 0x \phantom{+ 0} \\ -(-2x^2 - 4x) \phantom{+ 0} \\ \hline 4x + 0 \\ -(4x + 8) \\ \hline -8 \end{array}}
 \end{array}$$

$$\therefore \frac{x^3}{x+2} = x^2 - 2x + 4 - \frac{8}{x+2}$$

**b**

$$\begin{array}{r}
 2x^2 + 2x - 1 \\
 x - 1 \overline{) \begin{array}{l} 2x^3 + 0x^2 - 3x + 4 \\ -(2x^3 - 2x^2) \phantom{+ 0x + 4} \\ \hline 2x^2 - 3x \phantom{+ 4} \\ -(2x^2 - 2x) \phantom{+ 4} \\ \hline -x + 4 \\ -(-x + 1) \\ \hline 3 \end{array}}
 \end{array}$$

$$\therefore \frac{2x^3 - 3x + 4}{x-1} = 2x^2 + 2x - 1 + \frac{3}{x-1}$$

$$\frac{x^3}{(x+2)(x+3)} = \frac{x^3}{x^2 + 5x + 6}$$

$$\begin{array}{r}
 x - 5 \\
 x^2 + 5x + 6 \overline{) \begin{array}{l} x^3 + 0x^2 + 0x + 0 \\ -(x^3 + 5x^2 + 6x) \phantom{+ 0} \\ \hline -5x^2 - 6x + 0 \\ -(-5x^2 - 25x - 30) \\ \hline 19x + 30 \end{array}}
 \end{array}$$

$$\therefore \frac{x^3}{x^2 + 5x + 6} = x - 5 + \frac{19x + 30}{x^2 + 5x + 6}$$

$$\therefore \frac{x^3}{(x+2)(x+3)} = x - 5 + \frac{19x + 30}{(x+2)(x+3)}$$

**4 a** When  $x = 4$ ,  $x^3 - 3x^2 - 5x + 4 = 4^3 - 3(4)^2 - 5(4) + 4$   
 $= 64 - 48 - 20 + 4$   
 $= 0$

$\therefore 4$  is a zero of  $x^3 - 3x^2 - 5x + 4$ .

**b** When  $x = -3$ ,  $2x^3 + 5x^2 - x - 6 = 2(-3)^3 + 5(-3)^2 - (-3) - 6$   
 $= -54 + 45 + 3 - 6$   
 $= -12 \neq 0$

$\therefore -3$  is not a root of  $2x^3 + 5x^2 - x - 6 = 0$ .

**5** Let  $z^4 + 4 = (z^2 + az + b)(z^2 + cz + d)$  for some constants  $a, b, c, d$   
 $= z^4 + (a+c)z^3 + (ac+b+d)z^2 + (ad+bc)z + bd$

Equating coefficients gives

$$\begin{aligned}
 a + c &= 0 &\Rightarrow c &= -a \\
 ac + b + d &= 0 &\Rightarrow a^2 &= b + d \\
 ad + bc &= 0 &\Rightarrow a(d - b) &= 0 \\
 bd &= 4
 \end{aligned}$$

Now  $a \neq 0$  {since  $b + d = 0$ ,  $bd = 4$  has no solutions}

$\therefore d - b = 0$  and so  $b = d$

$\therefore b^2 = 4$

$\therefore b = d = 2$  {since  $a^2 = b + d$  must be positive}

$\therefore a^2 = 2 + 2 = 4$  and so  $a = \pm 2$  and  $c = \mp 2$

$\therefore z^4 + 4 = (z^2 + 2z + 2)(z^2 - 2z + 2)$

- 6** Since the quartic has real, rational coefficients,  $2 + i\sqrt{3}$  and  $-\sqrt{2} + 1$  are also zeros.  
 $\therefore$  the zeros are  $2 \pm i\sqrt{3}$  and  $1 \pm \sqrt{2}$ .

The zeros  $2 \pm i\sqrt{3}$  have sum = 4 and product =  $(2 + i\sqrt{3})(2 - i\sqrt{3}) = 7$ .

$\therefore$  the zeros  $2 \pm i\sqrt{3}$  come from the quadratic  $z^2 - 4z + 7$ .

The zeros  $1 \pm \sqrt{2}$  have sum = 2 and product =  $(1 + \sqrt{2})(1 - \sqrt{2}) = -1$ .

$\therefore$  the zeros  $1 \pm \sqrt{2}$  come from the quadratic  $z^2 - 2z - 1$ .

$$\begin{aligned}\therefore P(z) &= a(z^2 - 4z + 7)(z^2 - 2z - 1) \\ &= a(z^4 - 2z^3 - z^2 \\ &\quad - 4z^3 + 8z^2 + 4z \\ &\quad + 7z^2 - 14z - 7) \\ &= a(z^4 - 6z^3 + 14z^2 - 10z - 7), \quad a \in \mathbb{Q}, \quad a \neq 0\end{aligned}$$

- 7** Let  $P(z) = z^2 + az + (3 + a)$

if  $-2 + bi$  is a zero then  $P(-2 + bi) = 0$

$$\therefore (-2 + bi)^2 + a(-2 + bi) + 3 + a = 0$$

$$4 - 4bi + b^2i^2 - 2a + abi + 3 + a = 0$$

$$(4 - b^2 - 2a + 3 + a) + i(-4b + ab) = 0$$

$$\therefore 4 - b^2 - 2a + 3 + a = 0 \quad \text{and} \quad -4b + ab = 0$$

$$a = 7 - b^2 \quad \therefore b(a - 4) = 0$$

$$\therefore b = 0 \quad \text{or} \quad a = 4$$

If  $b = 0$  then  $a = 7 - 0 = 7$ .

If  $a = 4$  then  $b^2 = 3$  and so  $b = \pm\sqrt{3}$ .

- 8 a**  $4x^2 - 7x + 5 = 4x^2 + ax + (a + b)$

Since this is true for all  $x$ , we equate coefficients:

$$a = -7 \quad \text{and} \quad a + b = 5$$

$$\therefore -7 + b = 5$$

$$\therefore b = 12$$

So,  $a = -7$  and  $b = 12$ .

**b**  $6x^3 + 11x^2 - 9x - 4 = (2x^2 + 3x - 4)(ax + b)$

$$\therefore 6x^3 + 11x^2 - 9x - 4 = 2ax^3 + 2bx^2 + 3ax^2 + 3bx - 4ax - 4b$$

$$\therefore 6x^3 + 11x^2 - 9x - 4 = 2ax^3 + (2b + 3a)x^2 + (3b - 4a)x - 4b$$

Since this is true for all  $x$ , we equate coefficients:

$$2a = 6 \quad 2b + 3a = 11 \quad 3b - 4a = -9 \quad \text{and} \quad -4b = -4$$

So,  $a = 3$  and  $b = 1$ .

- 9** Since  $(2x - 3)$  is a factor of the cubic, then

$$\begin{aligned}2x^3 + 3x^2 - 29x + 30 &= (2x - 3)(x^2 + bx - 10) \\ &= 2x^3 + (2b - 3)x^2 - (20 + 3b)x + 30\end{aligned}$$

Equating coefficients:  $2b - 3 = 3$  and  $20 + 3b = 29$

$$\therefore b = 3$$

$$\begin{aligned}\therefore 2x^3 + 3x^2 - 29x + 30 &= (2x - 3)(x^2 + 3x - 10) \\ &= (2x - 3)(x + 5)(x - 2)\end{aligned}$$

- 10** Let  $P(x) = 6x^3 + ax^2 - 4ax + b$   
 $(3x + 2)$  and  $(x - 2)$  are factors

$$\therefore P(-\frac{2}{3}) = P(2) = 0 \quad \{\text{Factor theorem}\}$$

$$\therefore P(-\frac{2}{3}) = 6(-\frac{2}{3})^3 + a(-\frac{2}{3})^2 - 4a(-\frac{2}{3}) + b = 0$$

$$-\frac{48}{27} + \frac{4}{9}a + \frac{8a}{3} + b = 0$$

$$-48 + 12a + 72a + 27b = 0$$

$$84a + 27b = 48 \quad \dots (1)$$

$$\therefore P(2) = 6(2)^3 + a(2)^2 - 4a(2) + b = 0$$

$$48 + 4a - 8a + b = 0$$

$$-4a + b = -48$$

$$b = 4a - 48 \quad \dots (2)$$

Substituting (2) into (1),  $84a + 27(4a - 48) = 48$

$$\therefore 84a + 108a - 1296 = 48$$

$$\therefore 192a = 1344$$

$$\therefore a = 7 \text{ and so } b = -20$$

So,  $a = 7$  and  $b = -20$ .

- 11** The Remainder theorem: “When a polynomial  $P(x)$  is divided by  $x - k$  until a constant remainder  $R$  is obtained then  $R = P(k)$ .”

**Proof:** From the division process,  $P(x) = (x - k)Q(x) + R$

$$\text{Now, letting } x = k, \quad P(k) = (k - k) \times Q(k) + R$$

$$\therefore P(k) = 0 \times Q(k) + R$$

$$\therefore P(k) = R$$

$$\therefore R = P(k)$$

**12**  $\frac{P(x)}{x^2 - 3x + 2} = Q(x) + \frac{2x + 3}{x^2 - 3x + 2}$

$$\therefore P(x) = (x^2 - 3x + 2)Q(x) + 2x + 3$$

$$= (x - 1)(x - 2)Q(x) + 2x + 3$$

$$\therefore P(2) = (1)(0)Q(2) + 2(2) + 3$$

$$= 7$$

$\therefore$  when  $P(x)$  is divided by  $x - 2$ , the remainder is 7. {Remainder theorem}

**13** remainder  $= P(2)$  {Remainder theorem}

$$= 2(2)^{17} + 5(2)^{10} - 7(2)^3 + 6$$

$$= 267\,214$$

**14**  $(x + 2)$  and  $(x - 4)$  are factors of  $P(x) = x^3 + x^2 + ax + b$ .

$$\therefore P(-2) = 0 \quad \text{and} \quad P(4) = 0$$

$$\therefore (-2)^3 + (-2)^2 + a(-2) + b = 0$$

$$\therefore -8 + 4 - 2a + b = 0$$

$$\therefore 2a - b = -4 \quad \dots (1)$$

$$\text{and} \quad (4)^3 + (4)^2 + a(4) + b = 0$$

$$\text{and} \quad 64 + 16 + 4a + b = 0$$

$$\text{and} \quad 4a + b = -80 \quad \dots (2)$$

Solving (1) and (2),  $2a - b = -4$

$$4a + b = -80$$

$$\text{Adding, } \underline{6a = -84}$$

$$\therefore a = -14$$

$$\text{In (2): } 4(-14) + b = -80$$

$$\therefore -56 + b = -80$$

$$\therefore b = -24$$

So,  $a = -14$ ,  $b = -24$

**15** If  $f(x)$  has  $(x - k)^2$  as a factor, then  $f(x) = (x - k)^2(x + a)$  for some constant  $a$ .

$$\therefore x^3 - 3x^2 - 9x + b = (x - k)^2(x + a)$$

$$= (x^2 - 2kx + k^2)(x + a)$$

$$= x^3 + ax^2$$

$$- 2kx^2 - 2akx$$

$$+ k^2x + ak^2$$

$$= x^3 + (a - 2k)x^2 + (k^2 - 2ak)x + ak^2$$

Equating coefficients gives  $a - 2k = -3$ ,  $k^2 - 2ak = -9$ , and  $ak^2 = b$ .

Since  $a = 2k - 3$ , then as  $k^2 - 2ak = -9$

$$k^2 - 2k(2k - 3) = -9$$

$$\therefore k^2 - 4k^2 + 6k = -9$$

$$\therefore 3k^2 - 6k - 9 = 0$$

$$\therefore k^2 - 2k - 3 = 0$$

$$\therefore (k - 3)(k + 1) = 0$$

$$\therefore k = -1 \quad \text{or} \quad k = 3$$

If  $k = -1$ ,  $a = -5$  and  $b = ak^2 = -5$  and so  $f(x) = (x + 1)^2(x - 5)$ , and the solutions to  $f(x) = 0$  are  $x = -1$  and  $x = 5$ .

If  $k = 3$ ,  $a = 3$  and  $b = ak^2 = 3 \times 9 = 27$  and so  $f(x) = (x - 3)^2(x + 3)$ , and the solutions to  $f(x) = 0$  are  $x = 3$  and  $x = -3$ .

**16** Since  $P(z) = z^4 + kz^3 + 32z + 3k - 1$  is real, both  $3 - 2i$  and  $3 + 2i$  are zeros.

The zeros  $3 \pm 2i$  have sum = 6 and product =  $9 - 4i^2 = 13$

$\therefore$  the zeros  $3 \pm 2i$  come from the quadratic  $z^2 - 6z + 13$ .

$$\therefore P(z) = (z^2 - 6z + 13)(z^2 + az + b) \quad \text{for some constants } a \text{ and } b$$

$$= z^4 + az^3 + bz^2$$

$$- 6z^3 - 6az^2 - 6bz$$

$$+ 13z^2 + 13az + 13b$$

$$= z^4 + (a - 6)z^3 + (b - 6a + 13)z^2 + (13a - 6b)z + 13b$$



$$\text{Equating coefficients gives } \begin{cases} a - 6 = k \\ b - 6a + 13 = 0 \\ 13a - 6b = 32 \\ 3k - 1 = 13b \end{cases}$$

$$\therefore -6a + b = -13 \quad \dots (1)$$

$$\text{and } 13a - 6b = 32 \quad \dots (2)$$

$$(1) \times 6 \text{ gives } -36a + 6b = -78 \quad \dots (3)$$

$$\text{Adding (2) and (3) gives: } -23a = -46$$

$$\therefore a = 2$$

$$\begin{aligned} \therefore k &= a - 6 & \text{and} & & b &= 6a - 13 \\ &= 2 - 6 & & & &= 6(2) - 13 \\ &= -4 & & & &= -1 \end{aligned}$$

$$\therefore P(z) = (z^2 - 6z + 13)(z^2 + 2z - 1)$$

$$\text{The quadratic factor has zeros } \frac{-2 \pm \sqrt{4+4}}{2} = -1 \pm \sqrt{2}$$

$$\therefore k = -4, \text{ and the zeros are } 3 \pm 2i, -1 \pm \sqrt{2}.$$

**17** Consider  $P(z) = z^4 + 2z^3 + 6z^2 + 8z + 8$

Let  $ai$ ,  $a \in \mathbb{R}$  be the purely imaginary zero.

Since the quartic has real coefficients,  $-ai$  is also a zero.

The zeros  $\pm ai$  have sum = 0 and product =  $a^2$

$\therefore$  the zeros  $\pm ai$  come from the quadratic  $z^2 + a^2$ .

$\therefore z^2 + A$  is a factor.  $\{A = a^2\}$

$$\begin{aligned} \therefore P(z) &= (z^2 + A)(z^2 + Bz + C) \quad \text{for some constants } B \text{ and } C \\ &= z^4 + Bz^3 + Cz^2 \\ &\quad + Az^2 + ABz + AC \\ &= z^4 + Bz^3 + (A + C)z^2 + ABz + AC \end{aligned}$$

$$\text{Equating coefficients gives } B = 2, A + C = 6, AB = 8, \text{ and } AC = 8$$

$$\therefore B = 2 \text{ and so } A = 4 \text{ and } C = 2$$

$$\therefore P(z) = (z^2 + 4)(z^2 + 2z + 2)$$

$$\text{The quadratic factor has zeros } \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$$

$$\therefore \text{ the zeros are } \pm 2i, -1 \pm i.$$

**18 a**  $3x^4 - 4x^3 + 3x^2 + 8$

$$\therefore \text{ the sum of the zeros is } -\frac{(-4)}{3} = \frac{4}{3}$$

The polynomial has degree 4.

$\therefore$  the product of the zeros is

$$\frac{(-1)^4 8}{3} = \frac{8}{3}$$

**b**  $2x^6 + 2x^4 - x^3 + 7x - 10$   
 $= 2x^6 + (0)x^5 + 2x^4 - x^3 + 7x - 10$

$$\therefore \text{ the sum of the zeros is } -\frac{0}{2} = 0$$

The polynomial has degree 6.

$\therefore$  the product of the zeros is

$$\frac{(-1)^6(-10)}{2} = -5$$

- 19 a**  $P(x) = 2x^4 - 8x^3 + ax^2 + bx - 110$  has zeros  $m \pm 2i$  and  $1 \pm n\sqrt{3}$ .  
 $\therefore$  the sum of the zeros is

$$(m + 2i) + (m - 2i) + (1 + n\sqrt{3}) + (1 - n\sqrt{3}) = -\frac{-8}{2}$$

$$\therefore 2m + 2 = 4$$

$$\therefore 2m = 2$$

$$\therefore m = 1$$

$\therefore$  the product of the zeros is

$$(1 + 2i)(1 - 2i)(1 + n\sqrt{3})(1 - n\sqrt{3}) = \frac{(-1)^4(-110)}{2}$$

$$\therefore (1 + 4)(1 - 3n^2) = -55$$

$$\therefore 5(1 - 3n^2) = -55$$

$$\therefore 1 - 3n^2 = -11$$

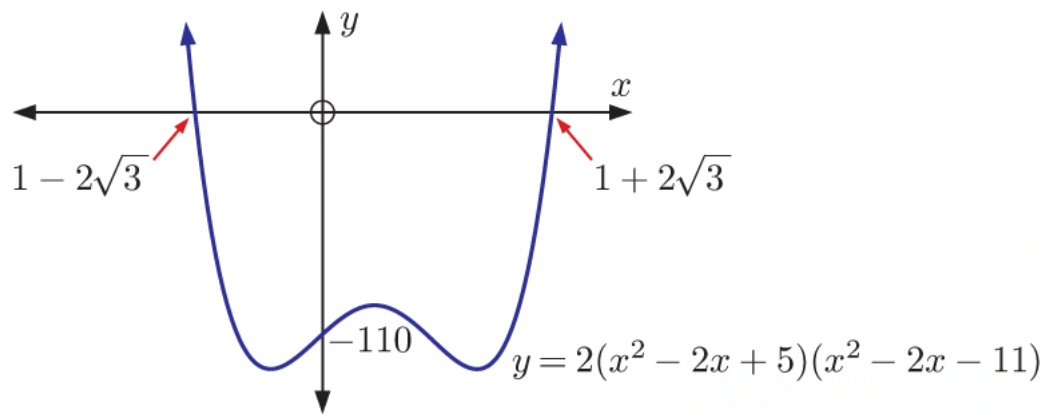
$$\therefore 3n^2 = 12$$

$$\therefore n^2 = 4$$

$$\therefore n = \pm 2$$

So,  $m = 1$  and  $n = \pm 2$ .

- b**  $P(x) = 2(x - 1 + 2i)(x - 1 - 2i)(x - 1 + 2\sqrt{3})(x - 1 - 2\sqrt{3})$   
 $= 2(x^2 - 2x + 5)(x^2 - 2x - 11)$



- 20** Let  $x^3 - x + 1 = (x - \alpha)(x - \beta)(x - \gamma)$   
 $= (x^2 - [\alpha + \beta]x + \alpha\beta)(x - \gamma)$   
 $= x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \beta\gamma + \alpha\gamma)x - \alpha\beta\gamma$

Equating coefficients gives 
$$\begin{cases} \alpha + \beta + \gamma = 0 \\ \alpha\beta + \beta\gamma + \alpha\gamma = -1 \\ \alpha\beta\gamma = -1 \end{cases}$$

Now  $\gamma = \frac{-1}{\alpha\beta}$  .... (1)

and  $\alpha\beta + \gamma(\alpha + \beta) = -1$  .... (2)

$\therefore \alpha\beta + \gamma(-\gamma) = -1$

$\therefore \alpha\beta - \gamma^2 = -1$

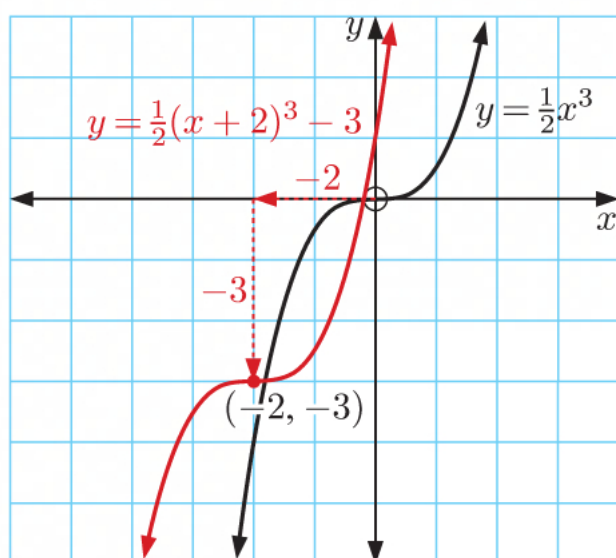
$\therefore \alpha\beta - \frac{1}{(\alpha\beta)^2} = -1$  {using (1)}

$\therefore (\alpha\beta)^3 - 1 = -(\alpha\beta)^2$

$\therefore (\alpha\beta)^3 + (\alpha\beta)^2 - 1 = 0$

$\therefore \alpha\beta$  is a root of  $x^3 + x^2 - 1 = 0$

- 21** If  $a + ai$  is a root of  $x^2 - 6x + b = 0$  then  $(a + ai)^2 - 6(a + ai) + b = 0$   
 $\therefore a^2 + 2a^2i - a^2 - 6a - 6ai + b = 0$   
 $\therefore (b - 6a) + (2a^2 - 6a)i = 0$   
 $\therefore b = 6a$  and  $2a^2 - 6a = 0$   
 $\therefore b = 6a$  and  $2a(a - 3) = 0$   
 $\therefore a = 0$  or  $3$ , and when  $a = 0$ ,  $b = 0$   
and when  $a = 3$ ,  $b = 18$

**22**

We translate  $y = \frac{1}{2}x^3$  2 units left and 3 units down.

- 23 a** The  $x$ -intercepts are  $-2$ ,  $2$ , and  $3$ .

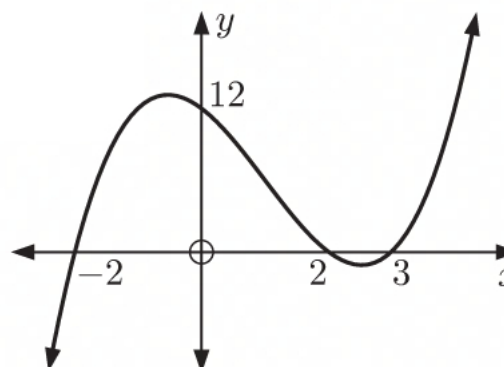
$$\therefore y = a(x + 2)(x - 2)(x - 3)$$

But when  $x = 0$ ,  $y = 12$

$$\therefore a(2)(-2)(-3) = 12$$

$$\therefore a = 1$$

So,  $y = (x + 2)(x - 2)(x - 3)$



- b** The graph touches the  $x$ -axis at  $-1$ , indicating a squared factor  $(x + 1)^2$ .

The other  $x$ -intercept is  $4$ , so

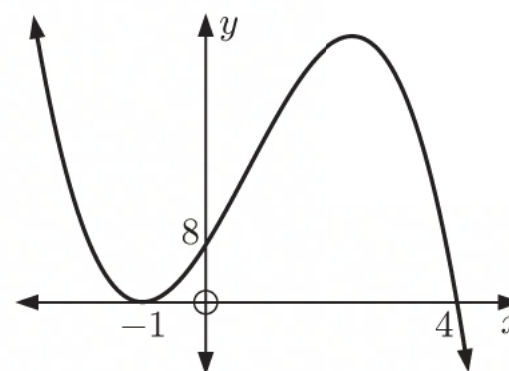
$$y = a(x + 1)^2(x - 4).$$

But when  $x = 0$ ,  $y = 8$

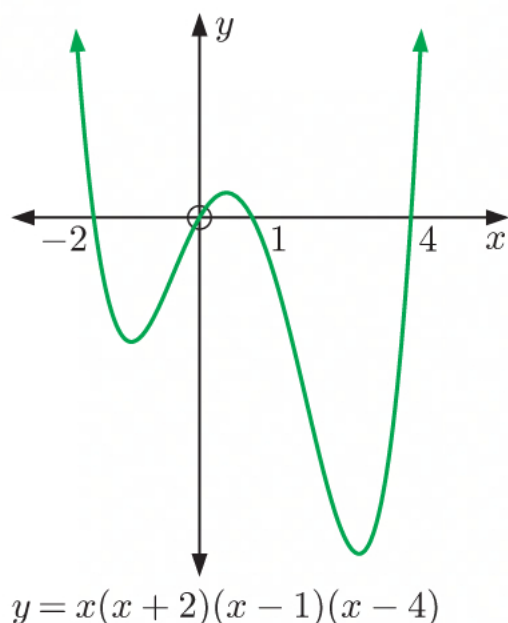
$$\therefore a(1)^2(-4) = 8$$

$$\therefore a = -2$$

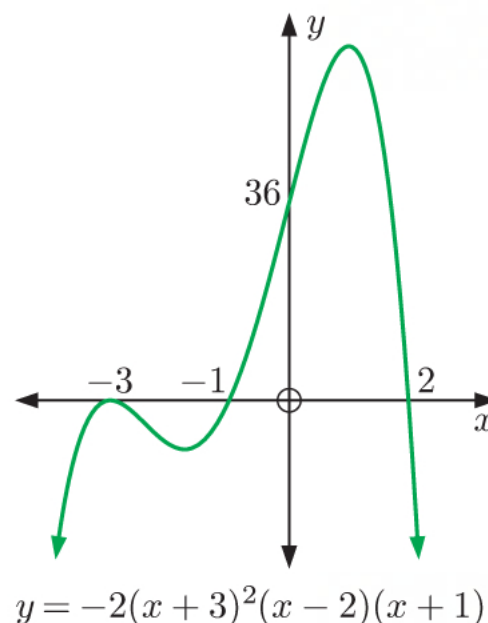
So,  $y = -2(x + 1)^2(x - 4)$



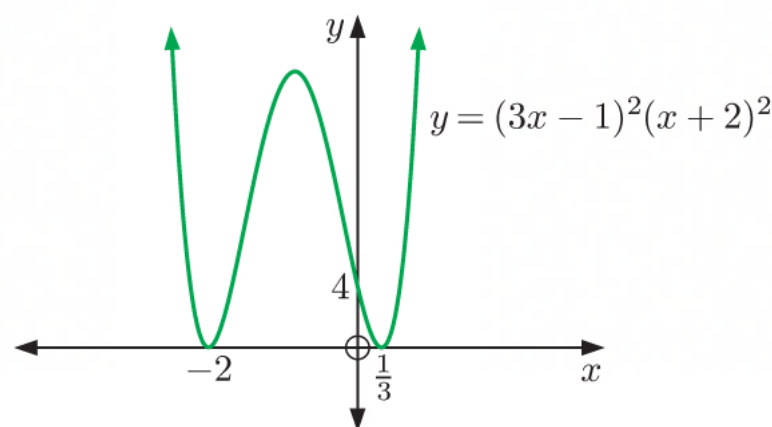
- 24 a**  $a > 0$ , so the graph opens upwards.  
 The graph cuts the  $x$ -axis at 0,  $-2$ , 1, and 4.  
 When  $x = 0$ ,  $y = 0(2)(-1)(-4) = 0$   
 $\therefore$  the  $y$ -intercept is 0.



- b**  $a < 0$ , so the graph opens downwards.  
 The graph touches the  $x$ -axis at  $-3$ , and cuts the  $x$ -axis at 2 and  $-1$ .  
 When  $x = 0$ ,  $y = -2(3)^2(-2)(1) = 36$   
 $\therefore$  the  $y$ -intercept is 36.



- c**  $a > 0$  so the graph opens upwards.  
 The graph touches the  $x$ -axis at  $\frac{1}{3}$  and  $-2$ .  
 When  $x = 0$ ,  $y = (-1)^2(2)^2 = 4$   
 $\therefore$  the  $y$ -intercept is 4.



- 25** The graph touches the  $x$ -axis at  $(-2, 0)$  and cuts the  $x$ -axis at  $(1, 0)$   
 $\therefore (x+2)^2$  and  $(x-1)$  are factors of  $P(x)$ .  
 $\therefore P(x) = (x+2)^2(x-1)(ax+b)$  for some constants  $a$  and  $b$ .

Now  $P(0) = 12$

$\therefore (2)^2(-1)(b) = 12$

$\therefore -4b = 12$

$\therefore b = -3$

$\therefore P(x) = (x+2)^2(x-1)(ax-3)$

Also  $P(2) = 80$

$\therefore (4)^2(1)(2a-3) = 80$

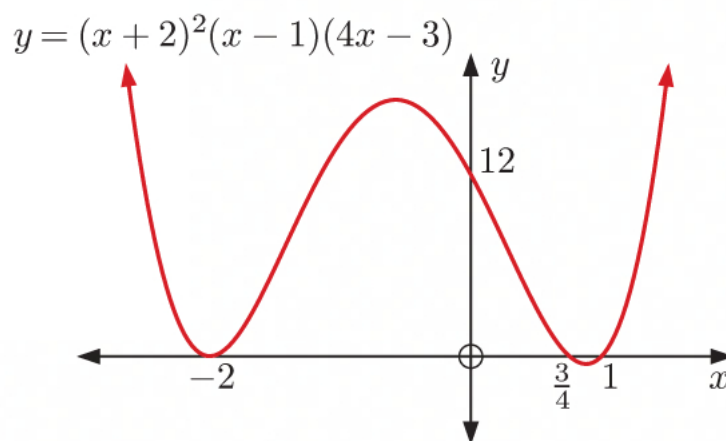
$\therefore 16(2a-3) = 80$

$\therefore 2a-3 = 5$

$\therefore 2a = 8$

$\therefore a = 4$

$\therefore P(x) = (x+2)^2(x-1)(4x-3)$





- 26** The graph cuts the  $x$ -axis at  $-4$  and  $3$ , and touches the  $x$ -axis at  $1$ .

$$\therefore P(x) = a(x+4)(x-1)^2(x-3), \quad a \neq 0$$

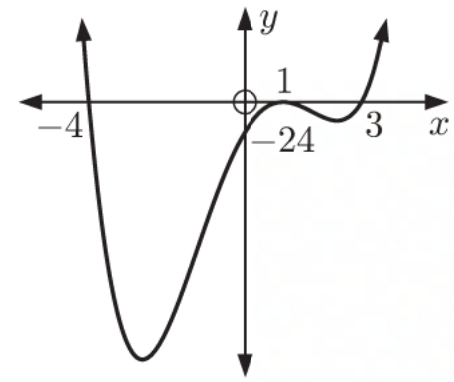
$$\text{Also, } P(0) = -24$$

$$\therefore a(4)(-1)^2(-3) = -24$$

$$\therefore -12a = -24$$

$$\therefore a = 2$$

$$\therefore P(x) = 2(x+4)(x-1)^2(x-3)$$



- 27 a**  $P(x) = 2x^4 + 7x^3 + 4x^2 - 4x$

Using technology,  $-2$ ,  $0$ , and  $\frac{1}{2}$  are roots.

The graph touches the  $x$ -axis at  $-2$ .

$\therefore (x+2)^2$ ,  $x$ , and  $(2x-1)$  are factors.

$$\therefore P(x) = ax(x+2)^2(2x-1) \quad \text{for some constant } a$$

$$\text{But } P(x) = ax(x^2 + 4x + 4)(2x - 1)$$

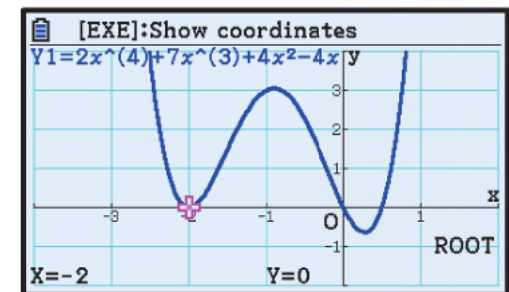
$$= ax(2x^3 - x^2 + 8x^2 - 4x + 8x - 4)$$

$$= ax(2x^3 + 7x^2 + 4x - 4)$$

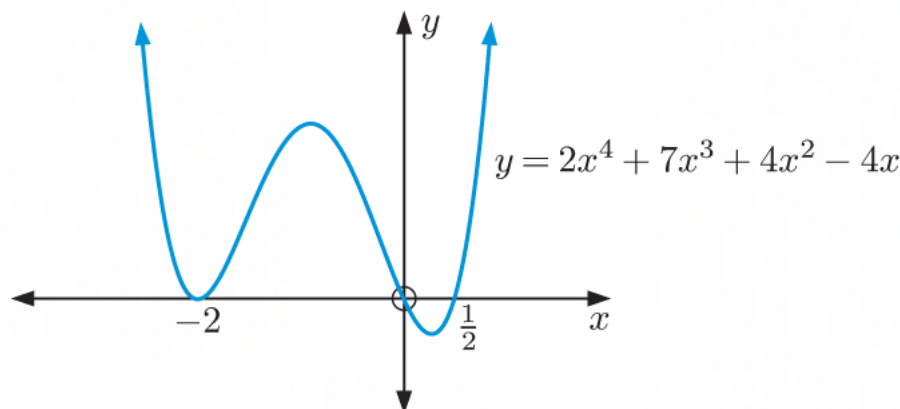
$$= a(2x^4 + 7x^3 + 4x^2 - 4x)$$

$$\therefore a = 1$$

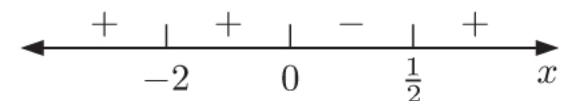
$$\therefore P(x) = x(x+2)^2(2x-1)$$



**b**



- c** Sign diagram of  $P(x)$ :



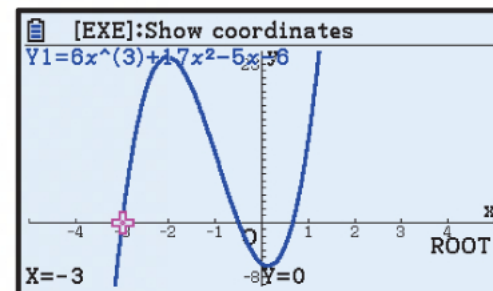
$$\therefore P(x) \geq 0 \quad \text{for } x \leq 0 \quad \text{and} \quad x \geq \frac{1}{2}.$$

- 28 a**  $6x^3 + 17x^2 = 5x + 6$

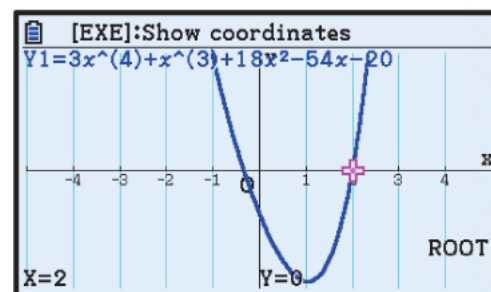
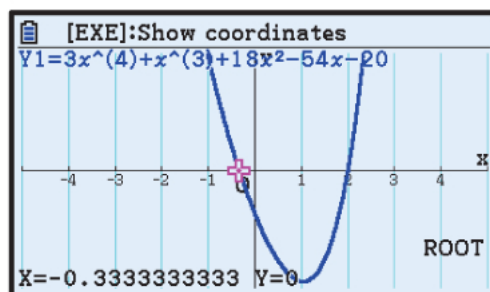
$$\therefore 6x^3 + 17x^2 - 5x - 6 = 0$$

Using technology,  $-3$ ,  $-\frac{1}{2}$ , and  $\frac{2}{3}$  are roots.

$$\therefore x = -3, -\frac{1}{2}, \text{ or } \frac{2}{3}.$$



**b**



$$3x^4 + x^3 + 18x^2 - 54x - 20 = 0$$

Using technology,  $-\frac{1}{3}$  and  $2$  are roots.

$\therefore (3x+1)$  and  $(x-2)$  are factors.

$$\begin{aligned}
 \therefore 3x^4 + x^3 + 18x^2 - 54x - 20 &= (3x + 1)(x - 2)(x^2 + ax + 10) \quad \text{for some constant } a \\
 &= (3x^2 - 5x - 2)(x^2 + ax + 10) \\
 &= 3x^4 + 3ax^3 + 30x^2 \\
 &\quad - 5x^3 - 5ax^2 - 50x \\
 &\quad - 2x^2 - 2ax - 20 \\
 &= 3x^4 + (3a - 5)x^3 + (28 - 5a)x^2 + (-50 - 2a)x - 20
 \end{aligned}$$

Equating coefficients of  $x^3$ :  $3a - 5 = 1$   
 $\therefore a = 2$

Equating coefficients of  $x^2$ :  $28 - 5a = 28 - 10 = 18$  ✓

Equating coefficients of  $x$ :  $-50 - 2a = -54$  ✓

$$\therefore (3x + 1)(x - 2)(x^2 + 2x + 10) = 0$$

$$\therefore x = -\frac{1}{3}, 2, \text{ or } \frac{-2 \pm \sqrt{4 - 4(1)(10)}}{2(1)} = -1 \pm 3i$$

**29 a**  $x^3 + x^2 - 12x - 18 < 0$

Using technology,  $x = -3$  is a root, so  $(x + 3)$  is a factor of the cubic.

$$\begin{aligned}
 \therefore x^3 + x^2 - 12x - 18 &= (x + 3)(x^2 + ax - 6) \quad \text{for some } a \\
 &= x^3 + ax^2 - 6x \\
 &\quad + 3x^2 + 3ax - 18 \\
 &= x^3 + (a + 3)x^2 + (3a - 6)x - 18
 \end{aligned}$$

Equating coefficients of  $x^2$ :  $a + 3 = 1$   
 $\therefore a = -2$

Equating coefficients of  $x$ :  $3a - 6 = -12$  ✓

$$\therefore x^3 + x^2 - 12x - 18 = (x + 3)(x^2 - 2x - 6)$$

$$\begin{aligned}
 x^2 - 2x - 6 = 0 \quad \text{when} \quad x &= \frac{2 \pm \sqrt{4 - 4(1)(-6)}}{2} \\
 &= \frac{2 \pm \sqrt{28}}{2} \\
 &= 1 \pm \sqrt{7}
 \end{aligned}$$

Sign diagram of LHS is:  $\begin{array}{ccccccc} & - & & + & & - & & + \\ & & -3 & & 1 - \sqrt{7} & & 1 + \sqrt{7} & & x \end{array}$

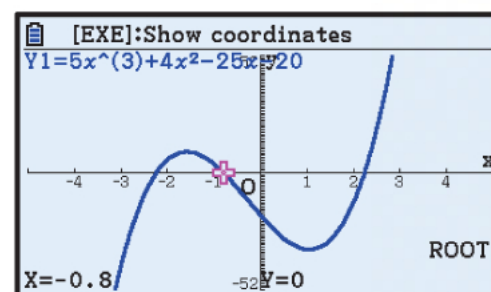
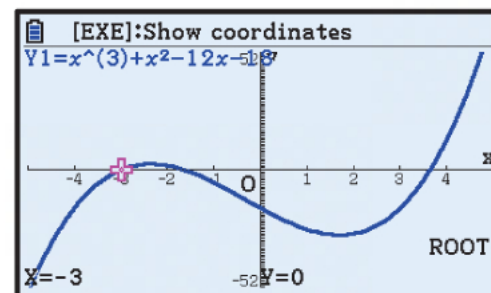
$$\therefore x < -3 \quad \text{or} \quad 1 - \sqrt{7} < x < 1 + \sqrt{7}$$

**b**

$$\begin{aligned}
 5x^3 - 20 &\geq 25x - 4x^2 \\
 \therefore 5x^3 + 4x^2 - 25x - 20 &\geq 0
 \end{aligned}$$

Using technology,  $x = -\frac{4}{5}$  is a root, so  $(5x + 4)$  is a factor of the cubic.

$$\begin{aligned}
 \therefore 5x^3 + 4x^2 - 25x - 20 &= (5x + 4)(x^2 + ax - 5) \quad \text{for some constant } a \\
 &= 5x^3 + 5ax^2 - 25x \\
 &\quad - 4x^2 + 4ax - 20 \\
 &= 5x^3 + (5a + 4)x^2 + (4a - 25)x - 20
 \end{aligned}$$



Equating coefficients of  $x^2$ :  $5a + 4 = 4$

$$\therefore a = 0$$

Equating coefficients of  $x$ :  $4a - 25 = -25$  ✓

$$\begin{aligned}\therefore 5x^3 + 4x^2 - 25x - 20 &= (5x + 4)(x^2 - 5) \\ &= (5x + 4)(x + \sqrt{5})(x - \sqrt{5})\end{aligned}$$

Sign diagram of LHS is: 

$$\therefore -\sqrt{5} \leq x \leq -\frac{4}{5} \text{ or } x \geq \sqrt{5}$$

## REVIEW SET 5B

**1**  $f(x) = x^4 - 5x^3 + 2x - 1$  and  $g(x) = -2x^3 + 3x^2 + 7$

**a** 
$$\begin{aligned}f(x) - g(x) &= x^4 - 5x^3 + 2x - 1 \\ &\quad - (-2x^3 + 3x^2 + 7) \\ &= x^4 - 3x^3 - 3x^2 + 2x - 8\end{aligned}$$

**b** 
$$\begin{aligned}f(x)g(x) &= (x^4 - 5x^3 + 2x - 1)(-2x^3 + 3x^2 + 7) \\ &= -2x^7 + 3x^6 + 7x^4 \\ &\quad + 10x^6 - 15x^5 - 35x^3 \\ &\quad - 4x^4 + 6x^3 + 14x \\ &\quad + 2x^3 - 3x^2 - 7 \\ &= -2x^7 + 13x^6 - 15x^5 + 3x^4 - 27x^3 - 3x^2 + 14x - 7\end{aligned}$$

**i** The highest power of the variable  $x$  is 7.

$\therefore$  the polynomial has degree 7.

**ii** The leading coefficient is  $-2$ .

**iii** The constant term is  $-7$ .

**2 a** 
$$\begin{aligned}(3x^2 - 2x + 1)(x^3 + 5x^2 - 2) &= 3x^5 + 15x^4 - 6x^2 \\ &\quad - 2x^4 - 10x^3 + 4x \\ &\quad + x^3 + 5x^2 - 2 \\ &= 3x^5 + 13x^4 - 9x^3 - x^2 + 4x - 2\end{aligned}$$

**b** 
$$\begin{aligned}(x^3 - 2x^2 + 1)^2 &= (x^3 - 2x^2 + 1)(x^3 - 2x^2 + 1) \\ &= x^6 - 2x^5 + x^3 \\ &\quad - 2x^5 + 4x^4 - 2x^2 \\ &\quad + x^3 - 2x^2 + 1 \\ &= x^6 - 4x^5 + 4x^4 + 2x^3 - 4x^2 + 1\end{aligned}$$

**3 a**  $3x^2 - 11x + 6 = (3x - 2)(x - 3)$

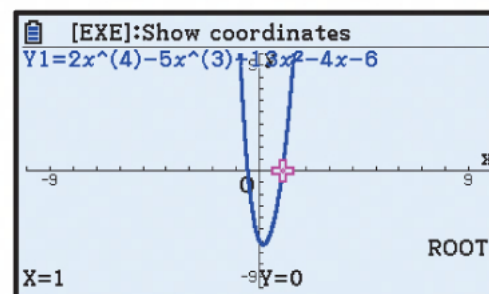
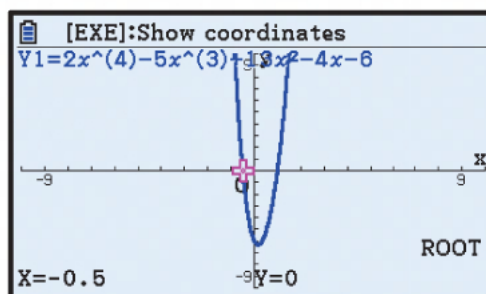
The zeros are  $\frac{2}{3}$  and 3.



- b** We need to find  $z$  such that  $2z^4 - 5z^3 + 13z^2 - 4z - 6 = 0$ .

Using technology,  $z = -\frac{1}{2}$   
and  $z = 1$  are roots.

$\therefore (2z + 1)$  and  $(z - 1)$   
are factors.



$$\begin{aligned}\therefore 2z^4 - 5z^3 + 13z^2 - 4z - 6 &= (2z + 1)(z - 1)(z^2 + az + 6) \quad \text{for some constant } a \\ &= (2z^2 - z - 1)(z^2 + az + 6) \\ &= 2z^4 + 2az^3 + 12z^2 \\ &\quad - z^3 - az^2 - 6z \\ &\quad - z^2 - az - 6 \\ &= 2z^4 + (2a - 1)z^3 + (11 - a)z^2 + (-6 - a)z - 6\end{aligned}$$

Equating coefficients of  $z^3$ :  $2a - 1 = -5$   
 $\therefore a = -2$

Equating coefficients of  $z^2$ :  $11 - a = 13$  ✓

Equating coefficients of  $z$ :  $-6 - a = -4$  ✓

$$\therefore 2z^4 - 5z^3 + 13z^2 - 4z - 6 = (2z + 1)(z - 1)(z^2 - 2z + 6)$$

$$z^2 - 2z + 6 = 0 \quad \text{when} \quad z = \frac{2 \pm \sqrt{4 - 4(1)(6)}}{2}$$

$$\therefore z = \frac{2 \pm \sqrt{-20}}{2}$$

$$\therefore z = 1 \pm i\sqrt{5}$$

$$\therefore \text{the zeros are } 1, -\frac{1}{2}, \text{ and } 1 \pm i\sqrt{5}.$$

**4**  $(3x - 2)(x + 2)(x - 3) = (3x^2 + 4x - 4)(x - 3)$   
 $= 3x^3 - 5x^2 - 16x + 12$  ✓

- 5 a**  $-4$  comes from the linear factor  $(x + 4)$ .  
 $1$  comes from the linear factor  $(x - 1)$ .  
 $6$  comes from the linear factor  $(x - 6)$ .  
 $\therefore P(x) = a(x + 4)(x - 1)(x - 6), a \neq 0$

- b**  $\frac{1}{3}$  comes from the linear factor  $(3x - 1)$ .  
The zeros  $3 \pm \sqrt{5}$  have sum  $= 3 + \sqrt{5} + 3 - \sqrt{5} = 6$   
and product  $= (3 + \sqrt{5})(3 - \sqrt{5}) = 4$   
 $\therefore$  they come from the quadratic factor  $(x^2 - 6x + 4)$ .  
 $\therefore P(x) = a(3x - 1)(x^2 - 6x + 4), a \neq 0$

- c** Since the polynomials have real coefficients, if  $i\sqrt{2}$  is a zero, then so is  $-i\sqrt{2}$ .  
 $\frac{1}{2}$  comes from the linear factor  $(2x - 1)$ .  
The zeros  $\pm i\sqrt{2}$  have sum  $= 0$   
and product  $= 2$   
 $\therefore$  they come from the quadratic factor  $(x^2 + 2)$ .  
 $\therefore P(x) = a(2x - 1)(x^2 + 2), a \neq 0$



**d** Since the polynomials have real coefficients, the zeros are  $1 \pm i$  and  $-3 \pm i$ .

The zeros  $1 \pm i$  have sum  $= 1 + i + 1 - i = 2$

and product  $= (1 + i)(1 - i) = 2$

$\therefore$  they come from the quadratic factor  $(x^2 - 2x + 2)$ .

The zeros  $-3 \pm i$  have sum  $= -3 + i - 3 - i = -6$

and product  $= (-3 + i)(-3 - i) = 10$

$\therefore$  they come from the quadratic factor  $(x^2 + 6x + 10)$ .

$\therefore P(x) = a(x^2 - 2x + 2)(x^2 + 6x + 10), a \neq 0$

$$\begin{aligned}
 \text{6 } (2x^2 + bx + c)(x^2 + 2x + 5) &= 2x^4 + 4x^3 + 10x^2 \\
 &\quad + bx^3 + 2bx^2 + 5bx \\
 &\quad + cx^2 + 2cx + 5c \\
 &= 2x^4 + (b + 4)x^3 + (2b + c + 10)x^2 + (5b + 2c)x + 5c
 \end{aligned}$$

$$\text{Equating coefficients gives } \begin{cases} b + 4 = 3 \\ 2b + c + 10 = 11 \\ 5b + 2c = a \\ 5c = 15 \end{cases}$$

$\therefore b = -1, c = 3$ , and so  $a = 1$ .

So  $a = 1, b = -1$ , and  $c = 3$ .

**7 a**

$$\begin{array}{r}
 \phantom{2x+3} \overline{x^2 + 3x - 2} \\
 2x + 3 \overline{) 2x^3 + 9x^2 + 5x - 1} \\
 \underline{-(2x^3 + 3x^2)} \phantom{-1} \downarrow \\
 6x^2 + 5x \phantom{-1} \downarrow \\
 \underline{-(6x^2 + 9x)} \phantom{-1} \downarrow \\
 -4x - 1 \phantom{-1} \downarrow \\
 \underline{-(-4x - 6)} \\
 5
 \end{array}$$

$$\therefore \frac{2x^3 - 9x^2 + 5x - 1}{2x + 3} = x^2 + 3x - 2 + \frac{5}{2x + 3}$$

**b**

$$\begin{array}{r}
 \phantom{x-1} \overline{x^3 + x^2 + 2x + 2} \\
 x - 1 \overline{) x^4 + 0x^3 + x^2 + 0x + 2} \\
 \underline{-(x^4 - x^3)} \phantom{+ 2} \downarrow \\
 x^3 + x^2 \phantom{+ 2} \downarrow \\
 \underline{-(x^3 - x^2)} \phantom{+ 2} \downarrow \\
 2x^2 + 0x \phantom{+ 2} \downarrow \\
 \underline{-(2x^2 - 2x)} \phantom{+ 2} \downarrow \\
 2x + 2 \phantom{+ 2} \downarrow \\
 \underline{-(2x - 2)} \\
 4
 \end{array}$$

$$\therefore \frac{x^4 + x^2 + 2}{x - 1} = x^3 + x^2 + 2x + 2 + \frac{4}{x - 1}$$

$$\begin{array}{r}
 \text{C} \quad \frac{x^4 + x^2}{(x-1)(x+2)} = \frac{x^4 + x^2}{x^2 + x - 2} \\
 x^2 + x - 2 \overline{) \begin{array}{r} x^4 + 0x^3 + x^2 + 0x + 0 \\ -(x^4 + x^3 - 2x^2) \phantom{+ 0x + 0} \\ \hline -x^3 + 3x^2 + 0x \\ -(-x^3 - x^2 + 2x) \phantom{+ 0} \\ \hline 4x^2 - 2x + 0 \\ -(4x^2 + 4x - 8) \\ \hline -6x + 8 \end{array}}
 \end{array}$$

$$\therefore \frac{x^4 + x^2}{(x-1)(x+2)} = x^2 - x + 4 + \frac{8 - 6x}{(x-1)(x+2)}$$

8 Since  $(3x - 2)$  is a factor of the cubic, then

$$\begin{aligned}
 6x^3 + 23x^2 - 33x + 10 &= (3x - 2)(2x^2 + ax - 5) \quad \text{for some constant } a \\
 &= 6x^3 + 3ax^2 - 15x \\
 &\quad - 4x^2 - 2ax + 10 \\
 &= 6x^3 + (3a - 4)x^2 + (-2a - 15)x + 10
 \end{aligned}$$

$$\begin{aligned}
 \text{Equating coefficients of } x^2: \quad 3a - 4 &= 23 \\
 \therefore a &= 9
 \end{aligned}$$

$$\begin{aligned}
 \text{Equating coefficients of } x: \quad -2a - 15 &= -33 \\
 \therefore -2(9) - 15 &= -33 \quad \checkmark
 \end{aligned}$$

$$\begin{aligned}
 \therefore 6x^3 + 23x^2 - 33x + 10 &= (3x - 2)(2x^2 + 9x - 5) \\
 &= (3x - 2)(2x - 1)(x + 5)
 \end{aligned}$$

9 Since  $2x + 1$  is a factor of the cubic, then

$$\begin{aligned}
 4x^3 - 8x^2 - 11x - 3 &= (2x + 1)(2x^2 + bx - 3) \\
 &= 4x^3 + (2b + 2)x^2 + (b - 6)x - 3 \quad \text{for some constant } b
 \end{aligned}$$

$$\begin{aligned}
 \text{Equating coefficients of } x^2: \quad 2b + 2 &= -8 \\
 \therefore b &= -5
 \end{aligned}$$

$$\text{Equating coefficients of } x: \quad b - 6 = -11 \quad \checkmark$$

$$\begin{aligned}
 \therefore 4x^3 - 8x^2 - 11x - 3 &= (2x + 1)(2x^2 - 5x - 3) \\
 &= (2x + 1)(2x + 1)(x - 3) \\
 &= (2x + 1)^2(x - 3)
 \end{aligned}$$

The solutions of  $4x^3 - 8x^2 - 11x - 3 = 0$  are  $x = -\frac{1}{2}$  (repeated) and  $x = 3$ .

$$\begin{aligned}
 \text{10 When } x = \frac{1}{4}, \quad 4x^3 - 13x^2 + 15x - 3 &= 4\left(\frac{1}{4}\right)^3 - 13\left(\frac{1}{4}\right)^2 + 15\left(\frac{1}{4}\right) - 3 \\
 &= \frac{1}{16} - \frac{13}{16} + \frac{3}{4} \\
 &= 0
 \end{aligned}$$

$$\therefore x = \frac{1}{4} \text{ is one solution of } 4x^3 - 13x^2 + 15x - 3 = 0.$$

The solution  $x = \frac{1}{4}$  comes from the linear factor  $(4x - 1)$ .

$$\begin{aligned}
 \therefore 4x^3 - 13x^2 + 15x - 3 &= (4x - 1)(x^2 + bx + 3) \\
 &= 4x^3 + (4b - 1)x^2 + (12 - b)x - 3 \quad \text{for some constant } b
 \end{aligned}$$

Equating coefficients of  $x^2$ :  $4b - 1 = -13$

$$\therefore b = -3$$

Equating coefficients of  $x$ :  $12 - b = 15$

$$\therefore 4x^3 - 13x^2 + 15x - 3 = (4x - 1)(x^2 - 3x + 3)$$

$$\begin{aligned} \text{For } x^2 - 3x + 3, \quad \Delta &= b^2 - 4ac \\ &= (-3)^2 - 4(1)(3) \\ &= -3 \text{ which is } < 0 \end{aligned}$$

$\therefore x^2 - 3x + 3$  has no real solutions.

$\therefore x = \frac{1}{4}$  is the only real solution of  $4x^3 - 13x^2 + 15x - 3 = 0$ .

$$\begin{aligned} \textbf{11} \quad \text{Let } P(z) &= z^3 + az^2 + kz + ka \\ \therefore P(z) &= z^2(z + a) + k(z + a) \\ &= (z + a)(z^2 + k) \end{aligned}$$

Alternatively, consider  $P(-a) = -a^3 + a^3 - ka + ka = 0$

$$\begin{aligned} \therefore z + a \text{ is a factor} \quad & -a \left| \begin{array}{cccc} 1 & a & k & ka \\ 0 & -a & 0 & -ka \\ \hline 1 & 0 & k & 0 \end{array} \right. \\ \therefore P(z) &= (z + a)(z^2 + k) \end{aligned}$$

**a**  $P(z) = 0$  has one real root if  $k > 0$ ,  $a \in \mathbb{R}$

**b**  $P(z) = 0$  has 3 real roots if  $k \leq 0$ ,  $a \in \mathbb{R}$

$$\begin{aligned} \textbf{12} \quad \text{Let } P(x) &= x^{47} - 3x^{26} + 5x^3 + 11 \\ \therefore \text{remainder} &= P(-1) \quad \{\text{Remainder theorem}\} \\ &= (-1)^{47} - 3(-1)^{26} + 5(-1)^3 + 11 \\ &= -1 - 3 - 5 + 11 \\ &= 2 \end{aligned}$$

$$\begin{aligned} \textbf{13} \quad & P(-2) = -54 \quad \{\text{Remainder theorem}\} \\ \therefore (-2)^m + 6(-2)^2 + 50 &= -54 \\ \therefore (-2)^m + 74 &= -54 \\ \therefore (-2)^m &= -128 \\ \therefore m &= 7 \end{aligned}$$

$$\textbf{14} \quad \text{Let } P(x) = x^4 + 3x^3 - 7x^2 + 11x - 1$$

$$\begin{array}{r} x^2 + 3x - 9 \\ x^2 + 2 \overline{) \begin{array}{l} x^4 + 3x^3 - 7x^2 + 11x - 1 \\ -(x^4 \phantom{+ 2x^2}) \phantom{+ 11x - 1} \\ \hline 3x^3 - 9x^2 + 11x \\ -(3x^3 \phantom{+ 2x^2} + 6x) \phantom{- 1} \\ \hline -9x^2 + 5x - 1 \\ -(-9x^2 \phantom{+ 5x} - 18) \\ \hline 5x + 17 \end{array}} \end{array}$$

$\therefore$  the quotient is  $Q(x) = x^2 + 3x - 9$  and the remainder is  $R(x) = 5x + 17$ .

The new polynomial is divisible by  $x^2 + 2$  if

$$\begin{aligned} x^4 + 3x^3 - 7x^2 + (2+a)x + b &= P(x) - R(x) \\ &= x^4 + 3x^3 - 7x^2 + 11x - 1 - (5x + 17) \\ &= x^4 + 3x^3 - 7x^2 + 6x - 18 \\ \therefore 2 + a &= 6 \text{ and } b = -18 \quad \{\text{equating coefficients}\} \\ \therefore a &= 4 \text{ and } b = -18 \end{aligned}$$

**15 a**  $x^2 + 1 = (x + i)(x - i)$

$$\begin{aligned} \text{Let } P(x) &= x^{10} + 1 \\ P(i) &= i^{10} + 1 \\ &= (i^2)^5 + 1 \\ &= (-1)^5 + 1 \\ &= 0 \end{aligned}$$

$\therefore (x - i)$  is a factor of  $P(x)$  {Factor theorem}

$$\begin{aligned} P(-i) &= (-i)^{10} + 1 \\ &= (-1)^{10} i^{10} + 1 \\ &= 1(-1) + 1 \\ &= 0 \end{aligned}$$

$\therefore (x + i)$  is a factor of  $P(x)$  {Factor theorem}

So both  $(x + i)$  and  $(x - i)$  are factors of  $x^{10} + 1$ .

$\therefore (x + i)(x - i) = x^2 + 1$  is a factor of  $x^{10} + 1$ .

**b** From **a**,  $x^{10} + 1 = (x^2 + 1)Q(x) \dots (*)$

$$\begin{aligned} \frac{x^{11} + 1}{x^2 + 1} &= \frac{x(x^{10} + 1) - x + 1}{x^2 + 1} \\ &= \frac{x(x^{10} + 1)}{x^2 + 1} + \frac{1 - x}{x^2 + 1} \\ &= \frac{\cancel{x(x^2 + 1)} Q(x)}{\cancel{x^2 + 1}} + \frac{1 - x}{x^2 + 1} \quad \{\text{using } (*)\} \\ &= xQ(x) + \frac{1 - x}{x^2 + 1} \end{aligned}$$

$\therefore$  the remainder is  $1 - x$ .

**16** Let  $P(z) = 2z^3 + z^2 + 10z + 5$

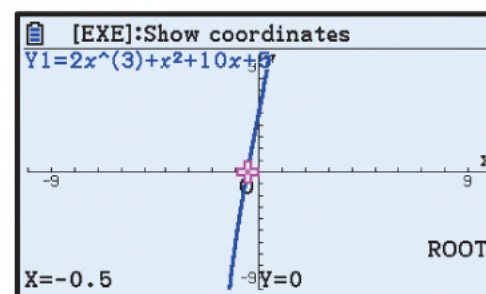
Using technology,  $-\frac{1}{2}$  is a zero.

$\therefore (2z + 1)$  is a factor.

$$\begin{aligned} \therefore 2z^3 + z^2 + 10z + 5 &= (2z + 1)(z^2 + az + 5) \text{ for some constant } a \\ &= 2z^3 + 2az^2 + 10z \\ &\quad + z^2 + az + 5 \\ &= 2z^3 + (2a + 1)z^2 + (10 + a)z + 5 \end{aligned}$$

Equating coefficients gives  $a = 0$ .

$$\begin{aligned} \therefore 2z^3 + z^2 + 10z + 5 &= (2z + 1)(z^2 + 5) \\ &= (2z + 1)(z + i\sqrt{5})(z - i\sqrt{5}) \end{aligned}$$





- 17** Since  $x^2 + ax + b = 0$  is real, if  $4 + i$  is a root, then  $4 - i$  is a root as well.

$$4 \pm i \text{ have sum} = 4 + i + 4 - i = 8$$

$$\text{and product} = (4 + i)(4 - i) = 17$$

$\therefore$  they come from the quadratic  $x^2 - 8x + 17$ .

$$\therefore a = -8, b = 17$$

- 18** Since  $2z^3 + az^2 + 62z + (a - 5)$  is real, both  $5 - i$  and  $5 + i$  are zeros.

The zeros  $5 \pm i$  have sum = 10 and product =  $25 - i^2 = 26$

$\therefore$  the zeros  $5 \pm i$  come from the quadratic  $(z^2 - 10z + 26)$

$$\therefore 2z^3 + az^2 + 62z + (a - 5) = (z^2 - 10z + 26)(2z + b) \text{ for some constant } b$$

$$= 2z^3 + bz^2$$

$$- 20z^2 - 10bz$$

$$+ 52z + 26b$$

$$= 2z^3 + (b - 20)z^2 + (52 - 10b)z + 26b$$

$$\text{Equating coefficients gives } \begin{cases} b - 20 = a \\ 52 - 10b = 62 \\ 26b = a - 5 \end{cases}$$

$$\therefore b = -1$$

$$\therefore a = -21$$

So,  $a = -21$  and the other two zeros are  $5 + i$  and  $\frac{1}{2}$ .

- 19 a**  $2x^3 + 3x^2 - 4x + 6 = 0$

$\therefore$  the sum of the roots is  $-\frac{3}{2}$

The polynomial equation has degree 3.

$\therefore$  the product of the roots is

$$\frac{(-1)^3 6}{2} = -3$$

- b**  $4x^4 = x^2 + 2x - 6$

$$\therefore 4x^4 + (0)x^3 - x^2 - 2x + 6 = 0$$

$\therefore$  the sum of the roots is  $-\frac{0}{4} = 0$

The polynomial equation has degree 4.

$\therefore$  the product of the roots is

$$\frac{(-1)^4 6}{4} = \frac{3}{2}$$

- 20 a** The polynomial  $P(x)$  of degree 5 has zeros  $m \pm 2i$ ,  $1 \pm mi$ , and 2.

The constant term is the  $y$ -intercept.  $\therefore$  the constant term is  $-56$ .

$$\therefore \text{the product of the zeros is } 2(m + 2i)(m - 2i)(1 + mi)(1 - mi) = \frac{(-1)^5(-56)}{1}$$

$$\therefore 2(m^2 + 4)(1 + m^2) = 56$$

$$\therefore m^4 + 5m^2 + 4 = 28$$

$$\therefore (m^2)^2 + 5(m^2) - 24 = 0$$

$$\therefore (m^2 + 8)(m^2 - 3) = 0$$

$$\therefore m^2 = -8 \text{ or } m^2 = 3$$

$$\therefore m^2 = 3 \quad \{m \in \mathbb{R}\}$$

$$\therefore m = \pm\sqrt{3}, \text{ as required}$$

- b** Let the coefficient of  $x^4$  be  $a$ .

If  $m = \sqrt{3}$ , then the sum of the zeros is

$$2 + (\sqrt{3} + 2i) + (\sqrt{3} - 2i) + (1 + i\sqrt{3}) + (1 - i\sqrt{3}) = -\frac{a}{1}$$

$$\therefore 2 + 2\sqrt{3} + 2 = -a$$

$$\therefore a = -4 - 2\sqrt{3}$$

If  $m = -\sqrt{3}$ , then the sum of the zeros is

$$\begin{aligned} 2 + (-\sqrt{3} + 2i) + (-\sqrt{3} - 2i) + (1 - i\sqrt{3}) + (1 + i\sqrt{3}) &= -a \\ \therefore 2 - 2\sqrt{3} + 2 &= -a \\ \therefore a &= -4 + 2\sqrt{3} \end{aligned}$$

So, the coefficient of  $x^4$  is  $-4 - 2\sqrt{3}$  if  $m = \sqrt{3}$ ,  
and  $-4 + 2\sqrt{3}$  if  $m = -\sqrt{3}$ .

**21** If  $k$  is the third root then  $x^3 + ax^2 + bx + c = (x - \alpha)(x - \beta)(x - k)$   
 $= x^3 - [\alpha + \beta + k]x^2 + [\alpha\beta + \alpha k + \beta k]x - \alpha\beta k$

Equating coefficients:  $a = -(\alpha + \beta + k)$  .... (1)

$b = \alpha\beta + \alpha k + \beta k$  .... (2)

$c = -\alpha\beta k$  and so  $k = -\frac{c}{\alpha\beta}$  .... (3)

So,  $(\alpha\beta)^3 - b(\alpha\beta)^2 + ac(\alpha\beta) - c^2$   
 $= (\alpha\beta)^3 - [\alpha\beta + \alpha k + \beta k](\alpha\beta)^2 - \alpha\beta c(\alpha + \beta + k) - c^2$  {using (1) and (2)}  
 $= (\alpha\beta)^3 - (\alpha\beta)^3 - (\alpha + \beta)k(\alpha\beta)^2 - \alpha\beta c(\alpha + \beta + k) - c^2$   
 $= -(\alpha + \beta)\left(-\frac{c}{\alpha\beta}\right)(\alpha\beta)^2 - \alpha\beta c\left(\alpha + \beta - \frac{c}{\alpha\beta}\right) - c^2$  {using (3)}  
 $= \cancel{-(\alpha + \beta)(-c\alpha\beta)} - \cancel{\alpha\beta c(\alpha + \beta)} + \cancel{c^2} - c^2$   
 $= 0$

$\therefore \alpha\beta$  is a root of  $x^3 - bx^2 + acx - c^2 = 0$ .

**22** The graph touches the  $x$ -axis at  $-1$  and cuts the  $x$ -axis at  $2$ .

$\therefore P(x) = a(x + 1)^2(x - 2)$ ,  $a \neq 0$

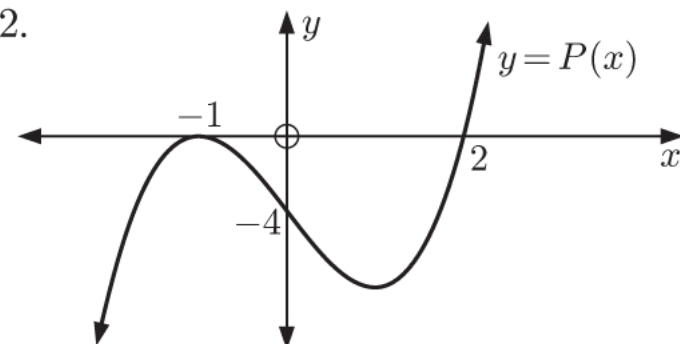
When  $x = 0$ ,  $P(x) = -4$

$\therefore a(1)^2(-2) = -4$

$\therefore -2a = -4$

$\therefore a = 2$

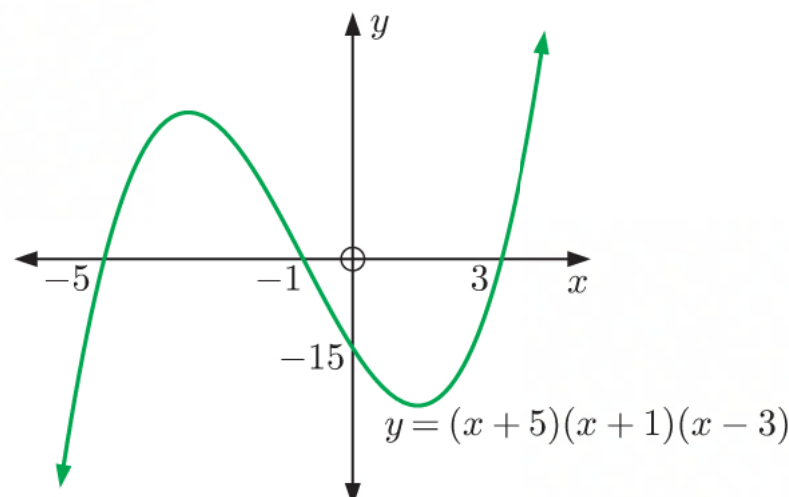
So,  $P(x) = 2(x + 1)^2(x - 2)$



**23 a** The graph cuts the  $x$ -axis at  $-5$ ,  $-1$ , and  $3$ .

When  $x = 0$ ,  $y = (5)(1)(-3) = -15$

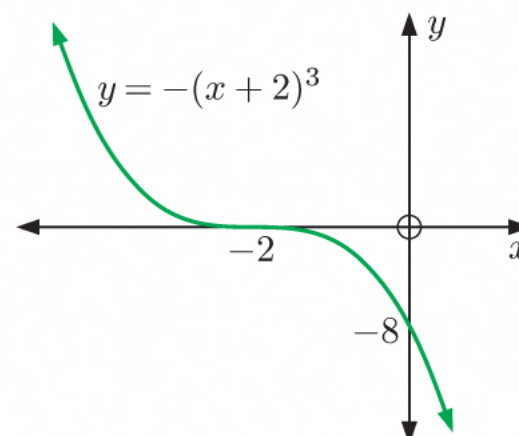
$\therefore$  the  $y$ -intercept is  $-15$ .



**b** The graph is horizontal at  $x = -2$

When  $x = 0$ ,  $y = -(2)^3 = -8$

$\therefore$  the  $y$ -intercept is  $-8$ .

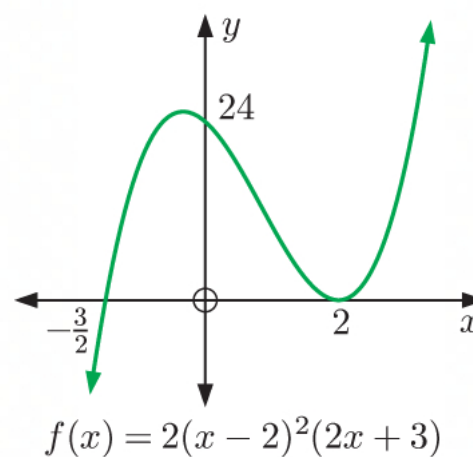


- c The graph touches the  $x$ -axis at 2, and cuts the  $x$ -axis at  $-\frac{3}{2}$ .

When  $x = 0$ ,

$$f(x) = 2(-2)^2(3) = 24$$

$\therefore$  the  $y$ -intercept is 24.



- 24 a  $-5$  is an  $x$ -intercept of the graph of  $f(x) = x^3 + 3x^2 - 14x - 20$

$\Leftrightarrow x = -5$  is a solution of  $f(x) = 0$ .

$$\begin{aligned}\text{Now, } f(-5) &= (-5)^3 + 3(-5)^2 - 14(-5) - 20 \\ &= -125 + 75 + 70 - 20 \\ &= 0 \quad \checkmark\end{aligned}$$

So,  $-5$  is an  $x$ -intercept of the graph of  $f(x)$ .

- b Since  $x = -5$  is a solution of  $f(x) = 0$ , then  $x + 5$  is a factor of  $f(x)$ .

$$\begin{aligned}\therefore x^3 + 3x^2 - 14x - 20 &= (x + 5)(x^2 + bx - 4) \\ &= x^3 + (b + 5)x^2 + (5b - 4)x - 20\end{aligned}$$

Equating coefficients of  $x^2$ :  $b + 5 = 3$

$$\therefore b = -2$$

Equating coefficients of  $x$ :  $5b - 4 = -14 \quad \checkmark$

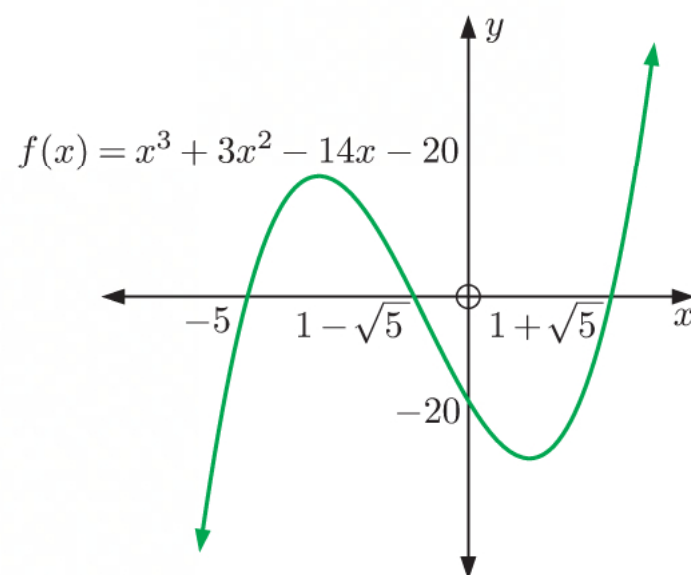
$$\therefore x^3 + 3x^2 - 14x - 20 = (x + 5)(x^2 - 2x - 4)$$

For  $x^2 - 2x - 4$ ,  $a = 1$ ,  $b = -2$ ,  $c = -4$

$$\begin{aligned}\therefore x &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-4)}}{2(1)} \\ &= \frac{2 \pm \sqrt{20}}{2} \\ &= 1 \pm \sqrt{5}\end{aligned}$$

So, the other  $x$ -intercepts are  $1 \pm \sqrt{5}$ .

c



- d as  $x \rightarrow \infty$ ,  $f(x) \rightarrow \infty$   
as  $x \rightarrow -\infty$ ,  $f(x) \rightarrow -\infty$

**25 a**  $P = 20v^3$

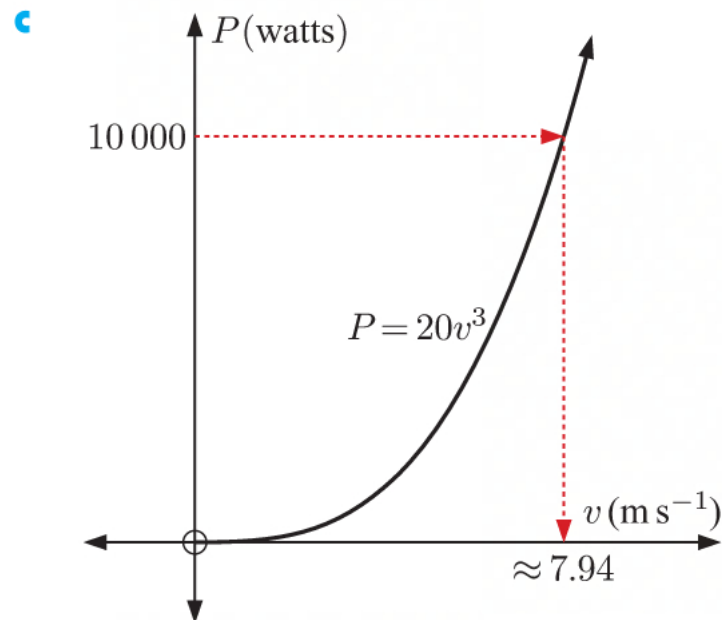
Since the power  $P$  must be  $\geq 0$ , then  $v$  must be  $\geq 0$ .

The highest wind speed ever recorded was  $\approx 100 \text{ m s}^{-1}$ , so it is reasonable to assume  $v < 100$ .

$$\therefore 0 \leq v < 100$$

**b** When  $v = 6$ ,  $P = 20(6)^3$   
 $= 4320$

4320 watts are generated at a wind speed of  $6 \text{ m s}^{-1}$ .



**d** When  $P = 10\,000$ ,  $20v^3 = 10\,000$   
 $\therefore v^3 = 500$   
 $\therefore v \approx 7.94$

A wind speed of  $\approx 7.94 \text{ m s}^{-1}$  is required to generate 10 000 watts of power.

**26 a** The graph cuts the  $x$ -axis at  $-2$ , and is horizontal at  $x = 3$ .

$$\therefore y = a(x+2)(x-3)^3, \quad a \neq 0$$

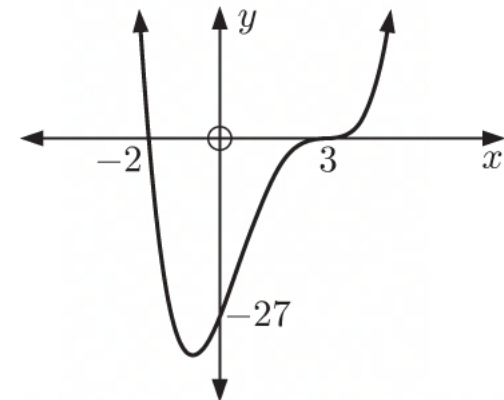
But when  $x = 0$ ,  $y = -27$

$$\therefore -27 = a(2)(-3)^3$$

$$\therefore -27 = -54a$$

$$\therefore a = \frac{1}{2}$$

$$\therefore y = \frac{1}{2}(x+2)(x-3)^3$$



**b** The graph touches the  $x$ -axis at  $-1$  and  $2$ .

$$\therefore y = a(x+1)^2(x-2)^2, \quad a \neq 0$$

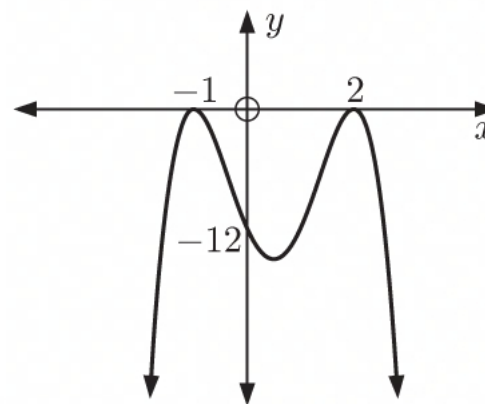
But when  $x = 0$ ,  $y = -12$

$$\therefore -12 = a(1)^2(-2)^2$$

$$\therefore -12 = 4a$$

$$\therefore a = -3$$

$$\therefore y = -3(x+1)^2(x-2)^2$$





- 27** Since the graph of  $f(x) = 2x^4 - 12x^3 + 15x^2 + 9x - 20$  cuts the  $x$ -axis at  $-1$  and  $4$ , then  $f(x) = (x+1)(x-4)(2x^2 + bx + 5)$  for some constant  $b$ .

$$\begin{aligned}\therefore 2x^4 - 12x^3 + 15x^2 + 9x - 20 &= (x+1)(x-4)(2x^2 + bx + 5) \\ &= (x^2 - 3x - 4)(2x^2 + bx + 5) \\ &= 2x^4 + bx^3 + 5x^2 \\ &\quad - 6x^3 - 3bx^2 - 15x \\ &\quad - 8x^2 - 4bx - 20 \\ &= 2x^4 + (b-6)x^3 - (3+3b)x^2 - (4b+15)x - 20\end{aligned}$$

Equating coefficients of  $x^3$ :  $b - 6 = -12$

$$\therefore b = -6$$

Equating coefficients of  $x^2$ :  $-(3+3b) = 15$  ✓

Equating coefficients of  $x$ :  $-(4b+15) = 9$  ✓

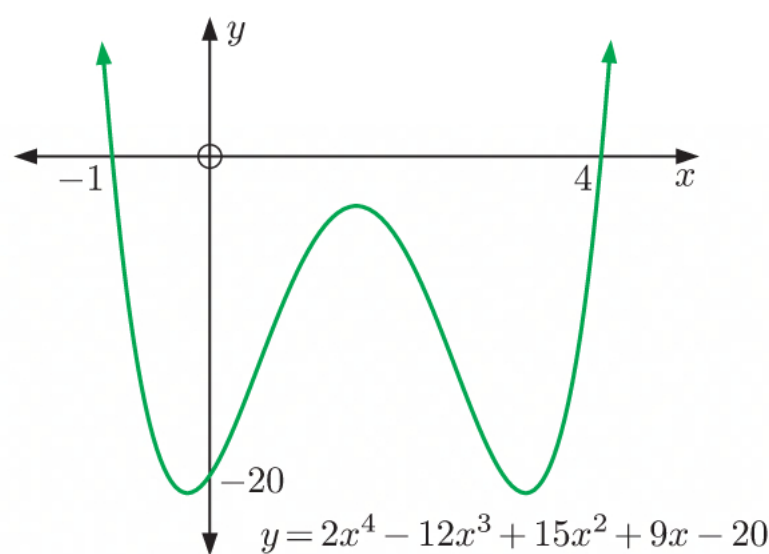
$$\therefore 2x^4 - 12x^3 + 15x^2 + 9x - 20 = (x+1)(x-4)(2x^2 - 6x + 5)$$

For  $2x^2 - 6x + 5$ ,  $\Delta = b^2 - 4ac$

$$\begin{aligned}&= (-6)^2 - 4(2)(5) \\ &= -4 \text{ which is } < 0\end{aligned}$$

$\therefore 2x^2 - 6x + 5$  has no real zeros.

$\therefore$  there are no other  $x$ -intercepts.



- 28 Note:** Earlier printings of this book erroneously repeat Review set 5B question 16.

**a** Let  $P(z) = 3z^3 - z^2 + 21z - 7$

Using technology,  $\frac{1}{3}$  is a zero.

$\therefore (3z - 1)$  is a factor.

$$\begin{aligned}\therefore 3z^3 - z^2 + 21z - 7 &= (3z - 1)(z^2 + az + 7) \text{ for some constant } a \\ &= 3z^3 + 3az^2 + 21z \\ &\quad - z^2 - az - 7 \\ &= 3z^3 + (3a - 1)z^2 + (21 - a)z - 7\end{aligned}$$

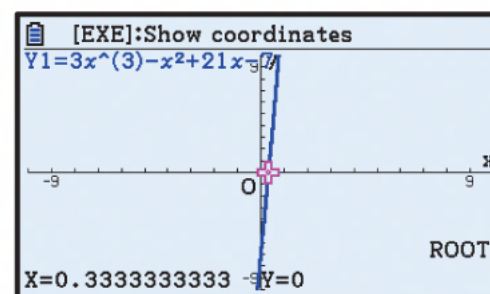
Equating coefficients of  $z^2$ :  $3a - 1 = -1$

$$\therefore a = 0$$

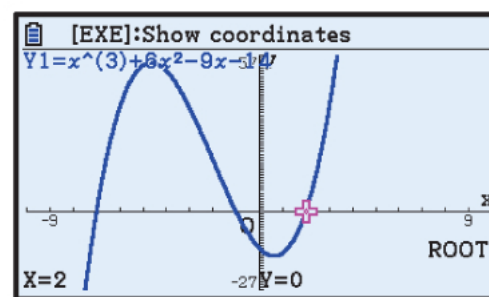
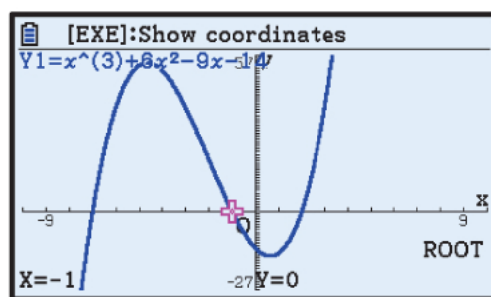
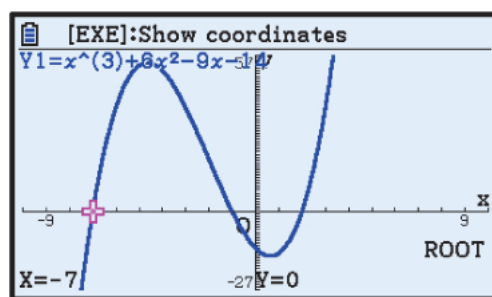
Equating coefficients of  $z$ :  $21 - a = 21$  ✓

$$\begin{aligned}\therefore 3z^3 - z^2 + 21z - 7 &= (3z - 1)(z^2 + 7) \\ &= (3z - 1)(z + i\sqrt{7})(z - i\sqrt{7})\end{aligned}$$

$\therefore$  the zeros are  $\frac{1}{3}$  and  $\pm i\sqrt{7}$ .



b



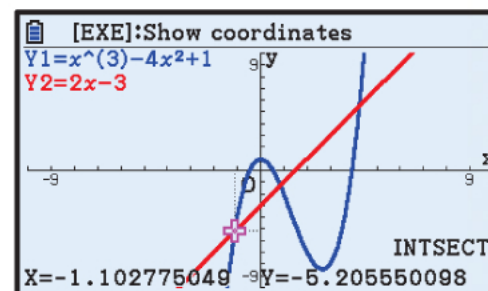
Using technology, the zeros of  $x^3 + 6x^2 - 9x - 14$  are  $-7$ ,  $-1$ , and  $2$ .

- 29 a** We graph  $y = x^3 - 4x^2 + 1$  and  $y = 2x - 3$  on the same set of axes.

Using technology, the graphs intersect at  $x \approx -1.10$ ,  $x \approx 0.854$ , and  $x \approx 4.25$ .

$x^3 - 4x^2 + 1 \leq 2x - 3$  whenever the graph of  $y = x^3 - 4x^2 + 1$  is on or below the graph of  $y = 2x - 3$ .

This occurs when  $x \leq -1.10$  or  $0.854 \leq x \leq 4.25$ .

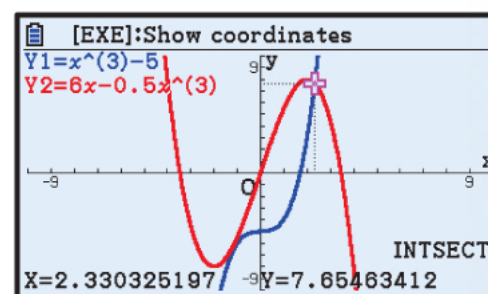


- b** We graph  $y = x^3 - 5$  and  $y = 6x - 0.5x^3$  on the same set of axes.

Using technology, the graphs intersect at  $x \approx 2.33$ .

$x^3 - 5 > 6x - 0.5x^3$  whenever the graph of  $y = x^3 - 5$  is above the graph of  $y = 6x - 0.5x^3$ .

This occurs when  $x > 2.33$ .



# Chapter 6

## FURTHER FUNCTIONS

### EXERCISE 6A

**1**  $f(x) = \frac{1}{x^2} + 2$   
 $\therefore f(-x) = \frac{1}{(-x)^2} + 2$   
 $= \frac{1}{x^2} + 2$   
 $= f(x)$   
 $\therefore f(x)$  is an even function.

**3**  $f(x) = 4x^3$   
 $\therefore f(-x) = 4(-x)^3$   
 $= -4x^3$   
 $= -f(x)$   
 $\therefore f(x)$  is an odd function.

The constant coefficient 4 does not affect whether the function is odd or even.

**4 a**  $f(x) = 5x$   
 $\therefore f(-x) = 5(-x)$   
 $= -5x$   
 $= -f(x)$   
 $\therefore f(x)$  is an odd function.

**c**  $f(x) = \frac{3}{x^2 - 4}$   
 $\therefore f(-x) = \frac{3}{(-x)^2 - 4}$   
 $= \frac{3}{x^2 - 4}$   
 $= f(x)$   
 $\therefore f(x)$  is an even function.

**e**  $f(x) = x^2 + \frac{7}{x^2} - 3$   
 $\therefore f(-x) = (-x)^2 + \frac{7}{(-x)^2} - 3$   
 $= x^2 + \frac{7}{x^2} - 3$   
 $= f(x)$   
 $\therefore f(x)$  is an even function.

**2**  $f(x) = x^3 - 3x$   
 $\therefore f(-x) = (-x)^3 - 3(-x)$   
 $= -x^3 + 3x$   
 $= -(x^3 - 3x)$   
 $= -f(x)$   
 $\therefore f(x)$  is an odd function.

**b**  $f(x) = -4x + 3$   
 $\therefore f(-x) = -4(-x) + 3$   
 $= 4x + 3$   
which is neither  $-f(x)$  nor  $f(x)$   
 $\therefore f(x)$  is neither even nor odd.

**d**  $f(x) = 2x^3 - \frac{5}{x}$   
 $\therefore f(-x) = 2(-x)^3 - \frac{5}{(-x)}$   
 $= -2x^3 + \frac{5}{x}$   
 $= -\left(2x^3 - \frac{5}{x}\right)$   
 $= -f(x)$   
 $\therefore f(x)$  is an odd function.

**f**  $f(x) = \sqrt{x}$   
 $\therefore f(-x) = \sqrt{-x}$   
which is neither  $-f(x)$  nor  $f(x)$   
 $\therefore f(x)$  is neither even nor odd.

**5**  $(1, 3)$  and  $(-5, -2)$  lie on the graph of  $y = f(x)$

$$\therefore f(1) = 3 \text{ and } f(-5) = -2$$

$f(x)$  is even, so  $f(-x) = f(x)$  for all  $x$

$$\therefore f(-1) = 3 \text{ and } f(5) = -2$$

$\therefore (-1, 3)$  and  $(5, -2)$  also lie on the graph of  $y = f(x)$ .

**6**  $(4, 6)$  and  $(-1, 2)$  lie on the graph of  $y = g(x)$

$$\therefore g(4) = 6 \text{ and } g(-1) = 2$$

$g(x)$  is odd, so  $g(-x) = -g(x)$  for all  $x$

$$\therefore g(-4) = -6 \text{ and } g(1) = -2$$

$\therefore (-4, -6)$  and  $(1, -2)$  also lie on the graph of  $y = g(x)$ .

**7**  $f(x) = (2x + 3)(x + a)$ ,  $a \in \mathbb{R}$  is an even function

$$\therefore f(-x) = f(x)$$

$$\therefore (2(-x) + 3)(-x + a) = (2x + 3)(x + a)$$

$$\therefore (-2x + 3)(-x + a) = (2x + 3)(x + a)$$

$$\therefore \cancel{2x^2} - 2ax - 3x + \cancel{3a} = \cancel{2x^2} + 2ax + 3x + \cancel{3a}$$

$$\therefore -4ax - 6x = 0$$

$$\therefore (-4a - 6)x = 0$$

$$\therefore -4a - 6 = 0$$

$$\therefore -4a = 6$$

$$\therefore a = -\frac{6}{4} = -\frac{3}{2}$$

**8**  $g(x) = (x + 1)\left(\frac{1}{x} + b\right)$  is an odd function

$$\therefore g(-x) = -g(x)$$

$$\therefore (-x + 1)\left(\frac{1}{-x} + b\right) = -(x + 1)\left(\frac{1}{x} + b\right)$$

$$\therefore \frac{x}{x} - bx - \frac{1}{x} + b = -\left(\frac{x}{x} + bx + \frac{1}{x} + b\right)$$

$$\therefore 1 - \cancel{bx} - \frac{1}{x} + b = -1 - \cancel{bx} - \frac{1}{x} - b$$

$$\therefore 1 + b = -1 - b$$

$$\therefore 2b = -2$$

$$\therefore b = -1$$

**9 a** Suppose  $f(x) = ax^2 + bx + c$ ,  $a \neq 0$  is an even function.

$$\therefore f(-x) = f(x)$$

$$\therefore a(-x)^2 + b(-x) + c = ax^2 + bx + c$$

$$\therefore \cancel{ax^2} - bx + \cancel{c} = \cancel{ax^2} + bx + \cancel{c}$$

$$\therefore -2bx = 0$$

$$\therefore -2b = 0$$

$$\therefore b = 0$$



**b** If  $g(x) = ax^3 + bx^2 + cx + d$ ,  $a \neq 0$  is an odd function, then  $g(-x) = -g(x)$ .

$$\therefore a(-x)^3 + b(-x)^2 + c(-x) + d = -(ax^3 + bx^2 + cx + d)$$

$$\therefore \cancel{ax^3} + bx^2 - \cancel{cx} + d = \cancel{-ax^3} - bx^2 - \cancel{cx} - d$$

$$\therefore 2bx^2 + 2d = 0$$

$$\text{Equating coefficients of } x^2: \quad 2b = 0$$

$$\therefore b = 0$$

$$\text{Equating constant terms:} \quad 2d = 0$$

$$\therefore d = 0$$

So, the cubic function  $g(x) = ax^3 + bx^2 + cx + d$ ,  $a \neq 0$  is an odd function when  $b = 0$ ,  $d = 0$ .

**c** If  $h(x) = ax^4 + bx^3 + cx^2 + dx + e$ ,  $a \neq 0$  is an even function, then  $h(-x) = h(x)$ .

$$\therefore a(-x)^4 + b(-x)^3 + c(-x)^2 + d(-x) + e = ax^4 + bx^3 + cx^2 + dx + e$$

$$\therefore \cancel{ax^4} - bx^3 + \cancel{cx^2} - dx + \cancel{e} = \cancel{ax^4} + bx^3 + \cancel{cx^2} + dx + \cancel{e}$$

$$\therefore -2bx^3 - 2dx = 0$$

$$\text{Equating coefficients of } x^3: \quad -2b = 0$$

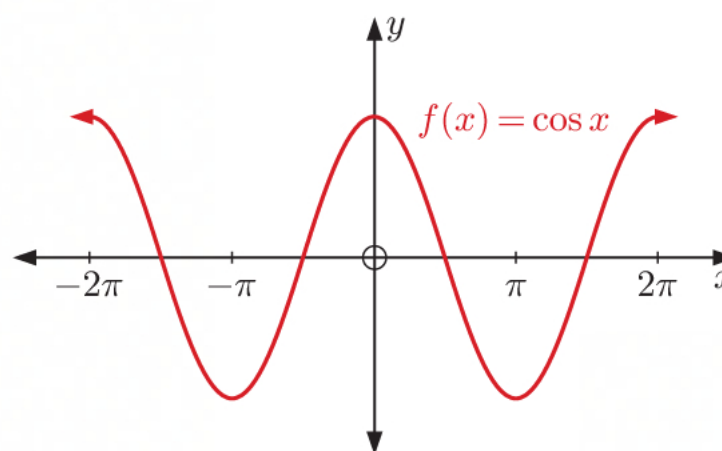
$$\therefore b = 0$$

$$\text{Equating coefficients of } x: \quad -2d = 0$$

$$\therefore d = 0$$

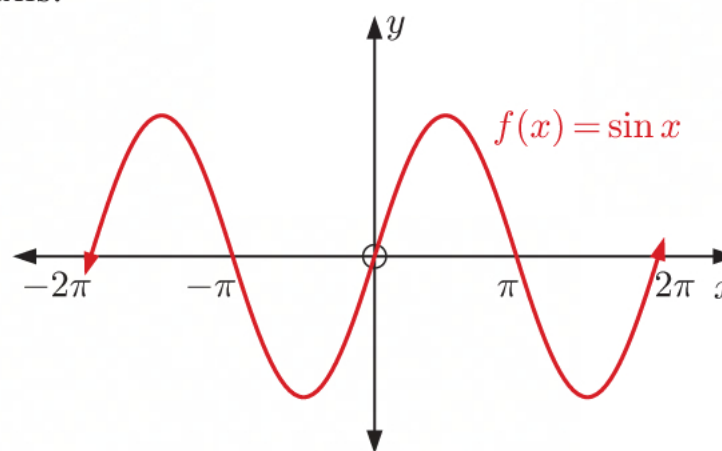
So, the quartic function  $h(x) = ax^4 + bx^3 + cx^2 + dx + e$ ,  $a \neq 0$  is an even function when  $b = 0$ ,  $d = 0$ .

**10 a**  $f(x) = \cos x$   
 $\therefore f(-x) = \cos(-x)$   
 $\quad = \cos x$   
 $\quad = f(x)$   
 $\therefore f(x) = \cos x$  is even.



The graph of  $y = \cos x$  is symmetric about the  $y$ -axis.

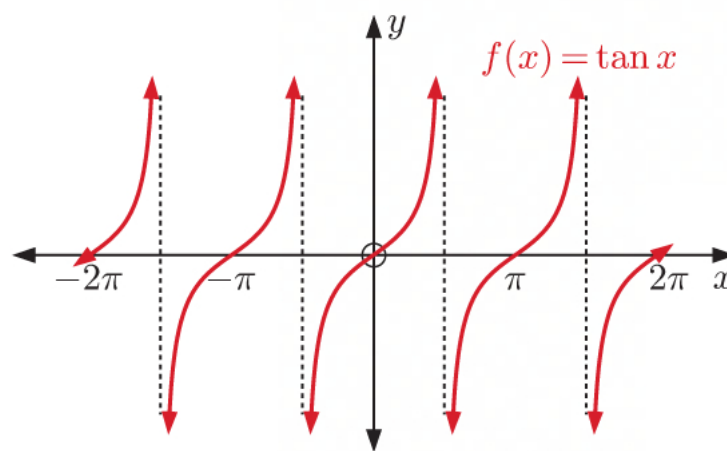
**b**  $f(x) = \sin x$   
 $\therefore f(-x) = \sin(-x)$   
 $\quad = -\sin x$   
 $\quad = -f(x)$   
 $\therefore f(x) = \sin x$  is odd.



The graph of  $y = \sin x$  has rotational symmetry about the origin.

$$\begin{aligned}
 \text{c} \quad & f(x) = \tan x \\
 \therefore & f(-x) = \tan(-x) \\
 &= \frac{\sin(-x)}{\cos(-x)} \\
 &= \frac{-\sin x}{\cos x} \\
 &= -\tan x \\
 &= -f(x)
 \end{aligned}$$

$\therefore f(x) = \tan x$  is odd.



The graph of  $y = \tan x$  has rotational symmetry about the origin.

$$\begin{aligned}
 \text{11 a} \quad & \text{If } f(x) = \cos(x - k) \text{ is even, then } f(-x) = f(x) \\
 & \therefore \cos(-x - k) = \cos(x - k) \\
 & \therefore \cos(-(x + k)) = \cos(x - k) \\
 & \therefore \cos(x + k) = \cos(x - k) \quad \{\cos(-x) = \cos x\} \\
 & \therefore \cancel{\cos x \cos k} - \sin x \sin k = \cancel{\cos x \cos k} + \sin x \sin k \\
 & \therefore -2 \sin x \sin k = 0 \\
 & \therefore \sin k = 0 \\
 & \therefore k = n\pi, \quad n \in \mathbb{Z}
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & \text{If } f(x) = \cos(x - k) \text{ is odd, then } f(-x) = -f(x) \\
 & \therefore \cos(-x - k) = -\cos(x - k) \\
 & \therefore \cos(-(x + k)) = -\cos(x - k) \\
 & \therefore \cos(x + k) = -\cos(x - k) \quad \{\cos(-x) = \cos x\} \\
 & \therefore \cos x \cos k - \sin x \sin k = -(\cos x \cos k + \sin x \sin k) \\
 & \therefore \cos x \cos k - \cancel{\sin x \sin k} = -\cos x \cos k - \cancel{\sin x \sin k} \\
 & \therefore 2 \cos x \cos k = 0 \\
 & \therefore \cos k = 0 \\
 & \therefore k = \frac{\pi}{2} + n\pi, \quad n \in \mathbb{Z}
 \end{aligned}$$

$$\text{c} \quad f(x) = \cos(x - k) \text{ is neither odd nor even if } k \neq \frac{n\pi}{2}, \quad n \in \mathbb{Z}.$$

$$\begin{aligned}
 \text{12 a} \quad & \text{Suppose } h(x) = f(x) + g(x) \text{ where } f(x) \text{ and } g(x) \text{ are even functions.} \\
 & \text{Now } h(-x) = f(-x) + g(-x) \\
 & \quad = f(x) + g(x) \quad \{f(x) \text{ and } g(x) \text{ are even functions}\} \\
 & \quad = h(x) \text{ for all } x \\
 & \therefore h(x) \text{ is even.}
 \end{aligned}$$

$\therefore$  the sum of two even functions is an even function.

$$\begin{aligned}
 \text{b} \quad & \text{Suppose } h(x) = f(x) - g(x) \text{ where } f(x) \text{ and } g(x) \text{ are odd functions.} \\
 & \text{Now } h(-x) = f(-x) - g(-x) \\
 & \quad = -f(x) - (-g(x)) \quad \{f(x) \text{ and } g(x) \text{ are odd functions}\} \\
 & \quad = -(f(x) - g(x)) \\
 & \quad = -h(x) \\
 & \therefore h(x) \text{ is odd.}
 \end{aligned}$$

$\therefore$  the difference between two odd functions is an odd function.

- Suppose  $h(x) = f(x)g(x)$  where  $f(x)$  and  $g(x)$  are odd functions.

$$\begin{aligned}\text{Now } h(-x) &= f(-x)g(-x) \\ &= -f(x) \times -g(x) \quad \{f(x) \text{ and } g(x) \text{ are odd functions}\} \\ &= f(x)g(x) \\ &= h(x)\end{aligned}$$

$\therefore h(x)$  is even.

$\therefore$  the product of two odd functions is an even function.

- 13**  $f(x)$  is an even function and  $g(x)$  is an odd function.

$$\begin{aligned}(f \circ g)(-x) &= f(g(-x)) \\ &= f(-g(x)) \quad \{g(x) \text{ is an odd function}\} \\ &= f(g(x)) \quad \{f(x) \text{ is an even function}\} \\ &= (f \circ g)(x)\end{aligned}$$

So  $(f \circ g)(x)$  is an even function.

- 14**  $f(x)$  is an odd, invertible function.

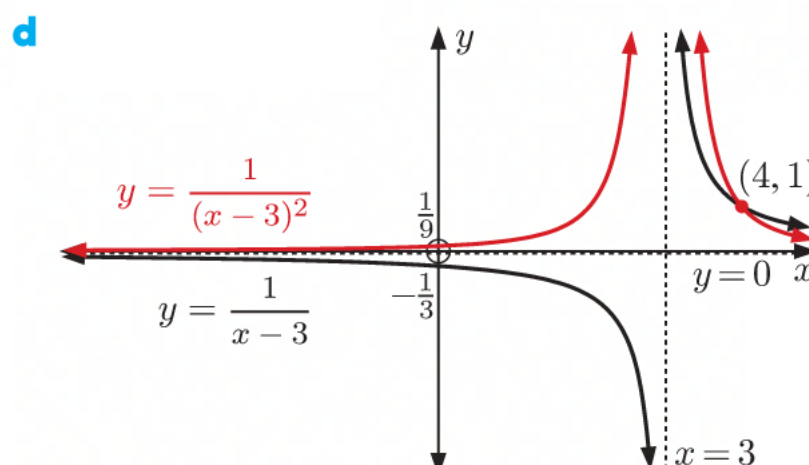
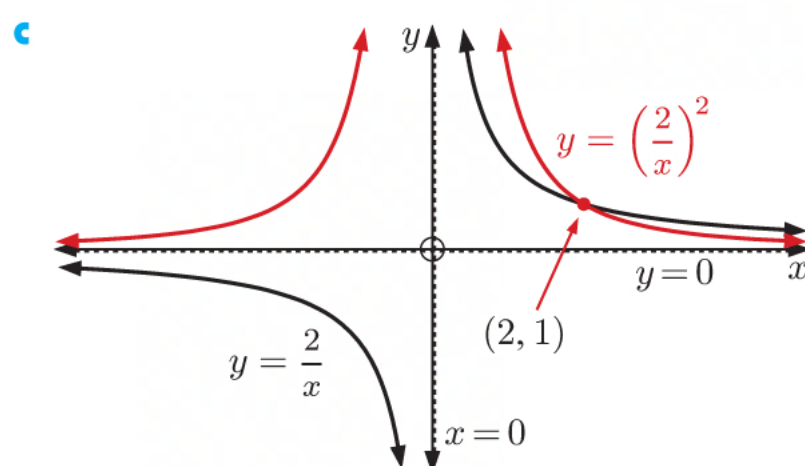
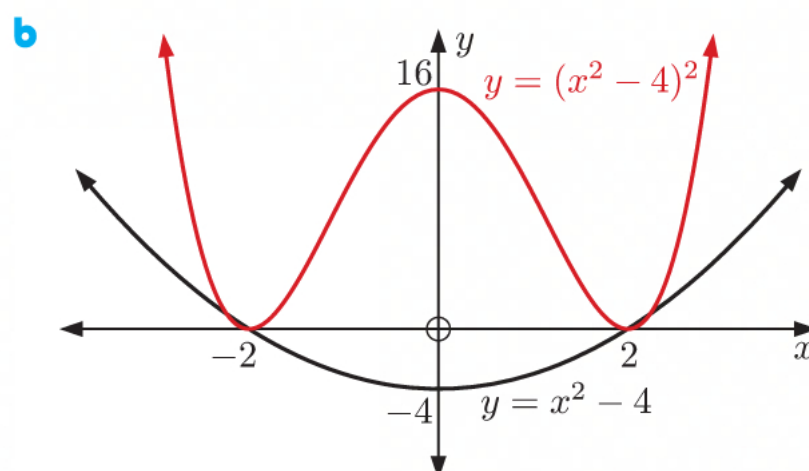
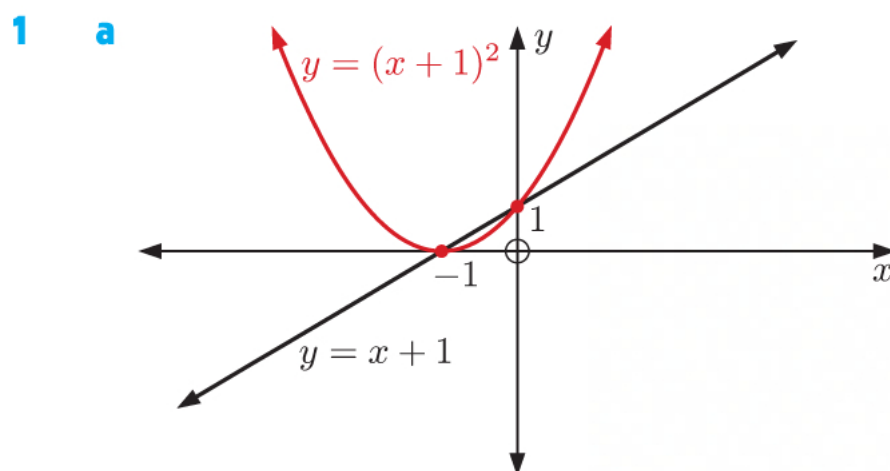
$$\begin{aligned}\text{Let } y &= f^{-1}(-x) \\ \therefore f(y) &= -x \\ \therefore -f(y) &= x \\ \therefore f(-y) &= x \quad \{f(x) \text{ is an odd function}\} \\ \therefore -y &= f^{-1}(x) \\ \therefore y &= -f^{-1}(x)\end{aligned}$$

So,  $f^{-1}(-x) = -f^{-1}(x)$  for all  $x$ .

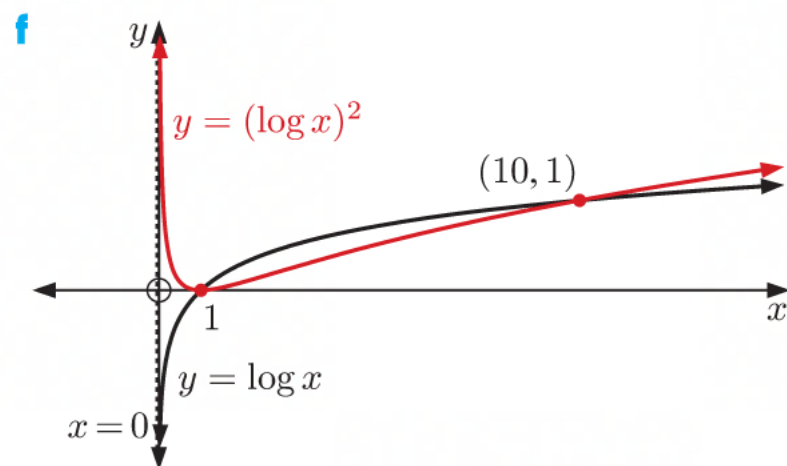
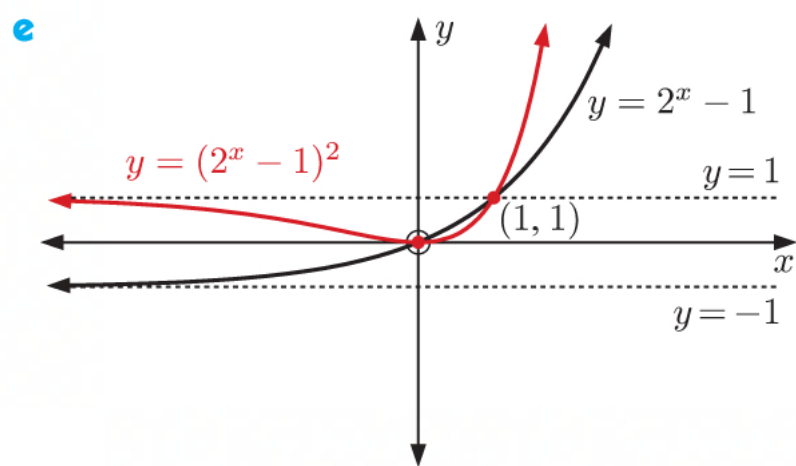
$\therefore f^{-1}(x)$  is an odd function.

## INVESTIGATION 1

## THE GRAPH OF $y = [f(x)]^2$







**2** The points where each pair of functions intersect each have  $y$ -coordinate 0 or 1.

**3** When  $y = [f(x)]^2$  is graphed from  $y = f(x)$ :

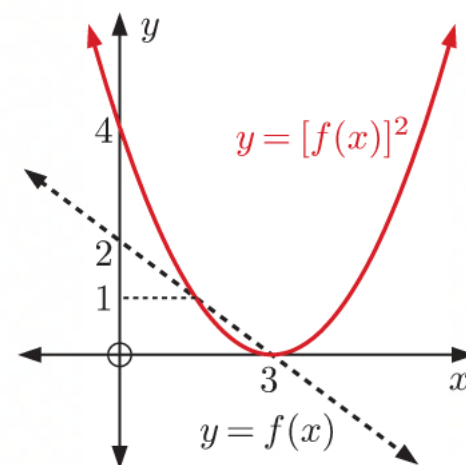
- The graph of  $y = [f(x)]^2$  touches the  $x$ -axis at its  $x$ -intercepts.
- The graph of  $y = [f(x)]^2$  lies above or on the  $x$ -axis for all  $x$ .
- The vertical asymptotes of  $y = f(x)$  are also vertical asymptotes of  $y = [f(x)]^2$ .
- When  $-1 < f(x) < 1$ ,  $y = [f(x)]^2$  is closer to the  $x$ -axis than  $y = f(x)$ .
- When  $|f(x)| > 1$ ,  $y = [f(x)]^2$  is further from the  $x$ -axis than  $y = f(x)$ .

## EXERCISE 6B

**1 a**  $y = f(x)$  cuts the  $x$ -axis at 3, so  $y = [f(x)]^2$  touches the  $x$ -axis at 3.

$y = f(x)$  and  $y = [f(x)]^2$  also intersect when  $y = 1$ .

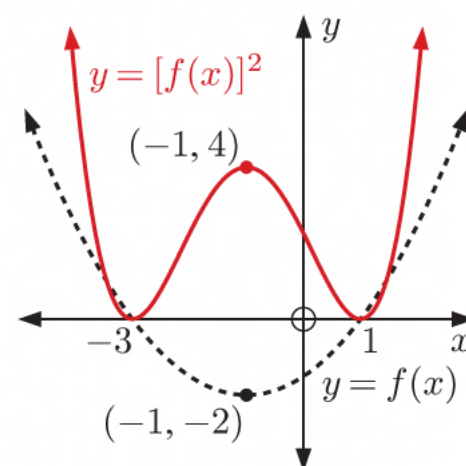
The  $y$ -intercept of  $y = [f(x)]^2$  is  $2^2 = 4$ .



**b**  $y = f(x)$  cuts the  $x$ -axis at  $-3$  and  $1$ , so  $y = [f(x)]^2$  touches the  $x$ -axis at  $-3$  and  $1$ .

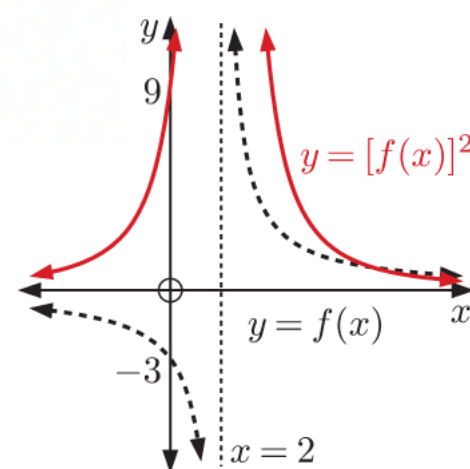
$y = f(x)$  and  $y = [f(x)]^2$  also intersect when  $y = 1$ .

$y = f(x)$  has a turning point at  $(-1, -2)$ , so  $y = [f(x)]^2$  has a turning point at  $(-1, 4)$ .

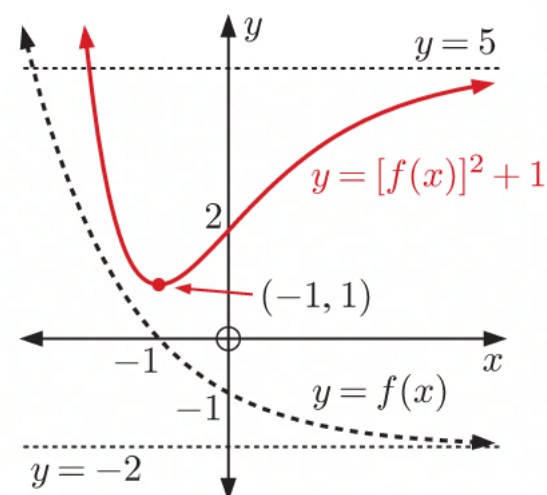




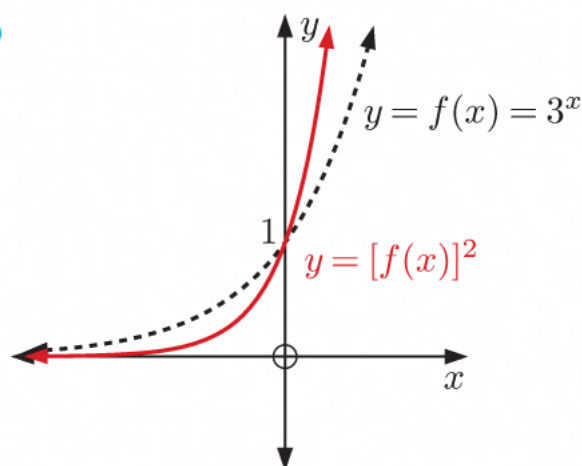
- $y = f(x)$  and  $y = [f(x)]^2$  intersect when  $y = 1$ .  
 $y = [f(x)]^2$  has  $y$ -intercept  $(-3)^2 = 9$ .



- 2 We draw the graph of  $y = [f(x)]^2$  translated up by 1 unit.

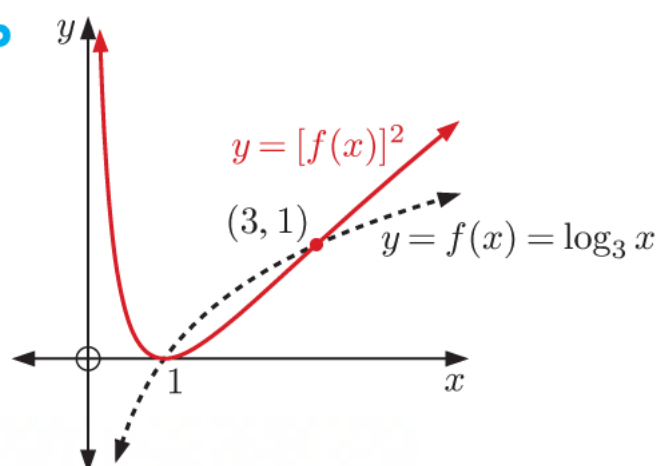


- 3 a, b



- $y = [f(x)]^2$   
 $\therefore y = (3^x)^2$   
 $\therefore y = 3^{2x} = f(2x)$   
 $\therefore y = f(x)$  is transformed to  $y = [f(x)]^2$  by a horizontal stretch with scale factor  $\frac{1}{2}$ .

- 4 a, b



- The points on  $y = \log_3 x$  with  $y$ -coordinate 0 or 1 are the invariant points when  $y = f(x)$  is transformed to  $y = [f(x)]^2$ . These are (1, 0) and (3, 1).

- 5 If  $f(x)$  is odd, then  $[f(-x)]^2 = [-f(x)]^2$   
 $= [f(x)]^2$   
 $\therefore [f(x)]^2$  is even.

- 6  $f(x)$  has domain  $\{x \mid 0 \leq x \leq 5\}$  and range  $\{y \mid -4 \leq y \leq 3\}$ .

$[f(x)]^2$  is defined when  $f(x)$  is defined.

$\therefore [f(x)]^2$  has domain  $\{x \mid 0 \leq x \leq 5\}$ .

Now, if  $-4 \leq y \leq 0$  and if  $0 \leq y \leq 3$   
 then  $0 \leq y^2 \leq 16$  then  $0 \leq y^2 \leq 9$

$\therefore$  if  $-4 \leq y \leq 3$ , then  $0 \leq y^2 \leq 16$ .

$\therefore [f(x)]^2$  has range  $\{y \mid 0 \leq y \leq 16\}$ .

7  $f(x) = \frac{2x+4}{x-1}$

a  $f(x) = 0$  when  $2x+4=0$   
 $\therefore 2x = -4$   
 $\therefore x = -2$

$\therefore$  the  $x$ -intercept is  $-2$ .

$f(x)$  is undefined when  $x-1=0$   
 $\therefore x = 1$

$\therefore$  the vertical asymptote is  $x = 1$ .

$$f(0) = \frac{4}{-1} = -4$$

$\therefore$  the  $y$ -intercept is  $-4$ .

$$\begin{aligned} f(x) &= \frac{2x+4}{x-1} \\ &= \frac{2(x-1)+4+2}{x-1} \\ &= 2 + \frac{6}{x-1} \end{aligned}$$

$\therefore$  the horizontal asymptote is  $y = 2$ .

- b The  $x$ -intercept and vertical asymptote are unchanged.

$y = [f(x)]^2$  has:

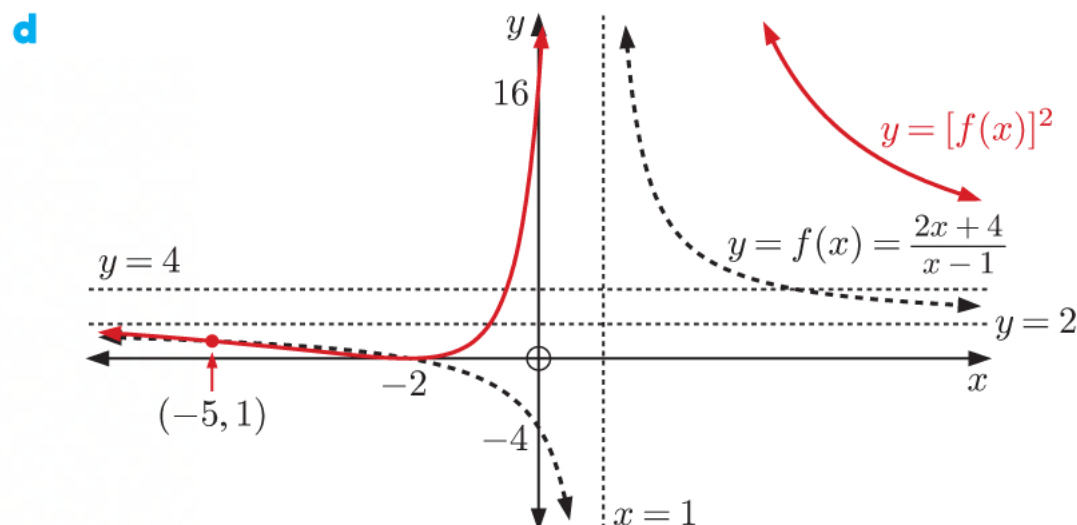
- $x$ -intercept  $-2$
- $y$ -intercept  $(-4)^2 = 16$
- vertical asymptote  $x = 1$
- horizontal asymptote  $y = 2^2 = 4$

c  $f(x) = \frac{2x+4}{x-1} = 1$

$$\therefore 2x+4 = x-1$$

$$\therefore x = -5$$

$\therefore$  the invariant points are  $(-5, 1)$ , and the  $x$ -intercept is  $(-2, 0)$ .

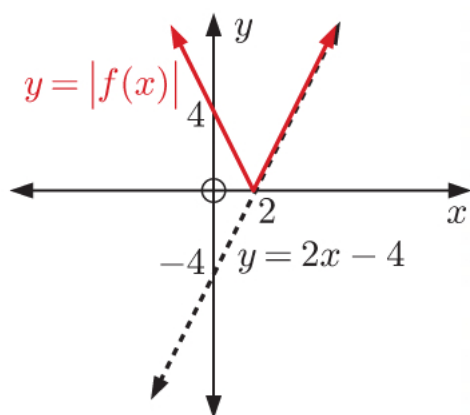


## EXERCISE 6C.1

1  $y = f(x) = 2x - 4$

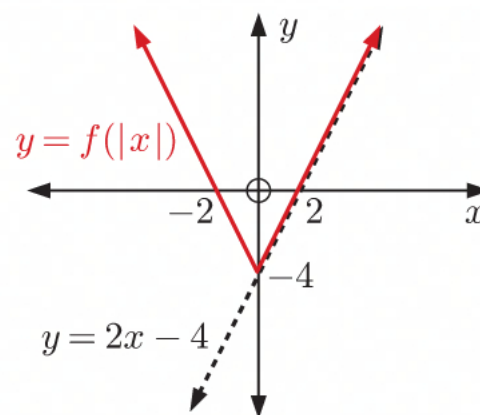
a  $y = |f(x)| = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ -f(x) & \text{if } f(x) < 0 \end{cases}$

The graph is unchanged for  $f(x) \geq 0$  and reflected in the  $x$ -axis for  $f(x) < 0$ .



b  $y = f(|x|) = \begin{cases} f(x) & \text{if } x \geq 0 \\ f(-x) & \text{if } x < 0 \end{cases}$

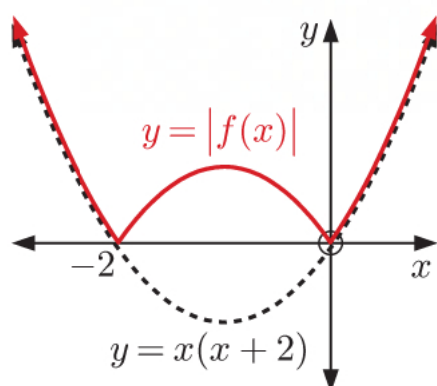
The graph is unchanged for  $x \geq 0$  and reflected in the  $y$ -axis for  $x < 0$ .



2  $y = f(x) = x(x + 2)$

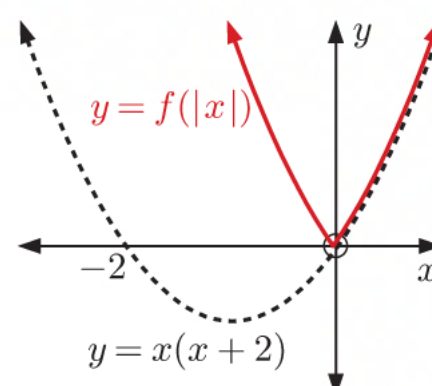
a  $y = |f(x)| = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ -f(x) & \text{if } f(x) < 0 \end{cases}$

The graph is unchanged for  $f(x) \geq 0$  and reflected in the  $x$ -axis for  $f(x) < 0$ .



b  $y = f(|x|) = \begin{cases} f(x) & \text{if } x \geq 0 \\ f(-x) & \text{if } x < 0 \end{cases}$

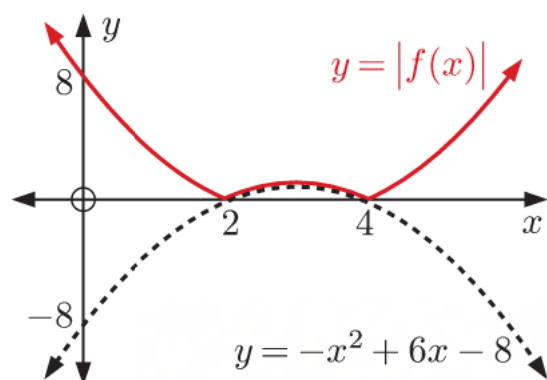
The graph is unchanged for  $x \geq 0$  and reflected in the  $y$ -axis for  $x < 0$ .



3  $y = f(x) = -x^2 + 6x - 8$

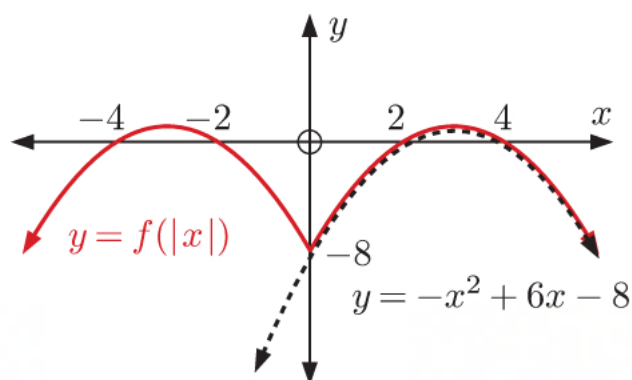
a  $y = |f(x)| = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ -f(x) & \text{if } f(x) < 0 \end{cases}$

The graph is unchanged for  $f(x) \geq 0$  and reflected in the  $x$ -axis for  $f(x) < 0$ .



b  $y = f(|x|) = \begin{cases} f(x) & \text{if } x \geq 0 \\ f(-x) & \text{if } x < 0 \end{cases}$

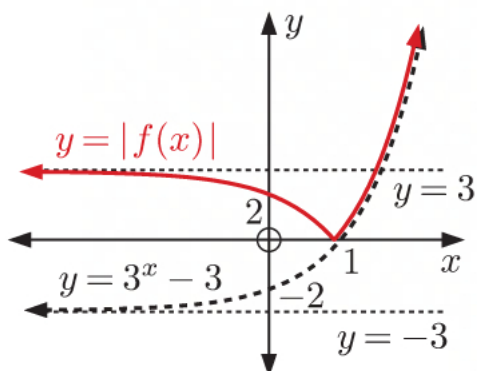
The graph is unchanged for  $x \geq 0$  and reflected in the  $y$ -axis for  $x < 0$ .



4  $y = f(x) = 3^x - 3$

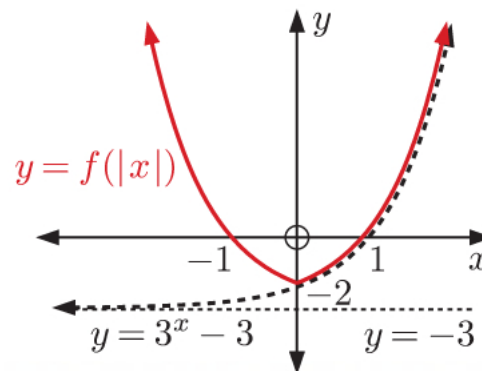
a  $y = |f(x)| = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ -f(x) & \text{if } f(x) < 0 \end{cases}$

The graph is unchanged for  $f(x) \geq 0$  and reflected in the  $x$ -axis for  $f(x) < 0$ .



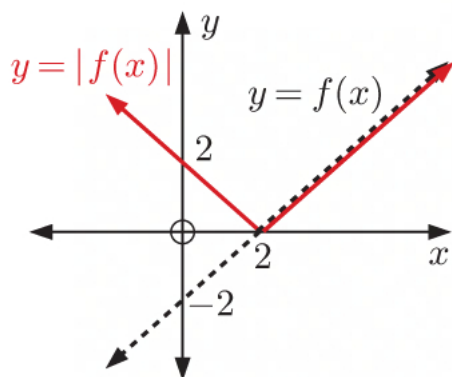
b  $y = f(|x|) = \begin{cases} f(x) & \text{if } x \geq 0 \\ f(-x) & \text{if } x < 0 \end{cases}$

The graph is unchanged for  $x \geq 0$  and reflected in the  $y$ -axis for  $x < 0$ .



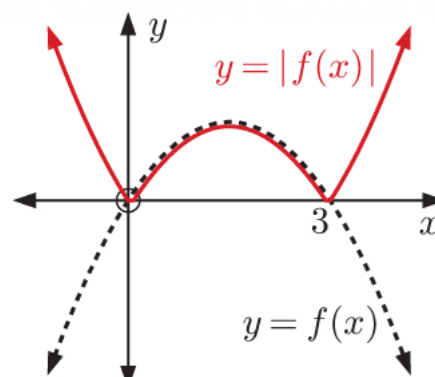
5 a i  $y = |f(x)| = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ -f(x) & \text{if } f(x) < 0 \end{cases}$

The graph is unchanged for  $f(x) \geq 0$  and reflected in the  $x$ -axis for  $f(x) < 0$ .



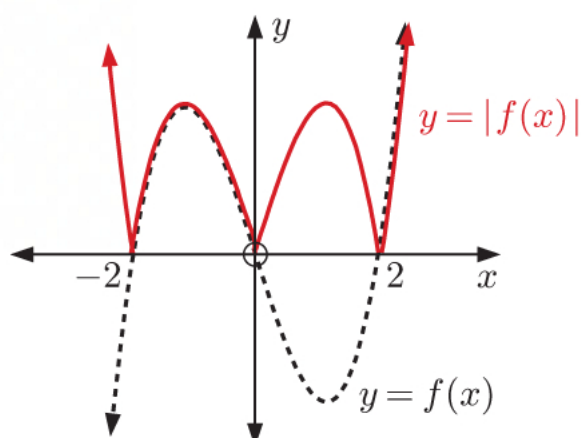
ii  $y = |f(x)| = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ -f(x) & \text{if } f(x) < 0 \end{cases}$

The graph is unchanged for  $f(x) \geq 0$  and reflected in the  $x$ -axis for  $f(x) < 0$ .



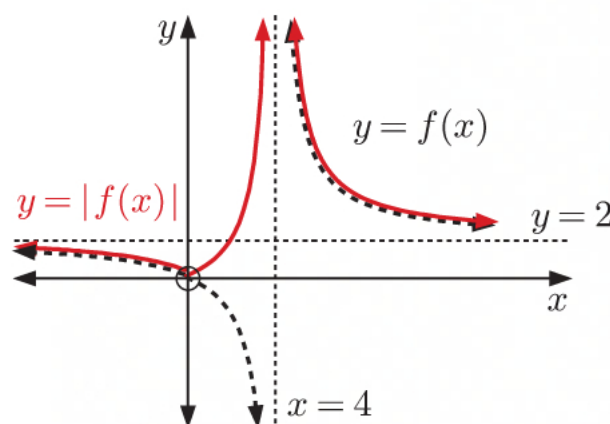
iii  $y = |f(x)| = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ -f(x) & \text{if } f(x) < 0 \end{cases}$

The graph is unchanged for  $f(x) \geq 0$  and reflected in the  $x$ -axis for  $f(x) < 0$ .



iv  $y = |f(x)| = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ -f(x) & \text{if } f(x) < 0 \end{cases}$

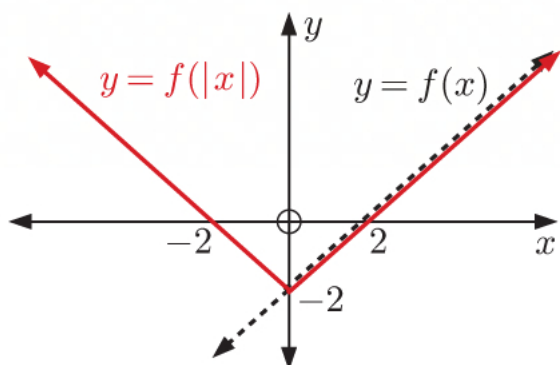
The graph is unchanged for  $f(x) \geq 0$  and reflected in the  $x$ -axis for  $f(x) < 0$ .





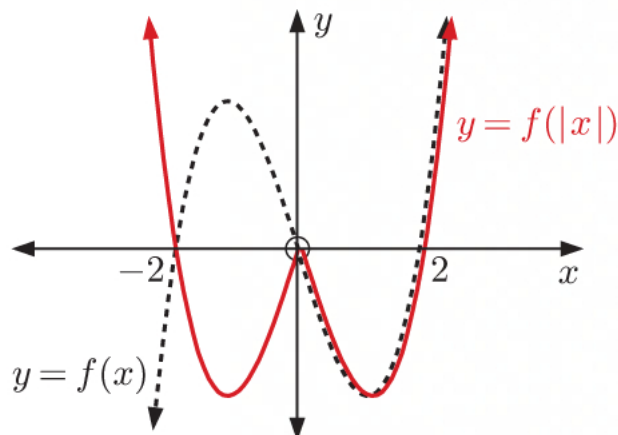
**b i**  $y = f(|x|) = \begin{cases} f(x) & \text{if } x \geq 0 \\ f(-x) & \text{if } x < 0 \end{cases}$

The graph is unchanged for  $x \geq 0$  and reflected in the  $y$ -axis for  $x < 0$ .



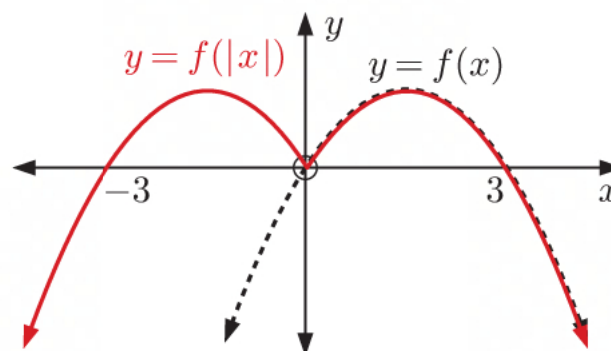
**iii**  $y = f(|x|) = \begin{cases} f(x) & \text{if } x \geq 0 \\ f(-x) & \text{if } x < 0 \end{cases}$

The graph is unchanged for  $x \geq 0$  and reflected in the  $y$ -axis for  $x < 0$ .



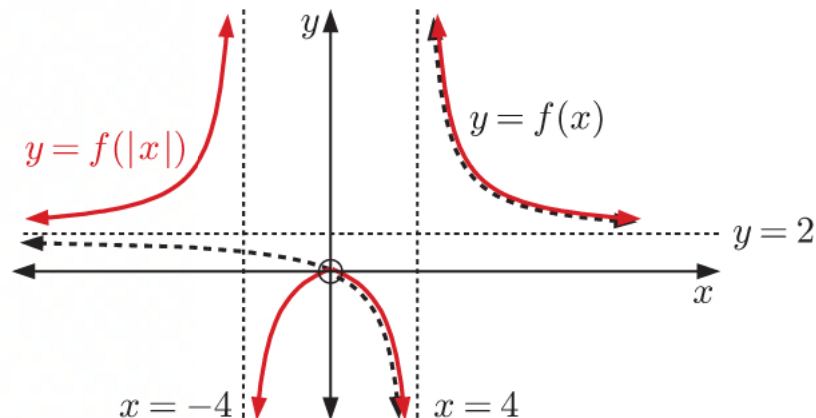
**ii**  $y = f(|x|) = \begin{cases} f(x) & \text{if } x \geq 0 \\ f(-x) & \text{if } x < 0 \end{cases}$

The graph is unchanged for  $x \geq 0$  and reflected in the  $y$ -axis for  $x < 0$ .



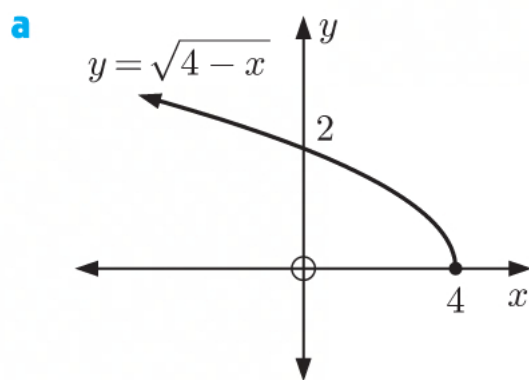
**iv**  $y = f(|x|) = \begin{cases} f(x) & \text{if } x \geq 0 \\ f(-x) & \text{if } x < 0 \end{cases}$

The graph is unchanged for  $x \geq 0$  and reflected in the  $y$ -axis for  $x < 0$ .



- 6** If  $f(x)$  is odd, then  $|f(-x)| = |-f(x)| = |f(x)|$   
 $\therefore |f(x)|$  is an even function.

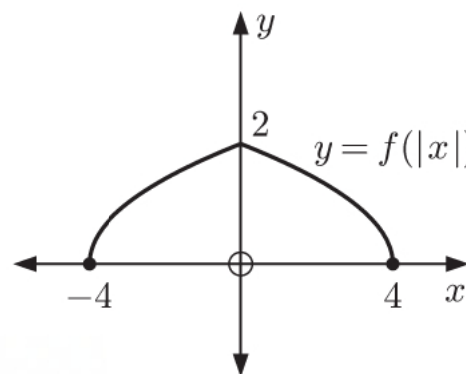
**7**  $f(x) = \sqrt{4-x}$



The domain is  $\{x \mid x \leq 4\}$ ,  
 and the range is  $\{y \mid y \geq 0\}$ .

**b**  $y = f(|x|) = \begin{cases} f(x) & \text{if } x \geq 0 \\ f(-x) & \text{if } x < 0 \end{cases}$

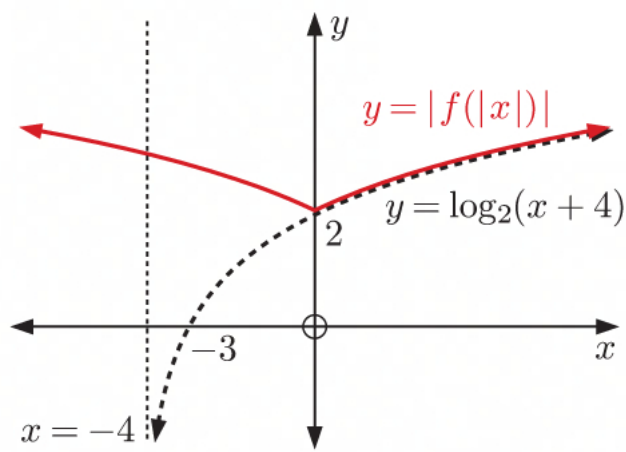
The graph is unchanged for  $x \geq 0$  and reflected in the  $y$ -axis for  $x < 0$ .



The domain is  $\{x \mid -4 \leq x \leq 4\}$ ,  
 and the range is  $\{y \mid 0 \leq y \leq 2\}$ .

- 8 a**  $f(x)$  has domain  $\{x \mid -2 \leq x \leq 6\}$  and range  $\{y \mid -7 \leq y \leq 5\}$ .
- i**  $f(x)$  on the domain  $0 \leq x \leq 6$  is reflected in the  $y$ -axis.  
 $\therefore f(|x|)$  has domain  $\{x \mid -6 \leq x \leq 6\}$ .
- ii** The smallest value of  $|f(x)|$  is  $|0| = 0$ , and the largest value is  $|-7| = 7$ .  
 $\therefore |f(x)|$  has range  $\{y \mid 0 \leq y \leq 7\}$ .
- b** No, we do not know the behaviour of  $f(x)$  on  $-2 \leq x < 0$ , which is discarded when we find  $f(|x|)$ . This may or may not affect the range of  $y = f(|x|)$ .
- 9**  $f(x)$  has  $x$ -intercepts  $-3$  and  $4$ , and  $y$ -intercept  $-2$ .
- a**  $|f(x)|$  has  $x$ -intercepts  $-3$  and  $4$ , and  $y$ -intercept  $|-2| = 2$ .
- b** The  $x$ -intercept  $4$  is reflected in the  $y$ -axis to the value  $-4$ . The  $y$ -intercept is unchanged.  
 $\therefore f(|x|)$  has  $x$ -intercepts  $4$  and  $-4$ , and  $y$ -intercept  $-2$ .

10



$$f(x) = \log_2(x+4) > 0 \text{ for } x \geq 0$$

$\therefore y = |f(|x|)|$  is the graph of  $y = f(x)$  unchanged for  $x \geq 0$ , and reflected in the  $y$ -axis for  $x < 0$ .

**11 a** 
$$\begin{aligned} |-x| &= \sqrt{(-x)^2} \\ &= \sqrt{x^2} \\ &= |x| \text{ for all } x \end{aligned}$$

**b** 
$$\begin{aligned} |x|^2 &= (\sqrt{x^2})^2 \\ &= x^2 \text{ for all } x \end{aligned}$$

**c** 
$$\begin{aligned} |xy| &= \sqrt{(xy)^2} \\ &= \sqrt{x^2 y^2} \\ &= \sqrt{x^2} \sqrt{y^2} \\ &= |x| |y| \text{ for all } x, y \end{aligned}$$

**d** 
$$\begin{aligned} \left| \frac{x}{y} \right| &= \sqrt{\left( \frac{x}{y} \right)^2} \\ &= \sqrt{\frac{x^2}{y^2}} \\ &= \frac{\sqrt{x^2}}{\sqrt{y^2}} \\ &= \frac{|x|}{|y|} \text{ for all } x \text{ and } y, y \neq 0 \end{aligned}$$

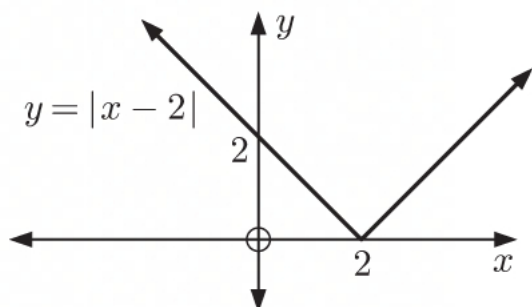
**e** 
$$\begin{aligned} |x-y| &= \sqrt{(x-y)^2} \\ &= \sqrt{(y-x)^2} \\ &= |y-x| \text{ for all } x, y \end{aligned}$$

**12 a**  $y = |x - 2|$

If  $x \geq 2$ ,  $y = x - 2$

If  $x < 2$ ,  $y = -(x - 2) = 2 - x$

$$\therefore y = \begin{cases} x - 2, & x \geq 2 \\ 2 - x, & x < 2 \end{cases}$$

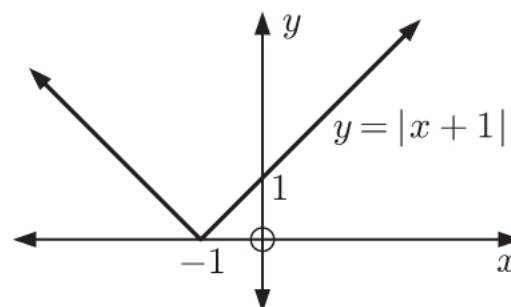


**b**  $y = |x + 1|$

If  $x \geq -1$ ,  $y = x + 1$

If  $x < -1$ ,  $y = -(x + 1) = -x - 1$

$$\therefore y = \begin{cases} x + 1, & x \geq -1 \\ -x - 1, & x < -1 \end{cases}$$

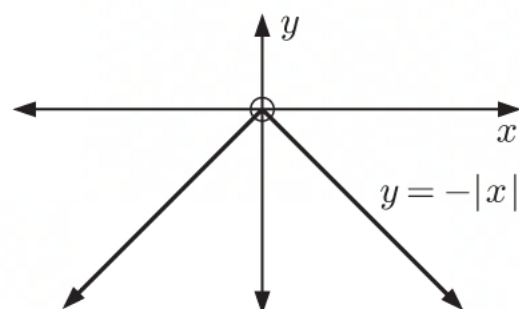


**c**  $y = -|x|$

If  $x \geq 0$ ,  $y = -x$

If  $x < 0$ ,  $y = x$

$$\therefore y = \begin{cases} -x, & x \geq 0 \\ x, & x < 0 \end{cases}$$

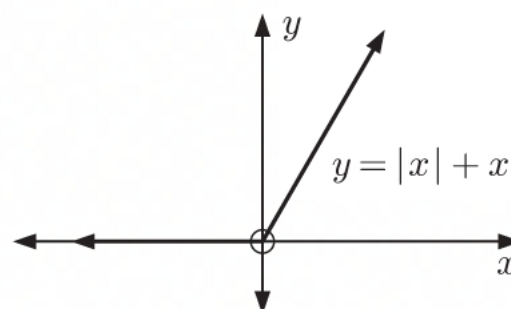


**d**  $y = |x| + x$

If  $x \geq 0$ ,  $y = x + x = 2x$

If  $x < 0$ ,  $y = -x + x = 0$

$$\therefore y = \begin{cases} 2x, & x \geq 0 \\ 0, & x < 0 \end{cases}$$



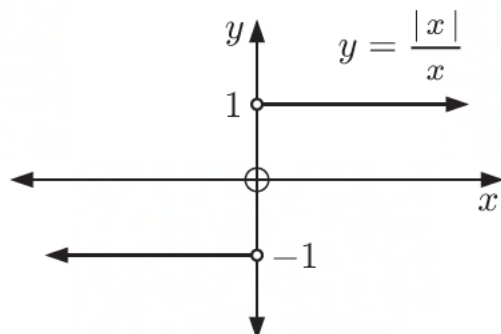
**e**  $y = \frac{|x|}{x}$

If  $x > 0$ ,  $y = \frac{x}{x} = 1$

If  $x = 0$ ,  $y = \frac{0}{0}$  which is undefined

If  $x < 0$ ,  $y = -\frac{x}{x} = -1$

$$\therefore y = \begin{cases} 1, & x > 0 \\ \text{undefined}, & x = 0 \\ -1, & x < 0 \end{cases}$$

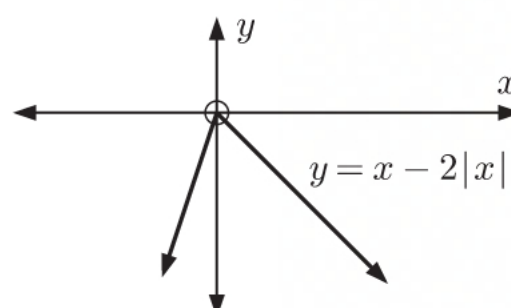


**f**  $y = x - 2|x|$

If  $x \geq 0$ ,  $y = x - 2x = -x$

If  $x < 0$ ,  $y = x + 2x = 3x$

$$\therefore y = \begin{cases} -x, & x \geq 0 \\ 3x, & x < 0 \end{cases}$$



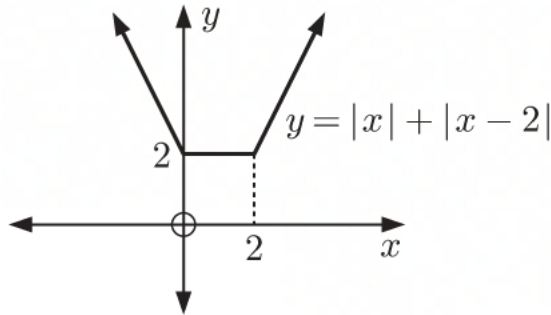
**g**  $y = |x| + |x - 2|$

If  $x \geq 2$ ,  $y = x + x - 2 = 2x - 2$

If  $0 \leq x < 2$ ,  $y = x - (x - 2) = 2$

If  $x < 0$ ,  $y = -x - (x - 2) = 2 - 2x$

$$\therefore y = \begin{cases} 2x - 2, & x \geq 2 \\ 2, & 0 \leq x < 2 \\ 2 - 2x, & x < 0 \end{cases}$$



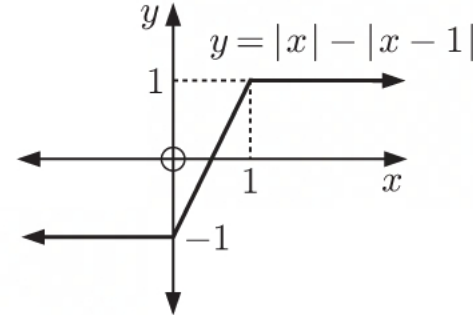
**h**  $y = |x| - |x - 1|$

If  $x \geq 1$ ,  $y = x - (x - 1) = 1$

If  $0 \leq x < 1$ ,  $y = x + (x - 1) = 2x - 1$

If  $x < 0$ ,  $y = -x + (x - 1) = -1$

$$\therefore y = \begin{cases} 1, & x \geq 1 \\ 2x - 1, & 0 \leq x < 1 \\ -1, & x < 0 \end{cases}$$



## EXERCISE 6C.2

**1 a**  $|x| = 3$   
 $\therefore x = \pm 3$

**b**  $|x| = -5$   
has no solutions as LHS  
is never negative.

**c**  $|x| = 0$   
 $\therefore x = 0$

**d**  $|x - 1| = 3$   
 $\therefore x - 1 = \pm 3$   
 $\therefore x - 1 = 3$  or  $x - 1 = -3$   
 $\therefore x = 4$  or  $x = -2$   
So,  $x = 4$  or  $-2$ .

**e**  $|3 - x| = 4$   
 $\therefore 3 - x = \pm 4$   
 $\therefore 3 - x = 4$  or  $3 - x = -4$   
 $\therefore x = -1$  or  $x = 7$   
So,  $x = -1$  or  $7$ .

**f**  $|x + 5| = -1$  has no solutions as  
LHS is never negative.

**g**  $|3x - 2| = 1$   
 $\therefore 3x - 2 = \pm 1$   
 $\therefore 3x - 2 = 1$  or  $3x - 2 = -1$   
 $\therefore 3x = 3$  or  $3x = 1$   
 $\therefore x = 1$  or  $x = \frac{1}{3}$   
So,  $x = 1$  or  $\frac{1}{3}$ .

**h**  $|3 - 2x| = 3$   
 $\therefore 3 - 2x = \pm 3$   
 $\therefore 3 - 2x = 3$  or  $3 - 2x = -3$   
 $\therefore -2x = 0$  or  $-2x = -6$   
 $\therefore x = 0$  or  $x = 3$   
So,  $x = 0$  or  $3$ .

**i**  $|2 - 5x| = 12$   
 $\therefore 2 - 5x = \pm 12$   
 $\therefore 2 - 5x = 12$  or  $2 - 5x = -12$   
 $\therefore -5x = 10$  or  $-5x = -14$   
 $\therefore x = -2$  or  $x = \frac{14}{5}$   
So,  $x = -2$  or  $\frac{14}{5}$ .



**2 a** If  $\left| \frac{x}{x-1} \right| = 3$ , then  $\frac{x}{x-1} = \pm 3$ .

$$\begin{aligned} \therefore \frac{x}{x-1} &= 3 & \text{or} & \frac{x}{x-1} = -3 \\ \therefore x &= 3(x-1) & \text{or} & x = -3(x-1) \\ \therefore x &= 3x-3 & \text{or} & x = -3x+3 \\ \therefore -2x &= -3 & \text{or} & 4x = 3 \\ \therefore x &= \frac{3}{2} & \text{or} & x = \frac{3}{4} \end{aligned}$$

So,  $x = \frac{3}{2}$  or  $\frac{3}{4}$ .

**b** If  $\left| \frac{2x-1}{x+1} \right| = 5$ , then  $\frac{2x-1}{x+1} = \pm 5$ .

$$\begin{aligned} \therefore \frac{2x-1}{x+1} &= 5 & \text{or} & \frac{2x-1}{x+1} = -5 \\ \therefore 2x-1 &= 5(x+1) & \text{or} & 2x-1 = -5(x+1) \\ \therefore 2x-1 &= 5x+5 & \text{or} & 2x-1 = -5x-5 \\ \therefore -3x &= 6 & \text{or} & 7x = -4 \\ \therefore x &= -2 & \text{or} & x = -\frac{4}{7} \end{aligned}$$

So,  $x = -2$  or  $-\frac{4}{7}$ .

**c** If  $\left| \frac{x+3}{1-3x} \right| = \frac{1}{2}$ , then  $\frac{x+3}{1-3x} = \pm \frac{1}{2}$ .

$$\begin{aligned} \therefore \frac{x+3}{1-3x} &= \frac{1}{2} & \text{or} & \frac{x+3}{1-3x} = -\frac{1}{2} \\ \therefore x+3 &= \frac{1}{2}(1-3x) & \text{or} & x+3 = -\frac{1}{2}(1-3x) \\ \therefore x+3 &= \frac{1}{2} - \frac{3}{2}x & \text{or} & x+3 = -\frac{1}{2} + \frac{3}{2}x \\ \therefore \frac{5}{2}x &= -\frac{5}{2} & \text{or} & -\frac{1}{2}x = -\frac{7}{2} \\ \therefore x &= -1 & \text{or} & x = 7 \end{aligned}$$

So,  $x = -1$  or  $7$ .

**3 a** If  $\left| \frac{3x+1}{x-1} \right| = 3$ , then  $\frac{3x+1}{x-1} = \pm 3$ .

$$\begin{aligned} \text{If } \frac{3x+1}{x-1} &= 3 \\ \text{then } 3x+1 &= 3(x-1) \\ \therefore 3x+1 &= 3x-3 \\ \therefore 1 &= -3 \end{aligned}$$

$$\therefore \frac{3x+1}{x-1} = 3 \text{ has no solutions.}$$

$$\therefore \left| \frac{3x+1}{x-1} \right| = 3 \text{ only has at most one solution.}$$

**b**  $\frac{3x+1}{x-1} = -3$

$$\begin{aligned} \therefore 3x+1 &= -3(x-1) \\ \therefore 3x+1 &= -3x+3 \\ \therefore 6x &= 2 \\ \therefore x &= \frac{1}{3} \end{aligned}$$

**4** If  $|x| = |a|$  then  $\sqrt{x^2} = \sqrt{a^2}$   
 $\therefore x^2 = a^2$   
 $\therefore x = \pm a$

**5 a** If  $|3x - 1| = |x + 2|$ , then  $3x - 1 = \pm(x + 2)$

$$\therefore 3x - 1 = x + 2 \quad \text{or} \quad 3x - 1 = -(x + 2)$$

$$\therefore 2x = 3 \quad \text{or} \quad 3x - 1 = -x - 2$$

$$\therefore x = \frac{3}{2} \quad \text{or} \quad 4x = -1$$

$$\therefore x = \frac{3}{2} \quad \text{or} \quad x = -\frac{1}{4}$$

So,  $x = \frac{3}{2}$  or  $-\frac{1}{4}$ .

**b** If  $|2x + 5| = |1 - x|$ , then  $2x + 5 = \pm(1 - x)$

$$\therefore 2x + 5 = 1 - x \quad \text{or} \quad 2x + 5 = -(1 - x)$$

$$\therefore 3x = -4 \quad \text{or} \quad 2x + 5 = -1 + x$$

$$\therefore x = -\frac{4}{3} \quad \text{or} \quad x = -6$$

So,  $x = -\frac{4}{3}$  or  $-6$ .

**c** If  $|x + 1| = |2 - x|$ , then  $x + 1 = \pm(2 - x)$

$$\therefore x + 1 = 2 - x \quad \text{or} \quad x + 1 = -(2 - x)$$

$$\therefore 2x = 1 \quad \text{or} \quad x + 1 = -2 + x$$

$$\therefore x = \frac{1}{2} \quad \text{or} \quad 1 = -2 \quad \text{which is false}$$

So,  $x = \frac{1}{2}$  is the only solution.

**d** If  $|x| = |5 - x|$ , then  $x = \pm(5 - x)$

$$\therefore x = 5 - x \quad \text{or} \quad x = -(5 - x)$$

$$\therefore 2x = 5 \quad \text{or} \quad x = -5 + x$$

$$\therefore x = \frac{5}{2} \quad \text{or} \quad 0 = -5 \quad \text{which is false}$$

So,  $x = \frac{5}{2}$  is the only solution.

**e** If  $|1 - 4x| = 2|x - 1|$ , then  $1 - 4x = \pm 2(x - 1)$

$$\therefore 1 - 4x = 2(x - 1) \quad \text{or} \quad 1 - 4x = -2(x - 1)$$

$$\therefore 1 - 4x = 2x - 2 \quad \text{or} \quad 1 - 4x = -2x + 2$$

$$\therefore -6x = -3 \quad \text{or} \quad -2x = 1$$

$$\therefore x = \frac{1}{2} \quad \text{or} \quad x = -\frac{1}{2}$$

So,  $x = \frac{1}{2}$  or  $-\frac{1}{2}$ .

**f** If  $|3x + 2| = 2|2 - x|$ , then  $3x + 2 = \pm 2(2 - x)$

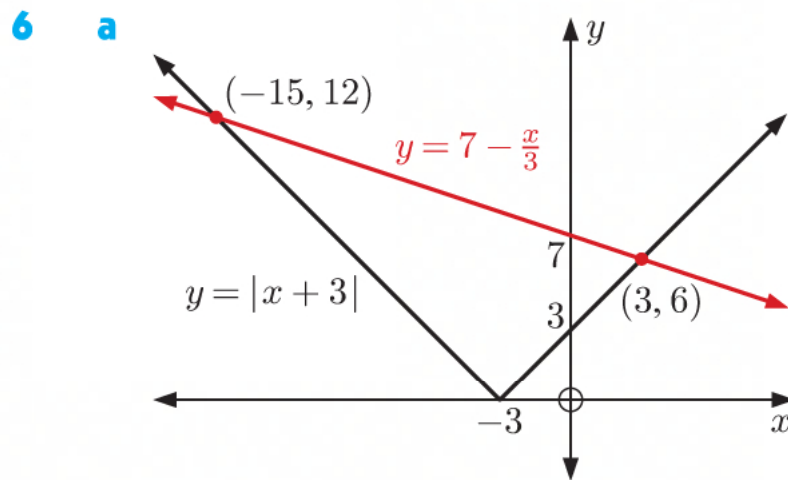
$$\therefore 3x + 2 = 2(2 - x) \quad \text{or} \quad 3x + 2 = -2(2 - x)$$

$$\therefore 3x + 2 = 4 - 2x \quad \text{or} \quad 3x + 2 = -4 + 2x$$

$$\therefore 5x = 2 \quad \text{or} \quad x = -6$$

$$\therefore x = \frac{2}{5} \quad \text{or} \quad x = -6$$

So,  $x = \frac{2}{5}$  or  $-6$ .



**b** If  $|x + 3| = 7 - \frac{x}{3}$ , then  $x + 3 = \pm \left(7 - \frac{x}{3}\right)$

$$\therefore x + 3 = 7 - \frac{x}{3} \quad \text{or} \quad x + 3 = -\left(7 - \frac{x}{3}\right)$$

$$\therefore \frac{4}{3}x = 4 \quad \text{or} \quad x + 3 = -7 + \frac{x}{3}$$

$$\therefore x = 3 \quad \text{or} \quad \frac{2}{3}x = -10$$

$$\therefore x = 3 \quad \text{or} \quad 2x = -30$$

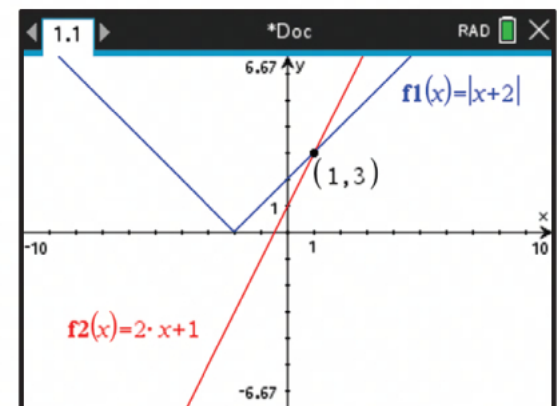
$$\therefore x = 3 \quad \text{or} \quad x = -15$$

So,  $x = 3$  or  $-15$ .

**7 a** We graph  $y = |x + 2|$  and  $y = 2x + 1$  on the same set of axes.

The graphs intersect at  $(1, 3)$ .

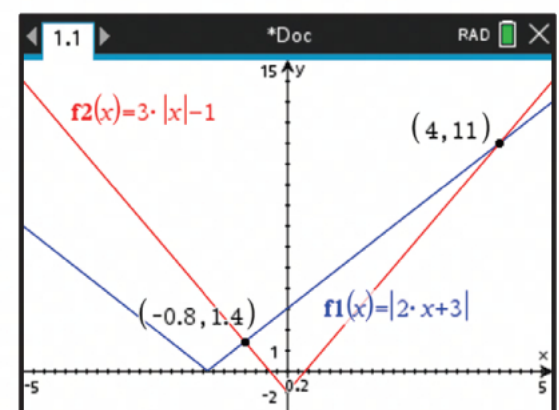
$\therefore$  the solution is  $x = 1$ .



**b** We graph  $y = |2x + 3|$  and  $y = 3|x| - 1$  on the same set of axes.

The graphs intersect at  $(-0.8, 1.4)$  and  $(4, 11)$ .

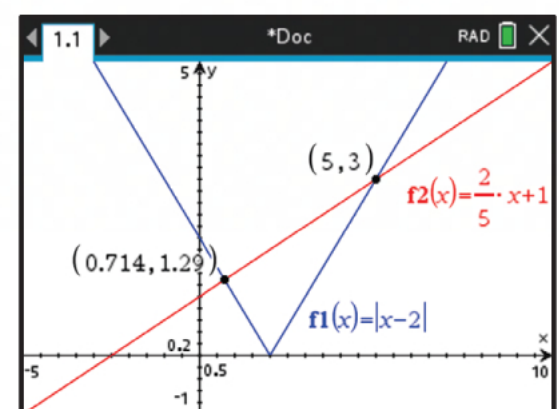
$\therefore$  the solutions are  $x = -\frac{4}{5}$  or  $4$ .



**c** We graph  $y = |x - 2|$  and  $y = \frac{2}{5}x + 1$  on the same set of axes.

The graphs intersect at about  $(0.714, 1.29)$  and  $(5, 3)$ .

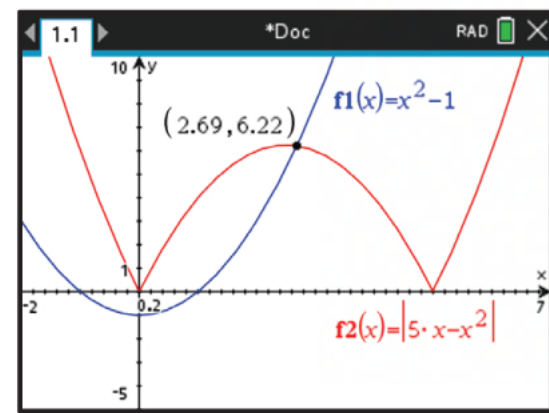
$\therefore$  the solutions are  $x \approx 0.714$  or  $x = 5$ .



- d** We graph  $y = x^2 - 1$  and  $y = |5x - x^2|$  on the same set of axes.

The graphs intersect at about  $(2.69, 6.22)$ .

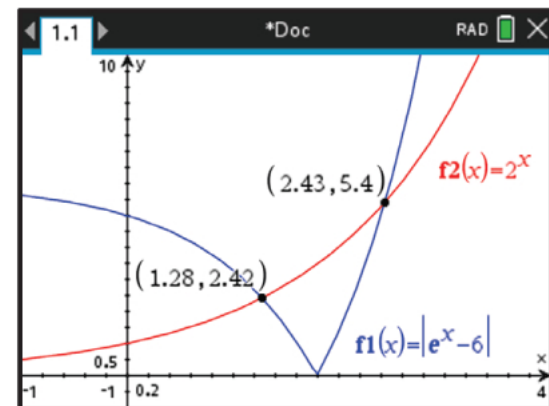
$\therefore$  the solution is  $x \approx 2.69$ .



- e** We graph  $y = |e^x - 6|$  and  $y = 2^x$  on the same set of axes.

The graphs intersect at about  $(1.28, 2.42)$  and  $(2.43, 5.40)$ .

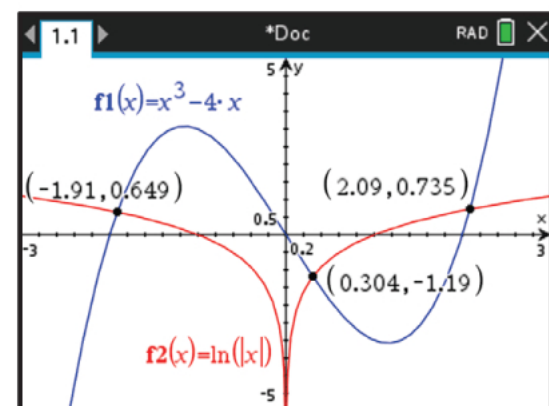
$\therefore$  the solutions are  $x \approx 1.28$  or  $2.43$ .



- f** We graph  $y = x^3 - 4x$  and  $y = \ln|x|$  on the same set of axes.

The graphs intersect at about  $(-1.91, 0.649)$ ,  $(0.304, -1.19)$ , and  $(2.09, 0.735)$ .

$\therefore$  the solutions are  $x \approx -1.91, 0.304$ , or  $2.09$ .



## EXERCISE 6C.3

**1 a**  $|x - 1| \leq 4$   
 $\therefore -4 \leq x - 1 \leq 4$   
 $\therefore -3 \leq x \leq 5$

**c**  $|2x - 3| < 1$   
 $\therefore -1 < 2x - 3 < 1$   
 $\therefore 2 < 2x < 4$   
 $\therefore 1 < x < 2$

**e**  $|1 - 2x| < 4$   
 $\therefore -4 < 1 - 2x < 4$   
 $\therefore -5 < -2x < 3$   
 $\therefore \frac{5}{2} > x > -\frac{3}{2}$   
 $\therefore -\frac{3}{2} < x < \frac{5}{2}$

**b**  $|x + 2| > 7$   
 $\therefore x + 2 < -7$  or  $x + 2 > 7$   
 $\therefore x < -9$  or  $x > 5$

**d**  $|3x + 5| \geq 2$   
 $\therefore 3x + 5 \leq -2$  or  $3x + 5 \geq 2$   
 $\therefore 3x \leq -7$  or  $3x \geq -3$   
 $\therefore x \leq -\frac{7}{3}$  or  $x \geq -1$

**f**  $|-5x - 4| > \frac{1}{2}$   
 $\therefore -5x - 4 < -\frac{1}{2}$  or  $-5x - 4 > \frac{1}{2}$   
 $\therefore -5x < \frac{7}{2}$  or  $-5x > \frac{9}{2}$   
 $\therefore x > -\frac{7}{10}$  or  $x < -\frac{9}{10}$



**2 a**

$$\begin{aligned}
 & |x+5| < |x-1| \\
 \therefore & |x+5|^2 < |x-1|^2 & \{\text{squaring both sides}\} \\
 \therefore & (x+5)^2 < (x-1)^2 & \{|a|^2 = a^2\} \\
 \therefore & \cancel{x^2} + 10x + 25 < \cancel{x^2} - 2x + 1 \\
 \therefore & 12x < -24 \\
 \therefore & x < -2
 \end{aligned}$$

**b**

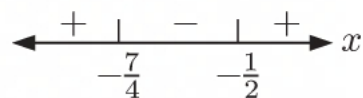
$$\begin{aligned}
 & |2x+1| > |x+2| \\
 \therefore & |2x+1|^2 > |x+2|^2 & \{\text{squaring both sides}\} \\
 \therefore & (2x+1)^2 > (x+2)^2 & \{|a|^2 = a^2\} \\
 \therefore & 4x^2 + 4x + 1 > x^2 + 4x + 4 \\
 \therefore & 3x^2 > 3 \\
 \therefore & x^2 > 1 \\
 \therefore & x < -1 \text{ or } x > 1
 \end{aligned}$$

**c**

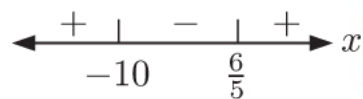
$$\begin{aligned}
 & |x-3| \leq |2x-6| \\
 \therefore & |x-3|^2 \leq |2x-6|^2 & \{\text{squaring both sides}\} \\
 \therefore & (x-3)^2 \leq (2x-6)^2 & \{|a|^2 = a^2\} \\
 \therefore & x^2 - 6x + 9 \leq 4x^2 - 24x + 36 \\
 \therefore & 3x^2 - 18x + 27 \geq 0 \\
 \therefore & x^2 - 6x + 9 \geq 0 \\
 \therefore & (x-3)^2 \geq 0
 \end{aligned}$$

which is true for all  $x \in \mathbb{R}$ .**d**

$$\begin{aligned}
 & |3x+4| \geq |x+3| \\
 \therefore & |3x+4|^2 \geq |x+3|^2 & \{\text{squaring both sides}\} \\
 \therefore & (3x+4)^2 \geq (x+3)^2 & \{|a|^2 = a^2\} \\
 \therefore & 9x^2 + 24x + 16 \geq x^2 + 6x + 9 \\
 \therefore & 8x^2 + 18x + 7 \geq 0 \\
 \therefore & (4x+7)(2x+1) \geq 0 \\
 \therefore & x \leq -\frac{7}{4} \text{ or } x \geq -\frac{1}{2}
 \end{aligned}$$

**e**

$$\begin{aligned}
 & 2|x-4| < |3x+2| \\
 \therefore & 4|x-4|^2 < |3x+2|^2 & \{\text{squaring both sides}\} \\
 \therefore & 4(x-4)^2 < (3x+2)^2 & \{|a|^2 = a^2\} \\
 \therefore & 4(x^2 - 8x + 16) < 9x^2 + 12x + 4 \\
 \therefore & 4x^2 - 32x + 64 < 9x^2 + 12x + 4 \\
 \therefore & 5x^2 + 44x - 60 > 0 \\
 \therefore & (5x-6)(x+10) > 0 \\
 \therefore & x < -10 \text{ or } x > \frac{6}{5}
 \end{aligned}$$



**f**

$$\frac{1}{3} |2x + 5| \geq |4 - x|$$

$$\therefore \frac{1}{9} |2x + 5|^2 \geq |4 - x|^2 \quad \{\text{squaring both sides}\}$$

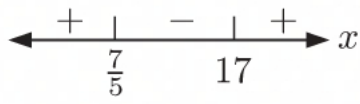
$$\therefore \frac{1}{9} (2x + 5)^2 \geq (4 - x)^2 \quad \{|a|^2 = a^2\}$$

$$\therefore \frac{1}{9} (4x^2 + 20x + 25) \geq 16 - 8x + x^2$$

$$\therefore 4x^2 + 20x + 25 \geq 144 - 72x + 9x^2$$

$$\therefore 5x^2 - 92x + 119 \leq 0$$

$$\therefore (5x - 7)(x - 17) \leq 0$$

$$\therefore \frac{7}{5} \leq x \leq 17$$


**3 a**

$$\left| \frac{x}{x-2} \right| \geq 3$$

$$\therefore \left| \frac{x}{x-2} \right|^2 \geq 3^2 \quad \{\text{squaring both sides}\}$$

$$\therefore \left( \frac{x}{x-2} \right)^2 \geq 9 \quad \{|a|^2 = a^2\}$$

$$\therefore \frac{x^2}{(x-2)^2} \geq 9$$

$$\therefore x^2 \geq 9(x-2)^2 \quad \text{provided } x \neq 2 \quad \{\text{since } (x-2)^2 \geq 0\}$$

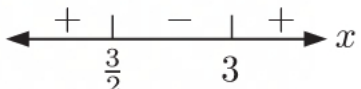
$$\therefore x^2 \geq 9(x^2 - 4x + 4) \quad \text{provided } x \neq 2$$

$$\therefore x^2 \geq 9x^2 - 36x + 36 \quad \text{provided } x \neq 2$$

$$\therefore 8x^2 - 36x + 36 \leq 0 \quad \text{provided } x \neq 2$$

$$\therefore 2x^2 - 9x + 9 \leq 0 \quad \text{provided } x \neq 2$$

$$\therefore (2x - 3)(x - 3) \leq 0 \quad \text{provided } x \neq 2$$

$$\therefore \frac{3}{2} \leq x \leq 3, \quad x \neq 2$$


**b**

$$\left| \frac{2x+3}{x-1} \right| \geq 2$$

$$\therefore \left| \frac{2x+3}{x-1} \right|^2 \geq 2^2 \quad \{\text{squaring both sides}\}$$

$$\therefore \left( \frac{2x+3}{x-1} \right)^2 \geq 4 \quad \{|a|^2 = a^2\}$$

$$\therefore \frac{(2x+3)^2}{(x-1)^2} \geq 4$$

$$\therefore (2x+3)^2 \geq 4(x-1)^2 \quad \text{provided } x \neq 1 \quad \{\text{since } (x-1)^2 \geq 0\}$$

$$\therefore 4x^2 + 12x + 9 \geq 4(x^2 - 2x + 1) \quad \text{provided } x \neq 1$$

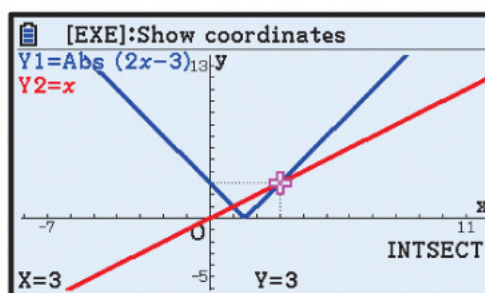
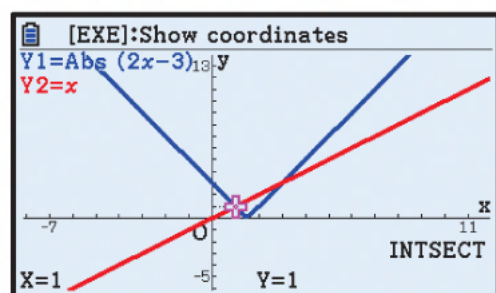
$$\therefore \cancel{4x^2} + 12x + 9 \geq \cancel{4x^2} - 8x + 4 \quad \text{provided } x \neq 1$$

$$\therefore 20x \geq -5 \quad \text{provided } x \neq 1$$

$$\therefore x \geq -\frac{1}{4}, \quad x \neq 1$$

$$\begin{aligned}
 & \left| \frac{x-4}{1-2x} \right| < \frac{2}{3} \\
 \therefore & \left| \frac{x-4}{1-2x} \right|^2 < \left( \frac{2}{3} \right)^2 && \{\text{squaring both sides}\} \\
 \therefore & \left( \frac{x-4}{1-2x} \right)^2 < \frac{4}{9} && \{|a|^2 = a^2\} \\
 \therefore & \frac{(x-4)^2}{(1-2x)^2} < \frac{4}{9} \\
 \therefore & 9(x-4)^2 < 4(1-2x)^2 && \text{provided } x \neq \frac{1}{2} \quad \{\text{since } (1-2x)^2 \geq 0\} \\
 \therefore & 9(x^2 - 8x + 16) < 4(1 - 4x + 4x^2) && \text{provided } x \neq \frac{1}{2} \\
 \therefore & 9x^2 - 72x + 144 < 4 - 16x + 16x^2 && \text{provided } x \neq \frac{1}{2} \\
 \therefore & 7x^2 + 56x - 140 > 0 && \text{provided } x \neq \frac{1}{2} \\
 \therefore & x^2 + 8x - 20 > 0 && \text{provided } x \neq \frac{1}{2} \\
 \therefore & (x+10)(x-2) > 0 && \text{provided } x \neq \frac{1}{2} \\
 \therefore & x < -10 \text{ or } x > 2
 \end{aligned}$$

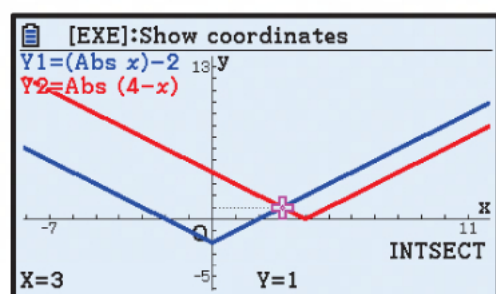
- 4 a We draw graphs of  $y = |2x - 3|$  and  $y = x$  on the same set of axes.



The graphs intersect at  $x = 1$  and  $x = 3$ .

$\therefore |2x - 3| < x$  when  $1 < x < 3$ .

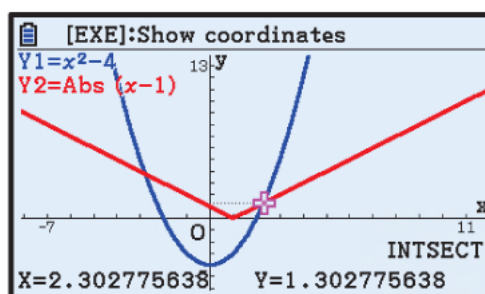
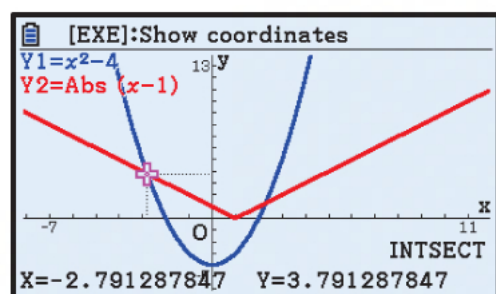
- b We draw graphs of  $y = |x| - 2$  and  $y = |4 - x|$  on the same set of axes.



The graphs intersect at  $x = 3$ .

$\therefore |x| - 2 \geq |4 - x|$  when  $x \geq 3$ .

- c We draw graphs of  $y = x^2 - 4$  and  $y = |x - 1|$  on the same set of axes.

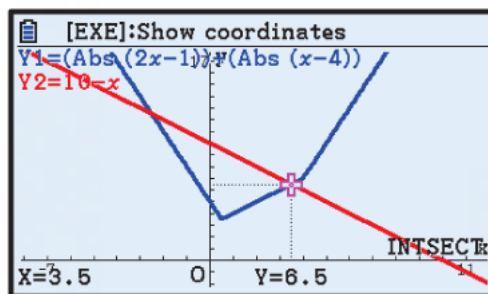
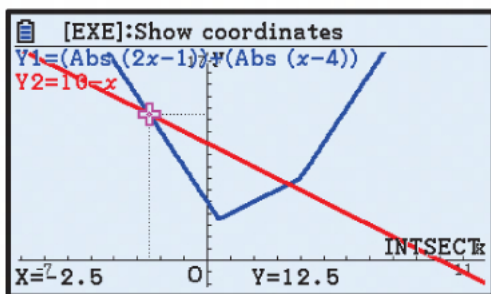


The graphs intersect at  $x \approx -2.79$  and  $x \approx 2.30$ .

$\therefore x^2 - 4 > |x - 1|$  when  $x < -2.79$  or  $x > 2.30$ .



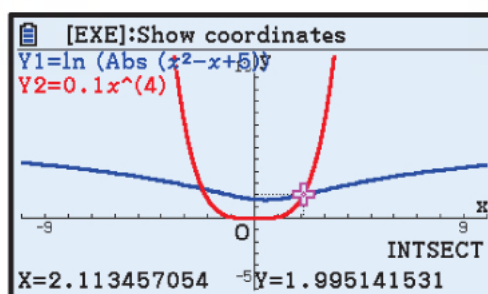
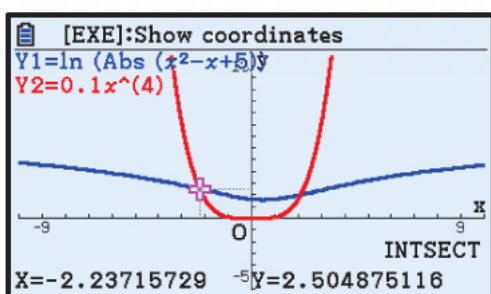
- d We draw graphs of  $y = |2x - 1| + |x - 4|$  and  $y = 10 - x$  on the same set of axes.



The graphs intersect at  $x = -\frac{5}{2}$  and  $x = \frac{7}{2}$ .

$\therefore |2x - 1| + |x - 4| \leq 10 - x$  when  $-\frac{5}{2} \leq x \leq \frac{7}{2}$ .

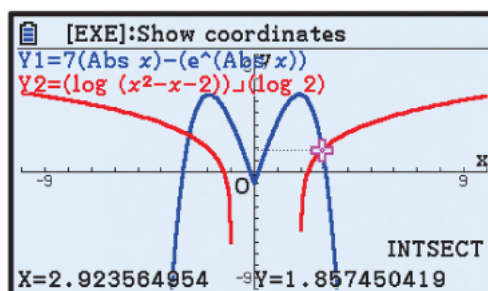
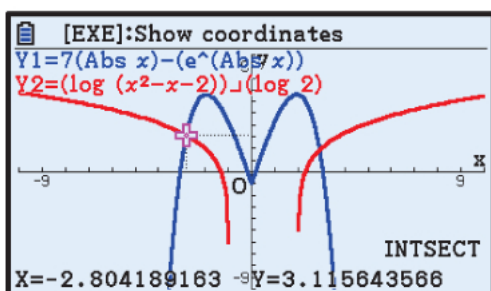
- e We draw graphs of  $y = \ln|x^2 - x + 5|$  and  $y = 0.1x^4$  on the same set of axes.



The graphs intersect at  $x \approx -2.24$  and  $x \approx 2.11$ .

$\therefore \ln|x^2 - x + 5| \geq 0.1x^4$  when  $-2.24 \leq x \leq 2.11$ .

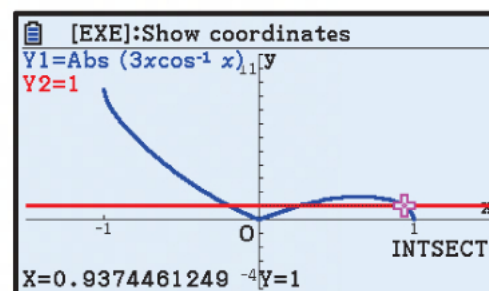
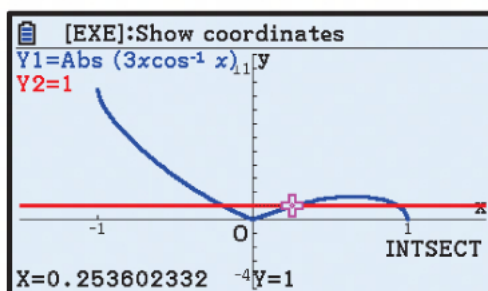
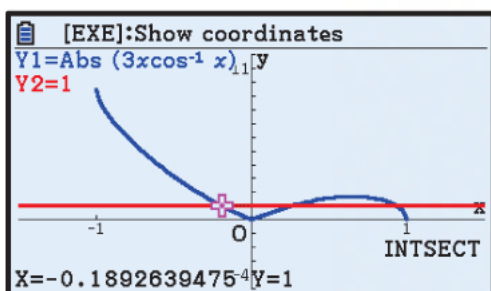
- f We draw graphs of  $y = 7|x| - e^{|x|}$  and  $y = \log_2(x^2 - x - 2)$  on the same set of axes.



The graphs intersect at  $x \approx -2.80$  and  $x \approx 2.92$ . The graph of  $y = \log_2(x^2 - x - 2)$  is undefined between  $x = -1$  and  $x = 2$ .

$\therefore 7|x| - e^{|x|} \geq \log_2(x^2 - x - 2)$  when  $-2.80 \leq x < -1$  or  $2 < x \leq 2.92$ .

- g We draw graphs of  $y = |3x \arccos x|$  and  $y = 1$  on the same set of axes.



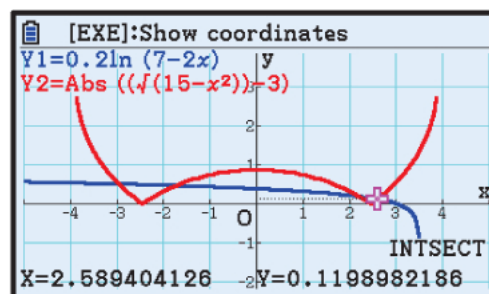
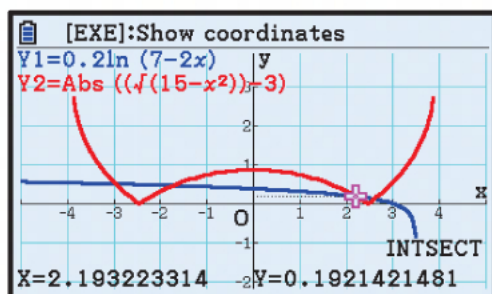
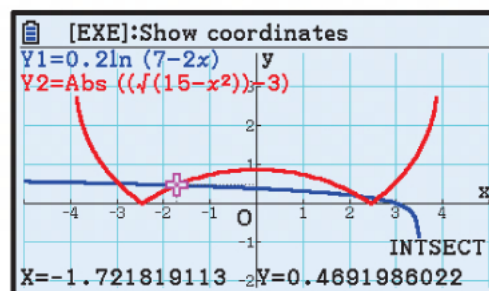
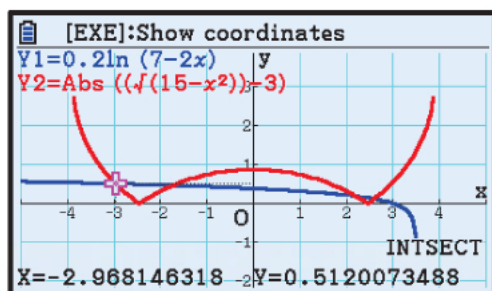
The graphs intersect at  $x \approx -0.189$ ,  $x \approx 0.254$ , and  $x \approx 0.937$ .

$y = |3x \arccos x|$  has domain  $-1 \leq x \leq 1$ .

$\therefore |3x \arccos x| > 1$  when  $-1 \leq x < -0.189$  or  $0.254 < x \leq 0.937$ .



**h** We draw graphs of  $y = 0.2 \ln(7 - 2x)$  and  $y = |\sqrt{15 - x^2} - 3|$  on the same set of axes.



The graphs intersect at  $x \approx -2.97$ ,  $x \approx -1.72$ ,  $x \approx 2.19$ , and  $x \approx 2.59$ .

$y = 0.2 \ln(7 - 2x)$  has domain  $x < 3.5$ .

$y = |\sqrt{15 - x^2} - 3|$  has domain  $-\sqrt{15} \leq x \leq \sqrt{15}$ .

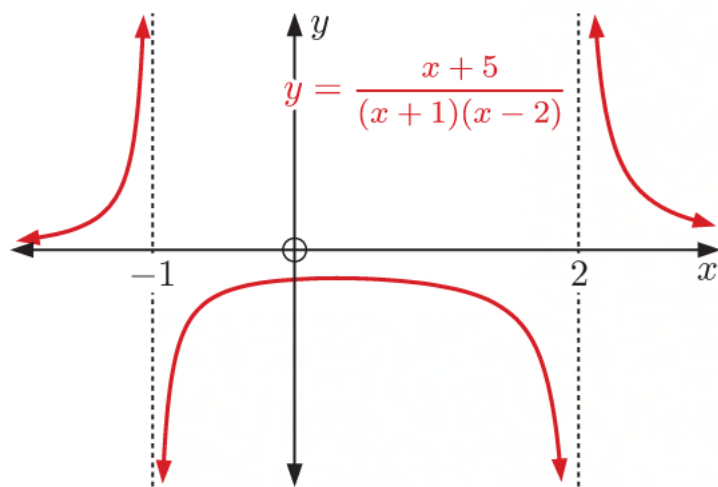
$\therefore 0.2 \ln(7 - 2x) < |\sqrt{15 - x^2} - 3|$  when  $-\sqrt{15} \leq x < -2.97$  or  $-1.72 < x < 2.19$  or  $2.59 < x < 3.5$ .

## INVESTIGATION 2

## RATIONAL FUNCTIONS OF THE FORM

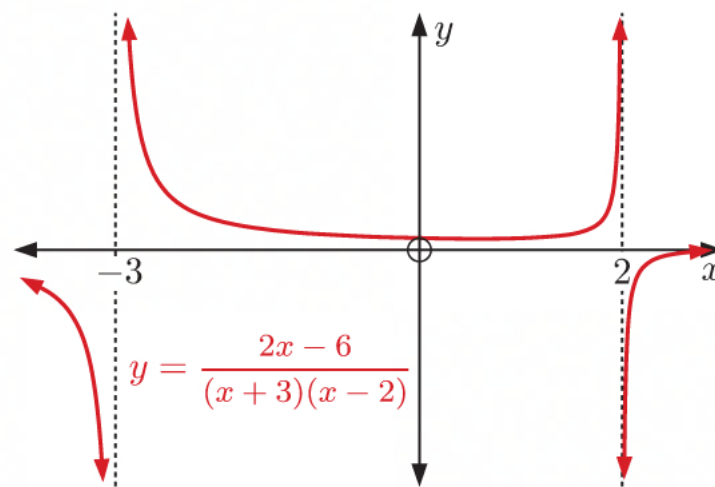
$$y = \frac{ax + b}{cx^2 + dx + e}, \quad c \neq 0$$

**1 a**

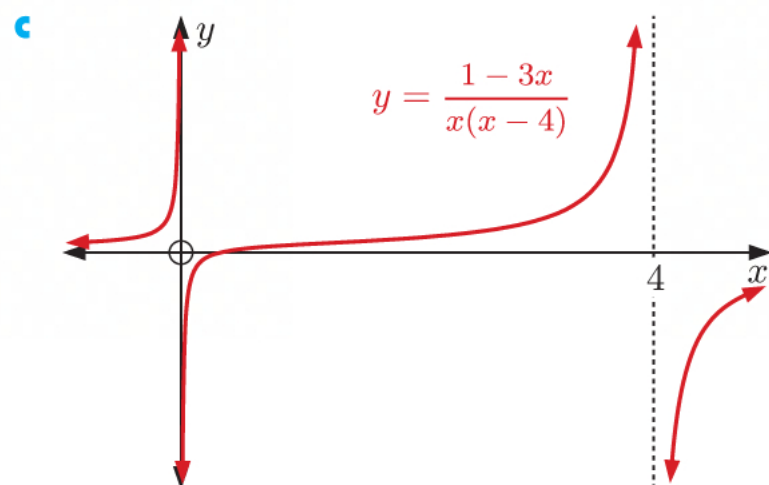


- i** Domain =  $\{x \mid x \neq -1 \text{ or } 2\}$
- ii** Vertical asymptotes  $x = -1$  and  $x = 2$ .  
Horizontal asymptote  $y = 0$ .

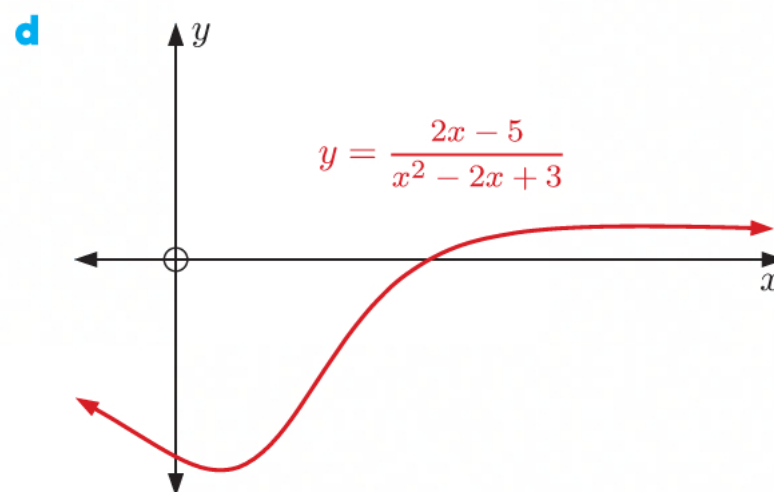
**b**



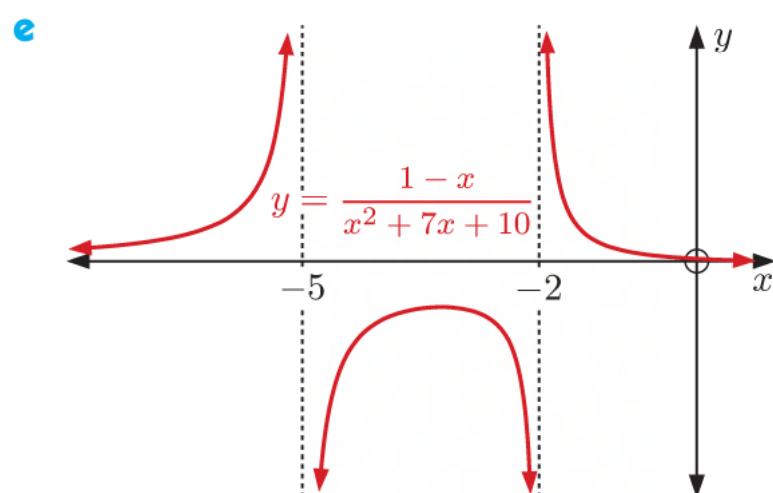
- i** Domain =  $\{x \mid x \neq -3 \text{ or } 2\}$
- ii** Vertical asymptotes  $x = -3$  and  $x = 2$ .  
Horizontal asymptote  $y = 0$ .



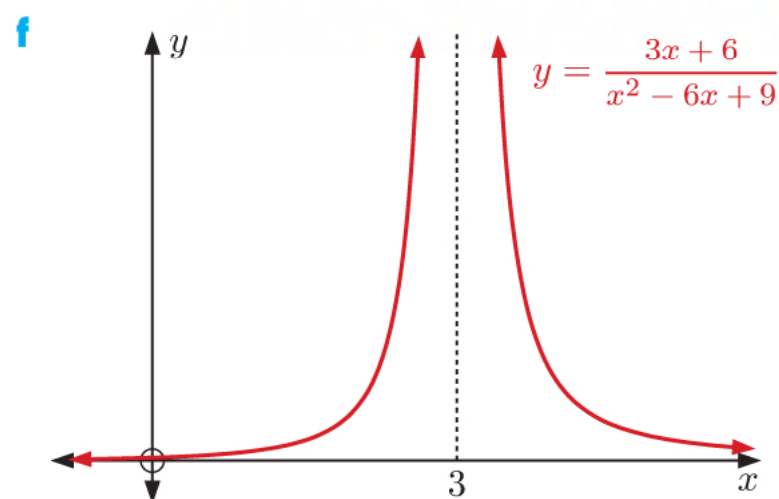
- i Domain =  $\{x \mid x \neq 0 \text{ or } 4\}$
- ii Vertical asymptotes  $x = 0$  and  $x = 4$ .  
Horizontal asymptote  $y = 0$ .



- i Domain =  $\{x \mid x \in \mathbb{R}\}$
- ii Horizontal asymptote  $y = 0$ .



- i Domain =  $\{x \mid x \neq -5 \text{ or } -2\}$
- ii Vertical asymptotes  $x = -5$  and  $x = -2$ .  
Horizontal asymptote  $y = 0$ .



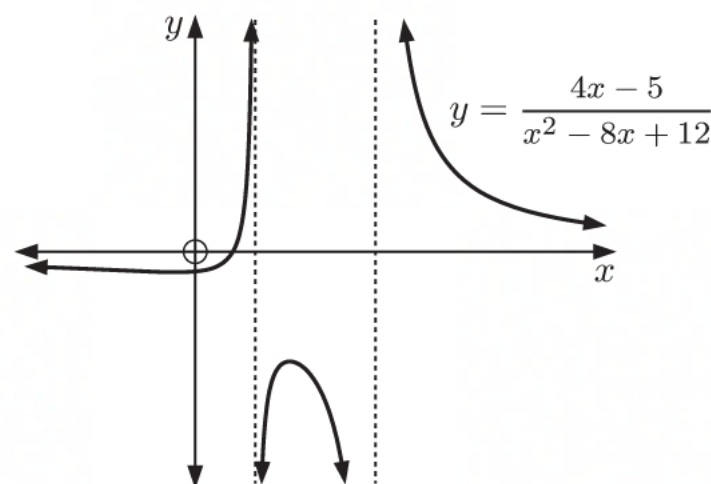
- i Domain =  $\{x \mid x \neq 3\}$
- ii Vertical asymptote  $x = 3$ .  
Horizontal asymptote  $y = 0$ .

**2**  $y = \frac{ax + b}{cx^2 + dx + e}, \quad c \neq 0$

- a The horizontal asymptote is  $y = 0$ .
- b The vertical asymptotes are the zeros of the denominator  $cx^2 + dx + e$ .

## EXERCISE 6D.1

- 1 a** The horizontal asymptote is  $y = 0$ .  
The function is undefined when
- $$x^2 - 8x + 12 = 0$$
- $$\therefore (x - 2)(x - 6) = 0$$
- $$\therefore x = 2 \text{ or } 6$$
- $\therefore$  the vertical asymptotes are  $x = 2$  and  $x = 6$ .



**b** When  $x = 0$ ,  $y = \frac{-5}{12}$ , so the  $y$ -intercept is  $-\frac{5}{12}$ .

When  $y = 0$ ,  $4x - 5 = 0$

$$\therefore 4x = 5$$

$$\therefore x = \frac{5}{4}$$

$\therefore$  the  $x$ -intercept is  $\frac{5}{4}$ .

**2 a i** The horizontal asymptote is  $y = 0$ .

The function is undefined when  $x = -3$  or  $2$ .

$\therefore$  the vertical asymptotes are  $x = -3$  and  $x = 2$ .

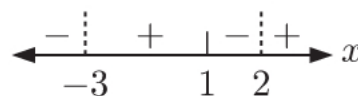
**ii** When  $x = 0$ ,  $y = \frac{-1}{(3)(-2)} = \frac{1}{6}$ , so the  $y$ -intercept is  $\frac{1}{6}$ .

When  $y = 0$ ,  $x - 1 = 0$

$$\therefore x = 1$$

$\therefore$  the  $x$ -intercept is  $1$ .

**iii**  $y = \frac{x-1}{(x+3)(x-2)}$  has sign diagram



**iv** As  $x \rightarrow -3^-$ ,  $y \rightarrow -\infty$

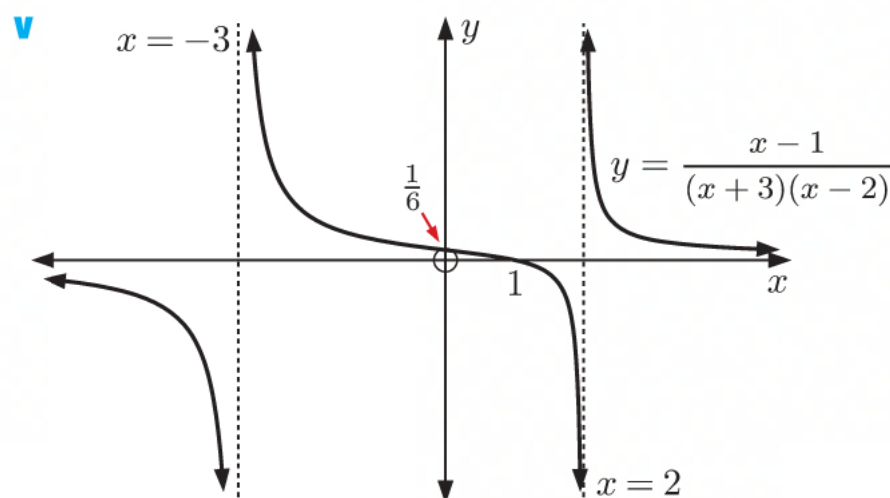
As  $x \rightarrow -3^+$ ,  $y \rightarrow \infty$

As  $x \rightarrow 2^-$ ,  $y \rightarrow -\infty$

As  $x \rightarrow 2^+$ ,  $y \rightarrow \infty$

As  $x \rightarrow -\infty$ ,  $y \rightarrow 0^-$

As  $x \rightarrow \infty$ ,  $y \rightarrow 0^+$



**b i** The horizontal asymptote is  $y = 0$ .

The function is undefined when  $x = 1$  or  $3$ .

$\therefore$  the vertical asymptotes are  $x = 1$  and  $x = 3$ .

**ii** When  $x = 0$ ,  $y = \frac{-8}{(-1)(-3)} = -\frac{8}{3}$ , so the  $y$ -intercept is  $-\frac{8}{3}$ .

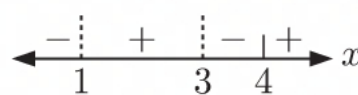
When  $y = 0$ ,  $2x - 8 = 0$

$$\therefore 2x = 8$$

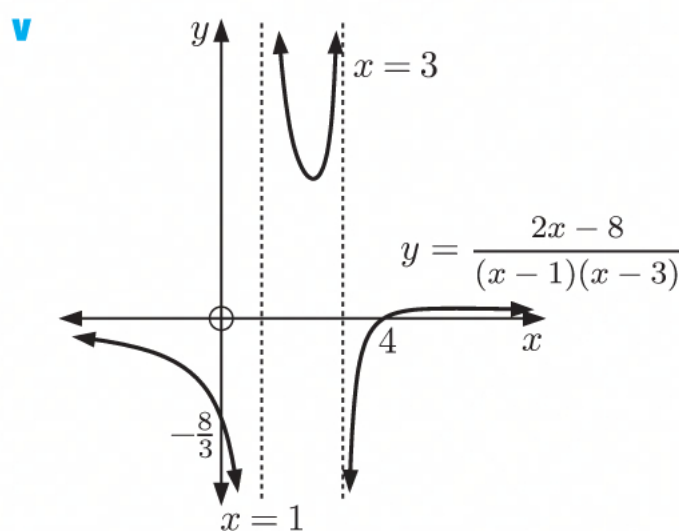
$$\therefore x = 4$$

$\therefore$  the  $x$ -intercept is  $4$ .

**iii**  $y = \frac{2x-8}{(x-1)(x-3)}$  has sign diagram



- iv As  $x \rightarrow 1^-$ ,  $y \rightarrow -\infty$   
 As  $x \rightarrow 1^+$ ,  $y \rightarrow \infty$   
 As  $x \rightarrow 3^-$ ,  $y \rightarrow \infty$   
 As  $x \rightarrow 3^+$ ,  $y \rightarrow -\infty$   
 As  $x \rightarrow -\infty$ ,  $y \rightarrow 0^-$   
 As  $x \rightarrow \infty$ ,  $y \rightarrow 0^+$



- c i The horizontal asymptote is  $y = 0$ .  
 The function is undefined when  $x = -2$  or  $4$ .  
 $\therefore$  the vertical asymptotes are  $x = -2$  and  $x = 4$ .

ii  $f(0) = \frac{5}{(2)(-4)} = -\frac{5}{8}$ , so the  $y$ -intercept is  $-\frac{5}{8}$ .

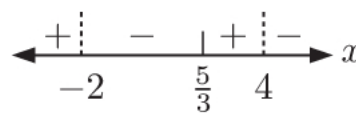
$$f(x) = 0 \text{ when } 5 - 3x = 0$$

$$\therefore 3x = 5$$

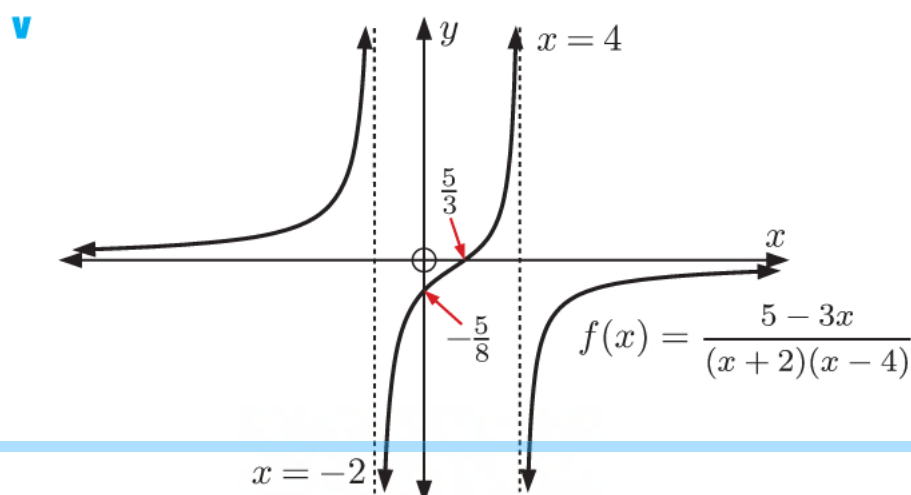
$$\therefore x = \frac{5}{3}$$

$\therefore$  the  $x$ -intercept is  $\frac{5}{3}$ .

iii  $f(x) = \frac{5 - 3x}{(x + 2)(x - 4)}$  has sign diagram

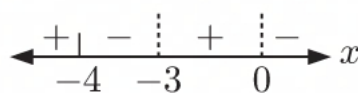


- iv As  $x \rightarrow -2^-$ ,  $f(x) \rightarrow \infty$   
 As  $x \rightarrow -2^+$ ,  $f(x) \rightarrow -\infty$   
 As  $x \rightarrow 4^-$ ,  $f(x) \rightarrow \infty$   
 As  $x \rightarrow 4^+$ ,  $f(x) \rightarrow -\infty$   
 As  $x \rightarrow -\infty$ ,  $f(x) \rightarrow 0^+$   
 As  $x \rightarrow \infty$ ,  $f(x) \rightarrow 0^-$



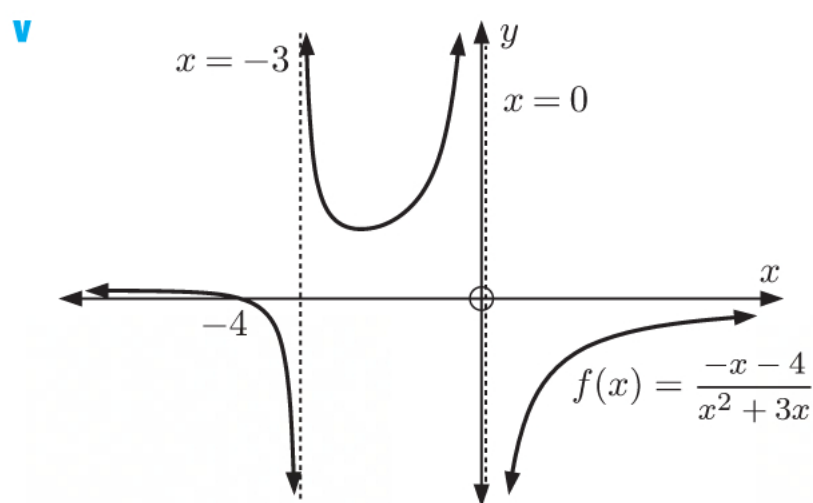
- d i The horizontal asymptote is  $y = 0$ .  
 The function is undefined when  $x^2 + 3x = 0$   
 $\therefore x(x + 3) = 0$   
 $\therefore x = 0$  or  $x = -3$   
 $\therefore$  the vertical asymptotes are  $x = 0$  and  $x = -3$ .  
 ii The function is undefined when  $x = 0$ , so there is no  $y$ -intercept.  
 $f(x) = 0$  when  $-x - 4 = 0$   
 $\therefore x = -4$   
 $\therefore$  the  $x$ -intercept is  $-4$ .

iii  $f(x) = \frac{-(x + 4)}{x(x + 3)}$  has sign diagram





- iv As  $x \rightarrow -3^-$ ,  $f(x) \rightarrow -\infty$   
 As  $x \rightarrow -3^+$ ,  $f(x) \rightarrow \infty$   
 As  $x \rightarrow 0^-$ ,  $f(x) \rightarrow \infty$   
 As  $x \rightarrow 0^+$ ,  $f(x) \rightarrow -\infty$   
 As  $x \rightarrow -\infty$ ,  $f(x) \rightarrow 0^+$   
 As  $x \rightarrow \infty$ ,  $f(x) \rightarrow 0^-$



- e i The horizontal asymptote is  $y = 0$ .

The function is undefined when  $x^2 - 4x - 5 = 0$

$$\therefore (x+1)(x-5) = 0$$

$$\therefore x = -1 \text{ or } 5$$

$\therefore$  the vertical asymptotes are  $x = -1$  and  $x = 5$ .

- ii When  $x = 0$ ,  $y = \frac{5}{-5} = -1$ , so the  $y$ -intercept is  $-1$ .

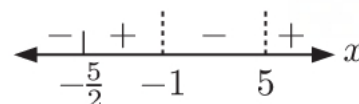
When  $y = 0$ ,  $2x + 5 = 0$

$$\therefore 2x = -5$$

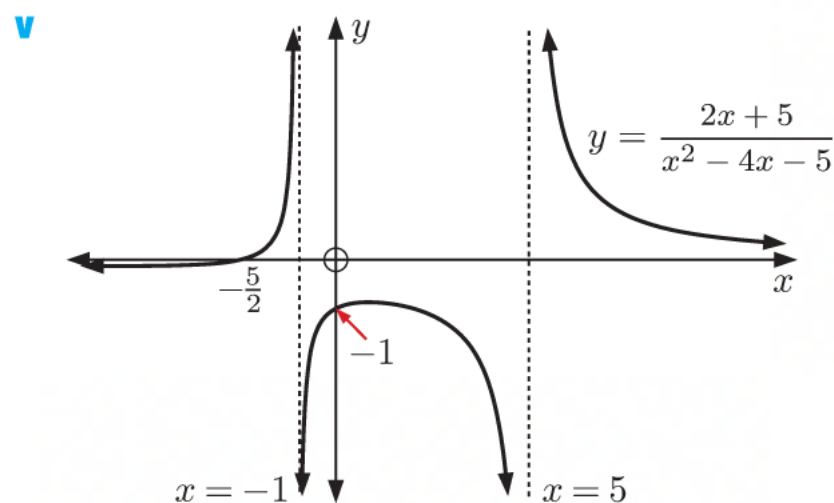
$$\therefore x = -\frac{5}{2}$$

$\therefore$  the  $x$ -intercept is  $-\frac{5}{2}$ .

- iii  $y = \frac{2x+5}{(x+1)(x-5)}$  has sign diagram



- iv As  $x \rightarrow -1^-$ ,  $y \rightarrow \infty$   
 As  $x \rightarrow -1^+$ ,  $y \rightarrow -\infty$   
 As  $x \rightarrow 5^-$ ,  $y \rightarrow -\infty$   
 As  $x \rightarrow 5^+$ ,  $y \rightarrow \infty$   
 As  $x \rightarrow -\infty$ ,  $y \rightarrow 0^-$   
 As  $x \rightarrow \infty$ ,  $y \rightarrow 0^+$



- f i The horizontal asymptote is  $y = 0$ .

The function is undefined when  $x^2 - 3x - 18 = 0$

$$\therefore (x+3)(x-6) = 0$$

$$\therefore x = -3 \text{ or } 6$$

$\therefore$  the vertical asymptotes are  $x = -3$  and  $x = 6$ .

- ii When  $x = 0$ ,  $y = \frac{12}{-18} = -\frac{2}{3}$ , so the  $y$ -intercept is  $-\frac{2}{3}$ .

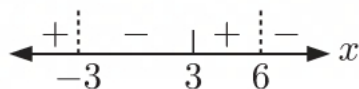
When  $y = 0$ ,  $12 - 4x = 0$

$$\therefore 4x = 12$$

$$\therefore x = 3$$

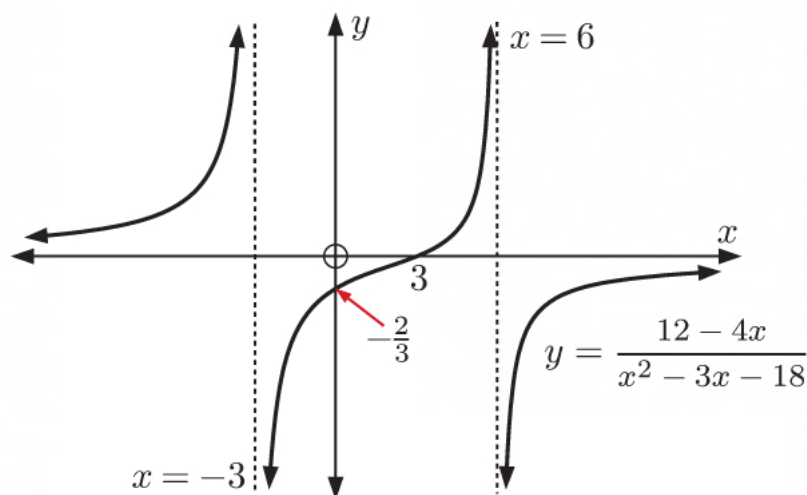
$\therefore$  the  $x$ -intercept is 3.

iii  $y = \frac{12 - 4x}{(x + 3)(x - 6)}$  has sign diagram



iv As  $x \rightarrow -3^-$ ,  $y \rightarrow \infty$   
 As  $x \rightarrow -3^+$ ,  $y \rightarrow -\infty$   
 As  $x \rightarrow 6^-$ ,  $y \rightarrow \infty$   
 As  $x \rightarrow 6^+$ ,  $y \rightarrow -\infty$   
 As  $x \rightarrow -\infty$ ,  $y \rightarrow 0^+$   
 As  $x \rightarrow \infty$ ,  $y \rightarrow 0^-$

v



g i The horizontal asymptote is  $y = 0$ .

The function is undefined when  $2x^2 + 10x - 12 = 0$

$$\therefore x^2 + 5x - 6 = 0$$

$$\therefore (x + 6)(x - 1) = 0$$

$$\therefore x = -6 \text{ or } 1$$

$\therefore$  the vertical asymptotes are  $x = -6$  and  $x = 1$ .

ii  $f(0) = \frac{-9}{-12} = \frac{3}{4}$ , so the  $y$ -intercept is  $\frac{3}{4}$ .

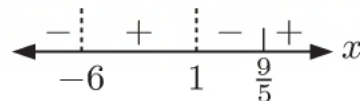
$f(x) = 0$  when  $5x - 9 = 0$

$$\therefore 5x = 9$$

$$\therefore x = \frac{9}{5}$$

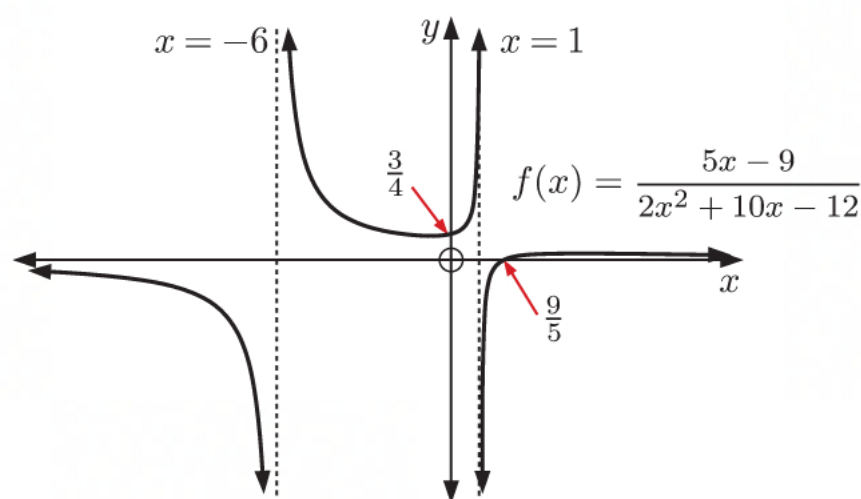
$\therefore$  the  $x$ -intercept is  $\frac{9}{5}$ .

iii  $f(x) = \frac{5x - 9}{2(x + 6)(x - 1)}$  has sign diagram



iv As  $x \rightarrow -6^-$ ,  $f(x) \rightarrow -\infty$   
 As  $x \rightarrow -6^+$ ,  $f(x) \rightarrow \infty$   
 As  $x \rightarrow 1^-$ ,  $f(x) \rightarrow \infty$   
 As  $x \rightarrow 1^+$ ,  $f(x) \rightarrow -\infty$   
 As  $x \rightarrow -\infty$ ,  $f(x) \rightarrow 0^-$   
 As  $x \rightarrow \infty$ ,  $f(x) \rightarrow 0^+$

v



h i The horizontal asymptote is  $y = 0$ .

The function is undefined when  $x^2 + 2x + 1 = 0$

$$\therefore (x + 1)^2 = 0$$

$$\therefore x = -1$$

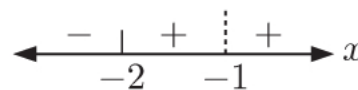
$\therefore$  the vertical asymptote is  $x = -1$ .

- ii When  $x = 0$ ,  $y = \frac{6}{1} = 6$ , so the  $y$ -intercept is 6.

$$\begin{aligned}\text{When } y = 0, \quad 3x + 6 &= 0 \\ \therefore 3x &= -6 \\ \therefore x &= -2\end{aligned}$$

$\therefore$  the  $x$ -intercept is  $-2$ .

- iii  $y = \frac{3x+6}{(x+1)^2}$  has sign diagram



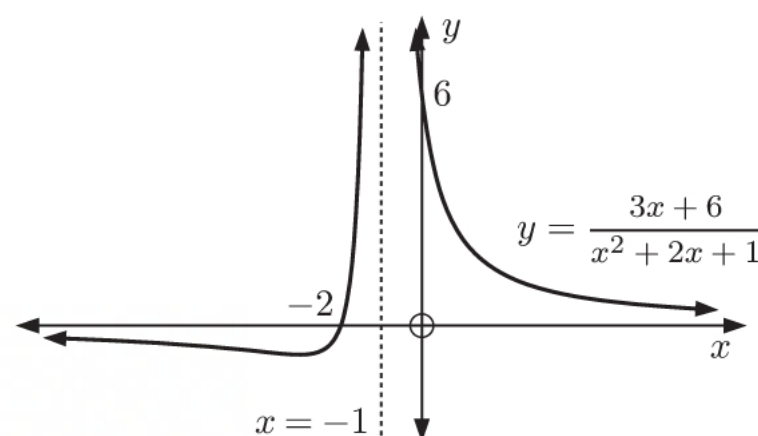
- iv As  $x \rightarrow -1^-$ ,  $y \rightarrow \infty$

$$\text{As } x \rightarrow -1^+, \quad y \rightarrow \infty$$

$$\text{As } x \rightarrow -\infty, \quad y \rightarrow 0^-$$

$$\text{As } x \rightarrow \infty, \quad y \rightarrow 0^+$$

v



- i i The horizontal asymptote is  $y = 0$ .

The function is undefined when  $2x^2 - 9x - 5 = 0$

$$\begin{aligned}\therefore (2x+1)(x-5) &= 0 \\ \therefore x &= -\frac{1}{2} \text{ or } 5\end{aligned}$$

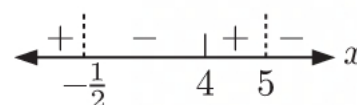
$\therefore$  the vertical asymptotes are  $x = -\frac{1}{2}$  and  $x = 5$ .

- ii  $f(0) = \frac{4}{-5}$ , so the  $y$ -intercept is  $-\frac{4}{5}$ .

$$\begin{aligned}f(x) = 0 \text{ when } 4 - x &= 0 \\ \therefore x &= 4\end{aligned}$$

$\therefore$  the  $x$ -intercept is 4.

- iii  $f(x) = \frac{4-x}{(2x+1)(x-5)}$  has sign diagram



- iv As  $x \rightarrow -\frac{1}{2}^-$ ,  $f(x) \rightarrow \infty$

$$\text{As } x \rightarrow -\frac{1}{2}^+, \quad f(x) \rightarrow -\infty$$

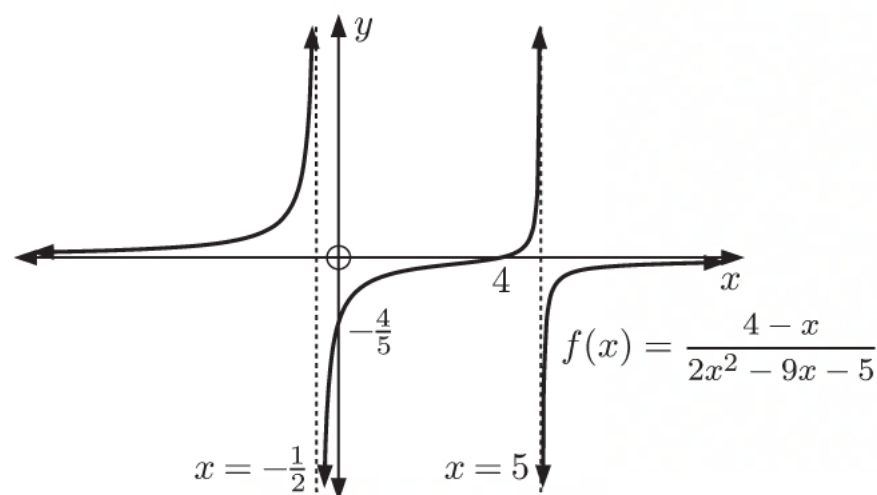
$$\text{As } x \rightarrow 5^-, \quad f(x) \rightarrow \infty$$

$$\text{As } x \rightarrow 5^+, \quad f(x) \rightarrow -\infty$$

$$\text{As } x \rightarrow -\infty, \quad f(x) \rightarrow 0^+$$

$$\text{As } x \rightarrow \infty, \quad f(x) \rightarrow 0^-$$

v



- j i** The horizontal asymptote is  $y = 0$ .

The function is undefined when  $3x^2 + 2x - 8 = 0$

$$\therefore (3x - 4)(x + 2) = 0$$

$$\therefore x = \frac{4}{3} \text{ or } -2$$

$\therefore$  the vertical asymptotes are  $x = \frac{4}{3}$  and  $x = -2$ .

- ii** When  $x = 0$ ,  $y = \frac{15}{-8}$ , so the  $y$ -intercept is  $-\frac{15}{8}$ .

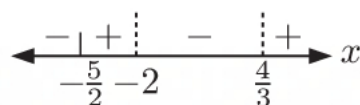
When  $y = 0$ ,  $6x + 15 = 0$

$$\therefore 6x = -15$$

$$\therefore x = -\frac{5}{2}$$

$\therefore$  the  $x$ -intercept is  $-\frac{5}{2}$ .

- iii**  $y = \frac{6x + 15}{(3x - 4)(x + 2)}$  has sign diagram



- iv** As  $x \rightarrow -2^-$ ,  $y \rightarrow \infty$

As  $x \rightarrow -2^+$ ,  $y \rightarrow -\infty$

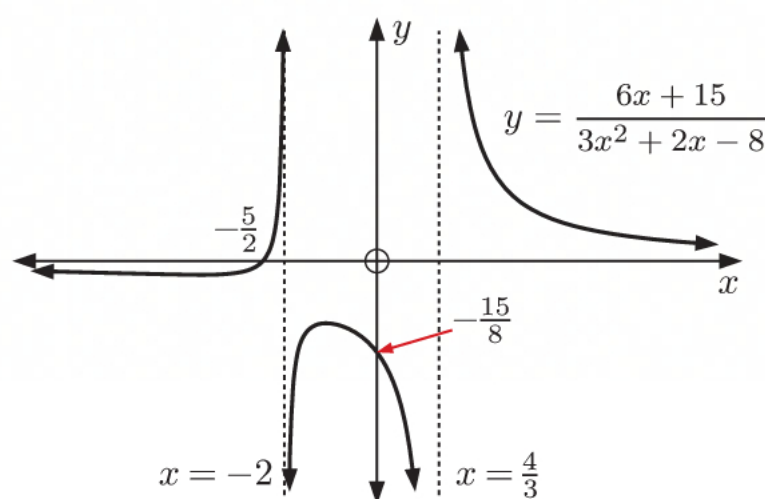
As  $x \rightarrow \frac{4}{3}^-$ ,  $y \rightarrow -\infty$

As  $x \rightarrow \frac{4}{3}^+$ ,  $y \rightarrow \infty$

As  $x \rightarrow -\infty$ ,  $y \rightarrow 0^-$

As  $x \rightarrow \infty$ ,  $y \rightarrow 0^+$

**v**



- k i** The horizontal asymptote is  $y = 0$ .

The function is undefined when  $21 + 8x - 4x^2 = 0$

$$\therefore 4x^2 - 8x - 21 = 0$$

$$\therefore (2x + 3)(2x - 7) = 0$$

$$\therefore x = -\frac{3}{2} \text{ or } \frac{7}{2}$$

$\therefore$  the vertical asymptotes are  $x = -\frac{3}{2}$  and  $x = \frac{7}{2}$ .

- ii**  $f(0) = \frac{7}{21} = \frac{1}{3}$ , so the  $y$ -intercept is  $\frac{1}{3}$ .

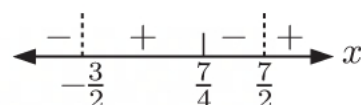
$f(x) = 0$  when  $7 - 4x = 0$

$$\therefore 4x = 7$$

$$\therefore x = \frac{7}{4}$$

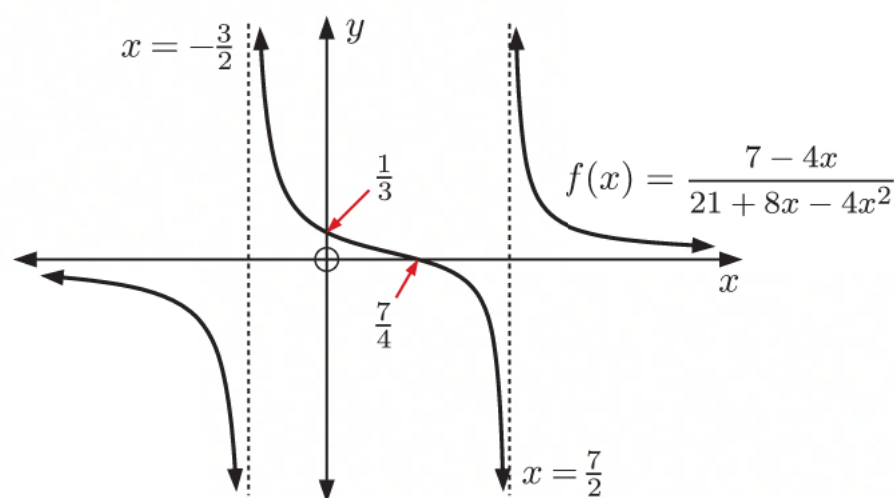
$\therefore$  the  $x$ -intercept is  $\frac{7}{4}$ .

- iii**  $f(x) = \frac{7 - 4x}{-(2x + 3)(2x - 7)}$  has sign diagram





- iv** As  $x \rightarrow -\frac{3}{2}^-$ ,  $f(x) \rightarrow -\infty$   
 As  $x \rightarrow -\frac{3}{2}^+$ ,  $f(x) \rightarrow \infty$   
 As  $x \rightarrow \frac{7}{2}^-$ ,  $f(x) \rightarrow -\infty$   
 As  $x \rightarrow \frac{7}{2}^+$ ,  $f(x) \rightarrow \infty$   
 As  $x \rightarrow -\infty$ ,  $f(x) \rightarrow 0^-$   
 As  $x \rightarrow \infty$ ,  $f(x) \rightarrow 0^+$



- i** **i** The horizontal asymptote is  $y = 0$ .

The function is undefined when  $9x^2 - 12x + 4 = 0$   
 $\therefore (3x - 2)^2 = 0$   
 $\therefore x = \frac{2}{3}$

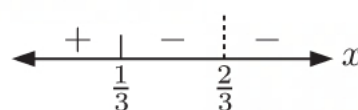
$\therefore$  the vertical asymptote is  $x = \frac{2}{3}$ .

- ii** When  $x = 0$ ,  $y = \frac{1}{4}$ , so the  $y$ -intercept is  $\frac{1}{4}$ .

When  $y = 0$ ,  $1 - 3x = 0$   
 $\therefore 3x = 1$   
 $\therefore x = \frac{1}{3}$

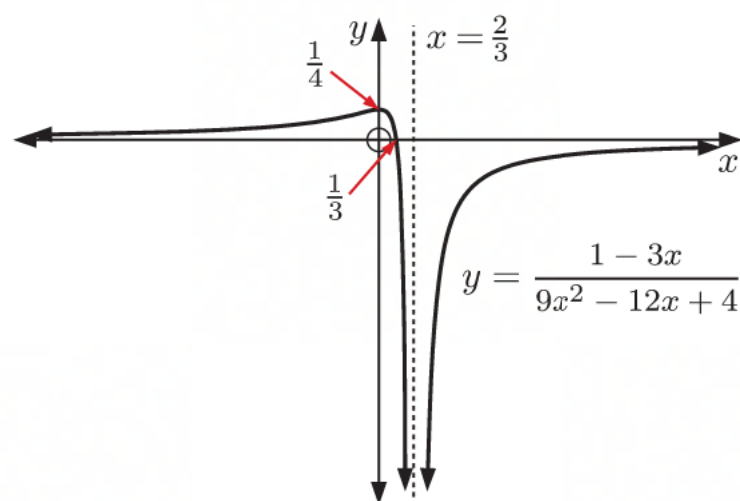
$\therefore$  the  $x$ -intercept is  $\frac{1}{3}$ .

- iii**  $y = \frac{1-3x}{(3x-2)^2}$  has sign diagram



- iv** As  $x \rightarrow \frac{2}{3}^-$ ,  $y \rightarrow -\infty$   
 As  $x \rightarrow \frac{2}{3}^+$ ,  $y \rightarrow -\infty$   
 As  $x \rightarrow -\infty$ ,  $y \rightarrow 0^+$   
 As  $x \rightarrow \infty$ ,  $y \rightarrow 0^-$

**v**



**3**  $y = \frac{4x-6}{x^2-5x+7}$

- a** The horizontal asymptote has equation  $y = 0$ .

- b** The function is undefined when  $x^2 - 5x + 7 = 0$ .

Now  $\Delta = (-5)^2 - 4(1)(7)$   
 $= 25 - 28$   
 $= -3 < 0$

$\therefore x^2 - 5x + 7 = 0$  has no real solutions.

$\therefore$  the function is defined for all  $x$ .

$\therefore$  the function has no vertical asymptotes.

- c** When  $x = 0$ ,  $y = \frac{-6}{7}$ , so the  $y$ -intercept is  $-\frac{6}{7}$ .

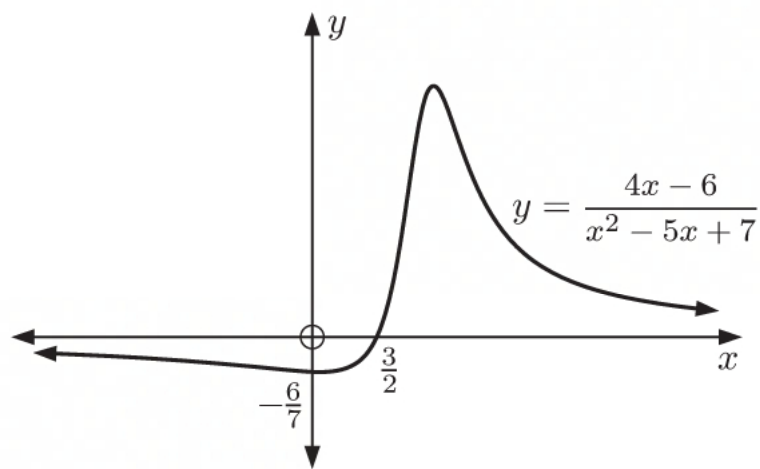
$$\text{When } y = 0, \quad 4x - 6 = 0$$

$$\therefore 4x = 6$$

$$\therefore x = \frac{3}{2}$$

$\therefore$  the  $x$ -intercept is  $\frac{3}{2}$ .

**d**



**4**  $f(x) = \frac{8x - 3}{cx^2 + dx + 9}$

- a**  $f(x)$  has one vertical asymptote, so  $cx^2 + dx + 9$  has exactly one zero.

$$\text{Thus, } \Delta = 0$$

$$\therefore d^2 - 4(c)(9) = 0$$

$$\therefore d^2 - 36c = 0 \quad \dots (1)$$

$$\text{Also, } f(1) = 5$$

$$\therefore \frac{8(1) - 3}{c(1)^2 + d(1) + 9} = 5$$

$$\therefore \frac{5}{c + d + 9} = 5$$

$$\therefore 5 = 5(c + d + 9)$$

$$\therefore 1 = c + d + 9$$

$$\therefore c = -d - 8 \quad \dots (2)$$

$$\text{Substituting (2) into (1), } d^2 - 36(-d - 8) = 0$$

$$\therefore d^2 + 36d + 288 = 0$$

$$\therefore (d + 12)(d + 24) = 0$$

$$\therefore d = -12 \text{ or } -24$$

$$\text{If } d = -12: \quad c = -(-12) - 8 \\ = 4$$

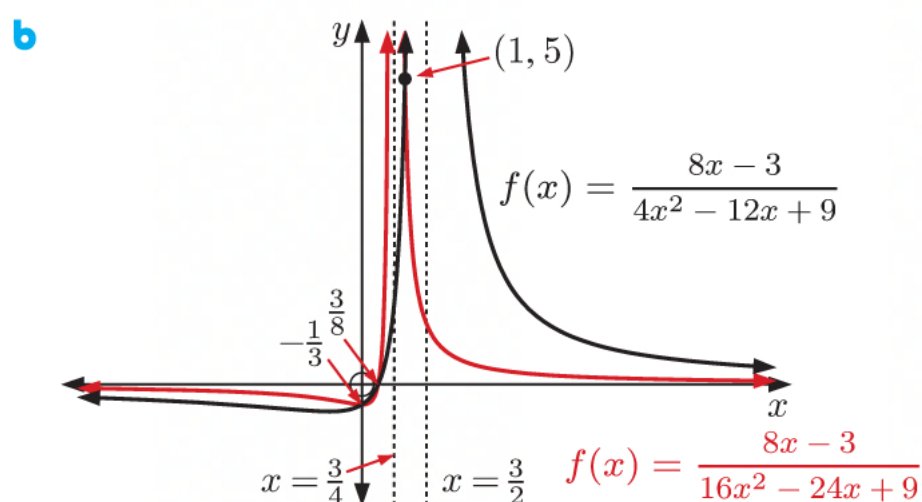
$$\therefore f(x) = \frac{8x - 3}{4x^2 - 12x + 9} \\ = \frac{8x - 3}{(2x - 3)^2}$$

which has vertical asymptote  $x = \frac{3}{2}$

$$\text{If } d = -24: \quad c = -(-24) - 8 \\ = 16$$

$$\therefore f(x) = \frac{8x - 3}{16x^2 - 24x + 9} \\ = \frac{8x - 3}{(4x - 3)^2}$$

which has vertical asymptote  $x = \frac{3}{4}$

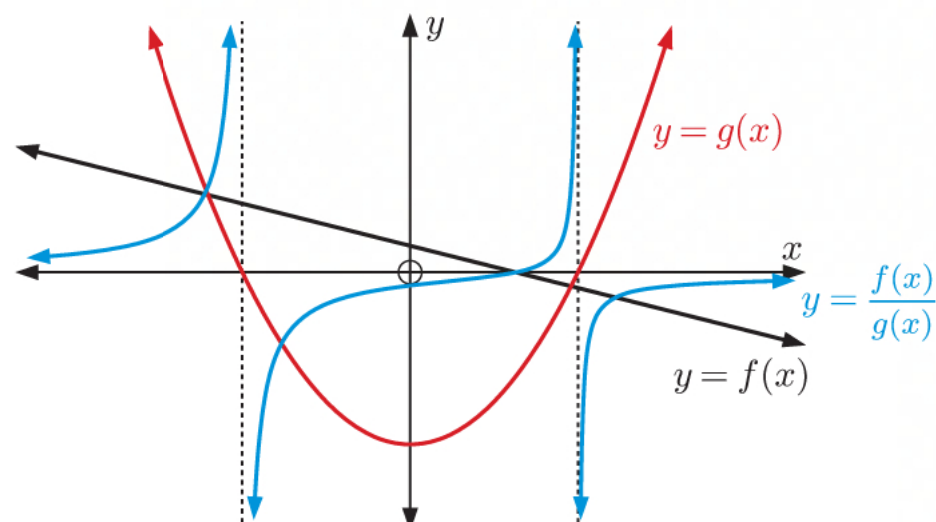


**5**  $y = \frac{f(x)}{g(x)}$  is a function of the form

$$y = \frac{\text{linear}}{\text{quadratic}} = \frac{ax+b}{cx^2+dx+e}.$$

So,  $y = \frac{f(x)}{g(x)}$  has horizontal asymptote  $y = 0$ .

The  $x$ -intercepts of  $y = g(x)$  become the vertical asymptotes of  $y = \frac{f(x)}{g(x)}$ .



### INVESTIGATION 3

### RATIONAL FUNCTIONS OF THE FORM

$$y = \frac{ax^2 + bx + c}{dx + e}, \quad d \neq 0$$

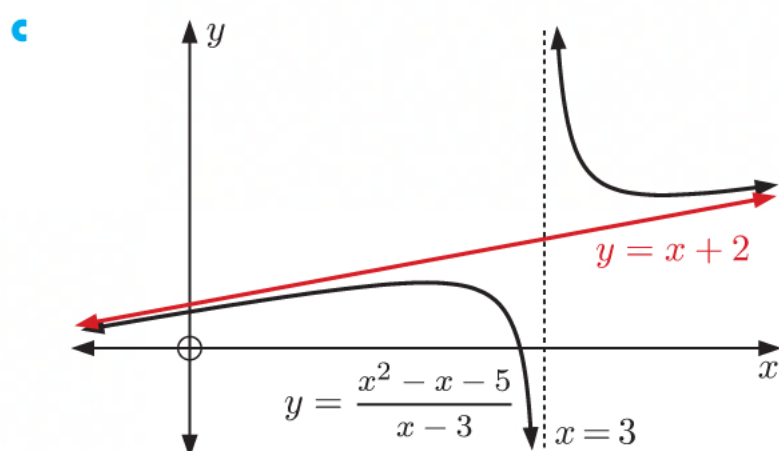
**1 a**  $y = \frac{x^2 - x - 5}{x - 3}$  is undefined when  $x = 3$ .

$\therefore$  the vertical asymptote is  $x = 3$ .

**b**

$$\frac{x^2 - x - 5}{x - 3} = x + 2 + \frac{1}{x - 3}$$

$$x - 3 \overline{) \begin{array}{r} x^2 - x - 5 \\ -(x^2 - 3x) \phantom{- 5} \\ \hline 2x - 5 \\ -(2x - 6) \\ \hline 1 \end{array}}$$



**d**  $y = \frac{x^2 - x - 5}{x - 3}$

$$= x + 2 + \frac{1}{x - 3}$$

As  $|x| \rightarrow \infty$ ,  $\frac{1}{x - 3} \rightarrow 0$

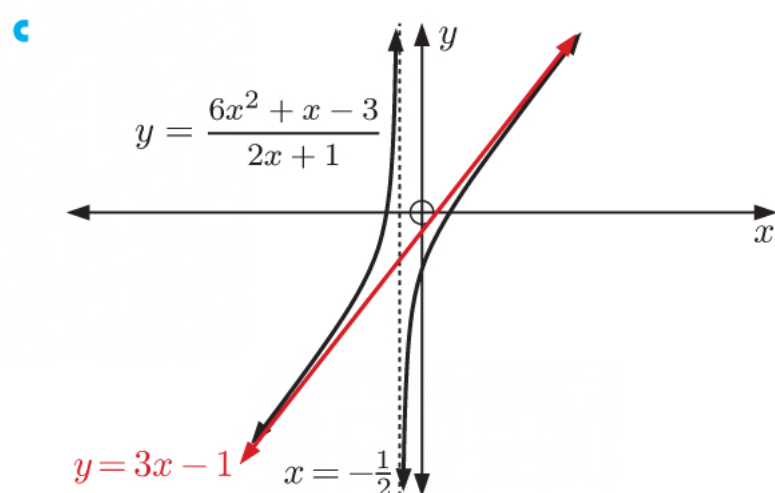
$\therefore$  as  $|x| \rightarrow \infty$ ,  $y \rightarrow x + 2$

- 2 a**  $y = \frac{6x^2 + x - 3}{2x + 1}$  is undefined when  $x = -\frac{1}{2}$ .  
 $\therefore$  the vertical asymptote is  $x = -\frac{1}{2}$ .

**b**

$$\frac{6x^2 + x - 3}{2x + 1} = 3x - 1 - \frac{2}{2x + 1}$$

$$2x + 1 \overline{) \begin{array}{r} 6x^2 + x - 3 \\ -(6x^2 + 3x) \phantom{-3} \\ \hline -2x - 3 \\ -(-2x - 1) \\ \hline -2 \end{array}}$$



**d**  $y = \frac{6x^2 + x - 3}{2x + 1}$   
 $= 3x - 1 - \frac{2}{2x + 1}$

As  $|x| \rightarrow \infty$ ,  $\frac{2}{2x + 1} \rightarrow 0$   
 $\therefore$  as  $|x| \rightarrow \infty$ ,  $y \rightarrow 3x - 1$

So, the graph of  $y = \frac{6x^2 + x - 3}{2x + 1}$  approaches the line  $y = 3x - 1$  as  $|x| \rightarrow \infty$ .

## EXERCISE 6D.2

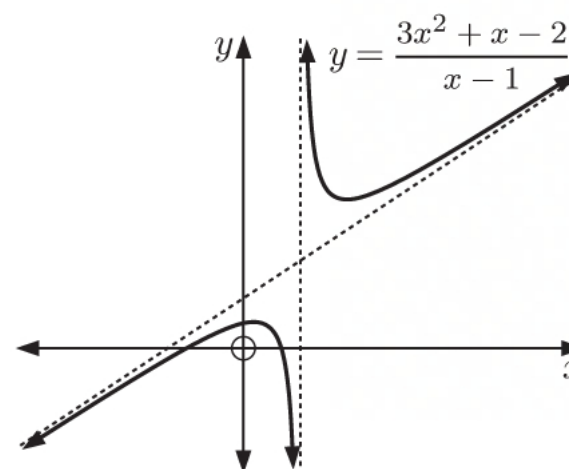
- 1 a**  $y = \frac{3x^2 + x - 2}{x - 1}$  is undefined when  $x - 1 = 0$   
 $\therefore x = 1$   
 $\therefore$  the vertical asymptote is  $x = 1$ .
- b** When  $x = 0$ ,  $y = \frac{-2}{-1} = 2$ , so the  $y$ -intercept is 2.  
 When  $y = 0$ ,  $3x^2 + x - 2 = 0$   
 $\therefore (3x - 2)(x + 1) = 0$   
 $\therefore x = \frac{2}{3}$  or  $-1$   
 $\therefore$  the  $x$ -intercepts are  $\frac{2}{3}$  and  $-1$ .

**c**

$$\frac{3x^2 + x - 2}{x - 1} = 3x + 4 + \frac{2}{x - 1}$$

$$x - 1 \overline{) \begin{array}{r} 3x^2 + x - 2 \\ -(3x^2 - 3x) \phantom{-2} \\ \hline 4x - 2 \\ -(4x - 4) \\ \hline 2 \end{array}}$$

$\therefore$  the oblique asymptote is  $y = 3x + 4$ .





**2**  $y = \frac{x^2 + x - 6}{x - 1}$

**a i** The vertical asymptote is  $x = 1$ .

**ii** When  $x = 0$ ,  $y = \frac{-6}{-1} = 6$ , so the  $y$ -intercept is 6.

When  $y = 0$ ,  $x^2 + x - 6 = 0$

$\therefore (x + 3)(x - 2) = 0$

$\therefore x = -3 \text{ or } 2$

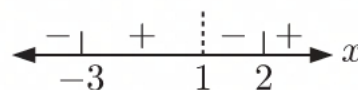
$\therefore$  the  $x$ -intercepts are  $-3$  and  $2$ .

**iii**  $y = \frac{x^2 + x - 6}{x - 1}$   
 $= x + 2 - \frac{4}{x - 1}$

$$x - 1 \overline{) \begin{array}{r} x^2 + x - 6 \\ -(x^2 - x) \phantom{- 6} \\ \hline 2x - 6 \\ -(2x - 2) \\ \hline -4 \end{array}}$$

$\therefore$  the oblique asymptote is  $y = x + 2$ .

**iv**  $y = \frac{(x + 3)(x - 2)}{x - 1}$  has sign diagram

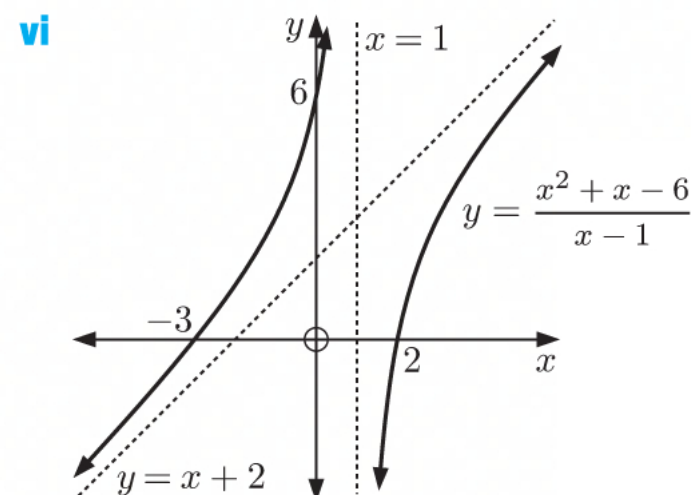


**v** As  $x \rightarrow 1^-$ ,  $y \rightarrow \infty$

As  $x \rightarrow 1^+$ ,  $y \rightarrow -\infty$

As  $x \rightarrow -\infty$ ,  $y \rightarrow x + 2^+$

As  $x \rightarrow \infty$ ,  $y \rightarrow x + 2^-$



**b**  $f(x) = \frac{-x^2 + 3x - 2}{x}$

**i** The vertical asymptote is  $x = 0$ .

**ii**  $f(0)$  is undefined, so there is no  $y$ -intercept.

$f(x) = 0$  when  $-x^2 + 3x - 2 = 0$

$\therefore x^2 - 3x + 2 = 0$

$\therefore (x - 2)(x - 1) = 0$

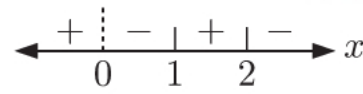
$\therefore x = 2 \text{ or } 1$

$\therefore$  the  $x$ -intercepts are  $2$  and  $1$ .

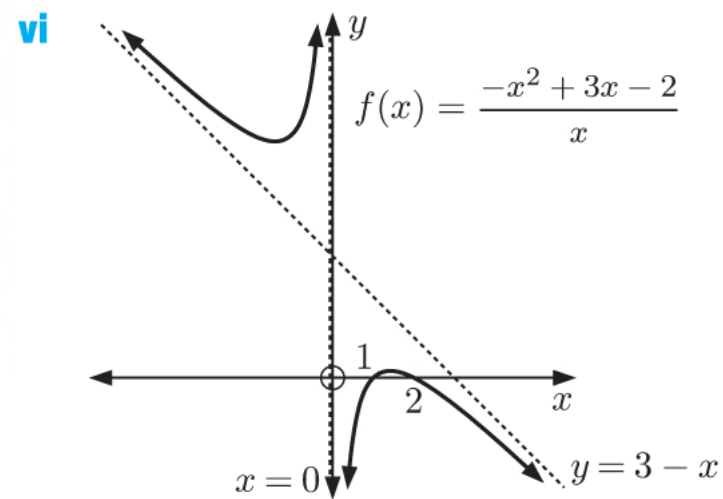
**iii**  $f(x) = \frac{-x^2 + 3x - 2}{x}$   
 $= -x + 3 - \frac{2}{x}$

$\therefore$  the oblique asymptote is  $y = 3 - x$ .

iv  $f(x) = \frac{-(x-2)(x-1)}{x}$  has sign diagram



- v As  $x \rightarrow 0^-$ ,  $f(x) \rightarrow \infty$   
 As  $x \rightarrow 0^+$ ,  $f(x) \rightarrow -\infty$   
 As  $x \rightarrow -\infty$ ,  $f(x) \rightarrow 3 - x^+$   
 As  $x \rightarrow \infty$ ,  $f(x) \rightarrow 3 - x^-$



c  $y = \frac{2x^2 - 8x + 8}{x - 3}$

i The vertical asymptote is  $x = 3$ .

ii When  $x = 0$ ,  $y = \frac{8}{-3}$ , so the  $y$ -intercept is  $-\frac{8}{3}$ .

$$\begin{aligned} \text{When } y = 0, \quad 2x^2 - 8x + 8 &= 0 \\ \therefore x^2 - 4x + 4 &= 0 \\ \therefore (x - 2)^2 &= 0 \\ \therefore x &= 2 \end{aligned}$$

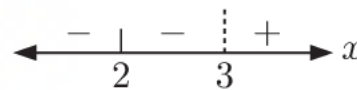
$\therefore$  the  $x$ -intercept is 2.

iii  $y = \frac{2x^2 - 8x + 8}{x - 3}$   
 $= 2x - 2 + \frac{2}{x - 3}$

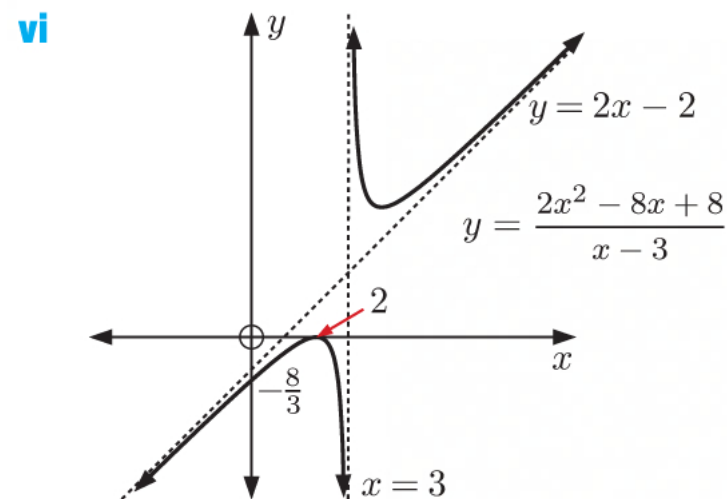
$$\begin{array}{r} 2x - 2 \\ x - 3 \overline{) 2x^2 - 8x + 8} \\ \underline{-(2x^2 - 6x)} \phantom{+ 8} \downarrow \\ -2x + 8 \\ \underline{-(-2x + 6)} \\ 2 \end{array}$$

$\therefore$  the oblique asymptote is  $y = 2x - 2$ .

iv  $y = \frac{2(x-2)^2}{x-3}$  has sign diagram



- v As  $x \rightarrow 3^-$ ,  $y \rightarrow -\infty$   
 As  $x \rightarrow 3^+$ ,  $y \rightarrow \infty$   
 As  $x \rightarrow -\infty$ ,  $y \rightarrow 2x - 2^-$   
 As  $x \rightarrow \infty$ ,  $y \rightarrow 2x - 2^+$



**d**  $f(x) = \frac{-6x^2 + 5x}{2x + 1}$

**i** The vertical asymptote is  $x = -\frac{1}{2}$ .

**ii**  $f(0) = \frac{0}{1} = 0$ , so the  $y$ -intercept is 0.

$$\begin{aligned} f(x) = 0 \text{ when } -6x^2 + 5x &= 0 \\ \therefore x(5 - 6x) &= 0 \\ \therefore x = 0 \text{ or } \frac{5}{6} \end{aligned}$$

$\therefore$  the  $x$ -intercepts are 0 and  $\frac{5}{6}$ .

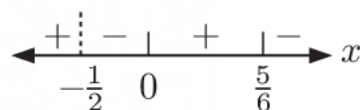
**iii**  $f(x) = \frac{-6x^2 + 5x}{2x + 1}$

$$= -3x + 4 - \frac{4}{2x + 1}$$

$$\begin{array}{r} 2x + 1 \overline{) \begin{array}{r} -6x^2 + 5x + 0 \\ -(-6x^2 - 3x) \quad \downarrow \\ \hline 8x + 0 \\ -(8x + 4) \\ \hline -4 \end{array}} \end{array}$$

$\therefore$  the oblique asymptote is  $y = 4 - 3x$ .

**iv**  $f(x) = \frac{x(5 - 6x)}{2x + 1}$  has sign diagram

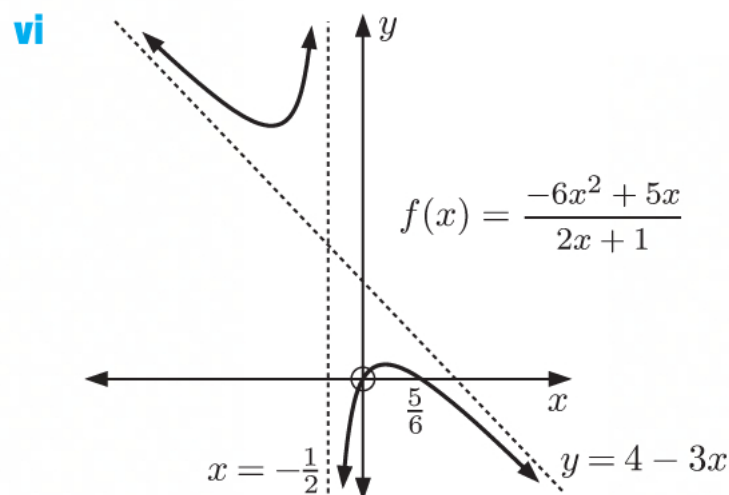


**v** As  $x \rightarrow -\frac{1}{2}^-$ ,  $f(x) \rightarrow \infty$

As  $x \rightarrow -\frac{1}{2}^+$ ,  $f(x) \rightarrow -\infty$

As  $x \rightarrow -\infty$ ,  $f(x) \rightarrow 4 - 3x^+$

As  $x \rightarrow \infty$ ,  $f(x) \rightarrow 4 - 3x^-$



**e**  $y = \frac{-6x^2 - 4x - 1}{3x + 2}$

**i** The vertical asymptote is  $x = -\frac{2}{3}$ .

**ii** When  $x = 0$ ,  $y = \frac{-1}{2}$ , so the  $y$ -intercept is  $-\frac{1}{2}$ .

$$\text{When } y = 0, \quad -6x^2 - 4x - 1 = 0$$

$$\therefore 6x^2 + 4x + 1 = 0$$

$$\text{which has } \Delta = (4)^2 - 4(6)(1)$$

$$= 16 - 24$$

$$= -8 < 0$$

$\therefore$  there are no  $x$ -intercepts.

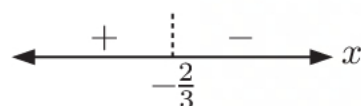
$$\text{iii } y = \frac{-6x^2 - 4x - 1}{3x + 2}$$

$$= -2x - \frac{1}{3x + 2}$$

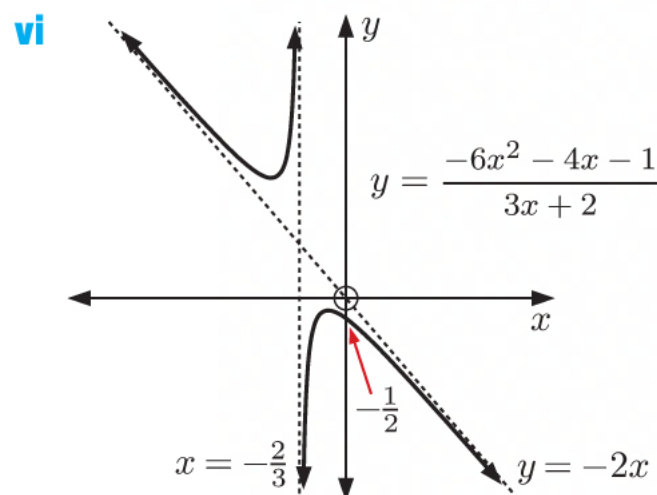
$$3x + 2 \overline{) \begin{array}{r} -6x^2 - 4x - 1 \\ -(-6x^2 - 4x) \quad \downarrow \\ \hline 0x - 1 \\ -(0x + 0) \\ \hline -1 \end{array}}$$

$\therefore$  the oblique asymptote is  $y = -2x$ .

iv  $y = \frac{-6x^2 - 4x - 1}{3x + 2}$  has sign diagram



v As  $x \rightarrow -\frac{2}{3}^-$ ,  $y \rightarrow \infty$   
 As  $x \rightarrow -\frac{2}{3}^+$ ,  $y \rightarrow -\infty$   
 As  $x \rightarrow -\infty$ ,  $y \rightarrow -2x^+$   
 As  $x \rightarrow \infty$ ,  $y \rightarrow -2x^-$



f  $f(x) = \frac{8x^2 - 19x - 15}{1 - 2x}$

i The vertical asymptote is  $x = \frac{1}{2}$ .

ii  $f(0) = \frac{-15}{1}$ , so the  $y$ -intercept is  $-15$ .

$$f(x) = 0 \text{ when } 8x^2 - 19x - 15 = 0$$

$$\therefore (8x + 5)(x - 3) = 0$$

$$\therefore x = -\frac{5}{8} \text{ or } 3$$

$\therefore$  the  $x$ -intercepts are  $-\frac{5}{8}$  and  $3$ .

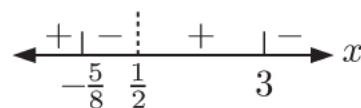
$$\text{iii } f(x) = \frac{8x^2 - 19x - 15}{1 - 2x}$$

$$= -4x + \frac{15}{2} - \frac{45}{2(1 - 2x)}$$

$$-2x + 1 \overline{) \begin{array}{r} 8x^2 - 19x - 15 \\ -(8x^2 - 4x) \quad \downarrow \\ \hline -15x - 15 \\ -(-15x + \frac{15}{2}) \\ \hline -\frac{45}{2} \end{array}}$$

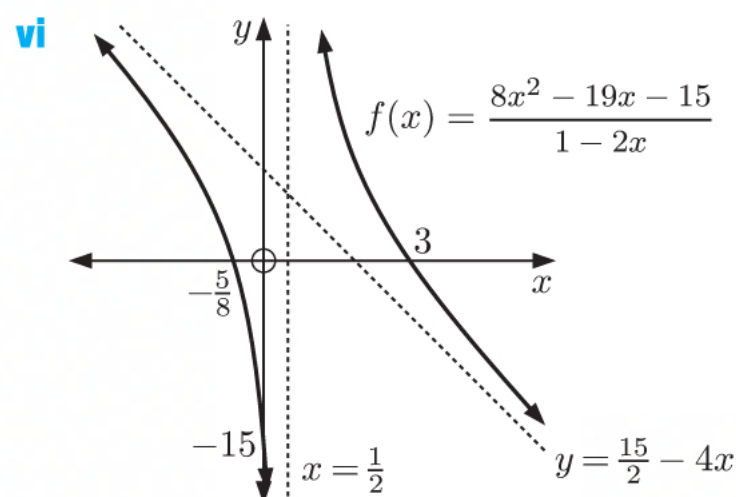
$\therefore$  the oblique asymptote is  $y = \frac{15}{2} - 4x$ .

iv  $f(x) = \frac{(8x + 5)(x - 3)}{(1 - 2x)}$  has sign diagram



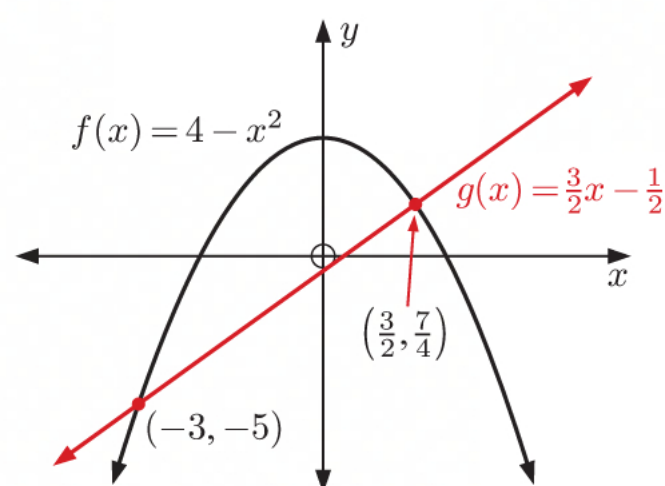


v As  $x \rightarrow \frac{1}{2}^-$ ,  $f(x) \rightarrow -\infty$   
 As  $x \rightarrow \frac{1}{2}^+$ ,  $f(x) \rightarrow \infty$   
 As  $x \rightarrow -\infty$ ,  $f(x) \rightarrow \frac{15}{2} - 4x^-$   
 As  $x \rightarrow \infty$ ,  $f(x) \rightarrow \frac{15}{2} - 4x^+$



3 a  $y = \frac{f(x)}{g(x)} = \frac{4 - x^2}{\frac{3}{2}x - \frac{1}{2}}$

i The vertical asymptote is  $\frac{3}{2}x - \frac{1}{2} = 0$   
 $\therefore \frac{3}{2}x = \frac{1}{2}$   
 $\therefore x = \frac{1}{3}$



ii  $y = \frac{4 - x^2}{\frac{3}{2}x - \frac{1}{2}}$   
 $= \frac{8 - 2x^2}{3x - 1}$   
 $= -\frac{2}{3}x - \frac{2}{9} + \frac{70}{9(3x - 1)}$

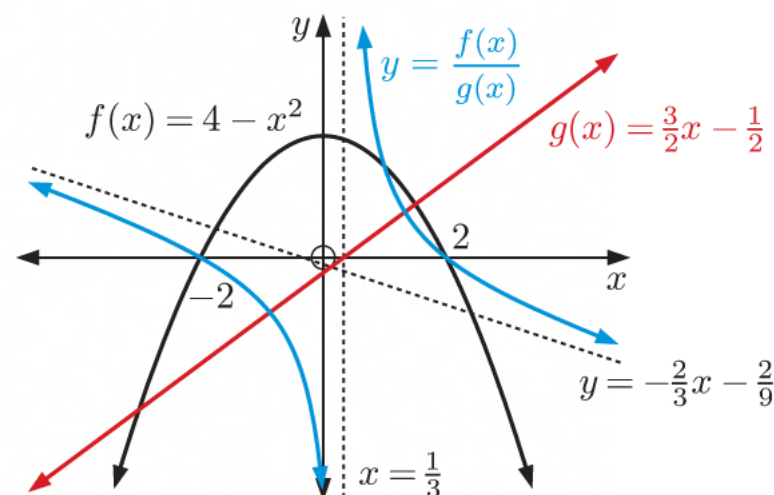
$$\begin{array}{r}
 -\frac{2}{3}x - \frac{2}{9} \\
 3x - 1 \overline{) \begin{array}{l} -2x^2 + 0x + 8 \\ -(-2x^2 + \frac{2}{3}x) \\ \hline -\frac{2}{3}x + 8 \\ -(-\frac{2}{3}x + \frac{2}{9}) \\ \hline \frac{70}{9} \end{array}}
 \end{array}$$

$\therefore$  the oblique asymptote is  $y = -\frac{2}{3}x - \frac{2}{9}$ .

b  $y = \frac{f(x)}{g(x)} = 0$  when  $f(x) = 0$   
 $\therefore 4 - x^2 = 0$   
 $\therefore x^2 = 4$   
 $\therefore x = \pm 2$

$\therefore$  the  $x$ -intercepts are  $-2$  and  $2$ .

The vertical asymptote is  $x = \frac{1}{3}$  and  
the oblique asymptote is  $y = -\frac{2}{3}x - \frac{2}{9}$ .



$$4 \quad y = \frac{ax^2 + bx + c}{dx + e}$$

$$= \frac{a}{d}x + \frac{1}{d}\left(b - \frac{ae}{d}\right) + \frac{c - \frac{e}{d}\left(b - \frac{ae}{d}\right)}{dx + e}$$

$$= \frac{a}{d}x + \frac{b}{d} - \frac{ae}{d^2} + \frac{c - \frac{be}{d} + \frac{ae^2}{d^2}}{dx + e}$$

$$\begin{array}{r} \frac{a}{d}x + \frac{1}{d}\left(b - \frac{ae}{d}\right) \\ dx + e \left| \begin{array}{r} ax^2 + bx + c \\ - \left(ax^2 + \frac{ae}{d}x\right) \quad \downarrow \\ \hline \left(b - \frac{ae}{d}\right)x + c \\ - \left(\left(b - \frac{ae}{d}\right)x + \frac{e}{d}\left(b - \frac{ae}{d}\right)\right) \\ \hline c - \frac{e}{d}\left(b - \frac{ae}{d}\right) \end{array} \right. \end{array}$$

$\therefore$  the oblique asymptote is  $y = \frac{a}{d}x + \frac{b}{d} - \frac{ae}{d^2}$ .

## EXERCISE 6E

1 a  $x^2 + 5x + 4 = (x + 4)(x + 1)$

Let  $\frac{3}{x^2 + 5x + 4} = \frac{A}{x + 4} + \frac{B}{x + 1}$

$$\therefore 3 = A(x + 1) + B(x + 4)$$

Substituting  $x = -1$ ,  $3 = B(-1 + 4)$

$$\therefore 3B = 3$$

$$\therefore B = 1$$

Substituting  $x = -4$ ,  $3 = A(-4 + 1)$

$$\therefore -3A = 3$$

$$\therefore A = -1$$

$$\therefore \frac{3}{x^2 + 5x + 4} = \frac{-1}{x + 4} + \frac{1}{x + 1} \quad \checkmark$$

b  $x^2 + 3x - 10 = (x + 5)(x - 2)$

Let  $\frac{9 - x}{x^2 + 3x - 10} = \frac{A}{x + 5} + \frac{B}{x - 2}$

$$\therefore 9 - x = A(x - 2) + B(x + 5)$$

Substituting  $x = 2$ ,  $9 - 2 = B(2 + 5)$

$$\therefore 7B = 7$$

$$\therefore B = 1$$

Substituting  $x = -5$ ,  $9 - (-5) = A(-5 - 2)$

$$\therefore -7A = 14$$

$$\therefore A = -2$$

$$\therefore \frac{9 - x}{x^2 + 3x - 10} = \frac{-2}{x + 5} + \frac{1}{x - 2} \quad \checkmark$$

$$\text{c } 2x^2 - 5x - 12 = (2x + 3)(x - 4)$$

$$\text{Let } \frac{3x - 23}{2x^2 - 5x - 12} = \frac{A}{2x + 3} + \frac{B}{x - 4}$$

$$\therefore 3x - 23 = A(x - 4) + B(2x + 3)$$

$$\text{Substituting } x = 4, \quad 3(4) - 23 = B(2(4) + 3)$$

$$\therefore 11B = -11$$

$$\therefore B = -1$$

$$\text{Substituting } x = -\frac{3}{2}, \quad 3(-\frac{3}{2}) - 23 = A(-\frac{3}{2} - 4)$$

$$\therefore -\frac{11}{2}A = -\frac{55}{2}$$

$$\therefore A = 5$$

$$\therefore \frac{3x - 23}{2x^2 - 5x - 12} = \frac{5}{2x + 3} - \frac{1}{x - 4} \quad \checkmark$$

$$\text{2 a } x^2 + 3x + 2 = (x + 1)(x + 2)$$

$$\text{Let } \frac{1}{x^2 + 3x + 2} = \frac{A}{x + 1} + \frac{B}{x + 2}$$

$$\therefore 1 = A(x + 2) + B(x + 1)$$

$$\text{Substituting } x = -2, \quad 1 = B(-2 + 1)$$

$$\therefore -B = 1$$

$$\therefore B = -1$$

$$\text{Substituting } x = -1, \quad 1 = A(-1 + 2)$$

$$\therefore A = 1$$

$$\therefore \frac{1}{x^2 + 3x + 2} = \frac{1}{x + 1} - \frac{1}{x + 2}$$

$$\text{c } x^2 - 2x - 8 = (x + 2)(x - 4)$$

$$\text{Let } \frac{3x}{x^2 - 2x - 8} = \frac{A}{x + 2} + \frac{B}{x - 4}$$

$$\therefore 3x = A(x - 4) + B(x + 2)$$

$$\text{Substituting } x = 4, \quad 3(4) = B(4 + 2)$$

$$\therefore 6B = 12$$

$$\therefore B = 2$$

$$\text{Substituting } x = -2,$$

$$3(-2) = A(-2 - 4)$$

$$\therefore -6A = -6$$

$$\therefore A = 1$$

$$\therefore \frac{3x}{x^2 - 2x - 8} = \frac{1}{x + 2} + \frac{2}{x - 4}$$

$$\text{b } x^2 - 5x + 6 = (x - 2)(x - 3)$$

$$\text{Let } \frac{x}{x^2 - 5x + 6} = \frac{A}{x - 2} + \frac{B}{x - 3}$$

$$\therefore x = A(x - 3) + B(x - 2)$$

$$\text{Substituting } x = 3, \quad 3 = B(3 - 2)$$

$$\therefore B = 3$$

$$\text{Substituting } x = 2, \quad 2 = A(2 - 3)$$

$$\therefore -A = 2$$

$$\therefore A = -2$$

$$\therefore \frac{x}{x^2 - 5x + 6} = \frac{-2}{x - 2} + \frac{3}{x - 3}$$

$$\text{d } x^2 - 4 = (x + 2)(x - 2)$$

$$\text{Let } \frac{3x + 2}{x^2 - 4} = \frac{A}{x + 2} + \frac{B}{x - 2}$$

$$\therefore 3x + 2 = A(x - 2) + B(x + 2)$$

$$\text{Substituting } x = 2, \quad 3(2) + 2 = B(2 + 2)$$

$$\therefore 4B = 8$$

$$\therefore B = 2$$

$$\text{Substituting } x = -2,$$

$$3(-2) + 2 = A(-2 - 2)$$

$$\therefore -4A = -4$$

$$\therefore A = 1$$

$$\therefore \frac{3x + 2}{x^2 - 4} = \frac{1}{x + 2} + \frac{2}{x - 2}$$

$$\text{e } x^2 - 2x - 3 = (x + 1)(x - 3)$$

$$\text{Let } \frac{2x - 1}{x^2 - 2x - 3} = \frac{A}{x + 1} + \frac{B}{x - 3}$$

$$\therefore 2x - 1 = A(x - 3) + B(x + 1)$$

$$\text{Substituting } x = 3, \quad 2(3) - 1 = B(3 + 1)$$

$$\therefore 4B = 5$$

$$\therefore B = \frac{5}{4}$$

$$\text{Substituting } x = -1,$$

$$2(-1) - 1 = A(-1 - 3)$$

$$\therefore -4A = -3$$

$$\therefore A = \frac{3}{4}$$

$$\begin{aligned} \therefore \frac{2x - 1}{x^2 - 2x - 3} &= \frac{\frac{3}{4}}{x + 1} + \frac{\frac{5}{4}}{x - 3} \\ &= \frac{3}{4(x + 1)} + \frac{5}{4(x - 3)} \end{aligned}$$

$$\text{g } 2x^2 + 5x + 3 = (2x + 3)(x + 1)$$

$$\text{Let } \frac{7x + 10}{2x^2 + 5x + 3} = \frac{A}{2x + 3} + \frac{B}{x + 1}$$

$$\therefore 7x + 10 = A(x + 1) + B(2x + 3)$$

$$\text{Substituting } x = -1,$$

$$7(-1) + 10 = B(2(-1) + 3)$$

$$\therefore B = 3$$

$$\text{Substituting } x = -\frac{3}{2},$$

$$7(-\frac{3}{2}) + 10 = A(-\frac{3}{2} + 1)$$

$$\therefore -\frac{1}{2}A = -\frac{1}{2}$$

$$\therefore A = 1$$

$$\therefore \frac{7x + 10}{2x^2 + 5x + 3} = \frac{1}{2x + 3} + \frac{3}{x + 1}$$

$$\text{f } x^2 + 4x - 12 = (x - 2)(x + 6)$$

$$\text{Let } \frac{2 - 5x}{x^2 + 4x - 12} = \frac{A}{x - 2} + \frac{B}{x + 6}$$

$$\therefore 2 - 5x = A(x + 6) + B(x - 2)$$

$$\text{Substituting } x = -6,$$

$$2 - 5(-6) = B(-6 - 2)$$

$$\therefore -8B = 32$$

$$\therefore B = -4$$

$$\text{Substituting } x = 2, \quad 2 - 5(2) = A(2 + 6)$$

$$\therefore 8A = -8$$

$$\therefore A = -1$$

$$\therefore \frac{2 - 5x}{x^2 + 4x - 12} = \frac{-1}{x - 2} - \frac{4}{x + 6}$$

$$\text{h } 3x^2 - 5x - 2 = (3x + 1)(x - 2)$$

$$\text{Let } \frac{x + 9}{3x^2 - 5x - 2} = \frac{A}{3x + 1} + \frac{B}{x - 2}$$

$$\therefore x + 9 = A(x - 2) + B(3x + 1)$$

$$\text{Substituting } x = 2, \quad 2 + 9 = B(3(2) + 1)$$

$$\therefore 7B = 11$$

$$\therefore B = \frac{11}{7}$$

$$\text{Substituting } x = -\frac{1}{3},$$

$$-\frac{1}{3} + 9 = A(-\frac{1}{3} - 2)$$

$$\therefore -\frac{7}{3}A = \frac{26}{3}$$

$$\therefore A = \frac{-26}{7}$$

$$\begin{aligned} \therefore \frac{x + 9}{3x^2 - 5x - 2} &= \frac{-\frac{26}{7}}{3x + 1} + \frac{\frac{11}{7}}{x - 2} \\ &= \frac{-26}{7(3x + 1)} + \frac{11}{7(x - 2)} \end{aligned}$$



$$\text{i } 8x^2 + 18x - 5 = (4x - 1)(2x + 5)$$

$$\text{Let } \frac{2x + 27}{8x^2 + 18x - 5} = \frac{A}{4x - 1} + \frac{B}{2x + 5}$$

$$\therefore 2x + 27 = A(2x + 5) + B(4x - 1)$$

$$\text{Substituting } x = -\frac{5}{2}, \quad 2(-\frac{5}{2}) + 27 = B(4(-\frac{5}{2}) - 1)$$

$$\therefore -11B = 22$$

$$\therefore B = -2$$

$$\text{Substituting } x = \frac{1}{4}, \quad 2(\frac{1}{4}) + 27 = A(2(\frac{1}{4}) + 5)$$

$$\therefore \frac{11}{2}A = \frac{55}{2}$$

$$\therefore A = 5$$

$$\therefore \frac{2x + 27}{8x^2 + 18x - 5} = \frac{5}{4x - 1} - \frac{2}{2x + 5}$$

$$\text{3 Let } \frac{5x^2 + 11x + 16}{(x - 5)(x + 2)^2} = \frac{A}{x - 5} + \frac{B}{x + 2} + \frac{C}{(x + 2)^2}$$

$$\therefore 5x^2 + 11x + 16 = A(x + 2)^2 + B(x - 5)(x + 2) + C(x - 5)$$

$$\text{Substituting } x = 5, \quad 5(5)^2 + 11(5) + 16 = A(5 + 2)^2$$

$$\therefore 196 = 49A$$

$$\therefore A = 4$$

$$\text{Substituting } x = -2, \quad 5(-2)^2 + 11(-2) + 16 = C(-2 - 5)$$

$$\therefore 14 = -7C$$

$$\therefore C = -2$$

$$\text{Substituting } x = 0, \quad 5(0)^2 + 11(0) + 16 = 4(2)^2 + B(-5)(2) - 2(-5)$$

$$\therefore 16 = 16 - 10B + 10$$

$$\therefore 10B = 10$$

$$\therefore B = 1$$

$$\therefore \frac{5x^2 + 11x + 16}{(x - 5)(x + 2)^2} = \frac{4}{x - 5} + \frac{1}{x + 2} - \frac{2}{(x + 2)^2} \quad \checkmark$$

$$\text{4 Let } \frac{-3x - 4}{(x + 2)(x + 3)^2} = \frac{A}{x + 2} + \frac{B}{x + 3} + \frac{C}{(x + 3)^2}$$

$$\therefore -3x - 4 = A(x + 3)^2 + B(x + 2)(x + 3) + C(x + 2)$$

$$\text{Substituting } x = -2, \quad -3(-2) - 4 = A(-2 + 3)^2$$

$$\therefore A = 2$$

$$\text{Substituting } x = -3, \quad -3(-3) - 4 = C(-3 + 2)$$

$$\therefore 5 = -C$$

$$\therefore C = -5$$

$$\text{Substituting } x = 0, \quad -3(0) - 4 = 2(3)^2 + B(2)(3) - 5(2)$$

$$\therefore -4 = 18 + 6B - 10$$

$$\therefore 6B = -12$$

$$\therefore B = -2$$

$$\therefore \frac{-3x - 4}{(x + 2)(x + 3)^2} = \frac{2}{x + 2} - \frac{2}{x + 3} - \frac{5}{(x + 3)^2}$$

$$\begin{aligned}
 \mathbf{5} \quad \mathbf{a} \quad (x+3)(x-1)^2 &= (x+3)(x^2-2x+1) \\
 &= x^3-2x^2+x \\
 &\quad + 3x^2-6x+3 \\
 &= x^3+x^2-5x+3 \quad \checkmark
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad \text{Let } \frac{19x+9}{x^3+x^2-5x+3} &= \frac{19x+9}{(x+3)(x-1)^2} = \frac{A}{x+3} + \frac{B}{x-1} + \frac{C}{(x-1)^2} \\
 \therefore 19x+9 &= A(x-1)^2 + B(x+3)(x-1) + C(x+3)
 \end{aligned}$$

$$\begin{aligned}
 \text{Substituting } x = -3, \quad 19(-3)+9 &= A(-3-1)^2 \\
 \therefore -48 &= 16A \\
 \therefore A &= -3
 \end{aligned}$$

$$\begin{aligned}
 \text{Substituting } x = 1, \quad 19(1)+9 &= C(1+3) \\
 \therefore 28 &= 4C \\
 \therefore C &= 7
 \end{aligned}$$

$$\begin{aligned}
 \text{Substituting } x = 0, \quad 19(0)+9 &= -3(-1)^2 + B(3)(-1) + 7(3) \\
 \therefore 9 &= -3 - 3B + 21 \\
 \therefore 3B &= 9 \\
 \therefore B &= 3
 \end{aligned}$$

$$\therefore \frac{19x+9}{x^3+x^2-5x+3} = \frac{-3}{x+3} + \frac{3}{x-1} + \frac{7}{(x-1)^2}$$

$$\begin{aligned}
 \mathbf{6} \quad \mathbf{a} \quad (2x-1)(x^2+x+4) &= 2x^3+2x^2+8x \\
 &\quad - x^2-x-4 \\
 &= 2x^3+x^2+7x-4 \quad \checkmark
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad \text{Let } \frac{10x^2-3x+18}{2x^3+x^2+7x-4} &= \frac{10x^2-3x+18}{(2x-1)(x^2+x+4)} = \frac{A}{2x-1} + \frac{Bx+C}{x^2+x+4} \\
 \therefore 10x^2-3x+18 &= A(x^2+x+4) + (Bx+C)(2x-1)
 \end{aligned}$$

$$\begin{aligned}
 \text{Substituting } x = \frac{1}{2}, \quad 10\left(\frac{1}{2}\right)^2 - 3\left(\frac{1}{2}\right) + 18 &= A\left(\left(\frac{1}{2}\right)^2 + \frac{1}{2} + 4\right) \\
 \therefore 19 &= \frac{19}{4}A \\
 \therefore A &= 4
 \end{aligned}$$

$$\begin{aligned}
 \text{Substituting } x = 0, \quad 18 &= 4(4) + C(-1) \\
 \therefore 2 &= -C \\
 \therefore C &= -2
 \end{aligned}$$

$$\begin{aligned}
 \text{Substituting } x = 1, \quad 10(1)^2 - 3(1) + 18 &= 4(1^2 + 1 + 4) + (B-2)(2(1)-1) \\
 \therefore 25 &= 24 + B - 2 \\
 \therefore B &= 3
 \end{aligned}$$

$$\therefore \frac{10x^2-3x+18}{2x^3+x^2+7x-4} = \frac{4}{2x-1} + \frac{3x-2}{x^2+x+4}$$

## REVIEW SET 6A

$$\begin{aligned}
 1 \quad a \quad f(x) &= -\frac{4}{x} \\
 \therefore f(-x) &= -\frac{4}{(-x)} \\
 &= \frac{4}{x} \\
 &= -f(x) \\
 \therefore f(x) &\text{ is odd.}
 \end{aligned}$$

$$\begin{aligned}
 c \quad f(x) &= \sqrt{x^2 - 5} \\
 \therefore f(-x) &= \sqrt{(-x)^2 - 5} \\
 &= \sqrt{x^2 - 5} \\
 &= f(x) \\
 \therefore f(x) &\text{ is even.}
 \end{aligned}$$

$$\begin{aligned}
 b \quad f(x) &= \frac{2x - 3}{x + 1} \\
 \therefore f(-x) &= \frac{2(-x) - 3}{-x + 1} \\
 &= \frac{-2x - 3}{-x + 1}
 \end{aligned}$$

which is neither  $f(x)$  nor  $-f(x)$ .  
 $\therefore f(x)$  is neither even nor odd.

$$\begin{aligned}
 2 \quad a \quad \text{If } f(x) = \sin(x - k) \text{ is even, then } f(-x) &= f(x) \\
 \therefore \sin(-x - k) &= \sin(x - k) \\
 \therefore \sin(-(x + k)) &= \sin(x - k) \\
 \therefore -\sin(x + k) &= \sin(x - k) \quad \{\sin(-x) = -\sin x\} \\
 \therefore -(\sin x \cos k + \sin k \cos x) &= \sin x \cos k - \sin k \cos x \\
 \therefore -\sin x \cos k - \sin k \cos x &= \sin x \cos k - \sin k \cos x \\
 \therefore 2 \sin x \cos k &= 0 \\
 \therefore \cos k &= 0 \\
 \therefore k &= \frac{\pi}{2} + n\pi, \quad n \in \mathbb{Z}
 \end{aligned}$$

$$\begin{aligned}
 b \quad \text{If } f(x) = \sin(x - k) \text{ is odd, then } f(-x) &= -f(x) \\
 \therefore \sin(-x - k) &= -\sin(x - k) \\
 \therefore \sin(-(x + k)) &= -\sin(x - k) \\
 \therefore -\sin(x + k) &= -\sin(x - k) \quad \{\sin(-x) = -\sin x\} \\
 \therefore \sin(x + k) &= \sin(x - k) \\
 \therefore \sin x \cos k + \sin k \cos x &= \sin x \cos k - \sin k \cos x \\
 \therefore 2 \sin k \cos x &= 0 \\
 \therefore \sin k &= 0 \\
 \therefore k &= n\pi, \quad n \in \mathbb{Z}
 \end{aligned}$$

$$c \quad f(x) = \sin(x - k) \text{ is neither even nor odd if } k \neq \frac{n\pi}{2}, \quad n \in \mathbb{Z}.$$

$$\begin{aligned}
 3 \quad (f \circ f)(-x) &= f(f(-x)) \\
 &= f(-f(x)) \quad \{f(x) \text{ is odd}\} \\
 &= -f(f(x)) \quad \{f(x) \text{ is odd}\} \\
 &= -(f \circ f)(x) \\
 \therefore (f \circ f)(x) &\text{ is odd.}
 \end{aligned}$$

- 4  $f(x) = (x^2 - 3x)(x + b)$ ,  $b \in \mathbb{R}$  is an odd function.

$$\therefore f(-x) = -f(x)$$

$$\therefore ((-x)^2 - 3(-x))(-x + b) = -(x^2 - 3x)(x + b)$$

$$\therefore (x^2 + 3x)(-x + b) = -(x^2 - 3x)(x + b)$$

$$\therefore -x^3 + bx^2 - 3x^2 + 3bx = -(x^3 + bx^2 - 3x^2 - 3bx)$$

$$\therefore \cancel{-x^3} + bx^2 - 3x^2 + \cancel{3bx} = \cancel{-x^3} - bx^2 + 3x^2 + \cancel{3bx}$$

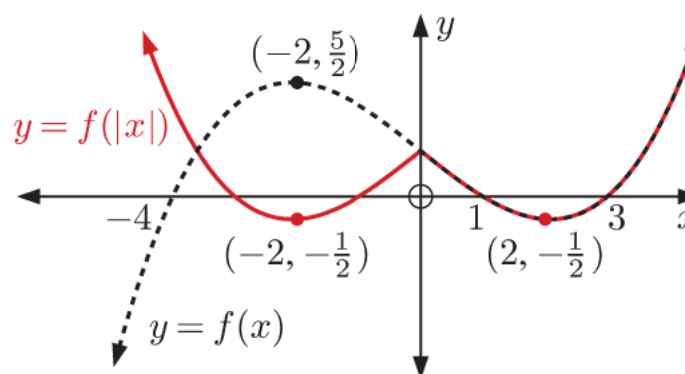
$$\therefore 2bx^2 = 6x^2 \quad \{\text{equating coefficients of } x^2\}$$

$$\therefore 2b = 6$$

$$\therefore b = 3$$

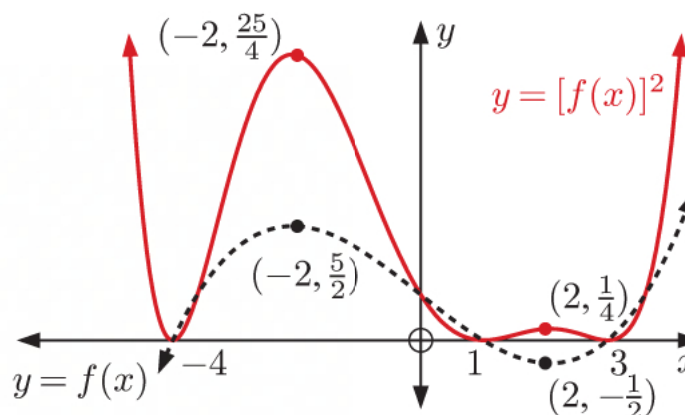
- 5 a  $y = f(|x|) = \begin{cases} f(x) & \text{if } x \geq 0 \\ f(-x) & \text{if } x < 0 \end{cases}$

The graph is unchanged for  $x \geq 0$  and reflected in the  $y$ -axis for  $x < 0$ .



- b  $y = f(x)$  cuts the  $x$ -axis at  $-4$ ,  $1$ , and  $3$ , so  $y = [f(x)]^2$  touches the  $x$ -axis at  $-4$ ,  $1$ , and  $3$ .  $y = f(x)$  and  $y = [f(x)]^2$  also intersect when  $y = 1$ .

$y = f(x)$  has turning points  $(-2, \frac{5}{2})$  and  $(2, -\frac{1}{2})$ , so  $y = [f(x)]^2$  has turning points  $(-2, \frac{25}{4})$  and  $(2, \frac{1}{4})$ .



- 6  $f(x)$  has domain  $\{x \mid 0 \leq x < 5\}$  and range  $\{y \mid -5 \leq y \leq 2\}$ .

- a  $f(x)$  on the domain  $0 \leq x < 5$  is reflected in the  $y$ -axis.

$\therefore f(|x|)$  has domain  $\{x \mid -5 < x < 5\}$ .

- b The smallest value of  $|f(x)|$  is  $|0| = 0$  and the largest value is  $|-5| = 5$ .

$\therefore |f(x)|$  has range  $\{y \mid 0 \leq y \leq 5\}$ .

- 7  $f(x)$  is odd, so  $f(-x) = -f(x)$  for all  $x$ .

- a  $f(2(-x)) = f(-(2x))$   
 $= -f(2x)$

$\therefore f(2x)$  is odd.

- c  $f(-x) + 2 = -f(x) + 2$

which is neither  $f(x)$  nor  $-f(x)$ .

$\therefore f(x) + 2$  is neither even nor odd.

- e  $|f(-x)| = |-f(x)|$   
 $= |-1||f(x)|$   
 $= |f(x)|$

$\therefore |f(x)|$  is even.

- b  $f((-x)^2) = f(x^2)$

$\therefore f(x^2)$  is even.

- d  $[f(-x)]^2 = [-f(x)]^2$   
 $= [f(x)]^2$

$\therefore [f(x)]^2$  is even.

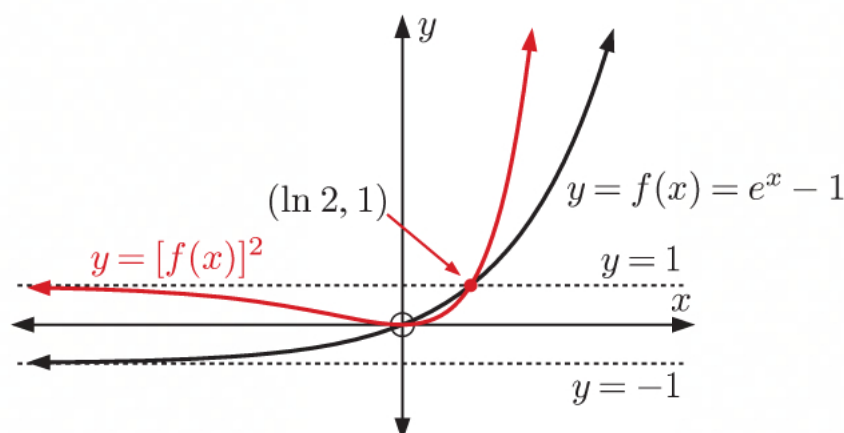
- f  $\frac{1}{f(-x)} = \frac{1}{-f(x)}$   
 $= -\frac{1}{f(x)}$

$\therefore \frac{1}{f(x)}$  is odd.



**8**  $f(x) = e^x - 1$

**a**



**b** The invariant points when  $y = f(x)$  is transformed to  $y = [f(x)]^2$  are the points on  $f(x) = e^x - 1$  with  $y$ -coordinate 0 or 1.

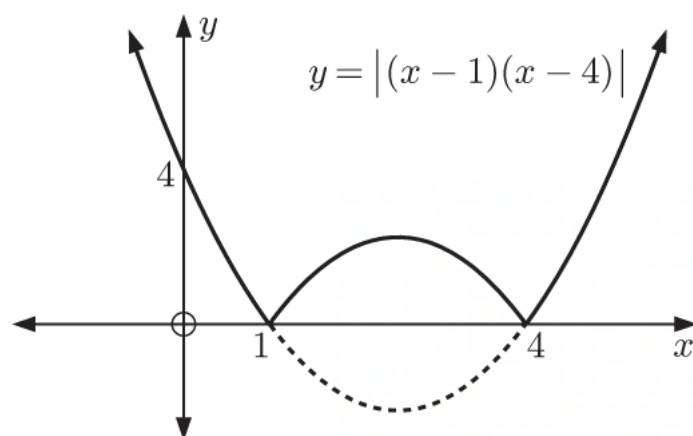
$$\begin{array}{ll} \text{When } y = 0, & e^x - 1 = 0 \\ & \therefore e^x = 1 \\ & \therefore x = 0 \end{array} \quad \begin{array}{ll} \text{and when } y = 1, & e^x - 1 = 1 \\ & \therefore e^x = 2 \\ & \therefore x = \ln 2 \end{array}$$

$\therefore$  the invariant points are  $(0, 0)$  and  $(\ln 2, 1)$ .

**9 a** Let  $f(x) = (x - 1)(x - 4)$

$$y = |f(x)| = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ -f(x) & \text{if } f(x) < 0 \end{cases}$$

The graph is unchanged for  $f(x) \geq 0$  and reflected in the  $x$ -axis for  $f(x) < 0$ .

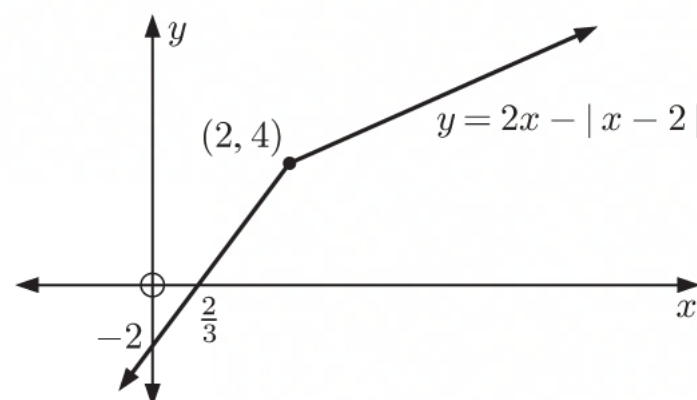


**b**  $y = 2x - |x - 2|$

If  $x \geq 2$ ,  $y = 2x - (x - 2) = x + 2$

If  $x < 2$ ,  $y = 2x + (x - 2) = 3x - 2$

$$\therefore y = \begin{cases} x + 2, & x \geq 2 \\ 3x - 2, & x < 2 \end{cases}$$



**10 a** If  $\left| \frac{2x+1}{x-2} \right| = 3$ , then  $\frac{2x+1}{x-2} = \pm 3$ .

$$\therefore \frac{2x+1}{x-2} = 3 \quad \text{or} \quad \frac{2x+1}{x-2} = -3$$

$$\therefore 2x+1 = 3(x-2) \quad \text{or} \quad 2x+1 = -3(x-2)$$

$$\therefore 2x+1 = 3x-6 \quad \text{or} \quad 2x+1 = -3x+6$$

$$\therefore x = 7 \quad \text{or} \quad 5x = 5$$

$$\therefore x = 7 \quad \text{or} \quad \therefore x = 1$$

So,  $x = 7$  or  $1$ .

**b** If  $|3x + 5| = |x - 7|$ , then  $3x + 5 = \pm(x - 7)$ .

$$\therefore 3x + 5 = x - 7 \quad \text{or} \quad 3x + 5 = -(x - 7)$$

$$\therefore 2x = -12 \quad \text{or} \quad 3x + 5 = -x + 7$$

$$\therefore x = -6 \quad \text{or} \quad 4x = 2$$

$$\therefore x = -6 \quad \text{or} \quad x = \frac{1}{2}$$

So,  $x = -6$  or  $\frac{1}{2}$ .

**11 a**  $|x + 3| \leq 6$

$$\therefore -6 \leq x + 3 \leq 6$$

$$\therefore -9 \leq x \leq 3$$

**b**  $|3 - 2x| > 1$

$$\therefore 3 - 2x < -1 \quad \text{or} \quad 3 - 2x > 1$$

$$\therefore 2x > 4 \quad \text{or} \quad 2x < 2$$

$$\therefore x > 2 \quad \text{or} \quad x < 1$$

**c**  $|3x + 2| < |x - 5|$

$$\therefore |3x + 2|^2 < |x - 5|^2 \quad \{\text{squaring both sides}\}$$

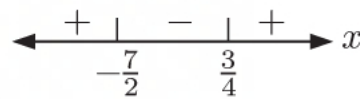
$$\therefore (3x + 2)^2 < (x - 5)^2 \quad \{|a|^2 = a^2\}$$

$$\therefore 9x^2 + 12x + 4 < x^2 - 10x + 25$$

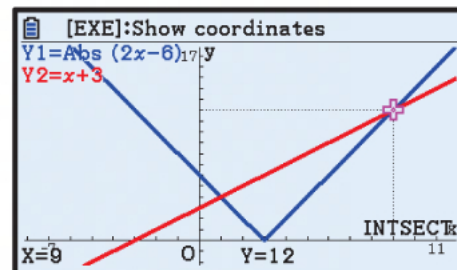
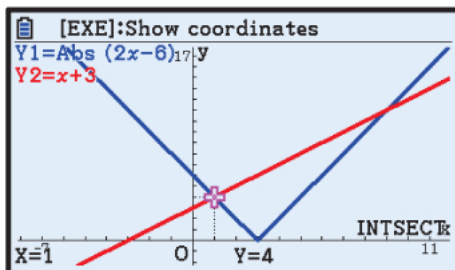
$$\therefore 8x^2 + 22x - 21 < 0$$

$$\therefore (4x - 3)(2x + 7) < 0$$

$$\therefore -\frac{7}{2} < x < \frac{3}{4}$$



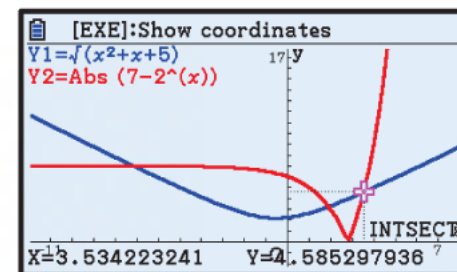
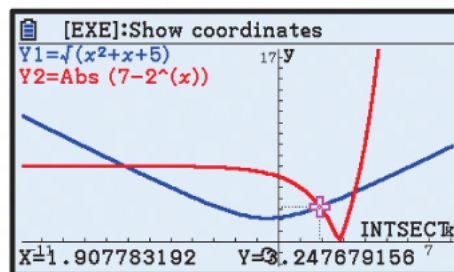
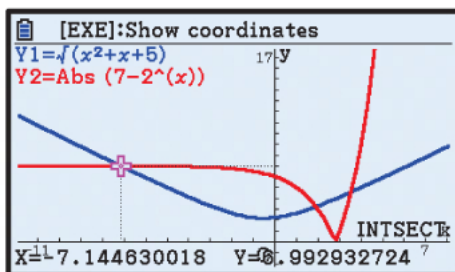
**12 a** We draw graphs of  $y = |2x - 6|$  and  $y = x + 3$  on the same set of axes.



The graphs intersect at  $x = 1$  and  $x = 9$ .

$\therefore |2x - 6| > x + 3$  when  $x < 1$  or  $x > 9$ .

**b** We draw graphs of  $y = \sqrt{x^2 + x + 5}$  and  $y = |7 - 2^x|$  on the same set of axes.



The graphs intersect at  $x \approx -7.14$ ,  $x \approx 1.91$ , and  $x \approx 3.53$ .

$\therefore \sqrt{x^2 + x + 5} \leq |7 - 2^x|$  when  $-7.14 \leq x \leq 1.91$  or  $x \geq 3.53$ .

**13**  $f(x) = \frac{x+1}{x^2-x-6}$

**a** The horizontal asymptote is  $y = 0$ .

The function is undefined when

$$x^2 - x - 6 = 0$$

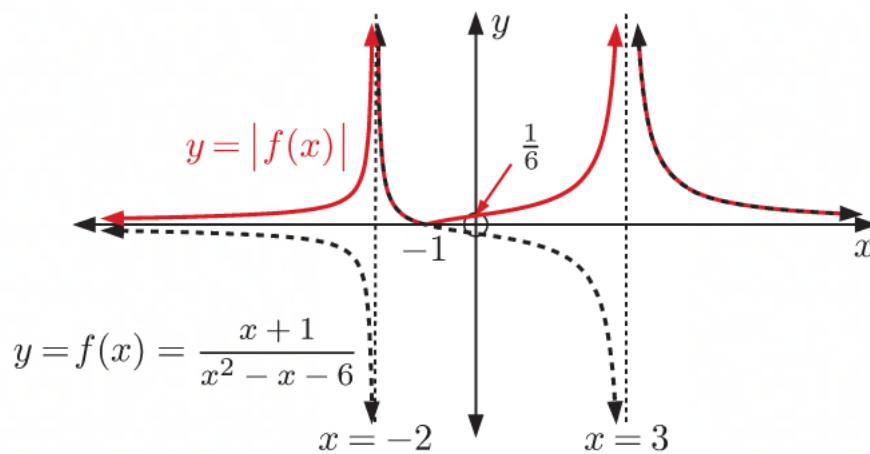
$$\therefore (x+2)(x-3) = 0$$

$$\therefore x = -2 \text{ or } 3$$

$\therefore$  the vertical asymptotes are  $x = -2$   
and  $x = 3$ .

**c**  $y = |f(x)| = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ -f(x) & \text{if } f(x) < 0 \end{cases}$

The graph is unchanged for  $f(x) \geq 0$  and reflected in the  $x$ -axis for  $f(x) < 0$ .



**b**  $f(0) = \frac{1}{-6}$ , so the  $y$ -intercept is  $-\frac{1}{6}$ .

$$f(x) = 0 \text{ when } x + 1 = 0$$

$$\therefore x = -1$$

$\therefore$  the  $x$ -intercept is  $-1$ .

**14**  $f(x) = \frac{2x^2+3x-5}{2x-3}$

**a** The vertical asymptote is  $x = \frac{3}{2}$ .

**b**  $f(0) = \frac{-5}{-3} = \frac{5}{3}$ , so the  $y$ -intercept is  $\frac{5}{3}$ .

$$f(x) = 0 \text{ when } 2x^2 + 3x - 5 = 0$$

$$\therefore (2x+5)(x-1) = 0$$

$$\therefore x = -\frac{5}{2} \text{ or } 1$$

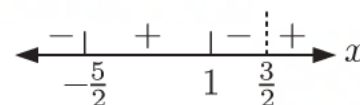
$\therefore$  the  $x$ -intercepts are  $-\frac{5}{2}$  and  $1$ .

**c**  $f(x) = \frac{2x^2+3x-5}{2x-3}$   
 $= x + 3 + \frac{4}{2x-3}$

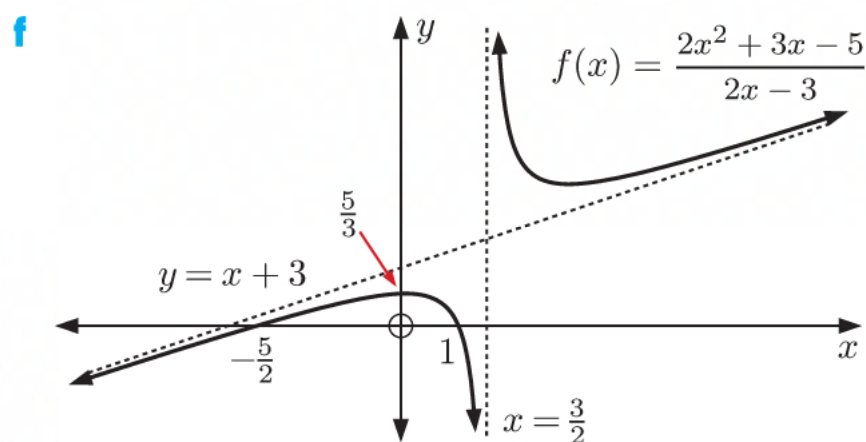
$$\begin{array}{r} 2x-3 \overline{) \begin{array}{r} 2x^2+3x-5 \\ -(2x^2-3x) \phantom{-5} \\ \hline 6x-5 \\ -(6x-9) \\ \hline 4 \end{array}} \\ x+3 \end{array}$$

$\therefore$  the oblique asymptote is  $y = x + 3$ .

**d**  $f(x) = \frac{(2x+5)(x-1)}{(2x-3)}$  has sign diagram



- e** As  $x \rightarrow \frac{3}{2}^-$ ,  $f(x) \rightarrow -\infty$   
 As  $x \rightarrow \frac{3}{2}^+$ ,  $f(x) \rightarrow \infty$   
 As  $x \rightarrow -\infty$ ,  $f(x) \rightarrow x + 3^-$   
 As  $x \rightarrow \infty$ ,  $f(x) \rightarrow x + 3^+$



**15 a**  $x^2 + 4x - 5 = (x + 5)(x - 1)$

Let  $\frac{x+1}{x^2+4x-5} = \frac{A}{x+5} + \frac{B}{x-1}$   
 $\therefore x+1 = A(x-1) + B(x+5)$

Substituting  $x = 1$ ,  $1+1 = B(1+5)$   
 $\therefore 6B = 2$   
 $\therefore B = \frac{1}{3}$

Substituting  $x = -5$ ,  
 $-5+1 = A(-5-1)$   
 $\therefore -6A = -4$   
 $\therefore A = \frac{2}{3}$

$\therefore \frac{x+1}{x^2+4x-5} = \frac{\frac{2}{3}}{x+5} + \frac{\frac{1}{3}}{x-1}$   
 $= \frac{2}{3(x+5)} + \frac{1}{3(x-1)}$

**b**  $2x^2 + 7x + 3 = (2x + 1)(x + 3)$

Let  $\frac{2x-4}{2x^2+7x+3} = \frac{A}{2x+1} + \frac{B}{x+3}$   
 $\therefore 2x-4 = A(x+3) + B(2x+1)$

Substituting  $x = -3$ ,  
 $2(-3) - 4 = B(2(-3) + 1)$   
 $\therefore -5B = -10$   
 $\therefore B = 2$

Substituting  $x = -\frac{1}{2}$ ,  
 $2(-\frac{1}{2}) - 4 = A(-\frac{1}{2} + 3)$   
 $\therefore \frac{5}{2}A = -5$   
 $\therefore A = -2$

$\therefore \frac{2x-4}{2x^2+7x+3} = \frac{-2}{2x+1} + \frac{2}{x+3}$

**16** Let  $\frac{13x+2}{(x+4)(x-1)^2} = \frac{A}{x+4} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$

$\therefore 13x+2 = A(x-1)^2 + B(x+4)(x-1) + C(x+4)$

Substituting  $x = -4$ ,  $13(-4) + 2 = A(-4-1)^2$   
 $\therefore -50 = 25A$   
 $\therefore A = -2$

Substituting  $x = 1$ ,  $13(1) + 2 = C(1+4)$   
 $\therefore 15 = 5C$   
 $\therefore C = 3$

Substituting  $x = 0$ ,  $2 = -2(-1)^2 + B(4)(-1) + 3(4)$   
 $\therefore 2 = -2 - 4B + 12$   
 $\therefore 4B = 8$   
 $\therefore B = 2$

$\therefore \frac{13x+2}{(x+4)(x-1)^2} = \frac{-2}{x+4} + \frac{2}{x-1} + \frac{3}{(x-1)^2}$



## REVIEW SET 6B

- 1  $(-2, 6)$  and  $(1, -4)$  lie on the graph of  $y = f(x)$

$$\therefore f(-2) = 6 \text{ and } f(1) = -4$$

$f(x)$  is even, so  $f(-x) = f(x)$  for all  $x$

$$\therefore f(2) = 6 \text{ and } f(-1) = -4$$

$\therefore (2, 6)$  and  $(-1, -4)$  also lie on the graph of  $y = f(x)$ .

- 2 a Let  $f(x)$  and  $g(x)$  be even functions, and  $h(x) = f(x)g(x)$ .

$$h(-x) = f(-x)g(-x)$$

$$= f(x)g(x)$$

$$= h(x)$$

$\therefore$  the product of two even functions is an even function.

- b Let  $f(x) = x + 3$ ,  $g(x) = x - 3$ , neither of which are odd nor even.

$$f(x) \times g(x) = x^2 - 9 \text{ which is even.}$$

- 3 If  $f(x) = \cos(2(x - a)) + b$  is odd, then  $f(-x) = -f(x)$

$$\therefore \cos(2(-x - a)) + b = -(\cos(2(x - a)) + b)$$

$$\therefore \cos(-2(x + a)) + b = -\cos(2(x - a)) - b$$

$$\therefore \cos(2x + 2a) + b = -\cos(2x - 2a) - b \quad \{\cos(-x) = \cos x\}$$

$$\therefore \cos 2x \cos 2a - \sin 2x \sin 2a + b = -(\cos 2x \cos 2a + \sin 2x \sin 2a) - b$$

$$\therefore \cos 2x \cos 2a - \sin 2x \sin 2a + b = -\cos 2x \cos 2a - \sin 2x \sin 2a - b$$

$$\therefore 2 \cos 2x \cos 2a = -2b$$

$$\therefore \cos 2x \cos 2a = -b$$

For this to be true for all  $x \in \mathbb{R}$ , we require  $\cos 2a = 0$  and  $b = 0$

$$\therefore 2a = \frac{\pi}{2} + n\pi, \quad n \in \mathbb{Z}$$

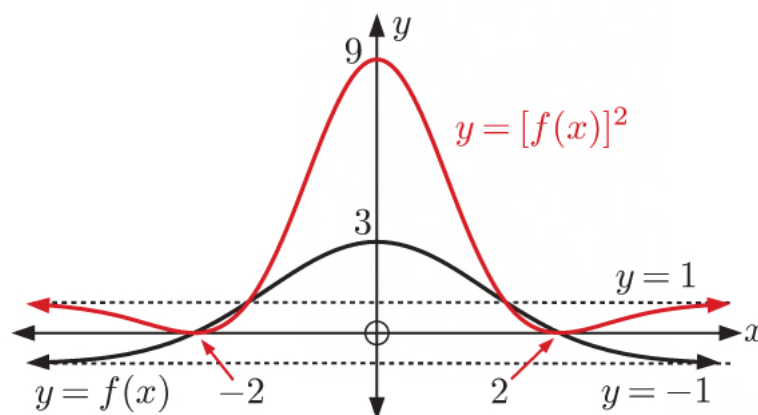
$$\therefore a = \frac{\pi}{4} + \frac{n\pi}{2}, \quad n \in \mathbb{Z}$$

- 4  $y = f(x)$  cuts the  $x$ -axis at  $-2$  and  $2$ , so

$y = [f(x)]^2$  touches the  $x$ -axis at  $-2$  and  $2$ .

$y = f(x)$  and  $y = [f(x)]^2$  also intersect when  $y = 1$ .

The  $y$ -intercept of  $y = [f(x)]^2$  is  $3^2 = 9$ .



- 5  $|-x| = |x|$

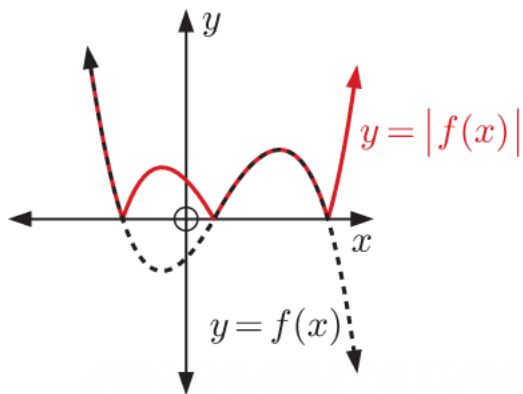
$$\therefore f(|-x|) = f(|x|)$$

So,  $f(|x|)$  is an even function for any function  $f(x)$ .

- 6 a** The  $x$ -intercepts of  $y = |f(x)|$  are the same as those for  $y = f(x)$ .  
 $y = |f(x)|$  will have the same number of turning points as  $y = f(x)$ , except that their  $y$ -coordinates will have positive sign.  
 $\therefore y = |f(x)|$  has 3 turning points, and 2  $x$ -intercepts.
- b** We cannot say anything about the  $x$ -intercepts or turning points of  $y = f(|x|)$ .  
 For example, if some or all of the  $x$ -intercepts and turning points of  $y = f(x)$  lie in the interval  $x < 0$ , this information is lost as we reflect in the  $y$ -axis.
- c** The 2  $x$ -intercepts of  $y = f(x)$  become turning points of  $y = [f(x)]^2$ , and remain the  $x$ -intercepts of  $y = [f(x)]^2$ .  
 If an  $x$ -intercept of  $y = f(x)$  is *also* a turning point, then it does not add an additional turning point to the graph of  $y = [f(x)]^2$ .  
 $\therefore y = [f(x)]^2$  has 3, 4, or 5 turning points, and 2  $x$ -intercepts.

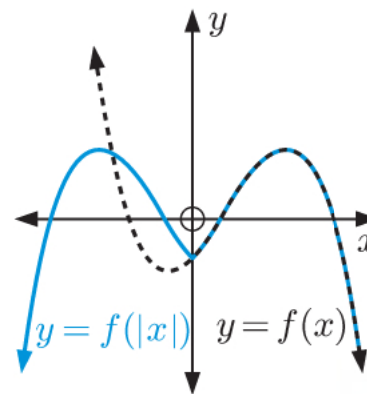
**7 a**  $y = |f(x)| = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ -f(x) & \text{if } f(x) < 0 \end{cases}$

The graph is unchanged for  $f(x) \geq 0$  and reflected in the  $x$ -axis for  $f(x) < 0$ .



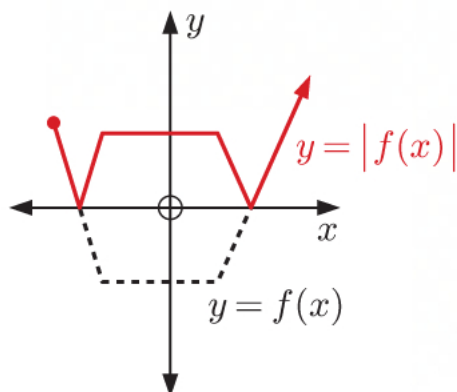
$y = f(|x|) = \begin{cases} f(x) & \text{if } x \geq 0 \\ f(-x) & \text{if } x < 0 \end{cases}$

The graph is unchanged for  $x \geq 0$  and reflected in the  $y$ -axis for  $x < 0$ .



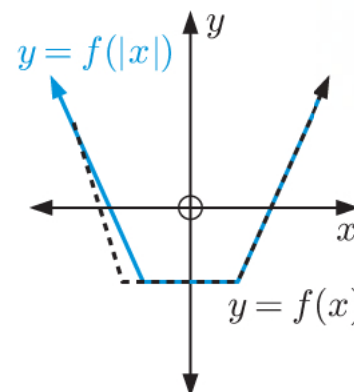
**b**  $y = |f(x)| = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ -f(x) & \text{if } f(x) < 0 \end{cases}$

The graph is unchanged for  $f(x) \geq 0$  and reflected in the  $x$ -axis for  $f(x) < 0$ .



$y = f(|x|) = \begin{cases} f(x) & \text{if } x \geq 0 \\ f(-x) & \text{if } x < 0 \end{cases}$

The graph is unchanged for  $x \geq 0$  and reflected in the  $y$ -axis for  $x < 0$ .



- 8 a**  $|2 - 5x| = 6$   
 $\therefore 2 - 5x = 6$  or  $2 - 5x = -6$   
 $\therefore 5x = -4$  or  $5x = 8$   
 $\therefore x = -\frac{4}{5}$  or  $x = \frac{8}{5}$   
 So,  $x = -\frac{4}{5}$  or  $\frac{8}{5}$ .

**b** If  $|4x - 2| = |x + 7|$ , then  $4x - 2 = \pm(x + 7)$ .

$$\therefore 4x - 2 = x + 7 \quad \text{or} \quad 4x - 2 = -x - 7$$

$$\therefore 3x = 9 \quad \text{or} \quad 5x = -5$$

$$\therefore x = 3 \quad \text{or} \quad x = -1$$

So,  $x = 3$  or  $-1$ .

**9 a**  $|3x + 2| > 3$

$$\therefore 3x + 2 < -3 \quad \text{or} \quad 3x + 2 > 3$$

$$\therefore 3x < -5 \quad \text{or} \quad 3x > 1$$

$$\therefore x < -\frac{5}{3} \quad \text{or} \quad x > \frac{1}{3}$$

**b**

$$\frac{1}{4}|2x + 5| \geq |x - 1|$$

$$\therefore \frac{1}{16}|2x + 5|^2 \geq |x - 1|^2 \quad \{\text{squaring both sides}\}$$

$$\therefore \frac{1}{16}(2x + 5)^2 \geq (x - 1)^2 \quad \{|a|^2 = a^2\}$$

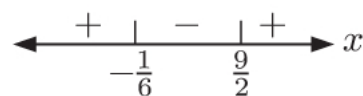
$$\therefore \frac{1}{16}(4x^2 + 20x + 25) \geq x^2 - 2x + 1$$

$$\therefore 4x^2 + 20x + 25 \geq 16x^2 - 32x + 16$$

$$\therefore 12x^2 - 52x - 9 \leq 0$$

$$\therefore (6x + 1)(2x - 9) \leq 0$$

$$\therefore -\frac{1}{6} \leq x \leq \frac{9}{2}$$



**c**

$$\left| \frac{x}{x+4} \right| \leq 2$$

$$\therefore \left| \frac{x}{x+4} \right|^2 \leq 2^2$$

{squaring both sides}

$$\therefore \left( \frac{x}{x+4} \right)^2 \leq 4$$

{|a|^2 = a^2}

$$\therefore \frac{x^2}{(x+4)^2} \leq 4$$

$$\therefore x^2 \leq 4(x+4)^2$$

provided  $x \neq -4$  {since  $(x+4)^2 \geq 0$ }

$$\therefore x^2 \leq 4(x^2 + 8x + 16)$$

provided  $x \neq -4$

$$\therefore x^2 \leq 4x^2 + 32x + 64$$

provided  $x \neq -4$

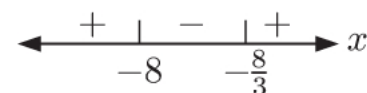
$$\therefore 3x^2 + 32x + 64 \geq 0$$

provided  $x \neq -4$

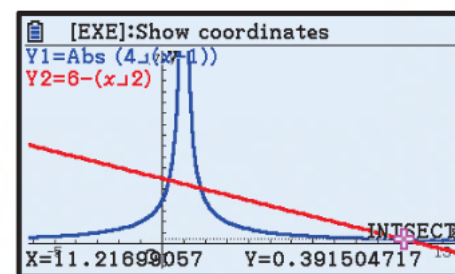
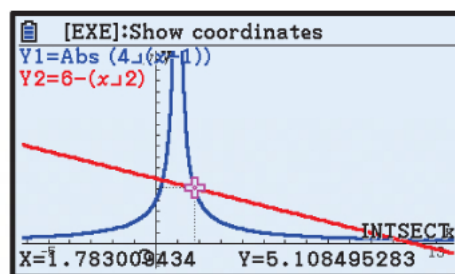
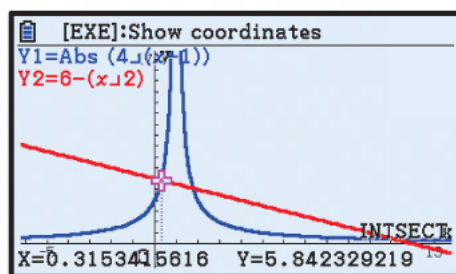
$$\therefore (3x + 8)(x + 8) \geq 0$$

provided  $x \neq -4$

$$\therefore x \leq -8 \quad \text{or} \quad x \geq -\frac{8}{3}$$



**10 a** We draw graphs of  $y = \left| \frac{4}{x-1} \right|$  and  $y = 6 - \frac{x}{2}$  on the same set of axes.

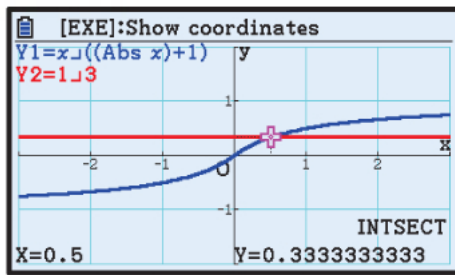


The graphs intersect at about  $(0.315, 5.84)$ ,  $(1.78, 5.11)$ , and  $(11.2, 0.392)$ .

$\therefore$  the solutions are  $x \approx 0.315, 1.78, \text{ or } 11.2$ .



- b** We draw graphs of  $y = \frac{x}{|x| + 1}$  and  $y = \frac{1}{3}$  on the same set of axes.



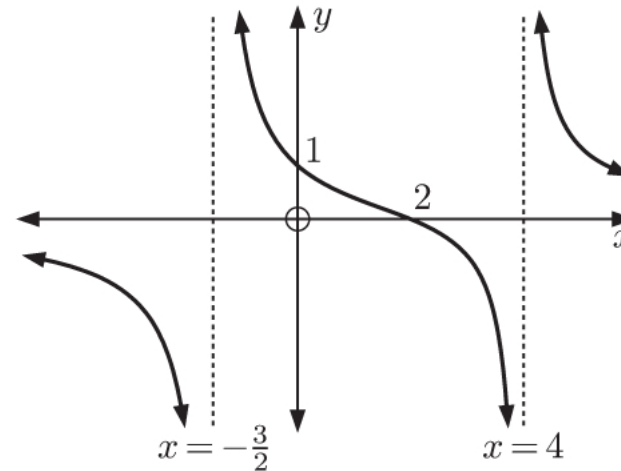
The graphs intersect at  $x = \frac{1}{2}$ .

$$\therefore \frac{x}{|x| + 1} \geq \frac{1}{3} \text{ when } x \geq \frac{1}{2}.$$

- 11**  $y = \frac{ax + b}{2x^2 + dx + e}$  is undefined when  $x = -\frac{3}{2}$  or 4

$$\begin{aligned} \therefore 2x^2 + dx + e &= (2x + 3)(x - 4) \\ &= 2x^2 - 5x - 12 \end{aligned}$$

$$\begin{aligned} \therefore d &= -5, \quad e = -12 \\ &\text{\{equating coefficients of } x, \\ &\text{\text{and constant coefficients}\}} \end{aligned}$$



Now, when  $x = 0$ ,  $y = 1$

$$\begin{aligned} \therefore \frac{a(0) + b}{2(0)^2 - 5(0) - 12} &= 1 \\ \therefore \frac{b}{-12} &= 1 \\ \therefore b &= -12 \end{aligned}$$

Also, when  $x = 2$ ,  $y = 0$

$$\begin{aligned} \therefore \frac{2a - 12}{2(2)^2 - 5(2) - 12} &= 0 \\ \therefore 2a - 12 &= 0 \\ \therefore 2a &= 12 \\ \therefore a &= 6 \end{aligned}$$

So,  $a = 6$ ,  $b = -12$ ,  $d = -5$ ,  $e = -12$ .

- 12**  $f(x) = \frac{2x - 5}{2x^2 - x - 6}$

- a** The horizontal asymptote is  $y = 0$ .

The function is undefined when  $2x^2 - x - 6 = 0$

$$\begin{aligned} \therefore (2x + 3)(x - 2) &= 0 \\ \therefore x &= -\frac{3}{2} \text{ or } 2 \end{aligned}$$

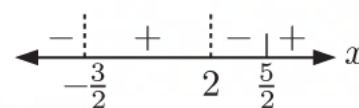
$\therefore$  the vertical asymptotes are  $x = -\frac{3}{2}$  and  $x = 2$ .

- b**  $f(0) = \frac{-5}{-6} = \frac{5}{6}$ , so the  $y$ -intercept is  $\frac{5}{6}$ .

$$\begin{aligned} f(x) = 0 \text{ when } 2x - 5 &= 0 \\ \therefore 2x &= 5 \\ \therefore x &= \frac{5}{2} \end{aligned}$$

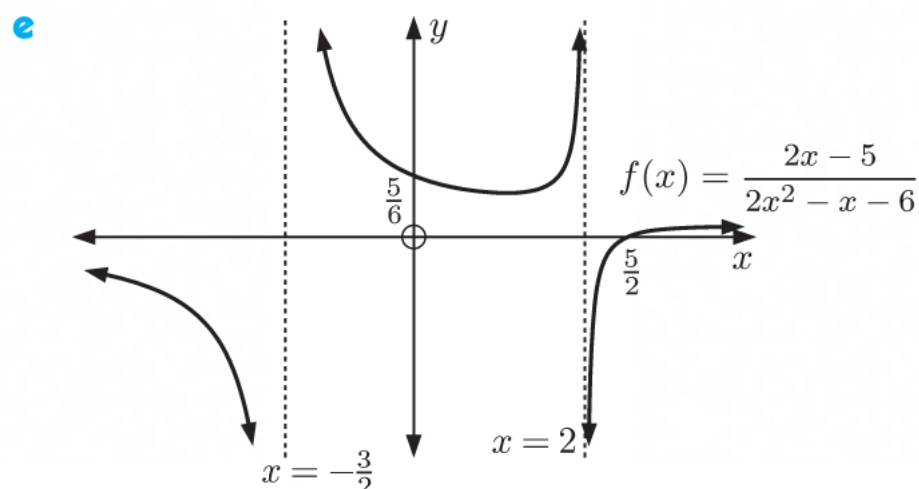
$\therefore$  the  $x$ -intercept is  $\frac{5}{2}$ .

- c**  $f(x) = \frac{2x - 5}{(2x + 3)(x - 2)}$  has sign diagram





- d** As  $x \rightarrow -\frac{3}{2}^-$ ,  $f(x) \rightarrow -\infty$   
 As  $x \rightarrow -\frac{3}{2}^+$ ,  $f(x) \rightarrow \infty$   
 As  $x \rightarrow 2^-$ ,  $f(x) \rightarrow \infty$   
 As  $x \rightarrow 2^+$ ,  $f(x) \rightarrow -\infty$   
 As  $x \rightarrow -\infty$ ,  $f(x) \rightarrow 0^-$   
 As  $x \rightarrow \infty$ ,  $f(x) \rightarrow 0^+$



**13**  $g(x) = \frac{3x^2 - 11x - 4}{3x - 2}$

- a** The vertical asymptote is  $x = \frac{2}{3}$ .

- b**  $g(0) = \frac{-4}{-2} = 2$ , so the  $y$ -intercept is 2.

$$g(x) = 0 \text{ when } 3x^2 - 11x - 4 = 0$$

$$\therefore (3x + 1)(x - 4) = 0$$

$$\therefore x = -\frac{1}{3} \text{ or } 4$$

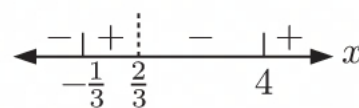
$\therefore$  the  $x$ -intercepts are  $-\frac{1}{3}$  and 4.

**c**  $g(x) = \frac{3x^2 - 11x - 4}{3x - 2}$   
 $= x - 3 - \frac{10}{3x - 2}$

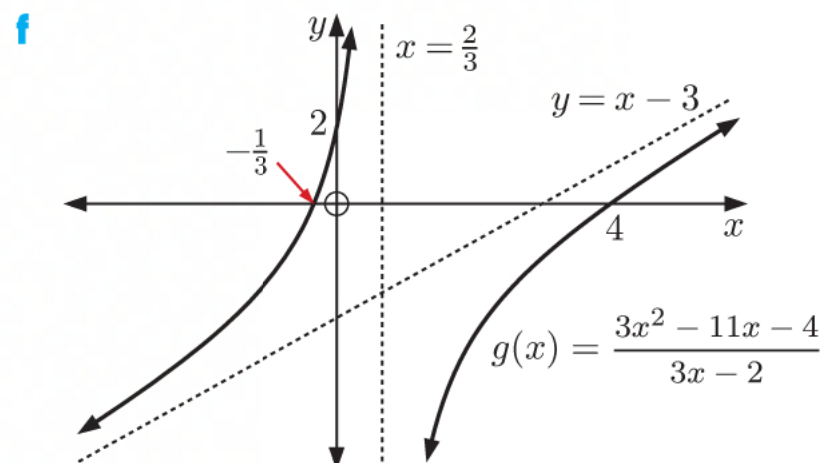
$$3x - 2 \overline{\begin{array}{r} x - 3 \\ 3x^2 - 11x - 4 \\ -(3x^2 - 2x) \phantom{- 4} \\ \hline -9x - 4 \\ -(-9x + 6) \\ \hline -10 \end{array}}$$

$\therefore$  the oblique asymptote is  $y = x - 3$ .

- d**  $g(x) = \frac{(3x + 1)(x - 4)}{3x - 2}$  has sign diagram



- e** As  $x \rightarrow \frac{2}{3}^-$ ,  $g(x) \rightarrow \infty$   
 As  $x \rightarrow \frac{2}{3}^+$ ,  $g(x) \rightarrow -\infty$   
 As  $x \rightarrow -\infty$ ,  $g(x) \rightarrow x - 3^+$   
 As  $x \rightarrow \infty$ ,  $g(x) \rightarrow x - 3^-$



$$\begin{aligned}
 14 \quad a \quad -x^2 + 4x - 3 &= -(x^2 - 4x + 3) \\
 &= -(x - 3)(x - 1)
 \end{aligned}$$

$$\begin{aligned}
 \text{Let } \frac{2x}{-x^2 + 4x - 3} &= \frac{-2x}{(x - 3)(x - 1)} = \frac{A}{x - 3} + \frac{B}{x - 1} \\
 \therefore -2x &= A(x - 1) + B(x - 3)
 \end{aligned}$$

$$\begin{aligned}
 \text{Substituting } x = 1, \quad -2(1) &= B(1 - 3) \\
 \therefore -2 &= -2B \\
 \therefore B &= 1
 \end{aligned}$$

$$\begin{aligned}
 \text{Substituting } x = 3, \quad -2(3) &= A(3 - 1) \\
 \therefore -6 &= 2A \\
 \therefore A &= -3
 \end{aligned}$$

$$\therefore \frac{2x}{-x^2 + 4x - 3} = \frac{-3}{x - 3} + \frac{1}{x - 1}$$

$$b \quad x^2 + 7x + 12 = (x + 3)(x + 4)$$

$$\begin{aligned}
 \text{Let } \frac{x - 7}{x^2 + 7x + 12} &= \frac{A}{x + 3} + \frac{B}{x + 4} \\
 \therefore x - 7 &= A(x + 4) + B(x + 3)
 \end{aligned}$$

$$\begin{aligned}
 \text{Substituting } x = -4, \quad -4 - 7 &= B(-4 + 3) \\
 \therefore -11 &= -B \\
 \therefore B &= 11
 \end{aligned}$$

$$\begin{aligned}
 \text{Substituting } x = -3, \quad -3 - 7 &= A(-3 + 4) \\
 \therefore A &= -10
 \end{aligned}$$

$$\therefore \frac{x - 7}{x^2 + 7x + 12} = \frac{-10}{x + 3} + \frac{11}{x + 4}$$

$$\begin{aligned}
 15 \quad a \quad (2x - 3)(x + 2)^2 &= (2x - 3)(x^2 + 4x + 4) \\
 &= 2x^3 + 8x^2 + 8x \\
 &\quad - 3x^2 - 12x - 12 \\
 &= 2x^3 + 5x^2 - 4x - 12 \quad \checkmark
 \end{aligned}$$

$$\begin{aligned}
 b \quad \frac{9x^2 - 2x - 5}{2x^3 + 5x^2 - 4x - 12} &= \frac{9x^2 - 2x - 5}{(2x - 3)(x + 2)^2} = \frac{A}{2x - 3} + \frac{B}{x + 2} + \frac{C}{(x + 2)^2} \\
 \therefore 9x^2 - 2x - 5 &= A(x + 2)^2 + B(2x - 3)(x + 2) + C(2x - 3)
 \end{aligned}$$

$$\begin{aligned}
 \text{Substituting } x = \frac{3}{2}, \quad 9\left(\frac{3}{2}\right)^2 - 2\left(\frac{3}{2}\right) - 5 &= A\left(\frac{3}{2} + 2\right)^2 \\
 \therefore \frac{49}{4} &= \frac{49}{4}A \\
 \therefore A &= 1
 \end{aligned}$$

$$\begin{aligned}
 \text{Substituting } x = -2, \quad 9(-2)^2 - 2(-2) - 5 &= C(2(-2) - 3) \\
 \therefore 36 + 4 - 5 &= -7C \\
 \therefore -7C &= 35 \\
 \therefore C &= -5
 \end{aligned}$$

$$\begin{aligned}\text{Substituting } x = 0, \quad -5 &= (2)^2 + B(-3)(2) - 5(-3) \\ \therefore -5 &= 4 - 6B + 15 \\ \therefore 6B &= 24 \\ \therefore B &= 4\end{aligned}$$

$$\therefore \frac{9x^2 - 2x - 5}{2x^3 + 5x^2 - 4x - 12} = \frac{1}{2x - 3} + \frac{4}{x + 2} - \frac{5}{(x + 2)^2}$$

**16 a i** Let  $g(x) = \frac{f(x) + f(-x)}{2}$

$$\begin{aligned}\therefore g(-x) &= \frac{f(-x) + f(x)}{2} \\ &= g(x) \\ \therefore \frac{f(x) + f(-x)}{2} &\text{ is even.}\end{aligned}$$

**ii** Let  $g(x) = \frac{f(x) - f(-x)}{2}$

$$\begin{aligned}\therefore g(-x) &= \frac{f(-x) - f(x)}{2} \\ &= -\frac{f(x) - f(-x)}{2} \\ &= -g(x) \\ \therefore \frac{f(x) - f(-x)}{2} &\text{ is odd.}\end{aligned}$$

**b**  $\underbrace{\frac{f(x) + f(-x)}{2}}_{\text{even \{from a i\}}} + \underbrace{\frac{f(x) - f(-x)}{2}}_{\text{odd \{from a ii\}}} = f(x)$

$\therefore f(x)$  can be written as the sum of an even function and an odd function.

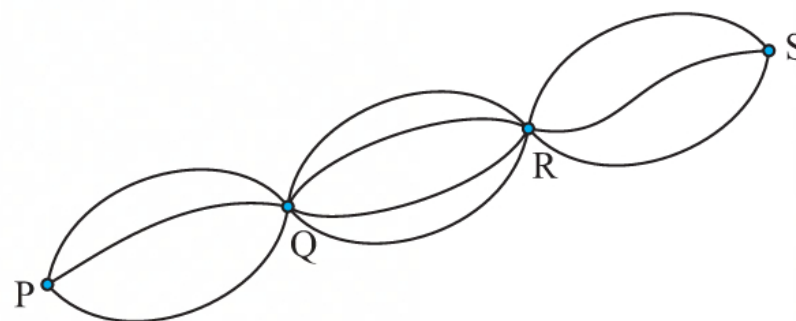
# Chapter 7

## COUNTING

### EXERCISE 7A

- 1 a For each of the 4 roads from Q to R, there are 3 ways of getting from R to S.

Using the product principle, there are  $4 \times 3 = 12$  different routes from Q to S via R.



- b Using a, there are 12 ways of getting from Q to S via R.

$\therefore$  there are 12 ways of getting from S to Q via R.

For each of these 12 ways, there are 3 ways of getting from Q to P.

Using the product principle, there are  $12 \times 3 = 36$  different routes from S to P via R and Q.

- 2 On the menu given, there are 2 entrées, 5 main courses, and 3 desserts available.

Using the product principle, there are  $2 \times 5 \times 3 = 30$  different combinations for a person to order one of each.

- 3 There are 26 choices for each of the 5 positions on the lock.

The letters chosen for each of the 5 positions are independent of each other.

Using the product principle, there are  $26 \times 26 \times 26 \times 26 \times 26$   
 $= 26^5$   
 $= 11\,881\,376$  different codes possible.

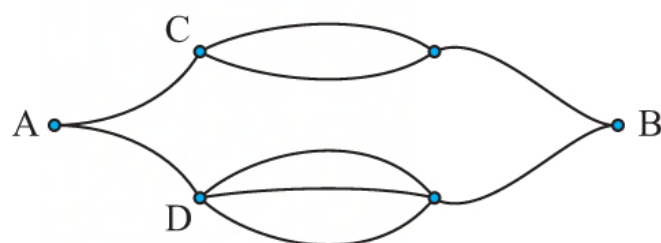
- 4 There are 26 choices for each of the 3 letters on the number plate.

There are 10 choices for each of the 4 numbers on the number plate.

Using the product principle, there are  $26 \times 26 \times 26 \times 10 \times 10 \times 10 \times 10$   
 $= 26^3 \times 10^4$   
 $= 175\,760\,000$  different number plates.

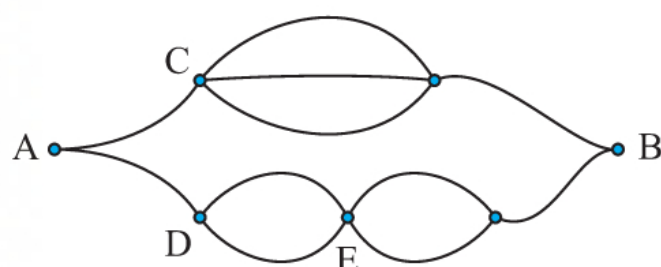
### EXERCISE 7B

- 1 a



From A we could go to C **or** D.  
 From C there are 2 paths to B.  
 From D there are 3 paths to B.  
 $\therefore$  there are  $2 + 3 = 5$  paths from A to B.

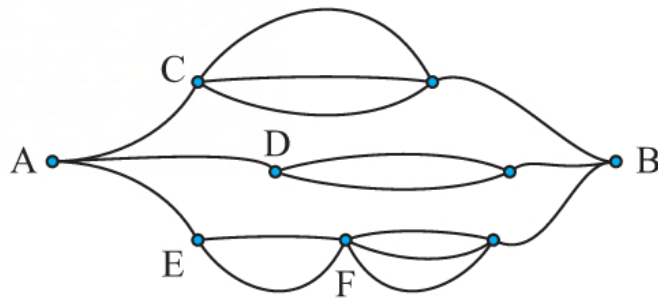
- b



From A we could go to C **or** D.  
 From C there are 3 paths to B.  
 From D there are 2 paths to E, and for each of these there are 2 paths to B.  
 $\therefore$  from D there are  $2 \times 2 = 4$  paths to B.  
 Overall, there are  $3 + 4 = 7$  paths from A to B.



c



From A we could go to C **or** D **or** E.

From C there are 3 paths to B.

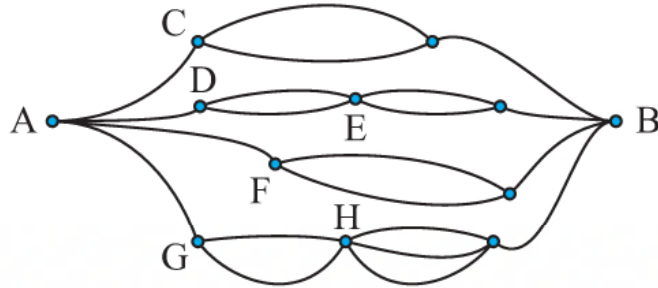
From D there are 2 paths to B.

From E there are 2 paths to F, and for each of these there are 3 paths to B.

$\therefore$  from E there are  $2 \times 3 = 6$  paths to B.

Overall, there are  $3 + 2 + 6 = 11$  paths from A to B.

d



From A we could go to C **or** D **or** F **or** G.

From C there are 2 paths to B.

From D there are 2 paths to E, and for each of these there are 2 paths to B.

$\therefore$  from D there are  $2 \times 2 = 4$  paths to B.

From F there are 2 paths to B.

From G there are 2 paths to H, and for each of these there are 3 paths to B.

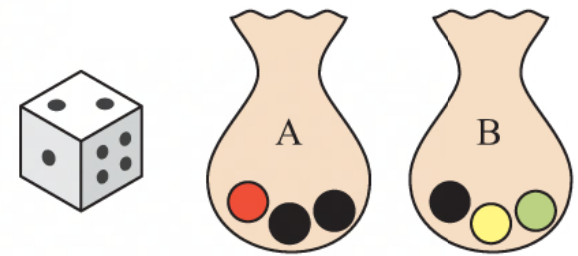
$\therefore$  from G there are  $2 \times 3 = 6$  paths to B.

Overall, there are  $2 + 4 + 2 + 6 = 14$  paths from A to B.

- 2 a To be able to pick a yellow ball, Michelle needs to roll an odd number on the die. There are 3 ways to do this (rolling a 1, 3, or 5).

There is only one yellow ball in bag B.

$\therefore$  there are  $3 \times 1 = 3$  ways that lead to a yellow ball being picked.



- b There are 3 ways to choose from bag A (rolling a 2, 4, or 6).

There are 2 black balls in bag A.

$\therefore$  there are  $3 \times 2 = 6$  ways to choose a black ball from bag A.

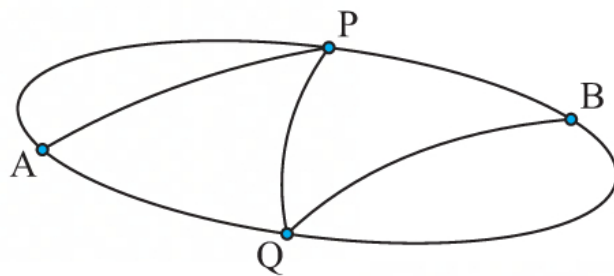
There are 3 ways to choose from bag B (rolling a 1, 3, or 5).

There is 1 black ball in bag B.

$\therefore$  there are  $3 \times 1 = 3$  ways to choose a black ball from bag B.

Overall, there are  $6 + 3 = 9$  ways that lead to a black ball being picked.

3



From A she could go to P **or** Q.

There are 2 routes from A to P and 1 route from A to Q.

From P there are 2 routes to B via Q, and 1 direct route to B.

$\therefore$  from A to B via P, there are  $2 \times (2 + 1) = 6$  routes.

From Q there is 1 route to B via P, and 2 direct routes to B.

$\therefore$  from A to B via Q, there are  $1 + 2 = 3$  routes.

Overall, there are  $6 + 3 = 9$  routes from A to B.

## EXERCISE 7C

$$1 \quad a \quad 2! = 2 \times 1 \\ = 2$$

$$c \quad 4! = 4 \times 3 \times 2 \times 1 \\ = 24$$

$$e \quad 6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 \\ = 720$$

$$b \quad 3! = 3 \times 2 \times 1 \\ = 6$$

$$d \quad 5! = 5 \times 4 \times 3 \times 2 \times 1 \\ = 120$$

$$f \quad 10! = 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 \\ = 3\,628\,800$$

$$2 \quad a \quad 4 \times 3 \times 2 \times 1 = 4!$$

$$c \quad 6 \times 5 = \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{4 \times 3 \times 2 \times 1} \\ = \frac{6!}{4!}$$

$$e \quad 10 \times 9 \times 8 \times 7 = \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{6 \times 5 \times 4 \times 3 \times 2 \times 1} \\ = \frac{10!}{6!}$$

$$f \quad 15 \times 14 \times 13 \times 12 = \frac{15 \times 14 \times 13 \times 12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1} \\ = \frac{15!}{11!}$$

$$g \quad \frac{9 \times 8 \times 7}{3 \times 2 \times 1} = \frac{9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1} \\ = \frac{9!}{3!6!}$$

$$h \quad \frac{13 \times 12 \times 11 \times 10}{4 \times 3 \times 2 \times 1} = \frac{13 \times 12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{4 \times 3 \times 2 \times 1 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1} \\ = \frac{13!}{4!9!}$$

$$i \quad \frac{15 \times 14 \times 13 \times 12 \times 11}{5 \times 4 \times 3 \times 2 \times 1} = \frac{15 \times 14 \times 13 \times 12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{5 \times 4 \times 3 \times 2 \times 1 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1} \\ = \frac{15!}{5!10!}$$

$$3 \quad a \quad \frac{7!}{6!} = \frac{7 \times \cancel{6!}}{\cancel{6!}} \\ = 7$$

$$b \quad \frac{8!}{6!} = \frac{8 \times 7 \times \cancel{6!}}{\cancel{6!}} \\ = 56$$

$$c \quad \frac{12!}{10!} = \frac{12 \times 11 \times \cancel{10!}}{\cancel{10!}} \\ = 132$$

$$d \quad \frac{120!}{119!} = \frac{120 \times \cancel{119!}}{\cancel{119!}} \\ = 120$$

$$e \quad \frac{10!}{8! \times 2!} = \frac{10 \times 9 \times \cancel{8!}}{\cancel{8!} \times 2 \times 1} \\ = \frac{90}{2} \\ = 45$$

$$f \quad \frac{100!}{98! \times 2!} = \frac{100 \times 99 \times \cancel{98!}}{\cancel{98!} \times 2 \times 1} \\ = \frac{9900}{2} \\ = 4950$$

$$\begin{aligned}
 4 \quad a \quad \frac{n!}{(n-1)!} &= \frac{n \times \cancel{(n-1)!}}{\cancel{(n-1)!}} \\
 &= n, \quad n \geq 1
 \end{aligned}$$

$$\begin{aligned}
 b \quad \frac{(n+2)!}{n!} &= \frac{(n+2) \times (n+1) \times \cancel{n!}}{\cancel{n!}} \\
 &= (n+2)(n+1), \quad n \geq 0
 \end{aligned}$$

$$\begin{aligned}
 c \quad \frac{(n+1)!}{(n-1)!} &= \frac{(n+1) \times n \times \cancel{(n-1)!}}{\cancel{(n-1)!}} \\
 &= (n+1)n, \quad n \geq 1
 \end{aligned}$$

$$\begin{aligned}
 5 \quad a \quad 5! + 4! \\
 &= 5 \times 4! + 4! \\
 &= 4!(5+1) \\
 &= 4! \times 6
 \end{aligned}$$

$$\begin{aligned}
 b \quad 11! - 10! \\
 &= 11 \times 10! - 10! \\
 &= 10!(11-1) \\
 &= 10! \times 10
 \end{aligned}$$

$$\begin{aligned}
 c \quad 5! + 7! \\
 &= 5! + 7 \times 6 \times 5! \\
 &= 5!(1+7 \times 6) \\
 &= 5! \times 43
 \end{aligned}$$

$$\begin{aligned}
 d \quad 12! - 10! \\
 &= 12 \times 11 \times 10! - 10! \\
 &= 10!(12 \times 11 - 1) \\
 &= 10! \times 131
 \end{aligned}$$

$$\begin{aligned}
 e \quad 9! + 8! + 7! \\
 &= 9 \times 8 \times 7! + 8 \times 7! + 7! \\
 &= 7!(9 \times 8 + 8 + 1) \\
 &= 7! \times 81
 \end{aligned}$$

$$\begin{aligned}
 f \quad 7! - 6! + 8! \\
 &= 7 \times 6! - 6! + 8 \times 7 \times 6! \\
 &= 6!(7 - 1 + 8 \times 7) \\
 &= 6! \times 62
 \end{aligned}$$

$$\begin{aligned}
 g \quad 12! - 2 \times 11! \\
 &= 12 \times 11! - 2 \times 11! \\
 &= 11!(12 - 2) \\
 &= 11! \times 10
 \end{aligned}$$

$$\begin{aligned}
 h \quad 3 \times 9! + 5 \times 8! \\
 &= 3 \times 9 \times 8! + 5 \times 8! \\
 &= 8!(3 \times 9 + 5) \\
 &= 8! \times 32
 \end{aligned}$$

$$\begin{aligned}
 6 \quad a \quad \frac{12! - 11!}{11} \\
 &= \frac{12 \times 11! - 11!}{11} \\
 &= \frac{11!(12 - 1)}{11} \\
 &= \frac{11! \times \cancel{11}}{\cancel{11}} \\
 &= 11!
 \end{aligned}$$

$$\begin{aligned}
 b \quad \frac{10! + 9!}{11} \\
 &= \frac{10 \times 9! + 9!}{11} \\
 &= \frac{9!(10 + 1)}{11} \\
 &= \frac{9! \times \cancel{11}}{\cancel{11}} \\
 &= 9!
 \end{aligned}$$

$$\begin{aligned}
 c \quad \frac{10! - 8!}{89} \\
 &= \frac{10 \times 9 \times 8! - 8!}{89} \\
 &= \frac{8!(90 - 1)}{89} \\
 &= \frac{8! \times \cancel{89}}{\cancel{89}} \\
 &= 8!
 \end{aligned}$$

$$\begin{aligned}
 d \quad \frac{10! - 9!}{9!} \\
 &= \frac{10 \times 9! - 9!}{9!} \\
 &= \frac{\cancel{9!}(10 - 1)}{\cancel{9!}} \\
 &= 9
 \end{aligned}$$

$$\begin{aligned}
 e \quad \frac{6! + 5! - 4!}{4!} \\
 &= \frac{6 \times 5 \times 4! + 5 \times 4! - 4!}{4!} \\
 &= \frac{\cancel{4!}(30 + 5 - 1)}{\cancel{4!}} \\
 &= 34
 \end{aligned}$$

$$\begin{aligned}
 f \quad \frac{n! + (n-1)!}{(n-1)!} \\
 &= \frac{n \times (n-1)! + (n-1)!}{(n-1)!} \\
 &= \frac{\cancel{(n-1)!}(n+1)}{\cancel{(n-1)!}} \\
 &= n+1
 \end{aligned}$$

$$\begin{aligned}
 g \quad \frac{n! - (n-1)!}{n-1} \\
 &= \frac{n \times (n-1)! - (n-1)!}{n-1} \\
 &= \frac{(n-1)! \cancel{(n-1)}}{\cancel{n-1}} \\
 &= (n-1)!
 \end{aligned}$$

$$\begin{aligned}
 h \quad \frac{(n+2)! + (n+1)!}{n+3} \\
 &= \frac{(n+2)(n+1)! + (n+1)!}{n+3} \\
 &= \frac{(n+1)! \cancel{(n+2+1)}}{\cancel{n+3}} \\
 &= (n+1)!
 \end{aligned}$$



**EXERCISE 7D.1**

- 1** **a** There are 4 permutations: W, X, Y, Z  
**b** There are 12 permutations: WX, WY, WZ, XW, XY, XZ, YW, YX, YZ, ZW, ZX, ZY  
**c** There are 24 permutations:  
 WXY, WXZ, WYX, WYZ, WZX, WZY, XWY, XWZ, XYW, XYZ, XZW, XZY, YWX, YWZ, YXW, YXZ, YZW, YZX, ZWX, ZWY, ZXW, ZXY, ZYW, ZYX
- 2** **a** There are 20 permutations:  
 AB, AC, AD, AE, BA, BC, BD, BE, CA, CB, CD, CE, DA, DB, DC, DE, EA, EB, EC, ED  
**b** There are 60 permutations:  
 ABC, ABD, ABE, ACB, ACD, ACE, ADB, ADC, ADE, AEB, AEC, AED, BAC, BAD, BAE, BCA, BCD, BCE, BDA, BDC, BDE, BEA, BEC, BED, CAB, CAD, CAE, CBA, CBD, CBE, CDA, CDB, CDE, CEA, CEB, CED, DAB, DAC, DAE, DBA, DBC, DBE, DCA, DCB, DCE, DEA, DEB, DEC, EAB, EAC, EAD, EBA, EBC, EBD, ECA, ECB, ECD, EDA, EDB, EDC

- 3** There are 5 books that could be placed first, 4 remaining books that could be placed second, and so on.

There are  $5! = 120$  ways of ordering 5 different books on a shelf.

5	4	3	2	1
1st	2nd	3rd	4th	5th

- 4** **a** There are 6 different symbols taken 3 at a time.

$$\begin{aligned}
 \therefore \text{the number of permutations} &= \frac{6!}{(6-3)!} \\
 &= \frac{6!}{3!} \\
 &= 120
 \end{aligned}$$

- b** There are 7 different symbols taken 5 at a time.

$$\begin{aligned}
 \therefore \text{the number of permutations} &= \frac{7!}{(7-5)!} \\
 &= \frac{7!}{2!} \\
 &= 2520
 \end{aligned}$$

- c** There are 26 different symbols taken 15 at a time.

$$\begin{aligned}
 \therefore \text{the number of permutations} &= \frac{26!}{(26-15)!} \\
 &= \frac{26!}{11!} \\
 &\approx 1.01 \times 10^{19}
 \end{aligned}$$

- 5** Any of the 9 teams could be in 1st place.  
 This leaves 8 teams which could be in 2nd place.

So, the total number of permutations  $= 9 \times 8$   
 $= 72$

9	8
1st	2nd



- 6** Any of the 8 teams could be in 1st place.  
 This leaves 7 teams which could be in 2nd place.  
 This leaves 6 teams which could be in 3rd place.  
 This leaves 5 teams which could be in 4th place.  
 So, the total number of permutations  $= 8 \times 7 \times 6 \times 5$   
 $= 1680$

8	7	6	5
1st	2nd	3rd	4th

- 7** Any of the 8 paintings could be placed first.  
 This leaves 7 paintings which could be placed second.  
 This leaves 6 paintings which could be placed third.  
 So, the total number of permutations  $= 8 \times 7 \times 6$   
 $= 336$

8	7	6
1st	2nd	3rd

- 8 a**

4	4	4
1st	2nd	3rd

 $\therefore 4 \times 4 \times 4 = 64$  numbers can be formed.

- b**

4	3	2
1st	2nd	3rd

 $\therefore 4 \times 3 \times 2 = 24$  numbers can be formed.

- 9 a**

4	3
1st	2nd

 $\therefore 4 \times 3 = 12$  different signals can be made.

- b**

4	3	2
1st	2nd	3rd

 $\therefore 4 \times 3 \times 2 = 24$  different signals can be made.

- c**  $12 + 24 = 36$  different signals can be made using either 2 or 3 flags.

- 10 a**

5	4	3	2	1
1st	2nd	3rd	4th	5th

 $\therefore$  there are  $5 \times 4 \times 3 \times 2 \times 1 = 120$  permutations.

- b**

3	2	1	1	1
1st	2nd	3rd	4th	5th

 $\therefore$  there are  $3 \times 2 \times 1 \times 1 \times 1 = 6$  permutations.  
 any others      R      Q

- c**

1	3	2	1	1
1st	2nd	3rd	4th	5th

 $\therefore$  there are  $1 \times 3 \times 2 \times 1 \times 1 = 6$  permutations.  
 S      any others      P

- 11 a**

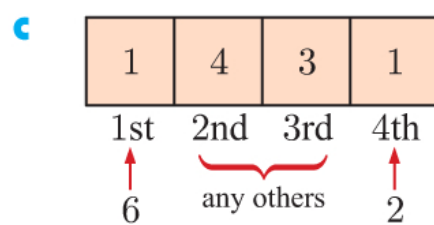
6	5	4	3
1st	2nd	3rd	4th

 $\therefore 6 \times 5 \times 4 \times 3 = 360$  different numbers can be formed.

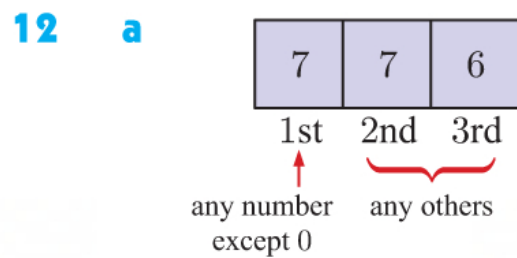
- b**

5	4	3	1
1st	2nd	3rd	4th

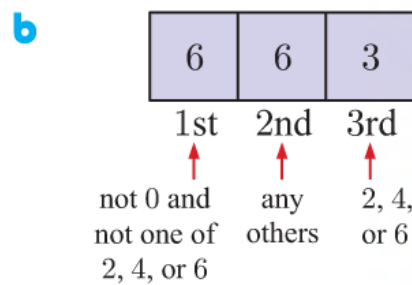
 $\therefore 5 \times 4 \times 3 \times 1 = 60$  different numbers can be formed.  
 any others      5



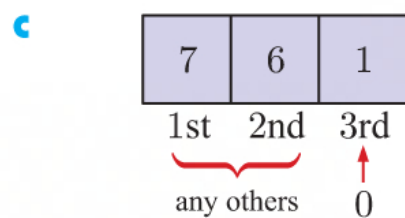
$\therefore 1 \times 4 \times 3 \times 1 = 12$  different numbers can be formed.



$\therefore 7 \times 7 \times 6 = 294$  different numbers can be formed.

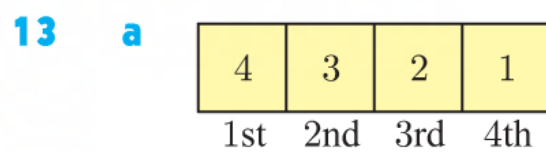


$\therefore 6 \times 6 \times 3 = 108$  different numbers can be formed.

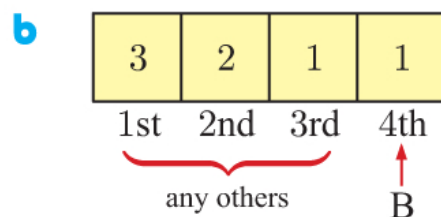


$\therefore 7 \times 6 \times 1 = 42$  different numbers can be formed.

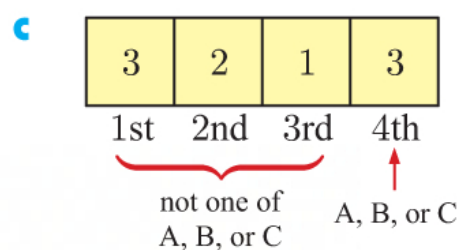
- d** Since the number must be even, then it must end in 0, 2, 4, or 6.  
 From **b**, there are 108 different numbers which end in 2, 4, or 6.  
 From **c**, there are 42 different numbers which end in 0.  
 $\therefore 108 + 42 = 150$  different even numbers can be formed.



$\therefore$  there are  $4 \times 3 \times 2 \times 1 = 24$  ways that the sprinters can be ordered.

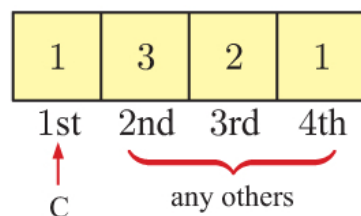


$\therefore$  there are  $3 \times 2 \times 1 \times 1 = 6$  ways that the sprinters can be ordered.



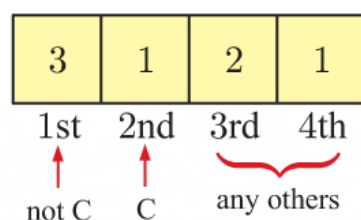
$\therefore$  there are  $3 \times 2 \times 1 \times 3 = 18$  ways that the sprinters can be ordered.

- d** If C sprints first:



$\therefore 1 \times 3 \times 2 \times 1 = 6$  different ways.

- If C sprints second:

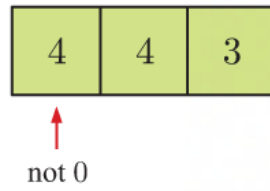


$\therefore 3 \times 1 \times 2 \times 1 = 6$  different ways.

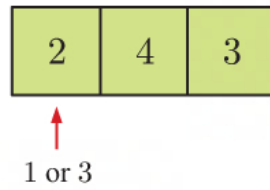
$\therefore$  there are  $6 + 6 = 12$  ways for C to sprint first or second.

- e** The different orderings are: ABCD, ABDC, CABD, CDAB, DABC, DCAB  
 $\therefore$  there are 6 different ways in which the sprinters can be ordered.

- 14 a**  $\therefore$  there are  $4 \times 4 \times 3 = 48$  different numbers.

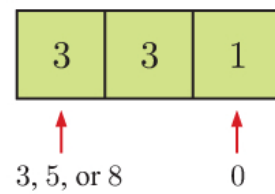


- b**  $\therefore$  there are  $2 \times 4 \times 3 = 24$  different numbers.



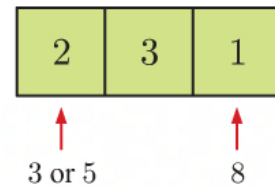
- c** The last digit must be a 0 or 8.

If it is 0:



$\therefore 3 \times 3 \times 1 = 9$  different numbers.

If it is 8:



$\therefore 2 \times 3 \times 1 = 6$  different numbers.

- 15 a** Books X and Y can be together in  $2!$  ways (XY or YX).

They, together with the 3 other books, can be ordered in  $4!$  ways.

$\therefore$  X and Y are together in  $2! \times 4!$   
 $= 48$  ways.

- b** If there were no restrictions, the 5 books could be arranged in  $5! = 120$  ways.

$\therefore$  X and Y are separated in  $120 - 48$  {from **a**}  
 $= 72$  ways.

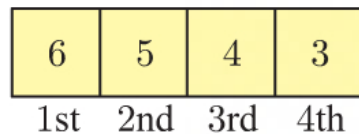
- 16 a** If there are no restrictions, the 10 students can sit in  $10! = 3\,628\,800$  ways.

- b** Students A, B, and C can be together in  $3!$  ways (ABC, ACB, BAC, BCA, CAB, or CBA).

They, together with the 7 other students, can be ordered in  $8!$  ways.

$\therefore$  A, B, and C can sit together in  $3! \times 8!$   
 $= 241\,920$  ways.

- 17 a**  $\therefore$  there are  $6 \times 5 \times 4 \times 3 = 360$  different permutations.



- b** If no vowels are used, there are 4 letters to choose from.

$\therefore$ 

4	3	2	1
1st	2nd	3rd	4th

 $\therefore$  there are  $4 \times 3 \times 2 \times 1 = 24$  different permutations.

$\therefore$  if at least one vowel must be used, there are  $360 - 24$  {from **a**}  
 $= 336$  different permutations.



- c** A and O can be together in  $2!$  ways (AO or OA).

These vowels can be placed in any one of 3 positions (1st and 2nd, 2nd and 3rd, or 3rd and 4th).

The remaining 2 places can be filled from the other 4 letters in  $4 \times 3$  different ways.

$\therefore$  two vowels are adjacent in  $2! \times 3 \times 4 \times 3 = 72$  ways

$\therefore$  no two vowels are adjacent in  $360 - 72 = 288$  ways {from **a**}

- 18 a**

9	8	7	6	5
1st	2nd	3rd	4th	5th

 $\therefore$  there are  $9 \times 8 \times 7 \times 6 \times 5 = 15\,120$  different ways if there are no restrictions.

- b**

4	3	2	6	5
1st	2nd	3rd	4th	4th
			<u>any others</u>	

 $\therefore$  there are  $4 \times 3 \times 2 \times 6 \times 5 = 720$  different ways if the first three children all take an even numbered cupcake.

- 19 a** If there are no restrictions, there are  $10! = 3\,628\,800$  different arrangements.

- b i**

10	9	8	7	6	5
1st	2nd	3rd	4th	5th	6th

 $\therefore$  there are  $10 \times 9 \times 8 \times 7 \times 6 \times 5 = 151\,200$  different seating arrangements.

- ii** Alice and her 5 friends as a group can sit together in  $6! = 720$  ways.

They can sit in seats 1 - 6, 2 - 7, 3 - 8, 4 - 9, or 5 - 10.

$\therefore$  there are  $720 \times 5 = 3600$  different seating arrangements.

- iii** Alice can sit in any of the 8 middle seats.

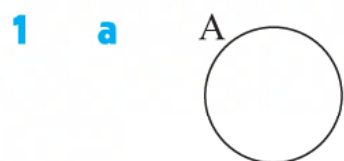
She can choose the two friends to sit next to her in  $5 \times 4$  different ways.

The remaining three friends can occupy the other 7 seats in  $7 \times 6 \times 5$  different ways.

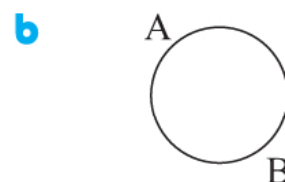
$\therefore$  there are  $8 \times 5 \times 4 \times 7 \times 6 \times 5 = 33\,600$  different seating arrangements.

## INVESTIGATION

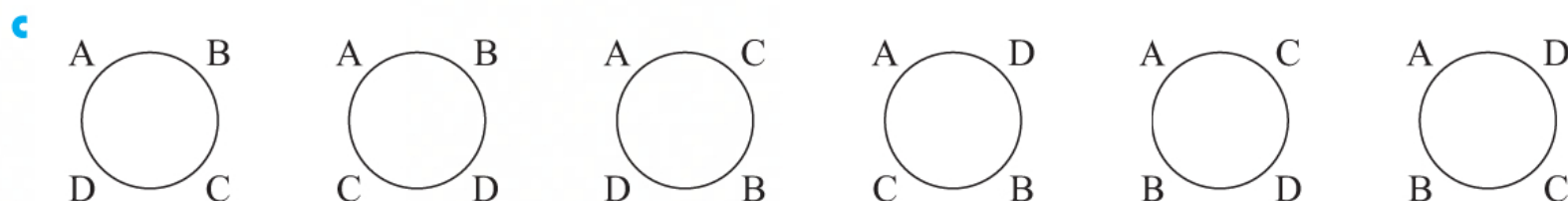
## PERMUTATIONS IN A CIRCLE



1 cyclic permutation



1 cyclic permutation



Number of symbols	Permutations in a line	Permutations in a circle
1	$1 = 1!$	$1 = 0!$
2	$2 = 2!$	$1 = 1!$
3	$6 = 3!$	$2 = 2!$
4	$24 = 4!$	$6 = 3!$

- 3** For  $n$  symbols, there are  $(n - 1)!$  cyclic permutations.



**EXERCISE 7D.2**

- 1**
- There are 3 As and 1 B.

The distinguishable permutations are AAAB, AABA, ABAA, and BAAA.

$$\frac{4!}{3!1!} = 4 \quad \checkmark$$

- 2**
- There are 2 As and 2 Bs.

The distinguishable permutations are AABB, ABAB, ABBA, BAAB, BABA, BBAA.

$$\frac{4!}{2!2!} = 6 \quad \checkmark$$

- 3 a**
- BOOKS has 1 B, 2 Os, 1 K, and 1 S.

$\therefore$  there are  $\frac{5!}{1!2!1!1!} = 60$  distinguishable permutations.

- b**
- LOTTO has 1 L, 2 Os, and 2 Ts.

$\therefore$  there are  $\frac{5!}{1!2!2!} = 30$  distinguishable permutations.

- c**
- ASSESS has 1 A, 4 Ss, and 1 E.

$\therefore$  there are  $\frac{6!}{1!4!1!} = 30$  distinguishable permutations.

- 4**
- There are 3 different books, and 2 copies each.

$\therefore$  there are  $\frac{6!}{2!2!2!} = 90$  distinguishable permutations.

- 5 a**
- There are 2 Chemistry books (C) and 3 Mathematics books (M).

If there are no restrictions, there are  $\frac{5!}{2!3!} = 10$  distinguishable permutations.

- b**
- If the books in each subject must be kept together, there are essentially only 2 “bundles” of books, and only 2 distinguishable permutations (CCMMM and MMMCC).

- 6 a**
- If there are white marbles on each end, then there is 1 red marble and 3 blue marbles to be arranged in between.

$\therefore$  there are  $\frac{4!}{1!3!} = 4$  distinguishable permutations.

- b**
- If there are blue marbles on each end, then there is 1 red marble, 2 white marbles, and 1 blue marble to be arranged in between.

$\therefore$  there are  $\frac{4!}{1!2!1!} = 12$  distinguishable permutations.

- c**
- There are
- $2!$
- ways to have a red marble at one end and a white marble at the other end. There is 1 white marble and 3 blue marbles to be arranged in between.

This can be done in  $\frac{4!}{1!3!} = 4$  distinguishable ways.

$\therefore$  there are  $2 \times 4 = 8$  distinguishable permutations.

- 7 There are  $9 \times 3 + 4 + 3 = 34$  tiles with 4 tiles each, and  $4 + 4 = 8$  tiles with 1 tile each.

$$\begin{aligned}\text{Total number of tiles} &= 34 \times 4 + 8 \\ &= 144\end{aligned}$$

$$\begin{aligned}\therefore \text{the number of distinguishable permutations is } & \frac{144!}{\underbrace{4! \times 4! \times \dots \times 4!}_{34 \text{ tiles with } 4 \text{ tiles each}} \times \underbrace{1! \times 1! \times \dots \times 1!}_{8 \text{ tiles with } 1 \text{ tile each}}} \\ &= \frac{144!}{(4!)^{34}}\end{aligned}$$

## EXERCISE 7E

$$\begin{array}{llll} \text{1 a } \binom{3}{2} = \frac{3!}{2!1!} & \text{b } \binom{6}{4} = \frac{6!}{4!2!} & \text{c } \binom{5}{1} = \frac{5!}{1!4!} & \text{d } \binom{7}{2} = \frac{7!}{2!5!} \\ & & & \\ & = \frac{3 \times \cancel{2!}}{\cancel{2!} \times 1} & = \frac{6 \times 5 \times \cancel{4!}}{\cancel{4!} \times 2 \times 1} & = \frac{7 \times 6 \times \cancel{5!}}{2 \times 1 \times \cancel{5!}} \\ & = 3 & = \frac{30}{2} = 15 & = \frac{42}{2} = 21 \end{array}$$

- 2 a The different teams are: PQ, PR, PS, PT, QR, QS, QT, RS, RT, ST

$$\begin{aligned}\text{b } \binom{5}{2} &= \frac{5!}{2!3!} \\ &= \frac{5 \times 4 \times \cancel{3!}}{2 \times 1 \times \cancel{3!}} \\ &= \frac{20}{2} \\ &= 10 \text{ different teams } \checkmark\end{aligned}$$

- 3 a The different teams are: ABCD, ABCE, ABCF, ABDE, ABDF, ABEF, ACDE, ACDF, ACEF, ADEF, BCDE, BCDF, BCEF, BDEF, CDEF

$$\begin{aligned}\text{b } \binom{6}{4} &= \frac{6!}{4!2!} \\ &= \frac{6 \times 5 \times \cancel{4!}}{\cancel{4!} \times 2 \times 1} \\ &= \frac{30}{2} \\ &= 15 \text{ different teams } \checkmark\end{aligned}$$

$$\begin{array}{lll} \text{4 a i } \binom{4}{1} = \frac{4!}{1!3!} & \text{ii } \binom{7}{1} = \frac{7!}{1!6!} & \text{iii } \binom{10}{1} = \frac{10!}{1!9!} \\ & & \\ & = \frac{4 \times \cancel{3!}}{1 \times \cancel{3!}} & = \frac{7 \times \cancel{6!}}{1 \times \cancel{6!}} & = \frac{10 \times \cancel{9!}}{1 \times \cancel{9!}} \\ & = 4 & = 7 & = 10 \end{array}$$

$$\begin{aligned}\text{b } \binom{n}{1} &= \frac{n!}{1!(n-1)!} \\ &= \frac{n \times \cancel{(n-1)!}}{1 \times \cancel{(n-1)!}} \\ &= n\end{aligned}$$

There are  $n$  different ways of picking 1 object from a set of  $n$  different objects, so  $\binom{n}{1} = n$ .

$$\begin{array}{llll}
 \text{5 a i} & \binom{5}{0} = \frac{5!}{0!5!} & \text{ii} & \binom{5}{5} = \frac{5!}{5!0!} \\
 & = \frac{\cancel{5!}}{1 \times \cancel{5!}} & & = \frac{\cancel{5!}}{\cancel{5!} \times 1} \\
 & = 1 & & = 1
 \end{array}
 \qquad
 \begin{array}{llll}
 \text{iii} & \binom{9}{0} = \frac{9!}{0!9!} & \text{iv} & \binom{9}{9} = \frac{9!}{9!0!} \\
 & = \frac{\cancel{9!}}{1 \times \cancel{9!}} & & = \frac{\cancel{9!}}{\cancel{9!} \times 1} \\
 & = 1 & & = 1
 \end{array}$$

$$\begin{array}{ll}
 \text{b i} & \binom{n}{0} = \frac{n!}{0!n!} \\
 & = \frac{\cancel{n!}}{1 \times \cancel{n!}} \\
 & = 1
 \end{array}
 \qquad
 \begin{array}{l}
 \text{ii} \quad \binom{n}{n} = \frac{n!}{n!0!} \\
 = \frac{\cancel{n!}}{\cancel{n!} \times 1} \\
 = 1
 \end{array}$$

There is only 1 way to select 0 objects from any set of objects, so  $\binom{n}{0} = 1$ .

If order is not important, then every set of  $n$  objects chosen from  $n$  objects is the same, so  $\binom{n}{n} = 1$ .

$$\begin{array}{ll}
 \text{6 a i} & \binom{7}{2} = \frac{7!}{2!5!} \\
 & = \frac{7 \times 6 \times \cancel{5!}}{2 \times 1 \times \cancel{5!}} \\
 & = \frac{42}{2} \\
 & = 21
 \end{array}
 \qquad
 \begin{array}{l}
 \binom{7}{5} = \frac{7!}{5!2!} \\
 = \frac{7 \times 6 \times \cancel{5!}}{\cancel{5!} \times 2 \times 1} \\
 = \frac{42}{2} \\
 = 21
 \end{array}$$

$$\begin{array}{ll}
 \text{ii} & \binom{10}{3} = \frac{10!}{3!7!} \\
 & = \frac{10 \times 9 \times 8 \times \cancel{7!}}{3 \times 2 \times 1 \times \cancel{7!}} \\
 & = \frac{720}{6} \\
 & = 120
 \end{array}
 \qquad
 \begin{array}{l}
 \binom{10}{7} = \frac{10!}{7!3!} \\
 = \frac{10 \times 9 \times 8 \times \cancel{7!}}{\cancel{7!} \times 3 \times 2 \times 1} \\
 = \frac{720}{6} \\
 = 120
 \end{array}$$

$$\begin{array}{l}
 \text{b} \quad \binom{n}{k} = \frac{n!}{k!(n-k)!} \\
 = \frac{n!}{(n-k)!k!} \\
 = \binom{n}{n-k}
 \end{array}$$

$$\text{c} \quad \binom{n}{k} = \frac{n!}{k!(n-k)!} \qquad \binom{n}{n-k} = \frac{n!}{(n-k)!k!} = \frac{n!}{k!(n-k)!}$$

Every time  $k$  objects are chosen from a set of  $n$  objects, the remaining objects form a possible choice of  $n - k$  objects. This means the number of different ways of choosing  $k$  objects is the same as the number of different ways of choosing  $n - k$  objects. So,  $\binom{n}{k} = \binom{n}{n-k}$ .



$$\begin{aligned}
7 \quad & \binom{9}{k} = 4 \binom{7}{k-1} \\
& \therefore \frac{9!}{k!(9-k)!} = \frac{4 \times 7!}{(k-1)!(7-(k-1))!} \\
& \therefore \frac{9 \times 8 \times \cancel{7!}}{4 \times \cancel{7!}} = \frac{k!(9-k)!}{(k-1)!(8-k)!} \\
& \therefore \frac{9 \times 8}{4} = \frac{k \times \cancel{(k-1)!} \times (9-k) \times \cancel{(8-k)!}}{\cancel{(k-1)!} \times \cancel{(8-k)!}} \\
& \therefore 18 = k(9-k) \\
& \therefore 18 = 9k - k^2 \\
& \therefore k^2 - 9k + 18 = 0 \\
& \therefore (k-3)(k-6) = 0 \\
& \therefore k = 3 \text{ or } 6
\end{aligned}$$

$$\begin{aligned}
8 \quad a \quad & \binom{n}{n-2} = \frac{n!}{(n-2)!(n-(n-2))!} \\
& = \frac{n!}{(n-2)!(2)!} \\
& = \frac{n \times (n-1) \times \cancel{(n-2)!}}{\cancel{(n-2)!} \times 2} \\
& = \frac{n^2 - n}{2} \\
& = \frac{1}{2}n^2 - \frac{1}{2}n, \quad n \in \mathbb{Z}^+, \quad n \geq 2
\end{aligned}$$

$$\begin{aligned}
b \quad & \binom{n}{4} = \frac{n!}{4!(n-4)!} \\
& = \frac{n \times (n-1) \times (n-2) \times (n-3) \times \cancel{(n-4)!}}{4! \times \cancel{(n-4)!}} \\
& = \frac{n^4 - 6n^3 + 11n^2 - 6n}{24} \\
& = \frac{1}{24}n^4 - \frac{1}{4}n^3 + \frac{11}{24}n^2 - \frac{1}{4}n, \quad n \in \mathbb{Z}^+, \quad n \geq 4
\end{aligned}$$

9 There are 15 members for selection and we can choose any 7 of them.

This can be done in  $\binom{15}{7} = 6435$  different ways. So, there are 6435 different teams.

10 a There are 27 members for selection and we can choose any 5 of them.

This can be done in  $\binom{27}{5} = 80\,730$  different ways. So, there are 80 730 different committees.

b The president must be included and we need any 4 of the other 26 members.

This can be done in  $\binom{1}{1} \times \binom{26}{4} = 14\,950$  different ways. So, there are 14 950 different committees that include the president.

11 The first two questions must be answered and students must answer any 3 of the other 5 questions.

This can be done in  $\binom{2}{2} \times \binom{5}{3} = 10$  different ways. So, there are 10 different possible selections.

12 a There are 12 members for selection and we can choose any 5 of them.

This can be done in  $\binom{12}{5} = 792$  different ways. So, there are 792 different teams.

b i The captain and vice-captain must be included and we need any 3 of the other 10 members.

This can be done in  $\binom{2}{2} \times \binom{10}{3} = 120$  different ways.

So, 120 teams contain both the captain and vice-captain.

ii Exactly one of the captain and vice-captain must be included and we need any 4 of the other 10 members.

This can be done in  $\binom{2}{1} \times \binom{10}{4} = 420$  different ways.

So, 420 teams have exactly one of the captain or vice-captain.



- 13** There are 3 players that must be included and we need any 6 of the other  $12 - 1 = 11$  healthy members.  
This can be done in  $\binom{3}{3} \times \binom{11}{6} = 462$  different ways.
- 14** **a** There are 6 male and 8 female members, or 14 members in total, and we can choose any 3 of them.  
This can be done in  $\binom{14}{3} = 364$  different ways.  
So, 364 different committees can be chosen.
- b** **i** There are 6 males in the club and we can choose any 3 of them.  
This can be done in  $\binom{6}{3} = 20$  different ways.  
So, 20 of the different possible committees consist of 3 males.
- ii** There are 6 males in the club and we can choose any 2 of them.  
There are 8 females in the club and we can choose any 1 of them.  
This can be done in  $\binom{6}{2} \times \binom{8}{1} = 120$  different ways.  
So, 120 of the different possible committees consist of 2 males and 1 female.
- iii** There are 6 males in the club and we can choose any 1 of them.  
There are 8 females in the club and we can choose any 2 of them.  
This can be done in  $\binom{6}{1} \times \binom{8}{2} = 168$  different ways.  
So, 168 of the different possible committees consist of 1 male and 2 females.
- iv** There are 8 females in the club and we can choose any 3 of them.  
This can be done in  $\binom{8}{3} = 56$  different ways.  
So, 56 of the different possible committees consist of 3 females.
- 15** **a** There are 7 tenors and 9 sopranos, or 16 singers in total, and we can choose any 4 of them.  
This can be done in  $\binom{16}{4} = 1820$  different ways.
- b** There are 7 tenors in the group and we can choose any 2 of them.  
There are 9 sopranos in the group and we can choose any 2 of them.  
This can be done in  $\binom{7}{2} \times \binom{9}{2} = 756$  different ways.
- c** We could have 2 sopranos and 2 tenors **or** 3 sopranos and 1 tenor **or** 4 sopranos.  
This can be done in  $\binom{9}{2} \times \binom{7}{2} + \binom{9}{3} \times \binom{7}{1} + \binom{9}{4} = 1470$  different ways.
- 16** **a** The answers for **1** are given an order, whereas the answers for **2** are ticks without specifying order.
- b** Since order is important when answering question **1** but not for question **2**, there will be more possible ways to answer question **1**.
- c** **i** The first preference can be made from 7 activities, the second preference can be made from 6 activities, and the third preference from 5 activities.  
Using the product principle, there are  $7 \times 6 \times 5 = 210$  different ways to answer question **1**.
- ii** For question **2**, there are 7 days in total, and students can choose any 3 of them.  
This can be done in  $\binom{7}{3} = 35$  different ways.  
So, there are 35 different ways to answer question **2**.
- d** There are 210 different ways to answer question **1**, and 35 different ways to answer question **2**.  
Using the product principle, there are  $210 \times 35 = 7350$  different ways to fill out the form.

- 17 a** There are 6 doctors in the group and we can choose any 2 of them.  
There are 3 dentists in the group and we can choose any 1 of them.  
Finally, we need any 2 of the 7 nurses.  
This can be done in  $\binom{6}{2} \times \binom{3}{1} \times \binom{7}{2} = 945$  different ways.
- b** There are 6 doctors in the group and we can choose any 2 of them.  
We then need any 3 of the  $3 + 7 = 10$  dentists and nurses.  
This can be done in  $\binom{6}{2} \times \binom{10}{3} = 1800$  different ways.
- c** If there were no restrictions, we could choose the committee in  $\binom{16}{5} = 4368$  different ways.  
Consider choosing a committee with *no* doctors or dentists, and 5 nurses.  
This can be done in  $\binom{9}{0} \binom{7}{5} = 21$  different ways.  
 $\therefore$  there are  $\binom{16}{5} - \binom{9}{0} \binom{7}{5}$   
 $= 4368 - 21$   
 $= 4347$  different committees with at least one doctor or dentist.
- 18** There are 20 vertices to choose from, and any 2 will form a line.  
This can be done in  $\binom{20}{2}$  different ways. However, this includes the 20 lines joining the vertices.  
 $\therefore$  the number of diagonals  $= \binom{20}{2} - 20 = 190 - 20$   
 $= 170$
- 19** Kristen could study 3 Group A, 2 Group B, and 1 Group C subjects **or** 2 Group A, 3 Group B, and 1 Group C subjects **or** 2 Group A, 2 Group B, and 2 Group C subjects.  
This can be done in  $\binom{6}{3} \times \binom{8}{2} \times \binom{5}{1} + \binom{6}{2} \times \binom{8}{3} \times \binom{5}{1} + \binom{6}{2} \times \binom{8}{2} \times \binom{5}{2}$   
 $= 11\,200$  different ways.
- 20 a** There are 11 players in the team, and we can choose any 3 of them to be attackers.  
This can be done in  $\binom{11}{3}$  different ways.  
There are now  $11 - 3 = 8$  players left in the team, and we can choose any 3 of them to be midfielders.  
This can be done in  $\binom{8}{3}$  different ways.  
There are now  $8 - 3 = 5$  players left in the team, and we can choose any 4 of them to be defenders.  
This can be done in  $\binom{5}{4}$  different ways.  
There is now  $5 - 4 = 1$  player left in the team, and we can choose this one player to be the goalkeeper.  
In total, the team can be sorted in  $\binom{11}{3} \times \binom{8}{3} \times \binom{5}{4} \times \binom{1}{1} = 46\,200$  different ways.



- b** Melissa is chosen to play goalkeeper.

This is done in  $\binom{1}{1}$  possible way.

There are now  $11 - 1 = 10$  players left to consider, none of whom can play goalkeeper.

Linda wants to play attacker.

This is done in  $\binom{1}{1}$  possible way.

There are now  $10 - 1 = 9$  players left to consider, only two of whom can play attacker.

Robyn does not want to play defender, so there are 8 players left who can play defender.

The 4 defenders can be chosen in  $\binom{8}{4}$  different ways.

There are now  $9 - 4 = 5$  players left to consider, 2 of whom can play attacker and 3 who can play midfielder.

The 2 attackers can be chosen in  $\binom{5}{2}$  different ways.

There are now  $5 - 2 = 3$  players left to consider, all of whom must play midfielder, as that is the only position left available.

The 3 midfielders can be chosen in  $\binom{3}{3}$  different ways.

In total, the team can be sorted in  $\binom{1}{1} \times \binom{1}{1} \times \binom{8}{4} \times \binom{5}{2} \times \binom{3}{3} = 700$  different ways.

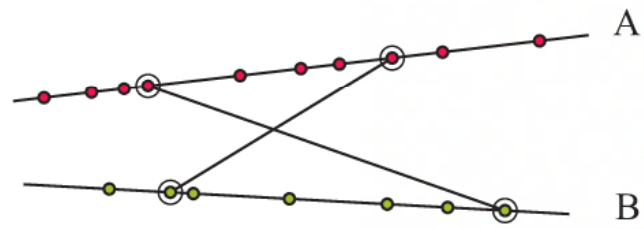
- 21 a i** There are 12 points and we can form a line by choosing any 2 of them.  
 $\therefore$  there are  $\binom{12}{2} = 66$  lines in total.
- ii** Every line which passes through B is made by choosing 1 point from the remaining 11, so  $\binom{11}{1} = 11$  pass through B.
- b i** We can form a triangle by choosing any 3 points.  
 $\therefore$  there are  $\binom{12}{3} = 220$  triangles in total.
- ii** Every triangle with vertex B is made by choosing 2 points from the remaining 11, so  $\binom{11}{2} = 55$  include vertex B.
- 22** The digits must be from 1 to 9. There are 9 numbers to choose from, and we want any 4 of them.  
 This can be done in  $\binom{9}{4} = 126$  ways.  
 For each selection of 4 different numbers, there is exactly one way to arrange them in ascending order.  
 $\therefore$  there are 126 4-digit numbers in which the digits are in ascending order from left to right.
- 23 a** The different committees of 4, selected from 5 men and 6 women in *all* possible ways are 0 men, 4 women **or** 1 man, 3 women **or** 2 men, 2 women **or** 3 men, 1 woman **or** 4 men, 0 women.  
 $\therefore \binom{5}{0} \times \binom{6}{4} + \binom{5}{1} \times \binom{6}{3} + \binom{5}{2} \times \binom{6}{2} + \binom{5}{3} \times \binom{6}{1} + \binom{5}{4} \times \binom{6}{0} = \binom{11}{4}$  since  $\binom{11}{4}$  is the number of ways of choosing a committee of 4 from 11 members with no restrictions.
- b** 
$$\sum_{i=0}^r \binom{m}{i} \binom{n}{r-i} = \binom{m}{0} \times \binom{n}{r} + \binom{m}{1} \times \binom{n}{r-1} + \dots + \binom{m}{r-1} \times \binom{n}{1} + \binom{m}{r} \times \binom{n}{0} = \binom{m+n}{r}$$
- 24 a** For 2 equal groups of 6, we choose 6 people from the group of 12, and then everyone from the remaining group of 6.  
 This can be done in  $\binom{12}{6} \times \binom{6}{6} = 924$  different ways.  
 However, this result assumes that the order in which the two groups are listed is important, which it is not. So, we need to divide this number by  $2! = 2$ .  
 $\therefore$  there are  $\frac{924}{2} = 462$  different ways to divide 12 people into 2 groups.

**b** For 3 equal groups of 4, the number of ways  $= \frac{1}{3!} \times \binom{12}{4} \times \binom{8}{4} \times \binom{4}{4}$   
 $= 5775$

- 25** There is one point of intersection for every combination of 4 points (2 from A, 2 from B) as shown.

There are  $\binom{10}{2} \times \binom{7}{2}$  ways to choose these points.

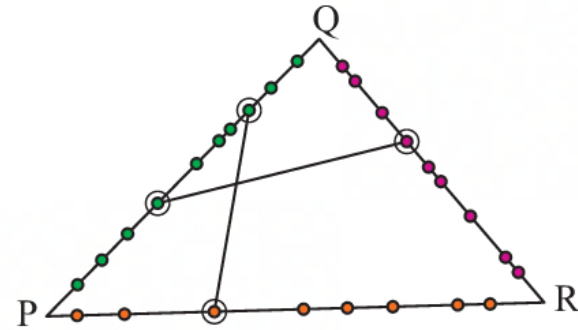
$\therefore$  the maximum number of points of intersection (when none of the intersection points coincide) is  $\binom{10}{2} \times \binom{7}{2} = 945$



- 26** There is one point of intersection for every combination of 4 points (no more than 2 from any one line) as shown.

$\therefore$  the maximum number of points of intersection (when none of the intersection points coincide) is

$$\begin{aligned} & \binom{10}{2} \binom{9}{2} \binom{8}{0} + \binom{10}{2} \binom{9}{0} \binom{8}{2} + \binom{10}{0} \binom{9}{2} \binom{8}{2} \\ & + \binom{10}{2} \binom{9}{1} \binom{8}{1} + \binom{10}{1} \binom{9}{2} \binom{8}{1} + \binom{10}{1} \binom{9}{1} \binom{8}{2} \\ & = 12\,528 \end{aligned}$$



- 27 a** There are 52 cards and we are dealt any 13 of them.

$\therefore$  there are  $\binom{52}{13} = 635\,013\,559\,600$  possible hands.

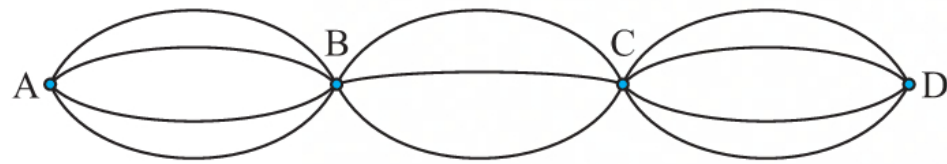
- b** There are  $\binom{13}{4}$  ways to be dealt 4 hearts, and  $\binom{39}{9}$  ways to be dealt 9 cards which are not hearts.

$\therefore$  there are  $\binom{13}{4} \times \binom{39}{9} = 151\,519\,319\,380$  hands which contain exactly 4 hearts.

- c** The probability that a hand containing exactly 4 hearts will be dealt is  $\frac{\binom{13}{4} \times \binom{39}{9}}{\binom{52}{13}} \approx 0.239$ .

## REVIEW SET 7A

- 1** For each of the 4 paths from A to B, there are 3 ways of getting from B to C, and 4 ways of getting from C to D.



Using the product principle, there are  $4 \times 3 \times 4 = 48$  paths which lead from A to D.

- 2 a**  $8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 8!$

**b**  $10 \times 9 \times 8 = \frac{10 \times 9 \times 8 \times \cancel{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}}{\cancel{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}}$   
 $= \frac{10!}{7!}$

**3 a**  $\frac{n!}{(n-2)!} = \frac{n \times (n-1) \times \cancel{(n-2)!}}{\cancel{(n-2)!}}$   
 $= n(n-1), n \geq 2$

**b**  $\frac{n! + (n+1)!}{n!} = \frac{n! + (n+1) \times n!}{n!}$   
 $= \frac{\cancel{n!} + (n+1) \times \cancel{n!}}{\cancel{n!}}$   
 $= n + 2$



4 There are 12 permutations: PQ, PR, PS, QP, QR, QS, RP, RQ, RS, SP, SQ, SR


5 a 

7	6	5	4
1st	2nd	3rd	4th

 $\therefore 7 \times 6 \times 5 \times 4 = 840$  numbers can be formed.

b 

6	5	4	3
1st	2nd	3rd	4th

 $\therefore 6 \times 5 \times 4 \times 3 = 360$  numbers can be formed.  


6 a The different teams are:  
JKL, JKM, JKN, JKO, JLM, JLN, JLO, JMN, JMO, JNO, KLM, KLN, KLO, KMN, KMO, KNO, LMN, LMO, LNO, MNO

b 
$$\begin{aligned} \binom{6}{3} &= \frac{6!}{3!3!} \\ &= \frac{\cancel{6} \times 5 \times 4 \times \cancel{3}}{\cancel{6} \times \cancel{3}} \\ &= 20 \text{ teams} \quad \checkmark \end{aligned}$$

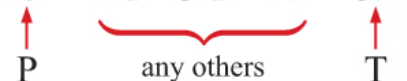
7 a 

5	4	3	2	1
1st	2nd	3rd	4th	5th

 $\therefore$  there are  $5 \times 4 \times 3 \times 2 \times 1 = 120$  different arrangements.


b i 

1	3	2	1	1
1st	2nd	3rd	4th	5th

 $\therefore$  there are  $1 \times 3 \times 2 \times 1 \times 1 = 6$  different arrangements.  


ii 

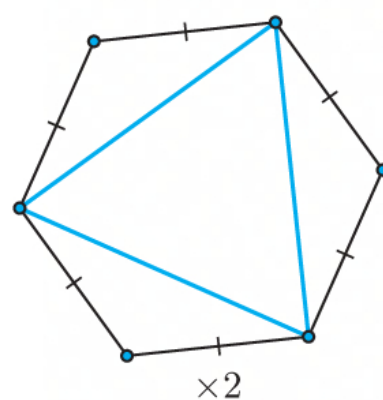
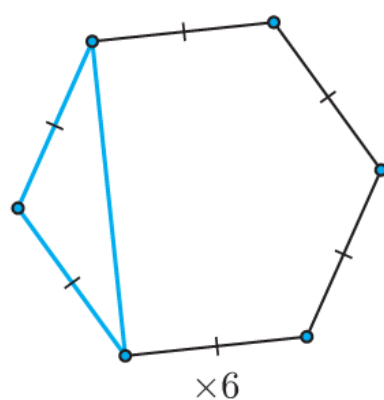
1	2	1	1	1
1st	2nd	3rd	4th	5th

 $\therefore$  there are  $1 \times 2 \times 1 \times 1 \times 1 = 2$  different arrangements.  


iii Q and S can be together in  $2!$  ways (QS or SQ).  
They, together with the 3 other letters, can be ordered in  $4!$  ways.  
 $\therefore$  there are  $2! \times 4! = 48$  different arrangements where Q and S are next to each other.

8 a There are 6 vertices and we can choose any 3 of them to form a triangle.  
 $\therefore$  there are  $\binom{6}{3} = 20$  different triangles.

b There are 6 isosceles triangles and 2 equilateral triangles that can be formed.

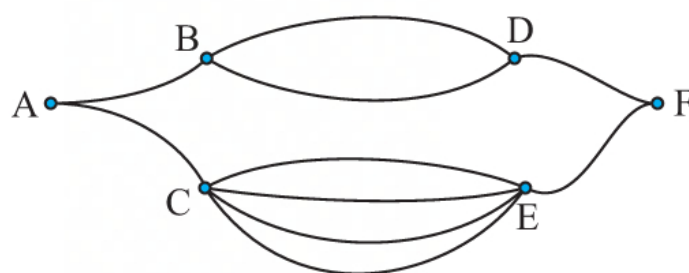


$\therefore 20 - 6 - 2 = 12$  of the triangles are scalene.

- 9 a** There are 3 As, 2 Bs, and 2 Cs.  
 $\therefore$  there are  $\frac{7!}{3!2!2!} = 210$  different orderings.
- b i** If there is an A at each end, then there is 1 A, 2 Bs, and 2 Cs to be arranged in between.  
 $\therefore \frac{5!}{1!2!2!} = 30$  orderings have an A at each end.
- ii** If the 3 As are together, we can consider them to be a single “bundle” to be taken with the 2 Bs and 2 Cs.  
 $\therefore \frac{5!}{1!2!2!} = 30$  orderings have the 3 As together.
- 10 a** If there are no restrictions, the committee of 4 can be chosen from the  $5 + 3 + 4 = 12$  teachers in  $\binom{12}{4} = 495$  different ways.
- b** The committee can consist of 2 science teachers, 1 language teacher, and 1 mathematics teacher **or** 2 language teachers, 1 science teacher, and 1 mathematics teacher **or** 2 mathematics teachers, 1 science teacher, and 1 language teacher.  
 This can be done in  $\binom{5}{2} \times \binom{3}{1} \times \binom{4}{1} + \binom{3}{2} \times \binom{5}{1} \times \binom{4}{1} + \binom{4}{2} \times \binom{5}{1} \times \binom{3}{1} = 270$  different ways.
- c** Science teachers X and Y must be included and we need any 2 of the 9 remaining teachers.  
 This can be done in  $\binom{9}{2} = 36$  different ways.
- 11 a** Each handshake occurs between 2 committee members. We need the number of ways to choose 2 people from a group of 10.  
 $\therefore$  the number of handshakes between committee members is  $\binom{10}{2} = 45$ .
- b** Similarly, the number of handshakes between delegates is  $\binom{273}{2} = 37\,128$ .
- c** The committee can line up in  $10! = 3\,628\,800$  different orders.
- 12 a** There are 52 cards and we are dealt any 8 of them.  
 $\therefore$  there are  $\binom{52}{8} = 752\,538\,150$  possible hands.
- b** There are  $\binom{13}{2}$  ways to be dealt 2 clubs,  $\binom{13}{2}$  ways to be dealt 2 diamonds,  $\binom{13}{2}$  ways to be dealt 2 hearts, and  $\binom{13}{2}$  ways to be dealt 2 spades.  
 $\therefore$  there are  $\left[\binom{13}{2}\right]^4 = 37\,015\,056$  hands which contain 2 of each of the four suits.
- c** The probability that a hand containing 2 of each suit will be dealt is  $\frac{\left[\binom{13}{2}\right]^4}{\binom{52}{8}} \approx 0.0492$ .

## REVIEW SET 7B

- 1** From A we could go to B **or** C.  
 From B there are 2 paths to F via D.  
 From C there are 4 paths to F via E.  
 $\therefore$  there are  $2 + 4 = 6$  paths from A to F.



**2 a**  $\frac{9!}{7!} = \frac{9 \times 8 \times \cancel{7!}}{\cancel{7!}}$   
 $= 72$

**b**  $\frac{8!}{3!5!} = \frac{8 \times 7 \times 6 \times \cancel{5!}}{3 \times 2 \times 1 \times \cancel{5!}}$   
 $= 56$



$$\begin{aligned}
 3 \quad a \quad \frac{7! + 8!}{9} &= \frac{7! + 8 \times 7!}{9} \\
 &= \frac{7!(1+8)}{9} \\
 &= 7!
 \end{aligned}$$

$$\begin{aligned}
 b \quad \frac{(n-1)! + (n+1)!}{(n-1)!} &= \frac{(n-1)! + (n+1) \times n \times (n-1)!}{(n-1)!} \\
 &= \frac{(n-1)!(1 + (n+1)n)}{(n-1)!} \\
 &= 1 + (n+1)n \\
 &= n^2 + n + 1
 \end{aligned}$$

$$\begin{aligned}
 4 \quad a \quad \binom{4}{2} &= \frac{4!}{2!2!} \\
 &= \frac{4 \times 3 \times \cancel{2!}}{2 \times 1 \times \cancel{2!}} \\
 &= 6
 \end{aligned}$$

$$\begin{aligned}
 b \quad \binom{6}{5} &= \frac{6!}{5!1!} \\
 &= \frac{6 \times \cancel{5!}}{\cancel{5!} \times 1} \\
 &= 6
 \end{aligned}$$

$$\begin{aligned}
 c \quad \binom{9}{2} &= \frac{9!}{2!7!} \\
 &= \frac{9 \times 8 \times \cancel{7!}}{2 \times 1 \times \cancel{7!}} \\
 &= 36
 \end{aligned}$$

5

8	7	6
Gold	Silver	Bronze

$\therefore$  there are  $8 \times 7 \times 6 = 336$  ways to award the medals.

6 a The total number of committees with no restrictions is  $\binom{21}{5} = 20\,349$  committees.

b i The total number of committees consisting of all engineers is

$$\binom{8}{5} \times \binom{7}{0} \times \binom{6}{0} = 56 \text{ committees.}$$

ii The total number of committees consisting of at least one of each profession is

$$\begin{aligned}
 &\binom{8}{1} \times \binom{7}{1} \times \binom{6}{3} + \binom{8}{1} \times \binom{7}{2} \times \binom{6}{2} + \binom{8}{1} \times \binom{7}{3} \times \binom{6}{1} \\
 &+ \binom{8}{2} \times \binom{7}{1} \times \binom{6}{2} + \binom{8}{2} \times \binom{7}{2} \times \binom{6}{1} + \binom{8}{3} \times \binom{7}{1} \times \binom{6}{1} \\
 &= 1120 + 2520 + 1680 + 2940 + 3528 + 2352 \\
 &= 14\,140 \text{ committees}
 \end{aligned}$$

$$7 \quad a \quad \binom{9}{5} = \frac{9!}{5!4!} = 126$$

$$\binom{9}{2} \times \binom{7}{3} = 36 \times 35 = 1260 \neq 126$$

b By multiplying the number of combinations of each group together, Miles is taking into account the number of ways that people from each group are ordered in the final group of 5, which is  $\binom{5}{2} = \binom{5}{3} = 10$ . He would need to divide the RHS by 10 to get the correct result.

8 Cathy, Robyn, and Jane as a group can sit together in  $3! = 6$  ways.

They, together with the 5 other people can be ordered in  $6! = 720$  ways.

$\therefore$  Cathy, Robyn, and Jane can sit together in  $6 \times 720 = 4320$  ways.

9 a If there are no restrictions, there are  $\binom{18}{8} = 43\,758$  different possible teams.

b If we must have 4 men and 4 women, there are  $\binom{11}{4} \times \binom{7}{4} = 11\,550$  different possible teams.

c The total number of teams with at least 2 women is

$$\begin{aligned}
 &\binom{11}{6} \times \binom{7}{2} + \binom{11}{5} \times \binom{7}{3} + \binom{11}{4} \times \binom{7}{4} + \binom{11}{3} \times \binom{7}{5} + \binom{11}{2} \times \binom{7}{6} + \binom{11}{1} \times \binom{7}{7} \\
 &= 9702 + 16\,170 + 11\,550 + 3465 + 385 + 11 \\
 &= 41\,283 \text{ teams}
 \end{aligned}$$

**10 a** There are  $\frac{8!}{3! 2! 2! 1!} = 1680$  distinguishable permutations.

**b i** If Petra is assigned a “villager” and Joel is assigned a “merchant” then there are two “villager” cards, two “sorcerer” cards, one “merchant” card, and one “queen” card to be assigned to the other 6 players.

This can be done in  $\frac{6!}{2! 2! 1! 1!} = 180$  different ways.

**ii** If Karen and Nick are both assigned “villager”, the remaining cards can be assigned to the other 6 players in  $\frac{6!}{1! 2! 2! 1!} = 180$  different ways.

If Karen and Nick are both assigned “sorcerer”, the remaining cards can be assigned to the other 6 players in  $\frac{6!}{3! 2! 1!} = 60$  different ways.

If Karen and Nick are both assigned “merchant”, the remaining cards can be assigned to the other 6 players in  $\frac{6!}{3! 2! 1!} = 60$  different ways.

$\therefore$  there are  $180 + 60 + 60 = 300$  different ways that Karen and Nick are assigned the same character.

**11 a** We decide both groups by choosing 8 people from the group of 16, which can be done in  $\binom{16}{8}$  different ways.

However, this result assumes that the order in which the two groups are listed is important, which it is not. So we need to divide this number by  $2! = 2$ .

$\therefore$  there are  $\frac{1}{2} \times \binom{16}{8} = 6435$  different ways to divide 16 people into two equal groups.

**b** We choose 4 people from the group of 16, then 4 people from the remaining group of 12, then 4 people from the remaining group of 8.

This can be done in  $\binom{16}{4} \times \binom{12}{4} \times \binom{8}{4} = 63\,063\,000$  different ways.

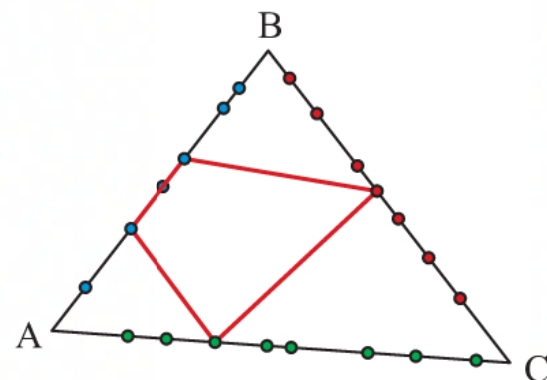
However, this result assumes that the order in which the 4 groups are listed is important, which it is not. So we need to divide this number by  $4! = 24$ .

$\therefore$  there are  $\frac{63\,063\,000}{24} = 2\,627\,625$  different ways to divide 16 people into 4 equal groups.

**12** There is one quadrilateral for every combination of 4 points (no more than 2 from any one line) as shown.

$\therefore$  the total number of different quadrilaterals that can be formed is

$$\begin{aligned} & \binom{6}{2} \binom{7}{2} \binom{8}{0} + \binom{6}{2} \binom{7}{0} \binom{8}{2} + \binom{6}{0} \binom{7}{2} \binom{8}{2} \\ & + \binom{6}{2} \binom{7}{1} \binom{8}{1} + \binom{6}{1} \binom{7}{2} \binom{8}{1} + \binom{6}{1} \binom{7}{1} \binom{8}{2} \\ & = 4347 \end{aligned}$$





# Chapter 8

## THE BINOMIAL THEOREM

### INVESTIGATION 1

### THE BINOMIAL EXPANSION

$$\begin{aligned}
 1 \quad (a+b)^4 &= (a+b)(a+b)^3 \\
 &= (a+b)(a^3 + 3a^2b + 3ab^2 + b^3) \\
 &= a^4 + 3a^3b + 3a^2b^2 + ab^3 + a^3b + 3a^2b^2 + 3ab^3 + b^4 \\
 &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4
 \end{aligned}$$

$$\begin{aligned}
 2 \quad (a+b)^5 &= (a+b)(a+b)^4 \\
 &= (a+b)(a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4) \\
 &= a^5 + 4a^4b + 6a^3b^2 + 4a^2b^3 + ab^4 + a^4b + 4a^3b^2 + 6a^2b^3 + 4ab^4 + b^5 \\
 &= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5
 \end{aligned}$$

3 a When the terms are written in this order, the powers of  $b$  increase.

b Yes, as the powers of  $a$  decrease, the powers of  $b$  increase.

$$\begin{array}{ccccccc}
 n=1 & & & & 1 & & 1 \\
 n=2 & & & & 1 & 2 & 1 \\
 n=3 & & & 1 & 3 & 3 & 1 \\
 n=4 & & 1 & 4 & 6 & 4 & 1 \\
 n=5 & 1 & 5 & 10 & 10 & 5 & 1
 \end{array}$$

4 a The values on the end of each row are always 1, and each of the remaining values is found by adding the two values diagonally above it.

$$\begin{array}{ccccccc}
 & 1 & 5 & 10 & 10 & 5 & 1 & \text{row 5} \\
 & \swarrow & \searrow & \swarrow & \searrow & \swarrow & \searrow & \\
 1 & 6 & 15 & 20 & 15 & 6 & 1 & \text{row 6}
 \end{array}$$

$$c \quad (a+b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$$

$$\begin{aligned}
 d \quad (a+b)^6 &= (a+b)(a+b)^5 \\
 &= (a+b)(a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5) \\
 &= a^6 + 5a^5b + 10a^4b^2 + 10a^3b^3 + 5a^2b^4 + ab^5 \\
 &\quad + a^5b + 5a^4b^2 + 10a^3b^3 + 10a^2b^4 + 5ab^5 + b^6 \\
 &= a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6 \quad \checkmark
 \end{aligned}$$

### EXERCISE 8A

$$1 \quad (a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$\begin{aligned}
 a \quad (p+q)^3 &= p^3 + 3p^2q + 3pq^2 + q^3
 \end{aligned}$$

$$\begin{aligned}
 b \quad (x+1)^3 &= x^3 + 3x^2(1)^1 + 3x(1)^2 + (1)^3 \\
 &= x^3 + 3x^2 + 3x + 1
 \end{aligned}$$

$$\begin{array}{ll}
 \text{c} & (x-3)^3 \\
 & = x^3 + 3x^2(-3) + 3x(-3)^2 + (-3)^3 \\
 & = x^3 - 9x^2 + 27x - 27
 \end{array}
 \qquad
 \begin{array}{ll}
 \text{d} & (2+x)^3 \\
 & = 2^3 + 3(2)^2x + 3(2)x^2 + x^3 \\
 & = 8 + 12x + 6x^2 + x^3
 \end{array}$$

$$\begin{array}{l}
 \text{e} \quad (4-x)^3 = 4^3 + 3(4)^2(-x) + 3(4)(-x)^2 + (-x)^3 \\
 \quad \quad = 64 - 48x + 12x^2 - x^3
 \end{array}$$

$$\begin{array}{l}
 \text{f} \quad (3x-1)^3 = (3x)^3 + 3(3x)^2(-1) + 3(3x)(-1)^2 + (-1)^3 \\
 \quad \quad = 27x^3 - 27x^2 + 9x - 1
 \end{array}$$

$$\begin{array}{l}
 \text{g} \quad (2x+5)^3 = (2x)^3 + 3(2x)^2(5) + 3(2x)(5)^2 + 5^3 \\
 \quad \quad = 8x^3 + 60x^2 + 150x + 125
 \end{array}$$

$$\begin{array}{l}
 \text{h} \quad (x^2-1)^3 = (x^2)^3 + 3(x^2)^2(-1) + 3(x^2)(-1)^2 + (-1)^3 \\
 \quad \quad = x^6 - 3x^4 + 3x^2 - 1
 \end{array}$$

$$\begin{array}{l}
 \text{i} \quad (2a-b)^3 = (2a)^3 + 3(2a)^2(-b) + 3(2a)(-b)^2 + (-b)^3 \\
 \quad \quad = 8a^3 - 12a^2b + 6ab^2 - b^3
 \end{array}$$

$$\begin{array}{l}
 \text{j} \quad (\sqrt{x}-1)^3 = (\sqrt{x})^3 + 3(\sqrt{x})^2(-1) + 3(\sqrt{x})(-1)^2 + (-1)^3 \\
 \quad \quad = x\sqrt{x} - 3x + 3\sqrt{x} - 1
 \end{array}$$

$$\begin{array}{l}
 \text{k} \quad (3x - \frac{1}{3})^3 = (3x)^3 + 3(3x)^2(-\frac{1}{3}) + 3(3x)(-\frac{1}{3})^2 + (-\frac{1}{3})^3 \\
 \quad \quad = 27x^3 - 9x^2 + x - \frac{1}{27}
 \end{array}$$

$$\begin{array}{l}
 \text{l} \quad \left(2x + \frac{1}{x}\right)^3 = (2x)^3 + 3(2x)^2\left(\frac{1}{x}\right) + 3(2x)\left(\frac{1}{x}\right)^2 + \left(\frac{1}{x}\right)^3 \\
 \quad \quad = 8x^3 + 12x + \frac{6}{x} + \frac{1}{x^3}
 \end{array}$$

$$\text{2} \quad (a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

$$\begin{array}{l}
 \text{a} \quad (1+x)^4 = 1^4 + 4(1)^3x + 6(1)^2x^2 + 4(1)x^3 + x^4 \\
 \quad \quad = 1 + 4x + 6x^2 + 4x^3 + x^4
 \end{array}$$

$$\begin{array}{l}
 \text{b} \quad (x-2)^4 = x^4 + 4x^3(-2) + 6x^2(-2)^2 + 4x(-2)^3 + (-2)^4 \\
 \quad \quad = x^4 - 8x^3 + 24x^2 - 32x + 16
 \end{array}$$

$$\begin{array}{l}
 \text{c} \quad (3-x)^4 = (3)^4 + 4(3)^3(-x) + 6(3)^2(-x)^2 + 4(3)(-x)^3 + (-x)^4 \\
 \quad \quad = 81 - 108x + 54x^2 - 12x^3 + x^4
 \end{array}$$

$$\begin{array}{l}
 \text{d} \quad (1+2x)^4 = (1)^4 + 4(1)^3(2x) + 6(1)^2(2x)^2 + 4(1)(2x)^3 + (2x)^4 \\
 \quad \quad = 1 + 8x + 24x^2 + 32x^3 + 16x^4
 \end{array}$$

$$\begin{array}{l}
 \text{e} \quad (2x-3)^4 = (2x)^4 + 4(2x)^3(-3) + 6(2x)^2(-3)^2 + 4(2x)(-3)^3 + (-3)^4 \\
 \quad \quad = 16x^4 - 12 \times 8x^3 + 54 \times 4x^2 - 108 \times 2x + 81 \\
 \quad \quad = 16x^4 - 96x^3 + 216x^2 - 216x + 81
 \end{array}$$

$$\begin{array}{l}
 \text{f} \quad (2x+b)^4 = (2x)^4 + 4(2x)^3b + 6(2x)^2b^2 + 4(2x)b^3 + b^4 \\
 \quad \quad = 16x^4 + 32x^3b + 24x^2b^2 + 8xb^3 + b^4
 \end{array}$$

$$\begin{aligned} \text{g} \quad \left(x + \frac{1}{x}\right)^4 &= x^4 + 4x^3\left(\frac{1}{x}\right) + 6x^2\left(\frac{1}{x}\right)^2 + 4x\left(\frac{1}{x}\right)^3 + \left(\frac{1}{x}\right)^4 \\ &= x^4 + 4x^2 + 6 + \frac{4}{x^2} + \frac{1}{x^4} \end{aligned}$$

$$\begin{aligned} \text{h} \quad \left(2x - \frac{1}{x}\right)^4 &= (2x)^4 + 4(2x)^3\left(-\frac{1}{x}\right) + 6(2x)^2\left(-\frac{1}{x}\right)^2 + 4(2x)\left(-\frac{1}{x}\right)^3 + \left(-\frac{1}{x}\right)^4 \\ &= 16x^4 - 32x^2 + 24 - \frac{8}{x^2} + \frac{1}{x^4} \end{aligned}$$

$$\begin{aligned} \text{i} \quad (x + \sqrt{x})^4 &= x^4 + 4x^3(\sqrt{x}) + 6x^2(\sqrt{x})^2 + 4x(\sqrt{x})^3 + (\sqrt{x})^4 \\ &= x^4 + 4x^3\sqrt{x} + 6x^3 + 4x^2\sqrt{x} + x^2 \end{aligned}$$

$$\begin{aligned} \text{3} \quad \text{a} \quad \text{i} \quad (a - b)^3 &= a^3 + 3a^2(-b) + 3a(-b)^2 + (-b)^3 \\ &= a^3 - 3a^2b + 3ab^2 - b^3 \end{aligned}$$

$$\begin{aligned} \text{ii} \quad (a - b)^4 &= a^4 + 4a^3(-b) + 6a^2(-b)^2 + 4a(-b)^3 + (-b)^4 \\ &= a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4 \end{aligned}$$

**b** The terms are the same, except for their signs. The signs in the expansions of  $(a + b)^3$  and  $(a + b)^4$  are all positive, whereas the first sign in the expansions of  $(a - b)^3$  and  $(a - b)^4$  is positive, and then they alternate ( $a > 0$ ,  $b > 0$ ).

$$\begin{array}{cccccc} \text{4} \quad \text{a} & 1 & 4 & 6 & 4 & 1 & \leftarrow \text{the 4th row} \\ & 1 & 5 & 10 & 10 & 5 & 1 & \leftarrow \text{the 5th row} \end{array}$$

$$\text{b} \quad (a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

$$\begin{aligned} \text{c} \quad \text{i} \quad (x + 2)^5 &= x^5 + 5x^4(2) + 10x^3(2)^2 + 10x^2(2)^3 + 5x(2)^4 + (2)^5 \\ &= x^5 + 10x^4 + 40x^3 + 80x^2 + 80x + 32 \end{aligned}$$

$$\begin{aligned} \text{ii} \quad (x - 2y)^5 &= x^5 + 5x^4(-2y) + 10x^3(-2y)^2 + 10x^2(-2y)^3 + 5x(-2y)^4 + (-2y)^5 \\ &= x^5 - 10x^4y + 40x^3y^2 - 80x^2y^3 + 80xy^4 - 32y^5 \end{aligned}$$

$$\begin{aligned} \text{iii} \quad (1 + 2x)^5 &= (1)^5 + 5(1)^4(2x) + 10(1)^3(2x)^2 + 10(1)^2(2x)^3 + 5(1)(2x)^4 + (2x)^5 \\ &= 1 + 10x + 40x^2 + 80x^3 + 80x^4 + 32x^5 \end{aligned}$$

$$\begin{aligned} \text{iv} \quad \left(x - \frac{1}{x}\right)^5 &= x^5 + 5x^4\left(-\frac{1}{x}\right) + 10x^3\left(-\frac{1}{x}\right)^2 + 10x^2\left(-\frac{1}{x}\right)^3 + 5x\left(-\frac{1}{x}\right)^4 + \left(-\frac{1}{x}\right)^5 \\ &= x^5 - 5x^3 + 10x - \frac{10}{x} + \frac{5}{x^3} - \frac{1}{x^5} \end{aligned}$$

$$\begin{array}{cccccc} \text{5} \quad \text{a} & 1 & 5 & 10 & 10 & 5 & 1 & \leftarrow \text{the 5th row} \\ & 1 & 6 & 15 & 20 & 15 & 6 & 1 & \leftarrow \text{the 6th row} \end{array}$$

$$\text{b} \quad (a + b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$$

$$\begin{aligned} \text{c} \quad \text{i} \quad (x + 2)^6 &= x^6 + 6x^5(2) + 15x^4(2)^2 + 20x^3(2)^3 + 15x^2(2)^4 + 6x(2)^5 + (2)^6 \\ &= x^6 + 12x^5 + 60x^4 + 160x^3 + 240x^2 + 192x + 64 \end{aligned}$$

$$\begin{aligned} \text{ii} \quad (2x - 1)^6 &= (2x)^6 + 6(2x)^5(-1) + 15(2x)^4(-1)^2 + 20(2x)^3(-1)^3 + 15(2x)^2(-1)^4 \\ &\quad + 6(2x)(-1)^5 + (-1)^6 \\ &= 64x^6 - 6 \times 32x^5 + 15 \times 16x^4 - 20 \times 8x^3 + 15 \times 4x^2 - 6 \times 2x + 1 \\ &= 64x^6 - 192x^5 + 240x^4 - 160x^3 + 60x^2 - 12x + 1 \end{aligned}$$



iii

6

**b**

C

7

**b**

8

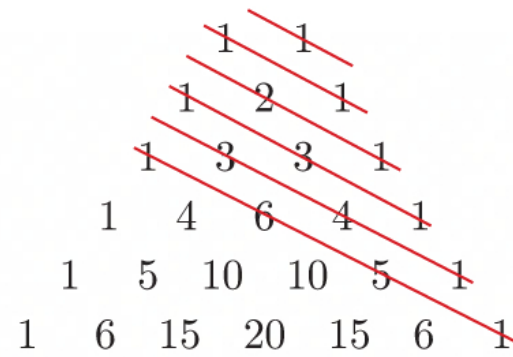
9

**b**



## ACTIVITY

- 1 Diagonal 1: 1  
 Diagonal 2:  $1 + 1 = 2$   
 Diagonal 3:  $2 + 1 = 3$   
 Diagonal 4:  $1 + 3 + 1 = 5$   
 Diagonal 5:  $3 + 4 + 1 = 8$   
 Diagonal 6:  $1 + 6 + 5 + 1 = 13$   
 $\vdots$



- 2 The sequence of numbers formed by the answer to 1 is the Fibonacci sequence. The sum of the terms in the  $n$ th shallow diagonal ( $n \geq 3$ ) is the sum of the terms in the  $(n-2)$ th and  $(n-1)$ th shallow diagonals.  
 For example, consider the term 6 in diagonal 6. It is obtained by adding together the 3 in diagonal 4 and the 3 in diagonal 5. We could repeat this process for the other terms in diagonal 6.

## INVESTIGATION 2

## THE BINOMIAL COEFFICIENT

- 1 Multiplying by  $(a+b)$  will double the amount of terms in the expression each time.  
 $\therefore$  there would be  $2^n$  terms.
- 2 a If we choose  $b$   $r$  times, then we choose  $a$   $(n-r)$  times.  
 b The order of each lot of  $r$  is not important, so there are  $\binom{n}{r} = \frac{n!}{(n-r)!r!}$  ways to choose  $r$  lots of  $b$  from  $n$  brackets.  
 c After collecting the “like” terms, there will be  $\frac{n!}{(n-r)!r!}$  terms of the form  $a^{n-r}b^r$ .

3 a

$$\begin{array}{ccccccc}
 & & \binom{1}{0} & \binom{1}{1} & & & \\
 & & \binom{2}{0} & \binom{2}{1} & \binom{2}{2} & & \\
 & \binom{3}{0} & \binom{3}{1} & \binom{3}{2} & \binom{3}{3} & & \\
 & \binom{4}{0} & \binom{4}{1} & \binom{4}{2} & \binom{4}{3} & \binom{4}{4} & \\
 \binom{5}{0} & \binom{5}{1} & \binom{5}{2} & \binom{5}{3} & \binom{5}{4} & \binom{5}{5} & \\
 \binom{6}{0} & \binom{6}{1} & \binom{6}{2} & \binom{6}{3} & \binom{6}{4} & \binom{6}{5} & \binom{6}{6}
 \end{array}
 =
 \begin{array}{ccccccc}
 & & 1 & 1 & & & \\
 & & 1 & 2 & 1 & & \\
 & 1 & 3 & 3 & 1 & & \\
 & 1 & 4 & 6 & 4 & 1 & \\
 1 & 5 & 10 & 10 & 5 & 1 & \\
 1 & 6 & 15 & 20 & 15 & 6 & 1
 \end{array}$$

- b The  $r$ th number in the  $n$ th row of Pascal's triangle is  $\binom{n}{r}$  where  $n, r \in \mathbb{N}$ ,  $n \geq 1$ ,  $r \leq n$ .  
 (The left most numbers are the zeroth numbers in the row.)

## EXERCISE 8B

- 1 a  $(1+2x)^{11} = 1^{11} + \binom{11}{1}(2x)^1 + \binom{11}{2}(2x)^2 + \dots + \binom{11}{10}(2x)^{10} + (2x)^{11}$   
 b  $\left(3x + \frac{2}{x}\right)^{15}$   
 $= (3x)^{15} + \binom{15}{1}(3x)^{14}\left(\frac{2}{x}\right)^1 + \binom{15}{2}(3x)^{13}\left(\frac{2}{x}\right)^2 + \dots + \binom{15}{14}(3x)^1\left(\frac{2}{x}\right)^{14} + \left(\frac{2}{x}\right)^{15}$

$$\begin{aligned} \text{c } \left(2x - \frac{3}{x}\right)^{20} &= (2x)^{20} + \binom{20}{1}(2x)^{19}\left(-\frac{3}{x}\right)^1 + \binom{20}{2}(2x)^{18}\left(-\frac{3}{x}\right)^2 + \dots \\ &\quad + \binom{20}{19}(2x)^1\left(-\frac{3}{x}\right)^{19} + \left(-\frac{3}{x}\right)^{20} \end{aligned}$$

**2 a** For  $(2x + 5)^{15}$ ,  $a = (2x)$ ,  $b = 5$ , and  $n = 15$ .

Now  $T_{r+1} = \binom{n}{r} a^{n-r} b^r$  and letting  $r = 5$  gives  $T_6 = \binom{15}{5} (2x)^{10} 5^5$ .

**b** For  $(x^2 + y)^9$ ,  $a = (x^2)$ ,  $b = y$ , and  $n = 9$ .

Now  $T_{r+1} = \binom{n}{r} a^{n-r} b^r$  and letting  $r = 3$  gives  $T_4 = \binom{9}{3} (x^2)^6 y^3$ .

**c** For  $(x + y^2)^{10}$ ,  $a = x$ ,  $b = y^2$ , and  $n = 10$ .

Now  $T_{r+1} = \binom{n}{r} a^{n-r} b^r$  and letting  $r = 3$  gives  $T_4 = \binom{10}{3} x^7 (y^2)^3$ .

**d** For  $\left(x - \frac{2}{x}\right)^{17}$ ,  $a = x$ ,  $b = \left(-\frac{2}{x}\right)$ , and  $n = 17$ .

Now  $T_{r+1} = \binom{n}{r} a^{n-r} b^r$  and letting  $r = 9$  gives  $T_{10} = \binom{17}{9} x^8 \left(-\frac{2}{x}\right)^9$ .

**e** For  $\left(2x^2 - \frac{1}{x}\right)^{21}$ ,  $a = (2x^2)$ ,  $b = \left(-\frac{1}{x}\right)$ , and  $n = 21$ .

Now  $T_{r+1} = \binom{n}{r} a^{n-r} b^r$  and letting  $r = 8$  gives  $T_9 = \binom{21}{8} (2x^2)^{13} \left(-\frac{1}{x}\right)^8$ .

**f** For  $(\sqrt{x} + 1)^{16}$ ,  $a = \sqrt{x}$ ,  $b = 1$ , and  $n = 16$ .

Now  $T_{r+1} = \binom{n}{r} a^{n-r} b^r$  and letting  $r = 6$  gives  $T_7 = \binom{16}{6} (\sqrt{x})^{10} 1^6$ .

$$\begin{aligned} \text{3 } (a - b)^n &= a^n + \binom{n}{1} a^{n-1}(-b) + \binom{n}{2} a^{n-2}(-b)^2 + \dots + \binom{n}{r} a^{n-r}(-b)^r + \dots + (-b)^n \\ &= \binom{n}{0} (-1)^0 a^n b^0 + \binom{n}{1} (-1)^1 a^{n-1} b^1 + \binom{n}{2} (-1)^2 a^{n-2} b^2 + \dots \\ &\quad + \binom{n}{r} (-1)^{n-r} a^r b^{n-r} + \dots + \binom{n}{n} (-1)^n a^0 b^n \\ &= \sum_{r=0}^n \binom{n}{r} (-1)^r a^{n-r} b^r \end{aligned}$$

**4 a** For  $(x + 2)^8$ ,  $a = x$ ,  $b = 2$ , and  $n = 8$

Now  $T_{r+1} = \binom{n}{r} a^{n-r} b^r$

$$\therefore T_{r+1} = \binom{8}{r} x^{8-r} 2^r$$

**b** If  $8 - r = 5$

then  $r = 3$

$$\therefore T_4 = \binom{8}{3} x^5 2^3$$

$\therefore$  the coefficient of  $x^5$  is  $\binom{8}{3} 2^3 = 448$ .

**5 a** For  $(x + b)^7$ ,  $a = x$ ,  $b = b$ , and  $n = 7$

$\therefore$  the general term  $T_{r+1} = \binom{7}{r} x^{7-r} b^r$

**b** If  $x^{7-r} = x^4$  then  $7 - r = 4$

$$\therefore r = 3$$

$$\text{Now } T_4 = \binom{7}{3} x^4 b^3$$

$\therefore$  the coefficient of  $x^4$  is  $\binom{7}{3} b^3 = 35b^3$

But the coefficient of  $x^4$  is  $-280$

$$\text{So, } 35b^3 = -280$$

$$\therefore b^3 = -8$$

$$\therefore b = \sqrt[3]{-8}$$

$$\therefore b = -2$$

**6 a** For  $\left(\frac{1}{2}x + \frac{1}{3}\right)^{12}$ ,  $a = \frac{1}{2}x$ ,  $b = \frac{1}{3}$ , and  $n = 12$

$\therefore$  the general term  $T_{r+1} = \binom{12}{r} \left(\frac{1}{2}x\right)^{12-r} \left(\frac{1}{3}\right)^r$ .

**b i** If  $12 - r = 3$   
then  $r = 9$

$$\therefore T_{10} = \binom{12}{9} \left(\frac{1}{2}x\right)^3 \left(\frac{1}{3}\right)^9$$

$\therefore$  the coefficient of  $x^3$  is  $\binom{12}{9} \left(\frac{1}{2}\right)^3 \left(\frac{1}{3}\right)^9 = \frac{55}{39\,366}$ .

**ii** If  $12 - r = 5$   
then  $r = 7$

$$\therefore T_8 = \binom{12}{7} \left(\frac{1}{2}x\right)^5 \left(\frac{1}{3}\right)^7$$

$\therefore$  the coefficient of  $x^5$  is  $\binom{12}{7} \left(\frac{1}{2}\right)^5 \left(\frac{1}{3}\right)^7 = \frac{11}{972}$ .

**iii** If  $12 - r = 9$   
then  $r = 3$

$$\therefore T_4 = \binom{12}{3} \left(\frac{1}{2}x\right)^9 \left(\frac{1}{3}\right)^3$$

$\therefore$  the coefficient of  $x^9$  is  $\binom{12}{3} \left(\frac{1}{2}\right)^9 \left(\frac{1}{3}\right)^3 = \frac{55}{3456}$ .

**7 a** For  $\left(x + \frac{2}{x^2}\right)^{15}$ ,  $a = x$ ,  $b = \left(\frac{2}{x^2}\right)$ , and  $n = 15$

$$\begin{aligned} \text{Now } T_{r+1} &= \binom{n}{r} a^{n-r} b^r \\ &= \binom{15}{r} x^{15-r} \left(\frac{2}{x^2}\right)^r \\ &= \binom{15}{r} x^{15-r} \frac{2^r}{x^{2r}} \\ &= \binom{15}{r} 2^r x^{15-3r} \end{aligned}$$

The constant term does not contain  $x$ .

$$\therefore 15 - 3r = 0$$

$$\therefore r = 5$$

$$\text{so } T_6 = \binom{15}{5} 2^5 x^0$$

$\therefore$  the constant term is  $\binom{15}{5} 2^5 = 96\,096$ .

**b** For  $\left(x - \frac{3}{x^2}\right)^9$ ,  $a = x$ ,  $b = \left(-\frac{3}{x^2}\right)$ , and  $n = 9$

$$\begin{aligned}\text{Now } T_{r+1} &= \binom{n}{r} a^{n-r} b^r \\ &= \binom{9}{r} x^{9-r} \left(-\frac{3}{x^2}\right)^r \\ &= \binom{9}{r} x^{9-r} \frac{(-3)^r}{x^{2r}} \\ &= \binom{9}{r} (-3)^r x^{9-3r}\end{aligned}$$

The constant term does not contain  $x$ .

$$\therefore 9 - 3r = 0$$

$$\therefore r = 3$$

$$\text{so } T_4 = \binom{9}{3} (-3)^3 x^0$$

$$\therefore \text{ the constant term is } \binom{9}{3} (-3)^3 = -2268.$$

**c** For  $\left(\frac{x}{3} - \frac{2}{\sqrt{x}}\right)^{15}$ ,  $a = \frac{x}{3}$ ,  $b = \left(-\frac{2}{\sqrt{x}}\right)$ , and  $n = 15$

$$\begin{aligned}\text{Now } T_{r+1} &= \binom{n}{r} a^{n-r} b^r \\ &= \binom{15}{r} \left(\frac{x}{3}\right)^{15-r} \left(-\frac{2}{\sqrt{x}}\right)^r \\ &= \binom{15}{r} \left(\frac{1}{3}\right)^{15-r} (-2)^r x^{15-r} x^{-\frac{1}{2}r} \\ &= \binom{15}{r} \left(\frac{1}{3}\right)^{15-r} (-2)^r x^{15-\frac{3}{2}r}\end{aligned}$$

The constant term does not contain  $x$ .

$$\therefore 15 - \frac{3}{2}r = 0$$

$$\therefore \frac{3}{2}r = 15$$

$$\therefore r = 10$$

$$\text{so } T_{11} = \binom{15}{10} \left(\frac{1}{3}\right)^5 (-2)^{10} x^0$$

$$\therefore \text{ the constant term is } \binom{15}{10} \left(\frac{1}{3}\right)^5 (-2)^{10} = \frac{1\,025\,024}{81}.$$

**8 a** In  $(3 + 2x^2)^{10}$ ,  $a = 3$ ,  $b = (2x^2)$ , and  $n = 10$

$$\begin{aligned}\text{Now } T_{r+1} &= \binom{n}{r} a^{n-r} b^r \\ &= \binom{10}{r} 3^{10-r} (2x^2)^r \\ &= \binom{10}{r} 3^{10-r} 2^r x^{2r}\end{aligned}$$

We now let  $2r = 10$

$$\therefore r = 5$$

$$\text{So, } T_6 = \binom{10}{5} 3^5 2^5 x^{10}$$

$$\therefore \text{ the coefficient of } x^{10} \text{ is } \binom{10}{5} 3^5 2^5 = 1\,959\,552.$$



**b** In  $\left(2x^2 - \frac{3}{x}\right)^6$ ,  $a = (2x^2)$ ,  $b = \left(-\frac{3}{x}\right)$ , and  $n = 6$

$$\begin{aligned}\text{Now } T_{r+1} &= \binom{n}{r} a^{n-r} b^r \\ &= \binom{6}{r} (2x^2)^{6-r} \left(-\frac{3}{x}\right)^r \\ &= \binom{6}{r} 2^{6-r} x^{12-2r} \frac{(-3)^r}{x^r} \\ &= \binom{6}{r} 2^{6-r} (-3)^r x^{12-3r}\end{aligned}$$

We now let  $12 - 3r = 3$

$$\therefore 3r = 9$$

$$\therefore r = 3$$

$$\text{So, } T_4 = \binom{6}{3} 2^3 (-3)^3 x^3$$

$\therefore$  the coefficient of  $x^3$  is  $\binom{6}{3} 2^3 (-3)^3 = -4320$ .

**c** In  $(2x^2 - 3y)^6$ ,  $a = (2x^2)$ ,  $b = (-3y)$ , and  $n = 6$

$$\begin{aligned}\text{Now } T_{r+1} &= \binom{n}{r} a^{n-r} b^r \\ &= \binom{6}{r} (2x^2)^{6-r} (-3y)^r \\ &= \binom{6}{r} 2^{6-r} x^{12-2r} (-3)^r y^r \\ &= \binom{6}{r} 2^{6-r} (-3)^r x^{12-2r} y^r\end{aligned}$$

We find  $r$  such that  $12 - 2r = 6$  and  $r = 3$

$\therefore r = 3$  is the solution

$$\text{So, } T_4 = \binom{6}{3} 2^3 (-3)^3 x^6 y^3$$

$\therefore$  the coefficient of  $x^6 y^3$  is  $\binom{6}{3} 2^3 (-3)^3 = -4320$ .

**d** In  $\left(2x^2 - \frac{1}{x}\right)^{12}$ ,  $a = (2x^2)$ ,  $b = \left(-\frac{1}{x}\right)$ , and  $n = 12$

$$\begin{aligned}\text{Now } T_{r+1} &= \binom{n}{r} a^{n-r} b^r \\ &= \binom{12}{r} (2x^2)^{12-r} \left(-\frac{1}{x}\right)^r \\ &= \binom{12}{r} 2^{12-r} x^{24-2r} \frac{(-1)^r}{x^r} \\ &= \binom{12}{r} 2^{12-r} (-1)^r x^{24-3r}\end{aligned}$$

We now let  $24 - 3r = 12$

$$\therefore 3r = 12$$

$$\therefore r = 4$$

$$\text{So, } T_5 = \binom{12}{4} 2^8 (-1)^4 x^{12}$$

$\therefore$  the coefficient of  $x^{12}$  is  $\binom{12}{4} 2^8 (-1)^4 = 126\,720$ .

- 9 In  $(x^2y - 2y^2)^6$ ,  $a = (x^2y)$ ,  $b = (-2y^2)$ , and  $n = 6$ .

$$\begin{aligned}\text{Now } T_{r+1} &= \binom{n}{r} a^{n-r} b^r \\ &= \binom{6}{r} (x^2y)^{6-r} (-2y^2)^r \\ &= \binom{6}{r} x^{12-2r} y^{6-r} (-2)^r y^{2r} \\ &= \binom{6}{r} (-2)^r x^{12-2r} y^{6+r}\end{aligned}$$

Since  $x$  and  $y$  are raised to the same power,

$$12 - 2r = 6 + r$$

$$\therefore 3r = 6$$

$$\therefore r = 2$$

$$\begin{aligned}T_3 &= \binom{6}{2} (-2)^2 x^8 y^8 \\ &= 60x^8 y^8\end{aligned}$$

- 10  $(1+x)^n$  has  $T_3 = \binom{n}{2} 1^{n-2} x^2 = \binom{n}{2} x^2$  and  $n \geq 2$

$$\text{But this term is } 36x^2 \therefore \binom{n}{2} = 36$$

$$\therefore \frac{n(n-1)}{2} = 36$$

$$\therefore n(n-1) = 72$$

$$\therefore n^2 - n - 72 = 0$$

$$\therefore (n-9)(n+8) = 0$$

$$\therefore n = 9 \text{ or } -8$$

$$\text{But } n \geq 2, \text{ so } n = 9$$

$$\text{and } T_4 = \binom{n}{3} 1^{n-3} x^3$$

$$\begin{aligned}\therefore T_4 &= \binom{9}{3} x^3 \\ &= 84x^3\end{aligned}$$

- 11 a  $T_{r+1} = \binom{n}{r} a^{n-r} b^r$  where  $n = 8$ ,  $a = (2x)$ ,  $b = a$

$$= \binom{8}{r} (2x)^{8-r} a^r$$

$$= \binom{8}{r} 2^{8-r} x^{8-r} a^r$$

$$\text{We let } 8 - r = 6 \text{ and } T_3 = \binom{8}{2} 2^6 x^6 a^2$$

$$\therefore r = 2$$

$$\text{So, } \binom{8}{2} 2^6 a^2 = 448$$

$$\therefore 1792a^2 = 448$$

$$\therefore a^2 = \frac{1}{4}$$

$$\therefore a = \pm \frac{1}{2}$$

$$\begin{aligned}
 \text{b } T_{r+1} &= \binom{n}{r} a^{n-r} b^r \quad \text{where } n = 8, \quad a = y, \quad b = \left(-\frac{a}{x}\right) \\
 &= \binom{8}{r} y^{8-r} \left(-\frac{a}{x}\right)^r \\
 &= \binom{8}{r} y^{8-r} \frac{(-a)^r}{x^r} \\
 &= \binom{8}{r} \frac{y^{8-r}}{x^r} (-a)^r
 \end{aligned}$$

$$\begin{aligned}
 \text{We let } 8 - r &= 2 \quad \text{and} \quad T_7 = \binom{8}{6} \frac{y^2}{x^6} (-a)^6 \\
 \therefore r &= 6
 \end{aligned}$$

$$\begin{aligned}
 \text{So, } \binom{8}{6} (-a)^6 &= 1792 \\
 \therefore 28a^6 &= 1792 \\
 \therefore a^6 &= 64 \\
 \therefore a &= \pm 2
 \end{aligned}$$

$$\begin{aligned}
 \text{c } T_{r+1} &= \binom{n}{r} a^{n-r} b^r \quad \text{where } n = 10, \quad a = (ax), \quad b = \sqrt{2} \\
 &= \binom{10}{r} (ax)^{10-r} (\sqrt{2})^r \\
 &= \binom{10}{r} a^{10-r} x^{10-r} 2^{\frac{1}{2}r}
 \end{aligned}$$

$$\begin{aligned}
 \text{We let } 10 - r &= 7 \quad \text{and} \quad T_4 = \binom{10}{3} a^7 x^7 2^{\frac{3}{2}} \\
 \therefore r &= 3
 \end{aligned}$$

$$\begin{aligned}
 \text{So, } \binom{10}{3} a^7 2^{\frac{3}{2}} &= 30 \\
 \therefore 120a^7 \times 2\sqrt{2} &= 30 \\
 \therefore a^7 &= \frac{1}{8\sqrt{2}} \\
 \therefore a &= \frac{1}{\sqrt{2}}
 \end{aligned}$$

$$\begin{aligned}
 \text{d } T_{r+1} &= \binom{n}{r} a^{n-r} b^r \quad \text{where } n = 10, \quad a = (x^2), \quad b = \left(\frac{1}{ax}\right) \\
 &= \binom{10}{r} (x^2)^{10-r} \left(\frac{1}{ax}\right)^r \\
 &= \binom{10}{r} x^{20-2r} \times \frac{1}{a^r x^r} \\
 &= \binom{10}{r} x^{20-3r} \times \frac{1}{a^r}
 \end{aligned}$$

$$\begin{aligned}
 \text{We let } 20 - 3r &= 11 \quad \text{and} \quad T_4 = \binom{10}{3} x^{11} \times \frac{1}{a^3} \\
 \therefore 3r &= 9 \\
 \therefore r &= 3 \qquad \qquad \qquad = \frac{\binom{10}{3}}{a^3} x^{11}
 \end{aligned}$$

$$\begin{aligned}
 \text{So, } \frac{\binom{10}{3}}{a^3} &= 15 \\
 \therefore \frac{120}{a^3} &= 15 \\
 \therefore a^3 &= 8 \\
 \therefore a &= 2
 \end{aligned}$$

**12 a**  $(x+4)(x-3)^6$

$$= (x+4)[x^6 + \binom{6}{1}x^5(-3) + \binom{6}{2}x^4(-3)^2 + \binom{6}{3}x^3(-3)^3 + \dots]$$

$$= (x+4)(x^6 - 3\binom{6}{1}x^5 + \binom{6}{2}(-3)^2x^4 + \binom{6}{3}(-3)^3x^3 + \dots)$$

So, the terms containing  $x^4$  are  $\binom{6}{3}(-3)^3x^4$  from (1)  
and  $4\binom{6}{2}(-3)^2x^4$  from (2)

$\therefore$  the coefficient of  $x^4$  is  $\binom{6}{3}(-3)^3 + 4\binom{6}{2}(-3)^2 = 0$

**b**  $(2-x)(3x+1)^9$

$$= (2-x)[(3x)^9 + \binom{9}{1}(3x)^8(1) + \binom{9}{2}(3x)^7(1)^2 + \binom{9}{3}(3x)^6(1)^3 + \binom{9}{4}(3x)^5(1)^4 + \dots]$$

$$= (2-x)(3^9x^9 + \binom{9}{1}3^8x^8 + \binom{9}{2}3^7x^7 + \binom{9}{3}3^6x^6 + \binom{9}{4}3^5x^5 + \dots)$$

So, the terms containing  $x^6$  are  $2\binom{9}{3}3^6x^6$  from (1)  
and  $-\binom{9}{4}3^5x^6$  from (2)

$\therefore$  the term containing  $x^6$  is  $2\binom{9}{3}3^6x^6 - \binom{9}{4}3^5x^6 = 91\,854x^6$ .

**c**  $(2+x)\left(\frac{1}{x} - 2x\right)^4$

$$= (2+x)\left[\left(\frac{1}{x}\right)^4 + \binom{4}{1}\left(\frac{1}{x}\right)^3(-2x) + \binom{4}{2}\left(\frac{1}{x}\right)^2(-2x)^2 + \binom{4}{3}\left(\frac{1}{x}\right)(-2x)^3 + (-2x)^4\right]$$

$$= (2+x)(x^{-4} + \binom{4}{1}(-2)x^{-2} + \binom{4}{2}(-2)^2 + \binom{4}{3}(-2)^3x^2 + (-2)^4x^4)$$

So, the term containing  $x$  is  $\binom{4}{2}(-2)^2x$

$\therefore$  the coefficient of  $x$  is  $4\binom{4}{2} = 24$ .

**d**  $(x-2)^2(1-2x)^3$

$$= (x^2 - 4x + 4)[1^3 + \binom{3}{1}(1)^2(-2x) + \binom{3}{2}(1)(-2x)^2 + (-2x)^3]$$

$$= (x^2 - 4x + 4)(1 + 3(-2)^1x + \binom{3}{2}(-2)^2x^2 + (-2)^3x^3)$$

So, the terms containing  $x^3$  are  $3(-2)^1x^3$  from (1)  
 $-4\binom{3}{2}(-2)^2x^3$  from (2)  
and  $4(-2)^3x^3$  from (3)

$\therefore$  the coefficient of  $x^3$  is  $3(-2)^1 - 4\binom{3}{2}(-2)^2 + 4(-2)^3 = -86$ .



$$\begin{aligned}
 \text{e} \quad & (x+1)^3 \left(\frac{1}{x} - 2x\right)^6 \\
 &= (x^3 + 3x^2 + 3x + 1) \left[ \left(\frac{1}{x}\right)^6 + \binom{6}{1} \left(\frac{1}{x}\right)^5 (-2x) + \binom{6}{2} \left(\frac{1}{x}\right)^4 (-2x)^2 + \dots \right] \\
 &= (x^3 + 3x^2 + 3x + 1) (x^{-6} + \binom{6}{1} (-2)x^{-4} + \binom{6}{2} (-2)^2 x^{-2} + \dots)
 \end{aligned}$$

So, the terms containing  $x^{-2}$  are  $3 \binom{6}{1} (-2)x^{-2}$  from (1)

and  $\binom{6}{2} (-2)^2 x^{-2}$  from (2)

$\therefore$  the coefficient of  $x^{-2}$  is  $3 \binom{6}{1} (-2) + \binom{6}{2} (-2)^2 = 24$ .

$$\begin{aligned}
 \text{f} \quad & \left(x^2 + \frac{1}{x}\right)^3 \left(x - \frac{2}{x}\right)^4 \\
 &= \left(x^6 + 3(x^2)^2 \left(\frac{1}{x}\right) + 3x^2 \left(\frac{1}{x}\right)^2 + \left(\frac{1}{x}\right)^3\right) \left[x^4 + \binom{4}{1} x^3 \left(-\frac{2}{x}\right) + \binom{4}{2} x^2 \left(-\frac{2}{x}\right)^2 \right. \\
 &\quad \left. + \binom{4}{3} x \left(-\frac{2}{x}\right)^3 + \left(-\frac{2}{x}\right)^4\right] \\
 &= (x^6 + 3x^3 + 3 + x^{-3}) [x^4 + \binom{4}{1} (-2)x^2 + \binom{4}{2} (-2)^2 + \binom{4}{3} (-2)^3 x^{-2} + (-2)^4 x^{-4}]
 \end{aligned}$$

So, the constant term is  $3 \binom{4}{2} (-2)^2 = 72$ .

$$\begin{aligned}
 \text{13} \quad & (1+kx)^n = 1^n + \binom{n}{1} 1^{n-1} (kx) + \binom{n}{2} 1^{n-2} (kx)^2 + \dots \\
 &= 1 + \binom{n}{1} kx + \binom{n}{2} k^2 x^2 + \dots
 \end{aligned}$$

$$\therefore \binom{n}{1} k = -12 \quad \text{and} \quad \binom{n}{2} k^2 = 60$$

$$\therefore nk = -12 \quad \text{and} \quad \frac{n(n-1)}{2} k^2 = 60$$

$$\therefore n(n-1)k^2 = 120$$

$$\text{But } k = -\frac{12}{n} \quad \therefore n(n-1) \frac{144}{n^2} = 120$$

$$\therefore 144(n-1) = 120n \quad \{n \geq 2\}$$

$$\therefore 144n - 120n = 144$$

$$\therefore 24n = 144$$

$$\therefore n = 6 \quad \text{and so } k = -2$$

$$\begin{array}{rcl}
 \text{14} \quad \text{a} & \begin{array}{cccc} 1 & 1 & & \\ 1 & 2 & 1 & \\ 1 & 3 & 3 & 1 \\ 1 & 4 & 6 & 4 & 1 \\ 1 & 5 & 10 & 10 & 5 & 1 \end{array} & \begin{array}{l} \leftarrow \text{row 1} \\ \leftarrow \text{row 2} \\ \leftarrow \text{row 3} \\ \leftarrow \text{row 4} \\ \leftarrow \text{row 5} \end{array}
 \end{array}$$

$$\begin{array}{rcl}
 \text{b i} & \text{sum} = 1 + 1 & = 2 = 2^1 \\
 \text{ii} & \text{sum} = 1 + 2 + 1 & = 4 = 2^2 \\
 \text{iii} & \text{sum} = 1 + 3 + 3 + 1 & = 8 = 2^3 \\
 \text{iv} & \text{sum} = 1 + 4 + 6 + 4 + 1 & = 16 = 2^4 \\
 \text{v} & \text{sum} = 1 + 5 + 10 + 10 + 5 + 1 & = 32 = 2^5
 \end{array}$$

**c** The sum of the numbers in row  $n$  of Pascal's triangle is  $2^n$ .

$$\begin{aligned}
 \text{d} \quad & (1+x)^n \\
 &= \binom{n}{0} 1^n + \binom{n}{1} 1^{n-1} x + \binom{n}{2} 1^{n-2} x^2 + \dots + \binom{n}{n-1} 1^1 x^{n-1} + \binom{n}{n} x^n \\
 &= \binom{n}{0} + \binom{n}{1} x + \binom{n}{2} x^2 + \dots + \binom{n}{n-1} x^{n-1} + \binom{n}{n} x^n \\
 &\quad \{\text{as all powers of 1 are 1}\} \quad \checkmark
 \end{aligned}$$

**e i** Letting  $x = 1$  in **d**, LHS  $= (1 + 1)^n = 2^n$

$$\text{and RHS} = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \dots + \binom{n}{n-1} + \binom{n}{n}$$

$$\therefore \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n-1} + \binom{n}{n} = 2^n \quad \checkmark$$

**ii** Letting  $x = -1$  in **d** gives LHS  $= (1 + (-1))^n = 0$

$$\begin{aligned} \text{and RHS} &= \binom{n}{0} + \binom{n}{1}(-1) + \binom{n}{2}(-1)^2 + \binom{n}{3}(-1)^3 + \dots \\ &\quad + \binom{n}{n-1}(-1)^{n-1} + \binom{n}{n}(-1)^n \\ &= \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \dots + (-1)^n \binom{n}{n} \end{aligned}$$

$$\therefore \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \dots + (-1)^n \binom{n}{n} = 0 \quad \checkmark$$

**iii** Letting  $x = 1$  and  $n = 2n + 1$  in **d**,

$$\text{LHS} = 2^{2n+1} = 2^{2n} \times 2^1 = 4^n \times 2$$

$$\begin{aligned} \text{and RHS} &= \binom{2n+1}{0} + \binom{2n+1}{1} + \binom{2n+1}{2} + \dots + \binom{2n+1}{2n} + \binom{2n+1}{2n+1} \\ &= 2 \left[ \binom{2n+1}{0} + \binom{2n+1}{1} + \binom{2n+1}{2} + \dots + \binom{2n+1}{n} \right] \end{aligned}$$

$$\left\{ \binom{2n+1}{2n+1} = \binom{2n+1}{0}, \binom{2n+1}{2n} = \binom{2n+1}{1}, \dots, \binom{2n+1}{n+1} = \binom{2n+1}{n} \right\}$$

$$\therefore 2 \left[ \binom{2n+1}{0} + \binom{2n+1}{1} + \binom{2n+1}{2} + \dots + \binom{2n+1}{n} \right] = 4^n \times 2$$

$$\therefore \binom{2n+1}{0} + \binom{2n+1}{1} + \binom{2n+1}{2} + \dots + \binom{2n+1}{n} = 4^n \quad \checkmark$$

**f** 
$$\sum_{r=0}^n 2^r \binom{n}{r} = 2^0 \binom{n}{0} + 2^1 \binom{n}{1} + 2^2 \binom{n}{2} + \dots + 2^{n-1} \binom{n}{n-1} + 2^n \binom{n}{n}$$

Using **d**,  $(1 + x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \dots + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n$

Letting  $x = 2$ ,  $(1 + 2)^n = \binom{n}{0} + \binom{n}{1}2 + \binom{n}{2}2^2 + \dots + \binom{n}{n-1}2^{n-1} + \binom{n}{n}2^n$

$$\therefore 3^n = 2^0 \binom{n}{0} + 2^1 \binom{n}{1} + 2^2 \binom{n}{2} + \dots + 2^{n-1} \binom{n}{n-1} + 2^n \binom{n}{n}$$

$$\therefore \sum_{r=0}^n 2^r \binom{n}{r} = 3^n$$

**15 a**  $(3 + x)^n = 3^n + \binom{n}{1}3^{n-1}x + \binom{n}{2}3^{n-2}x^2 + \binom{n}{3}3^{n-3}x^3 + \dots + \binom{n}{n-1}3x^{n-1} + x^n$

**b** Letting  $x = 1$  in **a**, LHS  $= (3 + 1)^n = 4^n$

$$\text{and RHS} = 3^n + \binom{n}{1}3^{n-1} + \binom{n}{2}3^{n-2} + \binom{n}{3}3^{n-3} + \dots + 3n + 1$$

$$\therefore 3^n + \binom{n}{1}3^{n-1} + \binom{n}{2}3^{n-2} + \binom{n}{3}3^{n-3} + \dots + 3n + 1 = 4^n$$

**16**  $(1 + 2x - x^2)^5$

$$= ([1 + 2x] - x^2)^5$$

$$= (1 + 2x)^5 + 5(1 + 2x)4(-x^2) + 10(1 + 2x)^3(-x^2)^2 + \dots$$

{all further terms contain higher powers of  $x$  than  $x^4$ }

$$= 1^5 + 5(1)^4(2x) + 10(1)^3(2x)^2 + 10(1)^2(2x)^3 + 5(1)(2x)^4 + \dots$$

$$- 5x^2(1^4 + 4(1)^3(2x) + 6(1)^2(2x)^2 + \dots) + 10x^4(1^3 + \dots) + \dots$$

$$= 1 + 10x + 40x^2 + 80x^3 + 80x^4 - 5x^2 - 40x^3 - 120x^4 + 10x^4 + \dots$$

$$= 1 + 10x + 35x^2 + 40x^3 - 30x^4 + \dots$$

$$\begin{aligned}
17 \quad \binom{n}{r} + \binom{n}{r+1} &= \frac{n!}{r!(n-r)!} + \frac{n!}{(r+1)!(n-r-1)!} \\
&= \frac{(r+1)n!}{(r+1)!(n-r)!} + \frac{(n-r)n!}{(r+1)!(n-r)!} \\
&= \frac{\cancel{r \times n!} + n! + n \times n! - \cancel{r \times n!}}{(r+1)!(n-r)!} \\
&= \frac{(n+1)n!}{(r+1)!(n-r)!} \\
&= \frac{(n+1)!}{(r+1)!(n-r)!} \\
&= \frac{(n+1)!}{(r+1)!(n+1-[r+1])!} \\
&= \binom{n+1}{r+1} \quad \text{for all } n \in \mathbb{Z}^+, r \in \mathbb{N}, r \leq n
\end{aligned}$$

This means that when we expand  $(a+b)(a+b)^n$  and collect like terms, we can use Pascal's rule  $\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$  to simplify the coefficients.

18 For any polynomial  $f(x)$ , the sum of its coefficients is  $f(1)$ .

$$\text{Let } f(x) = x^3 + 2x^2 + 3x - 7$$

$$\begin{aligned}
\therefore \text{ the sum of the coefficients of } f(x) &= f(1) \\
&= 1^3 + 2(1)^2 + 3(1) - 7 \\
&= 1 + 2 + 3 - 7 = -1
\end{aligned}$$

$$\begin{aligned}
\text{Now consider the function } g(x) &= (x^3 + 2x^2 + 3x - 7)^{100} \\
&= [f(x)]^{100}
\end{aligned}$$

$$\begin{aligned}
\text{The sum of the coefficients of } g(x) &= g(1) \\
&= [f(1)]^{100} \\
&= (-1)^{100} = 1
\end{aligned}$$

$\therefore$  the sum of the coefficients of  $(x^3 + 2x^2 + 3x - 7)^{100}$  is 1.

$$\begin{aligned}
19 \quad (1+x)^n &= \binom{n}{0} + \binom{n}{1}x + \dots + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n \\
\therefore (1+x)^{2n} &= \binom{2n}{0} + \binom{2n}{1}x + \dots + \binom{2n}{n-1}x^{n-1} + \binom{2n}{n}x^n + \dots + \binom{2n}{2n-1}x^{2n-1} + \binom{2n}{2n}x^{2n}
\end{aligned}$$

$$\text{Now } (1+x)^n(1+x)^n = (1+x)^{2n}$$

$$\begin{aligned}
\therefore & \left[ \binom{n}{0} + \binom{n}{1}x + \dots + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n \right] \left[ \binom{n}{0} + \binom{n}{1}x + \dots + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n \right] \\
&= \binom{2n}{0} + \binom{2n}{1}x + \dots + \binom{2n}{n-1}x^{n-1} + \binom{2n}{n}x^n + \dots + \binom{2n}{2n-1}x^{2n-1} + \binom{2n}{2n}x^{2n}
\end{aligned}$$

$$\text{Equating coefficients of } x^n, \quad \binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \dots + \binom{n}{n-1}\binom{n}{1} + \binom{n}{n}\binom{n}{0} = \binom{2n}{n}$$

But  $\binom{n}{n} = \binom{n}{0}$ ,  $\binom{n}{n-1} = \binom{n}{1}$ , and so on.

$$\therefore \binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \dots + \binom{n}{n-1}^2 + \binom{n}{n}^2 = \binom{2n}{n}$$

$$\begin{aligned}
 \text{20 a } n \binom{n-1}{r-1} &= n \frac{(n-1)!}{(r-1)!(n-1-[r-1])!} \\
 &= \frac{n \times (n-1)!}{(r-1)!(n-r)!} \\
 &= r \times \frac{n!}{r!(n-r)!} \\
 &= r \binom{n}{r}
 \end{aligned}$$

$$\begin{aligned}
 \text{b } &\binom{n}{1} + 2 \binom{n}{2} + 3 \binom{n}{3} + \dots + n \binom{n}{n} \\
 &= n \binom{n-1}{0} + n \binom{n-1}{1} + n \binom{n-1}{2} + \dots + n \binom{n-1}{n-1} \quad \{\text{using a}\} \\
 &= n \left[ \binom{n-1}{0} + \binom{n-1}{1} + \binom{n-1}{2} + \dots + \binom{n-1}{n-1} \right] \\
 &= n 2^{n-1} \quad \{\text{using 14 e i}\}
 \end{aligned}$$

$$\begin{aligned}
 \text{c i } &\sum_{r=0}^n P_r \\
 &= P_0 + P_1 + P_2 + \dots + P_n \\
 &= \binom{n}{0} p^0 (1-p)^n + \binom{n}{1} p^1 (1-p)^{n-1} + \binom{n}{2} p^2 (1-p)^{n-2} + \dots + \binom{n}{n} p^n (1-p)^0 \\
 &= [p + (1-p)]^n \quad \{\text{binomial expansion}\} \\
 &= 1^n = 1
 \end{aligned}$$

$$\begin{aligned}
 \text{ii } &\sum_{r=1}^n r P_r \\
 &= 1P_1 + 2P_2 + 3P_3 + \dots + nP_n \\
 &= 1 \binom{n}{1} p^1 (1-p)^{n-1} + 2 \binom{n}{2} p^2 (1-p)^{n-2} + 3 \binom{n}{3} p^3 (1-p)^{n-3} + \dots \\
 &\quad + n \binom{n}{n} p^n (1-p)^0 \\
 &= n \binom{n-1}{0} p^1 (1-p)^{n-1} + n \binom{n-1}{1} p^2 (1-p)^{n-2} + n \binom{n-1}{2} p^3 (1-p)^{n-3} + \dots \\
 &\quad + n \binom{n-1}{n-1} p^n \quad \{\text{using a}\} \\
 &= np \left[ \binom{n-1}{0} p^0 (1-p)^{n-1} + \binom{n-1}{1} p^1 (1-p)^{n-2} + \binom{n-1}{2} p^2 (1-p)^{n-3} + \dots \right. \\
 &\quad \left. + \binom{n-1}{n-1} p^{n-1} \right] \\
 &= np [(p + (1-p))^{n-1}] \\
 &= np \times 1^{n-1} \\
 &= np
 \end{aligned}$$



## EXERCISE 8C

**1 a**  $\binom{\frac{1}{2}}{1}$  has  $n = \frac{1}{2}$ ,  $r = 1$

So,  $n - r + 1 = \frac{1}{2}$

$$\begin{aligned}\therefore \binom{\frac{1}{2}}{1} &= \frac{\frac{1}{2}}{1!} \\ &= \frac{1}{2}\end{aligned}$$

**b**  $\binom{-\frac{1}{2}}{2}$  has  $n = -\frac{1}{2}$ ,  $r = 2$

So,  $n - r + 1 = -\frac{3}{2}$

$$\begin{aligned}\therefore \binom{-\frac{1}{2}}{2} &= \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!} \\ &= \frac{\frac{3}{4}}{2} \\ &= \frac{3}{8}\end{aligned}$$

**c**  $\binom{-2}{2}$  has  $n = -2$ ,  $r = 2$

So,  $n - r + 1 = -3$

$$\begin{aligned}\therefore \binom{-2}{2} &= \frac{(-2)(-3)}{2!} \\ &= \frac{6}{2} \\ &= 3\end{aligned}$$

**d**  $\binom{-\frac{1}{3}}{0} = 1$   $\{ \binom{n}{0} = 1 \text{ for any } n \in \mathbb{Q} \}$

**e**  $\binom{-1}{3}$  has  $n = -1$ ,  $r = 3$

So,  $n - r + 1 = -3$

$$\begin{aligned}\therefore \binom{-1}{3} &= \frac{(-1)(-2)(-3)}{3!} \\ &= \frac{-6}{6} \\ &= -1\end{aligned}$$

**f**  $\binom{\frac{1}{3}}{3}$  has  $n = \frac{1}{3}$ ,  $r = 3$

So,  $n - r + 1 = -\frac{5}{3}$

$$\begin{aligned}\therefore \binom{\frac{1}{3}}{3} &= \frac{(\frac{1}{3})(-\frac{2}{3})(-\frac{5}{3})}{3!} \\ &= \frac{\frac{10}{27}}{6} \\ &= \frac{5}{81}\end{aligned}$$

**2 a**  $\frac{1}{1+x} = (1+x)^{-1}$

$$\begin{aligned}&= \sum_{r=0}^{\infty} \binom{-1}{r} x^r \\ &= \binom{-1}{0} x^0 + \binom{-1}{1} x^1 + \binom{-1}{2} x^2 + \binom{-1}{3} x^3 + \dots \\ &= 1 + (-1)x + \frac{(-1)(-2)}{2!} x^2 + \frac{(-1)(-2)(-3)}{3!} x^3 + \dots \\ &= 1 - x + x^2 - x^3 + \dots\end{aligned}$$

**b** The series converges provided  $|x| < 1$ , which is the interval  $-1 < x < 1$ .

**c** Letting  $x = 0.1$ ,  $\frac{1}{1.1} \approx 1 - 0.1 + (0.1)^2 - (0.1)^3$   
 $\approx 0.909$

Using technology,  $\frac{1}{1.1} = 0.\overline{90}$

$$\begin{aligned}
3 \quad a \quad \frac{1}{\sqrt{4-x}} &= (4-x)^{-\frac{1}{2}} \\
&= 4^{-\frac{1}{2}} \left(1 - \frac{x}{4}\right)^{-\frac{1}{2}} \\
&= \frac{1}{2} \left(1 - \frac{x}{4}\right)^{-\frac{1}{2}} \\
&= \frac{1}{2} \sum_{r=0}^{\infty} \binom{-\frac{1}{2}}{r} \left(-\frac{x}{4}\right)^r \\
&= \frac{1}{2} \left( \binom{-\frac{1}{2}}{0} \left(-\frac{x}{4}\right)^0 + \binom{-\frac{1}{2}}{1} \left(-\frac{x}{4}\right)^1 + \binom{-\frac{1}{2}}{2} \left(-\frac{x}{4}\right)^2 + \binom{-\frac{1}{2}}{3} \left(-\frac{x}{4}\right)^3 + \dots \right) \\
&= \frac{1}{2} \left( 1 + \left(-\frac{1}{2}\right) \left(-\frac{x}{4}\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!} \left(-\frac{x}{4}\right)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!} \left(-\frac{x}{4}\right)^3 + \dots \right) \\
&= \frac{1}{2} \left( 1 + \frac{1}{8}x + \frac{3}{128}x^2 + \frac{5}{1024}x^3 + \dots \right) \\
&= \frac{1}{2} + \frac{1}{16}x + \frac{3}{256}x^2 + \frac{5}{2048}x^3 + \dots
\end{aligned}$$

**b** The series converges provided  $\left| -\frac{x}{4} \right| < 1$ , which is the interval  $-4 < x < 4$ .

$$\begin{aligned}
c \quad \text{Letting } x = 0.4, \quad \frac{1}{\sqrt{3.6}} &\approx \frac{1}{2} + \frac{1}{16}(0.4) + \frac{3}{256}(0.4)^2 + \frac{5}{2048}(0.4)^3 \\
&\approx 0.527031
\end{aligned}$$

$$\text{Using technology, } \frac{1}{\sqrt{3.6}} \approx 0.527046$$

$$\begin{aligned}
4 \quad a \quad \sqrt[3]{1+3x} &= (1+3x)^{\frac{1}{3}} \\
&= \sum_{r=0}^{\infty} \binom{\frac{1}{3}}{r} (3x)^r \\
&= \binom{\frac{1}{3}}{0} (3x)^0 + \binom{\frac{1}{3}}{1} (3x)^1 + \binom{\frac{1}{3}}{2} (3x)^2 + \binom{\frac{1}{3}}{3} (3x)^3 + \binom{\frac{1}{3}}{4} (3x)^4 + \dots \\
&= 1 + \frac{1}{3}(3x) + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2!} (3x)^2 + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{3!} (3x)^3 + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)\left(-\frac{8}{3}\right)}{4!} (3x)^4 + \dots \\
&= 1 + x - x^2 + \frac{5}{3}x^3 - \frac{10}{3}x^4 + \dots
\end{aligned}$$

**b** The series converges provided  $|3x| < 1$ , which is the interval  $-\frac{1}{3} < x < \frac{1}{3}$ .

$$\begin{aligned}
c \quad i \quad \text{Letting } x = 0.1, \quad \sqrt[3]{1.3} &\approx 1 + 0.1 - (0.1)^2 + \frac{5}{3}(0.1)^3 \\
&\approx 1.091667
\end{aligned}$$

$$\begin{aligned}
ii \quad \text{Letting } x = 0.1, \quad \sqrt[3]{1.3} &\approx 1 + 0.1 - (0.1)^2 + \frac{5}{3}(0.1)^3 - \frac{10}{3}(0.1)^4 \\
&\approx 1.091333
\end{aligned}$$

$$d \quad \text{Using technology, } \sqrt[3]{1.3} \approx 1.091393$$

$$\begin{aligned}
 \mathbf{5} \quad \mathbf{a} \quad \frac{1}{(1-2x)^2} &= (1-2x)^{-2} \\
 &= \sum_{r=0}^{\infty} \binom{-2}{r} (-2x)^r \\
 &= \binom{-2}{0} (-2x)^0 + \binom{-2}{1} (-2x)^1 + \binom{-2}{2} (-2x)^2 + \binom{-2}{3} (-2x)^3 + \dots \\
 &= 1 + (-2)(-2x) + \frac{(-2)(-3)}{2!} (-2x)^2 + \frac{(-2)(-3)(-4)}{3!} (-2x)^3 + \dots \\
 &= 1 + 4x + 12x^2 + 32x^3 + \dots
 \end{aligned}$$

**b** The series converges provided  $|-2x| < 1$ , which is the interval  $-\frac{1}{2} < x < \frac{1}{2}$ .

$$\begin{aligned}
 \mathbf{c} \quad \text{Letting } x = 0.02, \quad \frac{1}{(0.96)^2} &\approx 1 + 4(0.02) + 12(0.02)^2 + 32(0.02)^3 \\
 &\approx 1.085\,056
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{6} \quad \mathbf{a} \quad \sqrt{1+x} &= (1+x)^{\frac{1}{2}} \\
 &= \sum_{r=0}^{\infty} \binom{\frac{1}{2}}{r} x^r \\
 &= \binom{\frac{1}{2}}{0} x^0 + \binom{\frac{1}{2}}{1} x^1 + \binom{\frac{1}{2}}{2} x^2 + \binom{\frac{1}{2}}{3} x^3 + \dots \\
 &= 1 + \frac{1}{2}x + \frac{(\frac{1}{2})(-\frac{1}{2})}{2!} x^2 + \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})}{3!} x^3 + \dots \\
 &= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \dots
 \end{aligned}$$

**b** The series converges provided  $|x| < 1$ , which is the interval  $-1 < x < 1$ .

**c** We cannot estimate  $\sqrt{37}$  by substituting  $x = 36$  since the series only converges provided  $-1 < x < 1$ .

$$\begin{aligned}
 \mathbf{d} \quad \sqrt{37} &= \sqrt{36} \times \sqrt{\frac{37}{36}} \\
 &= \sqrt{36} \times \sqrt{1 + \frac{1}{36}} \\
 &\approx 6 \times \left(1 + \frac{1}{2}\left(\frac{1}{36}\right) - \frac{1}{8}\left(\frac{1}{36}\right)^2 + \frac{1}{16}\left(\frac{1}{36}\right)^3\right) \\
 &\approx 6.082\,763
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{7} \quad \mathbf{a} \quad (1+x)^{\frac{1}{5}} &= \sum_{r=0}^{\infty} \binom{\frac{1}{5}}{r} x^r \\
 &= \binom{\frac{1}{5}}{0} x^0 + \binom{\frac{1}{5}}{1} x^1 + \binom{\frac{1}{5}}{2} x^2 + \binom{\frac{1}{5}}{3} x^3 + \dots \\
 &= 1 + \frac{1}{5}x + \frac{(\frac{1}{5})(-\frac{4}{5})}{2!} x^2 + \frac{(\frac{1}{5})(-\frac{4}{5})(-\frac{9}{5})}{3!} x^3 + \dots \\
 &= 1 + \frac{1}{5}x - \frac{2}{25}x^2 + \frac{6}{125}x^3 - \dots
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad \text{Letting } x = 0.01, \quad (1.01)^{\frac{1}{5}} &\approx 1 + \frac{1}{5}(0.01) - \frac{2}{25}(0.01)^2 + \frac{6}{125}(0.01)^3 \\
 &\approx 1.001\,992
 \end{aligned}$$

**c** The series only converges provided  $-1 < x < 1$ , but to estimate  $\sqrt[5]{2}$  we would need to substitute  $x = 1$ .

$$\begin{aligned}
 \text{8 a } \frac{1}{(1+2x)^2} &= (1+2x)^{-2} \\
 &= \sum_{r=0}^{\infty} \binom{-2}{r} (2x)^r \\
 &= \binom{-2}{0} (2x)^0 + \binom{-2}{1} (2x)^1 + \binom{-2}{2} (2x)^2 + \binom{-2}{3} (2x)^3 + \dots \\
 &= 1 + (-2)(2x) + \frac{(-2)(-3)}{2!} (2x)^2 + \frac{(-2)(-3)(-4)}{3!} (2x)^3 + \dots \\
 &= 1 - 4x + 12x^2 - 32x^3 + \dots
 \end{aligned}$$

$$\begin{aligned}
 \text{b } \frac{(1-3x)^3}{(1+2x)^2} &= (1-3x)^3 (1-4x+12x^2-32x^3+\dots) \quad \{\text{from a}\} \\
 &= (1-9x+27x^2-27x^3)(1-4x+12x^2-32x^3+\dots)
 \end{aligned}$$

So, the terms containing  $x^2$  are  $12x^2$  from (1)  
 $36x^2$  from (2)  
 and  $27x^2$  from (3)

$\therefore$  the coefficient of  $x^2$  is  $12 + 36 + 27 = 75$ .

$$\text{9 a } \frac{x-8}{x^2-x-2} = \frac{x-8}{(x+1)(x-2)}$$

$$\text{Let } \frac{x-8}{x^2-x-2} = \frac{A}{x+1} + \frac{B}{x-2}$$

$$\therefore x-8 = A(x-2) + B(x+1)$$

$$\text{Letting } x=2, \quad 2-8 = B(2+1)$$

$$\therefore -6 = 3B$$

$$\therefore B = -2$$

$$\text{Letting } x=-1, \quad -1-8 = A(-1-2)$$

$$\therefore -9 = -3A$$

$$\therefore A = 3$$

$$\therefore \frac{x-8}{x^2-x-2} = \frac{3}{x+1} + \frac{-2}{x-2}$$



$$\begin{aligned}
\text{b } \frac{x-8}{x^2-x-2} &= \frac{3}{x+1} + \frac{-2}{x-2} \times \frac{-\frac{1}{2}}{-\frac{1}{2}} \quad \{\text{from a}\} \\
&= \frac{3}{x+1} + \frac{1}{1-\frac{x}{2}} \\
&= 3(1+x)^{-1} + \left(1 - \frac{x}{2}\right)^{-1} \\
&= 3 \sum_{r=0}^{\infty} \binom{-1}{r} x^r + \sum_{r=0}^{\infty} \binom{-1}{r} \left(-\frac{x}{2}\right)^r \\
&= 3 \left[ \binom{-1}{0} x^0 + \binom{-1}{1} x^1 + \binom{-1}{2} x^2 + \binom{-1}{3} x^3 + \dots \right] \\
&\quad + \left[ \binom{-1}{0} \left(-\frac{x}{2}\right)^0 + \binom{-1}{1} \left(-\frac{x}{2}\right)^1 + \binom{-1}{2} \left(-\frac{x}{2}\right)^2 + \binom{-1}{3} \left(-\frac{x}{2}\right)^3 + \dots \right] \\
&= 3 \left[ 1 + (-1)x + \frac{(-1)(-2)}{2!} x^2 + \frac{(-1)(-2)(-3)}{3!} x^3 + \dots \right] \\
&\quad + \left[ 1 + (-1) \left(-\frac{x}{2}\right) + \frac{(-1)(-2)}{2!} \left(-\frac{x}{2}\right)^2 + \frac{(-1)(-2)(-3)}{3!} \left(-\frac{x}{2}\right)^3 + \dots \right] \\
&= 3[1 - x + x^2 - x^3 + \dots] + \left[1 + \frac{1}{2}x + \frac{1}{4}x^2 + \frac{1}{8}x^3 + \dots\right] \\
&= [3 - 3x + 3x^2 - 3x^3 + \dots] + \left[1 + \frac{1}{2}x + \frac{1}{4}x^2 + \frac{1}{8}x^3 + \dots\right] \\
&= 4 - \frac{5}{2}x + \frac{13}{4}x^2 - \frac{23}{8}x^3 + \dots
\end{aligned}$$

c The series expansion for  $\frac{3}{x+1} = 3(1+x)^{-1}$  converges provided  $|x| < 1$ .

The series expansion for  $\frac{-2}{x-2} = (1 - \frac{1}{2}x)^{-1}$  converges provided  $|x| < 2$ .

$\therefore$  the complete expansion of  $\frac{x-8}{x^2-x-2}$  converges provided  $|x| < 1$ , which is the interval  $-1 < x < 1$ .

$$\begin{aligned}
\text{d } \text{When } x = 0.05, \quad \frac{x-8}{x^2-x-2} &\approx 4 - \frac{5}{2}(0.05) + \frac{13}{4}(0.05)^2 - \frac{23}{8}(0.05)^3 \\
&\approx 3.882766
\end{aligned}$$

$$\begin{aligned}
\text{10 a i } (1+3x)^{\frac{1}{4}} &= \sum_{r=0}^{\infty} \binom{\frac{1}{4}}{r} (3x)^r \\
&= \binom{\frac{1}{4}}{0} (3x)^0 + \binom{\frac{1}{4}}{1} (3x)^1 + \binom{\frac{1}{4}}{2} (3x)^2 + \binom{\frac{1}{4}}{3} (3x)^3 + \dots \\
&= 1 + \left(\frac{1}{4}\right)(3x) + \frac{\left(\frac{1}{4}\right)\left(-\frac{3}{4}\right)}{2!} (3x)^2 + \frac{\left(\frac{1}{4}\right)\left(-\frac{3}{4}\right)\left(-\frac{7}{4}\right)}{3!} (3x)^3 + \dots \\
&= 1 + \frac{3}{4}x - \frac{27}{32}x^2 + \frac{189}{128}x^3 - \dots \quad \text{provided } |3x| < 1, \text{ or } -\frac{1}{3} < x < \frac{1}{3}
\end{aligned}$$

$$\begin{aligned}
 \text{ii} \quad & (2-x)^{-\frac{1}{4}} \\
 &= 2^{-\frac{1}{4}} \left(1 - \frac{x}{2}\right)^{-\frac{1}{4}} \\
 &= \frac{1}{\sqrt[4]{2}} \left(1 - \frac{x}{2}\right)^{-\frac{1}{4}} \\
 &= \frac{1}{\sqrt[4]{2}} \sum_{r=0}^{\infty} \binom{-\frac{1}{4}}{r} \left(-\frac{x}{2}\right)^r \\
 &= \frac{1}{\sqrt[4]{2}} \left( \binom{-\frac{1}{4}}{0} \left(-\frac{x}{2}\right)^0 + \binom{-\frac{1}{4}}{1} \left(-\frac{x}{2}\right)^1 + \binom{-\frac{1}{4}}{2} \left(-\frac{x}{2}\right)^2 + \binom{-\frac{1}{4}}{3} \left(-\frac{x}{2}\right)^3 + \dots \right) \\
 &= \frac{1}{\sqrt[4]{2}} \left( 1 + \left(-\frac{1}{4}\right) \left(-\frac{x}{2}\right) + \frac{\left(-\frac{1}{4}\right)\left(-\frac{5}{4}\right)}{2!} \left(-\frac{x}{2}\right)^2 + \frac{\left(-\frac{1}{4}\right)\left(-\frac{5}{4}\right)\left(-\frac{9}{4}\right)}{3!} \left(-\frac{x}{2}\right)^3 + \dots \right) \\
 &= \frac{1}{\sqrt[4]{2}} \left( 1 + \frac{1}{8}x + \frac{5}{128}x^2 + \frac{15}{1024}x^3 + \dots \right) \quad \text{provided } \left| -\frac{x}{2} \right| < 1, \text{ or } -2 < x < 2
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & \sqrt[4]{\frac{1+3x}{2-x}} \\
 &= \frac{(1+3x)^{\frac{1}{4}}}{(2-x)^{\frac{1}{4}}} \\
 &= (2-x)^{-\frac{1}{4}} (1+3x)^{\frac{1}{4}} \\
 &= \frac{1}{\sqrt[4]{2}} \left( 1 + \frac{1}{8}x + \frac{5}{128}x^2 + \frac{15}{1024}x^3 + \dots \right) \left( 1 + \frac{3}{4}x - \frac{27}{32}x^2 + \frac{189}{128}x^3 - \dots \right) \quad \{\text{using a}\} \\
 &= \frac{1}{\sqrt[4]{2}} \left( 1 + \frac{3}{4}x - \frac{27}{32}x^2 + \frac{189}{128}x^3 + \frac{1}{8}x + \frac{3}{32}x^2 - \frac{27}{256}x^3 + \frac{5}{128}x^2 + \frac{15}{512}x^3 + \frac{15}{1024}x^3 + \dots \right) \\
 &= \frac{1}{\sqrt[4]{2}} \left( 1 + \frac{7}{8}x - \frac{91}{128}x^2 + \frac{1449}{1024}x^3 + \dots \right)
 \end{aligned}$$

$$\begin{aligned}
 \text{11 a} \quad & f(x) = \frac{1}{\sqrt[4]{1+x}} \\
 &= (1+x)^{-\frac{1}{4}} \\
 &= \sum_{r=0}^{\infty} \binom{-\frac{1}{4}}{r} x^r \\
 &= \binom{-\frac{1}{4}}{0} x^0 + \binom{-\frac{1}{4}}{1} x^1 + \binom{-\frac{1}{4}}{2} x^2 + \binom{-\frac{1}{4}}{3} x^3 + \dots \\
 &= 1 + \left(-\frac{1}{4}\right)x + \frac{\left(-\frac{1}{4}\right)\left(-\frac{5}{4}\right)}{2!} x^2 + \frac{\left(-\frac{1}{4}\right)\left(-\frac{5}{4}\right)\left(-\frac{9}{4}\right)}{3!} x^3 + \dots \\
 &= 1 - \frac{1}{4}x + \frac{5}{32}x^2 - \frac{15}{128}x^3 + \dots
 \end{aligned}$$

**b** The series converges provided  $|x| < 1$ , which is the interval  $-1 < x < 1$ .

$$\begin{aligned}
 \text{c} \quad f(x) &= \frac{1}{\sqrt[4]{1+x}} \\
 &= 1 - \frac{1}{4}x + \frac{5}{32}x^2 - \frac{15}{128}x^3 + \dots \\
 \text{So, } \frac{1}{\sqrt[4]{1+3x^2}} &= f(3x^2) \\
 &= 1 - \frac{1}{4}(3x^2) + \frac{5}{32}(3x^2)^2 - \frac{15}{128}(3x^2)^3 + \dots \\
 &= 1 - \frac{3}{4}x^2 + \frac{45}{32}x^4 - \frac{405}{128}x^6 + \dots
 \end{aligned}$$

## REVIEW SET 8A

$$\begin{aligned}
 \text{1 a} \quad (x+3)^3 &= x^3 + 3x^2(3) + 3x(3)^2 + 3^3 \\
 &= x^3 + 9x^2 + 27x + 27
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad (x-2)^5 &= x^5 + 5x^4(-2) + 10x^3(-2)^2 + 10x^2(-2)^3 + 5x(-2)^4 + (-2)^5 \\
 &= x^5 - 10x^4 + 40x^3 - 80x^2 + 80x - 32
 \end{aligned}$$

$$\text{2 a} \quad \text{For } (2x+3)^9, \quad a = (2x), \quad b = 3, \quad \text{and } n = 9$$

$$\text{Now } T_{r+1} = \binom{n}{r} a^{n-r} b^r \quad \text{and letting } r = 4 \quad \text{gives } T_5 = \binom{9}{4} (2x)^5 3^4.$$

$$\text{b} \quad \text{For } (y-3z)^9, \quad a = y, \quad b = (-3z), \quad \text{and } n = 9$$

$$\text{Now } T_{r+1} = \binom{n}{r} a^{n-r} b^r \quad \text{and letting } r = 3 \quad \text{gives } T_4 = \binom{9}{3} y^6 (-3z)^3.$$

$$\text{c} \quad \text{For } \left(3x - \frac{1}{x}\right)^{12}, \quad a = (3x), \quad b = \left(-\frac{1}{x}\right), \quad \text{and } n = 12$$

$$\text{Now } T_{r+1} = \binom{n}{r} a^{n-r} b^r \quad \text{and letting } r = 7 \quad \text{gives } T_8 = \binom{12}{7} (3x)^5 \left(-\frac{1}{x}\right)^7.$$

$$\text{d} \quad \text{For } \left(2x - \frac{1}{\sqrt{x}}\right)^{10}, \quad a = (2x), \quad b = \left(-\frac{1}{\sqrt{x}}\right), \quad \text{and } n = 10$$

$$\text{Now } T_{r+1} = \binom{n}{r} a^{n-r} b^r \quad \text{and letting } r = 4 \quad \text{gives } T_5 = \binom{10}{4} (2x)^6 \left(-\frac{1}{\sqrt{x}}\right)^4.$$

$$\begin{aligned}
 \text{3 a} \quad (5+\sqrt{3})^3 &= 5^3 + 3(5)^2(\sqrt{3}) + 3(5)(\sqrt{3})^2 + (\sqrt{3})^3 \\
 &= 125 + 75\sqrt{3} + 15 \times 3 + 3\sqrt{3} \\
 &= 125 + 75\sqrt{3} + 45 + 3\sqrt{3} \\
 &= 170 + 78\sqrt{3}
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad (x+3)(x-1)^4 &= (x+3)(x^4 - 4x^3 + 6x^2 - 4x + 1) \\
 &= x^5 - 4x^4 + 6x^3 - 4x^2 + x + 3x^4 - 12x^3 + 18x^2 - 12x + 3 \\
 &= x^5 - x^4 - 6x^3 + 14x^2 - 11x + 3
 \end{aligned}$$

$$\begin{aligned} 4 \quad (4+x)^3 &= 4^3 + 3(4)^2x + 3(4)x^2 + x^3 \\ &= 64 + 48x + 12x^2 + x^3 \end{aligned}$$

$(4.02)^3$  is obtained by letting  $x = 0.02$

$$\begin{aligned} \therefore (4.02)^3 &= 64 + 48 \times 0.02 + 12 \times (0.02)^2 + (0.02)^3 \\ &= 64.964\,808 \end{aligned}$$

$$\begin{array}{r} 64 \\ 0.96 \\ 0.0048 \\ + 0.000\,008 \\ \hline 64.964\,808 \end{array}$$

5 a The sixth row of Pascal's triangle is 1 6 15 20 15 6 1

$$\therefore (a+b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$$

$$\begin{aligned} \text{b i} \quad (x-3)^6 &= x^6 + 6x^5(-3) + 15x^4(-3)^2 + 20x^3(-3)^3 + 15x^2(-3)^4 + 6x(-3)^5 + (-3)^6 \\ &= x^6 - 18x^5 + 135x^4 - 540x^3 + 1215x^2 - 1458x + 729 \end{aligned}$$

$$\begin{aligned} \text{ii} \quad \left(2 + \frac{1}{x}\right)^6 &= (2)^6 + 6(2)^5\left(\frac{1}{x}\right) + 15(2)^4\left(\frac{1}{x}\right)^2 + 20(2)^3\left(\frac{1}{x}\right)^3 + 15(2)^2\left(\frac{1}{x}\right)^4 + 6(2)\left(\frac{1}{x}\right)^5 + \left(\frac{1}{x}\right)^6 \\ &= 64 + \frac{192}{x} + \frac{240}{x^2} + \frac{160}{x^3} + \frac{60}{x^4} + \frac{12}{x^5} + \frac{1}{x^6} \end{aligned}$$

6 a In  $\left(2x - \frac{3}{x^2}\right)^{12}$ ,  $a = (2x)$ ,  $b = \left(-\frac{3}{x^2}\right)$ ,  $n = 12$

$$\begin{aligned} \text{Now } T_{r+1} &= \binom{n}{r} a^{n-r} b^r \\ &= \binom{12}{r} (2x)^{12-r} \left(-\frac{3}{x^2}\right)^r \\ &= \binom{12}{r} 2^{12-r} x^{12-r} \frac{(-3)^r}{x^{2r}} \\ &= \binom{12}{r} 2^{12-r} (-3)^r x^{12-3r} \end{aligned}$$

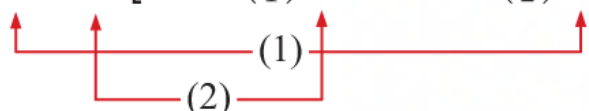
For the coefficient of  $x^{-6}$  we let  $12 - 3r = -6$

$$\therefore 3r = 18$$

$$\therefore r = 6$$

$$\text{So, } T_7 = \binom{12}{6} 2^6 (-3)^6 x^{-6}$$

$$\therefore \text{ the coefficient of } x^{-6} \text{ is } \binom{12}{6} 2^6 (-3)^6 = 43\,110\,144.$$

$$\begin{aligned} \text{b} \quad (2x+3)(x-2)^6 &= (2x+3) \left[ x^6 + \binom{6}{1} x^5(-2) + \binom{6}{2} x^4(-2)^2 + \dots \right] \end{aligned}$$


So, the terms containing  $x^5$  are  $2 \binom{6}{2} (-2)^2 x^5$  from (1)

and  $3 \binom{6}{1} (-2) x^5$  from (2)

$$\therefore \text{ the coefficient of } x^5 \text{ is } 8 \binom{6}{2} - 6 \binom{6}{1} = 84.$$



$$\begin{array}{c}
 \begin{array}{ccccccc}
 & & & & (3) & & \\
 & & & \swarrow & \longrightarrow & \searrow & \\
 & & (2) & \longrightarrow & & & \\
 & \swarrow & \longrightarrow & \searrow & \longrightarrow & \searrow & \\
 (1) & \longrightarrow & & & & & \\
 \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
 \end{array} \\
 = (x^4 + 8x^3 + 24x^2 + 32x + 16) \times (x^{-5} + \binom{5}{1}(-3)x^{-3} + \binom{5}{2}(-3)^2x^{-1} \\
 + \binom{5}{3}(-3)^3x + \binom{5}{4}(-3)^4x^3 + (-3)^5x^5)
 \end{array}$$

8  $(1 + cx)(1 + x)^4 = (1 + cx)\left(1^4 + \binom{4}{1} 1^3x + \binom{4}{2} 1^2x^2 + \binom{4}{3} 1x^3 + x^4\right)$

Diagram illustrating the multiplication of  $(1 + cx)$  by the binomial expansion of  $(1 + x)^4$ . The expansion is shown as  $1^4 + \binom{4}{1} 1^3x + \binom{4}{2} 1^2x^2 + \binom{4}{3} 1x^3 + x^4$ . Red arrows indicate the distribution of terms:

- Row (1) shows the distribution of the constant term  $1$  from  $(1 + cx)$  to each term in the expansion.
- Row (2) shows the distribution of the term  $cx$  from  $(1 + cx)$  to each term in the expansion.

$$\therefore a = \pm 4$$

$$\begin{aligned}
 \mathbf{11} \quad \mathbf{a} \quad \frac{1}{1-x} &= (1-x)^{-1} \\
 &= \sum_{r=0}^{\infty} \binom{-1}{r} (-x)^r \\
 &= \binom{-1}{0} (-x)^0 + \binom{-1}{1} (-x)^1 + \binom{-1}{2} (-x)^2 + \binom{-1}{3} (-x)^3 + \dots \\
 &= 1 + (-1)(-x) + \frac{(-1)(-2)}{2!} (-x)^2 + \frac{(-1)(-2)(-3)}{3!} (-x)^3 + \dots \\
 &= 1 + x + x^2 + x^3 + \dots
 \end{aligned}$$

**b** The series converges provided  $|-x| < 1$ , which is the interval  $-1 < x < 1$ .

$$\mathbf{c} \quad \text{Letting } x = 0.05, \quad \frac{1}{0.95} \approx 1 + 0.05 + (0.05)^2 + (0.05)^3 \\
 \approx 1.052625$$

$$\text{Using technology, } \frac{1}{0.95} \approx 1.052632$$

$$\begin{aligned}
 \mathbf{12} \quad \mathbf{a} \quad \frac{1}{\sqrt{x+3}} &= (x+3)^{-\frac{1}{2}} \\
 &= 3^{-\frac{1}{2}} \left(1 + \frac{x}{3}\right)^{-\frac{1}{2}} \\
 &= \frac{1}{\sqrt{3}} \left(1 + \frac{x}{3}\right)^{-\frac{1}{2}} \\
 &= \frac{1}{\sqrt{3}} \sum_{r=0}^{\infty} \binom{-\frac{1}{2}}{r} \left(\frac{x}{3}\right)^r \\
 &= \frac{1}{\sqrt{3}} \left( \binom{-\frac{1}{2}}{0} \left(\frac{x}{3}\right)^0 + \binom{-\frac{1}{2}}{1} \left(\frac{x}{3}\right)^1 + \binom{-\frac{1}{2}}{2} \left(\frac{x}{3}\right)^2 + \binom{-\frac{1}{2}}{3} \left(\frac{x}{3}\right)^3 + \dots \right) \\
 &= \frac{1}{\sqrt{3}} \left( 1 + \left(-\frac{1}{2}\right) \left(\frac{x}{3}\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!} \left(\frac{x}{3}\right)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!} \left(\frac{x}{3}\right)^3 + \dots \right) \\
 &= \frac{1}{\sqrt{3}} \left( 1 - \frac{1}{6}x + \frac{1}{24}x^2 - \frac{5}{432}x^3 + \dots \right) \\
 &= \frac{1}{\sqrt{3}} - \frac{1}{6\sqrt{3}}x + \frac{1}{24\sqrt{3}}x^2 - \frac{5}{432\sqrt{3}}x^3 + \dots
 \end{aligned}$$

**b** The series converges provided  $\left|\frac{x}{3}\right| < 1$ , which is the interval  $-3 < x < 3$ .

$$\mathbf{c} \quad \mathbf{i} \quad \text{Letting } x = 2, \quad \frac{1}{\sqrt{5}} \approx \frac{1}{\sqrt{3}} - \frac{1}{6\sqrt{3}}(2) + \frac{1}{24\sqrt{3}}(2)^2 - \frac{5}{432\sqrt{3}}(2)^3 \\
 \approx 0.427667$$

$$\text{Using technology, } \frac{1}{\sqrt{5}} \approx 0.447214$$

$$\mathbf{ii} \quad \text{Letting } x = 0.2, \quad \frac{1}{\sqrt{3.2}} \approx \frac{1}{\sqrt{3}} - \frac{1}{6\sqrt{3}}(0.2) + \frac{1}{24\sqrt{3}}(0.2)^2 - \frac{5}{432\sqrt{3}}(0.2)^3 \\
 \approx 0.559014$$

$$\text{Using technology, } \frac{1}{\sqrt{3.2}} \approx 0.559017$$

Our estimates are more accurate for  $x$  closer to 0.

**13 a**  $\sqrt{1-x} = (1-x)^{\frac{1}{2}}$

$$= \sum_{r=0}^{\infty} \binom{\frac{1}{2}}{r} (-x)^r$$

$$= \binom{\frac{1}{2}}{0} (-x)^0 + \binom{\frac{1}{2}}{1} (-x)^1 + \binom{\frac{1}{2}}{2} (-x)^2 + \binom{\frac{1}{2}}{3} (-x)^3 + \dots$$

$$= 1 + \left(\frac{1}{2}\right)(-x) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!} (-x)^2 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!} (-x)^3 + \dots$$

$$= 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3 - \dots$$

**b** The series converges provided  $|-x| < 1$ , which is the interval  $-1 < x < 1$ .

**c**  $\sqrt{15} = \sqrt{16} \times \sqrt{\frac{15}{16}}$

$$= \sqrt{16} \times \sqrt{1 - \frac{1}{16}}$$

$$= 4 \times \left(1 - \frac{1}{2}\left(\frac{1}{16}\right) - \frac{1}{8}\left(\frac{1}{16}\right)^2 - \frac{1}{16}\left(\frac{1}{16}\right)^3\right)$$

$$\approx 3.872986$$

**14 a**  $\frac{1}{(1-3x)^2} = (1-3x)^{-2}$

$$= \sum_{r=0}^{\infty} \binom{-2}{r} (-3x)^r$$

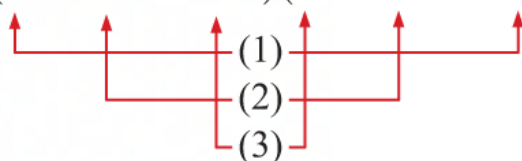
$$= \binom{-2}{0} (-3x)^0 + \binom{-2}{1} (-3x)^1 + \binom{-2}{2} (-3x)^2 + \binom{-2}{3} (-3x)^3 + \dots$$

$$= 1 + (-2)(-3x) + \frac{(-2)(-3)}{2!} (-3x)^2 + \frac{(-2)(-3)(-4)}{3!} (-3x)^3 + \dots$$

$$= 1 + 6x + 27x^2 + 108x^3 + \dots$$

**b**  $\frac{(1+4x)^2}{(1-3x)^2} = (1+4x^2)(1+6x+27x^2+108x^3+\dots)$  {using **a**}

$$= (1+8x+16x^2)(1+6x+27x^2+108x^3+\dots)$$



So, the terms containing  $x^2$  are  $27x^2$  from (1)

$48x^2$  from (2)

and  $16x^2$  from (3)

$\therefore$  the coefficient of  $x^2$  is  $27 + 48 + 16 = 91$ .

## REVIEW SET 8B

**1 a**  $(x-2y)^3 = x^3 + 3x^2(-2y) + 3x(-2y)^2 + (-2y)^3$

$$= x^3 - 6x^2y + 12xy^2 - 8y^3$$

**b**  $(3x+2)^4 = (3x)^4 + 4(3x)^3(2) + 6(3x)^2(2)^2 + 4(3x)(2)^3 + (2)^4$

$$= 81x^4 + 216x^3 + 216x^2 + 96x + 16$$



**2** In  $(2x + 5)^6$ ,  $a = (2x)$ ,  $b = 5$ , and  $n = 6$

$$\begin{aligned} \text{Now } T_{r+1} &= \binom{n}{r} a^{n-r} b^r & \text{If } 6 - r &= 3 \\ &= \binom{6}{r} (2x)^{6-r} 5^r & \text{then } r &= 3 \\ &= \binom{6}{r} 2^{6-r} x^{6-r} 5^r & \therefore T_4 &= \binom{6}{3} 2^3 5^3 x^3 \\ & & \therefore \text{the coefficient of } x^3 & \text{ is } \binom{6}{3} 2^3 5^3 = 20\,000. \end{aligned}$$

**3 a** In  $\left(2x^2 - \frac{1}{x}\right)^6$ ,  $a = (2x^2)$ ,  $b = \left(-\frac{1}{x}\right)$ , and  $n = 6$

$$\begin{aligned} \text{Now } T_{r+1} &= \binom{n}{r} a^{n-r} b^r & \text{For the constant term we let } 12 - 3r &= 0 \\ &= \binom{6}{r} (2x^2)^{6-r} \left(-\frac{1}{x}\right)^r & \therefore r &= 4 \\ &= \binom{6}{r} 2^{6-r} x^{12-2r} (-1)^r x^{-r} & \therefore T_5 &= \binom{6}{4} 2^2 (-1)^4 x^0 \\ &= \binom{6}{r} 2^{6-r} (-1)^r x^{12-3r} & \therefore \text{the constant term is } \binom{6}{4} 2^2 (-1)^4 &= 60. \end{aligned}$$

**b** In  $\left(x - \frac{6}{x^2}\right)^9$ ,  $a = x$ ,  $b = \left(-\frac{6}{x^2}\right)$ , and  $n = 9$

$$\begin{aligned} \text{Now } T_{r+1} &= \binom{n}{r} a^{n-r} b^r & \text{For the constant term we let } 9 - 3r &= 0 \\ &= \binom{9}{r} x^{9-r} \left(-\frac{6}{x^2}\right)^r & \therefore r &= 3 \\ &= \binom{9}{r} x^{9-r} (-6)^r x^{-2r} & \therefore T_4 &= \binom{9}{3} (-6)^3 x^0 \\ &= \binom{9}{r} (-6)^r x^{9-3r} & \therefore \text{the constant term is } \binom{9}{3} (-6)^3 &= -18\,144. \end{aligned}$$

**c** In  $\left(\sqrt{x} + \frac{6}{\sqrt{x}}\right)^5$ ,  $a = \sqrt{x}$ ,  $b = \left(\frac{6}{\sqrt{x}}\right)$ , and  $n = 5$

$$\begin{aligned} \text{Now } T_{r+1} &= \binom{n}{r} a^{n-r} b^r & \text{For the constant term we let } \frac{5}{2} - r &= 0 \\ &= \binom{5}{r} (\sqrt{x})^{5-r} \left(\frac{6}{\sqrt{x}}\right)^r & \therefore r &= \frac{5}{2} \\ &= \binom{5}{r} x^{\frac{5}{2}-\frac{1}{2}r} 6^r x^{-\frac{1}{2}r} & \text{However, } r & \text{ must be an integer.} \\ &= \binom{5}{r} 6^r x^{\frac{5}{2}-r} & \text{So, there is no term with } x^0. \\ & & \therefore \text{the constant term is } 0. \end{aligned}$$

**4 a**  $(2 - \sqrt{2})^6 = 2^6 + 6(2)^5(-\sqrt{2}) + 15(2)^4(-\sqrt{2})^2 + 20(2)^3(-\sqrt{2})^3$   
 $+ 15(2)^2(-\sqrt{2})^4 + 6(2)(-\sqrt{2})^5 + (-\sqrt{2})^6$   
 $= 64 - 192\sqrt{2} + 480 - 320\sqrt{2} + 240 - 48\sqrt{2} + 8$   
 $= 792 - 560\sqrt{2}$

**b**  $(x + 3)(2x + 1)^3 = (x + 3)[(2x)^3 + 3(2x)^2(1) + 3(2x)(1)^2 + (1)^3]$   
 $= (x + 3)(8x^3 + 12x^2 + 6x + 1)$   
 $= 8x^4 + 12x^3 + 6x^2 + x$   
 $+ 24x^3 + 36x^2 + 18x + 3$   
 $= 8x^4 + 36x^3 + 42x^2 + 19x + 3$



**5 a** In  $\left(\frac{3}{x^2} - 4x\right)^{10}$ ,  $a = \left(\frac{3}{x^2}\right)$ ,  $b = (-4x)$ , and  $n = 10$

$$\begin{aligned}\text{Now } T_{r+1} &= \binom{n}{r} a^{n-r} b^r \\ &= \binom{10}{r} \left(\frac{3}{x^2}\right)^{10-r} (-4x)^r \\ &= \binom{10}{r} \frac{3^{10-r}}{x^{20-2r}} (-4)^r x^r \\ &= \binom{10}{r} 3^{10-r} (-4)^r x^{3r-20}\end{aligned}$$

We now let  $3r - 20 = 10$

$$\therefore 3r = 30$$

$$\therefore r = 10$$

$$\begin{aligned}\text{So, } T_{11} &= \binom{10}{10} 3^0 (-4)^{10} x^{10} \\ &= (-4)^{10} x^{10}\end{aligned}$$

$\therefore$  the coefficient of  $x^{10}$  is  $(-4)^{10} = 1\,048\,576$ .

**b**  $(3x - 1)(5 + 2x)^6$

$$\begin{aligned}&= (3x - 1) \\ &\quad \times [5^6 + \binom{6}{1} 5^5(2x) + \binom{6}{2} 5^4(2x)^2 + \binom{6}{3} 5^3(2x)^3 + \binom{6}{4} 5^2(2x)^4 + \binom{6}{5} 5^1(2x)^5 + (2x)^6] \\ &= (3x - 1)[5^6 + 2\binom{6}{1} 5^5x + \binom{6}{2} 5^4 2^2 x^2 + \binom{6}{3} 5^3 2^3 x^3 + \binom{6}{4} 5^2 2^4 x^4 + 5\binom{6}{5} 2^5 x^5 + 2^6 x^6]\end{aligned}$$

So, the terms containing  $x^5$  are  $3\binom{6}{4} 5^2 2^4 x^5$  from (1)

and  $-5\binom{6}{5} 2^5 x^5$  from (2)

$\therefore$  the coefficient of  $x^5$  is  $3\binom{6}{4} 5^2 2^4 - 5\binom{6}{5} 2^5 = 15\,600$ .

**6** For  $\left(3x^2 + \frac{1}{x}\right)^9$ ,  $a = (3x^2)$ ,  $b = \left(\frac{1}{x}\right)$ , and  $n = 9$

$$\begin{aligned}T_{r+1} &= \binom{n}{r} a^{n-r} b^r \\ &= \binom{9}{r} (3x^2)^{9-r} \left(\frac{1}{x}\right)^r \\ &= \binom{9}{r} 3^{9-r} x^{18-2r} x^{-r} \\ &= \binom{9}{r} 3^{9-r} x^{18-3r}\end{aligned}$$

**a** If  $18 - 3r = 12$

then  $3r = 6$

$$\therefore r = 2$$

$$\therefore T_3 = \binom{9}{2} 3^7 x^{12}$$

$\therefore$  the coefficient of  $x^{12}$  is  $\binom{9}{2} 3^7 = 78\,732$ .

**b** If  $18 - 3r = 0$

then  $3r = 18$

$$\therefore r = 6$$

$$\therefore T_7 = \binom{9}{6} 3^3 x^0$$

$\therefore$  the constant term is  $\binom{9}{6} 3^3 = 2268$ .

$$\begin{aligned}
 & 7 \quad \left(2x + \frac{1}{x}\right)^3 \left(x^2 - \frac{2}{x}\right)^4 \\
 &= \left((2x)^3 + 3(2x)^2 \left(\frac{1}{x}\right) + 3(2x) \left(\frac{1}{x}\right)^2 + \left(\frac{1}{x}\right)^3\right) \\
 &\quad \times \left((x^2)^4 + \binom{4}{1} (x^2)^3 \left(-\frac{2}{x}\right) + \binom{4}{2} (x^2)^2 \left(-\frac{2}{x}\right)^2 + \binom{4}{3} (x^2) \left(-\frac{2}{x}\right)^3 + \left(-\frac{2}{x}\right)^4\right) \\
 &= (8x^3 + 12x + 6x^{-1} + x^{-3}) (x^8 + \binom{4}{1} (-2)x^5 + \binom{4}{2} (-2)^2 x^2 + \binom{4}{3} (-2)^3 x^{-1} + (-2)^4 x^{-4})
 \end{aligned}$$

So, the terms containing  $x^2$  are  $8 \binom{4}{3} (-2)^3 x^2$  from (1)

and  $\binom{4}{1} (-2)x^2$  from (2)

$\therefore$  the coefficient of  $x^2$  is  $8 \binom{4}{3} (-2)^3 + \binom{4}{1} (-2) = -264$ .

$$\begin{aligned}
 & 8 \quad (1 + kx)^n = 1^n + \binom{n}{1} 1^{n-1} (kx)^1 + \binom{n}{2} 1^{n-2} (kx)^2 + \dots \\
 & \quad = 1 + nkx + \binom{n}{2} k^2 x^2 + \dots
 \end{aligned}$$

$$\therefore nk = -4 \quad \text{and} \quad \binom{n}{2} k^2 = \frac{15}{2}$$

$$\therefore k = -\frac{4}{n} \quad \dots (*) \quad \therefore \frac{n(n-1)}{2} k^2 = \frac{15}{2}$$

$$\therefore n(n-1)k^2 = 15$$

$$\therefore n(n-1) \left(-\frac{4}{n}\right)^2 = 15 \quad \{\text{using } (*)\}$$

$$\therefore n(n-1) \left(\frac{16}{n^2}\right) = 15$$

$$\therefore 16(n-1) = 15n \quad \{n \geq 2\}$$

$$\therefore 16n - 16 = 15n$$

$$\therefore n = 16 \quad \text{and so} \quad k = -\frac{4}{16} = -\frac{1}{4}$$

$$\begin{aligned}
 & 9 \quad (m - 2n)^{10} = m^{10} + \binom{10}{1} m^9 (-2n) + \binom{10}{2} m^8 (-2n)^2 + \dots + (-2n)^{10} \\
 & \quad = m^{10} - 20m^9 n + 45m^8 (4n^2) - \dots + 1024n^{10} \\
 & \quad = m^{10} - 20m^9 n + 180m^8 n^2 - \dots + 1024n^{10} \\
 & \therefore k = 180
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{10} \quad \left(x^3 + \frac{q}{x^3}\right)^8 \text{ has } T_{r+1} &= \binom{8}{r} (x^3)^{8-r} \left(\frac{q}{x^3}\right)^r \\
 &= \binom{8}{r} x^{24-3r} \frac{q^r}{x^{3r}} \\
 &= \binom{8}{r} x^{24-6r} q^r
 \end{aligned}$$

which has constant term  $\binom{8}{4} q^4$   $\{24 - 6r = 0 \text{ when } r = 4\}$

$$\begin{aligned}
 \left(x^3 + \frac{q}{x^3}\right)^4 \text{ has } T_{r+1} &= \binom{4}{r} (x^3)^{4-r} \left(\frac{q}{x^3}\right)^r \\
 &= \binom{4}{r} x^{12-3r} q^r x^{-3r} \\
 &= \binom{4}{r} x^{12-6r} q^r
 \end{aligned}$$

which has constant term  $\binom{4}{2} q^2$   $\{12 - 6r = 0 \text{ when } r = 2\}$

$$\begin{aligned}
 \therefore \binom{8}{4} q^4 &= \binom{4}{2} q^2 \\
 \therefore 70q^4 - 6q^2 &= 0 \\
 \therefore q^2(70q^2 - 6) &= 0 \\
 \therefore 70q^2 - 6 &= 0 \quad \{q = 0 \text{ gives a trivial solution}\} \\
 \therefore q^2 &= \frac{6}{70} \\
 &= \frac{3}{35} \\
 \therefore q &= \pm \sqrt{\frac{3}{35}}
 \end{aligned}$$

$$\mathbf{11} \quad \mathbf{a} \quad (a - 2x)^7 \text{ has } a = a, \quad b = (-2x), \text{ and } n = 7$$

$$\begin{aligned}
 T_{r+1} &= \binom{n}{r} a^{n-r} b^r \\
 &= \binom{7}{r} a^{7-r} (-2x)^r \\
 &= \binom{7}{r} (-2)^r a^{7-r} x^r
 \end{aligned}$$

We let  $r = 5$ , so  $T_6 = \binom{7}{5} (-2)^5 a^2 x^5$ .

$$\text{So, } \binom{7}{5} (-2)^5 a^2 = -42$$

$$\therefore -672a^2 = -42$$

$$\therefore a^2 = \frac{1}{16}$$

$$\therefore a = \pm \frac{1}{4}$$

$$\text{b } \left( \frac{1}{x\sqrt{2}} - a \right)^8 \text{ has } a = \left( \frac{1}{x\sqrt{2}} \right), \text{ } b = (-a), \text{ and } n = 8$$

$$T_{r+1} = \binom{n}{r} a^{n-r} b^r$$

$$\therefore T_5 = \binom{8}{4} \left( \frac{1}{x\sqrt{2}} \right)^4 (-a)^4 = \frac{1120}{x^4}$$

$$\therefore 70 \times \frac{1}{4x^4} \times a^4 = \frac{1120}{x^4}$$

$$\therefore \frac{70a^4}{4} = 1120$$

$$\therefore 70a^4 = 4480$$

$$\therefore a^4 = 64$$

$$\therefore a = \pm 2\sqrt{2}$$

$$\begin{aligned} \text{12 a } (1+2x)^{-\frac{3}{2}} &= \sum_{r=0}^{\infty} \binom{-\frac{3}{2}}{r} (2x)^r \\ &= \binom{-\frac{3}{2}}{0} (2x)^0 + \binom{-\frac{3}{2}}{1} (2x)^1 + \binom{-\frac{3}{2}}{2} (2x)^2 + \binom{-\frac{3}{2}}{3} (2x)^3 + \dots \\ &= 1 + \left(-\frac{3}{2}\right)(2x) + \frac{\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{2!} (2x)^2 + \frac{\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)}{3!} (2x)^3 + \dots \\ &= 1 - 3x + \frac{15}{2}x^2 - \frac{35}{2}x^3 + \dots \end{aligned}$$

**b** The series converges provided  $|2x| < 1$ , so the interval of convergence is  $-\frac{1}{2} < x < \frac{1}{2}$ .

$$\begin{aligned} \text{c Letting } x = 0.05, \quad (1.1)^{-\frac{3}{2}} &\approx 1 - 3(0.05) + \frac{15}{2}(0.05)^2 - \frac{35}{2}(0.05)^3 \\ &\approx 0.866563 \end{aligned}$$

$$\begin{aligned} \text{13 } \frac{1}{\sqrt{1+x}} &= (1+x)^{-\frac{1}{2}} \\ &= \sum_{r=0}^{\infty} \binom{-\frac{1}{2}}{r} x^r \\ &= \binom{-\frac{1}{2}}{0} x^0 + \binom{-\frac{1}{2}}{1} x^1 + \binom{-\frac{1}{2}}{2} x^2 + \binom{-\frac{1}{2}}{3} x^3 + \dots \\ &= 1 + \left(-\frac{1}{2}\right)x + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!} x^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!} x^3 + \dots \\ &= 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \dots \end{aligned}$$

$$\begin{aligned} \therefore \frac{1+x^2}{\sqrt{1+x}} &= (1+x^2)(1+x)^{-\frac{1}{2}} \\ &= (1+x^2)\left(1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \dots\right) \\ &= 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + x^2 - \frac{1}{2}x^3 + \dots \\ &= 1 - \frac{1}{2}x + \frac{11}{8}x^2 - \frac{13}{16}x^3 + \dots \quad \text{provided } |x| < 1, \text{ or } -1 < x < 1 \end{aligned}$$



$$\begin{aligned}
 \text{14 a } \left(1 + \frac{x}{2}\right)^{\frac{1}{3}} &= \sum_{r=0}^{\infty} \binom{\frac{1}{3}}{r} \left(\frac{x}{2}\right)^r \\
 &= \binom{\frac{1}{3}}{0} \left(\frac{x}{2}\right)^0 + \binom{\frac{1}{3}}{1} \left(\frac{x}{2}\right)^1 + \binom{\frac{1}{3}}{2} \left(\frac{x}{2}\right)^2 + \binom{\frac{1}{3}}{3} \left(\frac{x}{2}\right)^3 + \dots \\
 &= 1 + \left(\frac{1}{3}\right) \left(\frac{x}{2}\right) + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2!} \left(\frac{x}{2}\right)^2 + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{3!} \left(\frac{x}{2}\right)^3 + \dots \\
 &= 1 + \frac{1}{6}x - \frac{1}{36}x^2 + \frac{5}{648}x^3 - \dots \quad \text{provided } \left|\frac{x}{2}\right| < 1, \text{ or } -2 < x < 2
 \end{aligned}$$

$$\begin{aligned}
 \text{b } \left(1 - \frac{x}{3}\right)^{-\frac{1}{3}} &= \sum_{r=0}^{\infty} \binom{-\frac{1}{3}}{r} \left(-\frac{x}{3}\right)^r \\
 &= \binom{-\frac{1}{3}}{0} \left(-\frac{x}{3}\right)^0 + \binom{-\frac{1}{3}}{1} \left(-\frac{x}{3}\right)^1 + \binom{-\frac{1}{3}}{2} \left(-\frac{x}{3}\right)^2 + \binom{-\frac{1}{3}}{3} \left(-\frac{x}{3}\right)^3 + \dots \\
 &= 1 + \left(-\frac{1}{3}\right) \left(-\frac{x}{3}\right) + \frac{\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)}{2!} \left(-\frac{x}{3}\right)^2 + \frac{\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)\left(-\frac{7}{3}\right)}{3!} \left(-\frac{x}{3}\right)^3 + \dots \\
 &= 1 + \frac{1}{9}x + \frac{2}{81}x^2 + \frac{14}{2187}x^3 + \dots \quad \text{provided } \left|-\frac{x}{3}\right| < 1, \text{ or } -3 < x < 3
 \end{aligned}$$

$$\begin{aligned}
 \text{c } (6 + 3x)^{\frac{1}{3}} &= 6^{\frac{1}{3}} \left(1 + \frac{x}{2}\right)^{\frac{1}{3}} \\
 &= \sqrt[3]{6} \left(1 + \frac{x}{2}\right)^{\frac{1}{3}} \\
 &= \sqrt[3]{6} \sum_{r=0}^{\infty} \binom{\frac{1}{3}}{r} \left(\frac{x}{2}\right)^r \\
 &= \sqrt[3]{6} \left( \binom{\frac{1}{3}}{0} \left(\frac{x}{2}\right)^0 + \binom{\frac{1}{3}}{1} \left(\frac{x}{2}\right)^1 + \binom{\frac{1}{3}}{2} \left(\frac{x}{2}\right)^2 + \binom{\frac{1}{3}}{3} \left(\frac{x}{2}\right)^3 + \dots \right) \\
 &= \sqrt[3]{6} \left( 1 + \left(\frac{1}{3}\right) \left(\frac{x}{2}\right) + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2!} \left(\frac{x}{2}\right)^2 + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{3!} \left(\frac{x}{2}\right)^3 + \dots \right) \\
 &= \sqrt[3]{6} \left( 1 + \frac{1}{6}x - \frac{1}{36}x^2 + \frac{5}{648}x^3 - \dots \right) \quad \text{provided } \left|\frac{x}{2}\right| < 1, \text{ or } -2 < x < 2
 \end{aligned}$$

$$\begin{aligned}
 &(6 - 2x)^{-\frac{1}{3}} \\
 &= 6^{-\frac{1}{3}} \left(1 - \frac{x}{3}\right)^{-\frac{1}{3}} \\
 &= \frac{1}{\sqrt[3]{6}} \left(1 - \frac{x}{3}\right)^{-\frac{1}{3}} \\
 &= \frac{1}{\sqrt[3]{6}} \sum_{r=0}^{\infty} \binom{-\frac{1}{3}}{r} \left(-\frac{x}{3}\right)^r \\
 &= \frac{1}{\sqrt[3]{6}} \left( \binom{-\frac{1}{3}}{0} \left(-\frac{x}{3}\right)^0 + \binom{-\frac{1}{3}}{1} \left(-\frac{x}{3}\right)^1 + \binom{-\frac{1}{3}}{2} \left(-\frac{x}{3}\right)^2 + \binom{-\frac{1}{3}}{3} \left(-\frac{x}{3}\right)^3 + \dots \right) \\
 &= \frac{1}{\sqrt[3]{6}} \left( 1 + \left(-\frac{1}{3}\right) \left(-\frac{x}{3}\right) + \frac{\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)}{2!} \left(-\frac{x}{3}\right)^2 + \frac{\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)\left(-\frac{7}{3}\right)}{3!} \left(-\frac{x}{3}\right)^3 + \dots \right) \\
 &= \frac{1}{\sqrt[3]{6}} \left( 1 + \frac{1}{9}x + \frac{2}{81}x^2 + \frac{14}{2187}x^3 + \dots \right) \quad \text{provided } \left|-\frac{x}{3}\right| < 1, \text{ or } -3 < x < 3
 \end{aligned}$$

$$\begin{aligned}\text{Now } & \left( \frac{6+3x}{6-2x} \right)^{\frac{1}{3}} \\ &= \frac{(6+3x)^{\frac{1}{3}}}{(6-2x)^{\frac{1}{3}}} \\ &= (6+3x)^{\frac{1}{3}} (6-2x)^{-\frac{1}{3}} \\ &= \sqrt[3]{6} \left( 1 + \frac{1}{6}x - \frac{1}{36}x^2 + \frac{5}{648}x^3 - \dots \right) \times \frac{1}{\sqrt[3]{6}} \left( 1 + \frac{1}{9}x + \frac{2}{81}x^2 + \frac{14}{2187}x^3 + \dots \right) \\ &= \left( 1 + \frac{1}{6}x - \frac{1}{36}x^2 + \frac{5}{648}x^3 - \dots \right) \left( 1 + \frac{1}{9}x + \frac{2}{81}x^2 + \frac{14}{2187}x^3 + \dots \right) \\ &= 1 + \frac{1}{9}x + \frac{2}{81}x^2 + \frac{14}{2187}x^3 + \frac{1}{6}x + \frac{1}{54}x^2 + \frac{1}{243}x^3 - \frac{1}{36}x^2 - \frac{1}{324}x^3 + \frac{5}{648}x^3 + \dots \\ &= 1 + \frac{5}{18}x + \frac{5}{324}x^2 + \frac{265}{17496}x^3 + \dots \quad \text{provided } -2 < x < 2\end{aligned}$$

# Chapter 9

## REASONING AND PROOF

### EXERCISE 9A

- 1**
  - a** The negation of “The cat is black” is “The cat is not black”.
  - b** The negation of “ $x$  is prime” is “ $x$  is not prime”, since  $x$  might be 1, or it might be a non-integer.
  - c** The negation of “The tree is deciduous” is “The tree is not deciduous”.
- 2**
  - a** The statement “If  $x^2 = 9$  then  $x = 3$ ” is false, since  $x$  may be  $-3$ .
  - b** The statement “If  $x = 3$  then  $x^2 = 9$ ” is true, since  $3^2 = 9$ .
  - c** The statement “ $x = 3$  if and only if  $x^2 = 9$ ” is false, since  $x^2 = 9$  does not imply that  $x = 3$  (from **a**).
- 3**
  - a** The statement “If  $x$  is positive then  $\sqrt{x} \in \mathbb{R}$ ” is true, as the square root of any positive number is real.
  - b** The statement “If  $\sqrt{x} \in \mathbb{R}$  then  $x$  is positive” is false, since  $\sqrt{0} = 0 \in \mathbb{R}$ , but 0 is not positive.
  - c** The statement “ $x$  is positive if and only if  $\sqrt{x} \in \mathbb{R}$ ” is false, since  $\sqrt{x} \in \mathbb{R}$  does not imply that  $x$  is positive (from **b**).
- 4**
  - a** The converse of the statement “If Socrates is a cat then Socrates is an animal” is “If Socrates is an animal, then Socrates is a cat”.
  - b** The converse is false. If Socrates is an animal he is not necessarily a cat.
- 5**
  - a**  $A: xyz = 0, \quad B: (x = 0) \vee (y = 0) \vee (z = 0)$   
If  $xyz = 0$ , then one of  $x, y$ , or  $z$  must be zero.  $\therefore A \Rightarrow B$   
If one of  $x, y$ , or  $z$  is 0, then  $xyz = 0$ .  $\therefore B \Rightarrow A$   
 $A \Rightarrow B$  and  $B \Rightarrow A$ ,  $\therefore A$  and  $B$  are equivalent.
  - b**  $A: x$  is even,  $B: x^2$  is even  
If  $x$  is even, then  $x^2$  is even.  $\therefore A \Rightarrow B$   
However, if  $x^2$  is even but not a perfect square, then  $x$  is irrational and hence not even.  
For example, if  $x^2 = 2$ , then  $x = \pm\sqrt{2}$  neither of which are even.  
 $\therefore B \not\Rightarrow A$   
 $\therefore A$  and  $B$  are not equivalent.



- 6 The statement which we are trying to establish the truth about is:

**“Every card which has a D on one side has a 7 on the other”.**

We are **only** concerned with cards that have (or might have) a D on one side.



We need to turn this card to make sure that the other side is a 7. Otherwise the statement would be false.



We need to turn this card to make sure that the other side is **not** a D. Otherwise the statement would be false.



We do not need to turn this card since the statement says nothing about cards with a letter other than D.



If we turn this card and the other side is a D, then the statement remains valid.

If it is **not** a D, it is of no concern to us.

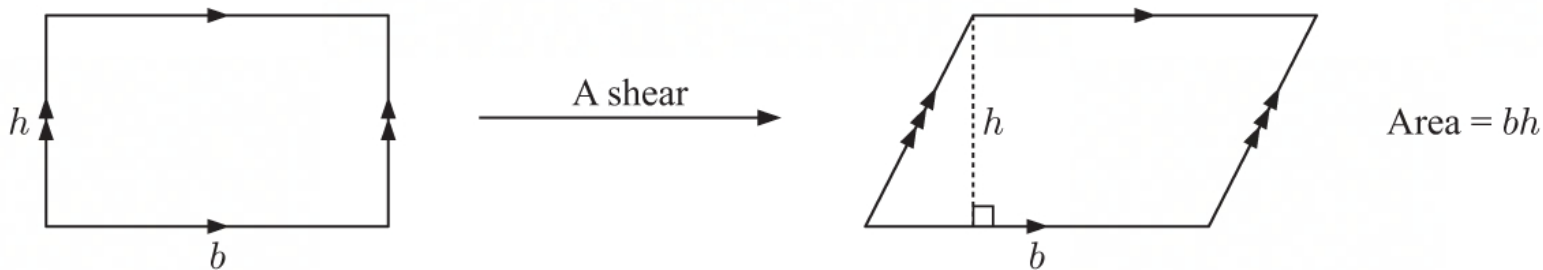
In either case, the statement remains valid, so we do not need to turn this card.

$\therefore$  we need to turn cards D and 3. We do not need to turn cards K and 7.

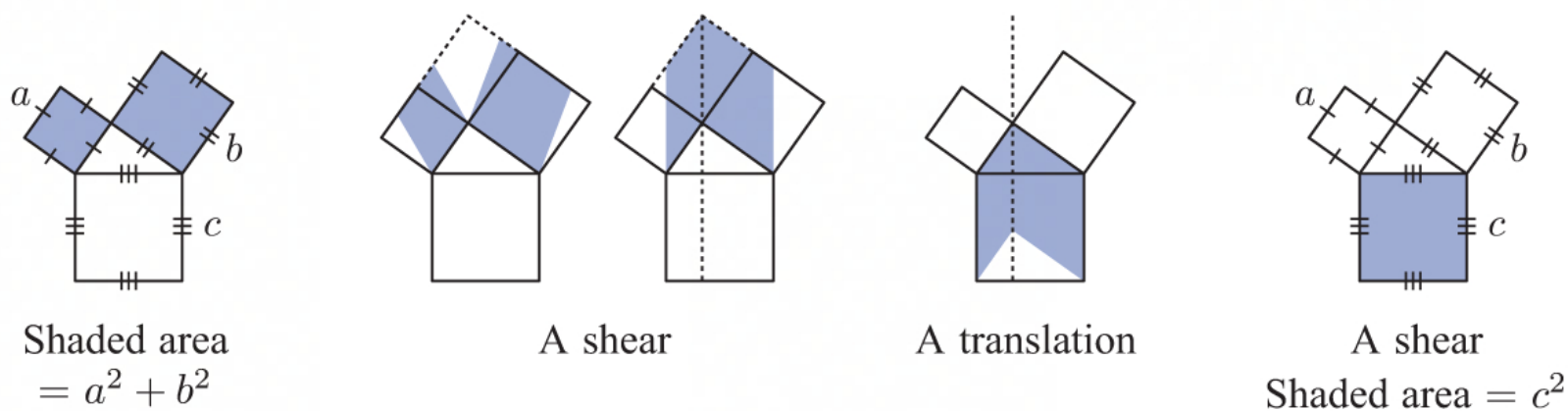
## INVESTIGATION

## PYTHAGORAS' THEOREM

- 1 When a shear is applied to a plane figure, the area does not change. For example, when a shear is applied to a parallelogram, the base length and height of the parallelogram do not change. Hence a shear does not change the area of a parallelogram.



When a translation is applied to a plane figure, the area does not change, as the object and image are congruent.



The shaded area does not change in the sequence of shears and translations above.

$$\therefore a^2 + b^2 = c^2$$

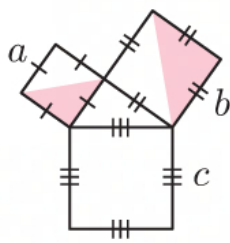


- 2 When a shear is applied to a triangle, the base length and height of the triangle do not change. Hence a shear does not change the area of a triangle.

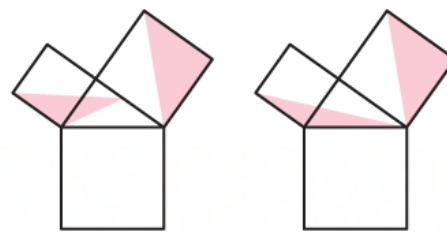


When a rotation is applied to a plane figure, the area does not change, as the object and image are congruent.

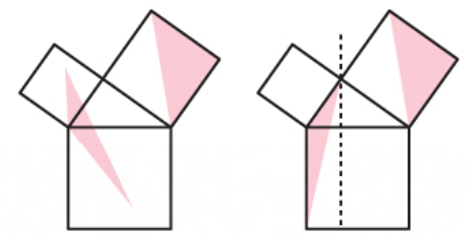
Using shears and rotations, we make the side lengths of the triangles match the side lengths of the area we are trying to fit them into.



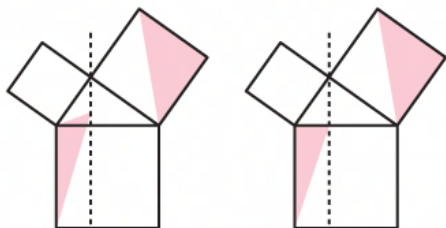
$$\text{Shaded area} = \frac{1}{2}a^2 + \frac{1}{2}b^2$$



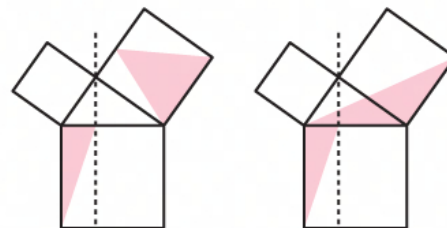
A shear



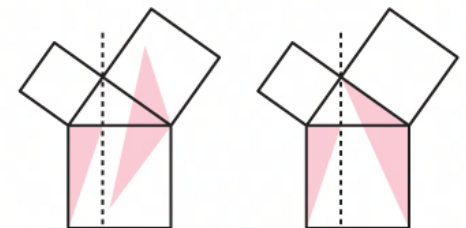
A rotation



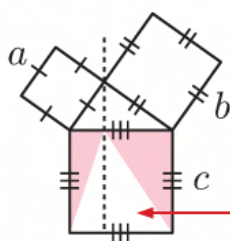
A shear



A shear



A rotation



$$\text{area of white triangle} = \frac{1}{2}c^2$$

A shear

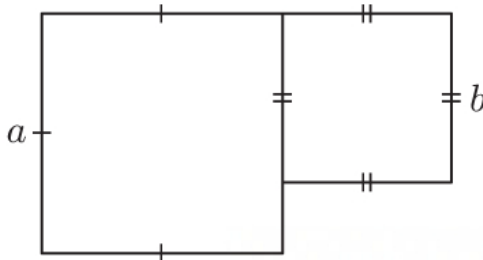
$$\begin{aligned} \text{Shaded area} &= c^2 - \frac{1}{2}c^2 \\ &= \frac{1}{2}c^2 \end{aligned}$$

The shaded area does not change in the sequence of shears and rotations above.

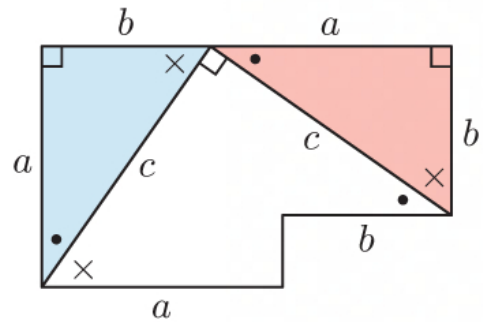
$$\therefore \frac{1}{2}a^2 + \frac{1}{2}b^2 = \frac{1}{2}c^2$$

$$\therefore a^2 + b^2 = c^2$$

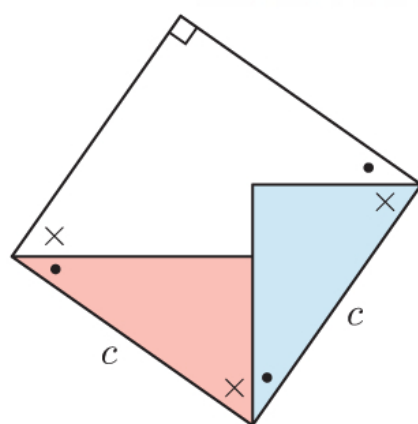
**3**



We have started with two squares with areas  $a^2$  and  $b^2$ . The total area is  $a^2 + b^2$ .



We have constructed the figure alongside (which has the same area as the figure above) with the side lengths shown. The hypotenuse of the right angled triangles is  $c$ , and  $\bullet + \times = 90^\circ$ .



We translate the red and blue triangles to the positions shown, which does not change the area of the figure.

We now have a square with side length  $c$ , and area  $c^2$ .

$\therefore a^2 + b^2 = c^2$

## EXERCISE 9B

**1 a** If  $x = -2$  then  $x^2 - x - 6 = (-2)^2 - (-2) - 6$   
 $= 4 + 2 - 6$   
 $= 0$

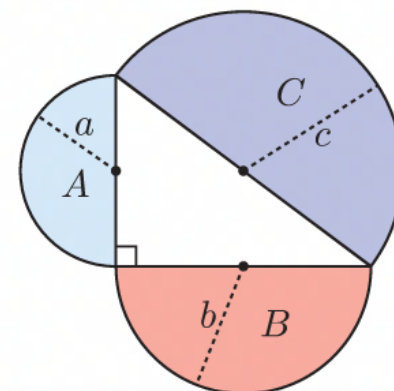
**b** The converse is “If  $x^2 - x - 6 = 0$  then  $x = -2$ .”  
 However if  $x^2 - x - 6 = 0$  then  $(x + 2)(x - 3) = 0$   
 $\therefore x = -2$  or  $3$   
 The converse is false, since  $x$  could be  $3$ .

**2** The lengths of the sides of the triangle are  $2a$ ,  $2b$ , and  $2c$ .

$$\begin{aligned} \therefore (2a)^2 + (2b)^2 &= (2c)^2 && \{\text{Pythagoras}\} \\ \therefore 4a^2 + 4b^2 &= 4c^2 \\ \therefore a^2 + b^2 &= c^2 \end{aligned}$$

Now  $A = \frac{\pi a^2}{2}$ ,  $B = \frac{\pi b^2}{2}$ ,  $C = \frac{\pi c^2}{2}$

$$\begin{aligned} \therefore A + B &= \frac{\pi a^2}{2} + \frac{\pi b^2}{2} \\ &= \frac{\pi}{2}(a^2 + b^2) \\ &= \frac{\pi}{2} \times c^2 \\ &= C \end{aligned}$$



**3**  $x = a^2 - b^2, \quad y = 2ab, \quad z = a^2 + b^2, \quad a, b \in \mathbb{N}$

$$\begin{aligned} x^2 + y^2 &= (a^2 - b^2)^2 + (2ab)^2 \\ &= a^4 - 2a^2b^2 + b^4 + 4a^2b^2 \\ &= a^4 + 2a^2b^2 + b^4 \\ &= (a^2 + b^2)^2 \\ &= z^2 \end{aligned}$$

**4** Let the middle number be  $x$ .

$\therefore$  the sum of the three consecutive integers is  $(x-1) + x + (x+1) = 3x$  which is divisible by 3.

**5** Let the middle number be  $x$ .

$\therefore$  the product of the three consecutive integers, increased by the middle integer is

$$\begin{aligned} (x-1)x(x+1) + x &= (x^2 - x)(x+1) + x \\ &= x^3 + x^2 - x^2 - x + x \\ &= x^3 \quad \text{which is a perfect cube.} \end{aligned}$$

**6**  $(a-b)^2 \geq 0$  for all  $a, b \in \mathbb{R}$

$$\begin{aligned} \therefore a^2 - 2ab + b^2 &\geq 0 \\ \therefore a^2 + b^2 &\geq 2ab \\ \therefore \frac{a^2 + b^2}{2} &\geq ab \end{aligned}$$

**7**  $\sin 2\theta \tan \theta = 2 \sin \theta \cos \theta \times \frac{\sin \theta}{\cos \theta} \quad \{\text{double angle formula, definition of } \tan \theta\}$   
 $= 2 \sin^2 \theta$

**8** Let the 3-digit number be “ $abc$ ” which has value  $100a + 10b + c$ .

This number written backwards is “ $cba$ ” which has value  $100c + 10b + a$ .

If “ $abc$ ”  $>$  “ $cba$ ”, then  $S = \text{“}abc\text{”} - \text{“}cba\text{”}$

$$\begin{aligned} &= 100a + 10b + c - (100c + 10b + a) \\ &= 100a + 10b + c - 100c - 10b - a \\ &= 99a - 99c \\ &= 99(a - c) \end{aligned}$$

Similarly, if “ $cba$ ”  $>$  “ $abc$ ”, then  $S = 99(c - a)$ .

$$\therefore S = 99|a - c|$$

Now  $0 < |a - c| \leq 9$  since  $a \neq c$ .

Let  $S'$  be  $S$  written backwards.

Consider the following table with each possible value of  $|a - c|$ :

$ a - c $	$S$	$S'$	$S + S'$
1	$99 \times 1 = 099$	990	1089
2	$99 \times 2 = 198$	891	1089
3	$99 \times 3 = 297$	792	1089
4	$99 \times 4 = 396$	693	1089
5	$99 \times 5 = 495$	594	1089
6	$99 \times 6 = 594$	495	1089
7	$99 \times 7 = 693$	396	1089
8	$99 \times 8 = 792$	297	1089
9	$99 \times 9 = 891$	198	1089

For each value of  $|a - c|$ ,  $S + S' = 1089$ .

$\therefore$  when  $S$  is written backwards and added to  $S$ , the result is always 1089.

**9 a**  $4x^2 = 3x$

$\therefore 4x = 3$   Incorrect step

$\therefore x = \frac{3}{4}$

Correct solution:

$$4x^2 = 3x$$

$$\therefore 4x^2 - 3x = 0$$


$$\therefore x(4x - 3) = 0$$

$$\therefore x = 0 \quad \text{or} \quad 4x - 3 = 0$$

$$\therefore x = 0 \quad \text{or} \quad 4x = 3$$

$$\therefore x = 0 \quad \text{or} \quad x = \frac{3}{4}$$

**b**  $(x + 3)(2 - x) = 4$

$\therefore x + 3 = 4 \quad \text{or} \quad 2 - x = 4$   Incorrect step

$\therefore x = 1 \quad \text{or} \quad x = -2$

Correct solution:

$$(x + 3)(2 - x) = 4$$

$$\therefore 2x - x^2 + 6 - 3x = 4$$

$$\therefore x^2 + x - 2 = 0$$

$$\therefore (x + 2)(x - 1) = 0$$

$$\therefore x = -2 \quad \text{or} \quad 1$$

## EXERCISE 9C

**1 a**  $(a + b)^2 - (a - b)^2 = [a + b + (a - b)][a + b - (a - b)] \quad \{\text{difference of two squares}\}$   
 $= 2a \times 2b$   
 $= 4ab$

**b**  $(a + b)^2 - 4(a - b)^2 = (a + b)^2 - [2(a - b)]^2$   
 $= [a + b + 2(a - b)][a + b - 2(a - b)] \quad \{\text{difference of two squares}\}$   
 $= (a + b + 2a - 2b)(a + b - 2a + 2b)$   
 $= (3a - b)(3b - a)$   
 $= (3b - a)(3a - b)$



$$\begin{aligned}
 & \mathbf{2} \quad x^2 + (a-3)x + 2a = 0 \\
 \Leftrightarrow & \left(x + \frac{a-3}{2}\right)^2 - \left(\frac{a-3}{2}\right)^2 + 2a = 0 \quad \{\text{completing the square}\} \\
 \Leftrightarrow & \left(x + \frac{a-3}{2}\right)^2 = \frac{(a-3)^2}{4} - 2a
 \end{aligned}$$

The equation has real solutions if and only if

$$\begin{aligned}
 & \frac{(a-3)^2}{4} - 2a \geq 0 \\
 \Leftrightarrow & (a-3)^2 - 8a \geq 0 \\
 \Leftrightarrow & a^2 - 6a + 9 - 8a \geq 0 \\
 \Leftrightarrow & a^2 - 14a + 9 \geq 0 \\
 \Leftrightarrow & (a-7)^2 - 49 + 9 \geq 0 \\
 \Leftrightarrow & (a-7)^2 \geq 40 \\
 \Leftrightarrow & a-7 \geq \sqrt{40} \quad \text{or} \quad a-7 \leq -\sqrt{40} \\
 \Leftrightarrow & a \geq 7 + \sqrt{40} \approx 13.3 \quad \text{or} \quad a \leq 7 - \sqrt{40} \approx 0.675
 \end{aligned}$$

$\therefore$  the smallest positive integer  $a$  for which the equation has real solutions is  $a = 14$ .

$$\begin{aligned}
 & \mathbf{3} \quad (x-y)^5 + (x-y)^3 = 0 \\
 \Leftrightarrow & (x-y)^3[(x-y)^2 + 1] = 0 \\
 \Leftrightarrow & (x-y)^3 = 0 \quad \text{or} \quad (x-y)^2 + 1 = 0 \\
 \Leftrightarrow & x = y \quad \text{or} \quad (x-y)^2 = -1 \\
 & \quad \quad \quad \text{which is not possible}
 \end{aligned}$$

$$\therefore (x-y)^5 + (x-y)^3 = 0 \Leftrightarrow x = y$$

$$\begin{aligned}
 & \mathbf{4} \quad \mathbf{a} \quad (n^2 - 2n + 2)(n^2 + 2n + 2) = n^4 + \cancel{2n^3} + \cancel{2n^2} - \cancel{2n^3} - \cancel{4n^2} - \cancel{4n} + \cancel{2n^2} + \cancel{4n} + 4 \\
 & \quad \quad \quad = n^4 + 4
 \end{aligned}$$

$$\begin{aligned}
 & \mathbf{b} \quad \text{Notice that } n^2 - 2n + 2 = n^2 - 2n + 1 + 1 = (n-1)^2 + 1 \\
 & \quad \quad \text{and } n^2 + 2n + 2 = n^2 + 2n + 1 + 1 = (n+1)^2 + 1
 \end{aligned}$$

$$\begin{aligned}
 \text{For } n \neq 1, \quad (n-1)^2 > 0 \quad \quad \quad \text{and for } n \neq -1, \quad (n+1)^2 > 0 \\
 \therefore (n-1)^2 + 1 > 1 \quad \quad \quad \therefore (n+1)^2 + 1 > 1 \\
 \therefore n^2 - 2n + 2 > 1 \quad \quad \quad \therefore n^2 + 2n + 2 > 1
 \end{aligned}$$

So, for  $n \neq \pm 1$ ,  $n^4 + 4 = (n^2 - 2n + 2)(n^2 + 2n + 2)$  is composite, since  $n^2 - 2n + 2 > 1$  and  $n^2 + 2n + 2 > 1$ .

Now, for  $n = 1$ ,  $n^4 + 4 = 1^4 + 4 = 5$  which is prime  
and for  $n = -1$ ,  $n^4 + 4 = (-1)^4 + 4 = 5$  which is also prime.

$$\therefore n^4 + 4 \text{ is prime} \Leftrightarrow n = \pm 1$$

$$\begin{aligned}
 & \mathbf{5} \quad \mathbf{a} \quad (k+1)^2 - (k-1)^2 = [k+1 + (k-1)][k+1 - (k-1)] \quad \{\text{difference of two squares}\} \\
 & \quad \quad \quad = 2k \times 2 \\
 & \quad \quad \quad = 4k
 \end{aligned}$$

$$\begin{aligned}
 & \mathbf{b} \quad \mathbf{i} \quad \text{Substituting } 4k = 40 \text{ or } k = 10 \text{ gives } (10+1)^2 - (10-1)^2 = 40 \\
 & \quad \quad \quad \therefore 11^2 - 9^2 = 40 \\
 & \quad \quad \quad \therefore 121 - 81 = 40
 \end{aligned}$$

So, 121 and 81 are two square numbers with difference 40.

- ii Substituting  $4k = 100$  or  $k = 25$  gives  $(25 + 1)^2 - (25 - 1)^2 = 100$   
 $\therefore 26^2 - 24^2 = 100$   
 $\therefore 676 - 576 = 100$

So, 676 and 576 are two square numbers with difference 100.

6 a

$$\begin{aligned}
 & a = b \\
 \Leftrightarrow & a^2 = ab \\
 \Leftrightarrow & a^2 - b^2 = ab - b^2 \\
 \Leftrightarrow & (a - b)(a + b) = b(a - b) \\
 \Leftrightarrow & a + b = b \quad \leftarrow \text{incorrect step, if } a = b \text{ then } a - b = 0, \text{ and we} \\
 \Leftrightarrow & 2a = a \quad \text{cannot divide both sides by } a - b. \\
 \Leftrightarrow & 2 = 1 \quad \leftarrow \text{incorrect step, if } a = 0 \text{ then we cannot divide} \\
 & \quad \text{both sides by } a.
 \end{aligned}$$

b

$$\begin{aligned}
 & \frac{x + 10}{x - 6} - 5 = \frac{4x - 40}{13 - x} \\
 \Leftrightarrow & \frac{x + 10 - 5(x - 6)}{x - 6} = \frac{4x - 40}{13 - x} \\
 \Leftrightarrow & \frac{4x - 40}{6 - x} = \frac{4x - 40}{13 - x} \\
 \Leftrightarrow & 6 - x = 13 - x \quad \leftarrow \text{incorrect step, if } 4x - 40 = 0 \text{ then we cannot} \\
 \Leftrightarrow & 6 = 13 \quad \text{divide both sides by } 4x - 40 \text{ and take the} \\
 & \quad \text{reciprocal.}
 \end{aligned}$$

Correct solution:

$$\begin{aligned}
 & \frac{4x - 40}{6 - x} = \frac{4x - 40}{13 - x} \\
 \Leftrightarrow & (4x - 40)(13 - x) = (6 - x)(4x - 40) \\
 \Leftrightarrow & 52x - \cancel{4x^2} - 520 + \cancel{40x} = 24x - 240 - \cancel{4x^2} + \cancel{40x} \\
 \Leftrightarrow & 28x = 280 \\
 \Leftrightarrow & x = 10
 \end{aligned}$$

7 a

$$\begin{aligned}
 & 6x - 12 = 3(x - 2) \\
 \Leftrightarrow & 6x - 12 + 3(x - 2) = 0 \quad \leftarrow \text{incorrect step, we should have subtracted} \\
 \Leftrightarrow & 12x - 24 = 0 \quad 3(x - 2) \text{ from both sides.} \\
 \Leftrightarrow & x = 2
 \end{aligned}$$

Correct solution:

$$\begin{aligned}
 & 6x - 12 = 3(x - 2) \\
 \Leftrightarrow & 6x - 12 - 3(x - 2) = 0 \\
 \Leftrightarrow & 6x - 12 - 3x + 6 = 0 \\
 \Leftrightarrow & 3x = 6 \\
 \Leftrightarrow & x = 2
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & x^2 - 6x + 9 = 0 \\
 \Leftrightarrow & x^2 - 6x = -9 \\
 \Leftrightarrow & x(x - 6) = 3(-3) \\
 \Leftrightarrow & x = 3 \text{ or } x - 6 = -3 \quad \leftarrow \text{incorrect step, in general this is not true.} \\
 \Leftrightarrow & x = 3
 \end{aligned}$$

Correct solution:

$$\begin{aligned}
 & x^2 - 6x + 9 = 0 \\
 \Leftrightarrow & (x - 3)^2 = 0 \\
 \Leftrightarrow & x = 3
 \end{aligned}$$

$$\begin{aligned}
 8 \quad \text{a} \quad \text{i} \quad & \frac{1}{m} + \frac{1}{3m} = \frac{3}{3m} + \frac{1}{3m} \\
 & = \frac{4}{3m}
 \end{aligned}$$

$$\begin{aligned}
 \text{ii} \quad \text{For } n \neq 0, -1, \quad & \frac{1}{n+1} + \frac{1}{n^2+n} = \frac{n}{n(n+1)} + \frac{1}{n(n+1)} \\
 & = \frac{n+1}{n(n+1)} \\
 & = \frac{1}{n}
 \end{aligned}$$

**b** No,  $\frac{1}{n+1} + \frac{1}{n^2+n}$  is undefined for  $n = 0, -1$  while  $\frac{1}{n}$  is only undefined for  $n = 0$ . It would be incorrect to say that  $\frac{1}{n+1} + \frac{1}{n^2+n}$  and  $\frac{1}{n}$  are equivalent even though  $\frac{1}{n+1} + \frac{1}{n^2+n} = \frac{1}{n}$  for  $n \neq 0, -1$ .

$$\begin{aligned}
 9 \quad & x + y + z = x - (-y - z) \text{ is a factor of } x^3 + y^3 + z^3 - 3xyz \\
 \Leftrightarrow & (-y - z)^3 + y^3 + z^3 - 3(-y - z)yz = 0 \quad \{\text{Factor theorem}\} \\
 \text{Now} \quad & (-y - z)^3 + y^3 + z^3 - 3(-y - z)yz \\
 & = -\cancel{y^3} - 3y^2z - 3yz^2 - \cancel{z^3} + \cancel{y^3} + \cancel{z^3} + (3y + 3z)yz \\
 & = -\cancel{3y^2z} - \cancel{3yz^2} + \cancel{3yz^2} + \cancel{3yz^2} \\
 & = 0
 \end{aligned}$$

$\therefore x + y + z$  is a factor of  $x^3 + y^3 + z^3 - 3xyz$

## EXERCISE 9D

$$\begin{aligned}
 1 \quad \text{Let } & x = 0.\overline{9} = 0.999\dots \\
 \Rightarrow & 10x = 9.999\dots \\
 \Rightarrow & 10x = 9 + x \\
 \Rightarrow & 9x = 9 \\
 \Rightarrow & x = 1 \\
 \Rightarrow & x \in \mathbb{Z} \\
 \Rightarrow & 0.\overline{9} \in \mathbb{Z}
 \end{aligned}$$

$$\mathbf{2} \quad \mathbf{a} \quad \text{Let } x = 0.\overline{4} = 0.444\dots$$

$$\Rightarrow 10x = 4.444\dots$$

$$\Rightarrow 10x = 4 + x$$

$$\Rightarrow 9x = 4$$

$$\Rightarrow x = \frac{4}{9}$$

$$\Rightarrow x \in \mathbb{Q}$$

$$\Rightarrow 0.\overline{4} \in \mathbb{Q}$$

$$\mathbf{b} \quad \text{Let } x = 0.\overline{23} = 0.2323\dots$$

$$\Rightarrow 100x = 23.2323\dots$$

$$\Rightarrow 100x = 23 + x$$

$$\Rightarrow 99x = 23$$

$$\Rightarrow x = \frac{23}{99}$$

$$\Rightarrow x \in \mathbb{Q}$$

$$\Rightarrow 0.\overline{23} \in \mathbb{Q}$$

$$\mathbf{c} \quad \text{Let } x = 0.0\overline{79} = 0.07979\dots$$

$$\Rightarrow 1000x = 79.7979\dots \quad \text{and} \quad 10x = 0.7979\dots$$

$$\Rightarrow 1000x = 79 + 10x$$

$$\Rightarrow 990x = 79$$

$$\Rightarrow x = \frac{79}{990}$$

$$\Rightarrow x \in \mathbb{Q}$$

$$\Rightarrow 0.0\overline{79} \in \mathbb{Q}$$

$$\mathbf{3} \quad (4 - \sqrt{2}) + \frac{4 - \sqrt{2}}{3 - \sqrt{2}} + \frac{4 - \sqrt{2}}{(3 - \sqrt{2})^2} + \frac{4 - \sqrt{2}}{(3 - \sqrt{2})^3} + \dots$$

is a geometric series with  $u_1 = 4 - \sqrt{2}$  and  $r = \frac{1}{3 - \sqrt{2}}$ .

$$\begin{aligned} \therefore (4 - \sqrt{2}) + \frac{4 - \sqrt{2}}{3 - \sqrt{2}} + \frac{4 - \sqrt{2}}{(3 - \sqrt{2})^2} + \frac{4 - \sqrt{2}}{(3 - \sqrt{2})^3} + \dots &= \frac{4 - \sqrt{2}}{1 - \left(\frac{1}{3 - \sqrt{2}}\right)} \quad \left\{ \frac{u_1}{1 - r} \right\} \\ &= \frac{4 - \sqrt{2}}{\frac{3 - \sqrt{2}}{3 - \sqrt{2}} - \left(\frac{1}{3 - \sqrt{2}}\right)} \\ &= \frac{4 - \sqrt{2}}{\left(\frac{2 - \sqrt{2}}{3 - \sqrt{2}}\right)} \\ &= (4 - \sqrt{2}) \times \frac{3 - \sqrt{2}}{2 - \sqrt{2}} \\ &= \frac{12 - 4\sqrt{2} - 3\sqrt{2} + 2}{2 - \sqrt{2}} \\ &= \frac{14 - 7\sqrt{2}}{2 - \sqrt{2}} \\ &= \frac{7(2 - \sqrt{2})}{2 - \sqrt{2}} \end{aligned}$$

$$\therefore (4 - \sqrt{2}) + \frac{4 - \sqrt{2}}{3 - \sqrt{2}} + \frac{4 - \sqrt{2}}{(3 - \sqrt{2})^2} + \frac{4 - \sqrt{2}}{(3 - \sqrt{2})^3} + \dots = 7 \in \mathbb{Z}$$

$$\therefore (4 - \sqrt{2}) + \frac{4 - \sqrt{2}}{3 - \sqrt{2}} + \frac{4 - \sqrt{2}}{(3 - \sqrt{2})^2} + \frac{4 - \sqrt{2}}{(3 - \sqrt{2})^3} + \dots \in \mathbb{Z}$$



**4** Let  $x$  and  $y$  be rational numbers.

By definition, there exists  $p, q \in \mathbb{Z}$ ,  $q \neq 0$  so that  $x = \frac{p}{q}$ .

By definition, there exists  $r, s \in \mathbb{Z}$ ,  $s \neq 0$  so that  $y = \frac{r}{s}$ .

$$\text{So, } x - y = \frac{p}{q} - \frac{r}{s} = \frac{ps - qr}{qs}$$

Since  $p, q, r, s$  are all integers,  $ps - qr$  is an integer which we call  $P$ .

Since  $q, s$  are non-zero integers,  $qs$  is a non-zero integer which we call  $Q$ .

$$\therefore x - y = \frac{ps - qr}{qs} = \frac{P}{Q} \text{ where } P, Q \in \mathbb{Z}, Q \neq 0$$

$\therefore$  by definition,  $x - y$  is a rational number.

$\therefore$  the difference between any two rational numbers is also a rational number.

**5** Let  $x$  and  $y$  be rational numbers.

By definition, there exists  $p, q \in \mathbb{Z}$ ,  $q \neq 0$  so that  $x = \frac{p}{q}$ .

By definition, there exists  $r, s \in \mathbb{Z}$ ,  $s \neq 0$  so that  $y = \frac{r}{s}$ .

$$\text{So, } xy = \frac{pr}{qs}$$

Since  $p$  and  $r$  are integers,  $pr$  is an integer which we call  $P$ .

Since  $q$  and  $s$  are non-zero integers,  $qs$  is a non-zero integer which we call  $Q$ .

$$\therefore xy = \frac{pr}{qs} = \frac{P}{Q} \text{ where } P, Q \in \mathbb{Z}, Q \neq 0$$

$\therefore$  by definition,  $xy$  is a rational number.

$\therefore$  the product of any two rational numbers is also a rational number.

**6** Let the two odd integers be  $2a + 1$  and  $2b + 1$ ,  $a, b \in \mathbb{Z}$ .

$$\begin{aligned} \text{Now } (2a + 1)(2b + 1) &= 4ab + 2a + 2b + 1 \\ &= 2(2ab + a + b) + 1 \end{aligned}$$

Since  $a$  and  $b$  are integers,  $2ab + a + b$  is an integer which we call  $c$ .

$$\therefore (2a + 1)(2b + 1) = 2c + 1 \text{ which is odd.}$$

$\therefore$  the product of two odd integers is odd.

**7** Let the two odd integers be  $p = 2a + 1$  and  $q = 2b + 1$ ,  $a, b \in \mathbb{Z}$ .

$$\begin{aligned} \text{Now } p^2 - q^2 &= (p + q)(p - q) \\ &= (2a + 1 + 2b + 1)(2a + 1 - 2b - 1) \\ &= (2a + 2b + 2)(2a - 2b) \\ &= 4(a + b + 1)(a - b) \\ &= 4(a^2 - ab + ba - b^2 + a - b) \\ &= 4(a^2 + a - b^2 - b) \\ &= 4[a(a + 1) - b(b + 1)] \end{aligned}$$

As  $a$  and  $a + 1$  are consecutive integers,  $a$  or  $a + 1$  is even, so  $a(a + 1)$  is even.

Similarly,  $b(b + 1)$  is even.

$$\text{Let } a(a + 1) = 2c, \quad c \in \mathbb{Z}$$

$$\text{and } b(b + 1) = 2d, \quad d \in \mathbb{Z}$$

$$\begin{aligned} \therefore p^2 - q^2 &= 4(2c - 2d) \\ &= 8(c - d) \text{ where } c - d \text{ is an integer} \end{aligned}$$

$\therefore p^2 - q^2$  is divisible by 8.

- 8 Let the two consecutive odd integers be  $p = 2a + 3$  and  $q = 2a + 1$ ,  $a \in \mathbb{Z}$ .

$$\begin{aligned}
 \text{Now } p^3 - q^3 - 2 &= (2a + 3)^3 - (2a + 1)^3 - 2 \\
 &= (2a)^3 + 3(2a)^2(3) + 3(2a)(3)^2 + (3)^3 - [(2a)^3 + 3(2a)^2(1) + 3(2a)(1)^2 + (1)^3] - 2 \\
 &= 8a^3 + 36a^2 + 54a + 27 - (8a^3 + 12a^2 + 6a + 1) - 2 \\
 &= \cancel{8a^3} + 36a^2 + 54a + 27 - \cancel{8a^3} - 12a^2 - 6a - 1 - 2 \\
 &= 24a^2 + 48a + 24 \\
 &= 24(a^2 + 2a + 1) \\
 &= 24(a + 1)^2 \quad \text{where } (a + 1)^2 \text{ is an integer.}
 \end{aligned}$$

$\therefore p^3 - q^3 - 2$  is divisible by 24.

- 9  $ax^2 + bx + c = 0$  has rational root  $\frac{r}{s}$ ,  $r, s \in \mathbb{Z}$ ,  $s \neq 0$ .

$\therefore$  the other root (which may be repeated) must also be rational, say  $\frac{t}{u}$ ,  $t, u \in \mathbb{Z}$ ,  $u \neq 0$ .

$$\begin{aligned}
 \therefore ax^2 + bx + c &= (sx - r)(ux - t), \quad r, s, t, u \in \mathbb{Z} \\
 &= (su)x^2 - (st + ru)x + rt
 \end{aligned}$$

Equating coefficients of  $x^2$ ,  $a = su$ ,  $u \in \mathbb{Z}$

$\therefore s$  is a factor of  $a$ .

Equating constant terms,  $c = rt$ ,  $t \in \mathbb{Z}$

$\therefore r$  is a factor of  $c$ .

## EXERCISE 9E

- 1 If  $n$  is odd, then  $n = 2k + 1$  for some  $k \in \mathbb{Z}$ .

$$\begin{aligned}
 \therefore n^2 &= (2k + 1)^2 \\
 &= 4k^2 + 4k + 1 \\
 &= 4(k^2 + k) + 1 \quad \text{which is odd.}
 \end{aligned}$$

If  $n$  is even, then  $n = 2k$  for some  $k \in \mathbb{Z}$ .

$$\begin{aligned}
 \therefore n^2 &= (2k)^2 \\
 &= 4k^2 \quad \text{which is even.}
 \end{aligned}$$

We can see that if  $n$  is an integer such that  $n^2$  is even, then  $n$  must be even.

### 2 Proof by exhaustion:

If we divide  $n$  by 3 then the remainder will be 0, 1, or 2. Hence every integer can be written in one of the forms  $3k$ ,  $3k + 1$ , or  $3k + 2$ , for some  $k \in \mathbb{Z}$ .

$$\text{Let } N = n^3 - n = n(n^2 - 1) = (n - 1)n(n + 1).$$

If  $n = 3k$  then the factor  $n$  is divisible by 3, and so  $N$  is divisible by 3.

If  $n = 3k + 1$  then the factor  $(n - 1) = 3k$  is divisible by 3, so  $N$  is divisible by 3.

$$\begin{aligned}
 \text{If } n = 3k + 2 \text{ then the factor } (n + 1) &= 3k + 2 + 1 \\
 &= 3k + 3 \\
 &= 3(k + 1)
 \end{aligned}$$

which is divisible by 3, so  $N$  is divisible by 3.

In all cases,  $N$  is divisible by 3, so  $n^3 - n$  is divisible by 3 for all  $n \in \mathbb{Z}$ .

**3**  $n^3 - n$  is divisible by 3 for all  $n \in \mathbb{Z}$  {from **2**}.  $\therefore n^3 - n = 3k$  for some  $k \in \mathbb{Z}$ .

$$\begin{aligned}\text{Now } n^3 + 2n &= (n^3 - n) + 3n \\ &= 3k + 3n, \quad k \in \mathbb{Z} \\ &= 3(k + n)\end{aligned}$$

$\therefore n^3 + 2n$  is divisible by 3 for all  $n \in \mathbb{Z}$ .

**4 Proof by exhaustion:**

If we divide an odd integer  $n$  by 8 then the remainder will be 1, 3, 5, or 7. Hence every odd integer can be written in one of the forms  $8k + 1$ ,  $8k + 3$ ,  $8k + 5$ , or  $8k + 7$ , for some  $k \in \mathbb{Z}$ .

$$\begin{aligned}\text{If } n = 8k + 1 \text{ then } (8k + 1)^2 - 1 &= 64k^2 + 16k + 1 - 1 \\ &= 64k^2 + 16k \\ &= 8(8k^2 + 2k) \text{ which is divisible by 8.}\end{aligned}$$

$$\begin{aligned}\text{If } n = 8k + 3 \text{ then } (8k + 3)^2 - 1 &= 64k^2 + 48k + 9 - 1 \\ &= 64k^2 + 48k + 8 \\ &= 8(8k^2 + 6k + 1) \text{ which is divisible by 8.}\end{aligned}$$

$$\begin{aligned}\text{If } n = 8k + 5 \text{ then } (8k + 5)^2 - 1 &= 64k^2 + 80k + 25 - 1 \\ &= 64k^2 + 80k + 24 \\ &= 8(8k^2 + 10k + 3) \text{ which is divisible by 8.}\end{aligned}$$

$$\begin{aligned}\text{If } n = 8k + 7 \text{ then } (8k + 7)^2 - 1 &= 64k^2 + 112k + 49 - 1 \\ &= 64k^2 + 112k + 48 \\ &= 8(8k^2 + 14k + 6) \text{ which is divisible by 8.}\end{aligned}$$

In all cases,  $n^2 - 1$  is divisible by 8.

$\therefore n^2 - 1$  is divisible by 8 for all odd integers  $n$ .

**5 Proof by exhaustion:**

If we divide  $n$  by 7 then the remainder will be 0, 1, 2, 3, 4, 5, or 6. Hence every integer can be written in one of the forms  $7k$ ,  $7k + 1$ ,  $7k + 2$ ,  $7k + 3$ ,  $7k + 4$ ,  $7k + 5$ , or  $7k + 6$ , for some  $k \in \mathbb{Z}$ .

$$\begin{aligned}\text{If } n = 7k \text{ then } (7k)^2 + 4 &= 49k^2 + 4 \\ &= 7(7k^2) + 4 \text{ which is not divisible by 7.}\end{aligned}$$

$$\begin{aligned}\text{If } n = 7k + 1 \text{ then } (7k + 1)^2 + 4 &= 49k^2 + 14k + 1 + 4 \\ &= 49k^2 + 14k + 5 \\ &= 7(7k^2 + 2k) + 5 \text{ which is not divisible by 7.}\end{aligned}$$

$$\begin{aligned}\text{If } n = 7k + 2 \text{ then } (7k + 2)^2 + 4 &= 49k^2 + 28k + 4 + 4 \\ &= 49k^2 + 28k + 8 \\ &= 7(7k^2 + 4k + 1) + 1 \text{ which is not divisible by 7.}\end{aligned}$$

$$\begin{aligned}\text{If } n = 7k + 3 \text{ then } (7k + 3)^2 + 4 &= 49k^2 + 42k + 9 + 4 \\ &= 49k^2 + 42k + 13 \\ &= 7(7k^2 + 6k + 1) + 6 \text{ which is not divisible by 7.}\end{aligned}$$

$$\begin{aligned}\text{If } n = 7k + 4 \text{ then } (7k + 4)^2 + 4 &= 49k^2 + 56k + 16 + 4 \\ &= 49k^2 + 56k + 20 \\ &= 7(7k^2 + 8k + 2) + 6 \text{ which is not divisible by 7.}\end{aligned}$$



If  $n = 7k + 5$  then  $(7k + 5)^2 + 4 = 49k^2 + 70k + 25 + 4$   
 $= 49k^2 + 70k + 29$   
 $= 7(7k^2 + 10k + 4) + 1$  which is not divisible by 7.

If  $n = 7k + 6$  then  $(7k + 6)^2 + 4 = 49k^2 + 84k + 36 + 4$   
 $= 49k^2 + 84k + 40$   
 $= 7(7k^2 + 12k + 5) + 5$  which is not divisible by 7.

In all cases,  $n^2 + 4$  is not divisible by 7.

$\therefore$  7 never divides  $n^2 + 4$  for all  $n \in \mathbb{Z}$ .

**6** Let  $n$  be a positive integer, and  $x = (n + 1)! + 2$ .

Now  $(n + 1)! = (n + 1) \times n \times \dots \times 2$  is divisible by  $i \in \mathbb{Z}$  such that  $2 \leq i \leq n + 1$

$\therefore (n + 1)! = ic_i$  for all  $i \in \mathbb{Z}$  such that  $2 \leq i \leq n + 1$ , where  $c_i \in \mathbb{Z}$  is a constant.

Let  $k \in \mathbb{Z}$  such that  $0 \leq k \leq n - 1$ .

Then  $x + k = (n + 1)! + 2 + k$   
 $= (k + 2)c_{k+2} + (k + 2) \quad \{0 \leq k \leq n - 1, \therefore 2 \leq k + 2 \leq n + 1\}$   
 $= (k + 2)(c_{k+2} + 1)$

which is divisible by  $k + 2$ , and hence composite.

So, none of  $x, x + 1, x + 2, \dots, x + n - 1$  are prime.

**7**

**A**

The prize is in this box.

**B**

The prize is not in this box.

**C**

The prize is in box A.

Suppose the prize is in box A.

The statements on each box are as follows:

**A** - true

**B** - true

**C** - true

However, we are told only one of the statements is true.

$\therefore$  the prize cannot be in box A.

Suppose the prize is in box B.

The statements on each box are as follows:

**A** - false

**B** - false

**C** - false

None of these statements are true.

$\therefore$  the prize cannot be in box B.

Suppose the prize is in box C.

The statements on each box are as follows:

**A** - false

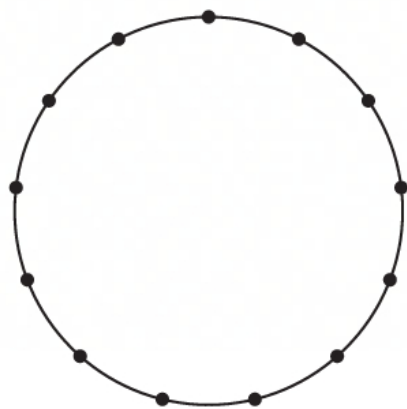
**B** - true

**C** - false

Only one of the statements is true.

$\therefore$  the prize is in box C.



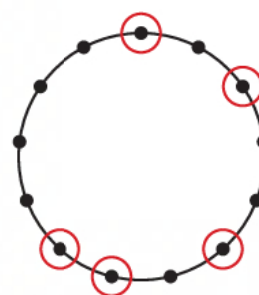
**8 Proof by exhaustion:**

Suppose the distance between consecutive points is 1 unit. For a given selection of 5 points, consider the distance measured clockwise around the circle to the next chosen point. We will have 5 distances, which sum to 13 units.

All possible sets of distances are given below:

$\{9, 1, 1, 1, 1\}$	$\{5, 3, 3, 1, 1\}$
$\{8, 2, 1, 1, 1\}$	$\{5, 3, 2, 2, 1\}$
$\{7, 3, 1, 1, 1\}$	$\{5, 2, 2, 2, 2\}$
$\{7, 2, 2, 1, 1\}$	$\{4, 4, 3, 1, 1\}$
$\{6, 4, 1, 1, 1\}$	$\{4, 4, 2, 2, 1\}$
$\{6, 3, 2, 1, 1\}$	$\{4, 3, 3, 2, 1\}$
$\{6, 2, 2, 2, 1\}$	$\{4, 3, 2, 2, 2\}$
$\{5, 5, 1, 1, 1\}$	$\{3, 3, 3, 3, 1\}$
$\{5, 4, 2, 1, 1\}$	$\{3, 3, 3, 2, 2\}$

For example, the set of distances for

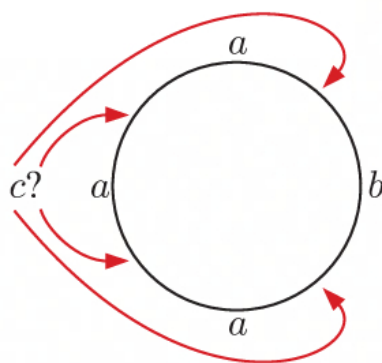


is  $\{2, 3, 2, 1, 5\}$  or  $\{5, 3, 2, 2, 1\}$ .

Every way of choosing 5 points must be a permutation of one of the 18 sets of distances.

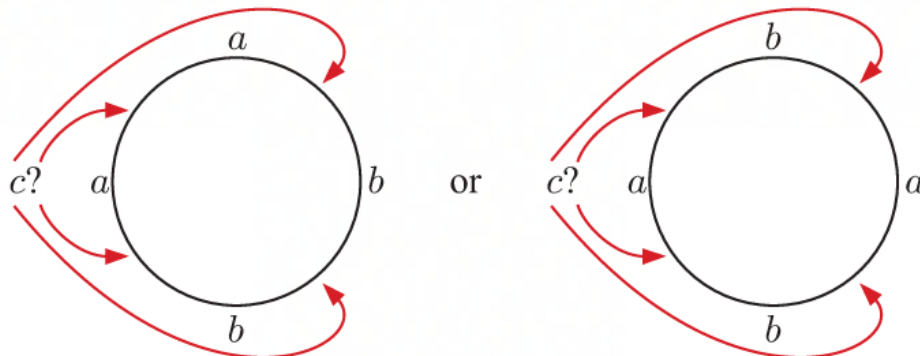
Every one of these combinations fits at least one of the following forms:

- 3 or 4 distances the same  $\{a, a, a, b, c\}$



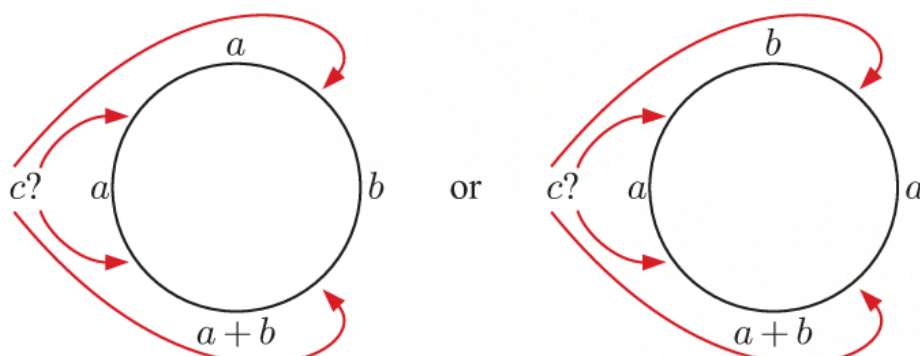
No matter where the  $c$  goes, we have  $a$  adjacent to  $a$ . So, an isosceles triangle can be formed.

- two pairs of two distances the same  $\{a, a, b, b, c\}$



No matter where  $c$  goes, we have  $a$  adjacent to  $a$ ,  $b$  adjacent to  $b$ , or  $a + b$  adjacent to  $a + b$ . So, an isosceles triangle can be formed.

- combinations of the form  $\{a, a, b, a + b, c\}$

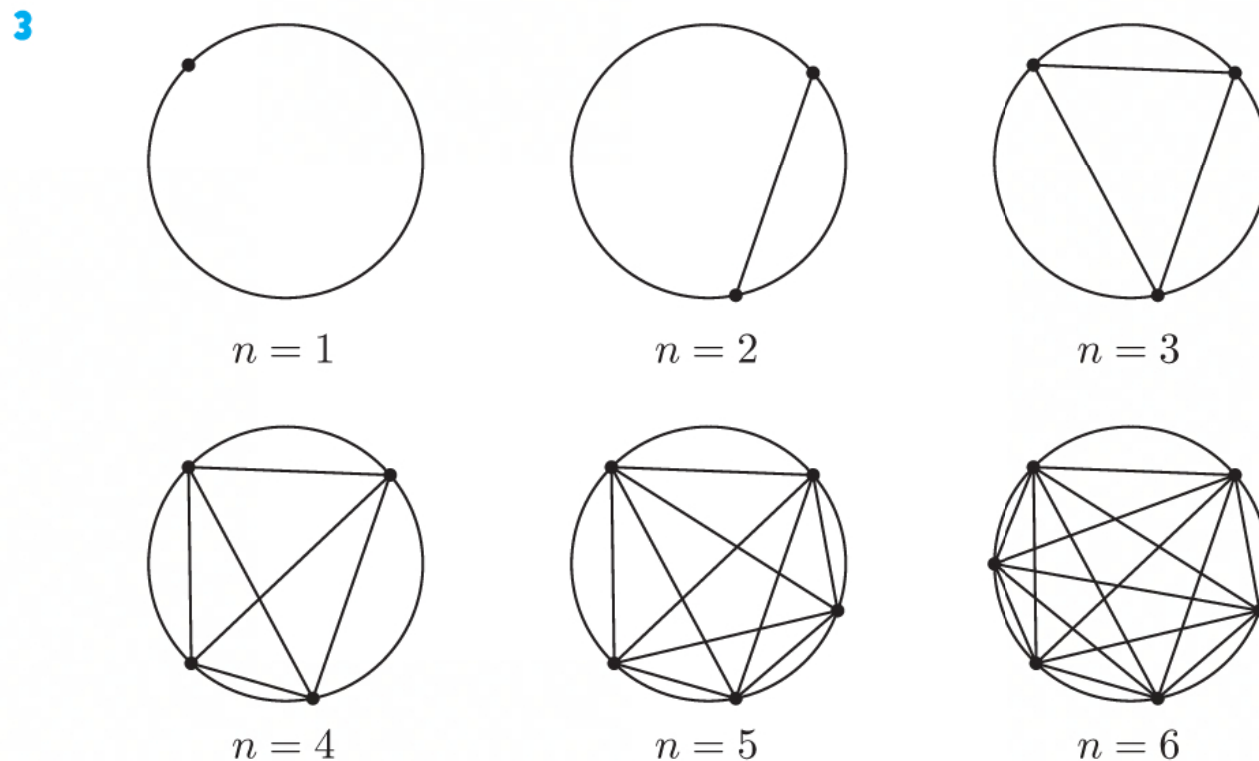


No matter where  $c$  goes, we have  $a$  adjacent to  $a$ , or  $a + b$  adjacent to  $a + b$ . So, an isosceles triangle can be formed.

So, if any 5 points are chosen, the set will include 3 points which form an isosceles triangle.

# EXERCISE 9F

- 1
  - a For  $a = 1$ ,  $b = 1$ , we have  
 $(a + b)^2 = 2^2 \neq 1 + 1 = a^2 + b^2$   
 $\therefore$  the conjecture is false.
  - b For  $p = 7$ , we have  
 $2p + 1 = 15$  which is not prime.  
 $\therefore$  the conjecture is false.
  - c For  $k = 31$ , we have  $6k - 1 = 185 = 5 \times 37$   
 $6k + 1 = 187 = 11 \times 17$  neither of which are prime.  
 $\therefore$  the conjecture is false.
- 2
  - a For  $p_1 = 2$ ,  $p_2 = 7$ , we have  $p = 2 \times 7 + 1 = 15$  which is not prime.  
 $\therefore$  the conjecture is false.
  - b For  $n = 6$ , we have  $p = 2 \times 3 \times 5 \times 7 \times 11 \times 13 + 1$   
 $= 30\,031$   
 $= 59 \times 509$  which is not prime.  
 $\therefore$  the conjecture is false.



For  $n = 6$ , there are 31 regions but  $31 \neq 2^5$ .  
 $\therefore$  the conjecture is false.

- 4  $333\,333\,331 = 17 \times 19\,607\,843$  which is not prime.  
 $\therefore$  the conjecture is false.

- 5

$n$	$2^n - 1$
1	1
2	3
3	7
4	15
5	31

$n$	$2^n - 1$
6	63
7	127
8	255
9	511
10	1023

$2^n - 1$  is prime for  $n = 2, 3, 5$ , and  $7$ .

- b For  $n = 11$ , we have  $2^{11} - 1 = 2047$   
 $= 23 \times 89$  which is not prime.  
 $\therefore$  the conjecture is false.

$$\begin{aligned}
 & \text{c} \quad (2^b - 1)(1 + 2^b + 2^{2b} + 2^{3b} + \dots + 2^{(a-1)b}) \\
 &= \cancel{2^b} + \cancel{2^{2b}} + \cancel{2^{3b}} + \dots + \cancel{2^{(a-1)b}} + 2^{ab} - 1 - \cancel{2^b} - \cancel{2^{2b}} - \cancel{2^{3b}} - \dots - \cancel{2^{(a-1)b}} \\
 &= 2^{ab} - 1
 \end{aligned}$$

So if  $n$  is composite, we can write  $n = ab$  where  $a, b > 1$  and hence write  $2^n - 1$  as a product of factors.

$\therefore$  if  $n$  is composite then  $2^n - 1$  is composite.

## EXERCISE 9G

- 1
  - a The contrapositive of “If it is raining then you get wet” is “If you do not get wet then it is not raining”.
  - b The contrapositive of “If penguins can fly then it is Thursday” is “If it is not Thursday then penguins cannot fly”.
  - c The contrapositive of “If you drink alcohol then you must be aged 18 or over” is “If you are not aged 18 or over then you do not drink alcohol”.
  - d The contrapositive of “If  $k$  is divisible by 5 then  $k$  is not prime” is “If  $k$  is prime then  $k$  is not divisible by 5”.
  - e The contrapositive of “If  $a^2 + b^2 = c^2$  then  $a$  or  $b$  is even” is “If  $a$  and  $b$  are not even then  $a^2 + b^2 \neq c^2$ ”.

### 2 a Proof by contrapositive:

The contrapositive is: If  $n$  is odd then  $n^2$  is odd.

If  $n$  is odd then there exists an integer  $k$  for which  $n = 2k + 1$ .

$$\begin{aligned}
 \text{Hence } n^2 &= (2k + 1)^2 \\
 &= 4k^2 + 4k + 1 \\
 &= 2(2k^2 + 2k) + 1 \quad \text{which is odd.}
 \end{aligned}$$

This proves the contrapositive and hence the original statement.

$\therefore$  if  $n^2$  is even then  $n$  is even.

### b Proof by contrapositive:

The contrapositive is: If  $m$  and  $n$  are odd then  $mn$  is odd.

If  $m$  and  $n$  are odd then there exist integers  $p$  and  $q$  such that  $m = 2p + 1$  and  $n = 2q + 1$ .

$$\begin{aligned}
 \text{Hence } mn &= (2p + 1)(2q + 1) \\
 &= 4pq + 2p + 2q + 1 \\
 &= 2(2pq + p + q) + 1 \quad \text{which is odd.}
 \end{aligned}$$

This proves the contrapositive and hence the original statement.

$\therefore$  if  $mn$  is even then at least one of  $m$  or  $n$  is even.

### c Proof by contrapositive:

The contrapositive is: If  $m$  and  $n$  do not have the same parity then  $m + n$  is odd.

If  $m$  and  $n$  do not have the same parity then one of  $m$  or  $n$  is even and the other is odd.

Hence  $m + n$  must be odd.

This proves the contrapositive and hence the original statement.

$\therefore$  if  $m + n$  is even then  $m$  and  $n$  have the same parity.



**3 Proof by contrapositive:**

The contrapositive is: If  $a$  and  $b$  are rational then the product  $ab$  is rational.

Let  $a = \frac{p}{q}$  and  $b = \frac{r}{s}$ ,  $p, q, r, s \in \mathbb{Z}$ ,  $q, s \neq 0$

$$\therefore ab = \frac{pr}{qs} \in \mathbb{Q} \quad \{pr, qs \in \mathbb{Z}, qs \neq 0\}$$

This proves the contrapositive and hence the original statement.

$\therefore$  if the product  $ab$  is irrational then either  $a$  or  $b$  must be irrational.

**4 Proof by contrapositive:**

The contrapositive is: If  $n$  is a perfect square then  $n \equiv 0$  or  $1 \pmod{4}$ .

{if the remainder when divided by 4 is not 2 or 3, then it must be 0 or 1}

Suppose  $n$  is a perfect square, so  $n = k^2$  for some integer  $k$ .

There are four cases to consider:

If  $k \equiv 0 \pmod{4}$  then  $k = 4q$  for some integer  $q$ .

$$\therefore n = k^2 = (4q)^2 = 16q^2 = 4(4q^2)$$

$$\therefore n \equiv 0 \pmod{4}$$

If  $k \equiv 1 \pmod{4}$  then  $k = 4q + 1$  for some integer  $q$ .

$$\begin{aligned} \therefore n = k^2 &= (4q + 1)^2 \\ &= 16q^2 + 8q + 1 \\ &= 4(4q^2 + 2q) + 1 \end{aligned}$$

$$\therefore n \equiv 1 \pmod{4}$$

If  $k \equiv 2 \pmod{4}$  then  $k = 4q + 2$  for some integer  $q$ .

$$\begin{aligned} \therefore n = k^2 &= (4q + 2)^2 \\ &= 16q^2 + 16q + 4 \\ &= 4(4q^2 + 4q + 1) \end{aligned}$$

$$\therefore n \equiv 0 \pmod{4}$$

If  $k \equiv 3 \pmod{4}$  then  $k = 4q + 3$  for some integer  $q$ .

$$\begin{aligned} \therefore n = k^2 &= (4q + 3)^2 \\ &= 16q^2 + 24q + 9 \\ &= 4(4q^2 + 6q + 2) + 1 \end{aligned}$$

$$\therefore n \equiv 1 \pmod{4}$$

$\therefore$  if  $n$  is a perfect square then  $n \equiv 0$  or  $1 \pmod{4}$

This proves the original statement: If  $n \equiv 2$  or  $3 \pmod{4}$  then  $n$  is not a perfect square.



**EXERCISE 9H****1 a Proof by contradiction:**

Suppose  $\log_2 5$  is rational

$$\therefore \log_2 5 = \frac{p}{q} \quad \text{where } p, q \in \mathbb{Z}, q \neq 0$$

$$\therefore 5 = 2^{\frac{p}{q}}$$

$$\therefore 5^q = 2^p$$

The left hand side is always odd and the right hand side is always even. This is a contradiction.

$\therefore$  our original supposition is false, and  $\log_2 5$  is irrational.

**b Proof by contradiction:**

Suppose  $\log_3 5$  is rational

$$\therefore \log_3 5 = \frac{p}{q} \quad \text{where } p, q \in \mathbb{Z}, q \neq 0$$

$$\therefore 5 = 3^{\frac{p}{q}}$$

$$\therefore 5^q = 3^p$$

The left hand side is always divisible by 5 and the right hand side is never divisible by 5.

This is a contradiction.

$\therefore$  our original supposition is false, and  $\log_3 5$  is irrational.

**c Proof by contradiction:**

Suppose  $\sqrt{3}$  is rational, so  $\sqrt{3} = \frac{p}{q}$ ,  $p, q \in \mathbb{Z}$ ,  $q \neq 0$ .

We assume this fraction has been written in lowest terms, so  $p$  and  $q$  have no common factors.

Squaring both sides,  $3 = \frac{p^2}{q^2}$

$$\therefore p^2 = 3q^2 \quad \dots (1)$$

$\therefore p^2$  is a multiple of 3, and so  $p$  must be a multiple of 3.

Thus  $p = 3k$  for some  $k \in \mathbb{Z}^+$ .

Substituting into (1),  $9k^2 = 3q^2$

$$\therefore q^2 = 3k^2$$

$\therefore q^2$  is a multiple of 3, and so  $q$  must be a multiple of 3.

This is a contradiction, as  $p$  and  $q$  have no common factors.

Thus our original supposition is false, and  $\sqrt{3}$  is irrational.

**d Proof by contradiction:**

Suppose  $\sqrt[3]{2}$  is rational, so  $\sqrt[3]{2} = \frac{p}{q}$ ,  $p, q \in \mathbb{Z}$ ,  $q \neq 0$ .

We assume this fraction has been written in lowest terms, so  $p$  and  $q$  have no common factors.

Cubing both sides,  $2 = \frac{p^3}{q^3}$

$$\therefore p^3 = 2q^3 \quad \dots (1)$$

$\therefore p^3$  is even, and so  $p$  must be even.

Thus  $p = 2k$  for some  $k \in \mathbb{Z}^+$ .

$$\begin{aligned}\text{Substituting into (1), } 8k^3 &= 2q^3 \\ \therefore q^3 &= 4k^3\end{aligned}$$

$\therefore q^3$  is even, and so  $q$  must be even.

This is a contradiction, as  $p$  and  $q$  have no common factors.

Thus our original supposition is false, and  $\sqrt[3]{2}$  is irrational.

**2** Following the method in **Example 10**:

Suppose  $\sqrt{4}$  is rational, so  $\sqrt{4} = \frac{p}{q}$  for some (positive) integers  $p$  and  $q$ ,  $q \neq 0$ .

We assume this fraction has been written in lowest terms, so  $p$  and  $q$  have no common factors.

$$\begin{aligned}\text{Squaring both sides, } 4 &= \frac{p^2}{q^2} \\ \therefore p^2 &= 4q^2\end{aligned}$$

However, even if  $p^2$  is a multiple of 4,  $p$  is not necessarily a multiple of 4.

For example,  $2^2 = 4$ , but 2 is not divisible by 4.

So the argument fails.

**3 a**  $a < b$

$$\begin{aligned}\therefore a + a &< a + b \\ \therefore 2a &< a + b \\ \therefore a &< \frac{a + b}{2}\end{aligned}$$

**b Proof by contradiction:**

$$\begin{aligned}\text{Suppose } a &\geq \frac{a + b}{2} \\ \therefore 2a &\geq a + b \\ \therefore a &\geq b\end{aligned}$$

This is a contradiction, as  $a < b$ .

Thus our supposition is false, and  $a < \frac{a + b}{2}$ .

**4 Proof by contradiction:**

Suppose both  $a$  and  $b$  are odd.

Let  $a = 2p + 1$ ,  $b = 2q + 1$ ,  $p, q \in \mathbb{Z}$ .

$$\begin{aligned}\therefore a^2 + b^2 &= (2p + 1)^2 + (2q + 1)^2 \\ &= 4p^2 + 4p + 1 + 4q^2 + 4q + 1 \\ &= 4(p^2 + q^2 + p + q) + 2 \\ &\equiv 2 \pmod{4}\end{aligned}$$

In **Exercise 9G** question **4** we proved that for a positive integer  $n$ , if  $n \equiv 2$  or  $3 \pmod{4}$ , then  $n$  is not a perfect square.

$\therefore a^2 + b^2$  is not a perfect square.

$\therefore a^2 + b^2 \neq c^2$

This is a contradiction so our original supposition is false, and  $a$  or  $b$  is even.

**5** The numbers shown in the diagram alongside are *not* part of the starting grid. We conjecture that they cannot be part of a solution to a well-posed Sudoku.

**Proof by contradiction:**

Suppose that the numbers are a part of a solution to a well-posed Sudoku.

	2	3					
	3	2					

We could then find *another* solution by simply reversing the 2s and 3s in the  $2 \times 2$  square.

This is a contradiction, as the Sudoku is well-posed and so has a unique solution.

Thus our supposition is false, and so the numbers cannot be part of a solution to a well-posed Sudoku.

	3	2						
	2	3						

- 6 Let Player 1 ( $X$ ) move first, and Player 2 ( $O$ ) move second.  
Suppose Player 2 has a winning strategy  $S$  which guarantees they will win.  
Consider the following strategy for Player 1:

- Select a random square on the grid, and assume Player 2 has placed an  $O$  on this square. We call this Player 2's "assumed move".

		$O$

- Use the same strategy  $S$  based on Player 2's "assumed move".

		$O$
	$X$	

- If, at any time, Player 2 places a  $O$  in the "assumed move" square, select another unused square at random for the "assumed move", and continue to use strategy  $S$ .

$O$	$X$	$O$
	$X$	

This is a winning strategy for Player 1, as  $S$  is a winning strategy for Player 2.

This is a contradiction, as strategy  $S$  guarantees a win for Player 2.

Thus the original supposition is false, and no such winning strategy  $S$  exists for Player 2.

- 7 For  $n = 1$ ,  $3 + 4(1) = 7$  which is prime.

So, there is at least one prime number of the form  $3 + 4n$ , where  $n \in \mathbb{Z}^+$ .

Suppose there is a largest prime number  $p$  of the form  $3 + 4n$ , where  $n \in \mathbb{Z}^+$ .

Consider  $N = 4(3 \times 7 \times 11 \times \dots \times p) - 1$

Notice that:

- $N > p$
- Since  $N = 4(3 \times 7 \times 11 \times \dots \times p - 1) + 3$ ,  $N$  is also of the form  $3 + 4n$ ,  $n \in \mathbb{Z}^+$  .... (\*)
- $N$  is not divisible by 2
- $N$  is not divisible by 3, 7, 11, ..., or  $p$ . So,  $N$  does not have any prime factors of the form  $3 + 4n$  which are less than or equal to  $p$ .

This means that either:

- (1) The only prime factors of  $N$  are of the form  $1 + 4n$ ,  $n \in \mathbb{Z}^+$ . However, if this were the case,  $N$  would also have the form  $1 + 4n$ ,  $n \in \mathbb{Z}^+$ , which contradicts (\*).



- (2)  $N$  is prime, which contradicts the original supposition.  
 (3)  $N$  has a prime factor of the form  $3 + 4n$  which is greater than  $p$ , which also contradicts the original supposition.

In each case, we reach a contradiction, so our supposition is false, and there are infinitely many prime numbers of the form  $3 + 4n$ ,  $n \in \mathbb{Z}^+$ .

## REVIEW SET 9A

- 1  $f(x) = x^2 + px + q$  has roots  $a$  and  $b$

$$\therefore x^2 + px + q = (x - a)(x - b)$$

$$\therefore x^2 + px + q = x^2 - bx - ax + ab$$

$$\therefore x^2 + px + q = x^2 - (a + b)x + ab$$

$$\text{Equating coefficients of } x, \quad p = -(a + b)$$

$$\text{Equating constant terms,} \quad q = ab$$

- 2 a Let  $x = 2.\overline{9} = 2.999 \dots$

$$\Rightarrow 10x = 29.999 \dots$$

$$\Rightarrow 10x = 27 + x$$

$$\Rightarrow 9x = 27$$

$$\Rightarrow x = 3$$

$$\Rightarrow x \in \mathbb{Z}$$

$$\Rightarrow 2.\overline{9} \in \mathbb{Z}$$

- b Let  $x = 0.\overline{38} = 0.3838 \dots$

$$\Rightarrow 100x = 38.3838 \dots$$

$$\Rightarrow 100x = 38 + x$$

$$\Rightarrow 99x = 38$$

$$\Rightarrow x = \frac{38}{99}$$

$$\Rightarrow x \in \mathbb{Q}$$

$$\Rightarrow 0.\overline{38} \in \mathbb{Q}$$

- 3 a The negation of “The boy has blue eyes” is “The boy does not have blue eyes”.  
 b The negation of “ $x$  is larger than 4” is “ $x$  is not larger than 4”.
- 4 a If a function  $f$  is periodic with period  $p$ , then  $f(x + p) = f(x)$  for all  $x$ , by definition.  
 $\therefore$  the statement is true.  
 b If  $f(x + p) = f(x)$  for all  $x$ , the period of  $f$  could be  $p$ ,  $\frac{p}{2}$ , or  $\frac{p}{3}$ , and so on.  
 $\therefore$  the statement is false.  
 c The statement “ $f$  is periodic with period  $p$  if and only if  $f(x + p) = f(x)$  for all  $x$ ” is false, since  $f(x + p) = f(x)$  for all  $x \not\Rightarrow f$  is periodic with period  $p$  (from b).

- 5  $(3a - b)^2 \geq 0, \quad a, b \in \mathbb{R}$

$$\Leftrightarrow (3a)^2 - 2(3a)(b) + (-b)^2 \geq 0$$

$$\Leftrightarrow 9a^2 - 6ab + b^2 \geq 0$$

$$\Leftrightarrow 9a^2 + b^2 \geq 6ab$$



$$\begin{aligned}
6 \quad a \quad & x = \tan^2 \theta - \sin^2 \theta \\
& \Leftrightarrow x = \frac{\sin^2 \theta}{\cos^2 \theta} - \sin^2 \theta \\
& \Leftrightarrow x = \frac{\sin^2 \theta - \sin^2 \theta \cos^2 \theta}{\cos^2 \theta} \\
& \Leftrightarrow x = \frac{\sin^2 \theta (1 - \cos^2 \theta)}{\cos^2 \theta} \\
& \Leftrightarrow x = \frac{\sin^2 \theta}{\cos^2 \theta} \times \sin^2 \theta \\
& \Leftrightarrow x = \tan^2 \theta \sin^2 \theta \\
& \therefore \tan^2 \theta - \sin^2 \theta = \tan^2 \theta \sin^2 \theta
\end{aligned}$$

$$\begin{aligned}
b \quad i \quad & \frac{1 - \sin^2 \theta}{\cos \theta} = \frac{\cos^2 \theta}{\cos \theta} \\
& = \cos \theta \\
ii \quad & \frac{1 - \sin^2 \theta}{\cos \theta} \text{ is undefined for } \theta = \frac{(2n+1)\pi}{2}, \quad n \in \mathbb{Z}, \text{ while } \cos \theta \text{ is defined for all } \\
& \theta \in \mathbb{R}. \text{ It would be incorrect to say that } \frac{1 - \sin^2 \theta}{\cos \theta} \text{ and } \cos \theta \text{ are equivalent, even} \\
& \text{though } \frac{1 - \sin^2 \theta}{\cos \theta} = \cos \theta \text{ for all } \theta \neq \frac{(2n+1)\pi}{2}, \quad n \in \mathbb{Z}.
\end{aligned}$$

7 Let  $x$  and  $y$  be non-zero rational numbers.

By definition, there exists  $p, q \in \mathbb{Z}$ ,  $p, q \neq 0$  so that  $x = \frac{p}{q}$ .

By definition, there exists  $r, s \in \mathbb{Z}$ ,  $r, s \neq 0$  so that  $y = \frac{r}{s}$ .

$$\begin{aligned}
\text{So, } \frac{x}{y} &= \frac{\frac{p}{q}}{\frac{r}{s}} \\
&= \frac{p}{q} \times \frac{s}{r} \\
&= \frac{ps}{qr}
\end{aligned}$$

Since  $p$  and  $s$  are non-zero integers,  $ps$  is a non-zero integer which we call  $P$ .

Since  $q$  and  $r$  are non-zero integers,  $qr$  is a non-zero integer which we call  $Q$ .

$$\therefore \frac{x}{y} = \frac{ps}{qr} = \frac{P}{Q} \text{ where } P, Q \in \mathbb{Z}, P, Q \neq 0$$

$\therefore$  by definition,  $\frac{x}{y}$  is a rational number.

$\therefore$  the quotient of two rational numbers is also a rational number.

$$\begin{aligned}
8 \quad a \quad i \quad & 1^3 + 2^3 = 1 + 8 \\
& = 9
\end{aligned}$$

which is composite since  $9 = 3 \times 3$ .

$$\begin{aligned}
iii \quad & 3^3 + 4^3 = 27 + 64 \\
& = 91
\end{aligned}$$

which is composite since  $91 = 7 \times 13$ .

$$\begin{aligned}
ii \quad & 2^3 + 3^3 = 8 + 27 \\
& = 35
\end{aligned}$$

which is composite since  $35 = 5 \times 7$ .

$$\begin{aligned}
iv \quad & 4^3 + 5^3 = 64 + 125 \\
& = 189
\end{aligned}$$

which is composite since  $189 = 9 \times 21$ .

$$\begin{aligned}
\text{b } k^3 + (k+1)^3 &= k^3 + k^3 + 3k^2 + 3k + 1 \\
&= 2k^3 + 3k^2 + 3k + 1 \\
&= 2k^3 + 2k^2 + 2k + k^2 + k + 1 \\
&= 2k(k^2 + k + 1) + k^2 + k + 1 \\
&= (2k+1)(k^2 + k + 1)
\end{aligned}$$

- c For all  $k \in \mathbb{Z}^+$ ,  $k^3 + (k+1)^3$  will always have factors  $(2k+1)$  and  $(k^2 + k + 1)$ .  
 $\therefore$  the sum of two consecutive positive cubes is always composite.

### 9 Proof by exhaustion:

If we divide  $n$  by 6 then the remainder will be 0, 1, 2, 3, 4, or 5. Hence every integer can be written in one of the forms  $6k$ ,  $6k+1$ ,  $6k+2$ ,  $6k+3$ ,  $6k+4$ , or  $6k+5$ , for some  $k \in \mathbb{Z}$ .

Let  $N = n^3 - n = n(n^2 - 1) = n(n+1)(n-1)$ .

If  $n = 6k$  then the factor  $n$  is divisible by 6, so  $N$  is divisible by 6.

If  $n = 6k+1$  then the factor  $(n-1) = 6k$  is divisible by 6, so  $N$  is divisible by 6.

If  $n = 6k+2$  then  $n(n+1) = (6k+2)(6k+3)$   
 $= 36k^2 + 30k + 6$   
 $= 6(6k^2 + 5k + 1)$  is divisible by 6, so  $N$  is divisible by 6.

If  $n = 6k+3$  then  $(n-1)n = (6k+2)(6k+3)$   
 $= 6(6k^2 + 5k + 1)$  is divisible by 6, so  $N$  is divisible by 6.

If  $n = 6k+4$  then  $(n-1)n = (6k+3)(6k+4)$   
 $= 36k^2 + 42k + 12$   
 $= 6(6k^2 + 7k + 2)$  is divisible by 6, so  $N$  is divisible by 6.

If  $n = 6k+5$  then  $(n+1) = 6k+6$   
 $= 6(k+1)$  is divisible by 6, so  $N$  is divisible by 6.

In all cases,  $N$  is divisible by 6, so  $n^3 - n$  is divisible by 6 for all  $n \in \mathbb{Z}$ .

- 10 a  $\therefore$  the fifth powers of the numbers  $k = 0, 1, 2, 3, \dots, 9$  all have last digit  $k$ .

$k$	$k^5$
0	0
1	1
2	32
3	243
4	1024
5	3125
6	7776
7	16 807
8	32 768
9	59 049

**b** Let  $n = 10m + k$ ,  $m \in \mathbb{Z}$ ,  $k \in \{0, 1, 2, 3, \dots, 9\}$

$$\begin{aligned}
 \therefore n^5 &= (10m + k)^5 \\
 &= (10m)^5 + \binom{5}{1}(10m)^4k + \binom{5}{2}(10m)^3k^2 + \binom{5}{3}(10m)^2k^3 + \binom{5}{4}(10m)k^4 + k^5 \\
 &\quad \text{\{binomial theorem\}} \\
 &= 100\,000m^5 + 50\,000m^4k + 10\,000m^3k^2 + 1000m^2k^3 + 50mk^4 + k^5 \\
 &= 10(10\,000m^5 + 5000m^4k + 1000m^3k^2 + 100m^2k^3 + 5k^4) + k^5 \\
 &\quad \underbrace{\hspace{15em}}_{\text{has last digit 0}} \quad \underbrace{\hspace{2em}}_{\text{has last digit } k \text{ \{from a\}}}
 \end{aligned}$$

So, if the last digit of  $n$  is  $k$ , the last digit of  $n^5$  is  $0 + k = k$ .

$\therefore n$  always has the same last digit as its 5th power  $n^5$  for all  $n \in \mathbb{Z}$ .

**11** For  $p = 5$ , we have  $p! + 1 = 5! + 1 = 121 = 11^2$  which is not prime.  
 $\therefore$  the conjecture is false.

**12 Proof by contrapositive:**

The contrapositive is: If either  $a$  or  $b$  are divisible by  $n$  then  $ab$  is divisible by  $n$ . ( $a, b, n \in \mathbb{Z}$ )

If  $n$  is a factor of  $a$  or  $b$  then it is also a factor of  $ab$ .

This proves the contrapositive and hence the original statement.

$\therefore$  if  $ab$  is not divisible by  $n$  then  $a$  is not divisible by  $n$  and  $b$  is not divisible by  $n$ .

**13 Proof by contradiction:**

Suppose there are only a finite number  $n$  of prime numbers, which we label  $p_1, p_2, \dots, p_n$ .

Every positive integer greater than 1 is either a member of this list, or is divisible by a member of this list.

Let  $N = p_1 \times p_2 \times \dots \times p_n + 1$ .

Notice that:

- $N > p_k$  for all  $k = 1, \dots, n$   
 $\therefore N$  is not a member of the list.
- If we divide  $N$  by any  $p_k$ ,  $k = 1, \dots, n$ , then the remainder is 1.  
 $\therefore N$  is not divisible by any  $p_k$ .

This is a contradiction, so our supposition is false, and there are infinitely many prime numbers.

**14 Proof by contradiction:**

Suppose  $\log_3 2$  is rational

$$\therefore \log_3 2 = \frac{p}{q} \quad \text{where } p, q \in \mathbb{Z}, q \neq 0$$

$$\therefore 2 = 3^{\frac{p}{q}}$$

$$\therefore 2^q = 3^p$$

The left hand side is always even and the right hand side is always odd. This is a contradiction.

$\therefore$  our original supposition is false, and  $\log_3 2$  is irrational.

**15 a** Suppose there exist  $m, n \in \mathbb{Z}$  such that  $14n + 7m = 1$

$$\therefore 7(2n + m) = 1 \quad \text{where } 2n + m \in \mathbb{Z}.$$

This is a contradiction, as 7 does not divide 1.

Thus our supposition is false, and there exist no integers  $n$  and  $m$  for which  $14n + 7m = 1$ .



- b** Suppose there exist  $m, n \in \mathbb{Z}$  such that  $4n^2 - m^2 = 1$   
 $\therefore (2n + m)(2n - m) = 1$

Since  $2n + m, 2n - m \in \mathbb{Z}$ , either  $2n + m = 1$  and  $2n - m = 1$  .... (1)  
 or  $2n + m = -1$  and  $2n - m = -1$  .... (2)

(1): Adding both equations we get  $4n = 2$   
 $\therefore n = \frac{1}{2}$  which is a contradiction, as  $n \in \mathbb{Z}$ .

(2): Adding both equations we get  $4n = -2$   
 $\therefore n = -\frac{1}{2}$  which is a contradiction, as  $n \in \mathbb{Z}$ .

In both cases, we arrive at a contradiction.

Thus our supposition is false, and there exist no integers  $n$  and  $m$  for which  $4n^2 - m^2 = 1$ .

## REVIEW SET 9B

$$\begin{aligned}
 \mathbf{1} \quad p(x) &= x^2 + 2bx + c, \quad q(x) = p(x - b) \\
 &= (x - b)^2 + 2b(x - b) + c \\
 &= x^2 - 2bx + b^2 + 2bx - 2b^2 + c \\
 &= x^2 - b^2 + c \\
 q(-x) &= p(-x - b) \\
 &= (-x - b)^2 + 2b(-x - b) + c \\
 &= (-x)^2 + 2(-x)(-b) + (-b)^2 - 2bx - 2b^2 + c \\
 &= x^2 + 2bx + b^2 - 2bx - 2b^2 + c \\
 &= x^2 - b^2 + c
 \end{aligned}$$

$\therefore q(x) = q(-x)$  for all  $x$ .

$$\begin{aligned}
 \mathbf{2} \quad &(\sqrt{a} - \sqrt{b})^2 \geq 0 \quad \text{for all } a, b \in \mathbb{R}^+ \\
 \Leftrightarrow &(\sqrt{a})^2 - 2\sqrt{a}\sqrt{b} + (\sqrt{b})^2 \geq 0 \\
 \Leftrightarrow &a - 2\sqrt{ab} + b \geq 0 \\
 \Leftrightarrow &a + b \geq 2\sqrt{ab} \\
 \Leftrightarrow &\frac{a + b}{2} \geq \sqrt{ab}
 \end{aligned}$$

- 3 a** The statement “If  $x$  is acute then  $\sin x$  is positive” is true as  $\sin x > 0$  whenever  $0 < x < \frac{\pi}{2}$ .  
**b** The converse of the statement “If  $x$  is acute then  $\sin x$  is positive” is “If  $\sin x$  is positive then  $x$  is acute”.  
**c** The converse is false. If  $\frac{\pi}{2} < x < \pi$ , then  $\sin x > 0$ . So,  $x$  could be obtuse and  $\sin x$  would still be positive.

- 4 a**  $A$ :  $x$  is not prime,  $B$ :  $x$  is composite  
 $A$  and  $B$  are not equivalent since  $x = 1$  is neither prime nor composite.

- b**  $A$ :  $x$  and  $y$  are both odd integers,  $B$ :  $xy$  is odd  
 If  $x$  and  $y$  are odd integers then  $xy$  is odd. {from **Exercise 9D 6**}  
 $\therefore A \Rightarrow B$   
 If, for example,  $xy = 3$ , then  $x = y = \sqrt{3}$  is a possible solution, but  $\sqrt{3}$  is not odd.  
 $\therefore B \not\Rightarrow A$   
 So  $A$  and  $B$  are not equivalent.

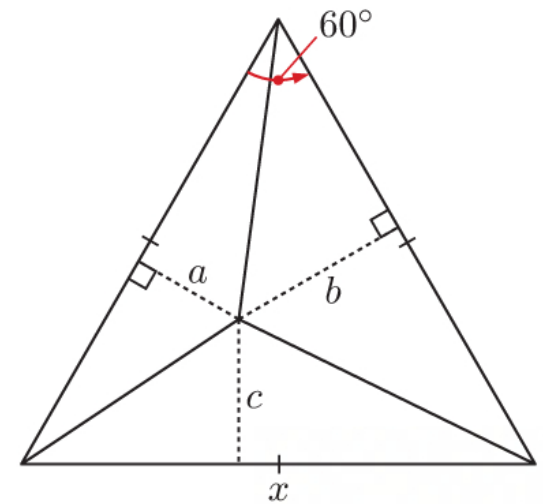


5 Let  $k = 2a + 1$ ,  $a \in \mathbb{Z}$

$$\begin{aligned}
 \therefore k^3 + k^2 - k - 1 &= (k^2 - 1)(k + 1) \\
 &= [(2a + 1)^2 - 1][2a + 1 + 1] \\
 &= (4a^2 + 4a + 1 - 1)(2a + 2) \\
 &= (4a^2 + 4a)(2a + 2) \\
 &= 8(a^2 + a)(a + 1) \quad \text{where } (a^2 + a) \text{ and } (a + 1) \text{ are integers.} \\
 \therefore k^3 + k^2 - k - 1 &\text{ is divisible by 8.}
 \end{aligned}$$

6 Given a point in an equilateral triangle, let  $a$ ,  $b$ , and  $c$  be the distances from the point to each side of the triangle, as shown alongside. We draw lines from the point to each corner which divides the triangle into three smaller triangles with heights  $a$ ,  $b$ , and  $c$ , and with total area

$$\begin{aligned}
 & \left(\frac{1}{2} \times x \times a\right) + \left(\frac{1}{2} \times x \times b\right) + \left(\frac{1}{2} \times x \times c\right) \\
 &= \frac{1}{2}x(a + b + c)
 \end{aligned}$$



$$\begin{aligned}
 \text{But the whole triangle has area } & \frac{1}{2} \times x \times x \times \sin 60^\circ \quad \{\text{sine rule}\} \\
 &= \frac{1}{2}x^2 \times \frac{\sqrt{3}}{2} \\
 &= \frac{\sqrt{3}}{4}x^2 \\
 \therefore \frac{1}{2}x(a + b + c) &= \frac{\sqrt{3}}{4}x^2 \\
 \therefore a + b + c &= \frac{\sqrt{3}}{2}x \quad \text{which is constant for a fixed value of } x.
 \end{aligned}$$

$\therefore$  in an equilateral triangle the sum of the distances from any point in the triangle to the three sides is a constant.

7 The two digit number “ $ab$ ” has value  $10a + b$  and the two digit number “ $ba$ ” has value  $10b + a$ .

$$\begin{aligned}
 \therefore \text{“}ab\text{”} - \text{“}ba\text{”} &= 10a + b - (10b + a) \\
 &= 10a + b - 10b - a \\
 &= 9a - 9b \\
 &= 9(a - b) \quad \text{where } (a - b) \text{ is an integer.}
 \end{aligned}$$

$\therefore$  the difference between the two digit numbers “ $ab$ ” and “ $ba$ ” is always divisible by 9.

8

$$\begin{aligned}
 & -6 = -6 \\
 \Leftrightarrow & 9 - 15 = 4 - 10 \\
 \Leftrightarrow & 3^2 - 3 \times 5 = 2^2 - 2 \times 5 \\
 \Leftrightarrow & 3^2 - 2 \times 3 \times \frac{5}{2} + \left(\frac{5}{2}\right)^2 = 2^2 - 2 \times 2 \times \frac{5}{2} + \left(\frac{5}{2}\right)^2 \\
 \Leftrightarrow & \left(3 - \frac{5}{2}\right)^2 = \left(2 - \frac{5}{2}\right)^2 \\
 \Leftrightarrow & 3 - \frac{5}{2} = 2 - \frac{5}{2} \quad \leftarrow \text{incorrect step, } a^2 = b^2 \not\Rightarrow a = b \\
 \Leftrightarrow & 3 = 2 \quad \text{If } a^2 = b^2, \text{ then } a^2 - b^2 = 0, \\
 & \quad \quad \quad (a + b)(a - b) = 0.
 \end{aligned}$$

9 Let the middle number be  $n$ .

The product of the 3 consecutive integers is  $(n-1)n(n+1) = n(n^2-1)$   
 $= n^3 - n$

In **Review set 9A** question 9 we proved that  $n^3 - n$  is divisible by 6 for all  $n \in \mathbb{Z}$ .  
 $\therefore$  the product of 3 consecutive integers is divisible by 6.

10 10 and 15 both divide 30, but 150 does not divide 30.  
 $\therefore$  the conjecture is false.

11 Suppose  $a < b$ , then  $|a - b| = b - a$ .

$$\begin{aligned}\text{Now } \max(a, b) &= \frac{a + b + |a - b|}{2} \\ &= \frac{\cancel{a} + b + b - \cancel{a}}{2} \\ &= b\end{aligned}$$

which is the greater of  $a$  and  $b$ .

$$\begin{aligned}\text{and } \min(a, b) &= \frac{a + b - |a - b|}{2} \\ &= \frac{a + b - (b - a)}{2} \\ &= \frac{a + \cancel{b} - \cancel{b} + a}{2} \\ &= a\end{aligned}$$

which is the lesser of  $a$  and  $b$ .

Suppose  $a \geq b$ , then  $|a - b| = a - b$ .

$$\begin{aligned}\text{Now } \max(a, b) &= \frac{a + b + |a - b|}{2} \\ &= \frac{a + \cancel{b} + a - \cancel{b}}{2} \\ &= a\end{aligned}$$

which is the greater of  $a$  and  $b$ .

$$\begin{aligned}\text{and } \min(a, b) &= \frac{a + b - |a - b|}{2} \\ &= \frac{a + b - (a - b)}{2} \\ &= \frac{\cancel{a} + b - \cancel{a} + b}{2} \\ &= b\end{aligned}$$

which is the lesser of  $a$  and  $b$ .

$\therefore$  for any  $a, b \in \mathbb{R}$ :

- the greater of  $a$  and  $b$  is  $\max(a, b) = \frac{a + b + |a - b|}{2}$
- the lesser of  $a$  and  $b$  is  $\min(a, b) = \frac{a + b - |a - b|}{2}$ .

## 12 Proof by contrapositive:

The contrapositive is: If  $n$  is odd then  $n^3$  is odd.

If  $n$  is odd then there exists an integer  $k$  for which  $n = 2k + 1$ .

$$\begin{aligned}\text{Hence } n^3 &= (2k + 1)^3 \\ &= (2k)^3 + 3(2k)^2 + 3(2k) + 1^3 \\ &= 8k^3 + 12k^2 + 6k + 1 \\ &= 2(4k^3 + 6k^2 + 3k) + 1 \quad \text{which is odd.}\end{aligned}$$

This proves the contrapositive and hence the original statement.

$\therefore$  if  $n^3$  is even then  $n$  is even.

**13 Proof by contradiction:**

Suppose  $\sqrt[3]{4}$  is rational, so  $\sqrt[3]{4} = \frac{p}{q}$  for some  $p, q \in \mathbb{Z}$ ,  $q \neq 0$ .

We assume this fraction has been written in lowest terms, so  $p$  and  $q$  have no common factors.

Cubing both sides,  $4 = \frac{p^3}{q^3}$

$$\therefore p^3 = 4q^3 \quad \dots (*)$$

$\therefore p^3$  is even, and so  $p$  must be even.

Thus  $p = 2k$  for some  $k \in \mathbb{Z}^+$ .

Substituting into (\*),  $8k^3 = 4q^3$

$$\therefore q^3 = 2k^3$$

$\therefore q^3$  is even, and so  $q$  must be even.

This is a contradiction, as  $p$  and  $q$  have no common factors.

Thus our original supposition is false, and  $\sqrt[3]{4}$  is irrational.

**14 a Proof by contrapositive:**

The contrapositive is: If  $n$  is not divisible by 6 then  $n^2$  is not divisible by 6.

If  $n$  is not divisible by 6 then it takes exactly one of the forms  $6k + 1$ ,  $6k + 2$ ,  $6k + 3$ ,  $6k + 4$ , or  $6k + 5$ , for some  $k \in \mathbb{Z}$ .

$$\begin{aligned} \text{If } n = 6k + 1 \text{ then } n^2 &= (6k + 1)^2 \\ &= 36k^2 + 12k + 1 \\ &= 6(6k^2 + 2k) + 1 \quad \text{which is not divisible by 6.} \end{aligned}$$

$$\begin{aligned} \text{If } n = 6k + 2 \text{ then } n^2 &= (6k + 2)^2 \\ &= 36k^2 + 24k + 4 \\ &= 6(6k^2 + 4k) + 4 \quad \text{which is not divisible by 6.} \end{aligned}$$

$$\begin{aligned} \text{If } n = 6k + 3 \text{ then } n^2 &= (6k + 3)^2 \\ &= 36k^2 + 36k + 9 \\ &= 6(6k^2 + 6k + 1) + 3 \quad \text{which is not divisible by 6.} \end{aligned}$$

$$\begin{aligned} \text{If } n = 6k + 4 \text{ then } n^2 &= (6k + 4)^2 \\ &= 36k^2 + 48k + 16 \\ &= 6(6k^2 + 8k + 2) + 4 \quad \text{which is not divisible by 6.} \end{aligned}$$

$$\begin{aligned} \text{If } n = 6k + 5 \text{ then } n^2 &= (6k + 5)^2 \\ &= 36k^2 + 60k + 25 \\ &= 6(6k^2 + 10k + 4) + 1 \quad \text{which is not divisible by 6.} \end{aligned}$$

In all cases,  $n^2$  is not divisible by 6.

This proves the contrapositive and hence the original statement.

$\therefore$  if  $n^2$  is divisible by 6 then  $n$  is divisible by 6.



**b Proof by contradiction:**

Suppose that  $\sqrt{6}$  is rational, so  $\sqrt{6} = \frac{p}{q}$  for some  $p, q \in \mathbb{Z}$ ,  $q \neq 0$ .

We assume this fraction has been written in lowest terms, so  $p$  and  $q$  have no common factors.

Squaring both sides,  $6 = \frac{p^2}{q^2}$

$$\therefore p^2 = 6q^2 \quad \dots (*) \quad \therefore p^2 \text{ is even and so } p \text{ must be even.}$$

Thus  $p = 2k$  for some  $k \in \mathbb{Z}^+$ .

Substituting into (\*),  $4k^2 = 6q^2$

$$\therefore 3q^2 = 2k^2$$

$\therefore 3q^2$  is even and so  $q^2$  and  $q$  must be even (odd  $\times$  even = even).

This is a contradiction, as  $p$  and  $q$  have no common factors.

Thus our original supposition is false, and  $\sqrt{6}$  is irrational.

**c Proof by contradiction:**

Suppose  $\sqrt{2} + \sqrt{3}$  is rational, so  $\sqrt{2} + \sqrt{3} = \frac{p}{q}$  for some  $p, q \in \mathbb{Z}$ ,  $q \neq 0$ .

Squaring both sides,  $(\sqrt{2} + \sqrt{3})^2 = \frac{p^2}{q^2}$

$$\therefore (\sqrt{2})^2 + 2\sqrt{2}\sqrt{3} + (\sqrt{3})^2 = \frac{p^2}{q^2}$$

$$\therefore 2 + 2\sqrt{6} + 3 = \frac{p^2}{q^2}$$

$$\therefore 5 + 2\sqrt{6} = \frac{p^2}{q^2}$$

$$\therefore \sqrt{6} = \frac{p^2}{2q^2} - \frac{5}{2} \in \mathbb{Z} \quad \{p, q \in \mathbb{Z}, q \neq 0\}$$

This is a contradiction as  $\sqrt{6}$  is irrational (from **b**).

Thus our original supposition is false, and  $\sqrt{2} + \sqrt{3}$  is irrational.

**15** The rule states:

If, and only if, a design includes either the colour I have written down, or the shape I have written down, but not both, then it is called a THOG.

We are told that the black diamond  $\blacklozenge$  is a THOG.

This tells us that the person has written down black square  $\blacksquare$  **or** white diamond  $\diamond$  since both of these share exactly *one* property with the black diamond  $\blacklozenge$ .

- a** The black diamond  $\blacklozenge$  is definitely a THOG, as this is given.
- b** If black square  $\blacksquare$  was written down then the white diamond  $\diamond$  is not a THOG.  
If white diamond  $\diamond$  was written down then the white diamond  $\diamond$  is not a THOG.  
 $\therefore$  the white diamond  $\diamond$  is definitely not a THOG.
- c** If black square  $\blacksquare$  was written down then the black square  $\blacksquare$  is not a THOG.  
If white diamond  $\diamond$  was written down then the black square  $\blacksquare$  is not a THOG.  
 $\therefore$  the black square  $\blacksquare$  is definitely not a THOG.
- d** If black square  $\blacksquare$  was written down then the white square  $\square$  is a THOG.  
If white diamond  $\diamond$  was written down then the white square  $\square$  is a THOG.  
 $\therefore$  the white square  $\square$  is definitely a THOG.



# Chapter 10

## PROOF BY MATHEMATICAL INDUCTION

### EXERCISE 10A

- 1 a The  $n$ th term of the sequence 3, 7, 11, 15, 19, ... is  $4n - 1$  for  $n \in \mathbb{Z}^+$ .  
b The  $n$ th term of the sequence 25, 22, 19, 16, ... is  $28 - 3n$  for  $n \in \mathbb{Z}^+$ .  
c The  $n$ th term of the sequence 8, 12, 18, 27, ... is  $8 \times \left(\frac{3}{2}\right)^{n-1}$  for  $n \in \mathbb{Z}^+$ .

- 2 a  $3^1 = 3$        $1 + 2(1) = 3$       Our proposition is:  
 $3^2 = 9$        $1 + 2(2) = 5$        $3^n > 1 + 2n$  for  $n = 2, 3, 4, 5, \dots$   
 $3^3 = 27$        $1 + 2(3) = 7$       or for all  $n \in \mathbb{Z}^+$ ,  $n \geq 2$ .  
 $3^4 = 81$        $1 + 2(4) = 9$

- b  $11^1 - 1 = 10$       Our proposition is:  
 $11^2 - 1 = 121 - 1 = 120$        $11^n - 1$  is divisible by 10 for all  $n \in \mathbb{Z}^+$ .  
 $11^3 - 1 = 1331 - 1 = 1330$   
 $11^4 - 1 = 14641 - 1 = 14640$

- c  $7^1 + 2 = 7 + 2 = 9 = 3 \times 3$       Our proposition is:  
 $7^2 + 2 = 49 + 2 = 51 = 3 \times 17$        $7^n + 2$  is divisible by 3 for all  $n \in \mathbb{Z}^+$ .  
 $7^3 + 2 = 343 + 2 = 345 = 3 \times 115$   
 $7^4 + 2 = 2401 + 2 = 2403 = 3 \times 801$

- d  $8^1 - 3^1 = 5$       Our proposition is:  
 $8^2 - 3^2 = 55$        $8^n - 3^n$  is divisible by 5 for all  $n \in \mathbb{Z}^+$ .  
 $8^3 - 3^3 = 485$   
 $8^4 - 3^4 = 4015$

- 3 a  $2 = 2 = 1 \times 2$       Our conjecture is:  
 $2 + 4 = 6 = 2 \times 3$        $2 + 4 + 6 + 8 + 10 + \dots + 2n = n(n+1)$  for all  $n \in \mathbb{Z}^+$   
 $2 + 4 + 6 = 12 = 3 \times 4$       or  $\sum_{i=1}^n 2i = n(n+1)$  for all  $n \in \mathbb{Z}^+$ .  
 $2 + 4 + 6 + 8 = 20 = 4 \times 5$   
 $2 + 4 + 6 + 8 + 10 = 30 = 5 \times 6$

- b  $1! = 1$   
 $1! + 2 \times 2! = 1 + 2(2) = 5$   
 $1! + 2 \times 2! + 3 \times 3! = 1 + (2)(2) + (3)(6) = 23$   
 $1! + 2 \times 2! + 3 \times 3! + 4 \times 4! = 1 + (2)(2) + (3)(6) + (4)(24) = 119$

We can see that the result of each addition is 1 less than a factorial number.

$$\begin{aligned} 1 &= 2! - 1 \\ 5 &= 3! - 1 \\ 23 &= 4! - 1 \\ 119 &= 5! - 1 \end{aligned}$$

Our conjecture is:

$$\begin{aligned} 1! + 2 \times 2! + 3 \times 3! + 4 \times 4! + \dots + n \times n! &= (n+1)! - 1 \\ \text{for all } n \in \mathbb{Z}^+ \\ \text{or } \sum_{i=1}^n i \times i! &= (n+1)! - 1 \text{ for all } n \in \mathbb{Z}^+. \end{aligned}$$

$$\text{c } \frac{1}{2!} = \frac{1}{2}$$

$$\frac{1}{2!} + \frac{2}{3!} = \frac{1}{2} + \frac{2}{6} = \frac{5}{6}$$

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} = \frac{1}{2} + \frac{2}{6} + \frac{3}{24} = \frac{23}{24}$$

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \frac{4}{5!} = \frac{1}{2} + \frac{2}{6} + \frac{3}{24} + \frac{4}{120} = \frac{119}{120}$$

We can see that in each fraction, the numerator is 1 less than a factorial number, and the denominator is a factorial number.

$$\frac{1}{2} = \frac{2! - 1}{2!}$$

$$\frac{5}{6} = \frac{3! - 1}{3!}$$

$$\frac{23}{24} = \frac{4! - 1}{4!}$$

$$\frac{119}{120} = \frac{5! - 1}{5!}$$

Our conjecture is:

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \frac{4}{5!} + \dots + \frac{n}{(n+1)!} = \frac{(n+1)! - 1}{(n+1)!} \quad \text{for all } n \in \mathbb{Z}^+$$

$$\text{or } \sum_{i=1}^n \frac{i}{(i+1)!} = \frac{(n+1)! - 1}{(n+1)!} \quad \text{for all } n \in \mathbb{Z}^+.$$

$$\text{d } \frac{1}{2 \times 5} = \frac{1}{10}$$

$$\frac{1}{2 \times 5} + \frac{1}{5 \times 8} = \frac{1}{10} + \frac{1}{40} = \frac{5}{40} = \frac{1}{8} = \frac{2}{16}$$

$$\frac{1}{2 \times 5} + \frac{1}{5 \times 8} + \frac{1}{8 \times 11} = \frac{1}{10} + \frac{1}{40} + \frac{1}{88} = \frac{3}{22}$$

$$\frac{1}{2 \times 5} + \frac{1}{5 \times 8} + \frac{1}{8 \times 11} + \frac{1}{11 \times 14} = \frac{1}{7} = \frac{4}{28}$$

10, 16, 22, 28, ... is arithmetic  
with  $u_1 = 10$ ,  $d = 6$

$$\begin{aligned} u_n &= u_1 + (n-1)d \\ &= 10 + 6(n-1) \\ &= 6n + 4 \end{aligned}$$

Our conjecture is:

$$\frac{1}{2 \times 5} + \frac{1}{5 \times 8} + \frac{1}{8 \times 11} + \frac{1}{11 \times 14} + \dots + \frac{1}{(3n-1)(3n+2)} = \frac{n}{6n+4} \quad \text{for all } n \in \mathbb{Z}^+.$$

$$\{2, 5, 8, 11, \dots \text{ is arithmetic with } u_1 = 2, d = 3 \therefore u_n = 2 + 3(n-1) = 3n-1$$

$$5, 8, 11, 14, \dots \text{ is arithmetic with } u_1 = 5, d = 3 \therefore u_n = 5 + 3(n-1) = 3n+2\}$$

$$\therefore \sum_{i=1}^n \frac{1}{(3i-1)(3i+2)} = \frac{n}{6n+4} \quad \text{for all } n \in \mathbb{Z}^+$$

$$\text{e } (1 - \frac{1}{2}) = \frac{1}{2}$$

$$(1 - \frac{1}{2})(1 - \frac{1}{3}) = \frac{1}{2} \times \frac{2}{3} = \frac{1}{3}$$

$$(1 - \frac{1}{2})(1 - \frac{1}{3})(1 - \frac{1}{4}) = \frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} = \frac{1}{4}$$

$$(1 - \frac{1}{2})(1 - \frac{1}{3})(1 - \frac{1}{4})(1 - \frac{1}{5}) = \frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} \times \frac{4}{5} = \frac{1}{5}$$

$$\text{Our conjecture is: } (1 - \frac{1}{2})(1 - \frac{1}{3})(1 - \frac{1}{4}) \dots \left(1 - \frac{1}{n+1}\right) = \frac{1}{n+1} \quad \text{for all } n \in \mathbb{Z}^+$$

$$\text{or } \prod_{i=1}^n \left(1 - \frac{1}{i+1}\right) = \frac{1}{n+1} \quad \text{for all } n \in \mathbb{Z}^+.$$

$$\begin{aligned}
 \mathbf{f} \quad & 2^1 = 2 = 2^2 - 2 \\
 & 2 + 2^2 = 6 = 2^3 - 2 \\
 & 2 + 2^2 + 2^3 = 14 = 2^4 - 2 \\
 & 2 + 2^2 + 2^3 + 2^4 = 30 = 2^5 - 2
 \end{aligned}$$

Our conjecture is:  $2 + 2^2 + 2^3 + \dots + 2^n = 2^{n+1} - 2$  for all  $n \in \mathbb{Z}^+$

$$\text{or } \sum_{i=1}^n 2^i = 2^{n+1} - 2 \text{ for all } n \in \mathbb{Z}^+.$$

$$\begin{aligned}
 \mathbf{g} \quad & \left(1 - \frac{2}{3}\right) = \frac{1}{3} = \frac{1}{1+2} \\
 & \left(1 - \frac{2}{3}\right) \left(1 - \frac{2}{4}\right) = \frac{1}{3} \times \frac{1}{2} = \frac{1}{6} = \frac{1}{1+2+3} \\
 & \left(1 - \frac{2}{3}\right) \left(1 - \frac{2}{4}\right) \left(1 - \frac{2}{5}\right) = \frac{1}{3} \times \frac{1}{2} \times \frac{2}{5} = \frac{1}{15} = \frac{1}{1+2+3+4} \\
 & \left(1 - \frac{2}{3}\right) \left(1 - \frac{2}{4}\right) \left(1 - \frac{2}{5}\right) \left(1 - \frac{2}{6}\right) = \frac{1}{3} \times \frac{1}{2} \times \frac{2}{5} \times \frac{2}{3} = \frac{1}{15} = \frac{1}{1+2+3+4+5}
 \end{aligned}$$

Our conjecture is:  $\left(1 - \frac{2}{3}\right) \left(1 - \frac{2}{4}\right) \left(1 - \frac{2}{5}\right) \dots \left(1 - \frac{2}{n+2}\right) = \frac{1}{1+2+\dots+(n+1)}$

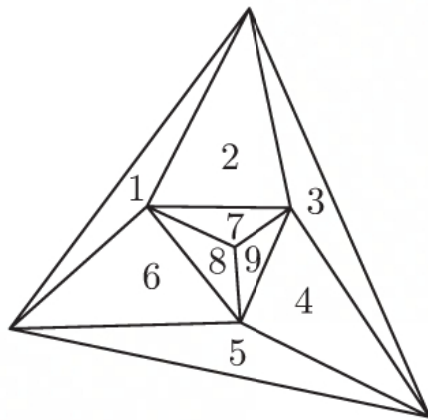
$$= \frac{1}{\frac{(n+1)(n+2)}{2}}$$

$$= \frac{2}{(n+1)(n+2)} \text{ for all } n \in \mathbb{Z}^+$$

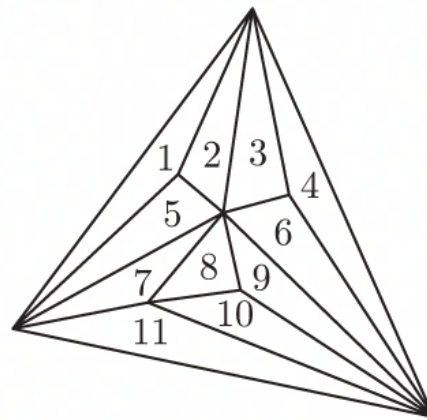
$$\text{or } \prod_{i=1}^n \left(1 - \frac{2}{i+2}\right) = \frac{2}{(n+1)(n+2)} \text{ for all } n \in \mathbb{Z}^+.$$

**4 a**

For  $n = 4$



For  $n = 5$



**b** Let  $T_n$  be the number of triangles obtained when  $n$  points are placed inside a triangle.

$$\text{When } n = 1, \quad T_1 = 3 = 2(1) + 1$$

$$n = 2, \quad T_2 = 5 = 2(2) + 1$$

$$n = 3, \quad T_3 = 7 = 2(3) + 1$$

$$n = 4, \quad T_4 = 9 = 2(4) + 1$$

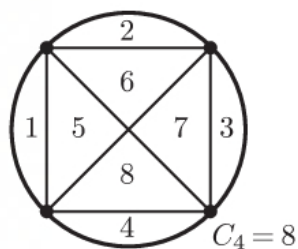
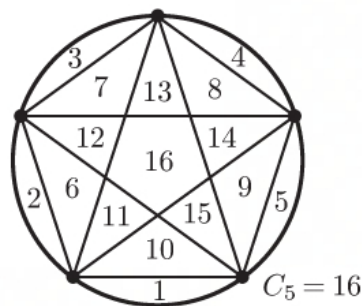
$$n = 5, \quad T_5 = 11 = 2(5) + 1$$

So, from the cases  $n = 1, 2, 3, 4, 5$ , our proposition is:

The maximum number of triangles for  $n$  points within the original triangle is given by

$$T_n = 2n + 1, \quad n \in \mathbb{Z}^+.$$



**5 a** For  $n = 4$ For  $n = 5$ **b** Let  $C_n$  be the number of regions formed when  $n$  points are placed around a circle.

$$\text{When } n = 1, C_1 = 1 = 2^0 = 2^{1-1}$$

$$n = 2, C_2 = 2 = 2^1 = 2^{2-1}$$

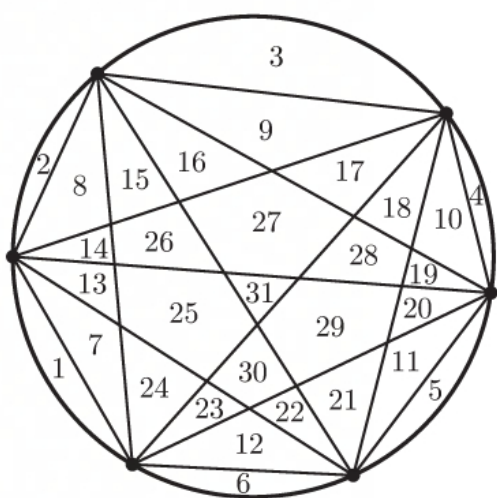
$$n = 3, C_3 = 4 = 2^2 = 2^{3-1}$$

$$n = 4, C_4 = 8 = 2^3 = 2^{4-1}$$

$$n = 5, C_5 = 16 = 2^4 = 2^{5-1}$$

So, from the cases  $n = 1, 2, 3, 4, 5$ , our conjecture is:The number of regions formed when  $n$  points are placed around a circle is given by

$$C_n = 2^{n-1}, n \in \mathbb{Z}^+.$$

**c** For  $n = 6$ 

By our conjecture we expect  $2^{6-1} = 2^5 = 32$  regions to be formed, but there are only 31. So, we no longer believe the conjecture.

## EXERCISE 10B

**1 a** If  $n = 0$ ,  $3^n + 1 = 3^0 + 1 = 2$  which is divisible by 2.

$$\text{For } n > 0, 3^n + 1 = (1 + 2)^n + 1$$

$$= 1^n + \binom{n}{1} 2 + \binom{n}{2} 2^2 + \binom{n}{3} 2^3 + \dots + \binom{n}{n-1} 2^{n-1} + \binom{n}{n} 2^n + 1$$

$$= 2 + \binom{n}{1} 2 + \binom{n}{2} 2^2 + \binom{n}{3} 2^3 + \dots + \binom{n}{n-1} 2^{n-1} + \binom{n}{n} 2^n$$

$$= 2 \left( 1 + \binom{n}{1} + \binom{n}{2} 2 + \binom{n}{3} 2^2 + \dots + \binom{n}{n-1} 2^{n-2} + \binom{n}{n} 2^{n-1} \right)$$

where the contents of the brackets is an integer.

 $\therefore 3^n + 1$  is divisible by 2 for all integers  $n \geq 0$ .



**b**  $P_n$  is:  $3^n + 1$  is divisible by 2 for all integers  $n \geq 0$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 0$ ,  $3^0 + 1 = 2 = 1 \times 2 \quad \therefore P_0$  is true.

(2) If  $P_k$  is true, then  $3^k + 1 = 2A$  where  $A \in \mathbb{Z}$ .

$$\begin{aligned} \text{Now } 3^{k+1} + 1 &= 3^1 3^k + 1 \\ &= 3(2A - 1) + 1 \quad \{\text{using } P_k\} \\ &= 6A - 3 + 1 \\ &= 6A - 2 \\ &= 2(3A - 1) \quad \text{where } (3A - 1) \in \mathbb{Z} \text{ as } A \in \mathbb{Z}. \end{aligned}$$

Thus  $3^{k+1} + 1$  is divisible by 2 if  $3^k + 1$  is divisible by 2.

Since  $P_0$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all integers  $n \geq 0$ . {principle of mathematical induction}

**2 a** If  $n = 0$ ,  $6^n - 1 = 6^0 - 1 = 0$  which is divisible by 5.

For  $n > 0$ ,  $6^n - 1 = (5 + 1)^n - 1$

$$\begin{aligned} &= 5^n + \binom{n}{1} 5^{n-1} + \binom{n}{2} 5^{n-2} + \binom{n}{3} 5^{n-3} + \dots + \binom{n}{n-1} 5 + \binom{n}{n} 1^n - 1 \\ &= 5^n + \binom{n}{1} 5^{n-1} + \binom{n}{2} 5^{n-2} + \binom{n}{3} 5^{n-3} + \dots + \binom{n}{n-1} 5 \\ &= 5 \left( 5^{n-1} + \binom{n}{1} 5^{n-2} + \binom{n}{2} 5^{n-3} + \binom{n}{3} 5^{n-4} + \dots + \binom{n}{n-1} \right) \end{aligned}$$

where the contents of the brackets is an integer.

$\therefore 6^n - 1$  is divisible by 5 for all integers  $n \geq 0$ .

**b**  $P_n$  is:  $6^n - 1$  is divisible by 5 for all integers  $n \geq 0$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 0$ ,  $6^0 - 1 = 0$  which is divisible by 5  $\therefore P_0$  is true.

(2) If  $P_k$  is true, then  $6^k - 1 = 5A$  where  $A \in \mathbb{Z}$ .

$$\begin{aligned} \text{Now } 6^{k+1} - 1 &= 6^1 6^k - 1 \\ &= 6(5A + 1) - 1 \quad \{\text{using } P_k\} \\ &= 30A + 6 - 1 \\ &= 30A + 5 \\ &= 5(6A + 1) \quad \text{where } (6A + 1) \in \mathbb{Z} \text{ as } A \in \mathbb{Z}. \end{aligned}$$

Thus,  $6^{k+1} - 1$  is divisible by 5 if  $6^k - 1$  is divisible by 5.

Since  $P_0$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all integers  $n \geq 0$ . {principle of mathematical induction}

**3 a**  $P_n$  is:  $n^3 + 2n$  is divisible by 3 for all  $n \in \mathbb{Z}^+$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ ,  $1^3 + 2(1) = 3$  which is divisible by 3  $\therefore P_1$  is true.

(2) If  $P_k$  is true, then  $k^3 + 2k = 3A$  where  $A \in \mathbb{Z}$ .

$$\begin{aligned}\text{Now } (k+1)^3 + 2(k+1) &= k^3 + 3k^2 + 3k + 1 + 2k + 2 \\ &= (k^3 + 2k) + 3k^2 + 3k + 3 \\ &= 3A + 3k^2 + 3k + 3 \quad \{\text{using } P_k\} \\ &= 3(A + k^2 + k + 1) \quad \text{where } (A + k^2 + k + 1) \in \mathbb{Z} \text{ as } A \in \mathbb{Z} \text{ and } k \in \mathbb{Z}^+.\end{aligned}$$

Thus  $(k+1)^3 + 2(k+1)$  is divisible by 3 if  $k^3 + 2k$  is divisible by 3.

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

**b**  $P_n$  is:  $7^n - 1$  is divisible by 3 for all  $n \in \mathbb{Z}^+$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ ,  $7^1 - 1 = 6$  which is divisible by 3  $\therefore P_1$  is true.

(2) If  $P_k$  is true, then  $7^k - 1 = 3A$  for some  $A \in \mathbb{Z}^+$ .

$$\begin{aligned}\text{Now } 7^{k+1} - 1 &= 7 \times 7^k - 1 \\ &= 7(3A + 1) - 1 \quad \{\text{using } P_k\} \\ &= 21A + 6 \\ &= 3(7A + 2) \quad \text{where } (7A + 2) \in \mathbb{Z} \text{ as } A \in \mathbb{Z}^+.\end{aligned}$$

Thus  $7^{k+1} - 1$  is divisible by 3 if  $7^k - 1$  is divisible by 3.

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

**c**  $P_n$  is:  $8^n - 3^n$  is divisible by 5 for all  $n \in \mathbb{Z}^+$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ ,  $8^1 - 3^1 = 5$  which is divisible by 5  $\therefore P_1$  is true.

(2) If  $P_k$  is true, then  $8^k - 3^k = 5A$  for some  $A \in \mathbb{Z}^+$ .

$$\begin{aligned}\text{Now } 8^{k+1} - 3^{k+1} &= 8(8^k - 3^k) + 8 \times 3^k - 3 \times 3^k \\ &= 8(5A) + 5 \times 3^k \quad \{\text{using } P_k\} \\ &= 5(8A + 3^k) \quad \text{where } (8A + 3^k) \in \mathbb{Z} \text{ as } A \in \mathbb{Z}^+ \text{ and } k \in \mathbb{Z}^+.\end{aligned}$$

Thus  $8^{k+1} - 3^{k+1}$  is divisible by 5 if  $8^k - 3^k$  is divisible by 5.

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

**d**  $P_n$  is:  $7^n - 4^n - 3^n$  is divisible by 12 for all  $n \in \mathbb{Z}^+$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ ,  $7^1 - 4^1 - 3^1 = 0$  which is divisible by 12  $\therefore P_1$  is true.

(2) If  $P_k$  is true, then  $7^k - 4^k - 3^k = 12A$  where  $A \in \mathbb{Z}$ .

$$\begin{aligned}
 \text{Now } 7^{k+1} - 4^{k+1} - 3^{k+1} &= 7(7^k) - 4(4^k) - 3(3^k) \\
 &= 7[12A + 4^k + 3^k] - 4(4^k) - 3(3^k) \quad \{\text{using } P_k\} \\
 &= 84A + 7(4^k) + 7(3^k) - 4(4^k) - 3(3^k) \\
 &= 84A + 3(4^k) + 4(3^k) \\
 &= 84A + 3 \times 4 \times 4^{k-1} + 4 \times 3 \times 3^{k-1} \\
 &= 12(7A + 4^{k-1} + 3^{k-1}) \quad \text{where } (7A + 4^{k-1} + 3^{k-1}) \in \mathbb{Z} \text{ as} \\
 &\quad A \in \mathbb{Z} \text{ and } k \in \mathbb{Z}^+.
 \end{aligned}$$

Thus  $7^{k+1} - 4^{k+1} - 3^{k+1}$  is divisible by 12 if  $7^k - 4^k - 3^k$  is divisible by 12.

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

**e**  $P_n$  is:  $7^n + 2 \times 4^n$  is divisible by 3 for all  $n \in \mathbb{Z}^+$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ ,  $7^1 + 2 \times 4^1 = 15$  which is divisible by 3  $\therefore P_1$  is true.

(2) If  $P_k$  is true, then  $7^k + 2 \times 4^k = 3A$  for some  $A \in \mathbb{Z}^+$ .

$$\begin{aligned}
 \text{Now } 7^{k+1} + 2 \times 4^{k+1} &= 7 \times 7^k + 2 \times 4 \times 4^k \\
 &= 7(3A - 2 \times 4^k) + 8 \times 4^k \quad \{\text{using } P_k\} \\
 &= 21A - 14 \times 4^k + 8 \times 4^k \\
 &= 21A - 6 \times 4^k \\
 &= 3(7A - 2 \times 4^k) \quad \text{where } (7A - 2 \times 4^k) \in \mathbb{Z} \\
 &\quad \text{as } A \in \mathbb{Z}^+ \text{ and } k \in \mathbb{Z}^+.
 \end{aligned}$$

Thus  $7^{k+1} + 2 \times 4^{k+1}$  is divisible by 3 if  $7^k + 2 \times 4^k$  is divisible by 3.

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

**f**  $P_n$  is:  $n^3 + 3n^2 + 2n$  is divisible by 6 for all  $n \in \mathbb{Z}^+$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ ,  $(1)^3 + 3(1)^2 + 2(1) = 6$  which is divisible by 6  $\therefore P_1$  is true.

(2) If  $P_k$  is true, then  $k^3 + 3k^2 + 2k = 6A$  for some  $A \in \mathbb{Z}^+$ .

$$\begin{aligned}
 \text{Now } (k+1)^3 + 3(k+1)^2 + 2(k+1) &= k^3 + 3k^2 + 3k + 1 + 3k^2 + 6k + 3 + 2k + 2 \\
 &= k^3 + 3k^2 + 2k + 3k^2 + 9k + 6 \\
 &= 6A + 3(k^2 + 3k + 2) \quad \{\text{using } P_k\} \\
 &= 6A + 3(k+1)(k+2)
 \end{aligned}$$

We notice that  $(k+1)(k+2)$  is the product of an even and an odd integer.

$\therefore (k+1)(k+2) = 2B$  where  $B \in \mathbb{Z}^+$ .



$$\begin{aligned}
&\therefore (k+1)^3 + 3(k+1)^2 + 2(k+1) \\
&= 6A + 3(2B) \\
&= 6(A+B) \quad \text{where } (A+B) \in \mathbb{Z} \text{ as } A, B \in \mathbb{Z}^+.
\end{aligned}$$

Thus  $(k+1)^3 + 3(k+1)^2 + 2(k+1)$  is divisible by 6 if  $k^3 + 3k^2 + 2k$  is divisible by 6.

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,  
 $P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

**4 a**  $P_n$  is:  $3^{2n+4} - 2^{2n}$  is divisible by 5 for all  $n \in \mathbb{Z}^+$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ ,  $3^{2+4} - 2^2 = 725$  which is divisible by 5  $\therefore P_1$  is true.

(2) If  $P_k$  is true, then  $3^{2k+4} - 2^{2k} = 5A$  where  $A \in \mathbb{Z}$ .

$$\begin{aligned}
\text{Now } &3^{2(k+1)+4} - 2^{2(k+1)} \\
&= 3^2 \times 3^{2k+4} - 2^2 \times 2^{2k} \\
&= 9(3^{2k+4} - 2^{2k}) + 5 \times 2^{2k} \\
&= 9(5A) + 5 \times 2^{2k} \quad \{\text{using } P_k\} \\
&= 5(9A + 2^{2k}) \quad \text{where } (9A + 2^{2k}) \in \mathbb{Z} \text{ as } A \in \mathbb{Z} \text{ and } k \in \mathbb{Z}^+.
\end{aligned}$$

Thus  $3^{2(k+1)+4} - 2^{2(k+1)}$  is divisible by 5 if  $3^{2k+4} - 2^{2k}$  is divisible by 5.

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,  
 $P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

**b**  $P_n$  is:  $3 \times 5^{2n+1} + 2^{3n+1}$  is divisible by 17 for all  $n \in \mathbb{Z}^+$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ ,  $3 \times 5^{2+1} + 2^{3+1} = 3 \times 125 + 16 = 391 = 17 \times 23$   $\therefore P_1$  is true.

(2) If  $P_k$  is true, then  $3 \times 5^{2k+1} + 2^{3k+1} = 17A$  where  $A \in \mathbb{Z}$ .

$$\begin{aligned}
\text{Now } &3 \times 5^{2(k+1)+1} + 2^{3(k+1)+1} \\
&= 5^2 \times 3 \times 5^{2k+1} + 2^3 \times 2^{3k+1} \\
&= 25(3 \times 5^{2k+1} + 2^{3k+1}) - 17 \times 2^{3k+1} \\
&= 25(17A) - 17 \times 2^{3k+1} \quad \{\text{using } P_k\} \\
&= 17(25A - 2^{3k+1}) \quad \text{where } (25A - 2^{3k+1}) \in \mathbb{Z} \text{ as } A \in \mathbb{Z} \text{ and } k \in \mathbb{Z}^+.
\end{aligned}$$

Thus  $3 \times 5^{2(k+1)+1} + 2^{3(k+1)+1}$  is divisible by 17 if  $3 \times 5^{2k+1} + 2^{3k+1}$  is divisible by 17.

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,  
 $P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

**5**  $P_n$  is:  $\frac{2^n - (-1)^n}{3}$  is an odd number for all  $n \in \mathbb{Z}^+$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ ,  $\frac{2^1 - (-1)^1}{3} = \frac{3}{3} = 1$  which is odd  $\therefore P_1$  is true.

(2) If  $P_k$  is true, then  $\frac{2^k - (-1)^k}{3} = 2A + 1$  where  $A \in \mathbb{Z}$ .



$$\begin{aligned}
\text{Now } \frac{2^{k+1} - (-1)^{k+1}}{3} &= \frac{2(2^k) - (-1)^{k+1}}{3} \\
&= \frac{2[6A + 3 + (-1)^k] - (-1)^{k+1}}{3} \quad \{\text{using } P_k\} \\
&= \frac{12A + 6 + 2(-1)^k - (-1)(-1)^k}{3} \\
&= \frac{12A + 6 + 2(-1)^k + (-1)^k}{3} \\
&= \frac{12A + 6 + 3(-1)^k}{3} \\
&= 4A + 2 + (-1)^k \\
&= 2(A + 1) + (-1)^k \quad \text{where } (A + 1) \in \mathbb{Z} \text{ as } A \in \mathbb{Z}.
\end{aligned}$$

Now  $(-1)^k$  is either  $+1$  or  $-1$  for any  $k \in \mathbb{Z}^+$ .

$\therefore 2(A + 1) + (-1)^k$  is odd.

Thus  $\frac{2^{k+1} - (-1)^{k+1}}{3}$  is odd whenever  $\frac{2^k - (-1)^k}{3}$  is odd.

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,  
 $P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

**6 a**  $P_n$  is:  $n^2 + 2$  is divisible by 3 for all  $n \in \mathbb{Z}^+$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ ,  $1^2 + 2 = 3 \quad \therefore P_1$  is true.

(2) If  $P_k$  is true, then  $k^2 + 2 = 3A$  where  $A \in \mathbb{Z}$ .

$$\begin{aligned}
\text{Now } (k + 1)^2 + 2 &= k^2 + 2k + 1 + 2 \\
&= 3A + 2k + 1 \quad \{\text{using } P_k\} \\
&= 3\left(A + \frac{2}{3}k + \frac{1}{3}\right)
\end{aligned}$$

But  $A + \frac{2}{3}k + \frac{1}{3}$  is not always an integer, for example, when  $k = 2$ .

$\therefore$  the statement is incorrect.

**b**  $P_n$  is:  $3^n + 4$  is divisible by 7 for all integers  $n \geq 1$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ ,  $3^1 + 4 = 7 \quad \therefore P_1$  is true.

(2) If  $P_k$  is true, then  $3^k + 4 = 7A$  where  $A \in \mathbb{Z}$ .

$$\begin{aligned}
\text{Now } 3^{k+1} + 4 &= 3 \times 3^k + 4 \\
&= 3 \times (3^k + 4) - 8 \\
&= 3(7A) - 8 \quad \{\text{using } P_k\} \\
&= 7(3A) - 8 \\
&= 7\left(3A - \frac{8}{7}\right)
\end{aligned}$$

But  $3A - \frac{8}{7}$  is never an integer.

$\therefore$  the statement is incorrect.

- $P_n$  is:  $2^{n+1} - 3^n - 1$  is divisible by 5 for all integers  $n \geq 0$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 0$ ,  $2^1 - 3^0 - 1 = 0$  which is divisible by 5  $\therefore P_1$  is true.

(2) If  $P_k$  is true, then  $2^{k+1} - 3^k - 1 = 5A$  where  $A \in \mathbb{Z}$ .

$$\begin{aligned} \text{Now } 2^{k+2} - 3^{k+1} - 1 &= 2 \times 2^{k+1} - 3 \times 3^k - 1 \\ &= 2(2^{k+1} - 3^k - 1) - 3^k + 1 \\ &= 2(5A) - 3^k + 1 \quad \{\text{using } P_k\} \\ &= 5(2A) - 3^k + 1 \\ &= 5\left(2A - \frac{3^k}{5} + \frac{1}{5}\right) \end{aligned}$$

But  $2A - \frac{3^k}{5} + \frac{1}{5}$  is not always an integer, for example, when  $k = 1$ .

$\therefore$  the statement is incorrect.

- 7 a  $P_n$  is:  $x^{2n+1} + a^{2n+1}$  has a factor  $(x + a)$  for all  $n \in \mathbb{Z}^+$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ ,  $x^{2+1} + a^{2+1} = x^3 + a^3 = (x + a)(x^2 - ax + a^2)$  which has  $(x + a)$  as a factor  $\therefore P_1$  is true.

(2) If  $P_k$  is true, then  $x^{2k+1} + a^{2k+1} = (x + a)P(x)$  where  $P(x)$  is some  $2k$  degree polynomial.

$$\begin{aligned} \text{Now } x^{2(k+1)+1} + a^{2(k+1)+1} &= x^2 \times x^{2k+1} + a^2 \times a^{2k+1} \\ &= x^2(x^{2k+1} + a^{2k+1}) - x^2 \times a^{2k+1} + a^2 \times a^{2k+1} \\ &= x^2(x + a)P(x) - a^{2k+1}(x + a)(x - a) \quad \{\text{using } P_k\} \\ &= (x + a)(x^2P(x) - a^{2k+1}(x - a)) \end{aligned}$$

Thus  $x^{2(k+1)+1} + a^{2(k+1)+1}$  has a factor  $(x + a)$  whenever  $x^{2k+1} + a^{2k+1}$  has a factor  $(x + a)$ .

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

- b  $P_n$  is:  $x^{2n} - 1$  has a factor  $(x + 1)$  for all  $n \in \mathbb{Z}^+$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ ,  $x^2 - 1 = (x + 1)(x - 1)$  which has  $(x + 1)$  as a factor  $\therefore P_1$  is true.

(2) If  $P_k$  is true, then  $x^{2k} - 1 = (x + 1)P(x)$  where  $P(x)$  is some  $2k - 1$  degree polynomial.

$$\begin{aligned} \text{Now } x^{2(k+1)} - 1 &= x^2 \times x^{2k} - 1 \\ &= x^2(x^{2k} - 1) + x^2 - 1 \\ &= x^2(x + 1)P(x) + (x - 1)(x + 1) \quad \{\text{using } P_k\} \\ &= (x + 1)(x^2P(x) + x - 1) \end{aligned}$$

Thus  $x^{2(k+1)} - 1$  has a factor  $(x + 1)$  whenever  $x^{2k} - 1$  has a factor  $(x + 1)$ .

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

**8 a**  $P_n$  is:  $\sum_{i=1}^n i = \frac{n(n+1)}{2}$  for all positive integers  $n$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ , LHS = 1 and RHS =  $\frac{1(2)}{2} = 1 \quad \therefore P_1$  is true.

(2) If  $P_k$  is true then  $\sum_{i=1}^k i = \frac{k(k+1)}{2}$

$$\begin{aligned} \text{Now } \sum_{i=1}^{k+1} i &= \left( \sum_{i=1}^k i \right) + (k+1) \\ &= \frac{k(k+1)}{2} + k+1 \quad \{\text{using } P_k\} \\ &= \frac{k(k+1)}{2} + \frac{2(k+1)}{2} \\ &= \frac{(k+1)(k+2)}{2} \\ &= \frac{(k+1)([k+1]+1)}{2} \end{aligned}$$

$\therefore P_{k+1}$  is also true.

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all positive integers  $n$ . {principle of mathematical induction}

**b**  $P_n$  is:  $\sum_{i=1}^n i(i+1) = \frac{n(n+1)(n+2)}{3}$  for all positive integers  $n$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ , LHS =  $1 \times 2 = 2$  and RHS =  $\frac{1(2)(3)}{3} = 2 \quad \therefore P_1$  is true.

(2) If  $P_k$  is true then  $\sum_{i=1}^k i(i+1) = \frac{k(k+1)(k+2)}{3}$

$$\begin{aligned} \text{Now } \sum_{i=1}^{k+1} i(i+1) &= \left( \sum_{i=1}^k i(i+1) \right) + (k+1)(k+2) \\ &= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2) \quad \{\text{using } P_k\} \\ &= \frac{k(k+1)(k+2) + 3(k+1)(k+2)}{3} \\ &= \frac{(k+1)(k+2)(k+3)}{3} \\ &= \frac{[k+1]([k+1]+1)([k+1]+2)}{3} \end{aligned}$$

$\therefore P_{k+1}$  is also true.

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all positive integers  $n$ . {principle of mathematical induction}



- c  $P_n$  is:  $\sum_{i=1}^n i^3 = \frac{n^2(n+1)^2}{4}$  for all positive integers  $n$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ , LHS =  $1^3 = 1$  and RHS =  $\frac{1^2(2)^2}{4} = 1 \quad \therefore P_1$  is true.

(2) If  $P_k$  is true, then  $\sum_{i=1}^k i^3 = \frac{k^2(k+1)^2}{4}$

$$\begin{aligned} \text{Now } \sum_{i=1}^{k+1} i^3 &= \left( \sum_{i=1}^k i^3 \right) + (k+1)^3 \\ &= \frac{k^2(k+1)^2}{4} + (k+1)^3 \quad \{\text{using } P_k\} \\ &= \frac{k^2(k+1)^2 + 4(k+1)^3}{4} \\ &= \frac{(k+1)^2[k^2 + 4(k+1)]}{4} \\ &= \frac{(k+1)^2(k^2 + 4k + 4)}{4} \\ &= \frac{(k+1)^2[(k+1) + 1]^2}{4} \end{aligned}$$

$\therefore P_{k+1}$  is also true.

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all positive integers  $n$ . {principle of mathematical induction}

- 9 a The sum of the first  $n$  odd numbers is  $1 + 3 + 5 + 7 + \dots + (2n - 1)$ .

This is the sum of the first  $n$  terms of an arithmetic series where  $u_1 = 1$  and  $d = 2$ .

$\therefore u_n = u_1 + (n - 1)d = 1 + 2(n - 1) = 2n - 1$ .

$$\begin{aligned} \text{Thus } S_n &= \frac{n}{2}(u_1 + u_n) \\ &= \frac{n}{2}(1 + 2n - 1) \\ &= \frac{n}{2}(2n) \\ &= n^2 \end{aligned}$$

So, the sum of the first  $n$  odd numbers is  $n^2$ .

- b  $P_n$  is:  $1 + 3 + 5 + 7 + \dots + (2n - 1) = n^2$  for all  $n \in \mathbb{Z}^+$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ , LHS = 1 and RHS =  $1^2 = 1 \quad \therefore P_1$  is true.

(2) If  $P_k$  is true, then  $1 + 3 + 5 + 7 + \dots + (2k - 1) = k^2$

$$\begin{aligned} \text{Now } 1 + 3 + 5 + 7 + \dots + (2k - 1) + [2(k + 1) - 1] \\ &= k^2 + [2(k + 1) - 1] \quad \{\text{using } P_k\} \\ &= k^2 + 2k + 2 - 1 \\ &= k^2 + 2k + 1 \\ &= (k + 1)^2 \end{aligned}$$

$\therefore P_{k+1}$  is also true.

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}



**10 a**  $P_n$  is:  $\sum_{i=1}^n 2^i = 2^{n+1} - 2$  for all  $n \in \mathbb{Z}^+$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ , LHS =  $2^1 = 2$  and RHS =  $2^2 - 2 = 2$   $\therefore P_1$  is true.

(2) If  $P_k$  is true, then  $\sum_{i=1}^k 2^i = 2^{k+1} - 2$ .

$$\begin{aligned} \text{Now } \sum_{i=1}^{k+1} 2^i &= \left( \sum_{i=1}^k 2^i \right) + 2^{k+1} \\ &= 2^{k+1} - 2 + 2^{k+1} \quad \{\text{using } P_k\} \\ &= 2 \times 2^{k+1} - 2 \\ &= 2^{k+2} - 2 \end{aligned}$$

$\therefore P_{k+1}$  is also true.

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,  
 $P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

**b**  $P_n$  is:  $\sum_{i=1}^n i \times 2^{i-1} = (n-1) \times 2^n + 1$  for all  $n \in \mathbb{Z}^+$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ , LHS = 1 and RHS =  $0 \times 2^0 + 1 = 1$   $\therefore P_1$  is true.

(2) If  $P_k$  is true, then  $\sum_{i=1}^k i \times 2^{i-1} = (k-1) \times 2^k + 1$ .

$$\begin{aligned} \text{Now } \sum_{i=1}^{k+1} i \times 2^{i-1} &= \left( \sum_{i=1}^k i \times 2^{i-1} \right) + (k+1) \times 2^k \\ &= (k-1) \times 2^k + 1 + (k+1) \times 2^k \quad \{\text{using } P_k\} \\ &= 2^k(k-1+k+1) + 1 \\ &= 2^k(2k) + 1 \\ &= k \times 2^{k+1} + 1 \\ &= ([k+1] - 1) \times 2^{k+1} + 1 \end{aligned}$$

$\therefore P_{k+1}$  is also true.

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,  
 $P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

•  $P_n$  is:  $\sum_{i=1}^n i \times 3^i = \frac{3}{4}[(2n-1)3^n + 1]$  for all  $n \in \mathbb{Z}^+$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ , LHS = 3 and RHS =  $\frac{3}{4}[(2-1)3^1 + 1] = 3 \quad \therefore P_1$  is true.

(2) If  $P_k$  is true, then  $\sum_{i=1}^k i \times 3^i = \frac{3}{4}[(2k-1)3^k + 1]$ .

$$\begin{aligned} \text{Now } \sum_{i=1}^{k+1} i \times 3^i &= \left( \sum_{i=1}^k i \times 3^i \right) + (k+1) \times 3^{k+1} \\ &= \frac{3}{4}[(2k-1)3^k + 1] + (k+1) \times 3^{k+1} \quad \{\text{using } P_k\} \\ &= \frac{3}{4} \left[ (2k-1)3^k + 1 + \frac{4(k+1)}{3} 3^{k+1} \right] \\ &= \frac{3}{4}[(2k-1+4(k+1))3^k + 1] \\ &= \frac{3}{4}[(6k+3)3^k + 1] \\ &= \frac{3}{4}[(2k+1)3^{k+1} + 1] \\ &= \frac{3}{4}[(2(k+1)-1)3^{k+1} + 1] \end{aligned}$$

$\therefore P_{k+1}$  is also true.

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

•  $P_n$  is:  $\sum_{i=1}^n \frac{i}{2^i} = 2 - \frac{n+2}{2^n}$  for all  $n \in \mathbb{Z}^+$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ , LHS =  $\frac{1}{2}$  and RHS =  $2 - \frac{1+2}{2^1} = 2 - \frac{3}{2} = \frac{1}{2} \quad \therefore P_1$  is true.

(2) If  $P_k$  is true, then  $\sum_{i=1}^k \frac{i}{2^i} = 2 - \frac{k+2}{2^k}$ .

$$\begin{aligned} \text{Now } \sum_{i=1}^{k+1} \frac{i}{2^i} &= \left( \sum_{i=1}^k \frac{i}{2^i} \right) + \frac{k+1}{2^{k+1}} \\ &= 2 - \frac{k+2}{2^k} + \frac{k+1}{2^{k+1}} \quad \{\text{using } P_k\} \\ &= 2 - \frac{1}{2^k} \left( k+2 - \frac{k+1}{2} \right) \\ &= 2 - \frac{1}{2^k} \left( \frac{2k+4-k-1}{2} \right) \\ &= 2 - \frac{1}{2^{k+1}}(k+3) \\ &= 2 - \frac{(k+1)+2}{2^{k+1}} \end{aligned}$$

$\therefore P_{k+1}$  is also true.

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

**11 a**  $P_n$  is:  $\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \dots + \frac{1}{n(n+1)} = \frac{n}{n+1}$  for all  $n \in \mathbb{Z}^+$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ , LHS =  $\frac{1}{1 \times 2} = \frac{1}{2}$  and RHS =  $\frac{1}{1+1} = \frac{1}{2} \quad \therefore P_1$  is true.

(2) If  $P_k$  is true, then  $\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \dots + \frac{1}{k(k+1)} = \frac{k}{k+1}$

$$\begin{aligned} \text{Now } & \frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \dots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} \\ &= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \quad \{\text{using } P_k\} \\ &= \frac{k(k+2) + 1}{(k+1)(k+2)} \\ &= \frac{k^2 + 2k + 1}{(k+1)(k+2)} \\ &= \frac{(k+1)^2}{(k+1)(k+2)} \\ &= \frac{k+1}{k+2} \\ &= \frac{k+1}{(k+1)+1} \end{aligned}$$

$\therefore P_{k+1}$  is also true.

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

**b**  $\frac{1}{10 \times 11} + \frac{1}{11 \times 12} + \frac{1}{12 \times 13} + \dots + \frac{1}{20 \times 21} = S_{20} - S_9 = \frac{20}{21} - \frac{9}{10} = \frac{11}{210}$

**12 a**  $P_n$  is:  $\frac{1}{1 \times 2 \times 3} + \frac{1}{2 \times 3 \times 4} + \frac{1}{3 \times 4 \times 5} + \dots + \frac{1}{n(n+1)(n+2)} = \frac{n(n+3)}{4(n+1)(n+2)}$   
for all  $n \in \mathbb{Z}^+$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ , LHS =  $\frac{1}{1 \times 2 \times 3} = \frac{1}{6}$  and RHS =  $\frac{1(4)}{4(2)(3)} = \frac{1}{6} \quad \therefore P_1$  is true.

(2) If  $P_k$  is true, then

$$\frac{1}{1 \times 2 \times 3} + \frac{1}{2 \times 3 \times 4} + \frac{1}{3 \times 4 \times 5} + \dots + \frac{1}{k(k+1)(k+2)} = \frac{k(k+3)}{4(k+1)(k+2)}$$

$$\begin{aligned} \text{Now } & \frac{1}{1 \times 2 \times 3} + \frac{1}{2 \times 3 \times 4} + \frac{1}{3 \times 4 \times 5} + \dots + \frac{1}{k(k+1)(k+2)} + \frac{1}{(k+1)(k+2)(k+3)} \\ &= \frac{k(k+3)}{4(k+1)(k+2)} + \frac{1}{(k+1)(k+2)(k+3)} \quad \{\text{using } P_k\} \\ &= \frac{k(k+3)^2 + 4}{4(k+1)(k+2)(k+3)} \\ &= \frac{k(k^2 + 6k + 9) + 4}{4(k+1)(k+2)(k+3)} \\ &= \frac{k^3 + 6k^2 + 9k + 4}{4(k+1)(k+2)(k+3)} \\ &= \frac{(k+1)(k^2 + 5k + 4)}{4(k+1)(k+2)(k+3)} \quad \{\text{we assume } (k+1) \text{ is a factor of } k^3 + 6k^2 + 9k + 4 \\ & \quad \text{and factorise the cubic}\} \\ &= \frac{(k+1)(k+4)}{4(k+2)(k+3)} \\ &= \frac{(k+1)([k+1] + 3)}{4([k+1] + 1)([k+1] + 2)} \end{aligned}$$

$\therefore P_{k+1}$  is also true.

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

**b**  $\frac{1}{4 \times 5 \times 6} + \frac{1}{5 \times 6 \times 7} + \dots + \frac{1}{8 \times 9 \times 10} = S_8 - S_3 = \frac{88}{360} - \frac{18}{80} = \frac{7}{360}$

**13 a**  $P_n$  is:  $1 \times 1! + 2 \times 2! + 3 \times 3! + 4 \times 4! + \dots + n \times n! = (n+1)! - 1$  for all  $n \in \mathbb{Z}^+$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ , LHS =  $1 \times 1! = 1$  and RHS =  $2! - 1 = 1 \quad \therefore P_1$  is true.

(2) If  $P_k$  is true, then  $1 \times 1! + 2 \times 2! + 3 \times 3! + \dots + k \times k! = (k+1)! - 1$

$$\begin{aligned} \text{Now } & 1 \times 1! + 2 \times 2! + 3 \times 3! + \dots + k \times k! + (k+1) \times (k+1)! \\ &= (k+1)! - 1 + (k+1) \times (k+1)! \quad \{\text{using } P_k\} \\ &= (k+1)!(1 + k+1) - 1 \\ &= (k+1)!(k+2) - 1 \\ &= (k+2)! - 1 \\ &= ([k+1] + 1)! - 1 \end{aligned}$$

$\therefore P_{k+1}$  is also true.

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}



**b**  $P_n$  is:  $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \frac{4}{5!} + \dots + \frac{n}{(n+1)!} = \frac{(n+1)! - 1}{(n+1)!}$  for all  $n \in \mathbb{Z}^+$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ , LHS =  $\frac{1}{2!} = \frac{1}{2}$  and RHS =  $\frac{2! - 1}{2!} = \frac{2 - 1}{2} = \frac{1}{2} \quad \therefore P_1$  is true.

(2) If  $P_k$  is true, then  $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \dots + \frac{k}{(k+1)!} = \frac{(k+1)! - 1}{(k+1)!}$

$$\begin{aligned} \text{Now } & \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \dots + \frac{k}{(k+1)!} + \frac{k+1}{(k+2)!} \\ &= \frac{(k+1)! - 1}{(k+1)!} + \frac{k+1}{(k+2)(k+1)!} \quad \{\text{using } P_k\} \\ &= \frac{(k+2)! - (k+2) + k+1}{(k+2)!} \\ &= \frac{(k+2)! - k - 2 + k + 1}{(k+2)!} \\ &= \frac{([k+1] + 1)! - 1}{([k+1] + 1)!} \end{aligned}$$

$\therefore P_{k+1}$  is also true.

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

**c**  $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \dots + \frac{9}{10!} = \frac{10! - 1}{10!} = \frac{3\,628\,799}{3\,628\,800}$

**14**  $P_n$  is:  $n + 2(n-1) + 3(n-2) + \dots + (n-2)3 + (n-1)2 + n = \frac{n(n+1)(n+2)}{6}$   
for all integers  $n \geq 1$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ , LHS = 1 and RHS =  $\frac{1(2)(3)}{6} = 1 \quad \therefore P_1$  is true.

(2) If  $P_k$  is true, then

$$k + 2(k-1) + 3(k-2) + \dots + (k-2)3 + (k-1)2 + k = \frac{k(k+1)(k+2)}{6}$$

$$\begin{aligned} \text{Now } & (k+1) + 2k + 3(k-1) + \dots + (k-1)3 + k2 + (k+1) \\ &= k + 2(k-1) + 3(k-2) + \dots + k + [1 + 2 + 3 + \dots + k + (k+1)] \quad \{\text{using the hint}\} \\ &= \frac{k(k+1)(k+2)}{6} + \frac{(k+1)(k+2)}{2} \quad \{\text{using } P_k \text{ and the sum of an arithmetic series}\} \\ &= \frac{k(k+1)(k+2) + 3(k+1)(k+2)}{6} \\ &= \frac{(k+1)(k+2)(k+3)}{6} \\ &= \frac{(k+1)[(k+1)+1][(k+1)+2]}{6} \end{aligned}$$

$\therefore P_{k+1}$  is also true.

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

$$15 \quad \mathbf{a} \quad u_1 = \frac{a}{3} + \frac{b}{2} = 3 \quad \therefore 2a + 3b = 18 \quad \dots (1)$$

$$u_2 = \frac{a}{9} + \frac{b}{4} = \frac{4}{3} \quad \therefore 4a + 9b = 48 \quad \dots (2)$$

$$\begin{array}{rcl} \therefore 4a + 6b = 36 & \{2 \times (1)\} \\ -4a - 9b = -48 & \{-1 \times (2)\} \\ \hline \end{array}$$

$$\text{Adding,} \quad \begin{array}{r} -3b = -12 \\ \therefore b = 4 \end{array}$$

$$\begin{array}{l} \text{Substituting } b = 4 \text{ into (1) gives } 2a + 3(4) = 18 \\ \therefore 2a = 6 \\ \therefore a = 3 \end{array}$$

$$\mathbf{b} \quad u_n = 3 \left(\frac{1}{3}\right)^n + 4 \left(\frac{1}{2}\right)^n \quad \{\text{from } \mathbf{a}\}$$

$$P_n \text{ is: } \sum_{i=1}^n u_i = \frac{3}{2} \left(1 - \frac{1}{3^n}\right) + 4 \left(1 - \frac{1}{2^n}\right) \quad \text{for all } n \in \mathbb{Z}^+.$$

**Proof:** (By the principle of mathematical induction)

$$\begin{aligned} (1) \quad \text{If } n = 1, \quad \text{LHS} = u_1 = 3 \quad \text{and} \quad \text{RHS} &= \frac{3}{2} \left(1 - \frac{1}{3^1}\right) + 4 \left(1 - \frac{1}{2^1}\right) \\ &= \frac{3}{2} \left(\frac{2}{3}\right) + 4 \left(\frac{1}{2}\right) \\ &= 3 \end{aligned}$$

$\therefore P_1$  is true.

$$(2) \quad \text{If } P_k \text{ is true, then } \sum_{i=1}^k u_i = \frac{3}{2} \left(1 - \frac{1}{3^k}\right) + 4 \left(1 - \frac{1}{2^k}\right).$$

$$\begin{aligned} \text{Now } \sum_{i=1}^{k+1} u_i &= \left(\sum_{i=1}^k u_i\right) + u_{k+1} \\ &= \frac{3}{2} \left(1 - \frac{1}{3^k}\right) + 4 \left(1 - \frac{1}{2^k}\right) + 3 \left(\frac{1}{3}\right)^{k+1} + 4 \left(\frac{1}{2}\right)^{k+1} \quad \{\text{using } P_k\} \\ &= \frac{3}{2} \left(1 - \frac{1}{3^k}\right) + 4 \left(1 - \frac{1}{2^k}\right) + \frac{3}{3^{k+1}} + \frac{2^2}{2^{k+1}} \\ &= \frac{3}{2} \left(1 - \frac{1}{3^k}\right) + 4 \left(1 - \frac{1}{2^k}\right) + \frac{1}{3^k} + \frac{2}{2^k} \\ &= \frac{3}{2} \left(1 - \frac{1}{3^k} + \frac{2}{3} \times \frac{1}{3^k}\right) + 4 \left(1 - \frac{1}{2^k} + \frac{1}{2} \times \frac{1}{2^k}\right) \\ &= \frac{3}{2} \left(1 - \frac{1}{3^k} \left(1 - \frac{2}{3}\right)\right) + 4 \left(1 - \frac{1}{2^k} \left(1 - \frac{1}{2}\right)\right) \\ &= \frac{3}{2} \left(1 - \frac{1}{3^k} \times \frac{1}{3}\right) + 4 \left(1 - \frac{1}{2^k} \times \frac{1}{2}\right) \\ &= \frac{3}{2} \left(1 - \frac{1}{3^{k+1}}\right) + 4 \left(1 - \frac{1}{2^{k+1}}\right) \end{aligned}$$

$\therefore P_{k+1}$  is also true.

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

$$\begin{aligned}
\text{c } \sum_{i=1}^{\infty} u_i &= \lim_{n \rightarrow \infty} \sum_{i=1}^n u_i \\
&= \lim_{n \rightarrow \infty} \left[ \frac{3}{2} \left( 1 - \frac{1}{3^n} \right) + 4 \left( 1 - \frac{1}{2^n} \right) \right] \quad \{\text{from b}\} \\
&= \frac{3}{2}(1 - 0) + 4(1 - 0) \quad \left\{ \frac{1}{k^n} \rightarrow 0 \text{ as } n \rightarrow \infty \text{ for all } k > 1 \right\} \\
&= \frac{11}{2}
\end{aligned}$$

**16 a**  $P_n$  is: if  $u_1 = 5$  and  $u_{n+1} = u_n + 8n + 5$  for all  $n \in \mathbb{Z}^+$ , then  $u_n = 4n^2 + n$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ ,  $u_1 = 4(1)^2 + 1 = 5$  which is true  $\therefore P_1$  is true.

(2) If  $P_k$  is true, then  $u_k = 4k^2 + k$

$$\begin{aligned}
\text{Now } u_{k+1} &= u_k + 8k + 5 \\
&= 4k^2 + k + 8k + 5 \quad \{\text{using } P_k\} \\
&= 4(k^2 + 2k + 1) + k + 1 \\
&= 4(k+1)^2 + (k+1)
\end{aligned}$$

$\therefore P_{k+1}$  is also true.

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

**b**  $P_n$  is: if  $u_1 = 1$  and  $u_{n+1} = 2 + 3u_n$  for all  $n \in \mathbb{Z}^+$ , then  $u_n = 2(3^{n-1}) - 1$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ ,  $u_1 = 2(3^{1-1}) - 1 = 1$  which is true  $\therefore P_1$  is true.

(2) If  $P_k$  is true, then  $u_k = 2(3^{k-1}) - 1$

$$\begin{aligned}
\text{Now } u_{k+1} &= 2 + 3u_k \\
&= 2 + 3(2[3^{k-1}] - 1) \quad \{\text{using } P_k\} \\
&= 2 + 2 \times 3^k - 3 \\
&= 2(3^k) - 1 \\
&= 2(3^{(k+1)-1}) - 1
\end{aligned}$$

$\therefore P_{k+1}$  is also true.

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

- $P_n$  is: if  $u_1 = 2$  and  $u_{n+1} = \frac{u_n}{2(n+1)}$  for all  $n \in \mathbb{Z}^+$ , then  $u_n = \frac{2^{2-n}}{n!}$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ ,  $u_1 = \frac{2^{2-1}}{1!} = 2^1 = 2$  which is true  $\therefore P_1$  is true.

(2) If  $P_k$  is true, then  $u_k = \frac{2^{2-k}}{k!}$

$$\begin{aligned} \text{Now } u_{k+1} &= \frac{u_k}{2(k+1)} \\ &= \frac{2^{2-k}}{k! \cdot 2(k+1)} \quad \{\text{using } P_k\} \\ &= \frac{2^{2-k-1}}{k!(k+1)} \\ &= \frac{2^{2-(k+1)}}{(k+1)!} \end{aligned}$$

$\therefore P_{k+1}$  is also true.

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

- $P_n$  is: if  $u_1 = 1$  and  $u_{n+1} = u_n + (-1)^n(n+1)^2$  for all  $n \in \mathbb{Z}^+$ ,  
then  $u_n = \frac{(-1)^{n-1}n(n+1)}{2}$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ ,  $u_1 = \frac{(-1)^0 \times 1 \times 2}{2} = 1$  which is true  $\therefore P_1$  is true.

(2) If  $P_k$  is true, then  $u_k = \frac{(-1)^{k-1}k(k+1)}{2}$

$$\begin{aligned} \text{Now } u_{k+1} &= u_k + (-1)^k(k+1)^2 \\ &= \frac{(-1)^{k-1}k(k+1)}{2} + (-1)^k(k+1)^2 \quad \{\text{using } P_k\} \\ &= \frac{(-1)^{k-1}k(k+1) + 2(-1)^k(k+1)^2}{2} \\ &= \frac{2(-1)^k(k+1)^2 - (-1)^k k(k+1)}{2} \\ &= \frac{(-1)^k(k+1)[2(k+1) - k]}{2} \\ &= \frac{(-1)^k(k+1)(k+2)}{2} \\ &= \frac{(-1)^k(k+1)([k+1] + 1)}{2} \end{aligned}$$

$\therefore P_{k+1}$  is also true.

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}



**e**  $P_n$  is: if  $u_{n+1} = \frac{u_n}{u_n + 1}$ , then  $u_n = \frac{u_0}{nu_0 + 1}$  for all  $n \in \mathbb{Z}^+$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ ,  $u_1 = \frac{u_0}{u_0 + 1} = \frac{u_0}{(1)u_0 + 1} \quad \therefore P_1$  is true.

(2) If  $P_k$  is true, then  $u_k = \frac{u_0}{ku_0 + 1}$ .

$$\begin{aligned} \text{Now } u_{k+1} &= \frac{u_k}{u_k + 1} \\ &= \frac{\left(\frac{u_0}{ku_0 + 1}\right)}{\left(\frac{u_0}{ku_0 + 1} + 1\right)} \quad \{\text{using } P_k\} \\ &= \frac{\left(\frac{u_0}{ku_0 + 1}\right)}{\left(\frac{u_0 + ku_0 + 1}{ku_0 + 1}\right)} \\ &= \frac{u_0}{ku_0 + 1} \times \frac{ku_0 + 1}{(k+1)u_0 + 1} \\ &= \frac{u_0}{(k+1)u_0 + 1} \end{aligned}$$

$\therefore P_{k+1}$  is also true.

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

**17 a**  $P_n$  is: if  $u_n = u_1 + (n-1)d$  then  $\sum_{i=1}^n u_i = \frac{n}{2}(2u_1 + (n-1)d)$  for all  $n \in \mathbb{Z}^+$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ , LHS =  $u_1$  and RHS =  $\frac{1}{2}(2u_1 + (1-1)d) = \frac{1}{2}(2u_1) = u_1 \quad \therefore P_1$  is true.

(2) If  $P_k$  is true, then  $\sum_{i=1}^k u_i = \frac{k}{2}(2u_1 + (k-1)d)$ .

$$\begin{aligned} \text{Now } \sum_{i=1}^{k+1} u_i &= \left(\sum_{i=1}^k u_i\right) + u_{k+1} \\ &= \frac{k}{2}(2u_1 + (k-1)d) + u_1 + kd \quad \{\text{using } P_k\} \\ &= ku_1 + \frac{k(k-1)}{2}d + u_1 + kd \\ &= (k+1)u_1 + \left(\frac{k(k-1)}{2} + k\right)d \\ &= (k+1)u_1 + \left(\frac{k^2 - k + 2k}{2}\right)d \\ &= \frac{1}{2}(k+1)2u_1 + \frac{1}{2}(k+1)kd \\ &= \frac{k+1}{2}(2u_1 + kd) \\ &= \frac{k+1}{2}(2u_1 + [(k+1) - 1]d) \end{aligned}$$

$\therefore P_{k+1}$  is also true.

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

- b**  $P_n$  is: if  $u_n = u_1 r^{n-1}$  then  $\sum_{i=1}^n u_i = \frac{u_1(1-r^n)}{1-r}$ ,  $r \neq 1$  for all  $n \in \mathbb{Z}^+$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ ,  $\text{LHS} = \frac{u_1(1-r^1)}{1-r} = u_1$  and  $\text{RHS} = \frac{u_1(1-r^1)}{1-r} = u_1$   
 $\therefore P_1$  is true.

(2) If  $P_k$  is true, then  $\sum_{i=1}^k u_i = \frac{u_1(1-r^k)}{1-r}$ .

$$\begin{aligned} \text{Now } \sum_{i=1}^{k+1} u_i &= \left( \sum_{i=1}^k u_i \right) + u_{k+1} \\ &= \frac{u_1(1-r^k)}{1-r} + u_1 r^k \quad \{\text{using } P_k\} \\ &= \frac{u_1(1-r^k) + (1-r)u_1 r^k}{1-r} \\ &= \frac{u_1(1-r^k + (1-r)r^k)}{1-r} \\ &= \frac{u_1(1-r^k + r^k - r^{k+1})}{1-r} \\ &= \frac{u_1(1-r^{k+1})}{1-r} \end{aligned}$$

$\therefore P_{k+1}$  is also true.

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

**18 a**  $u_1 = 1 = 1^2$

$$u_2 = u_1 + 2(1) + 1 = 1 + 2 + 1 = 4 = 2^2$$

$$u_3 = u_2 + 2(2) + 1 = 4 + 4 + 1 = 9 = 3^2$$

$$u_4 = u_3 + 2(3) + 1 = 9 + 6 + 1 = 16 = 4^2$$

We conjecture that  $u_n = n^2$  for all  $n \in \mathbb{Z}^+$ .

- b**  $P_n$  is: if  $u_1 = 1$  and  $u_{n+1} = u_n + (2n+1)$  for all  $n \in \mathbb{Z}^+$ , then  $u_n = n^2$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ , then  $u_1 = 1^2 = 1$  which is true  $\therefore P_1$  is true.

(2) If  $P_k$  is true, then  $u_k = k^2$ .

$$\begin{aligned} \text{Now } u_{k+1} &= u_k + (2k+1) \\ &= k^2 + 2k + 1 \quad \{\text{using } P_k\} \\ &= (k+1)^2 \end{aligned}$$

$\therefore P_{k+1}$  is also true.

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

**c**  $u_{20} = 20^2 = 400$  {using **b**}

**19 a**  $u_1 = \frac{1}{3}$

$$u_2 = u_1 + \frac{1}{(2(1)+1)(2(1)+3)} = \frac{1}{3} + \frac{1}{3 \times 5} = \frac{5+1}{15} = \frac{6}{15} = \frac{2}{5}$$

$$u_3 = u_2 + \frac{1}{(2(2)+1)(2(2)+3)} = \frac{2}{5} + \frac{1}{5 \times 7} = \frac{14+1}{35} = \frac{15}{35} = \frac{3}{7}$$

$$u_4 = u_3 + \frac{1}{(2(3)+1)(2(3)+3)} = \frac{3}{7} + \frac{1}{7 \times 9} = \frac{27+1}{63} = \frac{28}{63} = \frac{4}{9}$$

We conjecture that  $u_n = \frac{n}{2n+1}$  for all  $n \in \mathbb{Z}^+$ .

**b**  $P_n$  is: if  $u_1 = \frac{1}{3}$  and  $u_{n+1} = u_n + \frac{1}{(2n+1)(2n+3)}$  then  $u_n = \frac{n}{2n+1}$ , for all  $n \in \mathbb{Z}^+$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ , then  $u_1 = \frac{1}{2(1)+1} = \frac{1}{3}$  which is true  $\therefore P_1$  is true.

(2) If  $P_k$  is true, then  $u_k = \frac{k}{2k+1}$

$$\begin{aligned} \text{Now } u_{k+1} &= u_k + \frac{1}{(2k+1)(2k+3)} \\ &= \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)} \quad \{\text{using } P_k\} \\ &= \frac{k(2k+3) + 1}{(2k+1)(2k+3)} \\ &= \frac{2k^2 + 3k + 1}{(2k+1)(2k+3)} \\ &= \frac{(2k+1)(k+1)}{(2k+1)(2k+3)} \\ &= \frac{k+1}{2(k+1)+1} \end{aligned}$$

$\therefore P_{k+1}$  is also true.

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

**c**  $u_{50} = \frac{50}{2(50)+1} = \frac{50}{101}$  {using **b**}

**20 a** For  $n = 1$ ,  $(2 + \sqrt{3})^1 = 2 + \sqrt{3}$   $\therefore A_1 = 2, B_1 = 1$

For  $n = 2$ ,  $(2 + \sqrt{3})^2 = 4 + 4\sqrt{3} + 3$   
 $= 7 + 4\sqrt{3}$   $\therefore A_2 = 7, B_2 = 4$

For  $n = 3$ ,  $(2 + \sqrt{3})^3 = (7 + 4\sqrt{3})(2 + \sqrt{3})$   
 $= 14 + 7\sqrt{3} + 8\sqrt{3} + 4(3)$   
 $= 26 + 15\sqrt{3}$   $\therefore A_3 = 26, B_3 = 15$

For  $n = 4$ ,  $(2 + \sqrt{3})^4 = (7 + 4\sqrt{3})^2$   
 $= 49 + 56\sqrt{3} + 16(3)$   
 $= 97 + 56\sqrt{3}$   $\therefore A_4 = 97, B_4 = 56$



$$\begin{aligned}
 \text{b} \quad (2 + \sqrt{3})^n &= A_n + B_n\sqrt{3} \\
 \therefore (2 + \sqrt{3})^{n+1} &= (2 + \sqrt{3})^n(2 + \sqrt{3}) \\
 &= (A_n + B_n\sqrt{3})(2 + \sqrt{3}) \\
 &= 2A_n + A_n\sqrt{3} + 2B_n\sqrt{3} + B_n(3) \\
 &= 2A_n + 3B_n + (A_n + 2B_n)\sqrt{3}
 \end{aligned}$$

$$\therefore A_{n+1} = 2A_n + 3B_n, \quad B_{n+1} = A_n + 2B_n$$

$$\begin{aligned}
 \text{c} \quad (A_1)^2 - 3(B_1)^2 &= 2^2 - 3(1)^2 = 4 - 3 = 1 \\
 (A_2)^2 - 3(B_2)^2 &= 7^2 - 3(4)^2 = 49 - 3 \times 16 = 1 \\
 (A_3)^2 - 3(B_3)^2 &= 26^2 - 3(15)^2 = 676 - 3 \times 225 = 1 \\
 (A_4)^2 - 3(B_4)^2 &= 97^2 - 3(56)^2 = 9409 - 3 \times 3136 = 1
 \end{aligned}$$

We conjecture  $(A_n)^2 - 3(B_n)^2 = 1$  for all  $n \in \mathbb{Z}^+$ .

$$\text{d} \quad P_n \text{ is: if } (2 + \sqrt{3})^n = A_n + B_n\sqrt{3} \text{ then } (A_n)^2 - 3(B_n)^2 = 1, \text{ for all } n \in \mathbb{Z}^+.$$

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ ,  $A_1 = 2$ ,  $B_1 = 1$ , and  $(A_1)^2 - 3(B_1)^2 = 2^2 - 3(1)^2 = 1 \quad \therefore P_1$  is true.

(2) If  $P_k$  is true, then  $(A_k)^2 - 3(B_k)^2 = 1$ .

$$\begin{aligned}
 \text{Now } (A_{k+1})^2 - 3(B_{k+1})^2 &= (2A_k + 3B_k)^2 - 3(A_k + 2B_k)^2 \quad \{\text{using b}\} \\
 &= 4(A_k)^2 + 12A_kB_k + 9(B_k)^2 - 3[(A_k)^2 + 4A_kB_k + 4(B_k)^2] \\
 &= 4(A_k)^2 + 12A_kB_k + 9(B_k)^2 - 3(A_k)^2 - 12A_kB_k - 12(B_k)^2 \\
 &= (A_k)^2 - 3(B_k)^2 \\
 &= 1 \quad \{\text{using } P_k\}
 \end{aligned}$$

$\therefore P_{k+1}$  is also true.

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

$$\begin{aligned}
 \text{21 a} \quad P_n \text{ is: if } u_1 = 11, u_2 = 37, \text{ and } u_{n+2} = 5u_{n+1} - 6u_n \text{ for all } n \in \mathbb{Z}^+, \\
 \text{then } u_n = 5(3^n) - 2^{n+1}.
 \end{aligned}$$

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ ,  $5(3^1) - 2^{1+1} = 15 - 4 = 11 = u_1 \quad \therefore P_1$  is true.

If  $n = 2$ ,  $5(3^2) - 2^{2+1} = 5 \times 9 - 8 = 37 = u_2 \quad \therefore P_2$  is true.

(2) If  $P_k$  and  $P_{k+1}$  are true, then  $u_k = 5(3^k) - 2^{k+1}$  and  $u_{k+1} = 5(3^{k+1}) - 2^{k+2}$

$$\begin{aligned}
 \text{Now } u_{k+2} &= 5u_{k+1} - 6u_k \\
 &= 5[5(3^{k+1}) - 2^{k+2}] - 6[5(3^k) - 2^{k+1}] \quad \{\text{using } P_k \text{ and } P_{k+1}\} \\
 &= 25(3^{k+1}) - 5(2^{k+2}) - 30(3^k) + 6(2^{k+1}) \\
 &= 25(3^{k+1}) - 5(2^{k+2}) - 10(3^{k+1}) + 3(2^{k+2}) \\
 &= 15(3^{k+1}) - 2(2^{k+2}) \\
 &= 5(3^{k+2}) - 2^{k+3}
 \end{aligned}$$

$\therefore P_{k+1}$  is also true.

Since  $P_1$  and  $P_2$  are true, and  $P_{k+2}$  is true whenever  $P_k$  and  $P_{k+1}$  are true,

$P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}



**b**

$$u_{n+2} = au_{n+1} + bu_n$$

$$\therefore (3 + \sqrt{5})^{n+2} + (3 - \sqrt{5})^{n+2} = a[(3 + \sqrt{5})^{n+1} + (3 - \sqrt{5})^{n+1}] \\ + b[(3 + \sqrt{5})^n + (3 - \sqrt{5})^n]$$

$$\therefore (3 + \sqrt{5})^2(3 + \sqrt{5})^n + (3 - \sqrt{5})^2(3 - \sqrt{5})^n = a[(3 + \sqrt{5})(3 + \sqrt{5})^n \\ + (3 - \sqrt{5})(3 - \sqrt{5})^n] \\ + b[(3 + \sqrt{5})^n + (3 - \sqrt{5})^n]$$

$$\therefore (14 + 6\sqrt{5})(3 + \sqrt{5})^n + (14 - 6\sqrt{5})(3 - \sqrt{5})^n = (3a + a\sqrt{5})(3 + \sqrt{5})^n \\ + (3a - a\sqrt{5})(3 - \sqrt{5})^n \\ + b(3 + \sqrt{5})^n + b(3 - \sqrt{5})^n$$

$$\therefore (14 + 6\sqrt{5})(3 + \sqrt{5})^n + (14 - 6\sqrt{5})(3 - \sqrt{5})^n = (3a + a\sqrt{5} + b)(3 + \sqrt{5})^n \\ + (3a - a\sqrt{5} + b)(3 - \sqrt{5})^n$$

Equating coefficients of  $(3 + \sqrt{5})^n$ ,  $14 + 6\sqrt{5} = 3a + b + a\sqrt{5}$

Equating rational and irrational parts,

$$3a + b = 14 \quad \text{and} \quad a = 6$$

$$\therefore a = 6 \quad \text{and} \quad b = -4 \quad \{\text{this is consistent with the coefficients of } (3 - \sqrt{5})^n\}$$

$$\therefore u_{n+2} = 6u_{n+1} - 4u_n \quad \dots (*)$$

$P_n$  is: if  $u_n = (3 + \sqrt{5})^n + (3 - \sqrt{5})^n$  where  $n \in \mathbb{Z}^+$ , then  $u_n$  is a multiple of  $2^n$ .

**Proof:** (By the principle of mathematical induction)

$$(1) \quad \text{If } n = 1, \quad u_1 = (3 + \sqrt{5}) + (3 - \sqrt{5}) \\ = 6 \\ = 3 \times 2^1 \quad \therefore P_1 \text{ is true.}$$

$$\text{If } n = 2, \quad u_2 = (3 + \sqrt{5})^2 + (3 - \sqrt{5})^2 \\ = 9 + 6\sqrt{5} + 5 + 9 - 6\sqrt{5} + 5 \\ = 28 \\ = 7 \times 4 \\ = 7 \times 2^2 \quad \therefore P_2 \text{ is true.}$$

$$(2) \quad \text{If } P_k \text{ and } P_{k+1} \text{ are true, then } u_k = A \times 2^k, \text{ where } A \in \mathbb{Z} \\ \text{and } u_{k+1} = B \times 2^{k+1}, \text{ where } B \in \mathbb{Z}$$

$$\text{Now } u_{k+2} = 6u_{k+1} - 4u_k \quad \{\text{using } (*)\} \\ = 6(B \times 2^{k+1}) - 4(A \times 2^k) \quad \{\text{using } P_k \text{ and } P_{k+1}\} \\ = 3B \times 2^{k+2} - A \times 2^{k+2} \\ = (3B - A) \times 2^{k+2}$$

which is a multiple of  $2^{k+2}$  since  $3B - A \in \mathbb{Z}$  {as  $A, B \in \mathbb{Z}$ }

$\therefore P_{k+2}$  is also true.

Since  $P_1$  and  $P_2$  are true, and  $P_{k+2}$  is true whenever  $P_k$  and  $P_{k+1}$  are true,  $P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

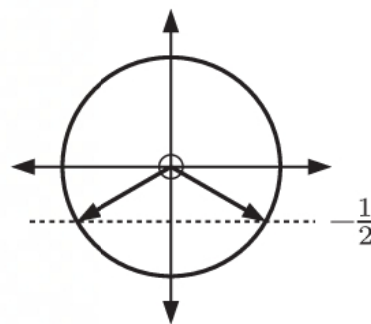
- 22 a**  $1 + \sin x + \sin^2 x + \sin^3 x + \dots + \sin^{n-1} x$  is a geometric series with  $u_1 = 1$ ,  $r = \sin x$

$$\begin{aligned}\therefore S_n &= \frac{u_1(1 - r^n)}{1 - r} \\ &= \frac{1(1 - \sin^n x)}{1 - \sin x} \\ &= \frac{1 - \sin^n x}{1 - \sin x}, \quad \sin x \neq 1\end{aligned}$$

**b**  $S = \frac{u_1}{1 - r} = \frac{1}{1 - \sin x}$  if  $-1 < \sin x < 1$ .

The series is not convergent for  $\sin x = \pm 1$

**c** If  $S = \frac{2}{3}$ ,  $\frac{1}{1 - \sin x} = \frac{2}{3}$   
 $\therefore 3 = 2 - 2 \sin x$   
 $\therefore 2 \sin x = -1$   
 $\therefore \sin x = -\frac{1}{2}$   
 $\therefore x = \frac{7\pi}{6} \text{ or } \frac{11\pi}{6}$



**23 a i**  $2 \sin x(\cos x + \cos 3x)$   
 $= 2 \sin x \cos x + 2 \sin x \cos 3x$   
 $= \sin 2x + \sin 4x + \sin(-2x) \quad \{2 \sin A \cos B = \sin(A + B) + \sin(A - B)\}$   
 $= \sin 2x + \sin 4x - \sin 2x \quad \{\sin(-\theta) = -\sin \theta\}$   
 $= \sin 4x$

**ii**  $2 \sin x(\cos x + \cos 3x + \cos 5x)$   
 $= 2 \sin x(\cos x + \cos 3x) + 2 \sin x \cos 5x$   
 $= \sin 4x + \sin 6x + \sin(-4x) \quad \{\text{from i}\}$   
 $= \sin 4x + \sin 6x - \sin 4x$   
 $= \sin 6x$

**b i**  $2 \sin x(\cos x + \cos 3x + \cos 5x + \cos 7x) = \sin 8x$

**ii**  $2 \sin x(\cos x + \cos 3x + \cos 5x + \dots + \cos 19x) = \sin 20x$

$$\therefore \cos x + \cos 3x + \cos 5x + \dots + \cos 19x = \frac{\sin 20x}{2 \sin x}$$

**iii** In general,  $\cos x + \cos 3x + \cos 5x + \dots + \cos[(2n - 1)x] = \frac{\sin 2nx}{2 \sin x}$

**c**  $P_n$  is:  $\cos \theta + \cos 3\theta + \cos 5\theta + \dots + \cos(2n - 1)\theta = \frac{\sin 2n\theta}{2 \sin \theta}$  for all  $n \in \mathbb{Z}^+$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ , LHS =  $\cos \theta$  and RHS =  $\frac{\sin 2\theta}{2 \sin \theta} = \frac{2 \sin \theta \cos \theta}{2 \sin \theta} = \cos \theta \quad \therefore P_1$  is true.

(2) If  $P_k$  is true, then  $\cos \theta + \cos 3\theta + \cos 5\theta + \dots + \cos(2k-1)\theta = \frac{\sin 2k\theta}{2 \sin \theta}$

Now  $\cos \theta + \cos 3\theta + \cos 5\theta + \dots + \cos(2k-1)\theta + \cos(2k+1)\theta$

$$\begin{aligned}
 &= \frac{\sin 2k\theta}{2 \sin \theta} + \cos(2k+1)\theta \quad \{\text{using } P_k\} \\
 &= \frac{\sin 2k\theta + 2 \sin \theta \cos(2k+1)\theta}{2 \sin \theta} \\
 &= \frac{\sin 2k\theta + \sin[\theta + (2k+1)\theta] + \sin[\theta - (2k+1)\theta]}{2 \sin \theta} \\
 &= \frac{\sin 2k\theta + \sin(\theta + 2k\theta + \theta) + \sin(\theta - 2k\theta - \theta)}{2 \sin \theta} \\
 &= \frac{\sin 2k\theta + \sin(2k\theta + 2\theta) + \sin(-2k\theta)}{2 \sin \theta} \\
 &= \frac{\sin 2k\theta + \sin 2(k+1)\theta - \sin 2k\theta}{2 \sin \theta} \\
 &= \frac{\sin 2(k+1)\theta}{2 \sin \theta}
 \end{aligned}$$

$\therefore P_{k+1}$  is also true.

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

**24 a**  $P_n$  is:  $\sin \theta + \sin 3\theta + \sin 5\theta + \dots + \sin(2n-1)\theta = \frac{1 - \cos 2n\theta}{2 \sin \theta}$  for all positive integers  $n$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ , LHS =  $\sin \theta$  and RHS =  $\frac{1 - \cos 2\theta}{2 \sin \theta} = \frac{1 - (1 - 2 \sin^2 \theta)}{2 \sin \theta} = \frac{2 \sin^2 \theta}{2 \sin \theta} = \sin \theta$   
 $\therefore P_1$  is true.

(2) If  $P_k$  is true, then  $\sin \theta + \sin 3\theta + \sin 5\theta + \dots + \sin(2k-1)\theta = \frac{1 - \cos 2k\theta}{2 \sin \theta}$

Now  $\sin \theta + \sin 3\theta + \sin 5\theta + \dots + \sin(2k-1)\theta + \sin(2k+1)\theta$

$$\begin{aligned}
 &= \frac{1 - \cos 2k\theta}{2 \sin \theta} + \sin(2k+1)\theta \quad \{\text{using } P_k\} \\
 &= \frac{1 - \cos 2k\theta + 2 \sin(2k+1)\theta \sin \theta}{2 \sin \theta} \\
 &= \frac{1 - \cos 2k\theta + \cos[(2k+1)\theta - \theta] - \cos[(2k+1)\theta + \theta]}{2 \sin \theta} \quad \left\{ \begin{array}{l} 2 \sin A \sin B \\ = \cos(A-B) - \cos(A+B) \end{array} \right\} \\
 &= \frac{1 - \cos 2k\theta + \cos 2k\theta - \cos[(2k+2)\theta]}{2 \sin \theta} \\
 &= \frac{1 - \cos 2(k+1)\theta}{2 \sin \theta}
 \end{aligned}$$

$\therefore P_{k+1}$  is also true.

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all positive integers  $n$ . {principle of mathematical induction}

**b** Thus  $\sin \frac{\pi}{7} + \sin \frac{3\pi}{7} + \sin \frac{5\pi}{7} + \dots + \sin \frac{13\pi}{7}$  has  $2n-1 = 13$  and  $\theta = \frac{\pi}{7}$   
 $\therefore n = 7$  and  $\theta = \frac{\pi}{7}$

$$\therefore \text{the sum is } \frac{1 - \cos(2 \times 7 \times \frac{\pi}{7})}{2 \sin \frac{\pi}{7}} = \frac{1 - \cos 2\pi}{2 \sin \frac{\pi}{7}} = \frac{1 - 1}{2 \sin \frac{\pi}{7}} = 0$$



**25 a**  $P_n$  is:  $\sum_{i=1}^n \frac{1}{2^i} \tan \frac{x}{2^i} = \frac{1}{2^n} \cot \frac{x}{2^n} - \cot x$  for all  $n \in \mathbb{Z}^+$ ,  $x \neq m\pi$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ ,  $\text{LHS} = \frac{1}{2^1} \tan \frac{x}{2^1} = \frac{1}{2} \tan \frac{x}{2}$

$$\begin{aligned} \text{and RHS} &= \frac{1}{2^1} \cot \frac{x}{2^1} - \cot x \\ &= \frac{1}{2} \cot \frac{x}{2} - \cot x \\ &= \frac{\cos \frac{x}{2}}{2 \sin \frac{x}{2}} - \frac{\cos x}{\sin x} \\ &= \frac{\cos^2(\frac{x}{2})}{2 \sin \frac{x}{2} \cos \frac{x}{2}} - \frac{\cos^2(\frac{x}{2}) - \sin^2(\frac{x}{2})}{2 \sin \frac{x}{2} \cos \frac{x}{2}} \quad \{\text{double angle formulae}\} \\ &= \frac{\sin^2(\frac{x}{2})}{2 \sin \frac{x}{2} \cos \frac{x}{2}} \\ &= \frac{\sin \frac{x}{2}}{2 \cos \frac{x}{2}} = \frac{1}{2} \tan \frac{x}{2} \end{aligned}$$

$\therefore P_1$  is true.

(2) If  $P_k$  is true, then  $\sum_{i=1}^k \frac{1}{2^i} \tan \frac{x}{2^i} = \frac{1}{2^k} \cot \frac{x}{2^k} - \cot x$

$$\begin{aligned} \text{Now } \sum_{i=1}^{k+1} \frac{1}{2^i} \tan \frac{x}{2^i} &= \left[ \sum_{i=1}^k \frac{1}{2^i} \tan \frac{x}{2^i} \right] + \frac{1}{2^{k+1}} \tan \frac{x}{2^{k+1}} \\ &= \frac{1}{2^k} \cot \frac{x}{2^k} - \cot x + \frac{1}{2^{k+1}} \tan \frac{x}{2^{k+1}} \quad \{\text{using } P_k\} \\ &= \frac{1}{2^{k+1}} \left[ \frac{2 \cos \frac{x}{2^k}}{\sin \frac{x}{2^k}} + \frac{\sin \frac{x}{2^{k+1}}}{\cos \frac{x}{2^{k+1}}} \right] - \cot x \\ &= \frac{1}{2^{k+1}} \left[ \frac{\cancel{2} \left[ \cos^2 \left( \frac{x}{2^{k+1}} \right) - \sin^2 \left( \frac{x}{2^{k+1}} \right) \right]}{\cancel{2} \sin \frac{x}{2^{k+1}} \cos \frac{x}{2^{k+1}}} + \frac{\sin \frac{x}{2^{k+1}}}{\cos \frac{x}{2^{k+1}}} \right] - \cot x \\ &\quad \{\text{double angle formulae}\} \\ &= \frac{1}{2^{k+1}} \left[ \frac{\cos^2 \left( \frac{x}{2^{k+1}} \right) - \cancel{\sin^2 \left( \frac{x}{2^{k+1}} \right)} + \cancel{\sin^2 \left( \frac{x}{2^{k+1}} \right)}}{\sin \frac{x}{2^{k+1}} \cancel{\cos \frac{x}{2^{k+1}}}} \right] - \cot x \\ &= \frac{1}{2^{k+1}} \frac{\cos \frac{x}{2^{k+1}}}{\sin \frac{x}{2^{k+1}}} - \cot x \\ &= \frac{1}{2^{k+1}} \cot \frac{x}{2^{k+1}} - \cot x \end{aligned}$$

$\therefore P_{k+1}$  is also true.

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all  $n \in \mathbb{Z}^+$ ,  $x \neq m\pi$ . {principle of mathematical induction}



- b**  $P_n$  is:  $\cos^2 \theta + \cos^2 2\theta + \cos^2 3\theta + \dots + \cos^2(n\theta) = \frac{1}{2} \left[ n + \frac{\cos[(n+1)\theta] \sin n\theta}{\sin \theta} \right]$   
for all  $n \in \mathbb{Z}^+$ .

**Proof:** (By the principle of mathematical induction)

$$\begin{aligned} (1) \text{ If } n = 1, \text{ LHS} &= \cos^2 \theta \text{ and } \text{RHS} = \frac{1}{2} \left[ 1 + \frac{\cos 2\theta \sin \theta}{\sin \theta} \right] \\ &= \frac{1}{2} + \frac{1}{2}(2 \cos^2 \theta - 1) \quad \{ \cos 2\theta = 2 \cos^2 \theta - 1 \} \\ &= \cos^2 \theta \quad \therefore P_1 \text{ is true.} \end{aligned}$$

(2) If  $P_k$  is true, then

$$\cos^2 \theta + \cos^2 2\theta + \cos^2 3\theta + \dots + \cos^2(k\theta) = \frac{1}{2} \left[ k + \frac{\cos[(k+1)\theta] \sin k\theta}{\sin \theta} \right]$$

$$\begin{aligned} \text{Now } \cos^2 \theta + \cos^2 2\theta + \cos^2 3\theta + \dots + \cos^2(k\theta) + \cos^2[(k+1)\theta] \\ &= \frac{1}{2} \left[ k + \frac{\cos[(k+1)\theta] \sin k\theta}{\sin \theta} \right] + \cos^2[(k+1)\theta] \quad \{ \text{using } P_k \} \\ &= \frac{1}{2} \left[ k + \frac{\cos[(k+1)\theta] \sin k\theta}{\sin \theta} \right] + \frac{1}{2} + \frac{1}{2} \cos[2(k+1)\theta] \quad \{ \cos^2 \theta = \frac{1}{2} + \frac{1}{2} \cos 2\theta \} \\ &= \frac{1}{2}(k+1) + \frac{\cos[(k+1)\theta] \sin k\theta + \cos[2(k+1)\theta] \sin \theta}{2 \sin \theta} \\ &= \frac{1}{2}(k+1) + \frac{\frac{1}{2} \sin[k\theta + (k+1)\theta] + \frac{1}{2} \sin[k\theta - (k+1)\theta] + \frac{1}{2} \sin[\theta + 2(k+1)\theta] + \frac{1}{2} \sin[\theta - 2(k+1)\theta]}{2 \sin \theta} \\ &\quad \{ \text{products to sums formula} \} \\ &= \frac{1}{2}(k+1) + \frac{\frac{1}{2} \sin[(2k+1)\theta] + \frac{1}{2} \sin(-\theta) + \frac{1}{2} \sin[(2k+3)\theta] + \frac{1}{2} \sin[(-2k-1)\theta]}{2 \sin \theta} \\ &= \frac{1}{2}(k+1) + \frac{\frac{1}{2} \sin[(2k+1)\theta] - \frac{1}{2} \sin \theta + \frac{1}{2} \sin[(2k+3)\theta] - \frac{1}{2} \sin[(2k+1)\theta]}{2 \sin \theta} \\ &= \frac{1}{2}(k+1) + \frac{\frac{1}{2} \sin[(2k+3)\theta] - \frac{1}{2} \sin \theta}{2 \sin \theta} \\ &= \frac{1}{2}(k+1) + \frac{\cos \left[ \frac{(2k+3)\theta + \theta}{2} \right] \sin \left[ \frac{(2k+3)\theta - \theta}{2} \right]}{2 \sin \theta} \quad \{ \text{factor formula} \} \\ &= \frac{1}{2}(k+1) + \frac{\cos[(k+2)\theta] \sin[(k+1)\theta]}{2 \sin \theta} \\ &= \frac{1}{2} \left[ (k+1) + \frac{\cos[(k+2)\theta] \sin[(k+1)\theta]}{\sin \theta} \right] \end{aligned}$$

$\therefore P_{k+1}$  is also true.

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

- 26 a**  $P_n$  is:  $(x+y)^n = \sum_{i=0}^n \binom{n}{i} x^{n-i} y^i$  for all  $n \in \mathbb{Z}^+$ .

**Proof:** (By the principle of mathematical induction)

$$\begin{aligned} (1) \text{ If } n = 1, \quad (x+y)^1 &= x+y \quad \text{and} \quad \sum_{i=0}^1 \binom{1}{i} x^{1-i} y^i = \binom{1}{0} x^1 y^0 + \binom{1}{1} x^0 y^1 = x+y \\ \therefore P_1 &\text{ is true.} \end{aligned}$$

(2) If  $P_k$  is true, then  $(x+y)^k = \sum_{i=0}^k \binom{k}{i} x^{k-i} y^i$ .

$$\begin{aligned}
 \text{Now } (x+y)^{k+1} &= (x+y)^k (x+y) \\
 &= \left( \sum_{i=0}^k \binom{k}{i} x^{k-i} y^i \right) \times (x+y) \quad \{\text{using } P_k\} \\
 &= \sum_{i=0}^k \binom{k}{i} x^{k+1-i} y^i + \sum_{i=0}^k \binom{k}{i} x^{k-i} y^{i+1} \\
 &= \sum_{i=0}^k \binom{k}{i} x^{k+1-i} y^i + \sum_{i=1}^{k+1} \binom{k}{i-1} x^{k+1-i} y^i \\
 &= \binom{k}{0} x^{k+1} y^0 + \left( \sum_{i=1}^k \left[ \binom{k}{i} + \binom{k}{i-1} \right] x^{k+1-i} y^i \right) + \binom{k}{k} x^0 y^{k+1} \\
 &= \binom{k+1}{0} x^{k+1} y^0 + \left( \sum_{i=1}^k \binom{k+1}{i} x^{k+1-i} y^i \right) + \binom{k+1}{k+1} x^0 y^{k+1} \\
 &= \sum_{i=0}^{k+1} \binom{k+1}{i} x^{k+1-i} y^i
 \end{aligned}$$

$\therefore P_{k+1}$  is also true.

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

**b**  $P_n$  is:  $n^p - n$  is divisible by  $p$  for any prime  $p$  and  $n \in \mathbb{Z}^+$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ ,  $1^p - 1 = 0$  which is divisible by  $p$

$\therefore P_1$  is true.

(2) If  $P_k$  is true, then  $k^p - k = pA$  for some  $A \in \mathbb{Z}$ .

$$\begin{aligned}
 \text{Now } (k+1)^p - (k+1) &= \left[ \sum_{i=0}^p \binom{p}{i} k^{p-i} 1^i \right] - (k+1) \quad \{\text{binomial theorem}\} \\
 &= k^p + \left[ \sum_{i=1}^{p-1} \binom{p}{i} k^{p-i} \right] + 1 - k - 1 \\
 &= (k^p - k) + \sum_{i=1}^{p-1} \binom{p}{i} k^{p-i} \\
 &= pA + \sum_{i=1}^{p-1} \frac{p!}{i!(p-i)!} k^{p-i} \quad \{\text{using } P_k\} \\
 &= pA + p \sum_{i=1}^{p-1} \frac{(p-1)!}{i!(p-i)!} k^{p-i} \\
 &= p \left( A + \sum_{i=1}^{p-1} \frac{(p-1)!}{i!(p-i)!} k^{p-i} \right)
 \end{aligned}$$

Since  $\binom{p}{i} = \frac{p!}{i!(p-i)!} = p \times \frac{(p-1)!}{i!(p-i)!} \in \mathbb{Z}^+$  for all  $i = 1, 2, \dots, p-2, p-1$ ,

$\frac{(p-1)!}{i!(p-i)!}$  must also be an integer for all  $i = 1, 2, \dots, p-2, p-1$  as  $p$  is prime.

$$\therefore \sum_{i=1}^{p-1} \frac{(p-1)!}{i!(p-i)!} k^{p-i} \in \mathbb{Z}^+ \quad \{\text{as } k^{p-i} \in \mathbb{Z} \text{ for all } k \geq 1\}$$

$$\therefore A + \sum_{i=1}^{p-1} \frac{(p-1)!}{i!(p-i)!} k^{p-i} \in \mathbb{Z} \quad \{\text{as } A \in \mathbb{Z}\}$$

Thus  $(k+1)^p - (k+1)$  is divisible by  $p$  if  $k^p - k$  is divisible by  $p$ .

$\therefore P_{k+1}$  is also true.

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

**27 a**  $P_n$  is:  $\prod_{i=1}^n \left(1 - \frac{1}{i+1}\right) = \left(1 - \frac{1}{2}\right) \left(1 - \frac{1}{3}\right) \left(1 - \frac{1}{4}\right) \dots \left(1 - \frac{1}{n+1}\right) = \frac{1}{n+1}$   
for all  $n \in \mathbb{Z}^+$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ , LHS =  $\left(1 - \frac{1}{2}\right) = \frac{1}{2}$  and RHS =  $\frac{1}{1+1} = \frac{1}{2} \therefore P_1$  is true.

(2) If  $P_k$  is true, then  $\left(1 - \frac{1}{2}\right) \left(1 - \frac{1}{3}\right) \left(1 - \frac{1}{4}\right) \dots \left(1 - \frac{1}{k+1}\right) = \frac{1}{k+1}$

$$\begin{aligned} \therefore & \left(1 - \frac{1}{2}\right) \left(1 - \frac{1}{3}\right) \left(1 - \frac{1}{4}\right) \dots \left(1 - \frac{1}{k+1}\right) \left(1 - \frac{1}{k+2}\right) \\ &= \frac{1}{k+1} \left(1 - \frac{1}{k+2}\right) \quad \{\text{using } P_k\} \\ &= \frac{1}{k+1} \left(\frac{k+2-1}{k+2}\right) \\ &= \frac{1}{k+1} \left(\frac{k+1}{k+2}\right) \\ &= \frac{1}{(k+1)+1} \end{aligned}$$

$\therefore P_{k+1}$  is also true.

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

**b**  $P_n$  is:  $\left(1 - \frac{1}{2^2}\right) \left(1 - \frac{1}{3^2}\right) \left(1 - \frac{1}{4^2}\right) \dots \left(1 - \frac{1}{n^2}\right) = \frac{n+1}{2n}$  for all integers  $n \geq 2$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 2$ , LHS =  $\left(1 - \frac{1}{2^2}\right) = 1 - \frac{1}{4} = \frac{3}{4}$  and RHS =  $\frac{2+1}{2(2)} = \frac{3}{4} \therefore P_2$  is true.

(2) If  $P_k$  is true, then  $\left(1 - \frac{1}{2^2}\right) \left(1 - \frac{1}{3^2}\right) \left(1 - \frac{1}{4^2}\right) \dots \left(1 - \frac{1}{k^2}\right) = \frac{k+1}{2k}$



$$\begin{aligned}
\text{Now } & \left(1 - \frac{1}{2^2}\right) \left(1 - \frac{1}{3^2}\right) \left(1 - \frac{1}{4^2}\right) \dots \left(1 - \frac{1}{k^2}\right) \left(1 - \frac{1}{(k+1)^2}\right) \\
&= \frac{k+1}{2k} \left(1 - \frac{1}{(k+1)^2}\right) \quad \{\text{using } P_k\} \\
&= \frac{k+1}{2k} \left(\frac{(k+1)^2 - 1}{(k+1)^2}\right) \\
&= \frac{k+1}{2k} \left(\frac{k^2 + 2k + 1 - 1}{(k+1)^2}\right) \\
&= \frac{k^2 + 2k}{2k(k+1)} \\
&= \frac{k(k+2)}{2k(k+1)} \\
&= \frac{([k+1] + 1)}{2(k+1)}
\end{aligned}$$

$\therefore P_{k+1}$  is also true.

Since  $P_2$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all integers  $n \geq 2$ . {principle of mathematical induction}

**28**  $P_n$  is: The product of  $n$  odd integers is odd for all  $n \in \mathbb{Z}$ ,  $n \geq 2$ .

**Proof:** (By the principle of mathematical induction)

(1) Let  $p_1$  and  $p_2$  be odd integers.

Then there exist  $q_1, q_2 \in \mathbb{Z}$  such that  $p_1 = 2q_1 + 1$  and  $p_2 = 2q_2 + 1$ .

$$\begin{aligned}
\text{Now } p_1 p_2 &= (2q_1 + 1)(2q_2 + 1) \\
&= 4q_1 q_2 + 2q_1 + 2q_2 + 1 \\
&= 2(2q_1 q_2 + q_1 + q_2) + 1 \quad \text{which is odd.}
\end{aligned}$$

$\therefore P_2$  is true.

(2) If  $P_k$  is true, then the product of  $k$  odd integers is odd.

Let  $p_1, p_2, \dots, p_k$ , and  $p_{k+1}$  be odd integers.

Then there exist  $q, r \in \mathbb{Z}$  such that  $p_1 p_2 \dots p_k = 2q + 1$  {using  $P_k$ }  
and  $p_{k+1} = 2r + 1$

$$\begin{aligned}
\text{Now } p_1 p_2 \dots p_k p_{k+1} &= (2q + 1)(2r + 1) \\
&= 4qr + 2q + 2r + 1 \\
&= 2(2qr + q + r) + 1 \quad \text{which is odd.}
\end{aligned}$$

$\therefore P_{k+1}$  is also true.

Since  $P_2$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all  $n \in \mathbb{Z}$ ,  $n \geq 2$ . {principle of mathematical induction}

**29 a i**

$$\begin{aligned}
& \sin x \cos x \cos 2x \\
&= \frac{1}{2}(2 \sin x \cos x) \cos 2x \\
&= \frac{1}{2} \sin 2x \cos 2x \\
&= \frac{1}{4}(2 \sin 2x \cos 2x) \\
&= \frac{1}{4} \sin 4x \quad \dots (1) \\
&= \frac{\sin(2^2 x)}{2^2}
\end{aligned}$$

**ii**

$$\begin{aligned}
& (\sin x \cos x \cos 2x) \cos 4x \\
&= \frac{1}{4} \sin 4x \cos 4x \quad \{\text{from (1)}\} \\
&= \frac{1}{8}(2 \sin 4x \cos 4x) \\
&= \frac{\sin 8x}{8} \\
&= \frac{\sin(2^3 x)}{2^3}
\end{aligned}$$



**b**   **i**  $\frac{\sin(2^4 x)}{2^4}$       **ii**  $\frac{\sin(2^6 x)}{2^6}$

**c**  $\sin x \cos x \cos 2x \cos 4x \dots \cos(2^n x) = \frac{\sin(2^{n+1} x)}{2^{n+1}}$

$P_n$  is:  $\sin x \cos x \cos 2x \cos 4x \dots \cos(2^n x) = \frac{\sin(2^{n+1} x)}{2^{n+1}}$  for all  $n \in \mathbb{Z}^+$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ , LHS =  $\sin x \cos x \cos 2x = \frac{\sin(2^2 x)}{2^2}$  {from **a i**}

and RHS =  $\frac{\sin(2^{1+1} x)}{2^{1+1}} = \frac{\sin(2^2 x)}{2^2}$

$\therefore P_1$  is true.

(2) If  $P_k$  is true, then  $\sin x \cos x \cos 2x \cos 4x \dots \cos(2^k x) = \frac{\sin(2^{k+1} x)}{2^{k+1}}$

Now  $\sin x \cos x \cos 2x \cos 4x \dots \cos(2^k x) \cos(2^{k+1} x)$

$= \frac{\sin(2^{k+1} x)}{2^{k+1}} \cos(2^{k+1} x)$  {using  $P_k$ }

$= \frac{1}{2} \frac{2 \sin(2^{k+1} x) \cos(2^{k+1} x)}{2^{k+1}}$

$= \frac{\sin 2(2^{k+1} x)}{2 \times 2^{k+1}}$

$= \frac{\sin(2^{k+2} x)}{2^{k+2}}$

$\therefore P_{k+1}$  is also true.

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

**30 a**  $P_n$  is:  $(z_1 + z_2 + \dots + z_n)^* = z_1^* + z_2^* + \dots + z_n^*$  for all  $n \in \mathbb{Z}^+$   
and complex  $z_1, z_2, \dots, z_n$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ ,  $(z_1)^* = z_1^* \therefore P_1$  is true.

(2) If  $P_k$  is true, then  $(z_1 + z_2 + \dots + z_k)^* = z_1^* + z_2^* + \dots + z_k^*$

Now  $(z_1 + z_2 + \dots + z_k + z_{k+1})^*$

$= ((z_1 + z_2 + \dots + z_k) + z_{k+1})^*$

$= (z_1 + z_2 + \dots + z_k)^* + z_{k+1}^*$  {using  $(z + w)^* = z^* + w^*$ }

$= z_1^* + z_2^* + \dots + z_k^* + z_{k+1}^*$  {using  $P_k$ }

$\therefore P_{k+1}$  is also true.

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

**b**  $P_n$  is:  $(z_1 z_2 \dots z_n)^* = z_1^* z_2^* \dots z_n^*$  for all  $n \in \mathbb{Z}^+$  and complex  $z_1, z_2, \dots, z_n$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ ,  $(z_1)^* = z_1^* \quad \therefore P_1$  is true.

(2) If  $P_k$  is true, then  $(z_1 z_2 \dots z_k)^* = z_1^* z_2^* \dots z_k^*$

$$\begin{aligned} \text{Now } (z_1 z_2 \dots z_k z_{k+1})^* &= ((z_1 z_2 \dots z_k) z_{k+1})^* \\ &= (z_1 z_2 \dots z_k)^* z_{k+1}^* \quad \{\text{using } (zw)^* = z^* w^*\} \\ &= z_1^* z_2^* \dots z_k^* z_{k+1}^* \quad \{\text{using } P_k\} \end{aligned}$$

$\therefore P_{k+1}$  is also true.

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

**c**  $P_n$  is:  $(z^n)^* = (z^*)^n$  for all  $n \in \mathbb{Z}^+$  and complex  $z$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ ,  $(z^1)^* = z^* = (z^*)^1 \quad \therefore P_1$  is true.

(2) If  $P_k$  is true, then  $(z^k)^* = (z^*)^k$

$$\begin{aligned} \text{Now } (z^{k+1})^* &= (z z^k)^* \\ &= z^* (z^k)^* \quad \{\text{using } (zw)^* = z^* w^*\} \\ &= z^* (z^*)^k \quad \{\text{using } P_k\} \\ &= (z^*)^{k+1} \end{aligned}$$

$\therefore P_{k+1}$  is also true.

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

**31 a**  $P_n$  is:  $3^n \geq 1 + 2n$  for integers  $n \geq 0$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 0$ , LHS =  $3^0 = 1$  and RHS =  $1 + 2(0) = 1 \quad \therefore P_0$  is true.

(2) If  $P_k$  is true, then  $3^k \geq 1 + 2k$

$$\begin{aligned} \text{Now } 3^{k+1} &= 3^k \times 3 \geq (1 + 2k) \times 3 \quad \{\text{using } P_k\} \\ &\geq 3 + 6k \\ &\geq 3 + 2k \quad \{k \geq 0\} \\ &\geq 1 + 2(k + 1) \end{aligned}$$

$\therefore P_{k+1}$  is also true.

Since  $P_0$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all integers  $n \geq 0$ . {principle of mathematical induction}

**b**  $P_n$  is:  $n! \geq 2^n$  for all  $n \in \mathbb{Z}$ ,  $n \geq 4$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 4$ , LHS =  $4! = 24$  and RHS =  $2^4 = 16$   $\therefore P_4$  is true.

(2) If  $P_k$  is true then  $k! \geq 2^k$

$$\begin{aligned} \text{Now } (k+1)! &= (k+1) \times k! \geq (k+1) \times 2^k \quad \{\text{using } P_k\} \\ &\geq 2 \times 2^k \quad \{k \geq 4\} \\ &\geq 2^{k+1} \end{aligned}$$

$\therefore P_{k+1}$  is also true.

Since  $P_4$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all  $n \in \mathbb{Z}$ ,  $n \geq 4$ . {principle of mathematical induction}

**c**  $P_n$  is:  $8^n \geq n^3$  for all  $n \in \mathbb{Z}^+$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ , LHS =  $8^1 = 8$  and RHS =  $1^3 = 1$   $\therefore P_1$  is true.

(2) If  $P_k$  is true then  $8^k \geq k^3$

$$\begin{aligned} \text{Now } 8^{k+1} &= 8 \times 8^k \geq 8 \times k^3 \quad \{\text{using } P_k\} \\ &\geq (2k)^3 \\ &\geq (k+1)^3 \quad \{k \geq 1, \text{ so } 2k \geq k+1\} \end{aligned}$$

$\therefore P_{k+1}$  is also true.

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

**d**  $P_n$  is:  $\sum_{i=1}^n \frac{1}{i} \leq \frac{n}{2} + 1$  for all  $n \in \mathbb{Z}^+$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ , LHS =  $\frac{1}{1} = 1$  and RHS =  $\frac{1}{2} + 1 = \frac{3}{2}$   $\therefore P_1$  is true.

(2) If  $P_k$  is true then  $\sum_{i=1}^k \frac{1}{i} \leq \frac{k}{2} + 1$ .

$$\begin{aligned} \text{Now } \sum_{i=1}^{k+1} \frac{1}{i} &= \left( \sum_{i=1}^k \frac{1}{i} \right) + \frac{1}{k+1} \\ &\leq \frac{k}{2} + \frac{1}{k+1} + 1 \quad \{\text{using } P_k\} \end{aligned}$$

$$\begin{aligned} \text{and } \frac{k}{2} + \frac{1}{k+1} - \frac{k+1}{2} &= \frac{1}{k+1} - \frac{1}{2} \\ &= \frac{2-k-1}{2(k+1)} \\ &= \frac{1-k}{2(k+1)} \leq 0 \quad \text{for all } k \in \mathbb{Z}^+ \end{aligned}$$

$$\therefore \frac{k}{2} + \frac{1}{k+1} \leq \frac{k+1}{2}$$

$$\text{So, } \sum_{i=1}^{k+1} \frac{1}{i} \leq \frac{k+1}{2} + 1$$

$\therefore P_{k+1}$  is also true.

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}



**e**  $P_n$  is:  $2^n > 3n^2 + 3n + 1$  for all  $n \in \mathbb{Z}^+$ ,  $n \geq 8$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 8$ , LHS =  $2^8 = 256$  and RHS =  $3(8)^2 + 3(8) + 1 = 217 \quad \therefore P_8$  is true.

(2) If  $P_k$  is true then  $2^k > 3k^2 + 3k + 1$ .

$$\begin{aligned} \text{Now } 2^{k+1} &= 2 \times 2^k \\ &> 2(3k^2 + 3k + 1) \quad \{\text{using } P_k\} \\ &> 6k^2 + 6k + 2 \end{aligned}$$

$$\begin{aligned} \text{and } 6k^2 + 6k + 2 - (3(k+1)^2 + 3(k+1) + 1) \\ &= 6k^2 + 6k + 2 - 3k^2 - 6k - 3 - 3k - 3 - 1 \\ &= 3k^2 - 3k - 5 \\ &> 0 \quad \text{for all } k \in \mathbb{Z}^+, k \geq 8 \end{aligned}$$

$$\therefore 6k^2 + 6k + 2 > 3(k+1)^2 + 3(k+1) + 1$$

$$\text{So, } 2^{k+1} > 3(k+1)^2 + 3(k+1) + 1$$

$\therefore P_{k+1}$  is also true.

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

**f**  $P_n$  is:  $\sum_{i=1}^n (-1)^{i-1} \frac{1}{i} > 0$  for all  $n \in \mathbb{Z}^+$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ ,  $\sum_{i=1}^1 (-1)^{i-1} \frac{1}{i} = (-1)^0 \frac{1}{1} = 1 > 0 \quad \therefore P_1$  is true.

If  $n = 2$ ,  $\sum_{i=1}^2 (-1)^{i-1} \frac{1}{i} = 1 - \frac{1}{2} = \frac{1}{2} > 0 \quad \therefore P_2$  is true.

(2) If  $P_k$  is true, then  $\sum_{i=1}^k (-1)^{i-1} \frac{1}{i} > 0$ .

If  $P_{k+1}$  is true, then  $\sum_{i=1}^{k+1} (-1)^{i-1} \frac{1}{i} > 0$ .

$$\text{Now } \sum_{i=1}^{k+2} (-1)^{i-1} \frac{1}{i} = \left[ \sum_{i=1}^{k+1} (-1)^{i-1} \frac{1}{i} \right] + (-1)^{k+1} \frac{1}{k+2} \quad \dots (1)$$

$$= \left[ \sum_{i=1}^k (-1)^{i-1} \frac{1}{i} \right] + (-1)^k \frac{1}{k+1} + (-1)^{k+1} \frac{1}{k+2} \quad \dots (2)$$

Case 1:  $k$  is odd

$$\begin{aligned} \text{In (1), } \sum_{i=1}^{k+2} (-1)^{i-1} \frac{1}{i} &= \left[ \sum_{i=1}^{k+1} (-1)^{i-1} \frac{1}{i} \right] + (-1)^{k+1} \frac{1}{k+2} \\ &= \left[ \sum_{i=1}^{k+1} (-1)^{i-1} \frac{1}{i} \right] + \frac{1}{k+2} \quad \{k \text{ is odd } \therefore k+1 \text{ is even}\} \\ &> 0 + \frac{1}{k+2} \quad \{\text{using } P_{k+1}\} \\ &> 0 \quad \text{for all } k \in \mathbb{Z}^+ \end{aligned}$$

$$\text{So, } \sum_{i=1}^{k+2} (-1)^{i-1} \frac{1}{i} > 0 \quad \text{when } k \text{ is odd.}$$



Case 2:  $k$  is even

$$\begin{aligned}
 \text{In (1), } \sum_{i=1}^{k+2} (-1)^{i-1} \frac{1}{i} &= \left[ \sum_{i=1}^k (-1)^{i-1} \frac{1}{i} \right] + (-1)^k \frac{1}{k+1} + (-1)^{k+1} \frac{1}{k+2} \\
 &= \left[ \sum_{i=1}^k (-1)^{i-1} \frac{1}{i} \right] + (-1)^k \left( \frac{1}{k+1} - \frac{1}{k+2} \right) \\
 &= \left[ \sum_{i=1}^k (-1)^{i-1} \frac{1}{i} \right] + (-1)^k \left( \frac{k+2 - (k+1)}{(k+1)(k+2)} \right) \\
 &= \left[ \sum_{i=1}^k (-1)^{i-1} \frac{1}{i} \right] + \frac{1}{(k+1)(k+2)} \quad \{k \text{ is even}\} \\
 &> \frac{1}{(k+1)(k+2)} \quad \{\text{using } P_k\} \\
 &> 0 \quad \text{for all } k \in \mathbb{Z}^+
 \end{aligned}$$

$$\text{So, } \sum_{i=1}^{k+2} (-1)^{i-1} \frac{1}{i} > 0 \quad \text{when } k \text{ is even.}$$

From Case 1 and Case 2,  $\sum_{i=1}^{k+2} (-1)^{i-1} \frac{1}{i} > 0$  regardless of whether  $k$  is odd or even.  
 $\therefore P_{k+2}$  is also true.

Since  $P_1$  is true, and  $P_{k+2}$  is true whenever  $P_k$  and  $P_{k+1}$  are true,  
 $P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

**32**  $P_n$  is:  $(1-h)^n \leq \frac{1}{1+nh}$  for  $0 \leq h \leq 1$  for all  $n \in \mathbb{Z}^+$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ , we have  $h^2 \geq 0$  {as  $0 \leq h \leq 1$ }

$$\therefore 1 - h^2 \leq 1$$

$$\therefore (1-h)(1+h) \leq 1$$

$$\therefore (1-h)^1 \leq \frac{1}{1-(1)h} \quad \{1+h \geq 1\} \quad \therefore P_1 \text{ is true.}$$

(2) If  $P_k$  is true then  $(1-h)^k \leq \frac{1}{1+kh}$  for  $0 \leq h \leq 1$ .

$$\text{Now } (1-h)^{k+1} = (1-h)(1-h)^k$$

$$\therefore (1-h)^{k+1} \leq (1-h) \left( \frac{1}{1+kh} \right) \quad \{\text{using } P_k\}$$

$$\therefore (1-h)^{k+1} \leq \left( \frac{1-h}{1+kh} \right) \times \left( \frac{1+kh+h}{1+kh+h} \right)$$

$$\therefore (1-h)^{k+1} \leq \frac{1+kh+h-h-kh^2-h^2}{(1+kh)(1+kh+h)}$$

$$\therefore (1-h)^{k+1} \leq \frac{(1+kh) - (kh^2+h^2)}{(1+kh)(1+(k+1)h)}$$

$$\therefore (1-h)^{k+1} \leq \frac{1 - \frac{kh^2+h^2}{1+kh}}{1+(k+1)h}$$

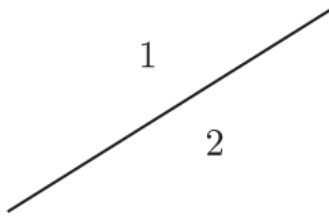
$$\therefore (1-h)^{k+1} \leq \frac{1}{1+(k+1)h} \quad \{k, h \geq 0, \text{ so } \frac{kh^2+h^2}{1+kh} \geq 0\}$$

$\therefore P_{k+1}$  is also true.

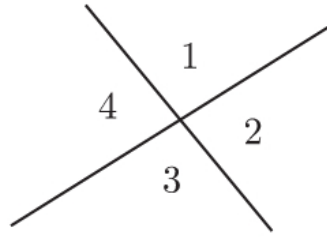
Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

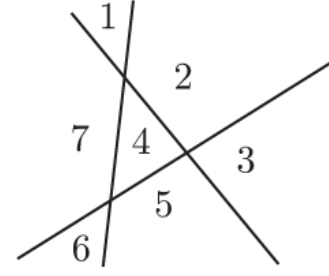
**33 a** Let  $R_n$  be the number of regions the plane is divided into.



1 line,  $R_1 = 2$



2 lines,  $R_2 = 4$



3 lines,  $R_3 = 7$ , and so on.

$P_n$  is: For  $n$  lines as described,  $R_n = \frac{n(n+1)}{2} + 1$  for all  $n \in \mathbb{Z}^+$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ ,  $R_1 = \frac{1(2)}{2} + 1 = 1 + 1 = 2 \quad \therefore P_1$  is true

(2) If  $P_k$  is true, then  $R_k = \frac{k(k+1)}{2} + 1$

The addition of another line creates another  $k + 1$  regions

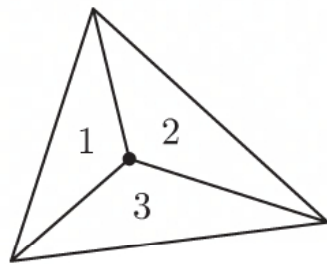
$$\begin{aligned} \therefore R_{k+1} &= \frac{k(k+1)}{2} + 1 + k + 1 \quad \{\text{using } P_k\} \\ &= \frac{k(k+1)}{2} + \frac{2(k+1)}{2} + 1 \\ &= \frac{k^2 + k + 2k + 2}{2} + 1 \\ &= \frac{k^2 + 3k + 2}{2} + 1 \\ &= \frac{(k+1)(k+2)}{2} + 1 \end{aligned}$$

$\therefore P_{k+1}$  is also true.

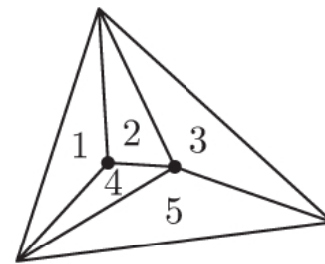
Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

**b** Let  $T_n$  be the number of smaller triangles the initial triangle is divided into.



$n = 1$ ,  $T_1 = 3$



$n = 2$ ,  $T_2 = 5$

$P_n$  is: For  $n$  points inside the triangle (as described) there are  $T_n = 2n + 1$  triangular partitions for all  $n \in \mathbb{Z}^+$ .

**Proof:** (By the principle of mathematical induction)

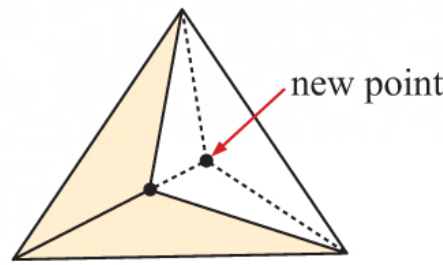
(1) If  $n = 1$ ,  $T_1 = 2(1) + 1 = 3 \quad \therefore P_1$  is true

(2) If  $P_k$  is true, then  $T_k = 2k + 1$

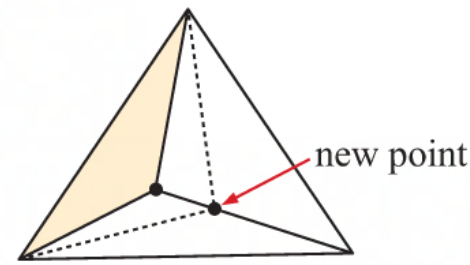
Adding an extra point within the triangle gives the  $(k + 1)$ th case.

This point could be either

- in an existing triangle      or      • on an existing line between 2 triangles



So, 1 triangle becomes 3,  
a net increase of 2.



So, 2 triangles become 4,  
a net increase of 2.

In each case 2 triangles are added

$$\begin{aligned}\therefore T_{k+1} &= T_k + 2 \\ &= 2k + 1 + 2 \quad \{\text{using } P_k\} \\ &= 2(k + 1) + 1\end{aligned}$$

$\therefore P_{k+1}$  is also true.

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

## INVESTIGATION 1

## SEQUENCES, SERIES, AND INDUCTION

**1**  $u_n = n \times n!$

$$\therefore u_1 = 1 \times 1! = 1$$

$$u_2 = 2 \times 2! = 4$$

$$u_3 = 3 \times 3! = 18$$

$$u_4 = 4 \times 4! = 96$$

$$u_5 = 5 \times 5! = 600$$

$$\therefore S_1 = u_1 = 1 = 2! - 1$$

$$S_2 = u_1 + u_2 = 5 = 3! - 1$$

$$S_3 = u_1 + u_2 + u_3 = 23 = 4! - 1$$

$$S_4 = u_1 + u_2 + u_3 + u_4 = 119 = 5! - 1$$

$$S_5 = u_1 + u_2 + u_3 + u_4 + u_5 = 719 = 6! - 1$$

**2** We conjecture that  $S_n = (n + 1)! - 1$  for all  $n \in \mathbb{Z}^+$ .

**3**  $P_n$  is:  $S_n = (n + 1)! - 1$  for all  $n \in \mathbb{Z}^+$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ ,  $S_1 = u_1 = 1 = 2! - 1 \therefore P_1$  is true.

(2) If  $P_k$  is true then  $S_k = (k + 1)! - 1$

$$\begin{aligned}\text{Now } S_{k+1} &= S_k + u_{k+1} \\ &= (k + 1)! - 1 + (k + 1)(k + 1)! \quad \{\text{using } P_k\} \\ &= (k + 1)![1 + (k + 1)] - 1 \\ &= (k + 1)!(k + 2) - 1 \\ &= (k + 2)! - 1\end{aligned}$$

$\therefore P_{k+1}$  is also true.

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}



$$\begin{aligned}
 4 \quad u_n &= n \times n! \\
 &= (n+1)n! - n! \\
 &= (n+1)! - n!
 \end{aligned}$$

$$\begin{aligned}
 \therefore S_n &= u_1 + u_2 + u_3 + \dots + u_n \\
 &= (\cancel{2!} - 1!) + (\cancel{3!} - \cancel{2!}) + (\cancel{4!} - \cancel{3!}) + \dots + ((n+1)! - \cancel{n!}) \\
 &= (n+1)! - 1
 \end{aligned}$$

$$\begin{aligned}
 5 \quad C_n &= u_n + u_{n+1} \\
 &= (\cancel{n+1})! - n! + (n+2)! - (\cancel{n+1})! \\
 &= (n+2)! - n!
 \end{aligned}$$

$$\begin{aligned}
 6 \quad T_n &= C_1 + C_2 + C_3 + \dots + C_n \\
 T_1 &= C_1 = (3! - 1!) = 3! + 2! - 2! - 1! \\
 T_2 &= C_1 + C_2 = (3! - 1!) + (4! - 2!) = 4! + 3! - 2! - 1! \\
 T_3 &= C_1 + C_2 + C_3 = (4! + 3! - 2! - 1!) + (5! - 3!) = 5! + 4! - 2! - 1! \\
 T_4 &= C_1 + C_2 + C_3 + C_4 = (5! + 4! - 2! - 1!) + (6! - 4!) = 6! + 5! - 2! - 1! \\
 T_5 &= C_1 + C_2 + C_3 + C_4 + C_5 = (6! + 5! - 2! - 1!) + (7! - 5!) = 7! + 6! - 2! - 1!
 \end{aligned}$$

7 We conjecture that  $T_n = (n+2)! + (n+1)! - 2! - 1!$  for all  $n \in \mathbb{Z}^+$ .

8  $P_n$  is:  $T_n = (n+2)! + (n+1)! - 2! - 1!$  for all  $n \in \mathbb{Z}^+$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ ,  $T_1 = 3! + 2! - 2! - 1! = 3! - 1!$   $\therefore P_1$  is true.

(2) If  $P_k$  is true then  $T_k = (k+2)! + (k+1)! - 2! - 1!$

$$\begin{aligned}
 \text{Now } T_{k+1} &= T_k + C_{k+1} \\
 &= (k+2)! + (\cancel{k+1})! - 2! - 1! + (k+3)! - (\cancel{k+1})! \quad \{\text{using } P_k\} \\
 &= (k+3)! + (k+2)! - 2! - 1!
 \end{aligned}$$

$\therefore P_{k+1}$  is also true.

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

## INVESTIGATION 2

## EQUIVALENT STATEMENTS

1  $P_n$  is:  $(n+2)^2 - n^2$  is a multiple of 8 for all odd integers  $n$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ ,  $(1+2)^2 - 1^2 = 8$  which is a multiple of 8.  $\therefore P_1$  is true.

(2) If  $P_k$  where  $k$  is odd is true then  $(k+2)^2 - k^2 = 8A$  for some  $A \in \mathbb{Z}$ .

$$\begin{aligned}
 \text{Now } (k+2+2)^2 - (k+2)^2 &= (k+4)^2 - (k+2)^2 \\
 &= k^2 + 8k + 16 - (k^2 + 4k + 4) \\
 &= k^2 + 4k + 4k + 4 + 12 - k^2 - 4k - 4 \\
 &= (k+2)^2 - k^2 + 8 \\
 &= 8A + 8 \quad \{\text{using } P_k\} \\
 &= 8(A+1)
 \end{aligned}$$

$\therefore P_{k+2}$  is also true.

Since  $P_1$  is true, and  $P_{k+2}$  is true whenever  $P_k$  is true,

$P_n$  is true for all odd integers  $n$ . {principle of mathematical induction}



**2**  $x$  is not divisible by 3  $\Leftrightarrow x^2 - 1$  is divisible by 3

$A \Rightarrow B$ : If  $x$  is not divisible by 3 then  $x$  is of the form  
 $x = 3k + 1$  or  $x = 3k + 2$  for some  $k \in \mathbb{Z}$ .

$$\begin{aligned} \text{If } x &= 3k + 1, \\ x^2 - 1 &= (3k + 1)^2 - 1 \\ &= 9k^2 + 6k + 1 - 1 \\ &= 3(3k^2 + 2k) \\ &\text{which is divisible by 3} \end{aligned}$$

$$\begin{aligned} \text{If } x &= 3k + 2, \\ x^2 - 1 &= (3k + 2)^2 - 1 \\ &= 9k^2 + 12k + 4 - 1 \\ &= 3(3k^2 + 4k + 1) \\ &\text{which is divisible by 3} \end{aligned}$$

$B \Rightarrow A$ : If  $x^2 - 1$  is divisible by 3 then  
 $x^2 - 1 = 3k$  for some integer  $k \in \mathbb{Z}$   
 $\therefore x^2 = 3k + 1$   
 $\therefore x^2$  is not a multiple of 3  
 $\therefore x$  is not divisible by 3.

**3**  $P_x$  is:  $x^2 - 1$  is divisible by 3 if  $x$  is not divisible by 3.

**Proof:** (By the principle of mathematical induction)

(1) If  $x = 1$ ,  $1^2 - 1 = 0$  which is divisible by 3  
 If  $x = 2$ ,  $2^2 - 1 = 3$  which is divisible by 3  
 $\therefore P_1$  and  $P_2$  are true.

(2) If  $P_k$  is true then  $k^2 - 1$  is divisible by 3 for some  $k$  which is not divisible by 3.

$$\begin{aligned} \text{Now } (k + 3)^2 - 1 &= k^2 + 6k + 9 - 1 \\ &= (k^2 - 1) + 6k + 9 \\ &= 3A + 3(2k + 3) \quad \{\text{using } P_k\} \\ &= 3(A + 2k + 3) \end{aligned}$$

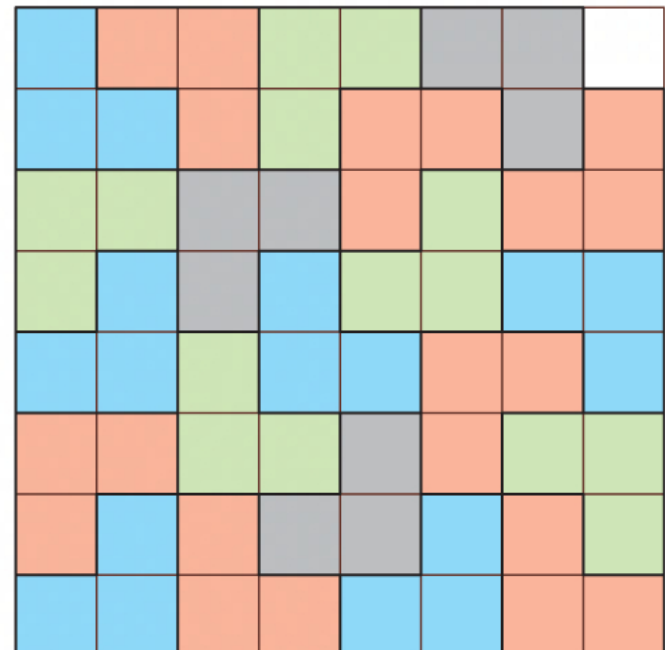
$\therefore P_{k+3}$  is also true.

Since  $P_1$  and  $P_2$  are true, and  $P_{k+3}$  is true whenever  $P_k$  is true,  
 $P_x$  is true for all positive integers  $x$  that are not divisible by 3.  
 {principle of mathematical induction}

## INVESTIGATION 3

## TRIOMINOES

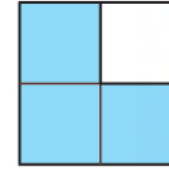
**1** An  $8 \times 8$  grid of squares has 64 total squares,  
 while each triomino has 3 squares.  
 64 leaves a remainder of 1 when divided by 3, so  
 it is impossible to exactly cover an  $8 \times 8$  grid of  
 squares with these triominoes.



- 2  $P_n$  is: Any  $2^n \times 2^n$  grid of squares can be covered by the triominoes, except for one square missing in the corner.

**Proof:** (By the principle of mathematical induction)

- (1) If  $n = 1$ , a  $2 \times 2$  grid can be covered by a single triomino, except for one corner square.  
 $\therefore P_1$  is true.

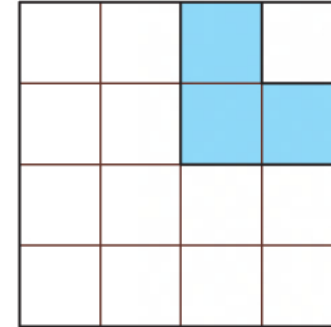


Consider as well the case  $n = 2$ , or a  $4 \times 4$  grid.

We essentially have four  $2 \times 2$  boards.

If we place a blue triomino in the corner shown, we have decided which corner square to leave blank.

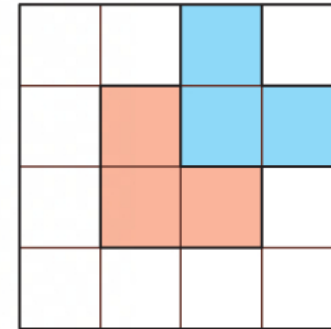
We are left with a large L-shape that is blank.



If we place the red triomino as shown, we can see how to place the rest of the triominoes.

**Notice how one square from the red triomino fills in the corner squares of the other  $2 \times 2$  grids.**

We now know how to fill a  $4 \times 4$  grid with one left over corner square.



- (2) If  $P_k$  is true, then any  $2^k \times 2^k$  grid of squares can be covered by the triominoes, except for one square missing in the corner.

Now, a  $2^{k+1} \times 2^{k+1}$  grid is just four  $2^k \times 2^k$  grids joined together.

Using  $P_k$ , we can fill in the top right grid, except for the one corner square.

We can *then* place a triomino that occupies exactly one corner square of the remaining three  $2^k \times 2^k$  grids.

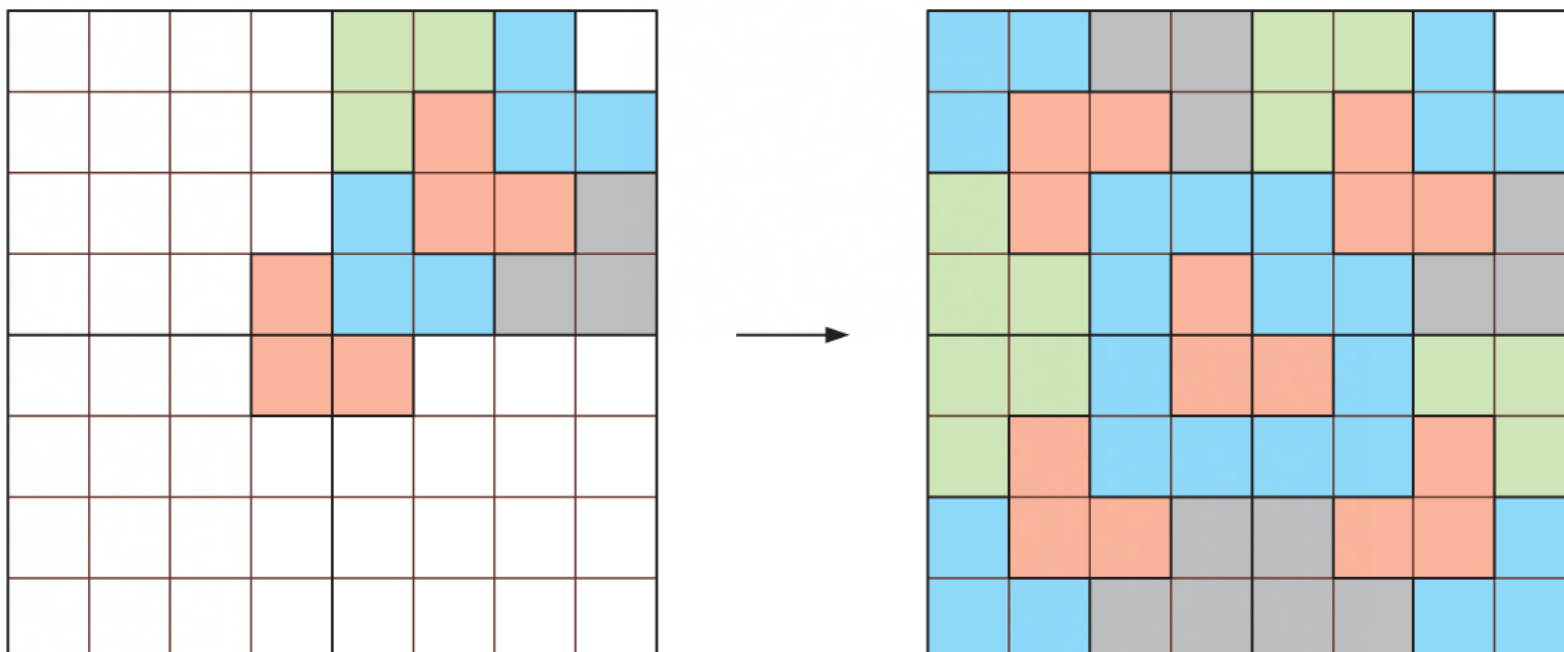
Using  $P_k$  again, we can then fill in the remaining squares in the three  $2^k \times 2^k$  grids.

$\therefore P_{k+1}$  is also true.

Since  $P_1$  is true and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

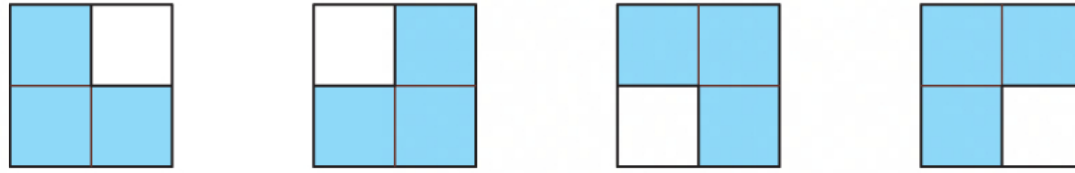
The case  $n = 3$  is shown below to help illustrate the general process.



- 3**  $P_n$  is: Any  $2^n \times 2^n$  grid of squares can be covered by the triominoes, except for any one particular square missing.

**Proof:** (By the principle of mathematical induction)

- (1) If  $n = 1$ , a  $2 \times 2$  grid can be covered by a single triomino, except for any one particular square missing.



$\therefore P_1$  is true.

- (2) If  $P_k$  is true, then any  $2^k \times 2^k$  grid of squares can be covered by the triominoes, except for any one particular square missing.

Now, a  $2^{k+1} \times 2^{k+1}$  board is just four  $2^k \times 2^k$  grids joined together.

The particular missing square must lie in exactly one of these grids.

We can place a triomino that occupies exactly one corner square of the three  $2^k \times 2^k$  grids that do *not* have the particular missing square. These grids can then be filled, using our conclusion from **2**.

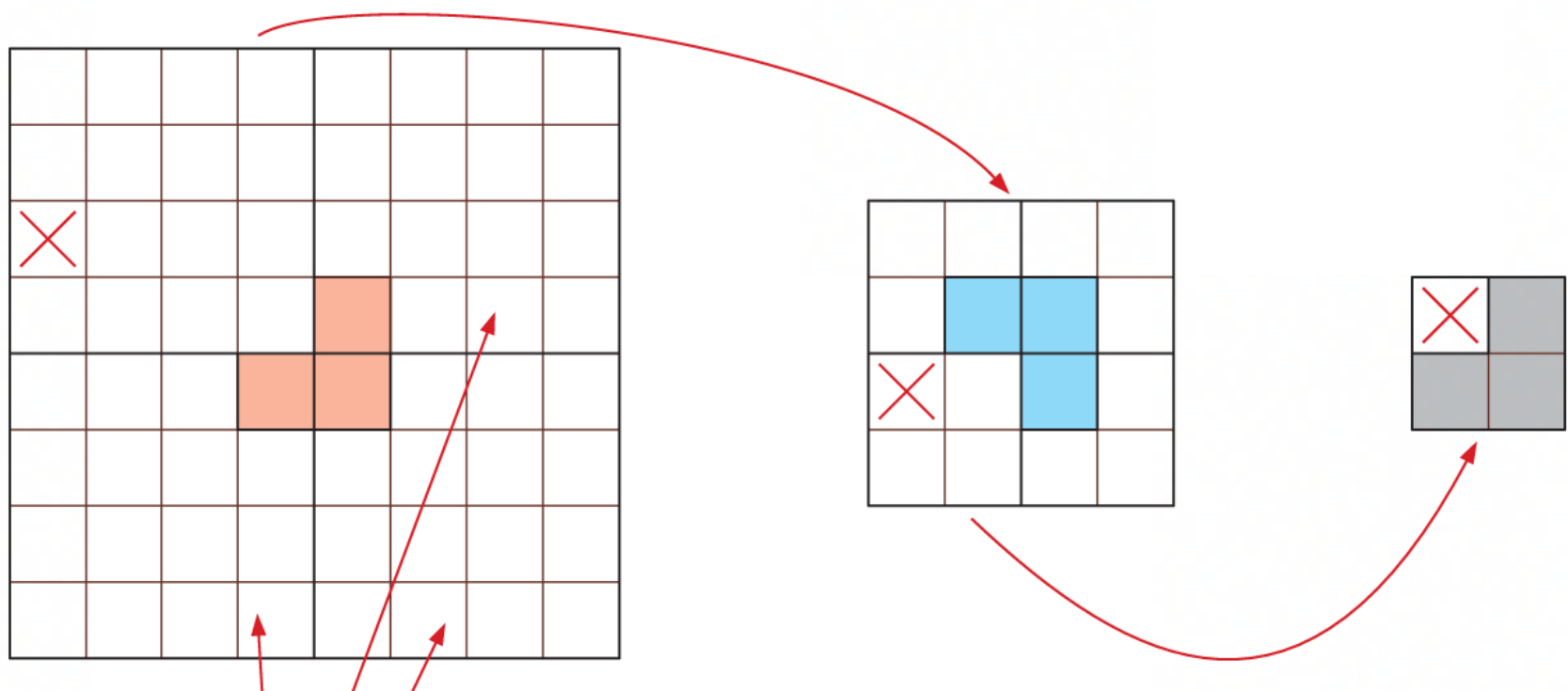
Using  $P_k$ , the remaining  $2^k \times 2^k$  grid that has the missing square can be covered, except for the missing square.

$\therefore P_{k+1}$  is also true.

Since  $P_1$  is true and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

An example for the case  $n = 3$  is shown below to help illustrate the general process.



These grids can all be filled {using **2**}.



# ACTIVITY

# FERMAT'S METHOD OF INFINITE DESCENT

- 1 Assume  $\sqrt{3}$  is rational, so  $\sqrt{3} = \frac{p_1}{q_1}$  for some  $p_1, q_1 \in \mathbb{Z}^+$ .

We know that  $\frac{9}{4} < 3 < 4$

$$\therefore \frac{3}{2} < \sqrt{3} < 2, \text{ so } 2p_1 > 3q_1 \text{ and } 2q_1 > p_1 \dots (*)$$

$$\text{Now } \sqrt{3} = \sqrt{3} \times \frac{\sqrt{3}-1}{\sqrt{3}-1}$$

$$= \frac{3 - \sqrt{3}}{\sqrt{3} - 1}$$

$$= \frac{3 - \frac{p_1}{q_1}}{\frac{p_1}{q_1} - 1}$$

$$= \frac{3q_1 - p_1}{p_1 - q_1}$$

Letting  $p_2 = 3q_1 - p_1$ , and  $q_2 = p_1 - q_1$ , we get  $\sqrt{3} = \frac{p_2}{q_2}$  where  $p_2, q_2 \in \mathbb{Z}^+$ .

Notice that  $p_2 = 3q_1 - p_1$

and  $q_2 = p_1 - q_1$

$$\therefore p_2 < 2p_1 - p_1 \text{ \{using (*)\}}$$

$$\therefore q_2 < 2q_1 - q_1 \text{ \{using (*)\}}$$

$$\therefore p_2 < p_1$$

$$\therefore q_2 < q_1$$

This process can be repeated an infinite number of times, so there is an infinite descent through the positive integers  $p, q$  satisfying  $\sqrt{3} = \frac{p}{q}$ .

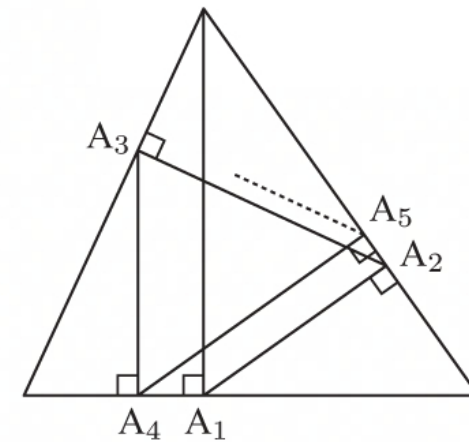
$\therefore$  no *smallest*  $p, q$  exist satisfying  $\sqrt{3} = \frac{p}{q}$

$\therefore \sqrt{3}$  cannot be rational.

- 2 Since the triangle is acute, the perpendicular drawn from any point which is not a vertex will *not* meet another side at a vertex.

Thus, the only point obtained from the perpendicular drawn from a vertex is  $A_1$ .

Assume that there exist  $m, n \in \mathbb{Z}^+$ ,  $m < n$ , such that points  $A_m$  and  $A_n$  coincide.



If  $m = 1$ , then  $n > 1$ , and  $A_m$  is the foot of the perpendicular which meets the opposite vertex. Since  $A_1$  is the only point with this property, and points  $A_m$  and  $A_n$  coincide, we must have  $n = 1$ . This is impossible as  $n > 1$ .

If  $m \neq 1$ , then  $A_m$  is the foot of a perpendicular which meets another side at  $A_{m-1}$ . Similarly,  $A_n$  is the foot of a perpendicular which meets another side at  $A_{n-1}$ . Since points  $A_m$  and  $A_n$  coincide, points  $A_{m-1}$  and  $A_{n-1}$  must also coincide.

This process can be repeated an infinite number of times, so there is an infinite descent through the positive integers  $m, n$  such that points  $A_m$  and  $A_n$  coincide.

$\therefore$  no *smallest*  $m, n$  exist such that points  $A_m$  and  $A_n$  coincide.

$\therefore$  each pair of points  $A_m$  and  $A_n$  are distinct.

$\therefore$  points  $A_1, A_2, A_3, \dots$  are all distinct.



**3** Assume  $a_1 \leq a_2 \leq \dots \leq a_{2n+1}$ .

We subtract  $a_1$  from every member of the set to generate a new set  $\{b_j\}$  where  $b_j = a_j - a_1$ . This set  $\{b_j\}$  has the same property.

$\therefore$  the sum of *any*  $2n$  members of the set must be even.

$\therefore$  the sum of the set  $\{b_j\}$  excluding  $b_1$  is even.

Since  $b_1 = 0$  is even, *all* members of  $\{b_j\}$  must be even. Otherwise we could swap  $b_1 = 0$  with any odd member from the set  $\{b_j\}$  excluding  $b_1$ , to generate a set of  $2n$  members of  $\{b_j\}$  which is odd.

We can now divide every member of  $\{b_j\}$  by 2 to generate another set  $\{b_j^*\}$  with first member  $b_1^* = 0$ , and which has the same property.

We can do this procedure again and again in infinite descent until the set members are no longer integers *unless* all  $b_j = 0$ .

Hence  $a_1 = a_2 = \dots = a_{2n+1}$ .

**4** For  $p = 2$ ,  $2^1 = 1^3 + 1^3$ .

For  $p = 3$ ,  $3^2 = 1^3 + 2^3$ .

After many trials, we found no more primes with this property.

So, let  $p \geq 5$  be prime, and assume that  $p^{n_1} = x_1^3 + y_1^3$  for some  $n_1, x_1, y_1 \in \mathbb{Z}^+$ .

Since  $p \geq 5$  is prime, and hence odd,  $p^{n_1} \geq 5$  is odd.

Also, if  $x_1 = y_1 = 1$ , then  $p^{n_1} = 1^3 + 1^3 = 2 \not\geq 5$ .

$\therefore$  at least one of  $x_1$  or  $y_1$  is greater than 1.

$\therefore x_1 + y_1 > 2$  and  $x_1 y_1 > 1$  .... (1)

If  $x_1 = y_1$ , then  $p^{n_1} = x_1^3 + x_1^3 = 2x_1^3$  which is even.

$\therefore x_1 \neq y_1$

$\therefore (x_1 - y_1)^2 \geq 1$  .... (2)

Now,  $p^{n_1} = x_1^3 + y_1^3$   
 $= (x_1 + y_1)(x_1^2 - x_1 y_1 + y_1^2)$

where  $x_1 + y_1 > 2$  {using (1)}

and  $x_1^2 - x_1 y_1 + y_1^2 = x_1^2 - x_1 y_1 + y_1^2 - x_1 y_1 + x_1 y_1$   
 $= x_1^2 - 2x_1 y_1 + y_1^2 + x_1 y_1$   
 $= (x_1 - y_1)^2 + x_1 y_1$

$\therefore x_1^2 - x_1 y_1 + y_1^2 > 2$  {using (1) and (2)}

Thus,  $x_1 + y_1$  and  $x_1^2 - x_1 y_1 + y_1^2$  are proper factors of  $p^{n_1}$ , and since  $p$  is prime,  $p$  must divide *both* factors.

$\therefore p$  must also divide  $(x_1 + y_1)^2 - (x_1^2 - x_1 y_1 + y_1^2) = x_1^2 + 2x_1 y_1 + y_1^2 - x_1^2 + x_1 y_1 - y_1^2$   
 $= 3x_1 y_1$

However, 3 is not divisible by  $p$ , so  $p$  must divide at least one of  $x_1$  or  $y_1$ . But  $p$  divides  $x_1 + y_1$ , so  $p$  must divide *both*  $x_1$  and  $y_1$ .

$\therefore \frac{x_1}{p}, \frac{y_1}{p} \in \mathbb{Z}^+$ ,  $x_1 \geq p$ , and  $y_1 \geq p$

$$\begin{aligned}
\text{Thus } p^{n_1} &= x_1^3 + y_1^3 \\
\therefore p^{n_1} &\geq p^3 + p^3 \\
\therefore p^{n_1} &\geq 2p^3 \\
\therefore p^{n_1} &> p^3 \\
\therefore n_1 &> 3
\end{aligned}$$

Letting  $n_2 = n_1 - 3$ ,  $x_2 = \frac{x_1}{p}$ ,  $y_2 = \frac{y_1}{p} \in \mathbb{Z}^+$ , we have

$$\begin{aligned}
p^{n_2} &= p^{n_1-3} \\
&= \frac{p^{n_1}}{p^3} \\
&= \frac{x_1^3 + y_1^3}{p^3} \\
&= \frac{x_1^3}{p^3} + \frac{y_1^3}{p^3} \\
&= \left(\frac{x_1}{p}\right)^3 + \left(\frac{y_1}{p}\right)^3
\end{aligned}$$

$$\therefore p^{n_2} = x_2^3 + y_2^3 \quad \text{where } n_2 < n_1, \quad x_2 < x_1, \quad \text{and } y_2 < y_1$$

This process can be repeated an infinite number of times, so there is an infinite descent through the positive integers  $n, x, y$  satisfying  $p^n = x^3 + y^3$  for primes  $p \geq 5$ .

$\therefore$  no *smallest*  $n, x, y$  exist satisfying  $p^n = x^3 + y^3$  for primes  $p \geq 5$ .

$\therefore$  there exist  $n, x, y$  satisfying  $p^n = x^3 + y^3$  for primes  $p = 2, 3$  only.

## REVIEW SET 10A

**1 a**  $P_n$  is:  $7^n + 2$  is divisible by 3 for all  $n \in \mathbb{Z}^+$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ ,  $7^1 + 2 = 9$  which is divisible by 3  $\therefore P_1$  is true.

(2) If  $P_k$  is true, then  $7^k + 2 = 3A$  where  $A \in \mathbb{Z}$ .

$$\begin{aligned}
\text{Now } 7^{k+1} + 2 &= 7 \times 7^k + 2 \\
&= 7(3A - 2) + 2 \quad \{\text{using } P_k\} \\
&= 21A - 14 + 2 \\
&= 21A - 12 \\
&= 3(7A - 4) \quad \text{where } (7A - 4) \in \mathbb{Z} \text{ as } A \in \mathbb{Z}.
\end{aligned}$$

Thus  $7^{k+1} + 2$  is divisible by 3 whenever  $7^k + 2$  is divisible by 3.

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

**b**  $P_n$  is:  $\sum_{i=1}^n i(i+2) = \frac{n(n+1)(2n+7)}{6}$  for all  $n \in \mathbb{Z}^+$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ , LHS =  $1 \times 3 = 3$  and RHS =  $\frac{1(2)(9)}{6} = \frac{18}{6} = 3 \quad \therefore P_1$  is true.

(2) If  $P_k$  is true, then  $\sum_{i=1}^k i(i+2) = \frac{k(k+1)(2k+7)}{6}$

$$\begin{aligned} \text{Now } \sum_{i=1}^{k+1} i(i+2) &= \sum_{i=1}^k i(i+2) + (k+1)(k+3) \\ &= \frac{k(k+1)(2k+7)}{6} + (k+1)(k+3) \quad \{\text{using } P_k\} \\ &= \frac{k(k+1)(2k+7)}{6} + \frac{6(k+1)(k+3)}{6} \\ &= \frac{(k+1)[k(2k+7) + 6(k+3)]}{6} \\ &= \frac{(k+1)[2k^2 + 13k + 18]}{6} \\ &= \frac{(k+1)(k+2)(2k+9)}{6} \\ &= \frac{(k+1)([k+1] + 1)(2[k+1] + 7)}{6} \end{aligned}$$

$\therefore P_{k+1}$  is also true.

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,  
 $P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

**c**  $P_n$  is:  $1 + r + r^2 + r^3 + \dots + r^{n-1} = \frac{1-r^n}{1-r}$  for all  $n \in \mathbb{Z}^+$ ,  $r \neq 1$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ , LHS = 1 and RHS =  $\frac{1-r}{1-r} = 1$  as  $r \neq 1 \quad \therefore P_1$  is true.

(2) If  $P_k$  is true, then  $1 + r + r^2 + r^3 + \dots + r^{k-1} = \frac{1-r^k}{1-r}$ .

Now  $1 + r + r^2 + r^3 + \dots + r^{k-1} + r^k$

$$\begin{aligned} &= \frac{1-r^k}{1-r} + r^k \quad \{\text{using } P_k\} \\ &= \frac{1-r^k}{1-r} + \frac{r^k(1-r)}{1-r} \\ &= \frac{1-r^k + r^k - r^{k+1}}{1-r} \\ &= \frac{1-r^{k+1}}{1-r} \end{aligned}$$

$\therefore P_{k+1}$  is also true.

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,  
 $P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}



**d**  $P_n$  is:  $5^{2n} - 1$  is divisible by 24 for all  $n \in \mathbb{Z}^+$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ ,  $5^2 - 1 = 25 - 1 = 24$  is divisible by 24  $\therefore P_1$  is true.

(2) If  $P_k$  is true, then  $5^{2k} - 1 = 24A$  where  $A \in \mathbb{Z}$ .

$$\begin{aligned}\text{Now } 5^{2(k+1)} - 1 &= 5^{2k}5^2 - 1 \\ &= 25(24A + 1) - 1 \quad \{\text{using } P_k\} \\ &= 25 \times 24A + 25 - 1 \\ &= 25 \times 24A + 24 \\ &= 24(25A + 1) \quad \text{where } (25A + 1) \in \mathbb{Z} \text{ as } A \in \mathbb{Z}.\end{aligned}$$

Thus  $5^{2(k+1)} - 1$  is divisible by 24 whenever  $5^{2k} - 1$  is divisible by 24.

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

**e**  $P_n$  is:  $5^n \geq 1 + 4n$  for all  $n \in \mathbb{Z}^+$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ , LHS =  $5^1 = 5$  and RHS =  $1 + 4(1) = 5$   $\therefore P_1$  is true.

(2) If  $P_k$  is true, then  $5^k \geq 1 + 4k$

$$\begin{aligned}\text{Now } 5^{k+1} &= 5 \times 5^k \geq 5 \times (1 + 4k) \quad \{\text{using } P_k\} \\ &\geq 5 + 20k \\ &\geq 5 + 4k \quad \{k \geq 0\} \\ &\geq 1 + 4(k + 1)\end{aligned}$$

$\therefore P_{k+1}$  is also true.

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

**f**  $P_n$  is:  $3^{3n+1} + 9 \times 2^{n+3}$  is divisible by 25 for all  $n \in \mathbb{Z}^+$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ ,  $3^{3+1} + 9 \times 2^{1+3} = 225 = 9 \times 25$  which is divisible by 25  $\therefore P_1$  is true.

(2) If  $P_k$  is true, then  $3^{3k+1} + 9 \times 2^{k+3} = 25A$  where  $A \in \mathbb{Z}$ .

$$\begin{aligned}\text{Now } 3^{3(k+1)+1} + 9 \times 2^{(k+1)+3} &= 3^3 \times 3^{3k+1} + 9 \times 2^1 \times 2^{k+3} \\ &= 27(3^{3k+1} + 9 \times 2^{k+3}) - 25 \times 9 \times 2^{k+3} \\ &= 27 \times 25A - 25 \times 9 \times 2^{k+3} \quad \{\text{using } P_k\} \\ &= 25(27A - 9 \times 2^{k+3}) \quad \text{where } (27A - 9 \times 2^{k+3}) \in \mathbb{Z} \text{ as } A \in \mathbb{Z} \text{ and } k \in \mathbb{Z}^+.\end{aligned}$$

Thus  $3^{3(k+1)+1} + 9 \times 2^{(k+1)+3}$  is divisible by 25 whenever  $3^{3k+1} + 9 \times 2^{k+3}$  is divisible by 25.

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}



- 2**  $P_n$  is: if  $u_1 = 1$  and  $u_{n+1} = 3u_n + 2^n$ , then  $u_n = 3^n - 2^n$  for all  $n \in \mathbb{Z}^+$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ ,  $u_1 = 3^1 - 2^1 = 1 \quad \therefore P_1$  is true.

(2) If  $P_k$  is true, then  $u_k = 3^k - 2^k$ .

$$\begin{aligned} \text{Now } u_{k+1} &= 3u_k + 2^k \\ &= 3(3^k - 2^k) + 2^k \quad \{\text{using } P_k\} \\ &= 3^{k+1} - 3 \times 2^k + 2^k \\ &= 3^{k+1} - 2 \times 2^k \\ &= 3^{k+1} - 2^{k+1} \end{aligned}$$

$\therefore P_{k+1}$  is also true.

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,  
 $P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

- 3 a**  $P_n$  is:  $7^n - 1$  is divisible by 6 for all  $n \in \mathbb{Z}^+$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ ,  $7^1 - 1 = 6$  which is divisible by 6  $\therefore P_1$  is true.

(2) If  $P_k$  is true, then  $7^k - 1 = 6A$  where  $A \in \mathbb{Z}$ .

$$\begin{aligned} \text{Now } 7^{k+1} - 1 &= 7 \times 7^k - 1 \\ &= 7(6A + 1) - 1 \quad \{\text{using } P_k\} \\ &= 42A + 7 - 1 \\ &= 42A + 6 \\ &= 6(7A + 1) \quad \text{where } (7A + 1) \in \mathbb{Z} \text{ as } A \in \mathbb{Z} \end{aligned}$$

Thus  $7^{k+1} - 1$  is divisible by 6 whenever  $7^k - 1$  is divisible by 6.

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,  
 $P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

- b**  $P_n$  is:  $\sum_{i=1}^n (2i - 1)^3 = n^2(2n^2 - 1)$  for all  $n \in \mathbb{Z}^+$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ , LHS =  $1^3 = 1$  and RHS =  $1^2(2 - 1) = 1 \quad \therefore P_1$  is true.

(2) If  $P_k$  is true, then  $1^3 + 3^3 + 5^3 + \dots + (2k - 1)^3 = k^2(2k^2 - 1)$ .

$$\begin{aligned} \text{Now } 1^3 + 3^3 + 5^3 + \dots + (2k - 1)^3 + (2k + 1)^3 & \\ &= k^2(2k^2 - 1) + (2k + 1)^3 \quad \{\text{using } P_k\} \\ &= 2k^4 - k^2 + (2k)^3 + 3(2k)^2 \cdot 1 + 3(2k) \cdot 1^2 + 1^3 \\ &= 2k^4 - k^2 + 8k^3 + 12k^2 + 6k + 1 \\ &= 2k^4 + 8k^3 + 11k^2 + 6k + 1 \\ &= (k + 1)^2(2k^2 + 4k + 1) \quad \{\text{we assume } (k + 1)^2 \text{ is a factor of } \\ &\quad 2k^4 + 8k^3 + 11k^2 + 6k + 1 \text{ and} \\ &\quad \text{factorise the quartic}\} \\ &= (k + 1)^2(2[k^2 + 2k + 1] - 1) \\ &= (k + 1)^2(2[k + 1]^2 - 1) \end{aligned}$$

$\therefore P_{k+1}$  is also true.

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,  
 $P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

•  $P_n$  is:  $\sum_{i=1}^n \frac{1}{(2i-1)(2i+1)} = \frac{n}{2n+1}$  for all  $n \in \mathbb{Z}^+$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ , LHS =  $\frac{1}{1 \times 3} = \frac{1}{3}$ , RHS =  $\frac{1}{2+1} = \frac{1}{3}$   $\therefore P_1$  is true.

(2) If  $P_k$  is true, then  $\frac{1}{1 \times 3} + \frac{1}{3 \times 5} + \frac{1}{5 \times 7} + \dots + \frac{1}{(2k-1)(2k+1)} = \frac{k}{2k+1}$ .

$$\begin{aligned} \text{Now } & \frac{1}{1 \times 3} + \frac{1}{3 \times 5} + \frac{1}{5 \times 7} + \dots + \frac{1}{(2k-1)(2k+1)} + \frac{1}{(2k+1)(2k+3)} \\ &= \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)} \quad \{\text{using } P_k\} \\ &= \frac{k(2k+3) + 1}{(2k+1)(2k+3)} \\ &= \frac{2k^2 + 3k + 1}{(2k+1)(2k+3)} \\ &= \frac{(k+1)(2k+1)}{(2k+1)(2k+3)} \\ &= \frac{(k+1)}{2(k+1)+1} \end{aligned}$$

$\therefore P_{k+1}$  is also true.

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

•  $\sqrt[n]{n!} \leq \frac{n+1}{2} \Leftrightarrow n! \leq \left(\frac{n+1}{2}\right)^n$

$\therefore P_n$  is:  $n! \leq \left(\frac{n+1}{2}\right)^n$  for all  $n \in \mathbb{Z}^+$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ , LHS =  $1! = 1$  and RHS =  $\left(\frac{1+1}{2}\right)^1 = 1$   $\therefore P_1$  is true.

(2) If  $P_k$  is true, then  $k! \leq \left(\frac{k+1}{2}\right)^k$ .

$$\begin{aligned} \text{Now } (k+1)! &= (k+1)k! \\ &\leq (k+1) \left(\frac{k+1}{2}\right)^k \quad \{\text{using } P_k\} \\ &\leq \frac{(k+1)^{k+1}}{2^k} \quad \dots (1) \end{aligned}$$

$$\begin{aligned} \text{Also, } (k+1)^k &= k^k + \binom{k}{1} k^{k-1} + \dots \\ &= k^k + k^k + \dots \\ &= 2k^k + \dots \end{aligned}$$

$$\therefore (k+1)^k \geq 2k^k \quad \text{for } k \in \mathbb{Z}^+$$

$$\therefore \left(\frac{k+1}{k}\right)^k \geq 2$$

$$\therefore \frac{1}{2} \left(\frac{k+1}{k}\right)^k \geq 1$$

$$\therefore \frac{1}{2} \left(\frac{k+2}{k+1}\right)^{k+1} \geq 1 \quad \dots (2) \quad \{\text{replacing } k \text{ with } k+1\}$$

$$\begin{aligned}
\text{Using (1) and (2), } (k+1)! &\leq \frac{(k+1)^{k+1}}{2^k} \times \frac{1}{2} \left( \frac{k+2}{k+1} \right)^{k+1} \\
&\leq \frac{(k+1)^{k+1}}{2^{k+1}} \times \frac{(k+2)^{k+1}}{(k+1)^{k+1}} \\
&\leq \frac{(k+2)^{k+1}}{2^{k+1}} \\
&\leq \left( \frac{[k+1] + 1}{2} \right)^{k+1}
\end{aligned}$$

$\therefore P_{k+1}$  is also true.

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,  
 $P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

**e**  $P_n$  is:  $3^{2n-1} + 2^{n+1}$  is divisible by 7 for all  $n \in \mathbb{Z}^+$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ ,  $3^{2-1} + 2^{1+1} = 7$  which is divisible by 7  $\therefore P_1$  is true.

(2) If  $P_k$  is true, then  $3^{2k-1} + 2^{k+1} = 7A$  where  $A \in \mathbb{Z}$ .

$$\begin{aligned}
\text{Now } 3^{2(k+1)-1} + 2^{(k+1)+1} &= 3^2 \times 3^{2k-1} + 2^1 \times 2^{k+1} \\
&= 9(3^{2k-1} + 2^{k+1}) - 7 \times 2^{k+1} \\
&= 9 \times 7A - 7 \times 2^{k+1} \quad \{\text{using } P_k\} \\
&= 7(9A - 2^{k+1}) \quad \text{where } (9A - 2^{k+1}) \in \mathbb{Z} \text{ since } A \in \mathbb{Z} \text{ and } k \in \mathbb{Z}^+.
\end{aligned}$$

Thus  $3^{2(k+1)-1} + 2^{(k+1)+1}$  is divisible by 7 whenever  $3^{2k-1} + 2^{k+1}$  is divisible by 7.

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,  
 $P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

**f**  $P_n$  is:  $\sum_{i=1}^n i(i+1)(i+2) = \frac{n(n+1)(n+2)(n+3)}{4}$  for all  $n \in \mathbb{Z}^+$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ , LHS =  $1 \times 2 \times 3 = 6$  and RHS =  $\frac{1(2)(3)(4)}{4} = 6$   $\therefore P_1$  is true.

(2) If  $P_k$  is true, then  $\sum_{i=1}^k i(i+1)(i+2) = \frac{k(k+1)(k+2)(k+3)}{4}$

$$\begin{aligned}
\therefore \sum_{i=1}^{k+1} i(i+1)(i+2) &= \sum_{i=1}^k i(i+1)(i+2) + (k+1)(k+2)(k+3) \\
&= \frac{k(k+1)(k+2)(k+3)}{4} + (k+1)(k+2)(k+3) \quad \{\text{using } P_k\} \\
&= \frac{k(k+1)(k+2)(k+3)}{4} + \frac{4(k+1)(k+2)(k+3)}{4} \\
&= \frac{(k+1)(k+2)(k+3)(k+4)}{4}
\end{aligned}$$

$\therefore P_{k+1}$  is also true.

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,  
 $P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

**4**  $P_n$  is:  $3^n - 1 - 2n$  is divisible by 4 for all non-negative integers  $n$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 0$ ,  $3^0 - 1 - 2(0) = 1 - 1 - 0 = 0$  which is divisible by 4  $\therefore P_0$  is true.



(2) If  $P_k$  is true, then  $3^k - 1 - 2k = 4A$  where  $A \in \mathbb{Z}$ .

$$\begin{aligned}
 \text{Now } 3^{k+1} - 1 - 2(k+1) &= 3 \times 3^k - 1 - 2k - 2 \\
 &= 3(4A + 1 + 2k) - 2k - 3 \quad \{\text{using } P_k\} \\
 &= 12A + 3 + 6k - 2k - 3 \\
 &= 12A + 4k \\
 &= 4(3A + k) \quad \text{where } (3A + k) \in \mathbb{Z} \text{ as } A \in \mathbb{Z} \\
 &\quad \text{and } k \in \mathbb{Z}, k \geq 0.
 \end{aligned}$$

Thus  $3^{k+1} - 1 - 2(k+1)$  is divisible by 4 whenever  $3^k - 1 - 2k$  is divisible by 4.

Since  $P_0$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all non-negative integers  $n$ . {principle of mathematical induction}

**5**  $P_n$  is:  $\sin^2 \theta + \sin^2 2\theta + \sin^2 3\theta + \dots + \sin^2(n\theta) = \frac{1}{2} \left[ n - \frac{\cos[(n+1)\theta] \sin n\theta}{\sin \theta} \right]$  for all  $n \in \mathbb{Z}^+$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ , LHS =  $\sin^2 \theta$

$$\text{and RHS} = \frac{1}{2} \left[ 1 - \frac{\cos 2\theta \sin \theta}{\sin \theta} \right] = \frac{1}{2} - \frac{1}{2}(1 - 2\sin^2 \theta) = \sin^2 \theta \quad \therefore P_1 \text{ is true.}$$

(2) If  $P_k$  is true then  $\sin^2 \theta + \sin^2 2\theta + \sin^2 3\theta + \dots + \sin^2(k\theta) = \frac{1}{2} \left[ k - \frac{\cos(k+1)\theta \sin k\theta}{\sin \theta} \right]$

Now  $\sin^2 \theta + \sin^2 2\theta + \sin^2 3\theta + \dots + \sin^2(k\theta) + \sin^2(k+1)\theta$

$$\begin{aligned}
 &= \frac{1}{2} \left[ k - \frac{\cos(k+1)\theta \sin k\theta}{\sin \theta} \right] + \sin^2(k+1)\theta \quad \{\text{using } P_k\} \\
 &= \frac{1}{2} \left[ k - \frac{\cos(k+1)\theta \sin k\theta}{\sin \theta} \right] + \frac{1}{2} - \frac{1}{2} \cos 2(k+1)\theta \quad \{\sin^2 \theta = \frac{1}{2} - \frac{1}{2} \cos 2\theta\} \\
 &= \frac{1}{2}(k+1) - \frac{\cos(k+1)\theta \sin k\theta + \cos 2(k+1)\theta \sin \theta}{2 \sin \theta} \\
 &= \frac{1}{2}(k+1) - \frac{\frac{1}{2} \sin[k\theta + (k+1)\theta] + \frac{1}{2} \sin[k\theta - (k+1)\theta] + \frac{1}{2} \sin[\theta + 2(k+1)\theta] + \frac{1}{2} \sin[\theta - 2(k+1)\theta]}{2 \sin \theta} \\
 &\quad \{\text{products to sums formula}\} \\
 &= \frac{1}{2}(k+1) - \frac{\frac{1}{2} \sin(2k+1)\theta + \frac{1}{2} \sin(-\theta) + \frac{1}{2} \sin(2k+3)\theta + \frac{1}{2} \sin(-2k-1)\theta}{2 \sin \theta} \\
 &= \frac{1}{2}(k+1) - \frac{\frac{1}{2} \sin(2k+1)\theta - \frac{1}{2} \sin \theta + \frac{1}{2} \sin(2k+3)\theta - \frac{1}{2} \sin(2k+1)\theta}{2 \sin \theta} \\
 &= \frac{1}{2}(k+1) - \frac{\frac{1}{2} \sin(2k+3)\theta - \frac{1}{2} \sin \theta}{2 \sin \theta} \\
 &= \frac{1}{2}(k+1) - \frac{\cos \left[ \frac{(2k+3)\theta + \theta}{2} \right] \sin \left[ \frac{(2k+3)\theta - \theta}{2} \right]}{2 \sin \theta} \quad \{\text{factor formula}\} \\
 &= \frac{1}{2}(k+1) - \frac{\cos(k+2)\theta \sin(k+1)\theta}{2 \sin \theta} \\
 &= \frac{1}{2} \left[ (k+1) - \frac{\cos(k+2)\theta \sin(k+1)\theta}{\sin \theta} \right]
 \end{aligned}$$

$\therefore P_{k+1}$  is also true.

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}



**6**  $P_n$  is:  $5^{2n} + 17 \times 7^n$  is divisible by 18 for all non-negative integers  $n$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 0$ ,  $5^0 + 17 \times 7^0 = 18$  which is divisible by 18  $\therefore P_0$  is true.

(2) If  $P_k$  is true then  $5^{2k} + 17 \times 7^k = 18A$  where  $A \in \mathbb{Z}$ .

$$\begin{aligned} \text{Now } 5^{2(k+1)} + 17 \times 7^{k+1} &= 5^2 \times 5^{2k} + 17 \times 7^1 \times 7^k \\ &= 25(5^{2k} + 17 \times 7^k) - 18 \times 17 \times 7^k \\ &= 25 \times 18A - 18 \times 17 \times 7^k \quad \{\text{using } P_k\} \\ &= 18(25A - 17 \times 7^k) \quad \text{where } (25A - 17 \times 7^k) \in \mathbb{Z} \text{ as } A \in \mathbb{Z} \text{ and } k \in \mathbb{Z}, k \geq 0. \end{aligned}$$

Thus  $5^{2(k+1)} + 17 \times 7^{k+1}$  is divisible by 18 whenever  $5^{2k} + 17 \times 7^k$  is divisible by 18.

Since  $P_0$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all non-negative integers  $n$ . {principle of mathematical induction}

<b>7 a</b>	$u_2 = 4u_1 - 3u_0$	$u_3 = 4u_2 - 3u_1$	$u_4 = 4u_3 - 3u_2$
	$= 4(3) - 3(1)$	$= 4(9) - 3(3)$	$= 4(27) - 3(9)$
	$= 12 - 3$	$= 36 - 9$	$= 108 - 27$
	$= 9$	$= 27$	$= 81$
	$= 3^2$	$= 3^3$	$= 3^4$

Our conjecture is  $u_n = 3^n$  for all  $n \in \mathbb{Z}$ ,  $n \geq 0$ .

**b**  $P_n$  is: if  $u_0 = 1$ ,  $u_1 = 3$ , and  $u_{n+2} = 4u_{n+1} - 3u_n$  for all  $n \in \mathbb{Z}$ ,  $n \geq 0$ , then  $u_n = 3^n$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 0$ ,  $u_0 = 1 = 3^0 \therefore P_0$  is true.

If  $n = 1$ ,  $u_1 = 3 = 3^1 \therefore P_1$  is true.

(2) If  $P_k$  is true, then  $u_k = 3^k$ .

If  $P_{k+1}$  is true, then  $u_{k+1} = 3^{k+1}$ .

$$\begin{aligned} \text{Now } u_{k+2} &= 4u_{k+1} - 3u_k \\ &= 4 \times 3^{k+1} - 3 \times 3^k \quad \{\text{using } P_k \text{ and } P_{k+1}\} \\ &= 4 \times 3^{k+1} - 3^{k+1} \\ &= 3 \times 3^{k+1} \\ &= 3^{k+2} \end{aligned}$$

$\therefore P_{k+2}$  is also true.

Since  $P_1$  is true, and  $P_{k+2}$  is true whenever  $P_k$  and  $P_{k+1}$  are true,

$P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

**8**  $P_n$  is:  $\prod_{i=1}^n \cos(2^{i-1}x) = \cos x \times \cos 2x \times \cos 4x \times \cos 8x \times \dots \times \cos(2^{n-1}x) = \frac{\sin(2^n x)}{2^n \times \sin x}$   
for all  $n \in \mathbb{Z}^+$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ , LHS =  $\cos x$ , RHS =  $\frac{\sin 2x}{2 \sin x} = \frac{2 \sin x \cos x}{2 \sin x} = \cos x \therefore P_1$  is true.

(2) If  $P_k$  is true, then  $\cos x \times \cos 2x \times \cos 4x \times \dots \times \cos(2^{k-1}x) = \frac{\sin(2^k x)}{2^k \sin x}$ .

$$\begin{aligned}
\text{Now } & \cos x \times \cos 2x \times \cos 4x \times \dots \times \cos(2^{k-1}x) \times \cos(2^k x) \\
&= \frac{\sin(2^k x)}{2^k \sin x} \times \cos(2^k x) \quad \{\text{using } P_k\} \\
&= \frac{2 \sin(2^k x) \cos(2^k x)}{2 \times 2^k \sin x} \\
&= \frac{\sin(2 \times 2^k x)}{2^{k+1} \sin x} \quad \{2 \sin \theta \cos \theta = \sin 2\theta\} \\
&= \frac{\sin(2^{k+1} x)}{2^{k+1} \sin x}
\end{aligned}$$

$\therefore P_{k+1}$  is also true.

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

## REVIEW SET 10B

**1 a**  $P_n$  is:  $\sum_{i=1}^n (2i-1)^2 = \frac{n(2n+1)(2n-1)}{3}$  for all  $n \in \mathbb{Z}^+$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ , LHS =  $1^2 = 1$  and RHS =  $\frac{1(3)(1)}{3} = 1 \quad \therefore P_1$  is true.

(2) If  $P_k$  is true, then  $\sum_{i=1}^k (2i-1)^2 = \frac{k(2k+1)(2k-1)}{3}$

$$\begin{aligned}
\text{Now } \sum_{i=1}^{k+1} (2i-1)^2 &= \left( \sum_{i=1}^k (2i-1)^2 \right) + (2k+1)^2 \\
&= \frac{k(2k+1)(2k-1)}{3} + (2k+1)^2 \quad \{\text{using } P_k\} \\
&= \frac{k(2k+1)(2k-1)}{3} + \frac{3(2k+1)^2}{3} \\
&= \frac{(2k+1)[k(2k-1) + 3(2k+1)]}{3} \\
&= \frac{(2k+1)[2k^2 - k + 6k + 3]}{3} \\
&= \frac{(2k+1)(2k^2 + 5k + 3)}{3} \\
&= \frac{(2k+1)(k+1)(2k+3)}{3} \\
&= \frac{(k+1)(2[k+1] + 1)(2[k+1] - 1)}{3}
\end{aligned}$$

$\therefore P_{k+1}$  is also true.

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

**b**  $P_n$  is:  $3^{2n+2} - 8n - 9$  is divisible by 64 for all  $n \in \mathbb{Z}^+$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ ,  $3^4 - 8 - 9 = 81 - 17 = 64$  which is divisible by 64  $\therefore P_1$  is true.

(2) If  $P_k$  is true, then  $3^{2k+2} - 8k - 9 = 64A$  where  $A \in \mathbb{Z}$ .

$$\begin{aligned} \text{Now } 3^{2(k+1)+2} - 8(k+1) - 9 &= 3^{2k+2} \times 3^2 - 8k - 8 - 9 \\ &= 9[64A + 8k + 9] - 8k - 17 \quad \{\text{using } P_k\} \\ &= 9 \times 64A + 72k + 81 - 8k - 17 \\ &= 9 \times 64A + 64k + 64 \\ &= 64(9A + k + 1) \quad \text{where } (9A + k + 1) \in \mathbb{Z} \text{ as } A \in \mathbb{Z} \text{ and } k \in \mathbb{Z}^+. \end{aligned}$$

Thus  $3^{2(k+1)+2} - 8(k+1) - 9$  is divisible by 64 whenever  $3^{2k+2} - 8k - 9$  is divisible by 64.

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

**c**  $P_n$  is:  $\sum_{i=1}^n (2i+1)2^{i-1} = 1 + (2n-1) \times 2^n$  for all  $n \in \mathbb{Z}^+$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ , LHS = 3 and RHS =  $1 + 1 \times 2^1 = 1 + 2 = 3$   
 $\therefore P_1$  is true.

(2) If  $P_k$  is true, then  $\sum_{i=1}^k (2i+1)2^{i-1} = 1 + (2k-1) \times 2^k$

$$\begin{aligned} \text{Now } \sum_{i=1}^{k+1} (2i+1)2^{i-1} &= \sum_{i=1}^k (2i+1)2^{i-1} + (2k+3)2^k \\ &= 1 + (2k-1)2^k + (2k+3)2^k \quad \{\text{using } P_k\} \\ &= 1 + 2^k(2k-1+2k+3) \\ &= 1 + 2^k(4k+2) \\ &= 1 + 2^k(2)(2k+1) \\ &= 1 + (2k+1)2^{k+1} \\ &= 1 + (2[k+1]-1)2^{k+1} \end{aligned}$$

$\therefore P_{k+1}$  is also true.

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

**d**  $P_n$  is:  $\sum_{i=1}^n i(i+1)^2 = \frac{n(n+1)(n+2)(3n+5)}{12}$  for all  $n \in \mathbb{Z}^+$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ , LHS =  $1 \times 2^2 = 4$  and RHS =  $\frac{1(2)(3)(8)}{12} = \frac{48}{12} = 4$   
 $\therefore P_1$  is true.

(2) If  $P_k$  is true, then  $\sum_{i=1}^k i(i+1)^2 = \frac{k(k+1)(k+2)(3k+5)}{12}$



$$\begin{aligned}
\text{Now } \sum_{i=1}^{k+1} i(i+1)^2 &= \sum_{i=1}^k i(i+1)^2 + (k+1)(k+2)^2 \\
&= \frac{k(k+1)(k+2)(3k+5)}{12} + (k+1)(k+2)^2 \quad \{\text{using } P_k\} \\
&= \frac{k(k+1)(k+2)(3k+5)}{12} + \frac{12(k+1)(k+2)^2}{12} \\
&= \frac{(k+1)(k+2)[k(3k+5) + 12(k+2)]}{12} \\
&= \frac{(k+1)(k+2)[3k^2 + 5k + 12k + 24]}{12} \\
&= \frac{(k+1)(k+2)(3k^2 + 17k + 24)}{12} \\
&= \frac{(k+1)(k+2)(k+3)(3k+8)}{12} \\
&= \frac{(k+1)([k+1]+1)([k+1]+2)(3[k+1]+5)}{12}
\end{aligned}$$

$\therefore P_{k+1}$  is also true.

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

**e**  $P_n$  is:  $5^n + 3^n \geq 2^{2n+1}$  for all  $n \in \mathbb{Z}^+$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ , LHS =  $5^1 + 3^1 = 8$  and RHS =  $2^{2+1} = 8 \quad \therefore P_1$  is true.

(2) If  $P_k$  is true, then  $5^k + 3^k \geq 2^{2k+1}$ , so  $5^k \geq 2^{2k+1} - 3^k$

$$\text{Now } 5^{k+1} + 3^{k+1} = 5 \times 5^k + 3 \times 3^k$$

$$\therefore 5^{k+1} + 3^{k+1} \geq 5(2^{2k+1} - 3^k) + 3 \times 3^k \quad \{\text{using } P_k\}$$

$$\therefore 5^{k+1} + 3^{k+1} \geq 5 \times 2^{2k+1} - 5 \times 3^k + 3 \times 3^k$$

$$\therefore 5^{k+1} + 3^{k+1} \geq 5 \times 2^{2k+1} - 2 \times 3^k$$

$$\therefore 5^{k+1} + 3^{k+1} \geq 5 \times 2^{2k+1} - 2 \times 4^k \quad \{4^k \geq 3^k \text{ as } k \in \mathbb{Z}^+\}$$

$$\therefore 5^{k+1} + 3^{k+1} \geq 5 \times 2^{2k+1} - 2^{2k+1}$$

$$\therefore 5^{k+1} + 3^{k+1} \geq 4 \times 2^{2k+1}$$

$$\therefore 5^{k+1} + 3^{k+1} \geq 2^{2k+3}$$

$$\therefore 5^{k+1} + 3^{k+1} \geq 2^{2(k+1)+1}$$

$\therefore P_{k+1}$  is also true.

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

**f**  $P_n$  is:  $5^{2n+1} + 13 \times 3^{n+1}$  is divisible by 22 for all  $n \in \mathbb{Z}^+$ .

**Proof:** (By the principle of mathematical induction)

$$\begin{aligned}
(1) \text{ If } n = 1, \text{ we have } 5^{2+1} + 13 \times 3^{1+1} &= 125 + 13 \times 9 \\
&= 242 \\
&= 11 \times 22
\end{aligned}$$

$\therefore P_1$  is true.



(2) If  $P_k$  is true, then  $5^{2k+1} + 13 \times 3^{k+1} = 22A$  where  $A \in \mathbb{Z}$ .

$$\begin{aligned}
 \text{Now } 5^{2(k+1)+1} + 13 \times 3^{(k+1)+1} &= 5^2 \times 5^{2k+1} + 3 \times 13 \times 3^{k+1} \\
 &= 25(5^{2k+1} + 13 \times 3^{k+1}) - 22 \times 13 \times 3^{k+1} \\
 &= 25(22A) - 22 \times 13 \times 3^{k+1} \\
 &= 22(25A - 13 \times 3^{k+1}) \quad \text{where } (25A - 13 \times 3^{k+1}) \in \mathbb{Z} \text{ as} \\
 &\quad A \in \mathbb{Z} \text{ and } k \in \mathbb{Z}^+.
 \end{aligned}$$

So  $5^{2(k+1)+1} + 13 \times 3^{(k+1)+1}$  is divisible by 22.

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

**2**  $P_n$  is:  $5^n + 3$  is divisible by 4 for all integers  $n \geq 0$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 0$ ,  $5^0 + 3 = 4$  which is divisible by 4

$\therefore P_0$  is true.

(2) If  $P_k$  is true, then  $5^k + 3 = 4A$  where  $A \in \mathbb{Z}$ .

$$\begin{aligned}
 \text{Now } 5^{k+1} + 3 &= 5 \times 5^k + 3 \\
 &= 5(4A - 3) + 3 \quad \{\text{using } P_k\} \\
 &= 20A - 15 + 3 \\
 &= 20A - 12 \\
 &= 4(5A - 3) \quad \text{where } (5A - 3) \in \mathbb{Z}, \text{ as } A \in \mathbb{Z}.
 \end{aligned}$$

So,  $5^{k+1} + 3$  is divisible by 4

Since  $P_0$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all integers  $n \geq 0$ . {principle of mathematical induction}

**3**  $P_n$  is: if  $u_1 = 9$  and  $u_{n+1} = 2u_n + 3(5^n)$  then  $u_n = 2^{n+1} + 5^n$  for all  $n \in \mathbb{Z}^+$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ ,  $u_1 = 2^2 + 5^1 = 9 \quad \therefore P_1$  is true.

(2) If  $P_k$  is true, then  $u_k = 2^{k+1} + 5^k$ .

$$\begin{aligned}
 \text{Now } u_{k+1} &= 2u_k + 3(5^k) \\
 &= 2(2^{k+1} + 5^k) + 3(5^k) \quad \{\text{using } P_k\} \\
 &= 2^{k+2} + 2(5^k) + 3(5^k) \\
 &= 2^{k+2} + 5(5^k) \\
 &= 2^{k+2} + 5^{k+1}
 \end{aligned}$$

$\therefore P_{k+1}$  is also true.

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

**4 a**  $P_n$  is:  $9^n - 3^n$  is divisible by 6 for all  $n \in \mathbb{Z}^+$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ ,  $9^1 - 3^1 = 6$  which is divisible by 6  $\therefore P_1$  is true.

(2) If  $P_k$  is true, then  $9^k - 3^k = 6A$  where  $A \in \mathbb{Z}$ .

$$\begin{aligned} \text{Now } 9^{k+1} - 3^{k+1} &= 9^1 \times 9^k - 3^1 \times 3^k \\ &= 9(9^k - 3^k) + 6 \times 3^k \\ &= 9 \times 6A + 6 \times 3^k \quad \{\text{using } P_k\} \\ &= 6(9A + 3^k) \quad \text{where } (9A + 3^k) \in \mathbb{Z} \text{ as } A \in \mathbb{Z} \text{ and } k \in \mathbb{Z}^+. \end{aligned}$$

Thus  $9^{k+1} - 3^{k+1}$  is divisible by 6 whenever  $9^k - 3^k$  is divisible by 6.

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

**b**  $P_n$  is:  $\frac{1^2}{1 \times 3} + \frac{2^2}{3 \times 5} + \frac{3^2}{5 \times 7} + \dots + \frac{n^2}{(2n-1)(2n+1)} = \frac{n(n+1)}{2(2n+1)}$  for all  $n \in \mathbb{Z}^+$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ , LHS =  $\frac{1^2}{1 \times 3} = \frac{1}{3}$  and RHS =  $\frac{1(1+1)}{2(2+1)} = \frac{1}{3}$   $\therefore P_1$  is true.

(2) If  $P_k$  is true, then  $\frac{1^2}{1 \times 3} + \frac{2^2}{3 \times 5} + \frac{3^2}{5 \times 7} + \dots + \frac{k^2}{(2k-1)(2k+1)} = \frac{k(k+1)}{2(2k+1)}$

$$\begin{aligned} \text{Now } &\frac{1^2}{1 \times 3} + \frac{2^2}{3 \times 5} + \frac{3^2}{5 \times 7} + \dots + \frac{k^2}{(2k-1)(2k+1)} + \frac{(k+1)^2}{(2(k+1)-1)(2(k+1)+1)} \\ &= \frac{k(k+1)}{2(2k+1)} + \frac{(k+1)^2}{(2k+1)(2k+3)} \quad \{\text{using } P_k\} \\ &= \frac{k(k+1)(2k+3)}{2(2k+1)(2k+3)} + \frac{2(k+1)^2}{2(2k+1)(2k+3)} \\ &= \frac{(k+1)[k(2k+3) + 2(k+1)]}{2(2k+1)(2k+3)} \\ &= \frac{(k+1)[2k^2 + 3k + 2k + 2]}{2(2k+1)(2k+3)} \\ &= \frac{(k+1)(2k^2 + 5k + 2)}{2(2k+1)(2k+3)} \\ &= \frac{(k+1)(2k+1)(k+2)}{2(2k+1)(2k+3)} \\ &= \frac{(k+1)(k+2)}{2(2k+3)} \end{aligned}$$

$\therefore P_{k+1}$  is also true.

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true

$P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

- $P_n$  is:  $6n^3 + 5n^2 + n$  is even for all integers  $n \geq 0$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 0$ ,  $6(0)^3 + 5(0)^2 + 0 = 0 = 2 \times 0$  which is even  $\therefore P_0$  is true.

(2) If  $P_k$  is true, then  $6k^3 + 5k^2 + k = 2A$  where  $A \in \mathbb{Z}$ .

$$\begin{aligned} \text{Now } 6(k+1)^3 + 5(k+1)^2 + (k+1) &= 6(k^3 + 3k^2 + 3k + 1) + 5(k^2 + 2k + 1) + k + 1 \\ &= 6k^3 + 18k^2 + 18k + 6 + 5k^2 + 10k + 5 + k + 1 \\ &= (6k^3 + 5k^2 + k) + 18k^2 + 28k + 12 \\ &= 2A + 18k^2 + 28k + 12 \quad \{\text{using } P_k\} \\ &= 2(A + 9k^2 + 14k + 6) \quad \text{where } (A + 9k^2 + 14k + 6) \in \mathbb{Z} \text{ as} \\ &\quad A \in \mathbb{Z} \text{ and } k \in \mathbb{Z}, k \geq 0. \end{aligned}$$

Thus  $6(k+1)^3 + 5(k+1)^2 + (k+1)$  is even whenever  $6k^3 + 5k^2 + k$  is even.

Since  $P_0$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all integers  $n \geq 0$ . {principle of mathematical induction}

- $P_n$  is:  $1^2 - 2^2 + 3^2 - 4^2 + \dots + (-1)^{n+1} \times n^2 = \frac{1}{2}(-1)^{n+1}n(n+1)$  for all integers  $n \geq 1$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ , LHS =  $1^2 = 1$  and RHS =  $\frac{1}{2}(-1)^{1+1}(1)(1+1) = 1 \therefore P_1$  is true.

(2) If  $P_k$  is true, then  $1^2 - 2^2 + 3^2 - 4^2 + \dots + (-1)^{k+1} \times k^2 = \frac{1}{2}(-1)^{k+1}k(k+1)$ .

$$\begin{aligned} \text{Now } 1^2 - 2^2 + 3^2 - 4^2 + \dots + (-1)^{k+1} \times k^2 + (-1)^{k+2} \times (k+1)^2 &= \frac{1}{2}(-1)^{k+1}k(k+1) + (-1)^{k+2} \times (k+1)^2 \quad \{\text{using } P_k\} \\ &= (-1)^{k+1}(k+1) \left[ \frac{k}{2} - (k+1) \right] \\ &= (-1)^{k+1}(k+1) \left( -\frac{k}{2} - 1 \right) \\ &= -\frac{1}{2}(-1)^{k+1}(k+1)(k+2) \\ &= \frac{1}{2}(-1)^{k+2}(k+1)([k+1] + 1) \end{aligned}$$

$\therefore P_{k+1}$  is also true.

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all integers  $n \geq 1$ . {principle of mathematical induction}

- $P_n$  is:  $8^{n+2} - 7n - 15$  is divisible by 49 for all  $n \in \mathbb{Z}^+$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ ,  $8^{1+2} - 7(1) - 15 = 490 = 49 \times 10$  which is divisible by 49

$\therefore P_1$  is true.

(2) If  $P_k$  is true, then  $8^{k+2} - 7k - 15 = 49A$  where  $A \in \mathbb{Z}$ .

$$\begin{aligned} \text{Now } 8^{(k+1)+2} - 7(k+1) - 15 &= 8^1 \times 8^{k+2} - 7k - 22 \\ &= 8(8^{k+2} - 7k - 15) + 49k + 98 \\ &= 8 \times 49A + 49k + 98 \quad \{\text{using } P_k\} \\ &= 49(8A + k + 2) \quad \text{where } (8A + k + 2) \in \mathbb{Z} \text{ as } A \in \mathbb{Z} \text{ and } k \in \mathbb{Z}^+. \end{aligned}$$

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}



**f**  $P_n$  is:  $\frac{3}{2^2 \times 1^2} + \frac{5}{3^2 \times 2^2} + \dots + \frac{2n-1}{n^2(n-1)^2} = 1 - \frac{1}{n^2}$  for all integers  $n \geq 2$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 2$ , LHS =  $\frac{3}{2^2 \times 1^2} = \frac{3}{4}$  and RHS =  $1 - \frac{1}{2^2} = \frac{3}{4} \quad \therefore P_2$  is true.

(2) If  $P_k$  is true, then  $\frac{3}{2^2 \times 1^2} + \frac{5}{3^2 \times 2^2} + \dots + \frac{2k-1}{k^2(k-1)^2} = 1 - \frac{1}{k^2}$ .

$$\begin{aligned}
 \text{Now } & \frac{3}{2^2 \times 1^2} + \frac{5}{3^2 \times 2^2} + \dots + \frac{2k-1}{k^2(k-1)^2} + \frac{2(k+1)-1}{(k+1)^2 k^2} \\
 &= 1 - \frac{1}{k^2} + \frac{2(k+1)-1}{(k+1)^2 k^2} \quad \{\text{using } P_k\} \\
 &= \frac{k^2(k+1)^2 - (k+1)^2 + 2(k+1) - 1}{k^2(k+1)^2} \\
 &= \frac{k^4 + 2k^3 + k^2 - k^2 - 2k - 1 + 2k + 1}{k^2(k+1)^2} \\
 &= \frac{k^3(k+2)}{k^2(k+1)^2} \\
 &= \frac{k(k+2)}{(k+1)^2} \\
 &= \frac{(k+1)^2 - 1}{(k+1)^2} \\
 &= 1 - \frac{1}{(k+1)^2}
 \end{aligned}$$

$\therefore P_{k+1}$  is also true.

Since  $P_2$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all integers  $n \geq 2$ . {principle of mathematical induction}

**g**  $P_n$  is:  $\prod_{i=1}^n \left(1 + \frac{1}{i}\right) = (1+1) \left(1 + \frac{1}{2}\right) \left(1 + \frac{1}{3}\right) \left(1 + \frac{1}{4}\right) \dots \left(1 + \frac{1}{n}\right) = n+1$   
for all  $n \in \mathbb{Z}^+$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ , LHS =  $1 + 1 = 2$  and RHS =  $1 + 1 = 2 \quad \therefore P_1$  is true.

(2) If  $P_k$  is true, then  $(1+1) \left(1 + \frac{1}{2}\right) \left(1 + \frac{1}{3}\right) \left(1 + \frac{1}{4}\right) \dots \left(1 + \frac{1}{k}\right) = k+1$ .

$$\begin{aligned}
 \text{Now } & (1+1) \left(1 + \frac{1}{2}\right) \left(1 + \frac{1}{3}\right) \left(1 + \frac{1}{4}\right) \dots \left(1 + \frac{1}{k}\right) \left(1 + \frac{1}{k+1}\right) \\
 &= (k+1) \left(1 + \frac{1}{k+1}\right) \quad \{\text{using } P_k\} \\
 &= k+1 + \frac{k+1}{k+1} \\
 &= k+1 + 1 \\
 &= (k+1) + 1
 \end{aligned}$$

$\therefore P_{k+1}$  is also true.

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}



- 5**  $P_n$  is:  $\frac{1}{\sin 2x} + \frac{1}{\sin 4x} + \frac{1}{\sin 8x} + \dots + \frac{1}{\sin(2^n x)} = \cot x - \cot(2^n x)$  for all  $n \in \mathbb{Z}^+$   
whenever these terms are defined.

**Proof:** (By the principle of mathematical induction)

$$\begin{aligned}
 (1) \quad \text{If } n = 1, \text{ LHS} &= \frac{1}{\sin 2x} \text{ and RHS} = \cot x - \cot(2^1 x) \\
 &= \frac{\cos x}{\sin x} - \frac{\cos 2x}{\sin 2x} \\
 &= \frac{\cos x \sin 2x - \cos 2x \sin x}{\sin x \sin 2x} \\
 &= \frac{\sin x}{\sin x \sin 2x} \\
 &\quad \{\sin A \cos B - \sin B \cos A = \sin(A - B)\} \\
 &= \frac{1}{\sin 2x}
 \end{aligned}$$

$\therefore P_1$  is true.

$$(2) \quad \text{If } P_k \text{ is true, then } \frac{1}{\sin 2x} + \frac{1}{\sin 4x} + \frac{1}{\sin 8x} + \dots + \frac{1}{\sin(2^k x)} = \cot x - \cot(2^k x).$$

$$\begin{aligned}
 \text{Now } &\frac{1}{\sin 2x} + \frac{1}{\sin 4x} + \frac{1}{\sin 8x} + \dots + \frac{1}{\sin(2^k x)} + \frac{1}{\sin(2^{k+1} x)} \\
 &= \cot x - \cot(2^k x) + \frac{1}{\sin(2^{k+1} x)} \quad \{\text{using } P_k\} \\
 &= \cot x - \frac{\cos(2^k x)}{\sin(2^k x)} + \frac{1}{\sin(2^{k+1} x)} \\
 &= \cot x - \left[ \frac{\cos(2^k x) \sin(2^{k+1} x) - \sin(2^k x)}{\sin(2^k x) \sin(2^{k+1} x)} \right] \\
 &= \cot x - \left[ \frac{2 \cos^2(2^k x) \sin(2^k x) - \sin(2^k x)}{\sin(2^k x) \sin(2^{k+1} x)} \right] \quad \{\sin 2\theta = 2 \sin \theta \cos \theta\} \\
 &= \cot x - \left[ \frac{2 \cos^2(2^k x) - 1}{\sin(2^{k+1} x)} \right] \\
 &= \cot x - \frac{\cos(2 \times 2^k x)}{\sin(2^{k+1} x)} \\
 &= \cot x - \cot(2^{k+1} x)
 \end{aligned}$$

$\therefore P_{k+1}$  is also true.

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

- 6  $P_n$  is: if  $u_1 = 5$  and  $u_{n+1} = 2u_n - 3(-1)^n$  then  $u_n = 3(2^n) + (-1)^n$ , for all  $n \in \mathbb{Z}^+$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ ,  $u_1 = 3(2^1) + (-1)^1 = 6 - 1 = 5$  which is true  $\therefore P_1$  is true.

(2) If  $P_k$  is true, then  $u_k = 3(2^k) + (-1)^k$ .

$$\begin{aligned}\text{Now } u_{k+1} &= 2u_k - 3(-1)^k \\ &= 2[3(2^k) + (-1)^k] - 3(-1)^k \quad \{\text{using } P_k\} \\ &= 6(2^k) + 2(-1)^k - 3(-1)^k \\ &= 3(2^{k+1}) - (-1)^k \\ &= 3(2^{k+1}) + (-1)^{k+1}\end{aligned}$$

$\therefore P_{k+1}$  is also true.

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

7 a  $\cos(\alpha - \beta) - \cos(\alpha + \beta)$   
 $= [\cos \alpha \cos \beta + \sin \alpha \sin \beta] - [\cos \alpha \cos \beta - \sin \alpha \sin \beta] \quad \{\text{compound angle formulae}\}$   
 $= \sin \alpha \sin \beta + \sin \alpha \sin \beta$   
 $= 2 \sin \alpha \sin \beta$

b  $2 \sin \alpha \sin \beta = \cos(\alpha - \beta) - \cos(\alpha + \beta)$   
 $\therefore \sin \alpha \sin \beta = \frac{1}{2}(\cos(\alpha - \beta) - \cos(\alpha + \beta))$

c  $\sin[(k+1)\theta] \sin \frac{\theta}{2} + \sin \frac{k\theta}{2} \sin \frac{(k+1)\theta}{2}$   
 $= \frac{1}{2} [\cos((k+1)\theta - \frac{\theta}{2}) - \cos((k+1)\theta + \frac{\theta}{2})]$   
 $+ \frac{1}{2} [\cos(\frac{k\theta}{2} - \frac{(k+1)\theta}{2}) - \cos(\frac{k\theta}{2} + \frac{(k+1)\theta}{2})]$   
 $= \frac{1}{2} [\cos(k\theta + \frac{\theta}{2}) - \cos((k+1)\theta + \frac{\theta}{2}) + \cos(-\frac{\theta}{2}) - \cos(k\theta + \frac{\theta}{2})]$   
 $= \frac{1}{2} [\cos \frac{\theta}{2} - \cos((k+1)\theta + \frac{\theta}{2})]$   
 $= \frac{1}{2} \left[ -2 \sin \left( \frac{\frac{\theta}{2} + (k+1)\theta + \frac{\theta}{2}}{2} \right) \sin \left( \frac{\frac{\theta}{2} - (k+1)\theta - \frac{\theta}{2}}{2} \right) \right] \quad \{\text{factor formulae}\}$   
 $= -\sin \left( \frac{(k+2)\theta}{2} \right) \sin \left( -\frac{(k+1)\theta}{2} \right)$   
 $= \sin \frac{(k+1)\theta}{2} \sin \frac{(k+2)\theta}{2}$

d  $P_n$  is:  $\sin \theta + \sin 2\theta + \sin 3\theta + \dots + \sin(n\theta) = \frac{\sin \left[ \frac{1}{2}(n+1)\theta \right] \sin \left( \frac{1}{2}n\theta \right)}{\sin \left( \frac{1}{2}\theta \right)}$  for all  $n \in \mathbb{Z}^+$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ , LHS =  $\sin \theta$  and  $\text{RHS} = \frac{\sin \left[ \frac{1}{2}(2)\theta \right] \sin \left( \frac{1}{2}\theta \right)}{\sin \left( \frac{1}{2}\theta \right)} = \sin \theta \quad \therefore P_1$  is true.

(2) If  $P_k$  is true, then  $\sin \theta + \sin 2\theta + \sin 3\theta + \dots + \sin(k\theta) = \frac{\sin \left[ \frac{1}{2}(k+1)\theta \right] \sin \left( \frac{1}{2}k\theta \right)}{\sin \left( \frac{1}{2}\theta \right)}$

$$\begin{aligned}
\text{Now } & \sin \theta + \sin 2\theta + \sin 3\theta + \dots + \sin(k\theta) + \sin(k+1)\theta \\
&= \frac{\sin \left[ \frac{1}{2}(k+1)\theta \right] \sin \left( \frac{1}{2}k\theta \right)}{\sin \left( \frac{1}{2}\theta \right)} + \sin(k+1)\theta \quad \{\text{using } P_k\} \\
&= \frac{1}{\sin \left( \frac{1}{2}\theta \right)} \left( \sin \left[ \frac{1}{2}(k+1)\theta \right] \sin \left( \frac{1}{2}k\theta \right) + \sin(k+1)\theta \sin \left( \frac{1}{2}\theta \right) \right) \\
&= \frac{1}{\sin \left( \frac{1}{2}\theta \right)} \left( \sin \frac{(k+1)\theta}{2} \sin \frac{(k+2)\theta}{2} \right) \quad \{\text{using } \clubsuit\} \\
&= \frac{\sin \left[ \frac{1}{2}([k+1]+1)\theta \right] \sin \left( \frac{1}{2}(k+1)\theta \right)}{\sin \left( \frac{1}{2}\theta \right)}
\end{aligned}$$

$\therefore P_{k+1}$  is also true.

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

- 8**  $P_n$  is: if  $n$  items are distributed amongst  $m$  pigeonholes with  $n, m \in \mathbb{Z}^+$  and  $n > m$ , then at least one pigeonhole will contain at least  $\frac{n}{m}$  items.

**Proof:** (By the principle of mathematical induction)

- (1) If  $n = m + 1$ , then  $\frac{n}{m} = \frac{m+1}{m} = 1 + \frac{1}{m}$ , so  $1 < \frac{n}{m} \leq 2$   $\{m, n \in \mathbb{Z}^+\}$

Now the first  $m$  items can be distributed by placing 1 item in each of the  $m$  pigeonholes. However, the pigeonhole in which the  $(m+1)$ th item is placed must contain 2 items.

Thus, at least one of the pigeonholes must contain at least 2 items, and  $\frac{n}{m} \leq 2$ .

$\therefore P_{m+1}$  is true.

- (2) If  $P_k$  is true, then if  $k$  items are distributed amongst  $m$  pigeonholes, then at least one pigeonhole will contain at least  $\frac{k}{m}$  items.

*Case 1:* If  $k$  is not a multiple of  $m$ , then  $\frac{k}{m}$  and  $\frac{k+1}{m}$  have the same value when rounded up to the nearest integer. Thus, if a  $(k+1)$ th item is added, there must still be at least one pigeonhole which contains at least  $\frac{k+1}{m}$  items.

*Case 2:* If  $k$  is a multiple of  $m$ , then  $\frac{k+1}{m}$  rounded up to the nearest integer is  $\frac{k}{m} + 1$ .

With  $k$  items distributed, the only way that none of the pigeonholes contain  $\frac{k}{m} + 1$

items is when each pigeonhole contains exactly  $\frac{k}{m}$  items. Thus, the pigeonhole in

which the  $(k+1)$ th item is placed must contain  $\frac{k}{m} + 1$  items.

$\therefore P_{k+1}$  is also true.

Since  $P_{m+1}$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all  $n \in \mathbb{Z}^+$ ,  $n > m$ . {principle of mathematical induction}



# Chapter 11

## LINEAR ALGEBRA

### EXERCISE 11A

- 1
  - a  $2x + y + xy = 3$  is not a linear equation as it contains the product of two variables,  $xy$ .
  - b  $x_1 - x_2 - 2x_3^2 = 0$  is not a linear equation as it contains the squared term  $-2x_3^2$ .
  - c  $x = 7 - \sqrt{y}$  is not a linear equation as it contains the square root term  $\sqrt{y}$ .
  
- 2
  - a  $8x - y = 3$   
 Let  $y = t$   
 $\therefore 8x - t = 3$   
 $\therefore 8x = t + 3$   
 $\therefore x = \frac{t+3}{8}$   
 $\therefore$  a solution set is  $x = \frac{t+3}{8}$ ,  $y = t$ ,  
 where  $t \in \mathbb{R}$ .
  - b  $x_1 - 2x_2 + x_3 = 10$   
 Let  $x_2 = s$  and  $x_3 = t$   
 $\therefore x_1 - 2s + t = 10$   
 $\therefore x_1 = 2s - t + 10$   
 $\therefore$  a solution set is  $x_1 = 2s - t + 10$ ,  
 $x_2 = s$ , and  $x_3 = t$ , where  $s, t \in \mathbb{R}$ .
  - c  $x + y - 2z = -2$   
 Let  $y = s$  and  $z = t$   
 $\therefore x + s - 2t = -2$   
 $\therefore x = -2 - s + 2t$   
 $\therefore$  a solution set is  $x = -2 - s + 2t$ ,  $y = s$ ,  $z = t$ , where  $s, t \in \mathbb{R}$ .
  
- 3
  - a  $\begin{cases} x + y = 2 \\ x + y = 3 \end{cases}$   
 This system is inconsistent as  $x + y$  cannot be equal to both 2 and 3 simultaneously.
  - b  $\begin{cases} 2x - y = 1 \\ 2x + y = 1 \end{cases}$   
 This system is consistent as  $x = \frac{1}{2}$ ,  $y = 0$  is a solution.  
 Check:  $2(\frac{1}{2}) - 0 = 1$  and  $2(\frac{1}{2}) + 0 = 1$  ✓
  - c  $\begin{cases} x - y + z = 3 \\ -x + y + z = 3 \end{cases}$   
 This system is consistent as  $x = 1$ ,  $y = 1$ ,  $z = 3$  is a solution.  
 Check:  $1 - 1 + 3 = 3$  and  $-1 + 1 + 3 = 3$  ✓
  
- 4
  - a The system  $\begin{cases} x + 2y = 0 \\ 2x - y = -1 \\ -x + y = 2 \end{cases}$  has augmented matrix  $\left( \begin{array}{cc|c} 1 & 2 & 0 \\ 2 & -1 & -1 \\ -1 & 1 & 2 \end{array} \right)$ .  
 This system is overspecified as there are 3 equations in 2 unknowns.
  - b The system  $\begin{cases} x + 2y - z = 1 \\ -2x + y + 2z = -4 \end{cases}$  has augmented matrix  $\left( \begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ -2 & 1 & 2 & -4 \end{array} \right)$ .  
 This system is underspecified as there are 2 equations in 3 unknowns.



c The system 
$$\begin{cases} x_1 + x_2 - x_3 = 4 \\ x_1 - x_2 + x_3 = 8 \\ 2x_1 + x_2 - 3x_3 = 0 \end{cases}$$
 has augmented matrix 
$$\left( \begin{array}{ccc|c} 1 & 1 & -1 & 4 \\ 1 & -1 & 1 & 8 \\ 2 & 1 & -3 & 0 \end{array} \right).$$

This system is neither underspecified nor overspecified as there are 3 equations in 3 unknowns.

5 a 
$$\left( \begin{array}{ccc|c} 2 & -1 & 3 & 7 \\ 1 & 3 & -1 & 5 \end{array} \right)$$
 is the augmented matrix for 
$$\begin{cases} 2x - y + 3z = 7 \\ x + 3y - z = 5 \end{cases}.$$

b 
$$\left( \begin{array}{ccc|c} 3 & 1 & -2 & -3 \\ 0 & -1 & 2 & 8 \\ 2 & 0 & -1 & 2 \end{array} \right)$$
 is the augmented matrix for 
$$\begin{cases} 3x + y - 2z = -3 \\ -y + 2z = 8 \\ 2x - z = 2 \end{cases}.$$

c 
$$\left( \begin{array}{ccc|c} 4 & -1 & 2 & -3 \\ -1 & 3 & 3 & 2 \\ 2 & 1 & -2 & 5 \end{array} \right)$$
 is the augmented matrix for 
$$\begin{cases} 4x - y + 2z = -3 \\ -x + 3y + 3z = 2 \\ 2x + y - 2z = 5 \end{cases}.$$

## EXERCISE 11B

1 
$$\begin{cases} x - 3y = 3 \\ 2x + y = -2 \end{cases}$$

a The system has augmented matrix 
$$\left( \begin{array}{cc|c} 1 & -3 & 3 \\ 2 & 1 & -2 \end{array} \right).$$

b i 
$$\left( \begin{array}{cc|c} 1 & -3 & 3 \\ 2 & 1 & -2 \end{array} \right) \sim \left( \begin{array}{cc|c} 2 & -6 & 6 \\ 2 & 1 & -2 \end{array} \right) \quad 2R_1 \rightarrow R_1$$

ii 
$$\left( \begin{array}{cc|c} 1 & -3 & 3 \\ 2 & 1 & -2 \end{array} \right) \sim \left( \begin{array}{cc|c} 0 & -7 & 8 \\ 2 & 1 & -2 \end{array} \right) \quad 2R_1 - R_2 \rightarrow R_1 \quad \left\{ \begin{array}{ccc} 2 & -6 & 6 \\ -2 & -1 & 2 \\ \hline 0 & -7 & 8 \end{array} \right\}$$

iii 
$$\left( \begin{array}{cc|c} 1 & -3 & 3 \\ 2 & 1 & -2 \end{array} \right) \sim \left( \begin{array}{cc|c} 1 & -3 & 3 \\ 7 & 0 & -3 \end{array} \right) \quad R_1 + 3R_2 \rightarrow R_2 \quad \left\{ \begin{array}{ccc} 1 & -3 & 3 \\ 6 & 3 & -6 \\ \hline 7 & 0 & -3 \end{array} \right\}$$

2 
$$\begin{cases} x + y = 4 \\ 3x + 3y = a \end{cases}$$

a The system has augmented matrix 
$$\left( \begin{array}{cc|c} 1 & 1 & 4 \\ 3 & 3 & a \end{array} \right).$$

b 
$$\left( \begin{array}{cc|c} 1 & 1 & 4 \\ 3 & 3 & a \end{array} \right) \sim \left( \begin{array}{cc|c} 3 & 3 & 12 \\ 3 & 3 & a \end{array} \right) \quad 3R_1 \rightarrow R_1$$

c The system is consistent for  $a = 12$  as  $x = 2, y = 2$  is a solution.

Check:  $3(2) + 3(2) = 12$  and  $3(2) + 3(2) = 12$  ✓

If  $a \neq 12$ , then the system is inconsistent as  $3x + 3y$  cannot be equal to both 12 and  $a$  simultaneously.

$$3 \quad \begin{cases} 2x - y = 5 \\ -x + ay = 3 \end{cases}$$

**a** The system has augmented matrix  $\left( \begin{array}{cc|c} 2 & -1 & 5 \\ -1 & a & 3 \end{array} \right)$ .

$$\mathbf{b} \quad \left( \begin{array}{cc|c} 2 & -1 & 5 \\ -1 & a & 3 \end{array} \right) \sim \left( \begin{array}{cc|c} 2 & -1 & 5 \\ 2 & -2a & -6 \end{array} \right) \quad -2R_2 \rightarrow R_2$$

**c** The system is inconsistent for  $a = \frac{1}{2}$  as  $2x - y$  cannot be equal to 5 and  $-6$  simultaneously.

$$4 \quad \begin{cases} x + y + z = p \\ x + 2z = q \\ 2x + y + 3z = r \end{cases}$$

**a** The system has augmented matrix  $\left( \begin{array}{ccc|c} 1 & 1 & 1 & p \\ 1 & 0 & 2 & q \\ 2 & 1 & 3 & r \end{array} \right)$ .

$$\mathbf{b} \quad \left( \begin{array}{ccc|c} 1 & 1 & 1 & p \\ 1 & 0 & 2 & q \\ 2 & 1 & 3 & r \end{array} \right) \sim \left( \begin{array}{ccc|c} 1 & 1 & 1 & p \\ 2 & 1 & 3 & p+q \\ 2 & 1 & 3 & r \end{array} \right) \quad R_1 + R_2 \rightarrow R_2 \quad \leftarrow \left\{ \begin{array}{ccc|c} 1 & 1 & 1 & p \\ 1 & 0 & 2 & q \\ \hline 2 & 1 & 3 & p+q \end{array} \right\}$$

**c** The system is consistent if  $p + q = r$ .

## EXERCISE 11C

$$1 \quad \begin{cases} x - 3y = 2 \\ 2x + y = -3 \end{cases}$$

**a** In augmented matrix form, the system is  $\left( \begin{array}{cc|c} 1 & -3 & 2 \\ 2 & 1 & -3 \end{array} \right)$ .

$$\mathbf{b} \quad \left( \begin{array}{cc|c} 1 & -3 & 2 \\ 2 & 1 & -3 \end{array} \right) \sim \left( \begin{array}{cc|c} 1 & -3 & 2 \\ 0 & 7 & -7 \end{array} \right) \quad R_2 - 2R_1 \rightarrow R_2 \quad \leftarrow \left\{ \begin{array}{cc|c} 2 & 1 & -3 \\ -2 & 6 & -4 \\ \hline 0 & 7 & -7 \end{array} \right\}$$

$$\mathbf{c} \quad \left( \begin{array}{cc|c} 1 & -3 & 2 \\ 2 & 1 & -3 \end{array} \right) \sim \left( \begin{array}{cc|c} 1 & -3 & 2 \\ 0 & 7 & -7 \end{array} \right) \quad \{\text{from } \mathbf{b}\} \\ \sim \left( \begin{array}{cc|c} 1 & -3 & 2 \\ 0 & 1 & -1 \end{array} \right) \quad \frac{1}{7}R_2 \rightarrow R_2$$

Using row 2,  $y = -1$

Substituting into row 1,  $x - 3(-1) = 2$

$$\therefore x + 3 = 2$$

$$\therefore x = -1$$

$\therefore$  the solution is  $x = -1, y = -1$ .

$$2 \quad a \quad \begin{cases} x + 2y = 1 \\ 3x + 6y = 3 \end{cases}$$

The second line has equation  $3x + 6y = 3$   
or  $x + 2y = 1$

So, the lines are coincident.

$\therefore$  the system has infinitely many solutions.

$$b \quad \begin{cases} 2x - y = -1 \\ x + 4y = 13 \end{cases}$$

The first line has equation  $2x - y = -1$   
or  $y = 2x + 1$

The second line has equation  $x + 4y = 13$   
or  $y = -\frac{1}{4}x + \frac{13}{4}$

Since the lines have different gradients, they are neither parallel nor coincident, and so must intersect at a point.

$\therefore$  the system has a unique solution.

$$c \quad \begin{cases} x - 5y = 8 \\ 2x = 10y + 14 \end{cases}$$

The second line has equation  $2x = 10y + 14$   
which is  $2x - 10y = 14$   
or  $x - 5y = 7$

So, the lines are parallel but do not intersect.

$\therefore$  the system has no solutions.

$$d \quad \begin{cases} x + y = 4 \\ x + y = a, \quad a \in \mathbb{R} \end{cases}$$

If  $a = 4$ , the lines are coincident.

$\therefore$  the system has infinitely many solutions if  $a = 4$ .

If  $a \neq 4$ , the lines are parallel but do not intersect.

$\therefore$  the system has no solutions if  $a \neq 4$ .

**3 a** 
$$\begin{cases} x - 3y = -8 \\ 4x + 5y = 19 \end{cases}$$

In augmented matrix form, the system is

$$\begin{aligned} & \left( \begin{array}{cc|c} 1 & -3 & -8 \\ 4 & 5 & 19 \end{array} \right) \\ & \sim \left( \begin{array}{cc|c} 1 & -3 & -8 \\ 0 & 17 & 51 \end{array} \right) \quad R_2 - 4R_1 \rightarrow R_2 \quad \leftarrow \left\{ \begin{array}{ccc} 4 & 5 & 19 \\ -4 & 12 & 32 \\ \hline 0 & 17 & 51 \end{array} \right\} \\ & \sim \left( \begin{array}{cc|c} 1 & -3 & -8 \\ 0 & 1 & 3 \end{array} \right) \quad \frac{1}{17}R_2 \rightarrow R_2 \end{aligned}$$

Using row 2,  $y = 3$

$$\begin{aligned} \text{Substituting into row 1, } x - 3(3) &= -8 \\ \therefore x - 9 &= -8 \\ \therefore x &= 1 \end{aligned}$$

$\therefore$  the solution is  $x = 1, y = 3$ .

**b** 
$$\begin{cases} x + 7y = -17 \\ 2x - y = 11 \end{cases}$$

In augmented matrix form, the system is

$$\begin{aligned} & \left( \begin{array}{cc|c} 1 & 7 & -17 \\ 2 & -1 & 11 \end{array} \right) \\ & \sim \left( \begin{array}{cc|c} 1 & 7 & -17 \\ 0 & -15 & 45 \end{array} \right) \quad R_2 - 2R_1 \rightarrow R_2 \quad \leftarrow \left\{ \begin{array}{ccc} 2 & -1 & 11 \\ -2 & -14 & 34 \\ \hline 0 & -15 & 45 \end{array} \right\} \\ & \sim \left( \begin{array}{cc|c} 1 & 7 & -17 \\ 0 & 1 & -3 \end{array} \right) \quad -\frac{1}{15}R_2 \rightarrow R_2 \end{aligned}$$

Using row 2,  $y = -3$

$$\begin{aligned} \text{Substituting into row 1, } x + 7(-3) &= -17 \\ \therefore x - 21 &= -17 \\ \therefore x &= 4 \end{aligned}$$

$\therefore$  the solution is  $x = 4, y = -3$ .



$$\text{c } \begin{cases} 2x + 3y = -8 \\ x + 4y = -9 \end{cases}$$

In augmented matrix form, the system is

$$\begin{aligned} & \left( \begin{array}{cc|c} 2 & 3 & -8 \\ 1 & 4 & -9 \end{array} \right) \\ & \sim \left( \begin{array}{cc|c} 2 & 3 & -8 \\ 0 & 5 & -10 \end{array} \right) \quad 2R_2 - R_1 \rightarrow R_2 \quad \left\{ \begin{array}{ccc} 2 & 8 & -18 \\ -2 & -3 & 8 \\ 0 & 5 & -10 \end{array} \right\} \\ & \sim \left( \begin{array}{cc|c} 1 & \frac{3}{2} & -4 \\ 0 & 1 & -2 \end{array} \right) \quad \begin{array}{l} \frac{1}{2}R_1 \rightarrow R_1 \\ \frac{1}{5}R_2 \rightarrow R_2 \end{array} \end{aligned}$$

Using row 2,  $y = -2$

$$\begin{aligned} \text{Substituting into row 1, } x + \frac{3}{2}(-2) &= -4 \\ \therefore x - 3 &= -4 \\ \therefore x &= -1 \end{aligned}$$

$\therefore$  the solution is  $x = -1, y = -2$ .

$$\text{d } \begin{cases} 3x - y = 9 \\ 4x + 3y = -1 \end{cases}$$

In augmented matrix form, the system is

$$\begin{aligned} & \left( \begin{array}{cc|c} 3 & -1 & 9 \\ 4 & 3 & -1 \end{array} \right) \\ & \sim \left( \begin{array}{cc|c} 3 & -1 & 9 \\ 0 & 13 & -39 \end{array} \right) \quad 3R_2 - 4R_1 \rightarrow R_2 \quad \left\{ \begin{array}{ccc} 12 & 9 & -3 \\ -12 & 4 & -36 \\ 0 & 13 & -39 \end{array} \right\} \\ & \sim \left( \begin{array}{cc|c} 1 & -\frac{1}{3} & 3 \\ 0 & 1 & -3 \end{array} \right) \quad \begin{array}{l} \frac{1}{3}R_1 \rightarrow R_1 \\ \frac{1}{13}R_2 \rightarrow R_2 \end{array} \end{aligned}$$

Using row 2,  $y = -3$

$$\begin{aligned} \text{Substituting into row 1, } x - \frac{1}{3}(-3) &= 3 \\ \therefore x + 1 &= 3 \\ \therefore x &= 2 \end{aligned}$$

$\therefore$  the solution is  $x = 2, y = -3$ .

$$\text{4 } \begin{cases} x + 3y = 4 \\ 2x + 6y = 8 \end{cases}$$

- a The second equation represents the line  $2x + 6y = 8$   
or  $x + 3y = 4$

So, the equations represent coincident lines.

$\therefore$  the system has infinitely many solutions.

**b** In augmented matrix form, the system is

$$\begin{pmatrix} 1 & 3 & | & 4 \\ 2 & 6 & | & 8 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & | & 4 \\ 0 & 0 & | & 0 \end{pmatrix} \quad R_2 - 2R_1 \rightarrow R_2 \quad \leftarrow \begin{Bmatrix} 2 & 6 & 8 \\ 2 & 6 & 8 \\ 0 & 0 & 0 \end{Bmatrix}$$

The second equation is a multiple of the first, so we obtain a row of zeros when we use row operations.

The second equation adds nothing to aid the solution to the system.

**c** Let  $y = t$ ,  $t \in \mathbb{R}$ .

Using row 1,  $x + 3t = 4$

$$\therefore x = 4 - 3t$$

$\therefore$  a solution set is  $x = 4 - 3t$ ,  $y = t$  where  $t \in \mathbb{R}$ .

**d** Let  $x = s$ ,  $s \in \mathbb{R}$ .

Using row 1,  $s + 3y = 4$

$$\therefore 3y = 4 - s$$

$$\therefore y = \frac{4-s}{3}$$

$\therefore$  a solution set is  $x = s$ ,  $y = \frac{4-s}{3}$  where  $s \in \mathbb{R}$ .

**e** Let  $(x, y)$  be in the solutions set of **d**, then  $x = s'$  and  $y = \frac{4-s'}{3}$  for some  $s' \in \mathbb{R}$ .

If  $t' = y = \frac{4-s'}{3}$ , then  $(4 - 3t', t')$  is in the solution set of **c**.

$$\begin{aligned} \text{But, } 4 - 3t' &= 4 - 3\left(\frac{4-s'}{3}\right) \\ &= 4 - (4 - s') \\ &= 4 - 4 + s' \\ &= s' \\ &= x \end{aligned}$$

So,  $(x, y) = (4 - 3t', t')$  is also in the solution set of **c**.

Similarly, if  $(x, y)$  is in the solution set of **c**, then it is also in the solution set of **d**.

$\therefore$  the solution sets of **c** and **d** are equivalent.

$$5 \quad \begin{cases} x - 5y = 8 \\ 2x - 10y = a \end{cases} \quad \text{where } a \in \mathbb{R}$$

**a** In augmented matrix form, the system is

$$\begin{pmatrix} 1 & -5 & | & 8 \\ 2 & -10 & | & a \end{pmatrix} \sim \begin{pmatrix} 1 & -5 & | & 8 \\ 0 & 0 & | & a - 16 \end{pmatrix} \quad R_2 - 2R_1 \rightarrow R_2 \quad \leftarrow \begin{Bmatrix} 2 & -10 & a \\ -2 & 10 & -16 \\ 0 & 0 & a - 16 \end{Bmatrix}$$

**b** When  $a \neq 16$ ,  $a - 16 \neq 0$  and the second row gives  $0x + 0y \neq 0$  which is not possible.  
 $\therefore$  the system has no solutions if  $a \neq 16$ .

**c** If  $a = 16$ , then  $\begin{pmatrix} 1 & -5 & | & 8 \\ 2 & -10 & | & 16 \end{pmatrix} \sim \begin{pmatrix} 1 & -5 & | & 8 \\ 0 & 0 & | & 0 \end{pmatrix}$  {from **a**}

Let  $y = t$ ,  $t \in \mathbb{R}$ .

Using row 1,  $x - 5t = 8$

$$\therefore x = 5t + 8$$

$\therefore$  a solution set is  $x = 5t + 8$ ,  $y = t$ , where  $t \in \mathbb{R}$ .

$$6 \quad \begin{cases} x + 3y = 4 \\ 2x + ay = b \end{cases} \quad \text{where } a, b \in \mathbb{R}$$

In augmented matrix form, the system is

$$\begin{pmatrix} 1 & 3 & | & 4 \\ 2 & a & | & b \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & | & 4 \\ 0 & a - 6 & | & b - 8 \end{pmatrix} \quad R_2 - 2R_1 \rightarrow R_2 \quad \leftarrow \begin{Bmatrix} 2 & a & b \\ -2 & -6 & -8 \\ 0 & a - 6 & b - 8 \end{Bmatrix}$$

There are 3 cases to consider:

- If  $a \neq 6$ , then using row 2,  $(a - 6)y = b - 8$

$$\therefore y = \frac{b - 8}{a - 6} \quad \{a \neq 6\}$$

Substituting into row 1,  $x + 3\left(\frac{b - 8}{a - 6}\right) = 4$

$$\therefore x + \frac{3b - 24}{a - 6} = 4$$

$$\begin{aligned} \therefore x &= 4 - \frac{3b - 24}{a - 6} \\ &= \frac{4(a - 6) - (3b - 24)}{a - 6} \\ &= \frac{4a - 24 - 3b + 24}{a - 6} \\ &= \frac{4a - 3b}{a - 6} \end{aligned}$$

$\therefore$  the system has a unique solution  $x = \frac{4a - 3b}{a - 6}$ ,  $y = \frac{b - 8}{a - 6}$ .

- If  $a = 6$  and  $b = 8$ , then the second equation represents the line  $2x + 6y = 8$   
or  $x + 3y = 4$

So, the equations represent coincident lines.

$\therefore$  the system has infinitely many solutions.

Let  $y = t$ ,  $t \in \mathbb{R}$

Using row 1,  $x + 3t = 4$

$$\therefore x = 4 - 3t$$

$\therefore$  a solution set is  $x = 4 - 3t$ ,  $y = t$ , where  $t \in \mathbb{R}$ .

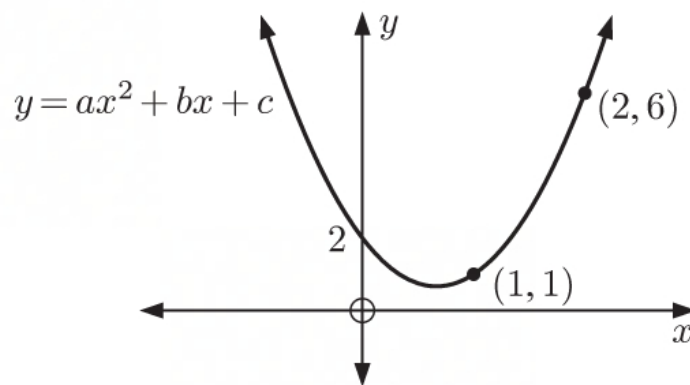
- If  $a = 6$  and  $b \neq 8$ , using row 2,  $0x + 0y \neq 0$  which is not possible.  
 $\therefore$  the system has no solutions.

- 7 a** From the graph, the  $y$ -intercept is 2.

So, when  $x = 0$ ,  $y = 2$

$$\therefore a(0)^2 + b(0) + c = 2$$

$$\therefore c = 2$$



- b** The graph passes through the points  $(1, 1)$  and  $(2, 6)$

$$\therefore a(1)^2 + b(1) + 2 = 1 \quad \text{and} \quad a(2)^2 + b(2) + 2 = 6$$

$$\therefore a + b = -1$$

$$\therefore 4a + 2b = 4$$

$$\therefore 2a + b = 2$$

**c** 
$$\begin{cases} a + b = -1 \\ 2a + b = 2 \end{cases}$$

In augmented matrix form, the system is

$$\begin{aligned} & \left( \begin{array}{cc|c} 1 & 1 & -1 \\ 2 & 1 & 2 \end{array} \right) \\ & \sim \left( \begin{array}{cc|c} 1 & 1 & -1 \\ 0 & -1 & 4 \end{array} \right) \quad R_2 - 2R_1 \rightarrow R_2 \quad \left\{ \begin{array}{ccc} 2 & 1 & 2 \\ -2 & -2 & 2 \\ \hline 0 & -1 & 4 \end{array} \right\} \\ & \sim \left( \begin{array}{cc|c} 1 & 1 & -1 \\ 0 & 1 & -4 \end{array} \right) \quad -R_2 \rightarrow R_2 \end{aligned}$$

Using row 2,  $b = -4$

Substituting into row 1,  $a + (-4) = -1$

$$\therefore a = 3$$

$\therefore$  the solution is  $a = 3$ ,  $b = -4$ .

$\therefore$  the equation of the quadratic function is  $y = 3x^2 - 4x + 2$ .



$$8 \quad \begin{cases} ax + bx = m \\ cx + dx = n \end{cases}$$

a In augmented matrix form, the system is

$$\begin{pmatrix} a & b & | & m \\ c & d & | & n \end{pmatrix} \sim \begin{pmatrix} a & b & | & m \\ 0 & bc - ad & | & cm - an \end{pmatrix} \quad cR_1 - aR_2 \rightarrow R_2 \quad \left\{ \begin{array}{ccc} ac & bc & cm \\ -ac & -ad & -an \\ \hline 0 & bc - ad & cm - an \end{array} \right\}$$

b For a unique solution, the left hand side of the second row must not all be zero.

$$\therefore bc - ad \neq 0$$

$$\therefore ad \neq bc$$

c For infinitely many solutions, the second row must be a row of zeros.

$$\therefore bc - ad = 0 \quad \text{and} \quad cm - an = 0$$

## EXERCISE 11D

$$1 \quad a \quad \begin{cases} x + 4y + 11z = 7 \\ x + 6y + 17z = 9 \\ x + 4y + 8z = 4 \end{cases}$$

The system has augmented matrix

$$\begin{pmatrix} 1 & 4 & 11 & | & 7 \\ 1 & 6 & 17 & | & 9 \\ 1 & 4 & 8 & | & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 4 & 11 & | & 7 \\ 0 & 2 & 6 & | & 2 \\ 0 & 0 & -3 & | & -3 \end{pmatrix} \quad \begin{array}{l} R_2 - R_1 \rightarrow R_2 \\ R_3 - R_1 \rightarrow R_3 \end{array} \quad \left\{ \begin{array}{cccc} 1 & 6 & 17 & 9 \\ -1 & -4 & -11 & -7 \\ \hline 0 & 2 & 6 & 2 \end{array} \right\}$$

$$\sim \begin{pmatrix} 1 & 4 & 11 & | & 7 \\ 0 & 1 & 3 & | & 1 \\ 0 & 0 & 1 & | & 1 \end{pmatrix} \quad \begin{array}{l} \frac{1}{2}R_2 \rightarrow R_2 \\ -\frac{1}{3}R_3 \rightarrow R_3 \end{array} \quad \left\{ \begin{array}{cccc} 1 & 4 & 8 & 4 \\ -1 & -4 & -11 & -7 \\ \hline 0 & 0 & -3 & -3 \end{array} \right\}$$

Using row 3,  $z = 1$

Substituting into row 2,  $y + 3(1) = 1$

$$\therefore y = -2$$

Substituting into row 1,  $x + 4(-2) + 11(1) = 7$

$$\therefore x = 4$$

$\therefore$  the unique solution is  $x = 4, y = -2, z = 1$ .

$$\text{b } \begin{cases} 2x - y + 3z = 17 \\ 2x - 2y - 5z = 4 \\ 3x + 2y + 2z = 10 \end{cases}$$

The system has augmented matrix

$$\begin{aligned} & \left( \begin{array}{ccc|c} 2 & -1 & 3 & 17 \\ 2 & -2 & -5 & 4 \\ 3 & 2 & 2 & 10 \end{array} \right) \\ & \sim \left( \begin{array}{ccc|c} 2 & -1 & 3 & 17 \\ 0 & -1 & -8 & -13 \\ 0 & 7 & -5 & -31 \end{array} \right) & \begin{array}{l} R_2 - R_1 \rightarrow R_2 \\ 2R_3 - 3R_1 \rightarrow R_3 \end{array} & \left\{ \begin{array}{ccc|c} 2 & -2 & -5 & 4 \\ -2 & 1 & -3 & -17 \\ \hline 0 & -1 & -8 & -13 \end{array} \right\} \\ & \sim \left( \begin{array}{ccc|c} 2 & -1 & 3 & 17 \\ 0 & -1 & -8 & -13 \\ 0 & 0 & -61 & -122 \end{array} \right) & R_3 + 7R_2 \rightarrow R_3 & \left\{ \begin{array}{ccc|c} 6 & 4 & 4 & 20 \\ -6 & 3 & -9 & -51 \\ \hline 0 & 7 & -5 & -31 \end{array} \right\} \\ & \sim \left( \begin{array}{ccc|c} 2 & -1 & 3 & 17 \\ 0 & 1 & 8 & 13 \\ 0 & 0 & 1 & 2 \end{array} \right) & \begin{array}{l} -R_2 \rightarrow R_2 \\ -\frac{1}{61}R_3 \rightarrow R_3 \end{array} & \left\{ \begin{array}{ccc|c} 0 & 7 & -5 & -31 \\ 0 & -7 & -56 & -91 \\ \hline 0 & 0 & -61 & -122 \end{array} \right\} \end{aligned}$$

Using row 3,  $z = 2$

$$\begin{aligned} \text{Substituting into row 2, } y + 8(2) &= 13 \\ \therefore y &= -3 \end{aligned}$$

$$\begin{aligned} \text{Substituting into row 1, } 2x - (-3) + 3(2) &= 17 \\ \therefore 2x &= 8 \\ \therefore x &= 4 \end{aligned}$$

$\therefore$  the unique solution is  $x = 4, y = -3, z = 2$ .

$$\text{c } \begin{cases} 2x + 3y + 4z = 1 \\ 5x + 6y + 7z = 2 \\ 8x + 9y + 10z = 4 \end{cases}$$

The system has augmented matrix

$$\begin{aligned} & \left( \begin{array}{ccc|c} 2 & 3 & 4 & 1 \\ 5 & 6 & 7 & 2 \\ 8 & 9 & 10 & 4 \end{array} \right) \\ & \sim \left( \begin{array}{ccc|c} 2 & 3 & 4 & 1 \\ 0 & -3 & -6 & -1 \\ 0 & -3 & -6 & 0 \end{array} \right) & \begin{array}{l} 2R_2 - 5R_1 \rightarrow R_2 \\ R_3 - 4R_1 \rightarrow R_3 \end{array} & \left\{ \begin{array}{ccc|c} 10 & 12 & 14 & 4 \\ -10 & -15 & -20 & -5 \\ \hline 0 & -3 & -6 & -1 \end{array} \right\} \\ & \sim \left( \begin{array}{ccc|c} 2 & 3 & 4 & 1 \\ 0 & -3 & -6 & -1 \\ 0 & 0 & 0 & 1 \end{array} \right) & R_3 - R_2 \rightarrow R_3 & \left\{ \begin{array}{ccc|c} 8 & 9 & 10 & 4 \\ -8 & -12 & -16 & -4 \\ \hline 0 & -3 & -6 & 0 \end{array} \right\} \\ & & & \left\{ \begin{array}{ccc|c} 0 & -3 & -6 & 0 \\ 0 & 3 & 6 & 1 \\ \hline 0 & 0 & 0 & 1 \end{array} \right\} \end{aligned}$$

Row 3 means that  $0x + 0y + 0z = 1$ , which is absurd.

$\therefore$  there is no solution, and the system is inconsistent.

$$\text{d } \begin{cases} x - 2y + 5z = 1 \\ 2x - y + 8z = 2 \\ -3x - 11z = -3 \end{cases}$$

The system has augmented matrix

$$\begin{aligned} & \left( \begin{array}{ccc|c} 1 & -2 & 5 & 1 \\ 2 & -1 & 8 & 2 \\ -3 & 0 & -11 & -3 \end{array} \right) \\ & \sim \left( \begin{array}{ccc|c} 1 & -2 & 5 & 1 \\ 0 & 3 & -2 & 0 \\ 0 & -6 & 4 & 0 \end{array} \right) \quad \begin{array}{l} R_2 - 2R_1 \rightarrow R_2 \\ R_3 + 3R_1 \rightarrow R_3 \end{array} \quad \left\{ \begin{array}{ccc|c} 2 & -1 & 8 & 2 \\ -2 & 4 & -10 & -2 \\ \hline 0 & 3 & -2 & 0 \end{array} \right\} \\ & \sim \left( \begin{array}{ccc|c} 1 & -2 & 5 & 1 \\ 0 & 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \quad \begin{array}{l} \frac{1}{3}R_2 \rightarrow R_2 \\ R_3 + 2R_2 \rightarrow R_3 \end{array} \quad \left\{ \begin{array}{ccc|c} -3 & 0 & -11 & -3 \\ 3 & -6 & 15 & 3 \\ \hline 0 & -6 & 4 & 0 \end{array} \right\} \\ & \quad \left\{ \begin{array}{ccc|c} 0 & -6 & 4 & 0 \\ 0 & 6 & -4 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array} \right\} \end{aligned}$$

Row 3 indicates there are infinitely many solutions.

If we let  $z = t$ , then using row 2,  $y - \frac{2}{3}t = 0$

$$\therefore y = \frac{2}{3}t$$

Substituting into row 1,  $x - 2(\frac{2}{3}t) + 5t = 1$

$$\therefore x = 1 - \frac{11}{3}t$$

$\therefore$  the solutions have the form  $x = 1 - \frac{11}{3}t$ ,  $y = \frac{2}{3}t$ ,  $z = t$ , where  $t \in \mathbb{R}$ .

$$\text{e } \begin{cases} x + 2y - z = 4 \\ 3x + 2y + z = 7 \\ 5x + 2y + 3z = 11 \end{cases}$$

The system has augmented matrix

$$\begin{aligned} & \left( \begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 3 & 2 & 1 & 7 \\ 5 & 2 & 3 & 11 \end{array} \right) \\ & \sim \left( \begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & -4 & 4 & -5 \\ 0 & -8 & 8 & -9 \end{array} \right) \quad \begin{array}{l} R_2 - 3R_1 \rightarrow R_2 \\ R_3 - 5R_1 \rightarrow R_3 \end{array} \quad \left\{ \begin{array}{ccc|c} 3 & 2 & 1 & 7 \\ -3 & -6 & 3 & -12 \\ \hline 0 & -4 & 4 & -5 \end{array} \right\} \\ & \sim \left( \begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & -4 & 4 & -5 \\ 0 & 0 & 0 & -1 \end{array} \right) \quad \begin{array}{l} R_3 - 2R_2 \rightarrow R_3 \end{array} \quad \left\{ \begin{array}{ccc|c} 5 & 2 & 3 & 11 \\ -5 & -10 & 5 & -20 \\ \hline 0 & -8 & 8 & -9 \end{array} \right\} \\ & \quad \left\{ \begin{array}{ccc|c} 0 & -8 & 8 & -9 \\ 0 & 8 & -8 & 10 \\ \hline 0 & 0 & 0 & -1 \end{array} \right\} \end{aligned}$$

Row 3 means that  $0x + 0y + 0z = -1$ , which is absurd.

$\therefore$  there is no solution, and the system is inconsistent.

$$\mathbf{f} \quad \begin{cases} 2x + 4y + z = 1 \\ 3x - 5y - 3z = 19 \\ 5x + 13y + 7z = 1 \end{cases}$$

The system has augmented matrix

$$\begin{aligned} & \left( \begin{array}{ccc|c} 2 & 4 & 1 & 1 \\ 3 & -5 & -3 & 19 \\ 5 & 13 & 7 & 1 \end{array} \right) \\ & \sim \left( \begin{array}{ccc|c} 2 & 4 & 1 & 1 \\ 0 & -22 & -9 & 35 \\ 0 & 6 & 9 & -3 \end{array} \right) \quad \begin{array}{l} 2R_2 - 3R_1 \rightarrow R_2 \\ 2R_3 - 5R_1 \rightarrow R_3 \end{array} \quad \left\{ \begin{array}{cccc} 6 & -10 & -6 & 38 \\ -6 & -12 & -3 & -3 \\ \hline 0 & -22 & -9 & 35 \end{array} \right\} \\ & \sim \left( \begin{array}{ccc|c} 2 & 4 & 1 & 1 \\ 0 & -22 & -9 & 35 \\ 0 & 0 & 72 & 72 \end{array} \right) \quad \begin{array}{l} 11R_3 + 3R_2 \rightarrow R_3 \end{array} \quad \left\{ \begin{array}{cccc} 10 & 26 & 14 & 2 \\ -10 & -20 & -5 & -5 \\ \hline 0 & 6 & 9 & -3 \end{array} \right\} \\ & \sim \left( \begin{array}{ccc|c} 2 & 4 & 1 & 1 \\ 0 & 1 & \frac{9}{22} & -\frac{35}{22} \\ 0 & 0 & 1 & 1 \end{array} \right) \quad \begin{array}{l} -\frac{1}{22}R_2 \rightarrow R_2 \\ \frac{1}{72}R_3 \rightarrow R_3 \end{array} \quad \left\{ \begin{array}{cccc} 0 & 66 & 99 & -33 \\ 0 & -66 & -27 & 105 \\ \hline 0 & 0 & 72 & 72 \end{array} \right\} \end{aligned}$$

Using row 3,  $z = 1$

$$\begin{aligned} \text{Substituting into row 2, } y + \frac{9}{22}(1) &= -\frac{35}{22} \\ \therefore y &= -\frac{44}{22} = -2 \end{aligned}$$

$$\begin{aligned} \text{Substituting into row 1, } 2x + 4(-2) + 1 &= 1 \\ \therefore 2x &= 8 \\ \therefore x &= 4 \end{aligned}$$

$\therefore$  the unique solution is  $x = 4$ ,  $y = -2$ ,  $z = 1$ .

**2** Let the quadratic function have equation  $y = ax^2 + bx + c$ , where  $a, b, c \in \mathbb{R}$ .

**a** The quadratic function passes through  $(1, -2)$ ,  $(2, 4)$ , and  $(3, 12)$ .

$$\begin{aligned} \therefore a(1)^2 + b(1) + c &= -2 \\ a(2)^2 + b(2) + c &= 4 \\ a(3)^2 + b(3) + c &= 12 \end{aligned}$$

$$\text{which gives the system of equations } \begin{cases} a + b + c = -2 \\ 4a + 2b + c = 4 \\ 9a + 3b + c = 12 \end{cases}$$

The system has augmented matrix

$$\begin{aligned} & \left( \begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 4 & 2 & 1 & 4 \\ 9 & 3 & 1 & 12 \end{array} \right) \\ & \sim \left( \begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 0 & -2 & -3 & 12 \\ 0 & -6 & -8 & 30 \end{array} \right) \quad \begin{array}{l} R_2 - 4R_1 \rightarrow R_2 \\ R_3 - 9R_1 \rightarrow R_3 \end{array} \quad \left\{ \begin{array}{cccc} 4 & 2 & 1 & 4 \\ -4 & -4 & -4 & 8 \\ \hline 0 & -2 & -3 & 12 \end{array} \right\} \\ & \sim \left( \begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 0 & 1 & \frac{3}{2} & -6 \\ 0 & 0 & 1 & -6 \end{array} \right) \quad \begin{array}{l} -\frac{1}{2}R_2 \rightarrow R_2 \\ R_3 - 3R_2 \rightarrow R_3 \end{array} \quad \left\{ \begin{array}{cccc} 9 & 3 & 1 & 12 \\ -9 & -9 & -9 & 18 \\ \hline 0 & -6 & -8 & 30 \end{array} \right\} \\ & \quad \left\{ \begin{array}{cccc} 0 & -6 & -8 & 30 \\ 0 & 6 & 9 & -36 \\ \hline 0 & 0 & 1 & -6 \end{array} \right\} \end{aligned}$$



Using row 3,  $c = -6$

Substituting into row 2,  $b + \frac{3}{2}(-6) = -6$

$$\therefore b = 3$$

Substituting into row 1,  $a + 3 + (-6) = -2$

$$\therefore a = 1$$

$\therefore$  the unique solution is  $a = 1$ ,  $b = 3$ ,  $c = -6$ .

$\therefore$  the quadratic function has equation  $y = x^2 + 3x - 6$ .

- b** The quadratic function passes through  $(-1, 3)$ ,  $(2, 9)$ , and  $(4, -7)$ .

$$\therefore a(-1)^2 + b(-1) + c = 3$$

$$a(2)^2 + b(2) + c = 9$$

$$a(4)^2 + b(4) + c = -7$$

which gives the system of equations 
$$\begin{cases} a - b + c = 3 \\ 4a + 2b + c = 9 \\ 16a + 4b + c = -7 \end{cases}$$

The system has augmented matrix

$$\begin{aligned} & \left( \begin{array}{ccc|c} 1 & -1 & 1 & 3 \\ 4 & 2 & 1 & 9 \\ 16 & 4 & 1 & -7 \end{array} \right) \\ & \sim \left( \begin{array}{ccc|c} 1 & -1 & 1 & 3 \\ 0 & 6 & -3 & -3 \\ 0 & 20 & -15 & -55 \end{array} \right) \quad \begin{array}{l} R_2 - 4R_1 \rightarrow R_2 \\ R_3 - 16R_1 \rightarrow R_3 \end{array} \quad \left\{ \begin{array}{ccc|c} 4 & 2 & 1 & 9 \\ -4 & 4 & -4 & -12 \\ \hline 0 & 6 & -3 & -3 \end{array} \right\} \\ & \sim \left( \begin{array}{ccc|c} 1 & -1 & 1 & 3 \\ 0 & 6 & -3 & -3 \\ 0 & 0 & -15 & -135 \end{array} \right) \quad \begin{array}{l} \\ \\ 3R_3 - 10R_2 \rightarrow R_3 \end{array} \quad \left\{ \begin{array}{ccc|c} 16 & 4 & 1 & -7 \\ -16 & 16 & -16 & -48 \\ \hline 0 & 20 & -15 & -55 \end{array} \right\} \\ & \sim \left( \begin{array}{ccc|c} 1 & -1 & 1 & 3 \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & 9 \end{array} \right) \quad \begin{array}{l} \frac{1}{6}R_2 \rightarrow R_2 \\ -\frac{1}{15}R_3 \rightarrow R_3 \end{array} \quad \left\{ \begin{array}{ccc|c} 0 & 60 & -45 & -165 \\ 0 & -60 & 30 & 30 \\ \hline 0 & 0 & -15 & -135 \end{array} \right\} \end{aligned}$$

Using row 3,  $c = 9$

Substituting into row 2,  $b - \frac{1}{2}(9) = -\frac{1}{2}$

$$\therefore b = 4$$

Substituting into row 1,  $a - 4 + 9 = 3$

$$\therefore a = -2$$

$\therefore$  the unique solution is  $a = -2$ ,  $b = 4$ ,  $c = 9$ .

$\therefore$  the quadratic function has equation  $y = -2x^2 + 4x + 9$ .

- 3** Suppose the quadratic function has equation  $y = ax^2 + bx + c$ , where  $a, b, c \in \mathbb{R}$ .

The quadratic function passes through  $(-1, 7)$ ,  $(2, 1)$ , and  $(3, -1)$ .

$$\therefore a(-1)^2 + b(-1) + c = 7$$

$$a(2)^2 + b(2) + c = 1$$

$$a(3)^2 + b(3) + c = -1$$

which gives the system of equations 
$$\begin{cases} a - b + c = 7 \\ 4a + 2b + c = 1 \\ 9a + 3b + c = -1 \end{cases}$$

The system has augmented matrix

$$\begin{aligned}
 & \left( \begin{array}{ccc|c} 1 & -1 & 1 & 7 \\ 4 & 2 & 1 & 1 \\ 9 & 3 & 1 & -1 \end{array} \right) \\
 & \sim \left( \begin{array}{ccc|c} 1 & -1 & 1 & 7 \\ 0 & 6 & -3 & -27 \\ 0 & 12 & -8 & -64 \end{array} \right) \quad \begin{array}{l} R_2 - 4R_1 \rightarrow R_2 \\ R_3 - 9R_1 \rightarrow R_3 \end{array} \quad \left\{ \begin{array}{cccc} 4 & 2 & 1 & 1 \\ -4 & 4 & -4 & -28 \\ \hline 0 & 6 & -3 & -27 \end{array} \right\} \\
 & \sim \left( \begin{array}{ccc|c} 1 & -1 & 1 & 7 \\ 0 & 6 & -3 & -27 \\ 0 & 0 & -2 & -10 \end{array} \right) \quad R_3 - 2R_2 \rightarrow R_3 \quad \left\{ \begin{array}{cccc} 9 & 3 & 1 & -1 \\ -9 & 9 & -9 & -63 \\ \hline 0 & 12 & -8 & -64 \end{array} \right\} \\
 & \sim \left( \begin{array}{ccc|c} 1 & -1 & 1 & 7 \\ 0 & 1 & -\frac{1}{2} & -\frac{9}{2} \\ 0 & 0 & 1 & 5 \end{array} \right) \quad \begin{array}{l} \frac{1}{6}R_2 \rightarrow R_2 \\ -\frac{1}{2}R_3 \rightarrow R_3 \end{array} \quad \left\{ \begin{array}{cccc} 0 & 12 & -8 & -64 \\ 0 & -12 & 6 & 54 \\ \hline 0 & 0 & -2 & -10 \end{array} \right\}
 \end{aligned}$$

Using row 3,  $c = 5$

Substituting into row 2,  $b - \frac{1}{2}(5) = -\frac{9}{2}$   
 $\therefore b = -2$

Substituting into row 1,  $a - (-2) + 5 = 7$   
 $\therefore a = 0$

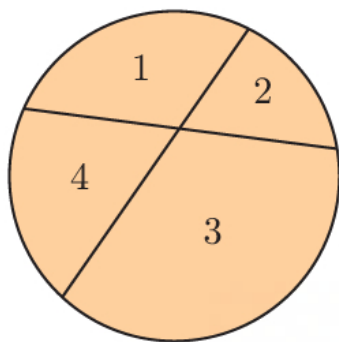
$\therefore$  the unique solution is  $a = 0$ ,  $b = -2$ ,  $c = 5$ .

$\therefore$  the function has equation  $y = -2x + 5$ .

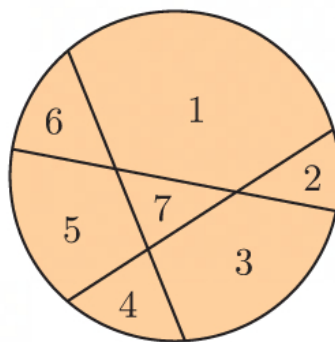
However, this is a straight line and *not* a quadratic. The three points are collinear and lie on the line  $y = -2x + 5$ .

4  $P(n) = an^2 + bn + c$ , where  $a, b, c \in \mathbb{R}$

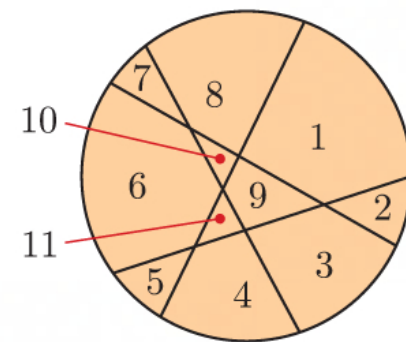
a • for  $n = 2$ , we can make a maximum of 4 pieces



• for  $n = 3$ , we can make a maximum of 7 pieces



• for  $n = 4$ , we can make a maximum of 11 pieces



$$\therefore a(2)^2 + b(2) + c = 4$$

$$a(3)^2 + b(3) + c = 7$$

$$a(4)^2 + b(4) + c = 11$$

which gives the system of equations

$$\begin{cases} 4a + 2b + c = 4 \\ 9a + 3b + c = 7 \\ 16a + 4b + c = 11 \end{cases}$$

The system has augmented matrix

$$\begin{aligned}
 & \left( \begin{array}{ccc|c} 4 & 2 & 1 & 4 \\ 9 & 3 & 1 & 7 \\ 16 & 4 & 1 & 11 \end{array} \right) \\
 & \sim \left( \begin{array}{ccc|c} 4 & 2 & 1 & 4 \\ 0 & -6 & -5 & -8 \\ 0 & -4 & -3 & -5 \end{array} \right) \quad \begin{array}{l} 4R_2 - 9R_1 \rightarrow R_2 \\ R_3 - 4R_1 \rightarrow R_3 \end{array} \quad \left\{ \begin{array}{cccc} 36 & 12 & 4 & 28 \\ -36 & -18 & -9 & -36 \\ \hline 0 & -6 & -5 & -8 \end{array} \right\} \\
 & \sim \left( \begin{array}{ccc|c} 4 & 2 & 1 & 4 \\ 0 & 1 & \frac{5}{6} & \frac{4}{3} \\ 0 & 0 & 1 & 1 \end{array} \right) \quad \begin{array}{l} -\frac{1}{6}R_2 \rightarrow R_2 \\ 3R_3 - 2R_2 \rightarrow R_3 \end{array} \quad \left\{ \begin{array}{cccc} 16 & 4 & 1 & 11 \\ -16 & -8 & -4 & -16 \\ \hline 0 & -4 & -3 & -5 \end{array} \right\} \\
 & \quad \left\{ \begin{array}{cccc} 0 & -12 & -9 & -15 \\ 0 & 12 & 10 & 16 \\ \hline 0 & 0 & 1 & 1 \end{array} \right\}
 \end{aligned}$$

Using row 3,  $c = 1$

Substituting into row 2,  $b + \frac{5}{6}(1) = \frac{4}{3}$

$$\therefore b = \frac{1}{2}$$

Substituting into row 1,  $4a + 2(\frac{1}{2}) + 1 = 4$

$$\therefore 4a = 2$$

$$\therefore a = \frac{1}{2}$$

$\therefore$  the unique solution is  $a = \frac{1}{2}$ ,  $b = \frac{1}{2}$ ,  $c = 1$ .

$\therefore P(n) = \frac{1}{2}n^2 + \frac{1}{2}n + 1$  where  $n = 0, 1, 2, \dots$

**b**  $P(12) = \frac{1}{2}(12)^2 + \frac{1}{2}(12) + 1 = 79$

$\therefore$  the maximum number of pieces that can be made using 12 cuts is 79.

**5** 
$$\begin{cases} x + 2y + z = 3 \\ 2x - y + 4z = 1 \\ x + 7y - z = k \end{cases} \quad \text{where } k \in \mathbb{R}$$

**a** The system has augmented matrix

$$\begin{aligned}
 & \left( \begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 2 & -1 & 4 & 1 \\ 1 & 7 & -1 & k \end{array} \right) \\
 & \sim \left( \begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -5 & 2 & -5 \\ 0 & 5 & -2 & k-3 \end{array} \right) \quad \begin{array}{l} R_2 - 2R_1 \rightarrow R_2 \\ R_3 - R_1 \rightarrow R_3 \end{array} \quad \left\{ \begin{array}{cccc} 2 & -1 & 4 & 1 \\ -2 & -4 & -2 & -6 \\ \hline 0 & -5 & 2 & -5 \end{array} \right\} \\
 & \sim \left( \begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -5 & 2 & -5 \\ 0 & 0 & 0 & k-8 \end{array} \right) \quad R_3 + R_2 \rightarrow R_3 \quad \left\{ \begin{array}{cccc} 1 & 7 & -1 & k \\ -1 & -2 & -1 & -3 \\ \hline 0 & 5 & -2 & k-3 \end{array} \right\} \\
 & \quad \left\{ \begin{array}{cccc} 0 & 5 & -2 & k-3 \\ 0 & -5 & 2 & -5 \\ \hline 0 & 0 & 0 & k-8 \end{array} \right\}
 \end{aligned}$$

**b** Using row 3, the system has no solutions if  $k \neq 8$ .

- c** The system has infinitely many solutions if the last row is all zeros. This occurs when  $k = 8$ . In this case we let  $z = t$ .

Using row 2,  $-5y + 2t = -5$

$$\therefore 5y = 2t + 5$$

$$\therefore y = \frac{2}{5}t + 1$$

Using row 1,  $x + 2(\frac{2}{5}t + 1) + t = 3$

$$\therefore x + \frac{9}{5}t + 2 = 3$$

$$\therefore x = 1 - \frac{9}{5}t$$

$\therefore$  the solutions have the form  $x = 1 - \frac{9}{5}t$ ,  $y = \frac{2}{5}t + 1$ ,  $z = t$ , where  $t \in \mathbb{R}$ .

- d** In row echelon form, row 3 reads  $0x + 0y + 0z = k - 8$ .

From **b** and **c**, the system has no solutions if  $k \neq 8$  and infinitely many solutions if  $k = 8$ .

$\therefore$  the system never has a unique solution.

$$6 \quad \begin{cases} x + 2y - 2z = 5 \\ x - y + 3z = -1 \\ x - 7y + kz = -k \end{cases} \quad \text{where } k \in \mathbb{R}$$

- a** The system has augmented matrix

$$\begin{aligned} & \left( \begin{array}{ccc|c} 1 & 2 & -2 & 5 \\ 1 & -1 & 3 & -1 \\ 1 & -7 & k & -k \end{array} \right) \\ & \sim \left( \begin{array}{ccc|c} 1 & 2 & -2 & 5 \\ 0 & -3 & 5 & -6 \\ 0 & -9 & k+2 & -k-5 \end{array} \right) & \begin{array}{l} R_2 - R_1 \rightarrow R_2 \\ R_3 - R_1 \rightarrow R_3 \end{array} & \left\{ \begin{array}{ccc|c} 1 & -1 & 3 & -1 \\ -1 & -2 & 2 & -5 \\ \hline 0 & -3 & 5 & -6 \end{array} \right\} \\ & \sim \left( \begin{array}{ccc|c} 1 & 2 & -2 & 5 \\ 0 & -3 & 5 & -6 \\ 0 & 0 & k-13 & -k+13 \end{array} \right) & \begin{array}{l} R_3 - 3R_2 \rightarrow R_3 \end{array} & \left\{ \begin{array}{ccc|c} 1 & -7 & k & -k \\ -1 & -2 & 2 & -5 \\ \hline 0 & -9 & k+2 & -k-5 \end{array} \right\} \\ & & & \left\{ \begin{array}{ccc|c} 0 & -9 & k+2 & -k-5 \\ 0 & 9 & -15 & 18 \\ \hline 0 & 0 & k-13 & -k+13 \end{array} \right\} \end{aligned}$$

- b** The system has infinitely many solutions if the last row is all zeros. This occurs when  $k = 13$ . In this case we let  $z = t$ .

Using row 2,  $-3y + 5t = -6$

$$\therefore 3y = 5t + 6$$

$$\therefore y = \frac{5}{3}t + 2$$

Using row 1,  $x + 2(\frac{5}{3}t + 2) - 2t = 5$

$$\therefore x + \frac{4}{3}t + 4 = 5$$

$$\therefore x = 1 - \frac{4}{3}t$$

$\therefore$  the solutions have the form  $x = 1 - \frac{4}{3}t$ ,  $y = \frac{5}{3}t + 2$ ,  $z = t$ , where  $t \in \mathbb{R}$ .



- Suppose  $k \neq 13$ .

Using row 3,  $(k - 13)z = -k + 13$

$$\begin{aligned}\therefore z &= \frac{-(k - 13)}{k - 13} \quad \{k \neq 13\} \\ &= -1\end{aligned}$$

Substituting into row 2,  $-3y + 5(-1) = -6$

$$\begin{aligned}\therefore 3y &= 1 \\ \therefore y &= \frac{1}{3}\end{aligned}$$

Substituting into row 1,  $x + 2(\frac{1}{3}) - 2(-1) = 5$

$$\therefore x = \frac{7}{3}$$

$\therefore$  the unique solution is  $x = \frac{7}{3}$ ,  $y = \frac{1}{3}$ ,  $z = -1$  for all  $k \neq 13$ ,  $k \in \mathbb{R}$ .

$$7 \quad \begin{cases} x + 3y + 3z = a - 1 \\ 2x - y + z = 7 \\ 3x - 5y + az = 16 \end{cases} \quad \text{where } a \in \mathbb{R}$$

- The system has augmented matrix

$$\begin{aligned}& \left( \begin{array}{ccc|c} 1 & 3 & 3 & a-1 \\ 2 & -1 & 1 & 7 \\ 3 & -5 & a & 16 \end{array} \right) \\ & \sim \left( \begin{array}{ccc|c} 1 & 3 & 3 & a-1 \\ 0 & -7 & -5 & 9-2a \\ 0 & -14 & a-9 & 19-3a \end{array} \right) \quad \begin{array}{l} R_2 - 2R_1 \rightarrow R_2 \\ R_3 - 3R_1 \rightarrow R_3 \end{array} \\ & \sim \left( \begin{array}{ccc|c} 1 & 3 & 3 & a-1 \\ 0 & -7 & -5 & 9-2a \\ 0 & 0 & a+1 & a+1 \end{array} \right) \quad \begin{array}{l} R_3 - 2R_2 \rightarrow R_3 \end{array} \\ & \quad \left\{ \begin{array}{ccc|c} 2 & -1 & 1 & 7 \\ -2 & -6 & -6 & -2(a-1) \\ 0 & -7 & -5 & 9-2a \end{array} \right\} \\ & \quad \left\{ \begin{array}{ccc|c} 3 & -5 & a & 16 \\ -3 & -9 & -9 & -3(a-1) \\ 0 & -14 & a-9 & 19-3a \end{array} \right\} \\ & \quad \left\{ \begin{array}{ccc|c} 0 & -14 & a-9 & 19-3a \\ 0 & 14 & 10 & -2(9-2a) \\ 0 & 0 & a+1 & a+1 \end{array} \right\}\end{aligned}$$

- The system has infinitely many solutions if the last row is all zeros. This occurs when  $a = -1$ . In this case we let  $z = t$ .

Using row 2,  $-7y - 5t = 9 - 2(-1)$

$$\begin{aligned}\therefore 7y &= -5t - 11 \\ \therefore y &= \frac{-5t - 11}{7}\end{aligned}$$

Substituting into row 1,  $x + 3\left(\frac{-5t - 11}{7}\right) + 3t = -1 - 1$

$$\begin{aligned}\therefore x + \frac{6}{7}t - \frac{33}{7} &= -2 \\ \therefore x &= \frac{19 - 6t}{7}\end{aligned}$$

$\therefore$  the solutions have the form  $x = \frac{19 - 6t}{7}$ ,  $y = \frac{-5t - 11}{7}$ ,  $z = t$ , where  $t \in \mathbb{R}$ .

- Suppose  $a \neq -1$ .

Using row 3,  $(a+1)z = a+1$

$$\begin{aligned}\therefore z &= \frac{a+1}{a+1} \quad \{a \neq -1\} \\ &= 1\end{aligned}$$

Substituting into row 2,  $-7y - 5(1) = 9 - 2a$

$$\therefore 7y = 2a - 14$$

$$\therefore y = \frac{2}{7}a - 2$$

Substituting into row 1,  $x + 3(\frac{2}{7}a - 2) + 3(1) = a - 1$

$$\therefore x + \frac{6}{7}a - 3 = a - 1$$

$$\therefore x = \frac{1}{7}a + 2$$

$\therefore$  the unique solution is  $x = \frac{1}{7}a + 2$ ,  $y = \frac{2}{7}a - 2$ ,  $z = 1$ , for  $a \neq -1$ ,  $a \in \mathbb{R}$ .

$$8 \quad \begin{cases} x + 4y + mz = -m \\ (m+1)x + 4y + z = 1 \\ 4x + 4y + z = 1 \end{cases} \quad \text{where } m \in \mathbb{R}, m \neq 0$$

- a The system has augmented matrix

$$\begin{aligned} & \left( \begin{array}{ccc|c} 1 & 4 & m & -m \\ m+1 & 4 & 1 & 1 \\ 4 & 4 & 1 & 1 \end{array} \right) & \left\{ \begin{array}{cccc} m+1 & 4 & 1 & 1 \\ -(m+1) & -4(m+1) & -m(m+1) & m(m+1) \\ 0 & -4m & 1-m^2-m & m^2+m+1 \end{array} \right\} \\ & \sim \left( \begin{array}{ccc|c} 1 & 4 & m & -m \\ 0 & -4m & 1-m^2-m & m^2+m+1 \\ 0 & -12 & 1-4m & 1+4m \end{array} \right) & \begin{array}{l} R_2 - (m+1)R_1 \rightarrow R_2 \\ R_3 - 4R_1 \rightarrow R_3 \end{array} \\ & & \left\{ \begin{array}{cccc} 4 & 4 & 1 & 1 \\ -4 & -16 & -4m & 4m \\ 0 & -12 & 1-4m & 1+4m \end{array} \right\} \\ & \sim \left( \begin{array}{ccc|c} 1 & 4 & m & -m \\ 0 & -4m & 1-m^2-m & m^2+m+1 \\ 0 & 0 & -(m-3)(m-1) & (m-3)(m+1) \end{array} \right) & mR_3 - 3R_2 \rightarrow R_3 \\ & & \left\{ \begin{array}{cccc} 0 & -12m & m(1-4m) & m(1+4m) \\ 0 & 12m & -3(1-m^2-m) & -3(m^2+m+1) \\ 0 & 0 & -m^2+4m-3 & m^2-2m-3 \\ & & = -(m-3)(m-1) & = (m-3)(m+1) \end{array} \right\} \end{aligned}$$

- b If  $m = 1$ , the last row gives  $0x + 0y + 0z = -4$  which is not possible.

$\therefore$  there are no solutions if  $m = 1$ .

- The system has infinitely many solutions if the last row is all zeros. This occurs when  $m = 3$ . In this case we let  $z = t$ .

Using row 2,  $-4(3)y + (1 - 3^2 - 3)t = 3^2 + 3 + 1$

$$\therefore -12y - 11t = 13$$

$$\therefore 12y = -13 - 11t$$

$$\therefore y = \frac{-13 - 11t}{12}$$

Substituting into row 1,  $x + 4\left(\frac{-13 - 11t}{12}\right) + 3t = -3$

$$\therefore x - \frac{2}{3}t - \frac{13}{3} = -3$$

$$\therefore x = \frac{4 + 2t}{3}$$

$\therefore$  the solutions have the form  $x = \frac{4 + 2t}{3}$ ,  $y = \frac{-13 - 11t}{12}$ ,  $z = t$ , where  $t \in \mathbb{R}$ .

- Suppose  $m \neq 1$  or  $3$ .

Using row 3,  $-(m - 3)(m - 1)z = (m - 3)(m + 1)$

$$\begin{aligned}\therefore z &= \frac{(m - 3)(m + 1)}{-(m - 3)(m - 1)} \quad \{m \neq 1 \text{ or } 3\} \\ &= \frac{m + 1}{1 - m}\end{aligned}$$

Substituting into row 2,  $-4my + (1 - m^2 - m)\left(\frac{m + 1}{1 - m}\right) = m^2 + m + 1$

$$\therefore -4m(1 - m)y + (1 - m^2 - m)(m + 1) = (m^2 + m + 1)(1 - m)$$

$$\therefore 4m(m - 1)y + m + 1 - m^3 - m^2 - m^2 - m = m^2 - m^3 + m - m^2 + 1 - m$$

$$\therefore 4m(m - 1)y = 2m^2$$

$$\begin{aligned}\therefore y &= \frac{2m^2}{4m(m - 1)} \quad \{m \neq 1\} \\ &= \frac{m}{2(m - 1)}\end{aligned}$$

Substituting into row 1,  $x + 4\left(\frac{m}{2(m - 1)}\right) + m\left(\frac{m + 1}{1 - m}\right) = -m$

$$\therefore x + \frac{2m - m(m + 1)}{m - 1} = -m$$

$$\therefore x + \frac{2m - m^2 - m}{m - 1} = -m$$

$$\therefore x + \frac{m - m^2}{m - 1} = -m$$

$$\therefore x - \frac{m(m - 1)}{m - 1} = -m$$

$$\therefore x - m = -m$$

$$\therefore x = 0$$

$\therefore$  the unique solution is  $x = 0$ ,  $y = \frac{m}{2(m - 1)}$ ,  $z = \frac{m + 1}{1 - m}$ , for  $m \in \mathbb{R}$ ,  $m \neq 1$  or  $3$ .

# ACTIVITY

1 a 
$$\begin{cases} 2x + 3y = 1 \\ x - 2y = 11 \end{cases}$$

	a	b	c
1	2	3	1
2	1	-2	11

11

	a	b	c
X	5		
Y	-3		

5

So, the system has unique solution  $x = 5$ ,  $y = -3$ .

b 
$$\begin{cases} 3x - y + 2z = 18 \\ y + 4z = 18 \\ 2x + 3y - 2z = -12 \end{cases}$$

	a	b	c	d
1	3	-1	2	18
2	0	1	4	18
3	2	3	-2	-12

-12

	a	b	c	d
X	2			
Y	-2			
Z	5			

2

So, the system has unique solution  $x = 2$ ,  $y = -2$ ,  $z = 5$ .

c 
$$\begin{cases} x - 2y - 4z = 3 \\ 5x + 3y + z = 4 \\ -2x + y - 3z = 15 \end{cases}$$

	a	b	c	d
1	1	-2	-4	3
2	5	3	1	4
3	-2	1	-3	15

15

	a	b	c	d
X	-1			
Y	4			
Z	-3			

-1

So, the system has unique solution  $x = -1$ ,  $y = 4$ ,  $z = -3$ .

d 
$$\begin{cases} x - 2y + 5z = -1 \\ 3x + y - 4z = 5 \\ 5x - 3y + 6z = -2 \end{cases}$$

	a	b	c	d
1	1	-2	5	-1
2	3	1	-4	5
3	5	-3	6	-2

-2

	a	b	c	d
No Solution				

So, the system has no solutions.

e 
$$\begin{cases} 3x + y + 2z = -7 \\ -2x + 8y - 7z = 0 \\ 2x - 3z = 10 \end{cases}$$

	a	b	c	d
1	3	1	2	-7
2	-2	8	-7	0
3	2	0	-3	10

10

	a	b	c	d
X	0.5			
Y	-2.5			
Z	-3			

$\frac{1}{2}$

So, the system has unique solution  $x = \frac{1}{2}$ ,  $y = -\frac{5}{2}$ ,  $z = -3$ .



$$\mathbf{f} \quad \begin{cases} -4x + y + 6z = 1 \\ x + 5y + 2z = -2 \\ 6x + 9y - 2z = -5 \end{cases}$$

	a	b	c	d
1	-4	1	6	1
2	1	5	2	-2
3	6	9	-2	-5

	a	b	c	d
1	-4	1	6	1
2	1	5	2	-2
3	6	9	-2	-5

	a	b	c	d
1	-4	1	6	1
2	1	5	2	-2
3	6	9	-2	-5

So, the system has infinitely many solutions. Letting  $z = t$ , the solutions are of the form  $x = -\frac{1}{3} + \frac{4}{3}t$ ,  $y = -\frac{1}{3} - \frac{2}{3}t$ ,  $z = t$ , where  $t \in \mathbb{R}$ .

- 2** The questions involving unknowns cannot directly be answered using a graphics calculator. These are questions **5** to **8** in the previous Exercise.
- 3** The augmented matrix of a system of equations is in reduced row echelon form if it is in row echelon form *and* every entry above a non-zero entry is zero.

For example, consider the equation 
$$\begin{cases} x + y + 3z = 1 \\ x + 2y + 5z = 3 \\ 2x + 2y + 7z = 5 \end{cases}$$

Using row operations, the augmented matrix is

$$\begin{pmatrix} 1 & 1 & 3 & 1 \\ 1 & 2 & 5 & 3 \\ 2 & 2 & 7 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 3 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 3 \end{pmatrix} \quad \text{which is in row echelon form but *not* in reduced row echelon form}$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 3 \end{pmatrix} \quad \text{which is in row echelon form *and* reduced row echelon form}$$

Reduced row echelon form is particularly useful for systems with a unique solution as we can read the solution from the right hand side. In the example above, the unique solution is  $x = -4$ ,  $y = -4$ ,  $z = 3$ .

## REVIEW SET 11A

**1 a**  $5x + y = -2$

Let  $y = t$

$\therefore 5x + t = -2$

$\therefore 5x = -2 - t$

$\therefore x = \frac{-2-t}{5}$

$\therefore$  a solution set is  $x = \frac{-2-t}{5}$ ,  $y = t$ ,  
where  $t \in \mathbb{R}$ .

**b**  $x - 4y + 3z = 1$

Let  $y = s$  and  $z = t$

$\therefore x - 4s + 3t = 1$

$\therefore x = 1 + 4s - 3t$

$\therefore$  a solution set is  $x = 1 + 4s - 3t$ ,  
 $y = s$ ,  $z = t$ , where  $s, t \in \mathbb{R}$ .

**2 a** The system  $\begin{cases} 3x - 2y = 4 \\ x + 5y = -8 \end{cases}$  has augmented matrix  $\left( \begin{array}{cc|c} 3 & -2 & 4 \\ 1 & 5 & -8 \end{array} \right)$ .

This system is neither underspecified nor overspecified as there are 2 equations in 2 unknowns.

**b** The system  $\begin{cases} 2x - y + 2z = -1 \\ 3x - 2y = 6 \end{cases}$  has augmented matrix  $\left( \begin{array}{ccc|c} 2 & -1 & 2 & -1 \\ 3 & -2 & 0 & 6 \end{array} \right)$ .

This system is underspecified as there are 2 equations in 3 unknowns.

**c** The system  $\begin{cases} x - 4y = 0 \\ 5x - y = 2 \\ -3x + y = -7 \end{cases}$  has augmented matrix  $\left( \begin{array}{cc|c} 1 & -4 & 0 \\ 5 & -1 & 2 \\ -3 & 1 & -7 \end{array} \right)$ .

This system is overspecified as there are 3 equations in 2 unknowns.

**3**  $\begin{cases} 4x - 6y = -1 \\ ax + 2y = 3 \end{cases}$

**a** The system has augmented matrix  $\left( \begin{array}{cc|c} 4 & -6 & -1 \\ a & 2 & 3 \end{array} \right)$ .

**b**  $\left( \begin{array}{cc|c} 4 & -6 & -1 \\ a & 2 & 3 \end{array} \right) \sim \left( \begin{array}{cc|c} 4 & -6 & -1 \\ -3a & -6 & -9 \end{array} \right) \quad -3R_2 \rightarrow R_2$

**c** The system is inconsistent for  $a = -\frac{4}{3}$  as  $4x - 6y$  cannot be equal to  $-1$  and  $-9$  simultaneously.

**4 a**  $\begin{cases} x + 2y = 3 \\ 3x + 5y = 11 \end{cases}$

In augmented matrix form, the system is

$$\begin{aligned} & \left( \begin{array}{cc|c} 1 & 2 & 3 \\ 3 & 5 & 11 \end{array} \right) \\ & \sim \left( \begin{array}{cc|c} 1 & 2 & 3 \\ 0 & -1 & 2 \end{array} \right) \quad R_2 - 3R_1 \rightarrow R_2 \quad \left\{ \begin{array}{ccc} 3 & 5 & 11 \\ -3 & -6 & -9 \\ 0 & -1 & 2 \end{array} \right\} \\ & \sim \left( \begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 1 & -2 \end{array} \right) \quad -R_2 \rightarrow R_2 \end{aligned}$$

Using row 2,  $y = -2$

Substituting into row 1,  $x + 2(-2) = 3$   
 $\therefore x = 7$

$\therefore$  the solution is  $x = 7, y = -2$ .

$$\mathbf{b} \quad \begin{cases} 5x + y = -1 \\ x - 2y = -9 \end{cases}$$

In augmented matrix form, the system is

$$\begin{aligned} & \left( \begin{array}{cc|c} 5 & 1 & -1 \\ 1 & -2 & -9 \end{array} \right) \\ & \sim \left( \begin{array}{cc|c} 5 & 1 & -1 \\ 0 & -11 & -44 \end{array} \right) \quad 5R_2 - R_1 \rightarrow R_2 \quad \left\{ \begin{array}{ccc} 5 & -10 & -45 \\ -5 & -1 & 1 \\ \hline 0 & -11 & -44 \end{array} \right\} \\ & \sim \left( \begin{array}{cc|c} 5 & 1 & -1 \\ 0 & 1 & 4 \end{array} \right) \quad -\frac{1}{11}R_2 \rightarrow R_2 \end{aligned}$$

Using row 2,  $y = 4$

Substituting into row 1,  $5x + 4 = -1$

$$\therefore 5x = -5$$

$$\therefore x = -1$$

$\therefore$  the solution is  $x = -1$ ,  $y = 4$ .

$$\mathbf{5} \quad \begin{cases} 2x - 5y = 3 \\ 6x + ky = 9 \end{cases} \quad \text{where } k \in \mathbb{R}$$

In augmented matrix form, the system is

$$\begin{aligned} & \left( \begin{array}{cc|c} 2 & -5 & 3 \\ 6 & k & 9 \end{array} \right) \\ & \sim \left( \begin{array}{cc|c} 2 & -5 & 3 \\ 0 & k+15 & 0 \end{array} \right) \quad R_2 - 3R_1 \rightarrow R_2 \quad \left\{ \begin{array}{ccc} 6 & k & 9 \\ -6 & 15 & -9 \\ \hline 0 & k+15 & 0 \end{array} \right\} \end{aligned}$$

**a** For a unique solution, the left hand side of the second row must not all be zero.

$$\therefore k + 15 \neq 0$$

$$\therefore k \neq -15$$

In this case, using row 2,  $(k + 15)y = 0$

$$\therefore y = 0 \quad \{k \neq -15\}$$

Substituting into row 1,  $2x - 5(0) = 3$

$$\therefore 2x = 3$$

$$\therefore x = \frac{3}{2}$$

$\therefore$  the unique solution is  $x = \frac{3}{2}$ ,  $y = 0$ .

**b** If  $k = -15$ , the last row is all zeros and so the system has infinitely many solutions.

In this case we let  $y = t$ .

Using row 1,  $2x - 5t = 3$

$$\therefore 2x = 3 + 5t$$

$$\therefore x = \frac{3 + 5t}{2}$$

$\therefore$  the solutions have the form  $x = \frac{3 + 5t}{2}$ ,  $y = t$ , where  $t \in \mathbb{R}$ .

$$6 \quad a \quad \begin{cases} x - 2y + 5z = 1 \\ 2x - 4y + 8z = 2 \\ -3x + 6y + 7z = -3 \end{cases}$$

The system has augmented matrix

$$\begin{aligned} & \left( \begin{array}{ccc|c} 1 & -2 & 5 & 1 \\ 2 & -4 & 8 & 2 \\ -3 & 6 & 7 & -3 \end{array} \right) \\ & \sim \left( \begin{array}{ccc|c} 1 & -2 & 5 & 1 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 22 & 0 \end{array} \right) \quad \begin{array}{l} R_2 - 2R_1 \rightarrow R_2 \\ R_3 + 3R_1 \rightarrow R_3 \end{array} \quad \left\{ \begin{array}{ccc|c} 2 & -4 & 8 & 2 \\ -2 & 4 & -10 & -2 \\ \hline 0 & 0 & -2 & 0 \end{array} \right\} \\ & \sim \left( \begin{array}{ccc|c} 1 & -2 & 5 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \quad \begin{array}{l} -\frac{1}{2}R_2 \rightarrow R_2 \\ R_3 + 11R_2 \rightarrow R_3 \end{array} \quad \left\{ \begin{array}{ccc|c} -3 & 6 & 7 & -3 \\ 3 & -6 & 15 & 3 \\ \hline 0 & 0 & 22 & 0 \end{array} \right\} \\ & \quad \left\{ \begin{array}{ccc|c} 0 & 0 & 22 & 0 \\ 0 & 0 & -22 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array} \right\} \end{aligned}$$

Row 3 indicates there are infinitely many solutions.

If we let  $y = t$ , then using row 2,  $z = 0$

Substituting into row 1,  $x - 2t + 5(0) = 1$

$$\therefore x = 1 + 2t$$

$\therefore$  the solutions have the form  $x = 1 + 2t$ ,  $y = t$ ,  $z = 0$ , where  $t \in \mathbb{R}$ .

$$b \quad \begin{cases} x + 4y + 11z = 7 \\ x + 6y + 17z = 9 \\ x + 4y + 8z = 4 \end{cases}$$

The system has augmented matrix

$$\begin{aligned} & \left( \begin{array}{ccc|c} 1 & 4 & 11 & 7 \\ 1 & 6 & 17 & 9 \\ 1 & 4 & 8 & 4 \end{array} \right) \\ & \sim \left( \begin{array}{ccc|c} 1 & 4 & 11 & 7 \\ 0 & 2 & 6 & 2 \\ 0 & 0 & -3 & -3 \end{array} \right) \quad \begin{array}{l} R_2 - R_1 \rightarrow R_2 \\ R_3 - R_1 \rightarrow R_3 \end{array} \quad \left\{ \begin{array}{ccc|c} 1 & 6 & 17 & 9 \\ -1 & -4 & -11 & -7 \\ \hline 0 & 2 & 6 & 2 \end{array} \right\} \\ & \sim \left( \begin{array}{ccc|c} 1 & 4 & 11 & 7 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 1 \end{array} \right) \quad \begin{array}{l} \frac{1}{2}R_2 \rightarrow R_2 \\ -\frac{1}{3}R_3 \rightarrow R_3 \end{array} \quad \left\{ \begin{array}{ccc|c} 1 & 4 & 8 & 4 \\ -1 & -4 & -11 & -7 \\ \hline 0 & 0 & -3 & -3 \end{array} \right\} \end{aligned}$$

Using row 3,  $z = 1$

Substituting into row 2,  $y + 3(1) = 1$

$$\therefore y = -2$$

Substituting into row 1,  $x + 4(-2) + 11(1) = 7$

$$\therefore x = 4$$

$\therefore$  the unique solution is  $x = 4$ ,  $y = -2$ ,  $z = 1$ .



$$\text{c } \begin{cases} 2x + 3y + 4z = 1 \\ 5x + 6y + 7z = 2 \\ 8x + 9y + 10z = 3 \end{cases}$$

The system has augmented matrix

$$\begin{aligned} & \left( \begin{array}{ccc|c} 2 & 3 & 4 & 1 \\ 5 & 6 & 7 & 2 \\ 8 & 9 & 10 & 3 \end{array} \right) \\ & \sim \left( \begin{array}{ccc|c} 2 & 3 & 4 & 1 \\ 0 & -3 & -6 & -1 \\ 0 & -3 & -6 & -1 \end{array} \right) \quad \begin{array}{l} 2R_2 - 5R_1 \rightarrow R_2 \\ R_3 - 4R_1 \rightarrow R_3 \end{array} \quad \left\{ \begin{array}{cccc} 10 & 12 & 14 & 4 \\ -10 & -15 & -20 & -5 \\ \hline 0 & -3 & -6 & -1 \end{array} \right\} \\ & \sim \left( \begin{array}{ccc|c} 2 & 3 & 4 & 1 \\ 0 & 1 & 2 & \frac{1}{3} \\ 0 & 0 & 0 & 0 \end{array} \right) \quad \begin{array}{l} -\frac{1}{3}R_2 \rightarrow R_2 \\ R_3 - R_2 \rightarrow R_3 \end{array} \quad \left\{ \begin{array}{cccc} 8 & 9 & 10 & 3 \\ -8 & -12 & -16 & -4 \\ \hline 0 & -3 & -6 & -1 \end{array} \right\} \\ & \quad \left\{ \begin{array}{cccc} 0 & -3 & -6 & -1 \\ 0 & 3 & 6 & 1 \\ \hline 0 & 0 & 0 & 0 \end{array} \right\} \end{aligned}$$

Row 3 indicates there are infinitely many solutions.

If we let  $z = t$ , then using row 2,  $y + 2t = \frac{1}{3}$   
 $\therefore y = \frac{1}{3} - 2t$

Substituting into row 1,  $2x + 3(\frac{1}{3} - 2t) + 4t = 1$   
 $\therefore 2x - 2t + 1 = 1$   
 $\therefore 2x = 2t$   
 $\therefore x = t$

$\therefore$  the unique solution is  $x = t$ ,  $y = \frac{1}{3} - 2t$ ,  $z = t$ , where  $t \in \mathbb{R}$ .

- 7 a** The rational function passes through  $(-3, -13)$ ,  $(0, 2)$ , and  $(1, \frac{1}{3})$ .

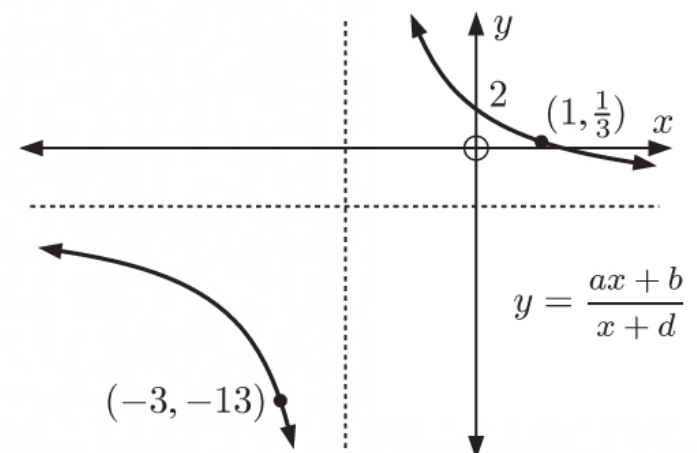
$$\therefore \frac{a(-3) + b}{-3 + d} = -13 \quad \text{or} \quad -3a + b + 13d = 39$$

$$\frac{a(0) + b}{0 + d} = 2 \quad \text{or} \quad b - 2d = 0$$

$$\frac{a(1) + b}{1 + d} = \frac{1}{3} \quad \text{or} \quad 3a + 3b - d = 1$$

So, we have the system of equations

$$\begin{cases} -3a + b + 13d = 39 \\ b - 2d = 0 \\ 3a + 3b - d = 1 \end{cases}$$



**b** This system has augmented matrix

$$\begin{aligned}
 & \left( \begin{array}{ccc|c} -3 & 1 & 13 & 39 \\ 0 & 1 & -2 & 0 \\ 3 & 3 & -1 & 1 \end{array} \right) \\
 & \sim \left( \begin{array}{ccc|c} -3 & 1 & 13 & 39 \\ 0 & 1 & -2 & 0 \\ 0 & 4 & 12 & 40 \end{array} \right) \quad R_3 + R_1 \rightarrow R_3 \leftarrow \left\{ \begin{array}{ccc|c} 3 & 3 & -1 & 1 \\ -3 & 1 & 13 & 39 \\ \hline 0 & 4 & 12 & 40 \end{array} \right\} \\
 & \sim \left( \begin{array}{ccc|c} -3 & 1 & 13 & 39 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 20 & 40 \end{array} \right) \quad R_3 - 4R_2 \rightarrow R_3 \leftarrow \left\{ \begin{array}{ccc|c} 0 & 4 & 12 & 40 \\ 0 & -4 & 8 & 0 \\ \hline 0 & 0 & 20 & 40 \end{array} \right\} \\
 & \sim \left( \begin{array}{ccc|c} -3 & 1 & 13 & 39 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 2 \end{array} \right) \quad \frac{1}{20}R_3 \rightarrow R_3
 \end{aligned}$$

Using row 3,  $d = 2$

Substituting into row 2,  $b - 2(2) = 0$   
 $\therefore b = 4$

Substituting into row 1,  $-3a + 4 + 13(2) = 39$   
 $\therefore -3a = 9$   
 $\therefore a = -3$

$\therefore$  the unique solution is  $a = -3$ ,  $b = 4$ ,  $d = 2$ .

$\therefore$  the rational function has equation  $y = \frac{-3x + 4}{x + 2}$ .

**8**  $\begin{cases} x - 3y + 2z = -7 \\ 2x + y - z = 5 \\ 5x - 8y + kz = 12 \end{cases} \quad \text{where } k \in \mathbb{R}$

**a** The system has augmented matrix

$$\begin{aligned}
 & \left( \begin{array}{ccc|c} 1 & -3 & 2 & -7 \\ 2 & 1 & -1 & 5 \\ 5 & -8 & k & 12 \end{array} \right) \\
 & \sim \left( \begin{array}{ccc|c} 1 & -3 & 2 & -7 \\ 0 & 7 & -5 & 19 \\ 0 & 7 & k-10 & 47 \end{array} \right) \quad \begin{array}{l} R_2 - 2R_1 \rightarrow R_2 \\ R_3 - 5R_1 \rightarrow R_3 \end{array} \leftarrow \left\{ \begin{array}{ccc|c} 2 & 1 & -1 & 5 \\ -2 & 6 & -4 & 14 \\ \hline 0 & 7 & -5 & 19 \end{array} \right\} \\
 & \sim \left( \begin{array}{ccc|c} 1 & -3 & 2 & -7 \\ 0 & 7 & -5 & 19 \\ 0 & 0 & k-5 & 28 \end{array} \right) \quad \begin{array}{l} R_3 - R_2 \rightarrow R_3 \end{array} \leftarrow \left\{ \begin{array}{ccc|c} 5 & -8 & k & 12 \\ -5 & 15 & -10 & 35 \\ \hline 0 & 7 & k-10 & 47 \end{array} \right\} \\
 & \quad \quad \quad \leftarrow \left\{ \begin{array}{ccc|c} 0 & 7 & k-10 & 47 \\ 0 & -7 & 5 & -19 \\ \hline 0 & 0 & k-5 & 28 \end{array} \right\}
 \end{aligned}$$

**b** If  $k = 5$ , the last row gives  $0x + 0y + 0z = 28$ , which is not possible.

$\therefore$  there are no solutions if  $k = 5$ .

- c i** Suppose  $k \neq 5$ .

Using row 3,  $(k-5)z = 28$

$$\therefore z = \frac{28}{k-5} \quad \{k \neq 5\}$$

Substituting into row 2,  $7y - 5\left(\frac{28}{k-5}\right) = 19$

$$\therefore 7y = 19 + \frac{140}{k-5}$$

$$\therefore y = \frac{19}{7} + \frac{20}{k-5}$$

Substituting into row 1,  $x - 3\left(\frac{19}{7} + \frac{20}{k-5}\right) + 2\left(\frac{28}{k-5}\right) = -7$

$$\therefore x - \frac{57}{7} - \frac{4}{k-5} = -7$$

$$\therefore x = \frac{8}{7} + \frac{4}{k-5}$$

$\therefore$  the unique solution is  $x = \frac{8}{7} + \frac{4}{k-5}$ ,  $y = \frac{19}{7} + \frac{20}{k-5}$ ,  $z = \frac{28}{k-5}$ , for  $k \neq 5$ ,  $k \in \mathbb{R}$ .

- ii** If  $k = 3$ , the unique solution is  $x = \frac{8}{7} + \frac{4}{3-5}$ ,  $y = \frac{19}{7} + \frac{20}{3-5}$ ,  $z = \frac{28}{3-5}$   
which is  $x = -\frac{6}{7}$ ,  $y = -\frac{51}{7}$ ,  $z = -14$ .

## REVIEW SET 11B

**1 a** 
$$\begin{cases} 3x + y = 2 \\ 3x - y = 2 \end{cases}$$

This system is consistent as  $x = \frac{2}{3}$ ,  $y = 0$  is a solution.

Check:  $3(\frac{2}{3}) + 0 = 2$  and  $3(\frac{2}{3}) - 0 = 2$  ✓

**b** 
$$\begin{cases} x + 4y - z = 1 \\ -x - 4y + z = 1 \end{cases} \text{ which is } \begin{cases} x + 4y - z = 1 \\ x + 4y - z = -1 \end{cases}$$

This system is inconsistent as  $x + 4y - z$  cannot be equal to both 1 and  $-1$  simultaneously.

**2 a** 
$$\left( \begin{array}{ccc|c} 5 & -1 & 4 & 3 \\ 1 & 0 & -2 & -1 \end{array} \right)$$
 is the augmented matrix for 
$$\begin{cases} 5x - y + 4z = 3 \\ x - 2z = -1 \end{cases}$$

**b** 
$$\left( \begin{array}{ccc|c} 2 & 0 & -3 & -2 \\ -1 & 6 & 0 & 3 \\ 3 & 3 & -4 & -7 \end{array} \right)$$
 is the augmented matrix for 
$$\begin{cases} 2x - 3z = -2 \\ -x + 6y = 3 \\ 3x + 3y - 4z = -7 \end{cases}$$

$$3 \quad \begin{cases} x - y + 3z = a \\ 5x + y - z = b \\ 2x + 4y - 10z = c \end{cases} \quad \text{where } a, b, c \in \mathbb{R}$$

$$a \quad \left( \begin{array}{ccc|c} 1 & -1 & 3 & a \\ 5 & 1 & -1 & b \\ 2 & 4 & -10 & c \end{array} \right)$$

$$b \quad \left( \begin{array}{ccc|c} 1 & -1 & 3 & a \\ 5 & 1 & -1 & b \\ 2 & 4 & -10 & c \end{array} \right) \sim \left( \begin{array}{ccc|c} 1 & -1 & 3 & a \\ 2 & 4 & -10 & b - 3a \\ 2 & 4 & -10 & c \end{array} \right) \quad R_2 - 3R_1 \rightarrow R_2 \quad \left\{ \begin{array}{ccc|c} 5 & 1 & -1 & b \\ -3 & 3 & -9 & -3a \\ \hline 2 & 4 & -10 & b - 3a \end{array} \right\}$$

c The system is consistent if  $b - 3a = c$ .

$$4 \quad \begin{cases} 4x + y = -1 \\ -8x - 2y = a \end{cases} \quad \text{where } a \in \mathbb{R}$$

In augmented matrix form, the system is

$$\left( \begin{array}{cc|c} 4 & 1 & -1 \\ -8 & -2 & a \end{array} \right) \sim \left( \begin{array}{cc|c} 4 & 1 & -1 \\ 0 & 0 & a - 2 \end{array} \right) \quad R_2 + 2R_1 \rightarrow R_2 \quad \left\{ \begin{array}{cc|c} -8 & -2 & a \\ 8 & 2 & -2 \\ \hline 0 & 0 & a - 2 \end{array} \right\}$$

a If  $a = 2$ , the last row is all zeros and so the system has infinitely many solutions. In this case we let  $y = t$ .

Using row 1,  $4x + t = -1$

$$\therefore 4x = -t - 1$$

$$\therefore x = -\left(\frac{t+1}{4}\right)$$

$\therefore$  the solutions have the form  $x = -\left(\frac{t+1}{4}\right)$ ,  $y = t$ , where  $t \in \mathbb{R}$ .

b When  $a \neq 2$ ,  $a - 2 \neq 0$  and the second row gives  $0x + 0y \neq 0$  which is not possible.  $\therefore$  the system has no solutions if  $a \neq 2$ .

$$5 \quad \begin{cases} x - 2y = 5 \\ 4x + ay = b \end{cases} \quad \text{where } a, b \in \mathbb{R}$$

In augmented matrix form, the system is

$$\left( \begin{array}{cc|c} 1 & -2 & 5 \\ 4 & a & b \end{array} \right) \sim \left( \begin{array}{cc|c} 1 & -2 & 5 \\ 0 & a+8 & b-20 \end{array} \right) \quad R_2 - 4R_1 \rightarrow R_2 \quad \left\{ \begin{array}{cc|c} 4 & a & b \\ -4 & 8 & -20 \\ \hline 0 & a+8 & b-20 \end{array} \right\}$$



There are 3 cases to consider:

- If  $a = -8$ ,  $b = 20$ , the last row is all zeros and so the system has infinitely many solutions. In this case we let  $y = t$ .

Using row 1,  $x - 2t = 5$

$$\therefore x = 5 + 2t$$

$\therefore$  the solutions have the form  $x = 5 + 2t$ ,  $y = t$ , where  $t \in \mathbb{R}$ .

- If  $a = -8$ ,  $b \neq 20$ ,  $b - 20 \neq 0$  and the last row gives  $0x + 0y \neq 0$ , which is not possible.  $\therefore$  the system has no solutions.

- If  $a \neq -8$ , then using row 2,  $(a + 8)y = b - 20$

$$\therefore y = \frac{b - 20}{a + 8} \quad \{a \neq -8\}$$

Substituting into row 1,  $x - 2\left(\frac{b - 20}{a + 8}\right) = 5$

$$\therefore x = 5 + 2\left(\frac{b - 20}{a + 8}\right)$$

$\therefore$  the system has a unique solution  $x = 5 + 2\left(\frac{b - 20}{a + 8}\right)$ ,  $y = \frac{b - 20}{a + 8}$ .

6 a 
$$\begin{cases} 2x - y + 3z = 17 \\ 2x - 2y - 5z = 4 \\ 3x + 2y + 2z = 10 \end{cases}$$

This system has augmented matrix

$$\begin{aligned} & \left( \begin{array}{ccc|c} 2 & -1 & 3 & 17 \\ 2 & -2 & -5 & 4 \\ 3 & 2 & 2 & 10 \end{array} \right) \\ & \sim \left( \begin{array}{ccc|c} 2 & -1 & 3 & 17 \\ 0 & -1 & -8 & -13 \\ 0 & 7 & -5 & -31 \end{array} \right) & \begin{array}{l} R_2 - R_1 \rightarrow R_2 \\ 2R_3 - 3R_2 \rightarrow R_3 \end{array} & \left\{ \begin{array}{ccc|c} 2 & -2 & -5 & 4 \\ -2 & 1 & -3 & -17 \\ 0 & -1 & -8 & -13 \end{array} \right\} \\ & \sim \left( \begin{array}{ccc|c} 2 & -1 & 3 & 17 \\ 0 & -1 & -8 & -13 \\ 0 & 0 & -61 & -122 \end{array} \right) & & \left\{ \begin{array}{ccc|c} 6 & 4 & 4 & 20 \\ -6 & 3 & -9 & -51 \\ 0 & 7 & -5 & -31 \end{array} \right\} \\ & \sim \left( \begin{array}{ccc|c} 2 & -1 & 3 & 17 \\ 0 & 1 & 8 & 13 \\ 0 & 0 & 1 & 2 \end{array} \right) & \begin{array}{l} R_3 + 7R_2 \rightarrow R_3 \\ -R_2 \rightarrow R_2 \\ -\frac{1}{61}R_3 \rightarrow R_3 \end{array} & \left\{ \begin{array}{ccc|c} 0 & 7 & -5 & -31 \\ 0 & -7 & -56 & -91 \\ 0 & 0 & -61 & -122 \end{array} \right\} \end{aligned}$$

Using row 3,  $z = 2$

Substituting into row 2,  $y + 8(2) = 13$

$$\therefore y = -3$$

Substituting into row 1,  $2x - (-3) + 3(2) = 17$

$$\therefore 2x = 8$$

$$\therefore x = 4$$

$\therefore$  the unique solution is  $x = 4$ ,  $y = -3$ ,  $z = 2$ .

$$\text{b } \begin{cases} x - 3y + z = -2 \\ 3x + y - 2z = -5 \\ 2x + 4y - 3z = 4 \end{cases}$$

This system has augmented matrix

$$\begin{aligned} & \left( \begin{array}{ccc|c} 1 & -3 & 1 & -2 \\ 3 & 1 & -2 & -5 \\ 2 & 4 & -3 & 4 \end{array} \right) \\ & \sim \left( \begin{array}{ccc|c} 1 & -3 & 1 & -2 \\ 0 & 10 & -5 & 1 \\ 0 & 10 & -5 & 8 \end{array} \right) \quad \begin{array}{l} R_2 - 3R_1 \rightarrow R_2 \\ R_3 - 2R_1 \rightarrow R_3 \end{array} \quad \left\{ \begin{array}{ccc|c} 3 & 1 & -2 & -5 \\ -3 & 9 & -3 & 6 \\ \hline 0 & 10 & -5 & 1 \end{array} \right\} \\ & \sim \left( \begin{array}{ccc|c} 1 & -3 & 1 & -2 \\ 0 & 10 & -5 & 1 \\ 0 & 0 & 0 & 7 \end{array} \right) \quad \begin{array}{l} R_3 - R_2 \rightarrow R_3 \end{array} \quad \left\{ \begin{array}{ccc|c} 2 & 4 & -3 & 4 \\ -2 & 6 & -2 & 4 \\ \hline 0 & 10 & -5 & 8 \end{array} \right\} \\ & \quad \left\{ \begin{array}{ccc|c} 0 & 10 & -5 & 8 \\ 0 & -10 & 5 & -1 \\ \hline 0 & 0 & 0 & 7 \end{array} \right\} \end{aligned}$$

Row 3 means that  $0x + 0y + 0z = 7$ , which is absurd.

$\therefore$  there is no solution, and the system is inconsistent.

**7 a** The points  $(-2, 4)$  and  $(1, 3)$  lie on the circle  $x^2 + y^2 + ax + by + c = 0$ .

$$\therefore (-2)^2 + 4^2 + a(-2) + b(4) + c = 0$$

$$\text{and } 1^2 + 3^2 + a(1) + b(3) + c = 0$$

which gives the system of equations  $\begin{cases} -2a + 4b + c = -20 \\ a + 3b + c = -10 \end{cases}$

This system has augmented matrix

$$\begin{aligned} & \left( \begin{array}{ccc|c} -2 & 4 & 1 & -20 \\ 1 & 3 & 1 & -10 \end{array} \right) \\ & \sim \left( \begin{array}{ccc|c} -2 & 4 & 1 & -20 \\ 0 & 10 & 3 & -40 \end{array} \right) \quad 2R_2 + R_1 \rightarrow R_2 \quad \left\{ \begin{array}{ccc|c} 2 & 6 & 2 & -20 \\ -2 & 4 & 1 & -20 \\ \hline 0 & 10 & 3 & -40 \end{array} \right\} \end{aligned}$$

If we let  $c = t$ , then using row 2,  $10b + 3t = -40$

$$\therefore 10b = -40 - 3t$$

$$\therefore b = -4 - \frac{3}{10}t$$

Substituting into row 1,  $-2a + 4(-4 - \frac{3}{10}t) + t = -20$

$$\therefore -2a - 16 - \frac{1}{5}t = -20$$

$$\therefore -2a = -4 + \frac{1}{5}t$$

$$\therefore a = 2 - \frac{1}{10}t$$

$\therefore$  the solutions are of the form  $a = 2 - \frac{1}{10}t$ ,  $b = -4 - \frac{3}{10}t$ ,  $c = t$ , where  $t \in \mathbb{R}$ .

**b** The system in **a** is underspecified as there are 2 equations in 3 unknowns.

$\therefore$  the system has infinitely many solutions.

- The point  $(4, q)$  lies on the circle  $x^2 + y^2 + ax + by + c = 0$ .

$$\therefore 4^2 + q^2 + a(4) + b(q) + c = 0$$

so the system of equations becomes 
$$\begin{cases} -2a + 4b + c = -20 \\ a + 3b + c = -10 \\ 4a + qb + c = -16 - q^2 \end{cases}$$

This system has augmented matrix

$$\begin{aligned} & \left( \begin{array}{ccc|c} -2 & 4 & 1 & -20 \\ 1 & 3 & 1 & -10 \\ 4 & q & 1 & -16 - q^2 \end{array} \right) \\ & \sim \left( \begin{array}{ccc|c} -2 & 4 & 1 & -20 \\ 0 & 10 & 3 & -40 \\ 0 & q+8 & 3 & -56 - q^2 \end{array} \right) & \begin{array}{l} \text{\{using \textcolor{blue}{a}\}} \\ R_3 + 2R_1 \rightarrow R_3 \end{array} & \left\{ \begin{array}{cccc} 4 & q & 1 & -16 - q^2 \\ -4 & 8 & 2 & -40 \\ \hline 0 & q+8 & 3 & -56 - q^2 \end{array} \right\} \\ & \sim \left( \begin{array}{ccc|c} -2 & 4 & 1 & -20 \\ 0 & 10 & 3 & -40 \\ 0 & 0 & 6-3q & 40q - 10q^2 - 240 \end{array} \right) & 10R_3 - (q+8)R_2 \rightarrow R_3 & \left\{ \begin{array}{cccc} 0 & 10(q+8) & 30 & 10(-56 - q^2) \\ 0 & -10(q+8) & -3(q+8) & 40(q+8) \\ \hline 0 & 0 & 6-3q & 40q - 10q^2 - 240 \end{array} \right\} \end{aligned}$$

- i If  $q = 2$ , then  $40q - 10q^2 - 240 = 40(2) - 10(2)^2 - 240 = -200$

and the last row gives  $0a + 0b + 0c = -200$ , which is not possible.

$\therefore$  the system has no solutions.

A circle passing through the points  $(-2, 4)$ ,  $(1, 3)$ , and  $(4, 2)$  cannot be constructed because the points are collinear.

- ii If  $q = 12$ , then using row 3,  $(6 - 3(12))c = 40(12) - 10(12)^2 - 240$   
 $\therefore -30c = -1200$   
 $\therefore c = 40$

Substituting into row 2,  $10b + 3(40) = -40$

$$\therefore 10b = -160$$

$$\therefore b = -16$$

Substituting into row 1,  $-2a + 4(-16) + 40 = -20$

$$\therefore -2a = 4$$

$$\therefore a = -2$$

$\therefore$  the unique solution is  $a = -2$ ,  $b = -16$ ,  $c = 40$ .

$\therefore$  the circle has equation  $x^2 + y^2 - 2x - 16y + 40 = 0$ .

$$8 \quad \begin{cases} x + 4y - z = k \\ -2x + z = 3 \\ 5x + 4y + (k-6)z = -3 \end{cases} \quad \text{where } k \in \mathbb{R}.$$

**a** This system has augmented matrix

$$\begin{aligned} & \left( \begin{array}{ccc|c} 1 & 4 & -1 & k \\ -2 & 0 & 1 & 3 \\ 5 & 4 & k-6 & -3 \end{array} \right) \\ & \sim \left( \begin{array}{ccc|c} 1 & 4 & -1 & k \\ 0 & 8 & -1 & 3+2k \\ 0 & -16 & k-1 & -3-5k \end{array} \right) \quad \begin{array}{l} R_2 + 2R_1 \rightarrow R_2 \\ R_3 - 5R_1 \rightarrow R_3 \end{array} \\ & \sim \left( \begin{array}{ccc|c} 1 & 4 & -1 & k \\ 0 & 8 & -1 & 3+2k \\ 0 & 0 & k-3 & 3-k \end{array} \right) \quad \begin{array}{l} R_3 + 2R_2 \rightarrow R_3 \end{array} \end{aligned}$$

$$\begin{aligned} & \left\{ \begin{array}{ccc|c} -2 & 0 & 1 & 3 \\ 2 & 8 & -2 & 2k \\ \hline 0 & 8 & -1 & 3+2k \end{array} \right\} \\ & \left\{ \begin{array}{ccc|c} 5 & 4 & k-6 & -3 \\ -5 & -20 & 5 & -5k \\ \hline 0 & -16 & k-1 & -3-5k \end{array} \right\} \\ & \left\{ \begin{array}{ccc|c} 0 & -16 & k-1 & -3-5k \\ 0 & 16 & -2 & 6+4k \\ \hline 0 & 0 & k-3 & 3-k \end{array} \right\} \end{aligned}$$

**b** The system has infinitely many solutions if the last row is all zeros. This occurs when  $k = 3$ . In this case we let  $z = t$ .

Using row 2,  $8y - t = 3 + 2(3)$

$$\therefore 8y = t + 9$$

$$\therefore y = \frac{t+9}{8}$$

Substituting into row 1,  $x + 4\left(\frac{t+9}{8}\right) - t = 3$

$$\therefore x - \frac{1}{2}t + \frac{9}{2} = 3$$

$$\therefore x = \frac{t-3}{2}$$

$\therefore$  the solutions have the form  $x = \frac{t-3}{2}$ ,  $y = \frac{t+9}{8}$ ,  $z = t$ , where  $t \in \mathbb{R}$ .

**c** Suppose  $k \neq 3$ .

Using row 3,  $(k-3)z = 3-k$

$$\begin{aligned} \therefore z &= \frac{-(k-3)}{k-3} \quad \{k \neq 3\} \\ &= -1 \end{aligned}$$

Substituting into row 2,  $8y - (-1) = 3 + 2k$

$$\therefore 8y = 2 + 2k$$

$$\therefore y = \frac{k+1}{4}$$

Substituting into row 1,  $x + 4\left(\frac{k+1}{4}\right) - (-1) = k$

$$\therefore x + k + 2 = k$$

$$\therefore x = -2$$

$\therefore$  the unique solution is  $x = -2$ ,  $y = \frac{k+1}{4}$ ,  $z = -1$ , for  $k \neq 3$ ,  $k \in \mathbb{R}$ .

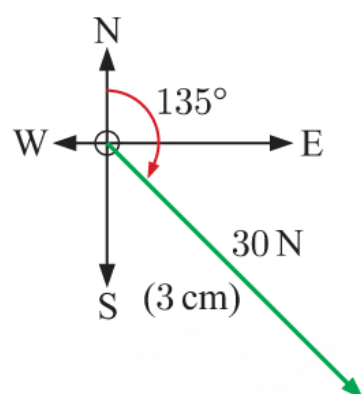


# Chapter 12

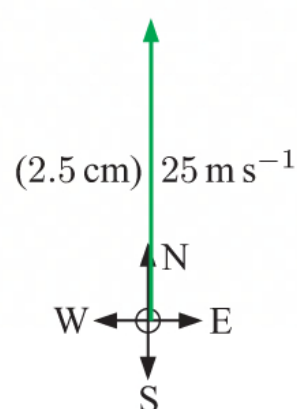
## VECTORS

### EXERCISE 12A.1

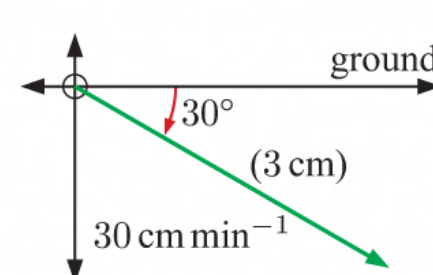
1 a



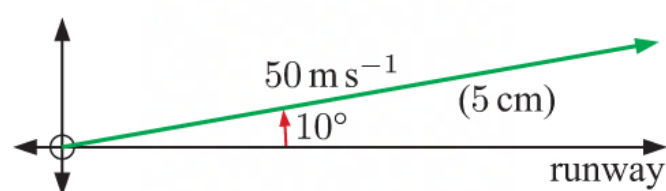
b



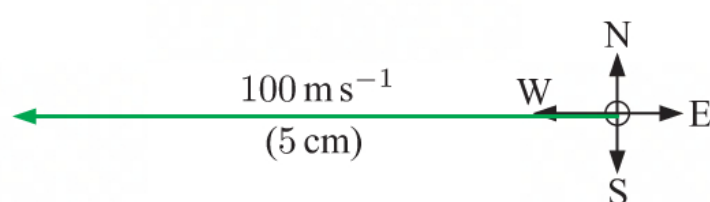
c



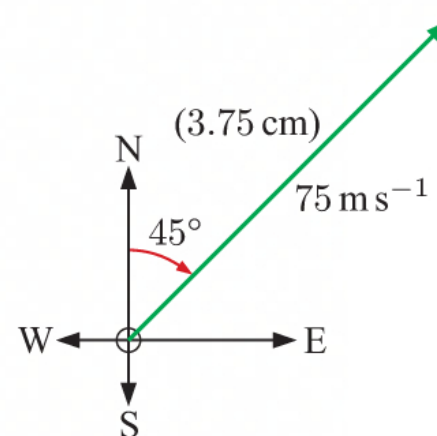
d



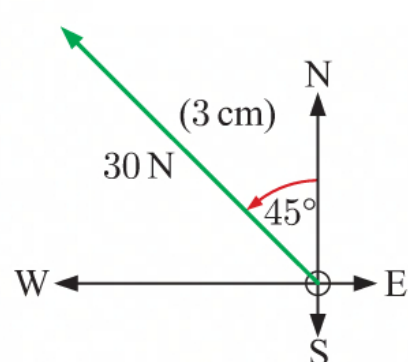
2 a



b

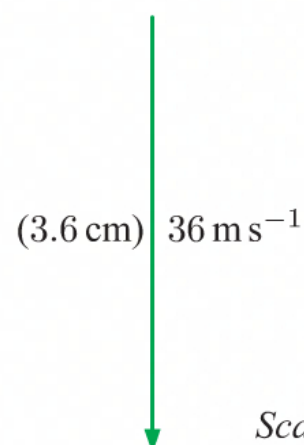


3 a



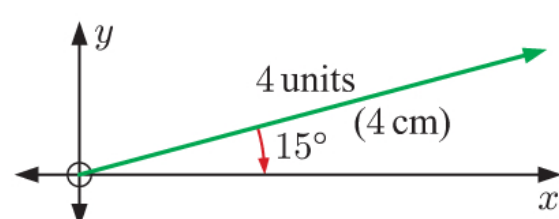
Scale: 1 cm  $\equiv$  10 N

b



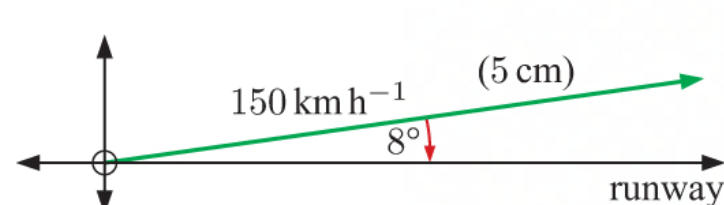
Scale: 1 cm  $\equiv$  10 m s<sup>-1</sup>

c



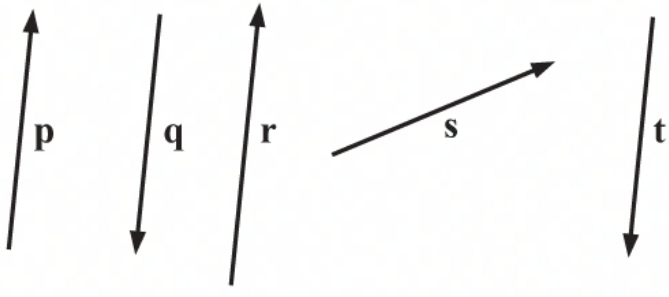
Scale: 1 cm  $\equiv$  1 unit

d

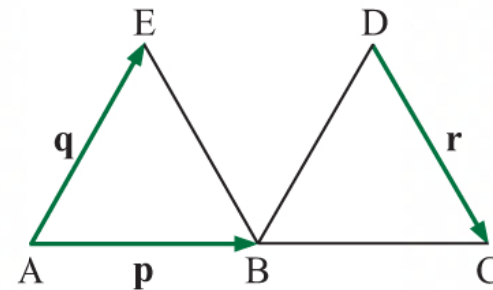


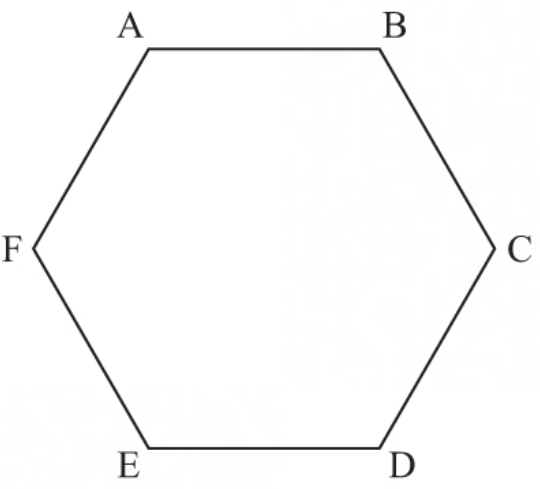
Scale: 1 cm  $\equiv$  30 km h<sup>-1</sup>

# EXERCISE 12A.2

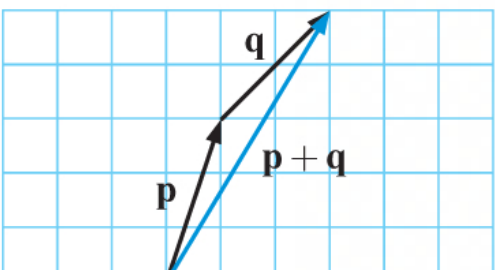
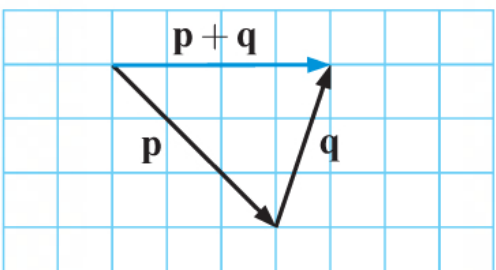
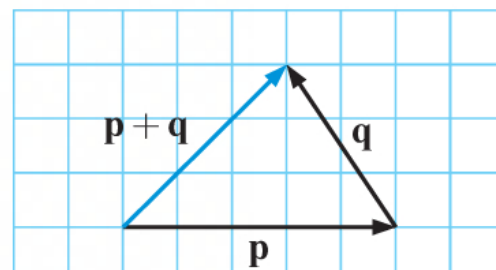
- 1
- 
- a If they are equal in magnitude, they have the same length. These are **p**, **q**, **s**, and **t**.
- b Those parallel are **p**, **q**, **r**, and **t**.
- c Those in the same direction are:  
**p** and **r**, **q** and **t**.
- d To be equal they must have the same direction and be equal in length  $\therefore \mathbf{q} = \mathbf{t}$ .
- e **p** and **q** are negatives (equal length, but opposite direction). Likewise, **p** and **t** are negatives. We write  $\mathbf{p} = -\mathbf{q}$  and  $\mathbf{p} = -\mathbf{t}$ .

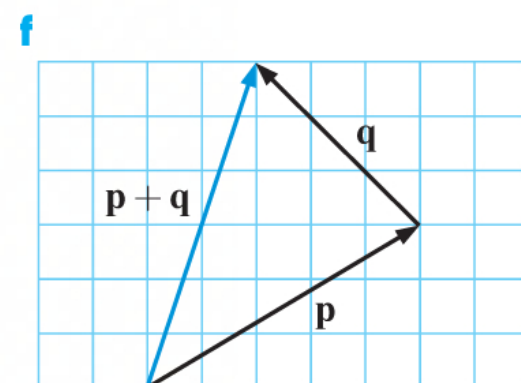
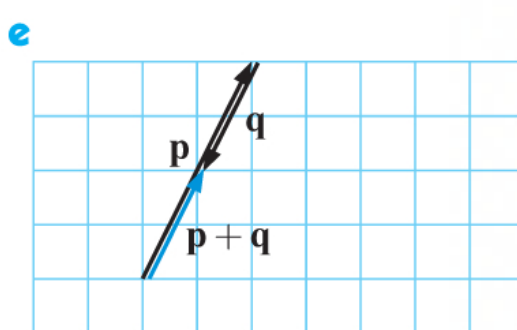
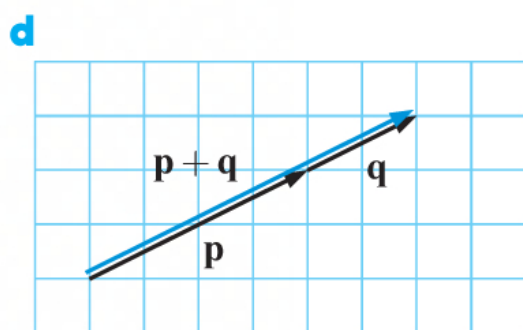
- 2
- a True, as they have the same length and direction.
- b True, as they are sides of an equilateral triangle.
- c False, as they do not have the same direction.
- d False, as they have opposite directions.
- e True, as they have the same length and direction.
- f False, as they do not have the same direction.



- 3
- 
- a
- i  $\overrightarrow{BC}$  is the vector which originates at B and terminates at C.
- ii  $\overrightarrow{ED} = \overrightarrow{AB}$ , as they have the same length and direction.
- b
- i  $\overrightarrow{FE}$  and  $\overrightarrow{BC}$  are negatives of  $\overrightarrow{EF}$ , as they both have the same length but opposite direction.
- ii All sides of the hexagon are equal in length  
 $\therefore$  the vectors with the same length as  $\overrightarrow{ED}$  are  
 $\overrightarrow{DE}$ ,  $\overrightarrow{EF}$ ,  $\overrightarrow{FE}$ ,  $\overrightarrow{FA}$ ,  $\overrightarrow{AF}$ ,  $\overrightarrow{AB}$ ,  $\overrightarrow{BA}$ ,  $\overrightarrow{BC}$ ,  
 $\overrightarrow{CB}$ ,  $\overrightarrow{CD}$ , and  $\overrightarrow{DC}$ .
- c The vector  $\overrightarrow{FC}$  is parallel to  $\overrightarrow{AB}$  and twice its length.  
 $\overrightarrow{CF}$  is also parallel to  $\overrightarrow{AB}$  and twice its length (but in the opposite direction).

# EXERCISE 12B.1

- 1
- a
- 
- b
- 
- c
- 



**2 a**  $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

**b**  $\overrightarrow{BC} + \overrightarrow{CD} = \overrightarrow{BD}$

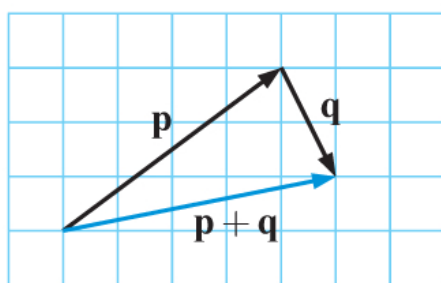
**c**  $\overrightarrow{AB} + \overrightarrow{BA} = \overrightarrow{AA}$   
 $= \mathbf{0}$

**d**  $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD}$   
 $= \overrightarrow{AC} + \overrightarrow{CD}$   
 $= \overrightarrow{AD}$

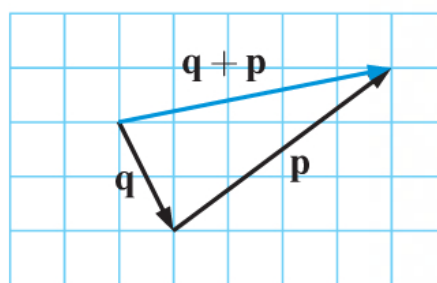
**e**  $\overrightarrow{AC} + \overrightarrow{CB} + \overrightarrow{BD}$   
 $= \overrightarrow{AB} + \overrightarrow{BD}$   
 $= \overrightarrow{AD}$

**f**  $\overrightarrow{BC} + \overrightarrow{CA} + \overrightarrow{AB}$   
 $= \overrightarrow{BA} + \overrightarrow{AB}$   
 $= \overrightarrow{BB}$   
 $= \mathbf{0}$

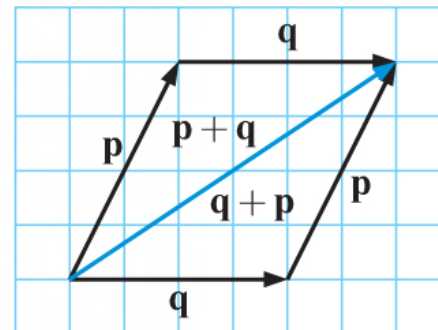
**3 a i**



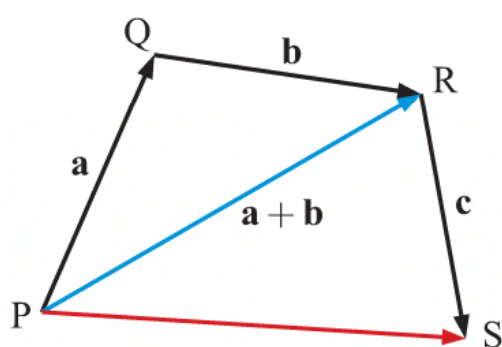
**ii**



**b** yes



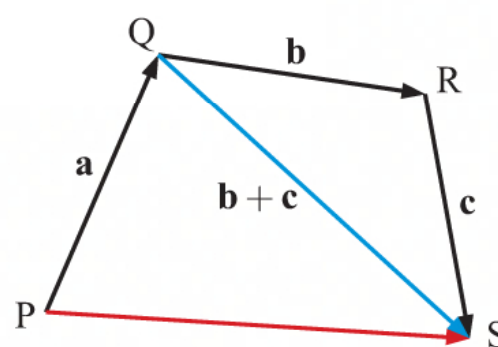
**4**



$$\overrightarrow{PS} = \overrightarrow{PR} + \overrightarrow{RS}$$

$$= (\mathbf{a} + \mathbf{b}) + \mathbf{c}$$

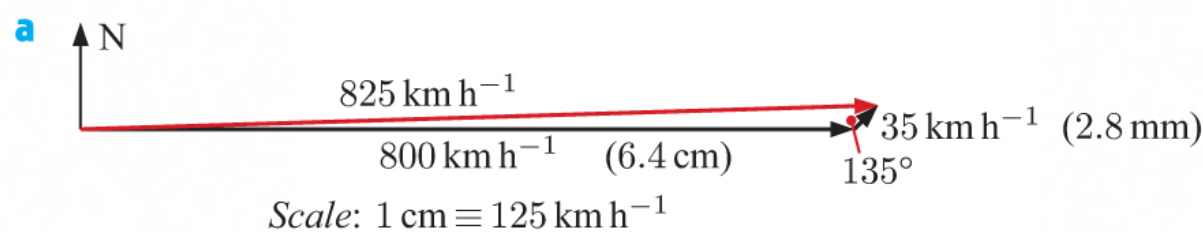
$$\therefore (\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$$



$$\text{But } \overrightarrow{PS} = \overrightarrow{PQ} + \overrightarrow{QS}$$

$$= \mathbf{a} + (\mathbf{b} + \mathbf{c})$$

**5**



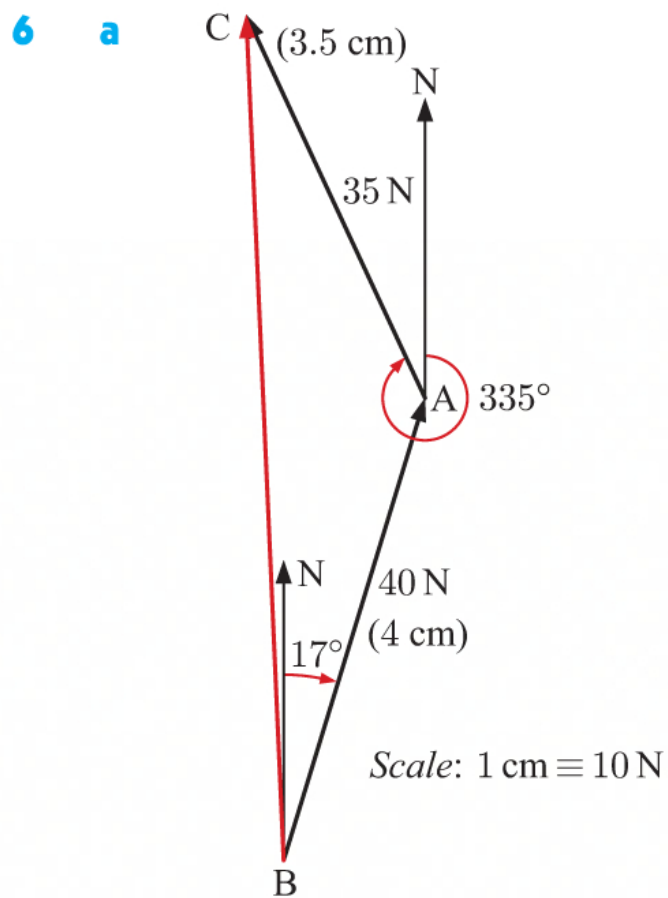
**b** We need to perform vector addition to find the effect of the wind on the aeroplane.

**c** Measuring the length of the resulting vector, we get 66 mm, or 6.6 cm.

$\therefore$  the resulting speed of the plane is  $6.6 \times 125 = 825 \text{ km h}^{-1}$ .

Using a protractor to measure the angle between 'true north' and the resulting vector, we get  $88^\circ$ .

$\therefore$  the direction of the aeroplane is  $88^\circ$  east of north.



Measuring the resulting vector, we get  $\approx 7$  cm.

$\therefore$  the resultant force is  $\approx 7 \times 10$  N  
 $\approx 70$  N

Using a protractor to measure the angle between 'true north' and the resulting vector, we get  $\approx 358^\circ$ .

$\therefore$  the direction of the force is  $\approx 358^\circ$  or  $2^\circ$  west of north.

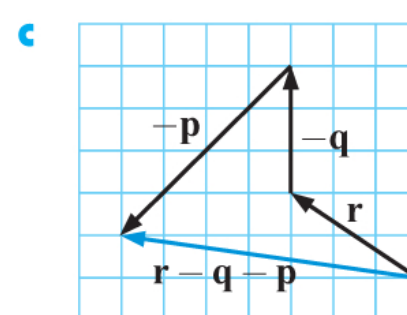
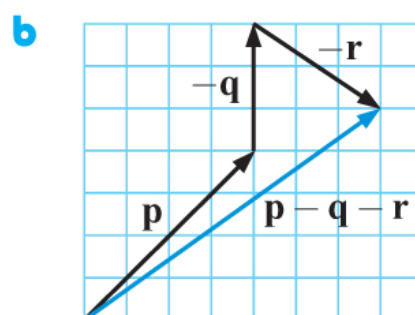
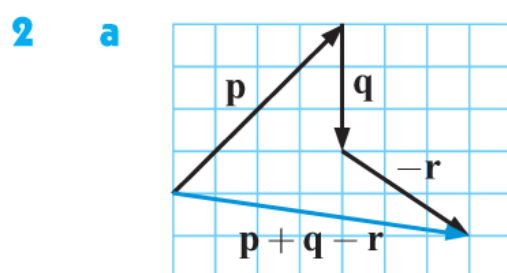
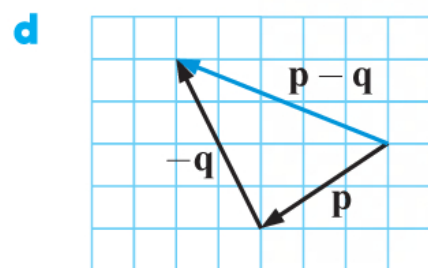
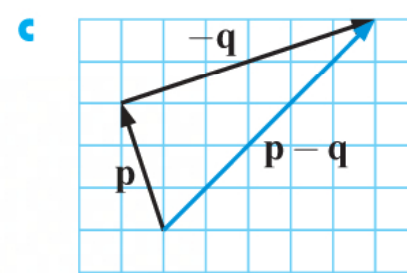
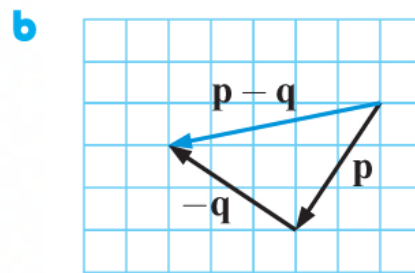
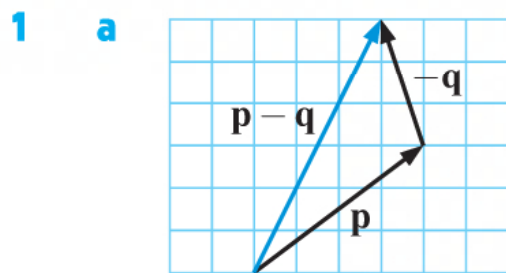
**b** Using the cosine rule,

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc \cos A \\ &= 35^2 + 40^2 - 2 \times 35 \times 40 \times \cos 138^\circ \\ &\approx 2825 - (-2080) \\ a &\approx \sqrt{4905} \\ &\approx 70.0 \text{ N} \end{aligned}$$

$$\begin{aligned} \cos B &= \frac{c^2 + a^2 - b^2}{2ac} \\ &\approx \frac{40^2 + 70^2 - 35^2}{2 \times 70 \times 40} \\ &\approx 0.942 \\ B &\approx 19.6^\circ \\ \text{and } 19.6^\circ - 17^\circ &= 2.6^\circ \end{aligned}$$

$\therefore$  the resultant force is  $\approx 70.0$  N in the direction  $\approx 357^\circ$  or  $\approx 3^\circ$  west of north.

## EXERCISE 12B.2





3 a  $\vec{AC} + \vec{CB} = \vec{AB}$

b  $\vec{AD} - \vec{BD} = \vec{AD} + \vec{DB}$   
 $= \vec{AB}$

c  $\vec{AC} + \vec{CA} = \vec{AA}$   
 $= \mathbf{0}$

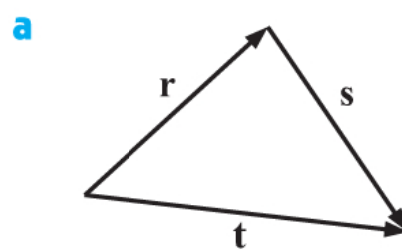
d  $\vec{AB} + \vec{BC} + \vec{CD}$   
 $= \vec{AC} + \vec{CD}$   
 $= \vec{AD}$

e  $\vec{BA} - \vec{CA} + \vec{CB}$   
 $= \vec{BA} + \vec{AC} + \vec{CB}$   
 $= \vec{BC} + \vec{CB}$   
 $= \vec{BB}$   
 $= \mathbf{0}$

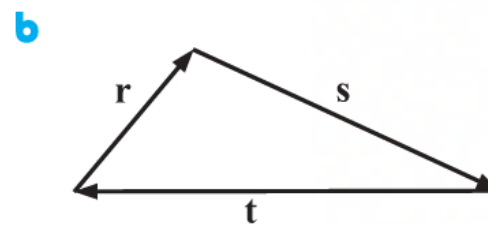
f  $\vec{AB} - \vec{CB} - \vec{DC}$   
 $= \vec{AB} + \vec{BC} + \vec{CD}$   
 $= \vec{AC} + \vec{CD}$   
 $= \vec{AD}$

### EXERCISE 12B.3

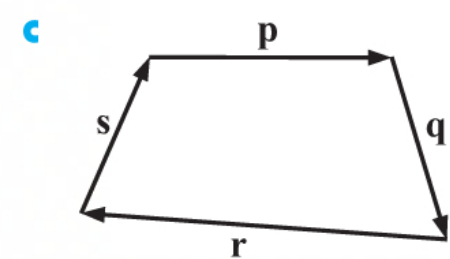
1 **Note:** Other answers are possible.



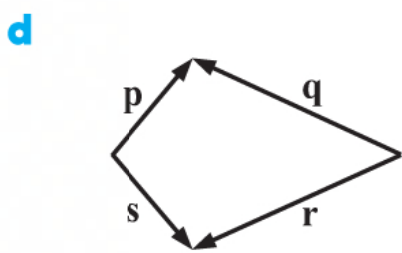
$$\mathbf{t} = \mathbf{r} + \mathbf{s}$$



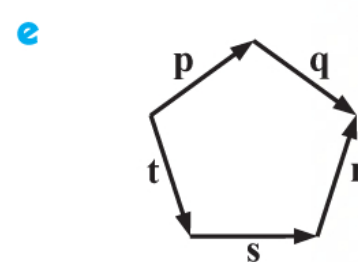
$$\begin{aligned}\mathbf{r} &= -\mathbf{t} - \mathbf{s} \\ &= -\mathbf{s} - \mathbf{t}\end{aligned}$$



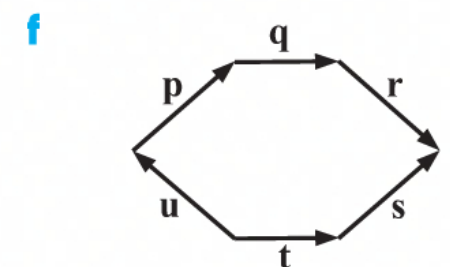
$$\begin{aligned}\mathbf{r} &= -\mathbf{q} - \mathbf{p} - \mathbf{s} \\ &= -\mathbf{p} - \mathbf{q} - \mathbf{s}\end{aligned}$$



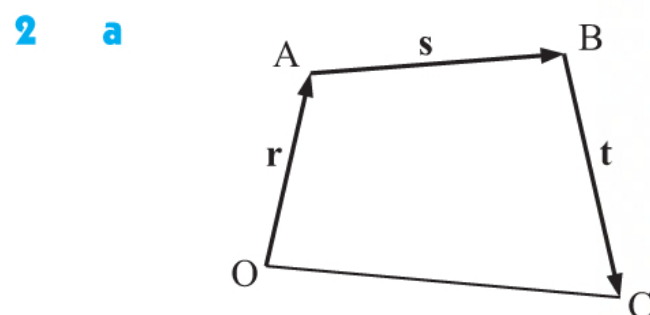
$$\mathbf{r} = \mathbf{q} - \mathbf{p} + \mathbf{s}$$



$$\mathbf{p} = \mathbf{t} + \mathbf{s} + \mathbf{r} - \mathbf{q}$$



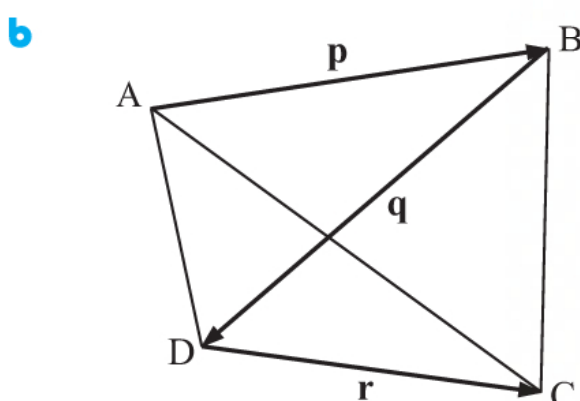
$$\mathbf{p} = -\mathbf{u} + \mathbf{t} + \mathbf{s} - \mathbf{r} - \mathbf{q}$$



i  $\vec{OB} = \vec{OA} + \vec{AB}$   
 $= \mathbf{r} + \mathbf{s}$

ii  $\vec{CA} = \vec{CB} + \vec{BA}$   
 $= -\vec{BC} - \vec{AB}$   
 $= -\mathbf{t} - \mathbf{s}$

iii  $\vec{OC} = \vec{OA} + \vec{AB} + \vec{BC}$   
 $= \mathbf{r} + \mathbf{s} + \mathbf{t}$

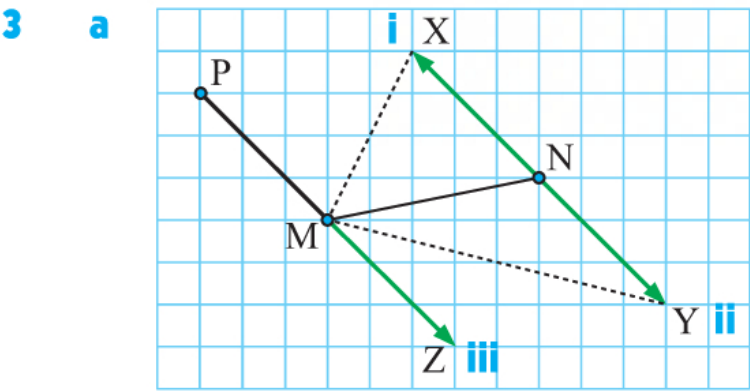
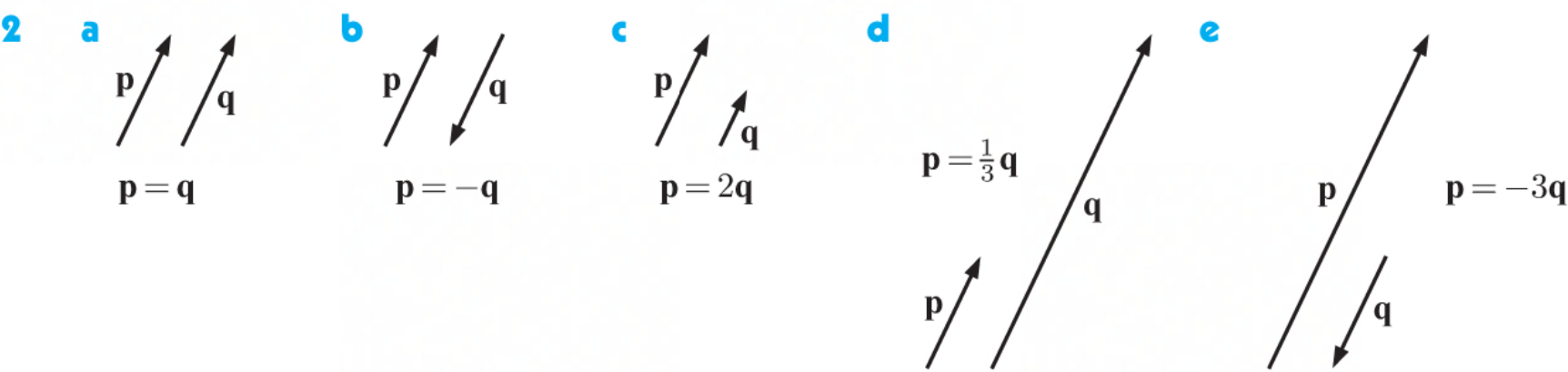
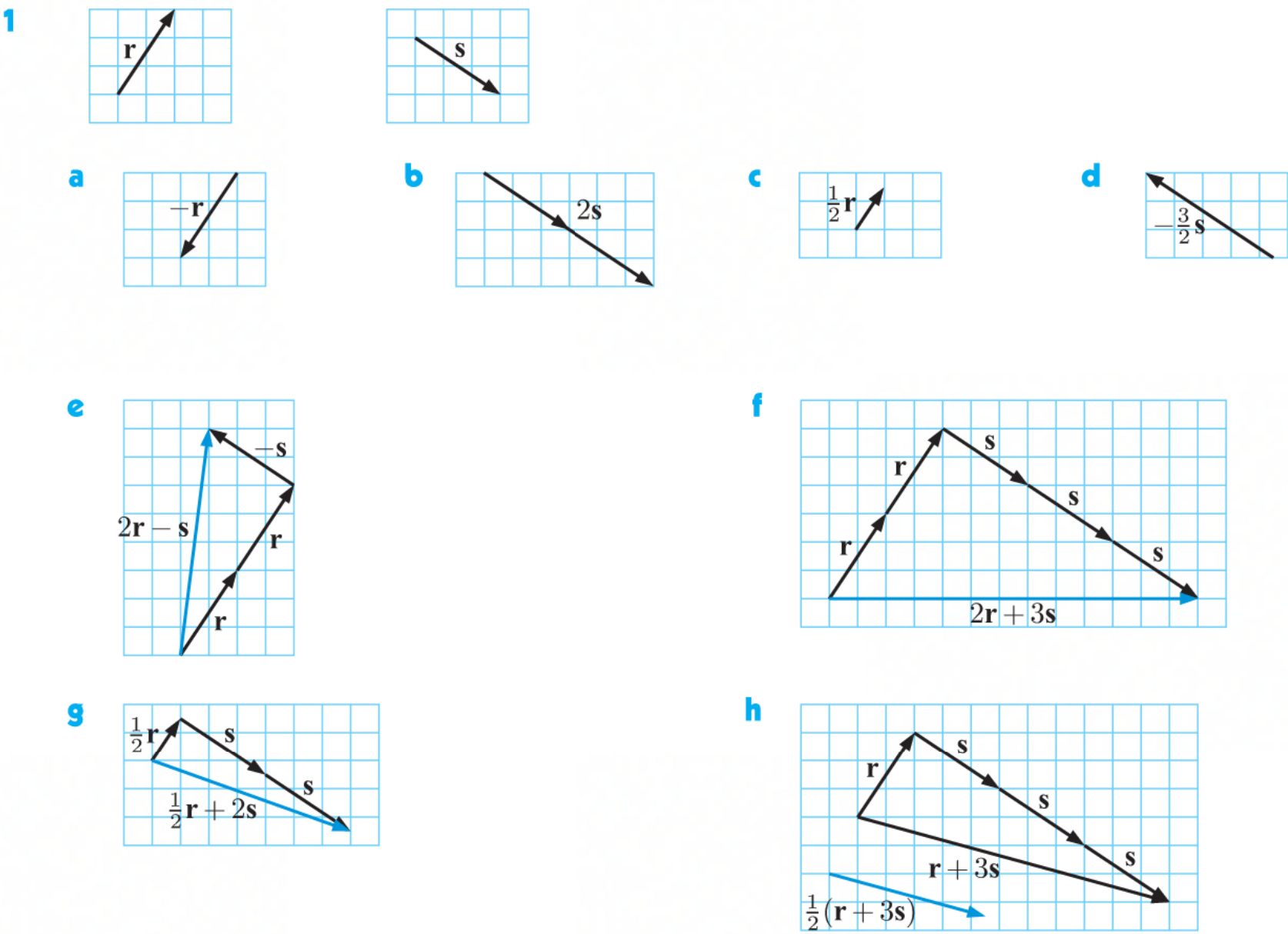


i  $\vec{AD} = \vec{AB} + \vec{BD}$   
 $= \mathbf{p} + \mathbf{q}$

ii  $\vec{BC} = \vec{BD} + \vec{DC}$   
 $= \mathbf{q} + \mathbf{r}$

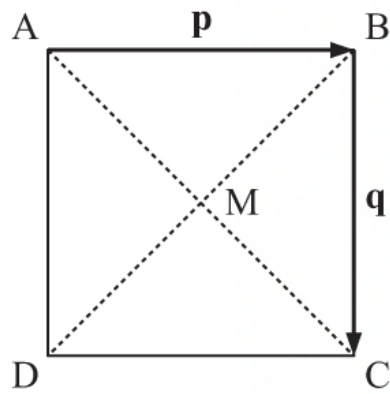
iii  $\vec{AC} = \vec{AB} + \vec{BD} + \vec{DC}$   
 $= \mathbf{p} + \mathbf{q} + \mathbf{r}$

EXERCISE 12B.4



b MNYZ is a parallelogram.

4



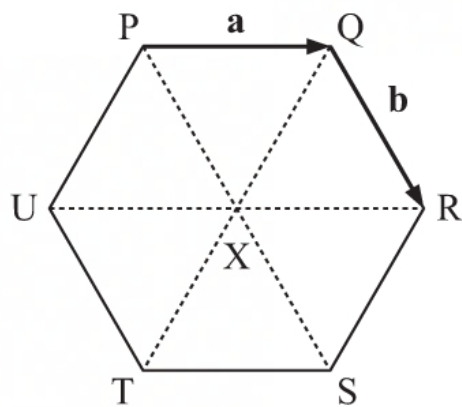
$$\begin{aligned} \text{a } \overrightarrow{CD} &= -\overrightarrow{AB} \\ &= -\mathbf{p} \end{aligned}$$

$$\begin{aligned} \text{c } \overrightarrow{AM} &= \frac{1}{2} \overrightarrow{AC} \\ &= \frac{1}{2} (\mathbf{p} + \mathbf{q}) \\ &\quad \{\text{using b}\} \end{aligned}$$

$$\begin{aligned} \text{b } \overrightarrow{AC} &= \overrightarrow{AB} + \overrightarrow{BC} \\ &= \mathbf{p} + \mathbf{q} \end{aligned}$$

$$\begin{aligned} \text{d } \overrightarrow{BD} &= \overrightarrow{BC} + \overrightarrow{CD} \\ &= \mathbf{q} + (-\mathbf{p}) \\ &\quad \{\text{using a}\} \\ &= \mathbf{q} - \mathbf{p} \\ \text{and } \overrightarrow{BM} &= \frac{1}{2} \overrightarrow{BD} \\ &= \frac{1}{2} (\mathbf{q} - \mathbf{p}) \end{aligned}$$

5



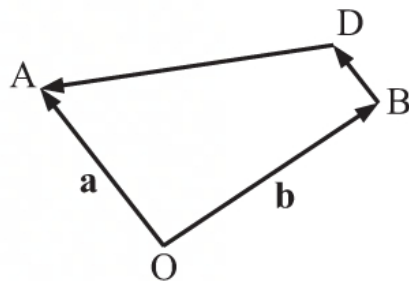
$$\begin{aligned} \text{a } \overrightarrow{PX} &= \overrightarrow{QR} \\ &= \mathbf{b} \end{aligned}$$

$$\begin{aligned} \text{c } \overrightarrow{QX} &= \overrightarrow{QR} + \overrightarrow{RX} \\ &= \mathbf{b} + (-\mathbf{a}) \\ &= \mathbf{b} - \mathbf{a} \end{aligned}$$

$$\begin{aligned} \text{b } \overrightarrow{PS} &= 2\overrightarrow{PX} \\ &= 2\mathbf{b} \\ &\quad \{\text{using a}\} \end{aligned}$$

$$\begin{aligned} \text{d } \overrightarrow{RS} &= \overrightarrow{QX} \\ &= \mathbf{b} - \mathbf{a} \\ &\quad \{\text{using c}\} \end{aligned}$$

6



$$\begin{aligned} \text{a } \overrightarrow{BD} &= \frac{1}{2} \overrightarrow{OA} \\ &= \frac{1}{2} \mathbf{a} \end{aligned}$$

$$\begin{aligned} \text{c } \overrightarrow{BA} &= -\overrightarrow{AB} \\ &= -(\mathbf{b} - \mathbf{a}) \\ &\quad \{\text{using b}\} \\ &= -\mathbf{b} + \mathbf{a} \end{aligned}$$

$$\begin{aligned} \text{b } \overrightarrow{AB} &= \overrightarrow{AO} + \overrightarrow{OB} \\ &= (-\mathbf{a}) + \mathbf{b} \\ &= \mathbf{b} - \mathbf{a} \end{aligned}$$

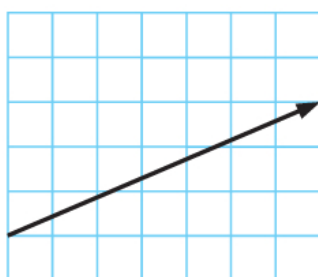
$$\begin{aligned} \text{e } \overrightarrow{AD} &= \overrightarrow{AO} + \overrightarrow{OB} + \overrightarrow{BD} \\ &= (-\mathbf{a}) + \mathbf{b} + \frac{1}{2} \mathbf{a} \\ &\quad \{\text{using a}\} \\ &= \mathbf{b} - \frac{1}{2} \mathbf{a} \end{aligned}$$

$$\begin{aligned} \text{d } \overrightarrow{OD} &= \overrightarrow{OB} + \overrightarrow{BD} \\ &= \mathbf{b} + \frac{1}{2} \mathbf{a} \\ &\quad \{\text{using a}\} \\ \text{f } \overrightarrow{DA} &= -\overrightarrow{AD} \\ &= -(\mathbf{b} - \frac{1}{2} \mathbf{a}) \\ &\quad \{\text{using e}\} \\ &= \frac{1}{2} \mathbf{a} - \mathbf{b} \end{aligned}$$

## EXERCISE 12C

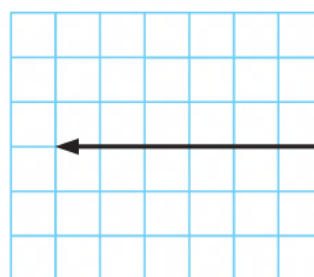
1

a



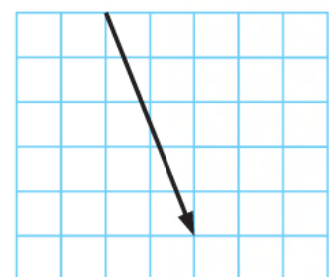
$$\begin{pmatrix} 7 \\ 3 \end{pmatrix} = 7\mathbf{i} + 3\mathbf{j}$$

b

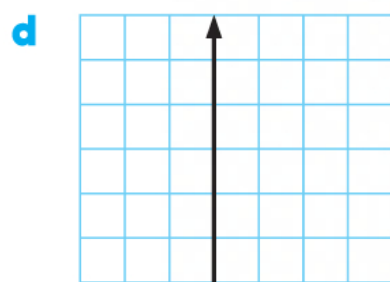


$$\begin{pmatrix} -6 \\ 0 \end{pmatrix} = -6\mathbf{i}$$

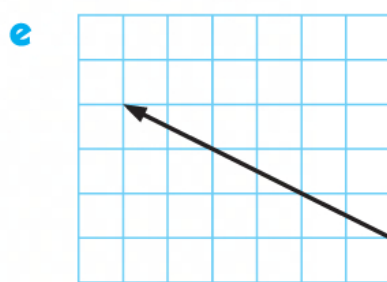
c



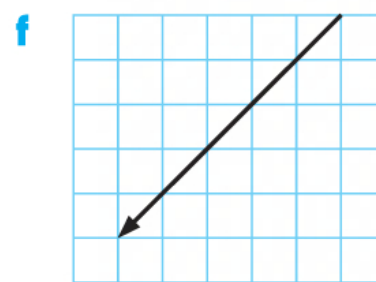
$$\begin{pmatrix} 2 \\ -5 \end{pmatrix} = 2\mathbf{i} - 5\mathbf{j}$$



$$\begin{pmatrix} 0 \\ 6 \end{pmatrix} = 6\mathbf{j}$$

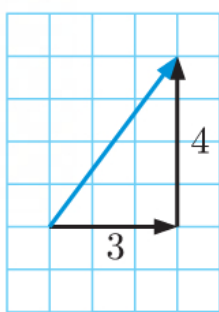


$$\begin{pmatrix} -6 \\ 3 \end{pmatrix} = -6\mathbf{i} + 3\mathbf{j}$$

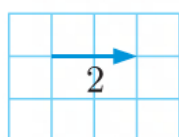


$$\begin{pmatrix} -5 \\ -5 \end{pmatrix} = -5\mathbf{i} - 5\mathbf{j}$$

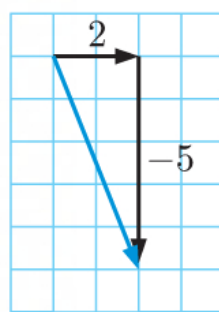
**2 a**  $\begin{pmatrix} 3 \\ 4 \end{pmatrix} = 3\mathbf{i} + 4\mathbf{j}$



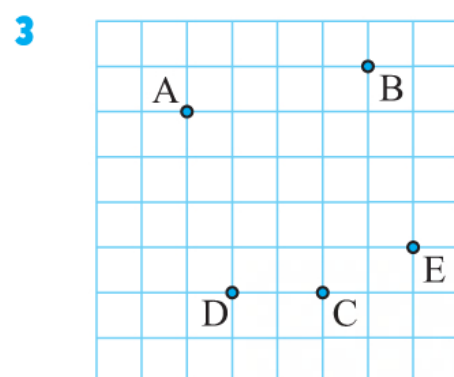
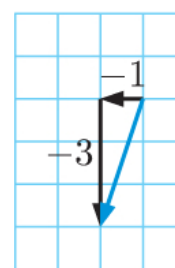
**b**  $\begin{pmatrix} 2 \\ 0 \end{pmatrix} = 2\mathbf{i}$



**c**  $\begin{pmatrix} 2 \\ -5 \end{pmatrix} = 2\mathbf{i} - 5\mathbf{j}$



**d**  $\begin{pmatrix} -1 \\ -3 \end{pmatrix} = -\mathbf{i} - 3\mathbf{j}$



**a i**  $\overrightarrow{BA} = \begin{pmatrix} -4 \\ -1 \end{pmatrix} = -4\mathbf{i} - \mathbf{j}$  **ii**  $\overrightarrow{BC} = \begin{pmatrix} -1 \\ -5 \end{pmatrix} = -\mathbf{i} - 5\mathbf{j}$

**iii**  $\overrightarrow{DC} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} = 2\mathbf{i}$  **iv**  $\overrightarrow{AC} = \begin{pmatrix} 3 \\ -4 \end{pmatrix} = 3\mathbf{i} - 4\mathbf{j}$

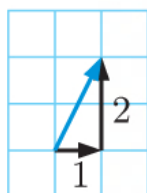
**v**  $\overrightarrow{CA} = \begin{pmatrix} -3 \\ 4 \end{pmatrix} = -3\mathbf{i} + 4\mathbf{j}$  **vi**  $\overrightarrow{DB} = \begin{pmatrix} 3 \\ 5 \end{pmatrix} = 3\mathbf{i} + 5\mathbf{j}$

**vii**  $\overrightarrow{AE} = \begin{pmatrix} 5 \\ -3 \end{pmatrix} = 5\mathbf{i} - 3\mathbf{j}$  **viii**  $\overrightarrow{CE} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 2\mathbf{i} + \mathbf{j}$

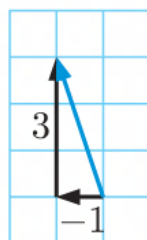
**ix**  $\overrightarrow{ED} = \begin{pmatrix} -4 \\ -1 \end{pmatrix} = -4\mathbf{i} - \mathbf{j}$

**b**  $\overrightarrow{BA} = \overrightarrow{ED}$ ,  $\overrightarrow{AB} = \overrightarrow{DE}$ ,  $\overrightarrow{AD} = \overrightarrow{BE}$ ,  $\overrightarrow{DA} = \overrightarrow{EB}$

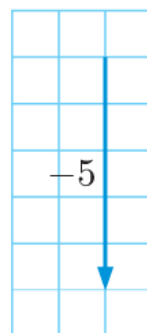
**4 a**  $\mathbf{i} + 2\mathbf{j} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$



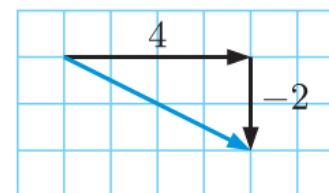
**b**  $-\mathbf{i} + 3\mathbf{j} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$



**c**  $-5\mathbf{j} = \begin{pmatrix} 0 \\ -5 \end{pmatrix}$



**d**  $4\mathbf{i} - 2\mathbf{j} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$



**5** The zero vector  $\mathbf{0}$  in component form is  $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ .



$$6 \quad a \quad \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

$$\therefore a = 3, \quad b = 5$$

$$b \quad \begin{pmatrix} a+1 \\ 2b-8 \end{pmatrix} = \begin{pmatrix} 9-a \\ a \end{pmatrix}$$

$$\therefore a+1 = 9-a \quad \text{and} \quad 2b-8 = a$$

$$\therefore 2a = 8 \quad \therefore 2b-8 = 4$$

$$\therefore a = 4 \quad \therefore 2b = 12$$

$$\therefore b = 6$$

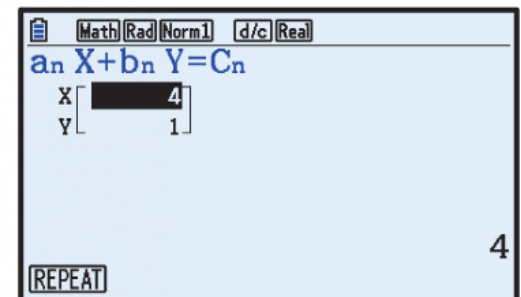
$$c \quad \begin{pmatrix} 2x+3y \\ x-2 \end{pmatrix} = \begin{pmatrix} 11 \\ 2y \end{pmatrix}$$

$$\therefore 2x+3y = 11 \quad \dots (1) \quad \text{and} \quad x-2 = 2y$$

$$\therefore x-2y = 2 \quad \dots (2)$$

Solving (1) and (2) simultaneously,

$$x = 4, \quad y = 1$$



## EXERCISE 12D

$$1 \quad a \quad \left| \begin{pmatrix} 3 \\ 4 \end{pmatrix} \right| = \sqrt{3^2 + 4^2} \\ = \sqrt{9 + 16} \\ = \sqrt{25} = 5 \text{ units}$$

$$b \quad \left| \begin{pmatrix} -4 \\ 3 \end{pmatrix} \right| = \sqrt{(-4)^2 + 3^2} \\ = \sqrt{16 + 9} \\ = \sqrt{25} = 5 \text{ units}$$

$$c \quad \left| \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right| = \sqrt{2^2 + 0^2} \\ = \sqrt{4} \\ = 2 \text{ units}$$

$$d \quad \left| \begin{pmatrix} -2 \\ 2 \end{pmatrix} \right| = \sqrt{(-2)^2 + 2^2} \\ = \sqrt{4 + 4} \\ = \sqrt{8} \text{ units}$$

$$e \quad \left| \begin{pmatrix} 0 \\ -3 \end{pmatrix} \right| = \sqrt{0^2 + (-3)^2} \\ = \sqrt{9} \\ = 3 \text{ units}$$

$$2 \quad a \quad \text{As } \mathbf{i} + \mathbf{j} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \\ |\mathbf{i} + \mathbf{j}| = \sqrt{1^2 + 1^2} \\ = \sqrt{2} \text{ units}$$

$$b \quad \text{As } 5\mathbf{i} - 12\mathbf{j} = \begin{pmatrix} 5 \\ -12 \end{pmatrix}, \\ |5\mathbf{i} - 12\mathbf{j}| = \sqrt{5^2 + (-12)^2} \\ = \sqrt{25 + 144} \\ = \sqrt{169} \\ = 13 \text{ units}$$

$$c \quad \text{As } -\mathbf{i} + 4\mathbf{j} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}, \\ |-\mathbf{i} + 4\mathbf{j}| = \sqrt{(-1)^2 + 4^2} \\ = \sqrt{1 + 16} \\ = \sqrt{17} \text{ units}$$

$$d \quad \text{As } 3\mathbf{i} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \\ |3\mathbf{i}| = \sqrt{3^2 + 0^2} \\ = \sqrt{9} \\ = 3 \text{ units}$$

**e** As  $k\mathbf{j} = \begin{pmatrix} 0 \\ k \end{pmatrix}$ ,

$$\begin{aligned} |k\mathbf{j}| &= \sqrt{0^2 + k^2} \\ &= \sqrt{k^2} \\ &= |k| \text{ units} \end{aligned}$$

**3 a**  $\left| \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right| = \sqrt{0^2 + (-1)^2}$   
 $= 1$   
 $\therefore \begin{pmatrix} 0 \\ -1 \end{pmatrix}$  is a unit vector.

**b**  $\left| \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \right| = \sqrt{\left(-\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2}$   
 $= \sqrt{\frac{1}{2} + \frac{1}{2}}$   
 $= 1$

$\therefore \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$  is a unit vector.

**c**  $\left| \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \end{pmatrix} \right| = \sqrt{\left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2}$   
 $= \sqrt{\frac{4}{9} + \frac{1}{9}}$   
 $= \frac{\sqrt{5}}{3}$   
 $\therefore \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}$  is not a unit vector.

**d**  $\left| \begin{pmatrix} -\frac{3}{5} \\ -\frac{4}{5} \end{pmatrix} \right| = \sqrt{\left(-\frac{3}{5}\right)^2 + \left(-\frac{4}{5}\right)^2}$   
 $= \sqrt{\frac{9}{25} + \frac{16}{25}}$   
 $= 1$

$\therefore \begin{pmatrix} -\frac{3}{5} \\ -\frac{4}{5} \end{pmatrix}$  is a unit vector.

**e**  $\left| \begin{pmatrix} \frac{2}{7} \\ -\frac{5}{7} \end{pmatrix} \right| = \sqrt{\left(\frac{2}{7}\right)^2 + \left(-\frac{5}{7}\right)^2}$   
 $= \sqrt{\frac{4}{49} + \frac{25}{49}}$   
 $= \frac{\sqrt{29}}{7}$   
 $\therefore \begin{pmatrix} \frac{2}{7} \\ -\frac{5}{7} \end{pmatrix}$  is not a unit vector.

**4 a** Since  $\begin{pmatrix} 0 \\ k \end{pmatrix}$  is a unit vector,

$$\begin{aligned} \left| \begin{pmatrix} 0 \\ k \end{pmatrix} \right| &= \sqrt{0^2 + k^2} = 1 \\ \therefore k^2 &= 1 \\ \therefore k &= \pm 1 \end{aligned}$$

**b** Since  $\begin{pmatrix} k \\ 0 \end{pmatrix}$  is a unit vector,

$$\begin{aligned} \left| \begin{pmatrix} k \\ 0 \end{pmatrix} \right| &= \sqrt{k^2 + 0} = 1 \\ \therefore k^2 &= 1 \\ \therefore k &= \pm 1 \end{aligned}$$

**c** Since  $\begin{pmatrix} k \\ 1 \end{pmatrix}$  is a unit vector,

$$\begin{aligned} \left| \begin{pmatrix} k \\ 1 \end{pmatrix} \right| &= \sqrt{k^2 + 1} = 1 \\ \therefore k^2 + 1 &= 1 \\ \therefore k^2 &= 0 \\ \therefore k &= 0 \end{aligned}$$

**e** Since  $\begin{pmatrix} \frac{1}{2} \\ k \end{pmatrix}$  is a unit vector,

$$\begin{aligned} \left| \begin{pmatrix} \frac{1}{2} \\ k \end{pmatrix} \right| &= \sqrt{\left(\frac{1}{2}\right)^2 + k^2} = 1 \\ \therefore \frac{1}{4} + k^2 &= 1 \\ \therefore k^2 &= \frac{3}{4} \\ \therefore k &= \pm\sqrt{\frac{3}{4}} = \pm\frac{\sqrt{3}}{2} \end{aligned}$$

**d** Since  $\begin{pmatrix} k \\ k \end{pmatrix}$  is a unit vector,

$$\begin{aligned} \left| \begin{pmatrix} k \\ k \end{pmatrix} \right| &= \sqrt{k^2 + k^2} = 1 \\ \therefore 2k^2 &= 1 \\ \therefore k^2 &= \frac{1}{2} \\ \therefore k &= \pm\sqrt{\frac{1}{2}} = \pm\frac{1}{\sqrt{2}} \end{aligned}$$

**5** If  $|\mathbf{v}| = \sqrt{73}$  units, then  $\sqrt{8^2 + p^2} = \sqrt{73}$   $\left\{ \mathbf{v} = \begin{pmatrix} 8 \\ p \end{pmatrix} \right\}$

$$\begin{aligned} \therefore 64 + p^2 &= 73 \\ \therefore p^2 &= 9 \\ \therefore p &= \pm 3 \end{aligned}$$

## EXERCISE 12E

**1 a**  $\mathbf{a} + \mathbf{b} = \begin{pmatrix} -3 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 4 \end{pmatrix}$

$$= \begin{pmatrix} -2 \\ 6 \end{pmatrix}$$

**b**  $\mathbf{b} + \mathbf{a} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} + \begin{pmatrix} -3 \\ 2 \end{pmatrix}$

$$= \begin{pmatrix} -2 \\ 6 \end{pmatrix}$$

**c**  $\mathbf{b} + \mathbf{c} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} + \begin{pmatrix} -2 \\ -5 \end{pmatrix}$

$$= \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

**d**  $\mathbf{c} + \mathbf{b} = \begin{pmatrix} -2 \\ -5 \end{pmatrix} + \begin{pmatrix} 1 \\ 4 \end{pmatrix}$

$$= \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

**e**  $\mathbf{a} + \mathbf{c} = \begin{pmatrix} -3 \\ 2 \end{pmatrix} + \begin{pmatrix} -2 \\ -5 \end{pmatrix}$

$$= \begin{pmatrix} -5 \\ -3 \end{pmatrix}$$

**f**  $\mathbf{c} + \mathbf{a} = \begin{pmatrix} -2 \\ -5 \end{pmatrix} + \begin{pmatrix} -3 \\ 2 \end{pmatrix}$

$$= \begin{pmatrix} -5 \\ -3 \end{pmatrix}$$

$$\begin{aligned} \text{g } \mathbf{a} + \mathbf{a} &= \begin{pmatrix} -3 \\ 2 \end{pmatrix} + \begin{pmatrix} -3 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} -6 \\ 4 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \text{h } \mathbf{b} + \mathbf{a} + \mathbf{c} &= \begin{pmatrix} 1 \\ 4 \end{pmatrix} + \begin{pmatrix} -3 \\ 2 \end{pmatrix} + \begin{pmatrix} -2 \\ -5 \end{pmatrix} \\ &= \begin{pmatrix} -2 \\ 6 \end{pmatrix} + \begin{pmatrix} -2 \\ -5 \end{pmatrix} \\ &= \begin{pmatrix} -4 \\ 1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \text{2 a } \mathbf{p} - \mathbf{q} &= \begin{pmatrix} -4 \\ 2 \end{pmatrix} - \begin{pmatrix} -1 \\ -5 \end{pmatrix} \\ &= \begin{pmatrix} -3 \\ 7 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \text{b } \mathbf{q} - \mathbf{r} &= \begin{pmatrix} -1 \\ -5 \end{pmatrix} - \begin{pmatrix} 3 \\ -2 \end{pmatrix} \\ &= \begin{pmatrix} -4 \\ -3 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \text{c } \mathbf{p} + \mathbf{q} - \mathbf{r} &= \begin{pmatrix} -4 \\ 2 \end{pmatrix} + \begin{pmatrix} -1 \\ -5 \end{pmatrix} - \begin{pmatrix} 3 \\ -2 \end{pmatrix} \\ &= \begin{pmatrix} -8 \\ -1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \text{d } \mathbf{p} - \mathbf{q} - \mathbf{r} &= \begin{pmatrix} -4 \\ 2 \end{pmatrix} - \begin{pmatrix} -1 \\ -5 \end{pmatrix} - \begin{pmatrix} 3 \\ -2 \end{pmatrix} \\ &= \begin{pmatrix} -6 \\ 9 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \text{e } \mathbf{q} - \mathbf{r} - \mathbf{p} &= \begin{pmatrix} -1 \\ -5 \end{pmatrix} - \begin{pmatrix} 3 \\ -2 \end{pmatrix} - \begin{pmatrix} -4 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ -5 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \text{f } \mathbf{r} + \mathbf{q} - \mathbf{p} &= \begin{pmatrix} 3 \\ -2 \end{pmatrix} + \begin{pmatrix} -1 \\ -5 \end{pmatrix} - \begin{pmatrix} -4 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} 6 \\ -9 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \text{3 a } \mathbf{a} + \mathbf{0} &= \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} a_1 + 0 \\ a_2 + 0 \end{pmatrix} \\ &= \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \\ &= \mathbf{a} \end{aligned}$$

$$\begin{aligned} \text{b } \mathbf{a} - \mathbf{a} &= \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} - \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \\ &= \begin{pmatrix} a_1 - a_1 \\ a_2 - a_2 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ &= \mathbf{0} \end{aligned}$$

$$\begin{aligned} \text{4 a } -3\mathbf{p} &= -3 \begin{pmatrix} 1 \\ 5 \end{pmatrix} \\ &= \begin{pmatrix} -3 \\ -15 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \text{b } \frac{1}{2}\mathbf{q} &= \frac{1}{2} \begin{pmatrix} -2 \\ 4 \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ 2 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \text{c } 2\mathbf{p} + \mathbf{q} &= 2 \begin{pmatrix} 1 \\ 5 \end{pmatrix} + \begin{pmatrix} -2 \\ 4 \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ 10 \end{pmatrix} + \begin{pmatrix} -2 \\ 4 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 14 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \text{d } \mathbf{p} - 2\mathbf{q} &= \begin{pmatrix} 1 \\ 5 \end{pmatrix} - 2 \begin{pmatrix} -2 \\ 4 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 5 \end{pmatrix} - \begin{pmatrix} -4 \\ 8 \end{pmatrix} \\ &= \begin{pmatrix} 5 \\ -3 \end{pmatrix} \end{aligned}$$

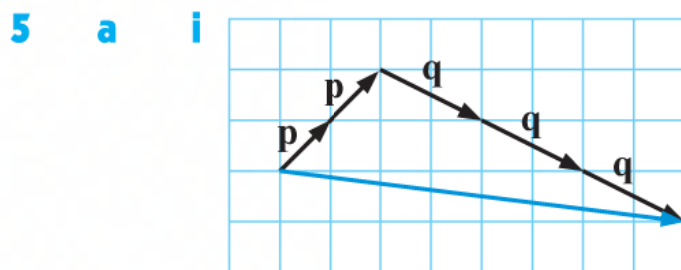


$$\begin{aligned}
 \text{e } \mathbf{p} - \frac{1}{2}\mathbf{r} &= \begin{pmatrix} 1 \\ 5 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} -3 \\ -1 \end{pmatrix} \\
 &= \begin{pmatrix} 1 \\ 5 \end{pmatrix} - \begin{pmatrix} -\frac{3}{2} \\ -\frac{1}{2} \end{pmatrix} \\
 &= \begin{pmatrix} \frac{5}{2} \\ \frac{11}{2} \end{pmatrix}
 \end{aligned}$$

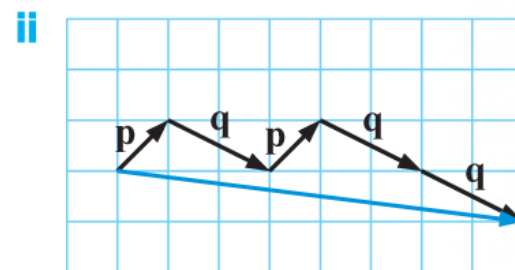
$$\begin{aligned}
 \text{f } 2\mathbf{p} + 3\mathbf{r} &= 2 \begin{pmatrix} 1 \\ 5 \end{pmatrix} + 3 \begin{pmatrix} -3 \\ -1 \end{pmatrix} \\
 &= \begin{pmatrix} 2 \\ 10 \end{pmatrix} + \begin{pmatrix} -9 \\ -3 \end{pmatrix} \\
 &= \begin{pmatrix} -7 \\ 7 \end{pmatrix}
 \end{aligned}$$

$$\begin{aligned}
 \text{g } 2\mathbf{q} - 3\mathbf{r} &= 2 \begin{pmatrix} -2 \\ 4 \end{pmatrix} - 3 \begin{pmatrix} -3 \\ -1 \end{pmatrix} \\
 &= \begin{pmatrix} -4 \\ 8 \end{pmatrix} - \begin{pmatrix} -9 \\ -3 \end{pmatrix} \\
 &= \begin{pmatrix} 5 \\ 11 \end{pmatrix}
 \end{aligned}$$

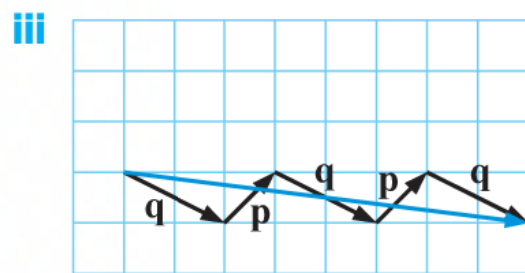
$$\begin{aligned}
 \text{h } 2\mathbf{p} - \mathbf{q} + \frac{1}{3}\mathbf{r} &= 2 \begin{pmatrix} 1 \\ 5 \end{pmatrix} - \begin{pmatrix} -2 \\ 4 \end{pmatrix} + \frac{1}{3} \begin{pmatrix} -3 \\ -1 \end{pmatrix} \\
 &= \begin{pmatrix} 2 \\ 10 \end{pmatrix} - \begin{pmatrix} -2 \\ 4 \end{pmatrix} + \begin{pmatrix} -1 \\ -\frac{1}{3} \end{pmatrix} \\
 &= \begin{pmatrix} 3 \\ \frac{17}{3} \end{pmatrix}
 \end{aligned}$$



$$\mathbf{p} + \mathbf{p} + \mathbf{q} + \mathbf{q} + \mathbf{q} = \begin{pmatrix} 8 \\ -1 \end{pmatrix}$$



$$\mathbf{p} + \mathbf{q} + \mathbf{p} + \mathbf{q} + \mathbf{q} = \begin{pmatrix} 8 \\ -1 \end{pmatrix}$$



$$\mathbf{q} + \mathbf{p} + \mathbf{q} + \mathbf{p} + \mathbf{q} = \begin{pmatrix} 8 \\ -1 \end{pmatrix}$$

- b** The vector expressions are equal, as each consists of the sum of 2  $\mathbf{p}$ s and 3  $\mathbf{q}$ s. Each expression is equal to  $2\mathbf{p} + 3\mathbf{q}$ .

$$6 \quad a \quad \begin{pmatrix} 2 \\ -5 \end{pmatrix} + k \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 20 \\ -11 \end{pmatrix}$$

$$\therefore \begin{pmatrix} 2+3k \\ -5-k \end{pmatrix} = \begin{pmatrix} 20 \\ -11 \end{pmatrix}$$

$$\therefore 2+3k=20$$

$$\therefore 3k=18$$

$$\therefore k=6$$

$$\text{Check: } -5-6=-11 \quad \checkmark$$

$$b \quad \frac{1}{3} \begin{pmatrix} -1 \\ k \end{pmatrix} + \frac{4}{3} \begin{pmatrix} 7 \\ k+1 \end{pmatrix} = \begin{pmatrix} 9 \\ -1 \end{pmatrix}$$

$$\therefore \begin{pmatrix} -\frac{1}{3} \\ \frac{1}{3}k \end{pmatrix} + \begin{pmatrix} \frac{28}{3} \\ \frac{4}{3}k + \frac{4}{3} \end{pmatrix} = \begin{pmatrix} 9 \\ -1 \end{pmatrix}$$

$$\therefore \begin{pmatrix} 9 \\ \frac{5}{3}k + \frac{4}{3} \end{pmatrix} = \begin{pmatrix} 9 \\ -1 \end{pmatrix}$$

$$\therefore \frac{5}{3}k + \frac{4}{3} = -1$$

$$\therefore \frac{5}{3}k = -\frac{7}{3}$$

$$\therefore k = -\frac{7}{5}$$

$$7 \quad a \quad |\mathbf{r}| = \sqrt{2^2 + 3^2} \\ = \sqrt{13} \text{ units}$$

$$b \quad |\mathbf{s}| = \sqrt{(-1)^2 + 4^2} \\ = \sqrt{17} \text{ units}$$

$$c \quad \mathbf{r} + \mathbf{s} \\ = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} -1 \\ 4 \end{pmatrix} \\ = \begin{pmatrix} 1 \\ 7 \end{pmatrix}$$

$$\therefore |\mathbf{r} + \mathbf{s}| \\ = \sqrt{1^2 + 7^2} \\ = \sqrt{1+49} \\ = \sqrt{50} \\ = 5\sqrt{2} \text{ units}$$

$$d \quad \mathbf{r} - \mathbf{s} \\ = \begin{pmatrix} 2 \\ 3 \end{pmatrix} - \begin{pmatrix} -1 \\ 4 \end{pmatrix} \\ = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

$$\therefore |\mathbf{r} - \mathbf{s}| \\ = \sqrt{3^2 + (-1)^2} \\ = \sqrt{9+1} \\ = \sqrt{10} \text{ units}$$

$$e \quad \mathbf{s} - 2\mathbf{r} \\ = \begin{pmatrix} -1 \\ 4 \end{pmatrix} - \begin{pmatrix} 4 \\ 6 \end{pmatrix} \\ = \begin{pmatrix} -5 \\ -2 \end{pmatrix}$$

$$\therefore |\mathbf{s} - 2\mathbf{r}| \\ = \sqrt{(-5)^2 + (-2)^2} \\ = \sqrt{25+4} \\ = \sqrt{29} \text{ units}$$

$$f \quad 2\mathbf{r} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$$

$$\therefore |2\mathbf{r}| = \sqrt{4^2 + 6^2} \\ = \sqrt{16+36} \\ = \sqrt{52} \\ = 2\sqrt{13} \text{ units}$$

$$g \quad \mathbf{r} + 2\mathbf{s} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} -2 \\ 8 \end{pmatrix} \\ = \begin{pmatrix} 0 \\ 11 \end{pmatrix}$$

$$\therefore |\mathbf{r} + 2\mathbf{s}| = 11 \text{ units}$$

$$h \quad 2\mathbf{r} - \mathbf{s} = \begin{pmatrix} 4 \\ 6 \end{pmatrix} - \begin{pmatrix} -1 \\ 4 \end{pmatrix} \\ = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

$$\therefore |2\mathbf{r} - \mathbf{s}| = \sqrt{5^2 + 2^2} \\ = \sqrt{25+4} \\ = \sqrt{29} \text{ units}$$

$$8 \quad a \quad i \quad |\mathbf{p}| = \sqrt{1^2 + 3^2} \\ = \sqrt{1+9} \\ = \sqrt{10} \text{ units}$$

$$ii \quad 2\mathbf{p} = \begin{pmatrix} 2 \\ 6 \end{pmatrix} \\ \therefore |2\mathbf{p}| = \sqrt{2^2 + 6^2} \\ = \sqrt{4+36} \\ = \sqrt{40} \\ = 2\sqrt{10} \text{ units}$$

$$\text{iii} \quad -2\mathbf{p} = \begin{pmatrix} -2 \\ -6 \end{pmatrix}$$

$$\begin{aligned} \therefore |-2\mathbf{p}| &= \sqrt{(-2)^2 + (-6)^2} \\ &= \sqrt{4 + 36} \\ &= \sqrt{40} \\ &= 2\sqrt{10} \text{ units} \end{aligned}$$

$$\text{iv} \quad 3\mathbf{p} = \begin{pmatrix} 3 \\ 9 \end{pmatrix}$$

$$\begin{aligned} \therefore |3\mathbf{p}| &= \sqrt{3^2 + 9^2} \\ &= \sqrt{9 + 81} \\ &= \sqrt{90} \\ &= 3\sqrt{10} \text{ units} \end{aligned}$$

$$\text{v} \quad -3\mathbf{p} = \begin{pmatrix} -3 \\ -9 \end{pmatrix}$$

$$\begin{aligned} \therefore |-3\mathbf{p}| &= \sqrt{(-3)^2 + (-9)^2} \\ &= \sqrt{9 + 81} \\ &= \sqrt{90} \\ &= 3\sqrt{10} \text{ units} \end{aligned}$$

$$\text{b} \quad k\mathbf{v} = \begin{pmatrix} kv_1 \\ kv_2 \end{pmatrix}$$

$$\begin{aligned} \therefore |k\mathbf{v}| &= \sqrt{(kv_1)^2 + (kv_2)^2} \\ &= \sqrt{k^2v_1^2 + k^2v_2^2} \\ &= \sqrt{k^2(v_1^2 + v_2^2)} \\ &= \sqrt{k^2} \sqrt{v_1^2 + v_2^2} \\ &= |k| \sqrt{v_1^2 + v_2^2} \\ &= |k| |\mathbf{v}| \end{aligned}$$

$$\text{9} \quad k\mathbf{x} = \mathbf{a}$$

$$\therefore k \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

$$\therefore kx_1 = a_1 \quad \text{and} \quad kx_2 = a_2$$

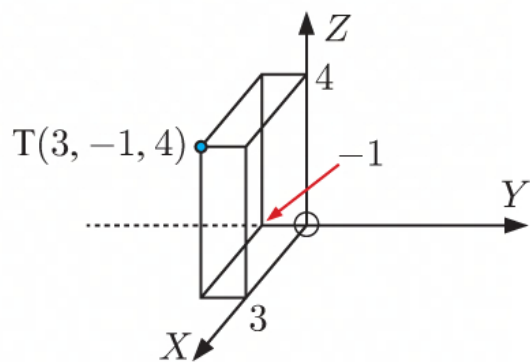
$$\therefore x_1 = \frac{1}{k}a_1 \quad \text{and} \quad x_2 = \frac{1}{k}a_2, \quad k \neq 0$$

$$\therefore \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{k}a_1 \\ \frac{1}{k}a_2 \end{pmatrix} = \frac{1}{k} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

$$\text{and so } \mathbf{x} = \frac{1}{k}\mathbf{a}, \quad k \neq 0$$

## EXERCISE 12F

1 a



$$\text{b } \overrightarrow{OT} = \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$$

$$\text{c } OT = \sqrt{3^2 + (-1)^2 + 4^2} \\ = \sqrt{26} \text{ units}$$

$$\text{2 a } \overrightarrow{OP} = \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix} = 2\mathbf{i} + 4\mathbf{k}$$

$$\text{c } \overrightarrow{OP} = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} = -2\mathbf{j}$$

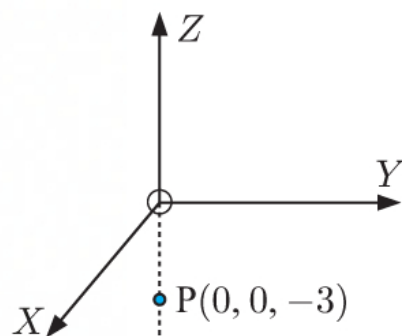
$$\text{3 a } \left| \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} \right| = 3 \text{ units}$$

$$\text{b } \left| \begin{pmatrix} -4 \\ 0 \\ 3 \end{pmatrix} \right| = \sqrt{(-4)^2 + 0^2 + 3^2} \\ = \sqrt{25} \\ = 5 \text{ units}$$

$$\text{c } \left| \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix} \right| = \sqrt{4^2 + 1^2 + (-2)^2} \\ = \sqrt{21} \text{ units}$$

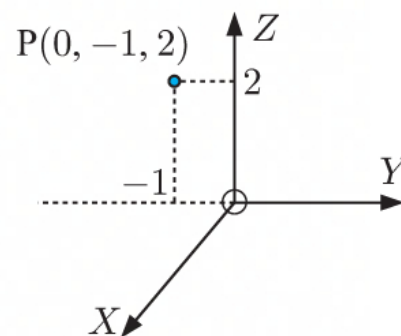
$$\text{d } \left| \begin{pmatrix} -2 \\ 2 \\ -1 \end{pmatrix} \right| = \sqrt{(-2)^2 + 2^2 + (-1)^2} \\ = \sqrt{9} \\ = 3 \text{ units}$$

4 a



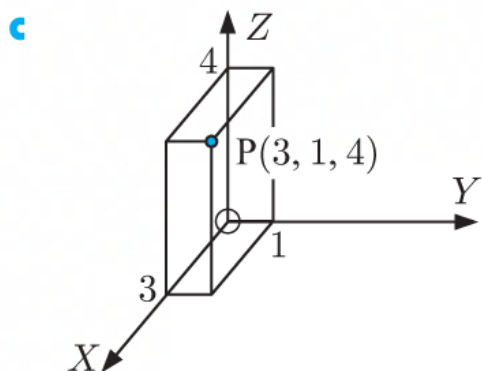
$$OP = \sqrt{0^2 + 0^2 + (-3)^2} \\ = \sqrt{9} \\ = 3 \text{ units}$$

b

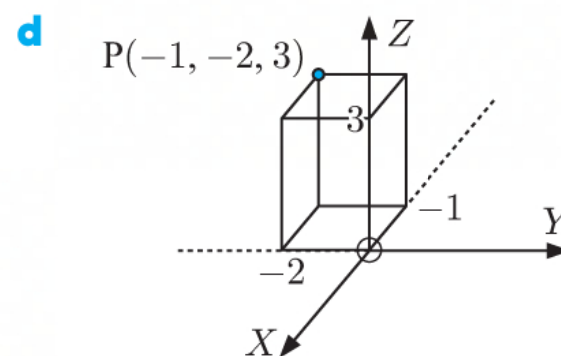


$$OP = \sqrt{0^2 + (-1)^2 + 2^2} \\ = \sqrt{5} \text{ units}$$





$$\begin{aligned} OP &= \sqrt{3^2 + 1^2 + 4^2} \\ &= \sqrt{26} \text{ units} \end{aligned}$$



$$\begin{aligned} OP &= \sqrt{(-1)^2 + (-2)^2 + 3^2} \\ &= \sqrt{14} \text{ units} \end{aligned}$$

**5 a**  $\mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}, \quad |\mathbf{a}| = \sqrt{2^2 + 1^2 + 4^2}$   
 $= \sqrt{21} \text{ units}$

**b**  $\mathbf{b} = \begin{pmatrix} -3 \\ 0 \\ 4 \end{pmatrix}, \quad |\mathbf{b}| = \sqrt{(-3)^2 + 0^2 + 4^2}$   
 $= \sqrt{25}$   
 $= 5 \text{ units}$

**c**  $\mathbf{c} = \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix}, \quad |\mathbf{c}| = \sqrt{0^2 + 2^2 + (-3)^2}$   
 $= \sqrt{13} \text{ units}$

**6 a**  $\begin{pmatrix} a-4 \\ b-3 \\ c+2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -4 \end{pmatrix}$   
 $\therefore a-4=1, \quad b-3=3, \quad c+2=-4$   
 $\therefore a=5, \quad b=6, \quad c=-6$

**b**  $\begin{pmatrix} a-5 \\ b-2 \\ c+3 \end{pmatrix} = \begin{pmatrix} 3-a \\ 2-b \\ 5-c \end{pmatrix}$   
 $\therefore a-5=3-a, \quad b-2=2-b, \quad \text{and} \quad c+3=5-c$   
 $\therefore 2a=8, \quad 2b=4, \quad \text{and} \quad 2c=2$   
 $\therefore a=4, \quad b=2, \quad \text{and} \quad c=1$   
 $\therefore a=4, \quad b=2, \quad c=1$

$$\text{c} \quad \begin{pmatrix} 3a+4 \\ b-c \\ 4-5b \end{pmatrix} = \begin{pmatrix} 12-a \\ a-4 \\ 5a-22 \end{pmatrix}$$

$$\therefore 3a+4 = 12-a$$

$$\therefore 4a = 8$$

$$\therefore a = 2$$

$$\text{Now } 4-5b = 5a-22$$

$$\therefore 4-5b = 5(2)-22$$

$$\therefore -5b = -16$$

$$\therefore b = \frac{16}{5}$$

$$\text{and } b-c = a-4$$

$$\therefore \frac{16}{5} - c = 2-4$$

$$\therefore c = \frac{26}{5}$$

$$\therefore a = 2, \quad b = \frac{16}{5}, \quad c = \frac{26}{5}$$

$$\text{7 a} \quad \left| \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \right| = 1 \text{ unit}$$

$$\therefore \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \text{ is a unit vector.}$$

$$\text{b} \quad \left| \begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \right| = \sqrt{0^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2}$$

$$= \sqrt{\frac{1}{4} + \frac{1}{4}}$$

$$= \sqrt{\frac{1}{2}}$$

$$= \frac{1}{\sqrt{2}} \text{ units}$$

$$\therefore \begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \text{ is not a unit vector.}$$

$$\text{c} \quad \left| \begin{pmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{pmatrix} \right| = \sqrt{\left(-\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2}$$

$$= \sqrt{\frac{4}{9} + \frac{1}{9} + \frac{4}{9}}$$

$$= \sqrt{1}$$

$$= 1 \text{ unit}$$

$$\therefore \begin{pmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{pmatrix} \text{ is a unit vector.}$$

$$\text{d} \quad \left| \begin{pmatrix} \frac{1}{3\sqrt{3}} \\ \frac{5}{3\sqrt{3}} \\ -\frac{1}{3\sqrt{3}} \end{pmatrix} \right|$$

$$= \sqrt{\left(\frac{1}{3\sqrt{3}}\right)^2 + \left(\frac{5}{3\sqrt{3}}\right)^2 + \left(-\frac{1}{3\sqrt{3}}\right)^2}$$

$$= \sqrt{\frac{1}{27} + \frac{25}{27} + \frac{1}{27}}$$

$$= \sqrt{1}$$

$$= 1 \text{ unit}$$

$$\therefore \begin{pmatrix} \frac{1}{3\sqrt{3}} \\ \frac{5}{3\sqrt{3}} \\ -\frac{1}{3\sqrt{3}} \end{pmatrix} \text{ is a unit vector.}$$

$$\begin{aligned}
 \mathbf{8} \quad \mathbf{a} \quad & \sqrt{\left(-\frac{1}{2}\right)^2 + k^2 + \left(\frac{1}{4}\right)^2} = 1 \\
 & \therefore \sqrt{\frac{1}{4} + k^2 + \frac{1}{16}} = 1 \\
 & \therefore \sqrt{k^2 + \frac{5}{16}} = 1 \\
 & \therefore k^2 + \frac{5}{16} = 1 \\
 & \therefore k^2 = \frac{11}{16} \\
 & \therefore k = \pm \frac{\sqrt{11}}{4}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \sqrt{k^2 + \left(\frac{2}{3}\right)^2 + \left(-\frac{1}{3}\right)^2} = 1 \\
 & \therefore \sqrt{k^2 + \frac{4}{9} + \frac{1}{9}} = 1 \\
 & \therefore \sqrt{k^2 + \frac{5}{9}} = 1 \\
 & \therefore k^2 + \frac{5}{9} = 1 \\
 & \therefore k^2 = \frac{4}{9} \\
 & \therefore k = \pm \frac{2}{3}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & \sqrt{k^2 + k^2 + k^2} = 1 \\
 & \therefore \sqrt{3k^2} = 1 \\
 & \therefore 3k^2 = 1 \\
 & \therefore k^2 = \frac{1}{3} \\
 & \therefore k = \pm \frac{1}{\sqrt{3}}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad & \sqrt{\left(\frac{1}{k}\right)^2 + \left(\frac{2}{k}\right)^2 + \left(\frac{3}{k}\right)^2} = 1 \\
 & \therefore \sqrt{\frac{1}{k^2} + \frac{4}{k^2} + \frac{9}{k^2}} = 1 \\
 & \therefore \sqrt{\frac{14}{k^2}} = 1 \\
 & \therefore \frac{14}{k^2} = 1 \\
 & \therefore k^2 = 14 \\
 & \therefore k = \pm \sqrt{14}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{9} \quad \mathbf{a} \quad & \left| \begin{pmatrix} -2 \\ m \\ 1 \end{pmatrix} \right| = \sqrt{14} \text{ units} \\
 & \therefore \sqrt{(-2)^2 + m^2 + 1^2} = \sqrt{14} \\
 & \therefore 4 + m^2 + 1 = 14 \\
 & \therefore m^2 = 9 \\
 & \therefore m = \pm 3
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \left| \begin{pmatrix} 4 \\ -3 \\ m \end{pmatrix} \right| = 6 \text{ units} \\
 & \therefore \sqrt{4^2 + (-3)^2 + m^2} = 6 \\
 & \therefore 16 + 9 + m^2 = 36 \\
 & \therefore m^2 = 11 \\
 & \therefore m = \pm \sqrt{11}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{10} \quad & \left| \begin{pmatrix} a \\ b \end{pmatrix} \right| = 5 \text{ units} \\
 & \therefore |a|, |b| \leq 5 \quad \text{and} \quad \sqrt{a^2 + b^2} = 5 \\
 & \therefore a^2 + b^2 = 25 \\
 & \therefore a^2 + 2b^2 + c^2 = 34
 \end{aligned}$$

$$\begin{aligned}
 & \left| \begin{pmatrix} b \\ c \end{pmatrix} \right| = 3 \text{ units} \\
 & \therefore |b|, |c| \leq 3 \quad \text{and} \quad \sqrt{b^2 + c^2} = 3 \\
 & \therefore b^2 + c^2 = 9
 \end{aligned}$$

$$\begin{aligned}
 \text{Now} \quad & \left| \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right| = \sqrt{a^2 + b^2 + c^2} \\
 & = \sqrt{a^2 + 2b^2 + c^2 - b^2} \\
 & = \sqrt{34 - b^2}
 \end{aligned}$$

which is maximised when  $b = 0$ , and minimised when  $b = \pm 3$   $\{-3 \leq b \leq 3\}$ .

$$\text{When } b = 0, \quad \left| \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right| = \sqrt{34} \text{ units}$$

$$\begin{aligned} \text{When } b = \pm 3, \quad \left| \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right| &= \sqrt{34 - (\pm 3)^2} \\ &= \sqrt{34 - 9} \\ &= \sqrt{25} \\ &= 5 \text{ units} \end{aligned}$$

$$\therefore 5 \leq \left| \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right| \leq \sqrt{34}$$

## EXERCISE 12G

$$\begin{aligned} \mathbf{1} \quad \mathbf{a} \quad \mathbf{a} + \mathbf{b} &= \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} \\ &= \begin{pmatrix} 2+1 \\ -1+2 \\ 1+(-3) \end{pmatrix} \\ &= \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \mathbf{a} - \mathbf{b} &= \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} \\ &= \begin{pmatrix} 2-1 \\ -1-2 \\ 1-(-3) \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad \mathbf{b} + 2\mathbf{c} &= \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} + 2 \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \\ -6 \end{pmatrix} \\ &= \begin{pmatrix} 1+0 \\ 2+2 \\ -3+(-6) \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 4 \\ -9 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad \mathbf{c} - \frac{1}{2}\mathbf{a} &= \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} - \begin{pmatrix} 1 \\ -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \\ &= \begin{pmatrix} 0-1 \\ 1-(-\frac{1}{2}) \\ -3-\frac{1}{2} \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ \frac{3}{2} \\ -\frac{7}{2} \end{pmatrix} \end{aligned}$$



$$\begin{aligned}
 \text{e} \quad \mathbf{a} - \mathbf{b} - \mathbf{c} &= \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} \\
 &= \begin{pmatrix} 2 - 1 - 0 \\ -1 - 2 - 1 \\ 1 - (-3) - (-3) \end{pmatrix} \\
 &= \begin{pmatrix} 1 \\ -4 \\ 7 \end{pmatrix}
 \end{aligned}$$

$$\begin{aligned}
 \text{2 a} \quad |\mathbf{a}| &= \sqrt{1^2 + 0^2 + 3^2} \\
 &= \sqrt{10} \text{ units}
 \end{aligned}$$

$$\text{c} \quad 2|\mathbf{a}| = 2\sqrt{10} \text{ units} \quad \{\text{using a}\}$$

$$\text{e} \quad -3|\mathbf{b}| = -3\sqrt{6} \text{ units} \quad \{\text{using b}\}$$

$$\begin{aligned}
 \text{g} \quad \mathbf{a} + \mathbf{b} &= \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \\
 &= \begin{pmatrix} 1 + (-2) \\ 0 + 1 \\ 3 + 1 \end{pmatrix} \\
 &= \begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix}
 \end{aligned}$$

$$\begin{aligned}
 \therefore |\mathbf{a} + \mathbf{b}| &= \sqrt{(-1)^2 + 1^2 + 4^2} \\
 &= \sqrt{18} \\
 &= 3\sqrt{2} \text{ units}
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad 2\mathbf{b} - \mathbf{c} + \mathbf{a} &= 2 \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \\
 &= \begin{pmatrix} 2 \\ 4 \\ -6 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \\
 &= \begin{pmatrix} 2 - 0 + 2 \\ 4 - 1 + (-1) \\ -6 - (-3) + 1 \end{pmatrix} \\
 &= \begin{pmatrix} 4 \\ 2 \\ -2 \end{pmatrix}
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad |\mathbf{b}| &= \sqrt{(-2)^2 + 1^2 + 1^2} \\
 &= \sqrt{6} \text{ units}
 \end{aligned}$$

$$\text{d} \quad 2\mathbf{a} = 2 \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 6 \end{pmatrix}$$

$$\begin{aligned}
 \therefore |2\mathbf{a}| &= \sqrt{2^2 + 0^2 + 6^2} \\
 &= \sqrt{40} \\
 &= 2\sqrt{10} \text{ units}
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad -3\mathbf{b} &= -3 \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \\ -3 \end{pmatrix} \\
 \therefore |-3\mathbf{b}| &= \sqrt{6^2 + (-3)^2 + (-3)^2} \\
 &= \sqrt{54} \\
 &= 3\sqrt{6} \text{ units}
 \end{aligned}$$

$$\begin{aligned}
 \text{h} \quad \mathbf{a} - \mathbf{b} &= \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} - \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \\
 &= \begin{pmatrix} 1 - (-2) \\ 0 - 1 \\ 3 - 1 \end{pmatrix} \\
 &= \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}
 \end{aligned}$$

$$\begin{aligned}
 \therefore |\mathbf{a} - \mathbf{b}| &= \sqrt{3^2 + (-1)^2 + 2^2} \\
 &= \sqrt{14} \text{ units}
 \end{aligned}$$

$$\begin{aligned}
 \text{3 a } 2\mathbf{a} - \mathbf{c} &= 2 \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} - \begin{pmatrix} -2 \\ 2 \\ 4 \end{pmatrix} \\
 &= \begin{pmatrix} -2 \\ 2 \\ 6 \end{pmatrix} - \begin{pmatrix} -2 \\ 2 \\ 4 \end{pmatrix} \\
 &= \begin{pmatrix} -2 - (-2) \\ 2 - 2 \\ 6 - 4 \end{pmatrix} \\
 &= \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}
 \end{aligned}$$

$$\begin{aligned}
 \text{b } 3\mathbf{b} + \frac{1}{2}\mathbf{c} &= 3 \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} -2 \\ 2 \\ 4 \end{pmatrix} \\
 &= \begin{pmatrix} 3 \\ -9 \\ 6 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \\
 &= \begin{pmatrix} 2 \\ -8 \\ 8 \end{pmatrix}
 \end{aligned}$$

$$\begin{aligned}
 \text{c } -\mathbf{a} + 3\mathbf{b} &= - \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} + 3 \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} \\
 &= \begin{pmatrix} 1 \\ -1 \\ -3 \end{pmatrix} + \begin{pmatrix} 3 \\ -9 \\ 6 \end{pmatrix} \\
 &= \begin{pmatrix} 1 + 3 \\ -1 + (-9) \\ -3 + 6 \end{pmatrix} \\
 &= \begin{pmatrix} 4 \\ -10 \\ 3 \end{pmatrix}
 \end{aligned}$$

$$\begin{aligned}
 \text{d } \mathbf{a} - 5\mathbf{b} + 4\mathbf{c} &= \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} - 5 \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} + 4 \begin{pmatrix} -2 \\ 2 \\ 4 \end{pmatrix} \\
 &= \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} - \begin{pmatrix} 5 \\ -15 \\ 10 \end{pmatrix} + \begin{pmatrix} -8 \\ 8 \\ 16 \end{pmatrix} \\
 &= \begin{pmatrix} -14 \\ 24 \\ 9 \end{pmatrix}
 \end{aligned}$$

$$\begin{aligned}
 \text{e } \mathbf{b} + \mathbf{c} &= \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} + \begin{pmatrix} -2 \\ 2 \\ 4 \end{pmatrix} \\
 &= \begin{pmatrix} -1 \\ -1 \\ 6 \end{pmatrix}
 \end{aligned}$$

$$\begin{aligned}
 \therefore |\mathbf{b} + \mathbf{c}| &= \sqrt{(-1)^2 + (-1)^2 + 6^2} \\
 &= \sqrt{38} \text{ units}
 \end{aligned}$$

$$\begin{aligned}
 \text{f } \mathbf{a} - \mathbf{c} &= \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} - \begin{pmatrix} -2 \\ 2 \\ 4 \end{pmatrix} \\
 &= \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}
 \end{aligned}$$

$$\begin{aligned}
 \therefore |\mathbf{a} - \mathbf{c}| &= \sqrt{1^2 + (-1)^2 + (-1)^2} \\
 &= \sqrt{3} \text{ units}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{g} \quad |\mathbf{a}| &= \left| \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} \right| \\
 &= \sqrt{(-1)^2 + 1^2 + 3^2} \\
 &= \sqrt{11} \text{ units} \\
 \therefore |\mathbf{a}| \mathbf{b} &= \sqrt{11} \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} \\
 &= \begin{pmatrix} \sqrt{11} \\ -3\sqrt{11} \\ 2\sqrt{11} \end{pmatrix}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{h} \quad \frac{1}{|\mathbf{a}|} \mathbf{a} &= \frac{1}{\sqrt{11}} \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} \quad \{\text{using } \mathbf{g}\} \\
 &= \begin{pmatrix} -\frac{1}{\sqrt{11}} \\ \frac{1}{\sqrt{11}} \\ \frac{3}{\sqrt{11}} \end{pmatrix}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{4} \quad \mathbf{a} \quad 2 \begin{pmatrix} 1 \\ 0 \\ 3a \end{pmatrix} &= \begin{pmatrix} b \\ c-1 \\ 2 \end{pmatrix} \\
 \therefore \begin{pmatrix} 2 \\ 0 \\ 6a \end{pmatrix} &= \begin{pmatrix} b \\ c-1 \\ 2 \end{pmatrix} \\
 \therefore 2 &= b, \quad 0 = c-1, \quad \text{and} \quad 6a = 2 \\
 \therefore a &= \frac{1}{3}, \quad b = 2, \quad \text{and} \quad c = 1
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad a \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} + c \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} &= \begin{pmatrix} -1 \\ 3 \\ 3 \end{pmatrix} \\
 \therefore \begin{pmatrix} a \\ a \\ 0 \end{pmatrix} + \begin{pmatrix} 2b \\ 0 \\ -b \end{pmatrix} + \begin{pmatrix} 0 \\ c \\ c \end{pmatrix} &= \begin{pmatrix} -1 \\ 3 \\ 3 \end{pmatrix} \\
 \therefore \begin{pmatrix} a+2b \\ a+c \\ -b+c \end{pmatrix} &= \begin{pmatrix} -1 \\ 3 \\ 3 \end{pmatrix}
 \end{aligned}$$

$$\text{So, } a + 2b = -1 \quad \dots (1)$$

$$a + c = 3$$

$$\therefore c = 3 - a \quad \dots (2)$$

$$-b + c = 3$$

$$\therefore c = b + 3 \quad \dots (3)$$

Equating (2) and (3), we get

$$3 - a = b + 3$$

$$\therefore -a = b$$

$$\therefore a = 1, \quad b = -1, \quad \text{and} \quad c = 2$$

Substituting into (1), we get

$$a + 2(-a) = -1$$

$$\therefore -a = -1$$

$$\therefore a = 1$$

$$\therefore b = -1$$

$$\text{and } c = -1 + 3 = 2 \quad \{\text{using (3)}\}$$

$$\begin{aligned} \text{c} \quad & a \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} + b \begin{pmatrix} 1 \\ 7 \\ 2 \end{pmatrix} = \begin{pmatrix} 7 \\ c \\ 2 \end{pmatrix} \\ \therefore & \begin{pmatrix} 2a \\ -3a \\ a \end{pmatrix} + \begin{pmatrix} b \\ 7b \\ 2b \end{pmatrix} = \begin{pmatrix} 7 \\ c \\ 2 \end{pmatrix} \\ \therefore & \begin{pmatrix} 2a + b \\ -3a + 7b \\ a + 2b \end{pmatrix} = \begin{pmatrix} 7 \\ c \\ 2 \end{pmatrix} \end{aligned}$$

$$\text{So, } 2a + b = 7$$

$$\therefore b = 7 - 2a \quad \dots (1)$$

$$-3a + 7b = c \quad \dots (2)$$

$$a + 2b = 2 \quad \dots (3)$$

Substituting (1) into (3), we get

$$a + 2(7 - 2a) = 2$$

$$\therefore a + 14 - 4a = 2$$

$$\therefore -3a = -12$$

$$\therefore a = 4$$

$$\text{and so } b = 7 - 2(4) = -1$$

$$\therefore a = 4, \quad b = -1, \quad c = -19$$

Substituting  $a = 4$  and  $b = -1$  into (2),

$$\text{we get } -3(4) + 7(-1) = c$$

$$\therefore -12 - 7 = c$$

$$\therefore c = -19$$

$$\begin{aligned} \text{d} \quad & 2 \begin{pmatrix} a \\ 1 \\ -2 \end{pmatrix} + \begin{pmatrix} b \\ c \\ 4 \end{pmatrix} = \begin{pmatrix} c \\ -c \\ ab \end{pmatrix} \\ \therefore & \begin{pmatrix} 2a \\ 2 \\ -4 \end{pmatrix} + \begin{pmatrix} b \\ c \\ 4 \end{pmatrix} = \begin{pmatrix} c \\ -c \\ ab \end{pmatrix} \\ \therefore & \begin{pmatrix} 2a + b \\ 2 + c \\ 0 \end{pmatrix} = \begin{pmatrix} c \\ -c \\ ab \end{pmatrix} \end{aligned}$$

$$\text{So, } 2a + b = c \quad \dots (1)$$

$$2 + c = -c$$

$$\therefore 2c = -2$$

$$\therefore c = -1 \quad \dots (2)$$

$$ab = 0$$

$$\therefore a = 0 \quad \text{or} \quad b = 0$$

If  $a = 0$ , then

$$b = c = -1 \quad \{\text{using (1) and (2)}\}$$

If  $b = 0$ , then

$$2a = c = -1 \quad \{\text{using (1) and (2)}\}$$

$$\therefore a = -\frac{1}{2}$$

$$\therefore a = 0, \quad b = -1, \quad c = -1 \quad \text{or} \quad a = -\frac{1}{2}, \quad b = 0, \quad c = -1$$



**INVESTIGATION 1****PROPERTIES OF VECTORS IN SPACE**

$$\mathbf{1} \quad \mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

$$\begin{aligned} \mathbf{a} \quad \mathbf{a} + \mathbf{b} &= \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \\ &= \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{pmatrix} \\ &= \begin{pmatrix} b_1 + a_1 \\ b_2 + a_2 \\ b_3 + a_3 \end{pmatrix} \quad \{a + b = b + a\} \\ &= \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} + \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \\ &= \mathbf{b} + \mathbf{a} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \mathbf{a} + \mathbf{0} &= \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} a_1 + 0 \\ a_2 + 0 \\ a_3 + 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 + a_1 \\ 0 + a_2 \\ 0 + a_3 \end{pmatrix} \\ &= \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \quad = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \\ &= \mathbf{a} \quad = \mathbf{0} + \mathbf{a} \end{aligned}$$

$$\therefore \mathbf{a} + \mathbf{0} = \mathbf{0} + \mathbf{a} = \mathbf{a}$$

$$\begin{aligned} \mathbf{c} \quad \mathbf{a} + (-\mathbf{a}) &= \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \begin{pmatrix} -a_1 \\ -a_2 \\ -a_3 \end{pmatrix} \\ &= \begin{pmatrix} a_1 - a_1 \\ a_2 - a_2 \\ a_3 - a_3 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} -a_1 + a_1 \\ -a_2 + a_2 \\ -a_3 + a_3 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad = \begin{pmatrix} -a_1 \\ -a_2 \\ -a_3 \end{pmatrix} + \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \\ &= \mathbf{0} \quad = (-\mathbf{a}) + \mathbf{a} \end{aligned}$$

$$\therefore \mathbf{a} + (-\mathbf{a}) = (-\mathbf{a}) + \mathbf{a} = \mathbf{0}$$

$$\begin{aligned}
 \text{d } (\mathbf{a} + \mathbf{b}) + \mathbf{c} &= \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \\
 &= \begin{pmatrix} a_1 + b_1 + c_1 \\ a_2 + b_2 + c_2 \\ a_3 + b_3 + c_3 \end{pmatrix} \\
 &= \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \begin{pmatrix} b_1 + c_1 \\ b_2 + c_2 \\ b_3 + c_3 \end{pmatrix} \\
 &= \mathbf{a} + (\mathbf{b} + \mathbf{c})
 \end{aligned}$$

$$\begin{aligned}
 \text{2 a } |\mathbf{ka}| &= \left| k \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \right| \\
 &= \left| \begin{pmatrix} ka_1 \\ ka_2 \\ ka_3 \end{pmatrix} \right| \\
 &= \sqrt{(ka_1)^2 + (ka_2)^2 + (ka_3)^2} \\
 &= \sqrt{k^2 a_1^2 + k^2 a_2^2 + k^2 a_3^2} \\
 &= \sqrt{k^2(a_1^2 + a_2^2 + a_3^2)} \\
 &= \sqrt{k^2} \sqrt{a_1^2 + a_2^2 + a_3^2} \\
 &= |k| |\mathbf{a}|
 \end{aligned}$$

$$\begin{aligned}
 \text{b } k(\mathbf{a} + \mathbf{b}) &= k \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{pmatrix} \\
 &= \begin{pmatrix} ka_1 + kb_1 \\ ka_2 + kb_2 \\ ka_3 + kb_3 \end{pmatrix} \\
 &= \begin{pmatrix} ka_1 \\ ka_2 \\ ka_3 \end{pmatrix} + \begin{pmatrix} kb_1 \\ kb_2 \\ kb_3 \end{pmatrix} \\
 &= k \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + k \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \\
 &= k\mathbf{a} + k\mathbf{b}
 \end{aligned}$$

## EXERCISE 12H

$$\begin{aligned}
 \text{1 a } 2\mathbf{x} &= \mathbf{q} \\
 \therefore \mathbf{x} &= \frac{1}{2}\mathbf{q}
 \end{aligned}$$

$$\begin{aligned}
 \text{b } \frac{1}{2}\mathbf{x} &= \mathbf{n} \\
 \therefore \mathbf{x} &= 2\mathbf{n}
 \end{aligned}$$

$$\begin{aligned}
 \text{c } -3\mathbf{x} &= \mathbf{p} \\
 \therefore 3\mathbf{x} &= -\mathbf{p} \\
 \therefore \mathbf{x} &= -\frac{1}{3}\mathbf{p}
 \end{aligned}$$

$$\begin{aligned}
 \text{d } \mathbf{q} + 2\mathbf{x} &= \mathbf{r} \\
 \therefore 2\mathbf{x} &= \mathbf{r} - \mathbf{q} \\
 \therefore \mathbf{x} &= \frac{1}{2}(\mathbf{r} - \mathbf{q})
 \end{aligned}$$

$$\begin{aligned}
 \text{e } 4\mathbf{s} - 5\mathbf{x} &= \mathbf{t} \\
 \therefore -5\mathbf{x} &= \mathbf{t} - 4\mathbf{s} \\
 \therefore 5\mathbf{x} &= 4\mathbf{s} - \mathbf{t} \\
 \therefore \mathbf{x} &= \frac{1}{5}(4\mathbf{s} - \mathbf{t})
 \end{aligned}$$

$$\begin{aligned}
 \text{f } 4\mathbf{m} - \frac{1}{3}\mathbf{x} &= \mathbf{n} \\
 \therefore 4\mathbf{m} - \mathbf{n} &= \frac{1}{3}\mathbf{x} \\
 \therefore \mathbf{x} &= 3(4\mathbf{m} - \mathbf{n}) \\
 &= 12\mathbf{m} - 3\mathbf{n}
 \end{aligned}$$

$$\mathbf{2} \quad \mathbf{a} \quad 2\mathbf{x} = \mathbf{p}$$

$$\begin{aligned} \therefore \mathbf{x} &= \frac{1}{2}\mathbf{p} \\ &= \frac{1}{2} \begin{pmatrix} 2 \\ -5 \\ 4 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ -\frac{5}{2} \\ 2 \end{pmatrix} \end{aligned}$$

$$\mathbf{b} \quad \frac{1}{3}\mathbf{x} = \mathbf{p}$$

$$\begin{aligned} \therefore \mathbf{x} &= 3\mathbf{p} \\ &= 3 \begin{pmatrix} 2 \\ -5 \\ 4 \end{pmatrix} \\ &= \begin{pmatrix} 6 \\ -15 \\ 12 \end{pmatrix} \end{aligned}$$

$$\mathbf{c} \quad 4\mathbf{x} + \mathbf{p} = \mathbf{0}$$

$$\begin{aligned} \therefore 4\mathbf{x} &= -\mathbf{p} \\ \therefore \mathbf{x} &= -\frac{1}{4}\mathbf{p} \\ &= -\frac{1}{4} \begin{pmatrix} 2 \\ -5 \\ 4 \end{pmatrix} \\ &= \begin{pmatrix} -\frac{1}{2} \\ \frac{5}{4} \\ -1 \end{pmatrix} \end{aligned}$$

$$\mathbf{3} \quad \mathbf{a} \quad 2\mathbf{a} + \mathbf{x} = \mathbf{b}$$

$$\begin{aligned} \therefore \mathbf{x} &= \mathbf{b} - 2\mathbf{a} \\ &= \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} - 2 \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} - \begin{pmatrix} -2 \\ 4 \\ 6 \end{pmatrix} \\ &= \begin{pmatrix} 4 \\ -6 \\ -5 \end{pmatrix} \end{aligned}$$

$$\mathbf{b} \quad 3\mathbf{x} - \mathbf{a} = 2\mathbf{b}$$

$$\begin{aligned} \therefore 3\mathbf{x} &= \mathbf{a} + 2\mathbf{b} \\ \therefore \mathbf{x} &= \frac{1}{3}(\mathbf{a} + 2\mathbf{b}) \\ &= \frac{1}{3} \left[ \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} + 2 \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \right] \\ &= \frac{1}{3} \left[ \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 4 \\ -4 \\ 2 \end{pmatrix} \right] \\ &= \frac{1}{3} \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ -\frac{2}{3} \\ \frac{5}{3} \end{pmatrix} \end{aligned}$$

$$\mathbf{c} \quad 2\mathbf{b} - 2\mathbf{x} = -\mathbf{a}$$

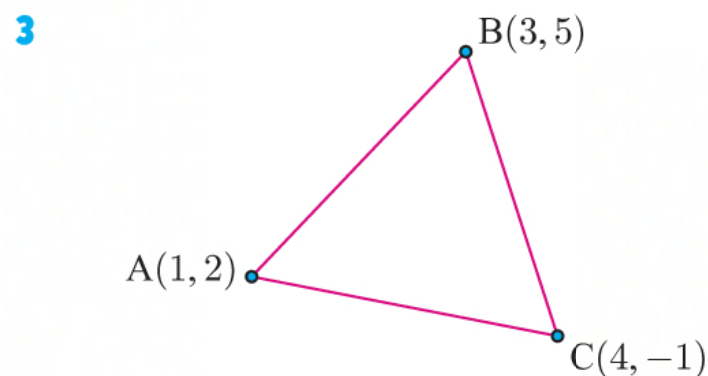
$$\therefore \mathbf{a} + 2\mathbf{b} = 2\mathbf{x}$$

$$\begin{aligned} \therefore \mathbf{x} &= \frac{1}{2}(\mathbf{a} + 2\mathbf{b}) \\ &= \frac{1}{2} \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix} \quad \{\text{using } \mathbf{b}\} \\ &= \begin{pmatrix} \frac{3}{2} \\ -1 \\ \frac{5}{2} \end{pmatrix} \end{aligned}$$

## EXERCISE 12I

$$1 \quad \begin{array}{lll} \text{a} \quad \overrightarrow{AB} = \begin{pmatrix} 4-2 \\ 7-3 \end{pmatrix} & \text{b} \quad \overrightarrow{AB} = \begin{pmatrix} 1-3 \\ 4-(-1) \end{pmatrix} & \text{c} \quad \overrightarrow{AB} = \begin{pmatrix} 1-(-2) \\ 4-7 \end{pmatrix} \\ & = \begin{pmatrix} -2 \\ 5 \end{pmatrix} & = \begin{pmatrix} 3 \\ -3 \end{pmatrix} \\ & = \begin{pmatrix} 2 \\ 4 \end{pmatrix} & \end{array}$$

$$2 \quad \begin{array}{lll} \text{a} \quad \overrightarrow{AB} = \begin{pmatrix} 6-2 \\ 2-3 \end{pmatrix} & \text{b} \quad \overrightarrow{BA} = -\overrightarrow{AB} & \text{c} \quad |\overrightarrow{AB}| = \left| \begin{pmatrix} -4 \\ 1 \end{pmatrix} \right| \quad \{\text{using a}\} \\ & = -\begin{pmatrix} 4 \\ -1 \end{pmatrix} \quad \{\text{using a}\} & = \sqrt{4^2 + (-1)^2} \\ & = \begin{pmatrix} -4 \\ 1 \end{pmatrix} & = \sqrt{17} \text{ units} \\ & & \end{array}$$



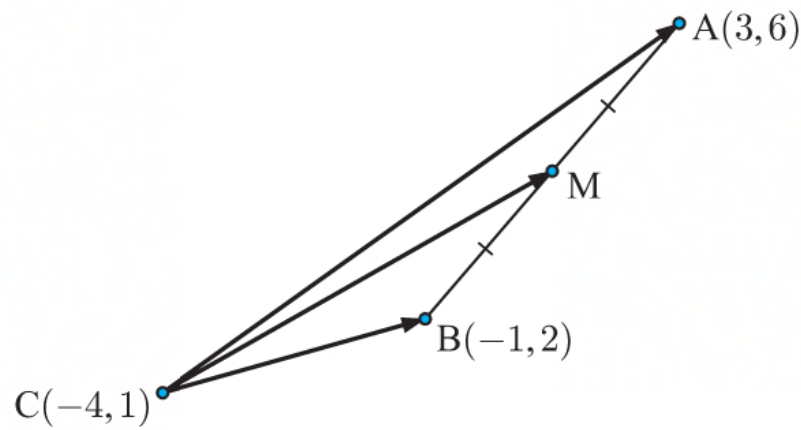
$$\begin{array}{llll} \text{a} \quad \overrightarrow{AB} = \begin{pmatrix} 3-1 \\ 5-2 \end{pmatrix} & \text{b} \quad \overrightarrow{BC} & \text{c} \quad \overrightarrow{BC} & \text{d} \quad \overrightarrow{BC} = \begin{pmatrix} 4-3 \\ -1-5 \end{pmatrix} \\ & = \overrightarrow{BA} + \overrightarrow{AC} & = -\overrightarrow{AB} + \overrightarrow{AC} & = \begin{pmatrix} 1 \\ -6 \end{pmatrix} \quad \checkmark \\ & = -\overrightarrow{AB} + \overrightarrow{AC} & = -\begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 3 \\ -3 \end{pmatrix} & \\ \overrightarrow{AC} = \begin{pmatrix} 4-1 \\ -1-2 \end{pmatrix} & & = \begin{pmatrix} -2 \\ -3 \end{pmatrix} + \begin{pmatrix} 3 \\ -3 \end{pmatrix} & \\ & & = \begin{pmatrix} 1 \\ -6 \end{pmatrix} & \\ & & & \end{array}$$

$$4 \quad \begin{array}{ll} \text{a} \quad \overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} & \text{b} \quad \overrightarrow{CB} = \overrightarrow{CA} + \overrightarrow{AB} \\ & = -\overrightarrow{BA} + \overrightarrow{BC} \\ & = -\begin{pmatrix} 2 \\ -3 \end{pmatrix} + \begin{pmatrix} -3 \\ 1 \end{pmatrix} \\ & = \begin{pmatrix} -5 \\ 4 \end{pmatrix} \\ & \\ & = \begin{pmatrix} 2 \\ -1 \end{pmatrix} + \begin{pmatrix} -1 \\ 3 \end{pmatrix} \\ & = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \end{array}$$



$$\begin{aligned}
 \text{c } \overrightarrow{SP} &= \overrightarrow{SR} + \overrightarrow{RQ} + \overrightarrow{QP} \\
 &= -\overrightarrow{RS} + \overrightarrow{RQ} - \overrightarrow{PQ} \\
 &= -\begin{pmatrix} -3 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 \\ 4 \end{pmatrix} \\
 &= \begin{pmatrix} 6 \\ -5 \end{pmatrix}
 \end{aligned}$$

5



$$\text{a } M \text{ is } \left( \frac{3 + (-1)}{2}, \frac{6 + 2}{2} \right)$$

$$\therefore M \text{ is } (1, 4)$$

$$\text{b } \overrightarrow{CA} = \begin{pmatrix} 3 - (-4) \\ 6 - 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$$

$$\overrightarrow{CM} = \begin{pmatrix} 1 - (-4) \\ 4 - 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$$

$$\overrightarrow{CB} = \begin{pmatrix} -1 - (-4) \\ 2 - 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$\begin{aligned}
 \text{c } &\frac{1}{2} \overrightarrow{CA} + \frac{1}{2} \overrightarrow{CB} \\
 &= \frac{1}{2} \begin{pmatrix} 7 \\ 5 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad \{\text{from b}\} \\
 &= \begin{pmatrix} \frac{7}{2} \\ \frac{5}{2} \end{pmatrix} + \begin{pmatrix} \frac{3}{2} \\ \frac{1}{2} \end{pmatrix} \\
 &= \begin{pmatrix} 5 \\ 3 \end{pmatrix} \\
 &= \overrightarrow{CM} \quad \{\text{from b}\}
 \end{aligned}$$

$$\begin{aligned}
 \text{6 a } \overrightarrow{PQ} &= \begin{pmatrix} 3 - 1 \\ 5 - 0 \\ 4 - 2 \end{pmatrix} \\
 &= \begin{pmatrix} 2 \\ 5 \\ 2 \end{pmatrix}
 \end{aligned}$$

$$\begin{aligned}
 \text{b } \overrightarrow{PQ} &= \begin{pmatrix} 6 - 5 \\ 2 - 2 \\ -1 - 3 \end{pmatrix} \\
 &= \begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix}
 \end{aligned}$$

$$\begin{aligned}
 \text{c } \overrightarrow{PQ} &= \begin{pmatrix} 1 - (-2) \\ 4 - 3 \\ -3 - 0 \end{pmatrix} \\
 &= \begin{pmatrix} 3 \\ 1 \\ -3 \end{pmatrix}
 \end{aligned}$$

$$\begin{aligned}
 \text{d } \overrightarrow{PQ} &= \begin{pmatrix} -1 - 4 \\ -5 - (-1) \\ 3 - 5 \end{pmatrix} \\
 &= \begin{pmatrix} -5 \\ -4 \\ -2 \end{pmatrix}
 \end{aligned}$$

$$\begin{aligned}
 \text{7 } \overrightarrow{AB} &= \overrightarrow{AO} + \overrightarrow{OB} = -\overrightarrow{OA} + \overrightarrow{OB} \\
 &= -\begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} \\
 &= \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} \\
 &= \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix}
 \end{aligned}$$

$$\therefore AB = \sqrt{3^2 + 4^2 + (-2)^2} = \sqrt{29} \text{ units}$$

$$\text{8 a } \overrightarrow{AB} = \begin{pmatrix} 1 - (-3) \\ 0 - 1 \\ -1 - 2 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix}, \quad \overrightarrow{BA} = \begin{pmatrix} -3 - 1 \\ 1 - 0 \\ 2 - (-1) \end{pmatrix} = \begin{pmatrix} -4 \\ 1 \\ 3 \end{pmatrix}$$

$$\begin{aligned}
 \text{b } |\overrightarrow{AB}| &= \sqrt{4^2 + (-1)^2 + (-3)^2} = \sqrt{26} \text{ units,} \\
 |\overrightarrow{BA}| &= \sqrt{(-4)^2 + 1^2 + 3^2} = \sqrt{26} \text{ units}
 \end{aligned}$$

$$\text{9 a The displacement vector of M relative to N} \\ = \overrightarrow{NM}$$

$$\begin{aligned}
 &= \begin{pmatrix} 4 - (-1) \\ -2 - 2 \\ -1 - 0 \end{pmatrix} \\
 &= \begin{pmatrix} 5 \\ -4 \\ -1 \end{pmatrix}
 \end{aligned}$$

$$\text{c } MN = \sqrt{(-5)^2 + 4^2 + 1^2} = \sqrt{42} \text{ units}$$

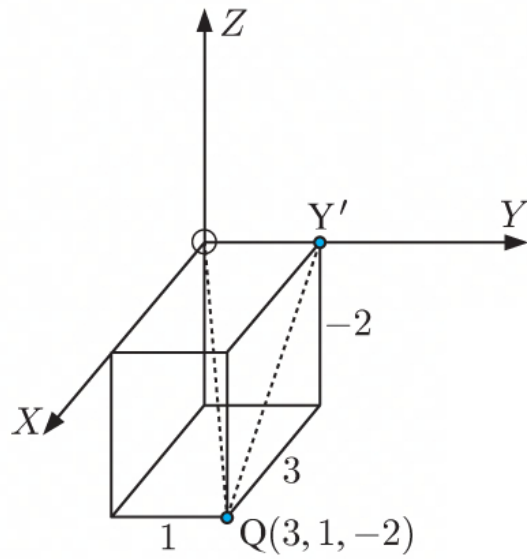
$$\text{b The displacement vector of N relative to M} \\ = \overrightarrow{MN}$$

$$\begin{aligned}
 &= \begin{pmatrix} -1 - 4 \\ 2 - (-2) \\ 0 - (-1) \end{pmatrix} \\
 &= \begin{pmatrix} -5 \\ 4 \\ 1 \end{pmatrix}
 \end{aligned}$$

$$\begin{aligned}
 \text{10 a } \overrightarrow{AB} &= \begin{pmatrix} 3 - (-1) \\ -2 - 3 \\ 1 - (-2) \end{pmatrix} \\
 &= \begin{pmatrix} 4 \\ -5 \\ 3 \end{pmatrix} \\
 &= 4\mathbf{i} - 5\mathbf{j} + 3\mathbf{k}
 \end{aligned}$$

$$\begin{aligned}
 \text{b } |\overrightarrow{AB}| &= \sqrt{4^2 + (-5)^2 + 3^2} \\
 &= \sqrt{50} \\
 &= 5\sqrt{2} \text{ units}
 \end{aligned}$$

11



- a** The distance from Q to the Y-axis is the distance from Q to  $Y'(0, 1, 0)$ .

$$\begin{aligned}\therefore QY' &= \sqrt{(3-0)^2 + (1-1)^2 + (-2-0)^2} \\ &= \sqrt{9+0+4} \\ &= \sqrt{13} \text{ units}\end{aligned}$$

- b** The distance from Q to the origin is

$$\begin{aligned}QO &= \sqrt{(3-0)^2 + (1-0)^2 + (-2-0)^2} \\ &= \sqrt{9+1+4} \\ &= \sqrt{14} \text{ units}\end{aligned}$$

- c** The distance from Q to the YZ-plane is the distance from Q to  $(0, 1, -2)$ , which is 3 units.

$$\begin{aligned}\mathbf{12} \quad \vec{AC} &= \vec{AB} + \vec{BC} \\ &= (\mathbf{i} - \mathbf{j} + \mathbf{k}) + (-2\mathbf{i} + \mathbf{j} - 3\mathbf{k}) \\ &= -\mathbf{i} - 2\mathbf{k}\end{aligned}$$

$$\begin{aligned}\mathbf{13} \quad \mathbf{a} \quad \vec{AD} &= \vec{AB} + \vec{BD} \\ &= \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ 5 \\ -1 \end{pmatrix}\end{aligned}$$

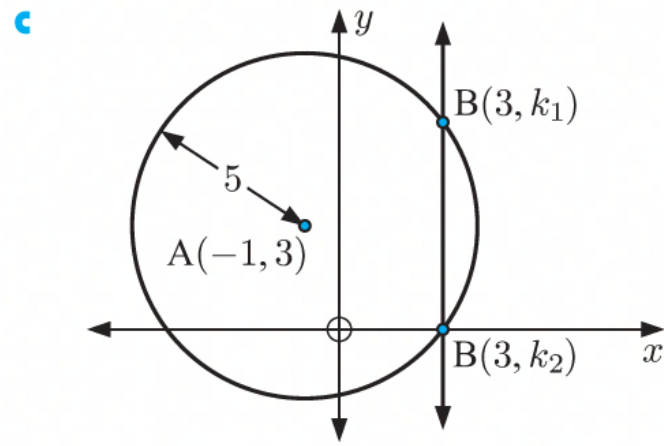
$$\begin{aligned}\mathbf{b} \quad \vec{CB} &= \vec{CA} + \vec{AB} \\ &= -\vec{AC} + \vec{AB} \\ &= -\begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} + \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} -2 \\ 1 \\ -4 \end{pmatrix} + \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} -3 \\ 4 \\ -2 \end{pmatrix}\end{aligned}$$

$$\begin{aligned}\mathbf{c} \quad \vec{CD} &= \vec{CB} + \vec{BD} \\ &= \begin{pmatrix} -3 \\ 4 \\ -2 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix} \quad \{\text{using } \mathbf{b}\} \\ &= \begin{pmatrix} -3 \\ 6 \\ -5 \end{pmatrix}\end{aligned}$$

$$\begin{aligned}\mathbf{14} \quad \mathbf{a} \quad \vec{AB} &= \begin{pmatrix} 3 - (-1) \\ k - 3 \end{pmatrix} \\ &= \begin{pmatrix} 4 \\ k - 3 \end{pmatrix}\end{aligned}$$

$$\begin{aligned}\therefore |\vec{AB}| &= \sqrt{4^2 + (k-3)^2} \\ &= \sqrt{16 + (k-3)^2} \text{ units}\end{aligned}$$

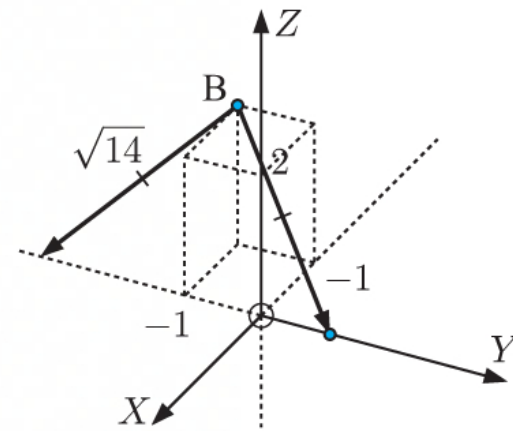
$$\begin{aligned}
 \text{b} \quad & |\overrightarrow{AB}| = 5 \\
 \therefore & \sqrt{16 + (k-3)^2} = 5 \quad \{\text{using a}\} \\
 \therefore & 16 + k^2 - 6k + 9 = 25 \\
 \therefore & k^2 - 6k = 0 \\
 \therefore & k(k-6) = 0 \\
 \therefore & k = 0 \quad \text{or} \quad k - 6 = 0 \\
 \therefore & k = 0 \quad \text{or} \quad 6
 \end{aligned}$$



15 a A is  $(0, y, 0)$  for any  $y \in \mathbb{R}$

$$\begin{aligned}
 \text{b} \quad & \overrightarrow{AB} = \begin{pmatrix} -1 - 0 \\ -1 - y \\ 2 - 0 \end{pmatrix} \\
 \therefore & AB = \sqrt{(-1)^2 + (-1 - y)^2 + 2^2} \\
 \text{Now} \quad & \sqrt{1 + (y+1)^2 + 4} = \sqrt{14} \\
 \therefore & (y+1)^2 = 9 \\
 \therefore & y+1 = \pm 3 \\
 \therefore & y = -1 \pm 3 \\
 \therefore & y = 2 \quad \text{or} \quad -4
 \end{aligned}$$

$\therefore$  the two points are  $(0, 2, 0)$  and  $(0, -4, 0)$ .



16 a Let B have coordinates  $(b_1, b_2)$ .

$$\begin{aligned}
 \therefore & \overrightarrow{AB} = \begin{pmatrix} b_1 - 1 \\ b_2 - 4 \end{pmatrix} \\
 \therefore & \begin{pmatrix} b_1 - 1 \\ b_2 - 4 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \\
 \therefore & b_1 - 1 = 3 \quad \text{and} \quad b_2 - 4 = -2 \\
 \therefore & b_1 = 4 \quad \text{and} \quad b_2 = 2 \\
 \therefore & \text{B has coordinates } (4, 2).
 \end{aligned}$$

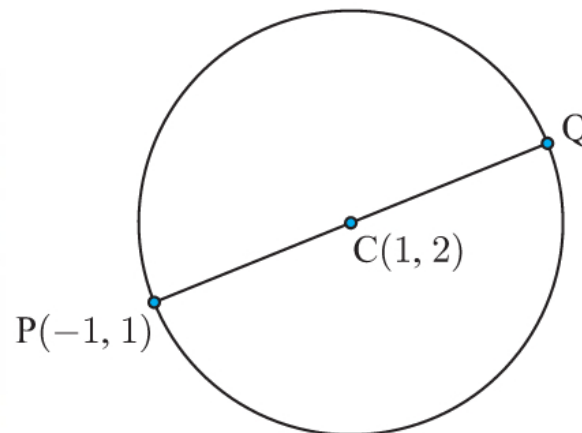
b Let C have coordinates  $(c_1, c_2)$ .

$$\begin{aligned}
 \therefore & \overrightarrow{CA} = \begin{pmatrix} 1 - c_1 \\ 4 - c_2 \end{pmatrix} \\
 \therefore & \begin{pmatrix} 1 - c_1 \\ 4 - c_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \\
 \therefore & 1 - c_1 = -1 \quad \text{and} \quad 4 - c_2 = 2 \\
 \therefore & c_1 = 2 \quad \text{and} \quad c_2 = 2 \\
 \therefore & \text{C has coordinates } (2, 2).
 \end{aligned}$$

$$\text{17 a } \overrightarrow{PC} = \begin{pmatrix} 1 - (-1) \\ 2 - 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

b Let Q have coordinates  $(q_1, q_2)$ .

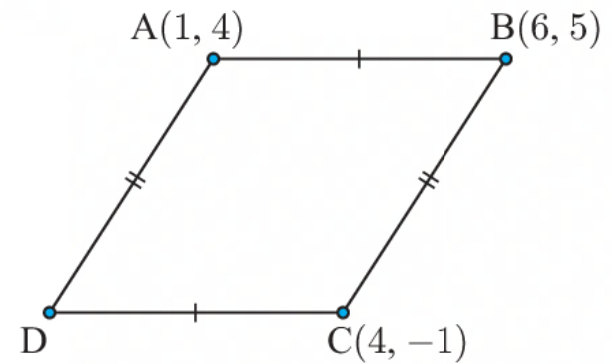
$$\begin{aligned}
 \therefore & \overrightarrow{CQ} = \begin{pmatrix} q_1 - 1 \\ q_2 - 2 \end{pmatrix} \\
 \text{But } & \overrightarrow{CQ} = \overrightarrow{PC} \\
 \therefore & \begin{pmatrix} q_1 - 1 \\ q_2 - 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\
 \therefore & q_1 - 1 = 2 \quad \text{and} \quad q_2 - 2 = 1 \\
 \therefore & q_1 = 3 \quad \text{and} \quad q_2 = 3 \\
 \therefore & \text{Q has coordinates } (3, 3).
 \end{aligned}$$





$$\begin{aligned} \text{18 a } \overrightarrow{AB} &= \begin{pmatrix} 6-1 \\ 5-4 \end{pmatrix} \\ &= \begin{pmatrix} 5 \\ 1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \text{b } \overrightarrow{CD} &= -\overrightarrow{AB} \\ &= -\begin{pmatrix} 5 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} -5 \\ -1 \end{pmatrix} \end{aligned}$$



c Let D have coordinates  $(d_1, d_2)$ .

$$\therefore \overrightarrow{CD} = \begin{pmatrix} d_1 - 4 \\ d_2 - (-1) \end{pmatrix} = \begin{pmatrix} d_1 - 4 \\ d_2 + 1 \end{pmatrix}$$

$$\text{But } \overrightarrow{CD} = \begin{pmatrix} -5 \\ -1 \end{pmatrix}$$

$$\therefore \begin{pmatrix} d_1 - 4 \\ d_2 + 1 \end{pmatrix} = \begin{pmatrix} -5 \\ -1 \end{pmatrix}$$

$$\therefore d_1 - 4 = -5 \quad \text{and} \quad d_2 + 1 = -1$$

$$\therefore d_1 = -1 \quad \text{and} \quad d_2 = -2$$

$\therefore$  D has coordinates  $(-1, -2)$ .

$$\text{19 } A(2, 1, -2), \quad B(0, 3, -4), \quad C(1, -2, 1), \quad D(-2, -3, 2)$$

$$\overrightarrow{AC} = \begin{pmatrix} 1-2 \\ -2-1 \\ 1-(-2) \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ 3 \end{pmatrix}$$

$$\overrightarrow{BD} = \begin{pmatrix} -2-0 \\ -3-3 \\ 2-(-4) \end{pmatrix} = \begin{pmatrix} -2 \\ -6 \\ 6 \end{pmatrix} = 2 \begin{pmatrix} -1 \\ -3 \\ 3 \end{pmatrix} = 2\overrightarrow{AC}$$

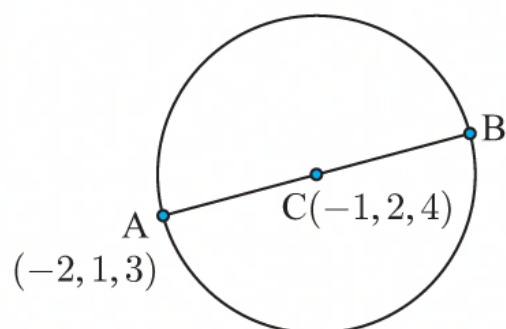
$$\text{20 } \overrightarrow{AB} = \begin{pmatrix} 2-(-1) \\ 3-5 \\ -3-2 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ -5 \end{pmatrix}$$

$\therefore$  C is  $(2+3, 3-2, -3-5)$ , or  $(5, 1, -8)$ .

$\therefore$  D is  $(5+3, 1-2, -8-5)$ , or  $(8, -1, -13)$ .

$\therefore$  E is  $(8+3, -1-2, -13-5)$ , or  $(11, -3, -18)$ .

21



If B is  $(a, b, c)$  then  $\frac{a-2}{2} = -1$ ,  $\frac{b+1}{2} = 2$ ,  $\frac{c+3}{2} = 4$

$$\therefore a = 0, \quad b = 3, \quad c = 5$$

$\therefore$  B is  $(0, 3, 5)$

$$\begin{aligned} r = AC &= \sqrt{(-1-(-2))^2 + (2-1)^2 + (4-3)^2} \\ &= \sqrt{1+1+1} \\ &= \sqrt{3} \text{ units} \end{aligned}$$

$$\mathbf{22} \quad \mathbf{a} \quad \overrightarrow{AB} = \begin{pmatrix} 2 - (-1) \\ 0 - 2 \\ 3 - 5 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ -2 \end{pmatrix}$$

$$\therefore AB = \sqrt{3^2 + (-2)^2 + (-2)^2} \\ = \sqrt{17} \text{ units}$$

$$\mathbf{c} \quad \overrightarrow{CB} = \begin{pmatrix} 2 - (-3) \\ 0 - 1 \\ 3 - 0 \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix}$$

$$\therefore CB = \sqrt{5^2 + (-1)^2 + 3^2} \\ = \sqrt{35} \text{ units}$$

$$\mathbf{b} \quad \overrightarrow{AC} = \begin{pmatrix} -3 - (-1) \\ 1 - 2 \\ 0 - 5 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ -5 \end{pmatrix}$$

$$\therefore AC = \sqrt{(-2)^2 + (-1)^2 + (-5)^2} \\ = \sqrt{30} \text{ units}$$

$\mathbf{d}$  Triangle ABC has  $AC = \sqrt{30}$  units,  $CB = \sqrt{35}$  units, and  $AB = \sqrt{17}$  units. All the side lengths are different, and  $(\sqrt{17})^2 + (\sqrt{30})^2 \neq (\sqrt{35})^2$ . So, triangle ABC is not right angled.  $\therefore$  triangle ABC is scalene.

$$\mathbf{23} \quad P(0, 4, 4), Q(2, 6, 5), R(1, 4, 3)$$

$$PQ = \sqrt{(2-0)^2 + (6-4)^2 + (5-4)^2} \\ = \sqrt{4+4+1} \\ = 3$$

$$QR = \sqrt{(1-2)^2 + (4-6)^2 + (3-5)^2} \\ = \sqrt{1+4+4} \\ = 3$$

$\therefore PQ = QR$  and so triangle PQR is isosceles.

$$PR = \sqrt{(1-0)^2 + (4-4)^2 + (3-4)^2} \\ = \sqrt{1+0+1} \\ = \sqrt{2}$$

$$\mathbf{24} \quad \mathbf{a} \quad A(0, 0, 3), B(2, 8, 1), C(-9, 6, 18)$$

$$AB = \sqrt{(2-0)^2 + (8-0)^2 + (1-3)^2} \\ = \sqrt{4+64+4} \\ = \sqrt{72}$$

$$BC = \sqrt{(-9-2)^2 + (6-8)^2 + (18-1)^2} \\ = \sqrt{121+4+289} \\ = \sqrt{414}$$

Since  $BC^2 = AB^2 + AC^2$ , triangle ABC is right angled.

$$AC = \sqrt{(-9-0)^2 + (6-0)^2 + (18-3)^2} \\ = \sqrt{81+36+225} \\ = \sqrt{342}$$

$$\mathbf{b} \quad A(1, 0, -3), B(2, 2, 0), C(4, 6, 6)$$

$$AB = \sqrt{(2-1)^2 + (2-0)^2 + (0-(-3))^2} \\ = \sqrt{1+4+9} \\ = \sqrt{14}$$

$$BC = \sqrt{(4-2)^2 + (6-2)^2 + (6-0)^2} \\ = \sqrt{4+16+36} \\ = \sqrt{56} = 2\sqrt{14}$$

$$AC = \sqrt{(4-1)^2 + (6-0)^2 + (6-(-3))^2} \\ = \sqrt{9+36+81} \\ = \sqrt{126} = 3\sqrt{14}$$

Since  $AB + BC = AC$ , the points A, B, and C lie on a straight line, and do not form a triangle.

$$\mathbf{25} \quad \mathbf{a} \quad \overrightarrow{AB} = \begin{pmatrix} 6-5 \\ 12-6 \\ 9-(-2) \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \\ 11 \end{pmatrix}$$

$$\therefore AB = \sqrt{1^2 + 6^2 + 11^2} \\ = \sqrt{158} \text{ units}$$

$$\overrightarrow{AC} = \begin{pmatrix} 2-5 \\ 4-6 \\ 2-(-2) \end{pmatrix} = \begin{pmatrix} -3 \\ -2 \\ 4 \end{pmatrix}$$

$$\therefore AC = \sqrt{(-3)^2 + (-2)^2 + 4^2} \\ = \sqrt{29} \text{ units}$$

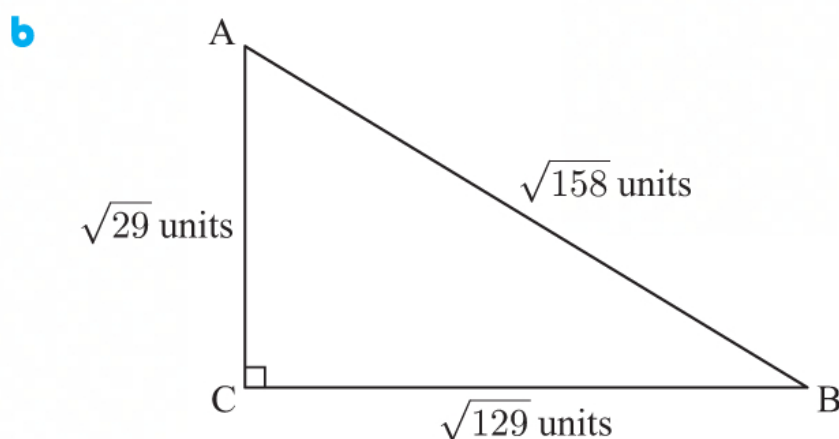
$$\overrightarrow{BC} = \begin{pmatrix} 2-6 \\ 4-12 \\ 2-9 \end{pmatrix} = \begin{pmatrix} -4 \\ -8 \\ -7 \end{pmatrix}$$

$$\therefore BC = \sqrt{(-4)^2 + (-8)^2 + (-7)^2} \\ = \sqrt{129} \text{ units}$$

$$\text{Now, } (\sqrt{29})^2 + (\sqrt{129})^2 = 29 + 129 \\ = 158 \\ = (\sqrt{158})^2$$

$$\text{So, } AC^2 + BC^2 = AB^2$$

$\therefore$  triangle ABC is right angled with the right angle at C.



$$\text{Area} = \frac{1}{2} \times \text{base} \times \text{height} \\ = \frac{1}{2} \times \sqrt{129} \times \sqrt{29} \\ \approx 30.6 \text{ units}^2$$

$$\mathbf{26} \quad A(-1, 3, 4), \quad B(2, 5, -1), \quad C(-1, 2, -2), \quad D(r, s, t)$$

$$\mathbf{a} \quad \text{If } \overrightarrow{AC} = \overrightarrow{BD} \text{ then } \begin{pmatrix} -1-(-1) \\ 2-3 \\ -2-4 \end{pmatrix} = \begin{pmatrix} r-2 \\ s-5 \\ t-(-1) \end{pmatrix}$$

$$\therefore r-2=0, \quad s-5=-1, \quad \text{and } t+1=-6$$

$$\therefore r=2, \quad s=4, \quad \text{and } t=-7$$

$$\mathbf{b} \quad \text{If } \overrightarrow{AB} = \overrightarrow{DC} \text{ then } \begin{pmatrix} 2-(-1) \\ 5-3 \\ -1-4 \end{pmatrix} = \begin{pmatrix} -1-r \\ 2-s \\ -2-t \end{pmatrix}$$

$$\therefore -1-r=3, \quad 2-s=2, \quad \text{and } -2-t=-5$$

$$\therefore r=-4, \quad s=0, \quad \text{and } t=3$$

$$\mathbf{27} \quad A(1, 2, 3), \quad B(3, -3, 2), \quad C(7, -4, 5), \quad D(5, 1, 6)$$

$$\mathbf{a} \quad \overrightarrow{AB} = \begin{pmatrix} 3-1 \\ -3-2 \\ 2-3 \end{pmatrix} = \begin{pmatrix} 2 \\ -5 \\ -1 \end{pmatrix} \quad \text{and} \quad \overrightarrow{DC} = \begin{pmatrix} 7-5 \\ -4-1 \\ 5-6 \end{pmatrix} = \begin{pmatrix} 2 \\ -5 \\ -1 \end{pmatrix}.$$

**b** ABCD is a parallelogram since its opposite sides are parallel and equal in length.

**28 a** Suppose S is  $(x, y, z)$ .

$$\overrightarrow{PQ} = \overrightarrow{SR} \quad \{\text{opposite sides are parallel and equal in length}\}$$

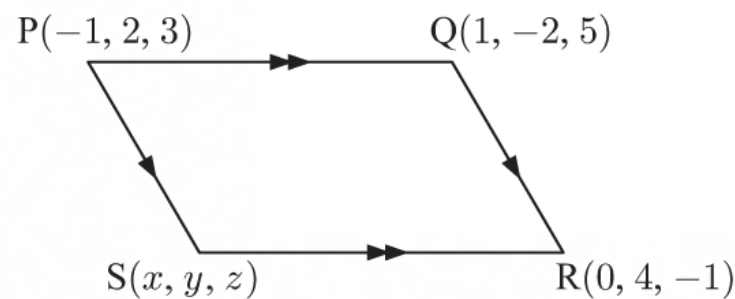
$$\therefore \begin{pmatrix} 1 - (-1) \\ -2 - 2 \\ 5 - 3 \end{pmatrix} = \begin{pmatrix} 0 - x \\ 4 - y \\ -1 - z \end{pmatrix}$$

$$\therefore \begin{pmatrix} 2 \\ -4 \\ 2 \end{pmatrix} = \begin{pmatrix} -x \\ 4 - y \\ -1 - z \end{pmatrix}$$

$$\therefore -x = 2, \quad 4 - y = -4, \quad \text{and} \quad -1 - z = 2$$

$$\therefore x = -2, \quad y = 8, \quad \text{and} \quad z = -3$$

$$\therefore S \text{ is } (-2, 8, -3).$$

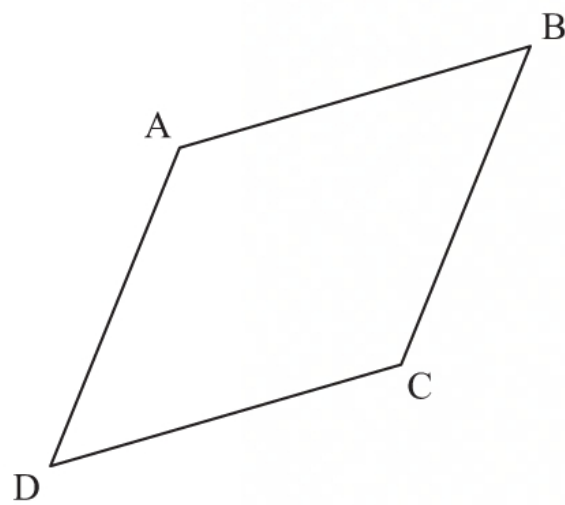


**b** The midpoint of [PR] is  $\left(\frac{-1+0}{2}, \frac{2+4}{2}, \frac{3+(-1)}{2}\right)$  which is  $\left(-\frac{1}{2}, 3, 1\right)$ .

The midpoint of [QS] is  $\left(\frac{1+(-2)}{2}, \frac{-2+8}{2}, \frac{5+(-3)}{2}\right)$  which is  $\left(-\frac{1}{2}, 3, 1\right)$ .

So, [PR] and [QS] have the same midpoint. ✓

**29**



$$\text{a } \overrightarrow{AB} = \begin{pmatrix} -1 - 5 \\ 2 - 0 \\ 4 - 3 \end{pmatrix} = \begin{pmatrix} -6 \\ 2 \\ 1 \end{pmatrix} \quad \text{and}$$

$$\overrightarrow{DC} = \begin{pmatrix} 4 - 10 \\ -3 - (-5) \\ 6 - 5 \end{pmatrix} = \begin{pmatrix} -6 \\ 2 \\ 1 \end{pmatrix}$$

$$\text{So, } \overrightarrow{AB} = \overrightarrow{DC}$$

Sides [AB] and [DC] are equal in length and parallel.

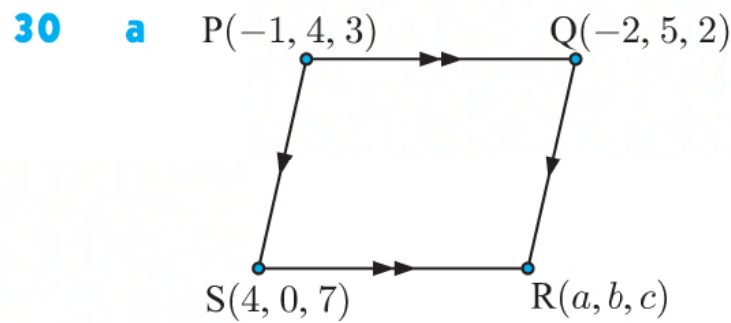
$\therefore$  ABCD is a parallelogram.

$$\text{b } \overrightarrow{AB} = \begin{pmatrix} 1 - 2 \\ 4 - (-3) \\ -1 - 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 7 \\ -3 \end{pmatrix} \quad \text{and} \quad \overrightarrow{DC} = \begin{pmatrix} -2 - (-1) \\ 6 - (-1) \\ -2 - 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 7 \\ -4 \end{pmatrix}$$

$$\text{So, } \overrightarrow{AB} \neq \overrightarrow{DC}$$

$\therefore$  ABCD is not a parallelogram.





Let R be  $(a, b, c)$ .

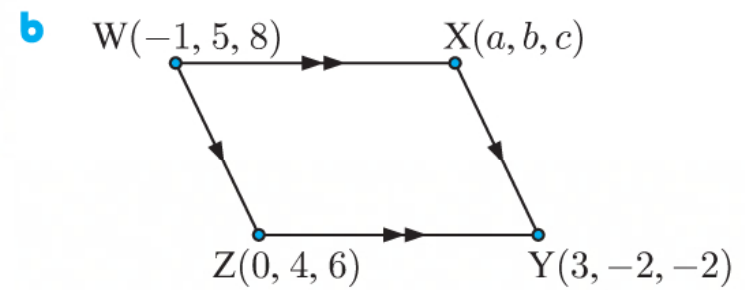
$$\text{Now } \overrightarrow{SR} = \overrightarrow{PQ}$$

$$\therefore \begin{pmatrix} a-4 \\ b-0 \\ c-7 \end{pmatrix} = \begin{pmatrix} -2-(-1) \\ 5-4 \\ 2-3 \end{pmatrix}$$

$$\therefore \begin{pmatrix} a-4 \\ b \\ c-7 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$$

$$\therefore a=3, b=1, c=6$$

So, R is  $(3, 1, 6)$ .



Let X be  $(a, b, c)$ .

$$\text{Now } \overrightarrow{WX} = \overrightarrow{ZY}$$

$$\therefore \begin{pmatrix} a-(-1) \\ b-5 \\ c-8 \end{pmatrix} = \begin{pmatrix} 3-0 \\ -2-4 \\ -2-6 \end{pmatrix}$$

$$\therefore \begin{pmatrix} a+1 \\ b-5 \\ c-8 \end{pmatrix} = \begin{pmatrix} 3 \\ -6 \\ -8 \end{pmatrix}$$

$$\therefore a=2, b=-1, c=0$$

So, X is  $(2, -1, 0)$ .

## EXERCISE 12J

**1 a**  $\begin{pmatrix} 2 \\ 5 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 6 \\ 15 \end{pmatrix}$   
 $\therefore \begin{pmatrix} 2 \\ 5 \end{pmatrix}$  and  $\begin{pmatrix} 6 \\ 15 \end{pmatrix}$  are parallel.

**c**  $\begin{pmatrix} 6 \\ -12 \end{pmatrix} = -\frac{3}{2} \begin{pmatrix} -4 \\ 8 \end{pmatrix}$   
 $\therefore \begin{pmatrix} 6 \\ -12 \end{pmatrix}$  and  $\begin{pmatrix} -4 \\ 8 \end{pmatrix}$  are parallel.

**e**  $\begin{pmatrix} 8 \\ 0 \\ -20 \end{pmatrix} = -\frac{4}{3} \begin{pmatrix} -6 \\ 0 \\ 15 \end{pmatrix}$   
 $\therefore \begin{pmatrix} 8 \\ 0 \\ -20 \end{pmatrix}$  and  $\begin{pmatrix} -6 \\ 0 \\ 15 \end{pmatrix}$  are parallel.

**b** There is no scalar  $k$  such that  
 $\begin{pmatrix} 1 \\ -4 \end{pmatrix} = k \begin{pmatrix} 2 \\ -10 \end{pmatrix}$   
 $\therefore \begin{pmatrix} 1 \\ -4 \end{pmatrix}$  and  $\begin{pmatrix} 2 \\ -10 \end{pmatrix}$  are not parallel.

**d**  $\begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} -2 \\ -8 \\ 6 \end{pmatrix}$   
 $\therefore \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}$  and  $\begin{pmatrix} -2 \\ -8 \\ 6 \end{pmatrix}$  are parallel.

**f** There is no scalar  $k$  such that  
 $\begin{pmatrix} 6 \\ 10 \\ -2 \end{pmatrix} = k \begin{pmatrix} 24 \\ 40 \\ 8 \end{pmatrix}$   
 $\therefore \begin{pmatrix} 6 \\ 10 \\ -2 \end{pmatrix}$  and  $\begin{pmatrix} 24 \\ 40 \\ 8 \end{pmatrix}$  are not parallel.

**2 a**  $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$  and  $\begin{pmatrix} -6 \\ a \end{pmatrix}$  are parallel.

$$\therefore \begin{pmatrix} 2 \\ -1 \end{pmatrix} = k \begin{pmatrix} -6 \\ a \end{pmatrix} \text{ for some scalar } k.$$

$$\therefore 2 = -6k \quad \text{and} \quad -1 = ka$$

$$\therefore k = -\frac{1}{3} \quad \text{and} \quad -1 = -\frac{1}{3}a$$

$$\therefore a = 3$$

**b**  $\begin{pmatrix} a \\ 2 \end{pmatrix}$  and  $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$  are parallel.

$$\therefore \begin{pmatrix} a \\ 2 \end{pmatrix} = k \begin{pmatrix} 3 \\ -1 \end{pmatrix} \text{ for some scalar } k.$$

$$\therefore a = 3k \quad \text{and} \quad 2 = -k$$

$$\therefore a = -6$$

**3 a** Since **a** and **b** are parallel, **b** = *ka*.

$$\therefore \begin{pmatrix} -6 \\ r \\ s \end{pmatrix} = k \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2k \\ -k \\ 3k \end{pmatrix}$$

$$\therefore 2k = -6$$

$$\therefore k = -3$$

$$\therefore r = 3, \quad s = -9$$

**b** Since **a** and **b** are parallel, **b** = *ka*.

$$\therefore \begin{pmatrix} r \\ 2 \\ s \end{pmatrix} = k \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3k \\ -k \\ 2k \end{pmatrix}$$

$$\therefore 2 = -k$$

$$\therefore k = -2$$

$$\therefore r = -6, \quad s = -4$$

**4 a**  $\overrightarrow{AB} = 3\overrightarrow{CD}$  means that  $\overrightarrow{AB}$  is parallel to  $\overrightarrow{CD}$  and 3 times its length.

**b**  $\overrightarrow{RS} = -\frac{1}{2}\overrightarrow{KL}$  means that  $\overrightarrow{RS}$  is parallel to  $\overrightarrow{KL}$ , half its length, and in the opposite direction.

**c**



$\overrightarrow{AB} = 2\overrightarrow{BC}$  means that A, B, and C are collinear and the length of  $\overrightarrow{AB}$  is twice the length of  $\overrightarrow{BC}$ .

**5**  $\overrightarrow{OP} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$ ,  $\overrightarrow{OQ} = \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}$ ,  $\overrightarrow{OR} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$ ,  $\overrightarrow{OS} = \begin{pmatrix} -1 \\ -2 \\ 3 \end{pmatrix}$

$$\overrightarrow{PR} = \overrightarrow{PO} + \overrightarrow{OR} = -\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ 3 \end{pmatrix}$$

$$\overrightarrow{QS} = \overrightarrow{QO} + \overrightarrow{OS}$$

$$= -\begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix} + \begin{pmatrix} -1 \\ -2 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} -2 \\ -6 \\ 6 \end{pmatrix}$$

$$= 2 \begin{pmatrix} -1 \\ -3 \\ 3 \end{pmatrix} = 2\overrightarrow{PR}$$

Thus [PR] and [QS] are parallel and  $PR : QS = 1 : 2$ .

- 6 a** The vector in the same direction as **a** and twice its length is  $2\mathbf{a}$ .

$$2\mathbf{a} = 2 \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \end{pmatrix}$$

- b** The vector in the opposite direction to **a** and half its length is  $-\frac{1}{2}\mathbf{a}$ .

$$-\frac{1}{2}\mathbf{a} = -\frac{1}{2} \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} -\frac{2}{2} \\ -\frac{4}{2} \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$$

- 7 a**  $\mathbf{i} + 2\mathbf{j}$  has length  $\sqrt{1^2 + 2^2} = \sqrt{5}$  units.

$$\therefore \text{the unit vector in the same direction as } \mathbf{i} + 2\mathbf{j} \text{ is } \frac{1}{\sqrt{5}}(\mathbf{i} + 2\mathbf{j}) = \frac{1}{\sqrt{5}}\mathbf{i} + \frac{2}{\sqrt{5}}\mathbf{j}.$$

- b**  $2\mathbf{i} - 3\mathbf{j}$  has length  $\sqrt{2^2 + (-3)^2} = \sqrt{13}$  units.

$$\therefore \text{the unit vector in the same direction as } 2\mathbf{i} - 3\mathbf{j} \text{ is } \frac{1}{\sqrt{13}}(2\mathbf{i} - 3\mathbf{j}) = \frac{2}{\sqrt{13}}\mathbf{i} - \frac{3}{\sqrt{13}}\mathbf{j}.$$

- c**  $-2\mathbf{i} + 2\mathbf{j}$  has length  $\sqrt{(-2)^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$  units.

$$\therefore \text{the unit vector in the same direction as } -2\mathbf{i} + 2\mathbf{j} \text{ is } \frac{1}{2\sqrt{2}}(-2\mathbf{i} + 2\mathbf{j}) = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}.$$

- 8 a**  $2\mathbf{i} + 3\mathbf{k}$  has length  $\sqrt{2^2 + 3^2} = \sqrt{13}$  units

$$\therefore \text{the unit vector} = \frac{1}{\sqrt{13}}(2\mathbf{i} + 3\mathbf{k}) = \frac{2}{\sqrt{13}}\mathbf{i} + \frac{3}{\sqrt{13}}\mathbf{k}$$

- b**  $-\mathbf{i} + 2\mathbf{j} - \mathbf{k}$  has length  $\sqrt{(-1)^2 + 2^2 + (-1)^2} = \sqrt{6}$  units

$$\therefore \text{the unit vector} = \frac{1}{\sqrt{6}}(-\mathbf{i} + 2\mathbf{j} - \mathbf{k}) = -\frac{1}{\sqrt{6}}\mathbf{i} + \frac{2}{\sqrt{6}}\mathbf{j} - \frac{1}{\sqrt{6}}\mathbf{k}$$

- c**  $2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$  has length  $\sqrt{2^2 + (-2)^2 + 1^2} = \sqrt{9} = 3$  units

$$\therefore \text{the unit vector is } \frac{1}{3}(2\mathbf{i} - 2\mathbf{j} + \mathbf{k}) = \frac{2}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}$$

- 9 a**  $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$  has length  $\sqrt{2^2 + (-1)^2} = \sqrt{5}$  units

$$\therefore \text{the unit vector in the same direction is } \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$\therefore \text{the vector of length 3 units in the same direction is } \frac{3}{\sqrt{5}} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{6}{\sqrt{5}} \\ -\frac{3}{\sqrt{5}} \end{pmatrix}$$

- b**  $\begin{pmatrix} -1 \\ -4 \end{pmatrix}$  has length  $\sqrt{(-1)^2 + (-4)^2} = \sqrt{17}$  units

$$\therefore \text{the unit vector in the opposite direction is } -\frac{1}{\sqrt{17}} \begin{pmatrix} -1 \\ -4 \end{pmatrix} = \frac{1}{\sqrt{17}} \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

$$\therefore \text{the vector of length 2 units in the opposite direction is } \frac{2}{\sqrt{17}} \begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt{17}} \\ \frac{8}{\sqrt{17}} \end{pmatrix}$$

**c**  $\begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$  has length  $\sqrt{(-1)^2 + 4^2 + 1^2} = \sqrt{18} = 3\sqrt{2}$  units

$\therefore$  the unit vector in the same direction is  $\frac{1}{3\sqrt{2}} \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$

$\therefore$  the vector of length 6 units in the same direction is

$$\frac{6}{3\sqrt{2}} \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix} = \sqrt{2} \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} -\sqrt{2} \\ 4\sqrt{2} \\ \sqrt{2} \end{pmatrix}$$

**d**  $\begin{pmatrix} -1 \\ -2 \\ -2 \end{pmatrix}$  has length  $\sqrt{(-1)^2 + (-2)^2 + (-2)^2} = \sqrt{9} = 3$  units

$\therefore$  the unit vector in the opposite direction is  $-\frac{1}{3} \begin{pmatrix} -1 \\ -2 \\ -2 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

$\therefore$  the vector of length 5 units in the opposite direction is  $\frac{5}{3} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{5}{3} \\ \frac{10}{3} \\ \frac{10}{3} \end{pmatrix}$

**10 a**  $|\mathbf{a}| = \sqrt{2^2 + (-1)^2 + (-2)^2}$   
 $= \sqrt{4 + 1 + 4}$   
 $= 3$  units

$\therefore$  the vectors of length 1 unit parallel to  $\mathbf{a}$  are  $\pm \frac{1}{3}\mathbf{a}$ .

$\therefore$  the vectors are

$$\begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ -\frac{2}{3} \end{pmatrix} \text{ and } \begin{pmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}.$$

**b**  $|\mathbf{b}| = \sqrt{(-2)^2 + (-1)^2 + 2^2}$   
 $= \sqrt{4 + 1 + 4}$   
 $= 3$  units

$\therefore$  the vectors of length 2 units parallel to  $\mathbf{b}$  are  $\pm \frac{2}{3}\mathbf{b}$ .

$\therefore$  the vectors are

$$\begin{pmatrix} -\frac{4}{3} \\ -\frac{2}{3} \\ \frac{4}{3} \end{pmatrix} \text{ and } \begin{pmatrix} \frac{4}{3} \\ \frac{2}{3} \\ -\frac{4}{3} \end{pmatrix}.$$

**11 a**  $\overrightarrow{\text{AB}}$  is a vector in the same direction as  $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$  with length 4 units.

Now,  $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$  has length  $\sqrt{1^2 + (-1)^2} = \sqrt{2}$  units

$\therefore$  the unit vector in the same direction is  $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\therefore$  the vector of length 4 units in the same direction is  $\frac{4}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 2\sqrt{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$   
 $= \begin{pmatrix} 2\sqrt{2} \\ -2\sqrt{2} \end{pmatrix}$

$\therefore \overrightarrow{\text{AB}} = \begin{pmatrix} 2\sqrt{2} \\ -2\sqrt{2} \end{pmatrix}$



$$\text{b } \vec{OA} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \quad \vec{AB} = \begin{pmatrix} 2\sqrt{2} \\ -2\sqrt{2} \end{pmatrix}$$

$$\text{Now } \vec{OB} = \vec{OA} + \vec{AB}$$

$$\begin{aligned} \therefore \vec{OB} &= \begin{pmatrix} 3 \\ 2 \end{pmatrix} + \begin{pmatrix} 2\sqrt{2} \\ -2\sqrt{2} \end{pmatrix} \\ &= \begin{pmatrix} 3 + 2\sqrt{2} \\ 2 - 2\sqrt{2} \end{pmatrix} \end{aligned}$$

$$\text{c } \text{If } \vec{OB} = \begin{pmatrix} 3 + 2\sqrt{2} \\ 2 - 2\sqrt{2} \end{pmatrix}, \text{ then the coordinates of B are } (3 + 2\sqrt{2}, 2 - 2\sqrt{2}).$$

$$\text{12 a } \begin{pmatrix} 3 \\ 4 \end{pmatrix} \text{ has length } \sqrt{3^2 + 4^2} = 5 \text{ units.}$$

$$\therefore \text{ the vector of length 5 units in the same direction is } \begin{pmatrix} 3 \\ 4 \end{pmatrix}.$$

$$\text{Now } \begin{pmatrix} 1 \\ -5 \end{pmatrix} + \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$$

$$\therefore \text{ the point 5 units from } (1, -5) \text{ in the direction } \begin{pmatrix} 3 \\ 4 \end{pmatrix} \text{ is } X(4, -1).$$

$$\text{b } \begin{pmatrix} -3 \\ 0 \\ 4 \end{pmatrix} \text{ has length } \sqrt{(-3)^2 + 0^2 + 4^2} = \sqrt{25} = 5 \text{ units}$$

$$\therefore \text{ the unit vector in the same direction is } \frac{1}{5} \begin{pmatrix} -3 \\ 0 \\ 4 \end{pmatrix}.$$

$$\therefore \text{ the vector of length 6 units in the same direction is } \frac{6}{5} \begin{pmatrix} -3 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} -\frac{18}{5} \\ 0 \\ \frac{24}{5} \end{pmatrix}.$$

$$\text{Now } \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} + \begin{pmatrix} -\frac{18}{5} \\ 0 \\ \frac{24}{5} \end{pmatrix} = \begin{pmatrix} -\frac{13}{5} \\ 3 \\ \frac{14}{5} \end{pmatrix}$$

$$\therefore \text{ the point 6 units from } (1, 3, -2) \text{ in the direction } \begin{pmatrix} -3 \\ 0 \\ 4 \end{pmatrix} \text{ is } X\left(-\frac{13}{5}, 3, \frac{14}{5}\right).$$

$$\text{c } \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \text{ has length } \sqrt{1^2 + (-1)^2 + 2^2} = \sqrt{6} \text{ units}$$

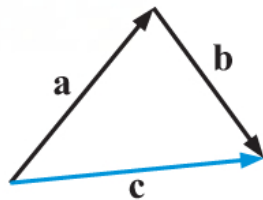
$$\therefore \text{ the unit vector in the same direction is } \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}.$$

$$\therefore \text{ the vector of length 4 units in the same direction is } \frac{4}{\sqrt{6}} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{4}{\sqrt{6}} \\ -\frac{4}{\sqrt{6}} \\ \frac{8}{\sqrt{6}} \end{pmatrix}.$$

$$\text{Now } \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} + \begin{pmatrix} \frac{4}{\sqrt{6}} \\ -\frac{4}{\sqrt{6}} \\ \frac{8}{\sqrt{6}} \end{pmatrix} = \begin{pmatrix} 2 + \frac{4}{\sqrt{6}} \\ -1 - \frac{4}{\sqrt{6}} \\ 4 + \frac{8}{\sqrt{6}} \end{pmatrix}$$

$\therefore$  the point 4 units from  $(2, -1, 4)$  in the direction  $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$  is  
 $X\left(2 + \frac{4}{\sqrt{6}}, -1 - \frac{4}{\sqrt{6}}, 4 + \frac{8}{\sqrt{6}}\right).$

**13** Case:  $\mathbf{a}$  and  $\mathbf{b}$  are not parallel.



Since  $\mathbf{a}$ ,  $\mathbf{b}$ , and  $\mathbf{c}$  form a triangle,

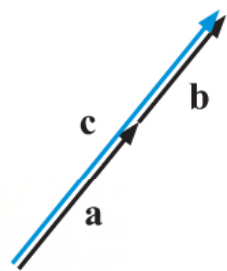
length of  $\mathbf{a}$  + length of  $\mathbf{b}$  > length of  $\mathbf{c}$  {triangle inequality}

But  $\mathbf{c} = \mathbf{a} + \mathbf{b}$ , so length of  $\mathbf{c}$  = length of  $(\mathbf{a} + \mathbf{b})$

$\therefore$  length of  $\mathbf{a}$  + length of  $\mathbf{b}$  > length of  $(\mathbf{a} + \mathbf{b})$

$$\therefore |\mathbf{a}| + |\mathbf{b}| > |\mathbf{a} + \mathbf{b}|$$

Case:  $\mathbf{a}$  and  $\mathbf{b}$  are parallel.



length of  $\mathbf{a}$  + length of  $\mathbf{b}$  = length of  $(\mathbf{a} + \mathbf{b})$

$$\therefore |\mathbf{a}| + |\mathbf{b}| = |\mathbf{a} + \mathbf{b}|$$

Case:  $\mathbf{a} = \mathbf{0}$

$$|\mathbf{a} + \mathbf{b}| = |\mathbf{0} + \mathbf{b}| = |\mathbf{b}|, \text{ and } |\mathbf{a}| + |\mathbf{b}| = |\mathbf{0}| + |\mathbf{b}| = |\mathbf{b}|$$

$$\therefore |\mathbf{a} + \mathbf{b}| = |\mathbf{a}| + |\mathbf{b}|$$

The same applies if  $\mathbf{b} = \mathbf{0}$ , or both are zero.

$\therefore$  for any  $\mathbf{a}$ ,  $\mathbf{b}$ ,  $|\mathbf{a}| + |\mathbf{b}| \geq |\mathbf{a} + \mathbf{b}|$ .

**14 a**  $\overrightarrow{AB} = \begin{pmatrix} 4 - (-1) \\ 6 - 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$

$$\overrightarrow{BC} = \begin{pmatrix} 1 - 4 \\ 3 - 6 \end{pmatrix} = \begin{pmatrix} -3 \\ -3 \end{pmatrix}$$

$$\text{So, } \overrightarrow{AB} = -\frac{5}{3} \overrightarrow{BC}$$

$\therefore$  A, B, and C are collinear.

**c**  $\overrightarrow{AB} = \begin{pmatrix} 4 - (-2) \\ 3 - 1 \\ 0 - 4 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ -4 \end{pmatrix}$

$$\overrightarrow{BC} = \begin{pmatrix} 19 - 4 \\ 8 - 3 \\ -10 - 0 \end{pmatrix} = \begin{pmatrix} 15 \\ 5 \\ -10 \end{pmatrix}$$

$$\text{So, } \overrightarrow{AB} = \frac{2}{5} \overrightarrow{BC}$$

$\therefore$  A, B, and C are collinear.

**b**  $\overrightarrow{PQ} = \begin{pmatrix} 6 - 3 \\ -3 - 3 \end{pmatrix} = \begin{pmatrix} 3 \\ -6 \end{pmatrix}$

$$\overrightarrow{QR} = \begin{pmatrix} 1 - 6 \\ 7 - (-3) \end{pmatrix} = \begin{pmatrix} -5 \\ 10 \end{pmatrix}$$

$$\text{So, } \overrightarrow{PQ} = -\frac{3}{5} \overrightarrow{QR}$$

$\therefore$  P, Q, and R are collinear.

**d**  $\overrightarrow{PQ} = \begin{pmatrix} 5 - 2 \\ -5 - 1 \\ -2 - 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -6 \\ -3 \end{pmatrix}$

$$\overrightarrow{QR} = \begin{pmatrix} -1 - 5 \\ 7 - (-5) \\ 4 - (-2) \end{pmatrix} = \begin{pmatrix} -6 \\ 12 \\ 6 \end{pmatrix}$$

$$\text{So, } \overrightarrow{PQ} = -\frac{1}{2} \overrightarrow{QR}$$

$\therefore$  P, Q, and R are collinear.

$$15 \quad a \quad \overrightarrow{AB} = \begin{pmatrix} 11-2 \\ -9-(-3) \\ 7-4 \end{pmatrix} = \begin{pmatrix} 9 \\ -6 \\ 3 \end{pmatrix}$$

$$\overrightarrow{BC} = \begin{pmatrix} -13-11 \\ a-(-9) \\ b-7 \end{pmatrix} = \begin{pmatrix} -24 \\ a+9 \\ b-7 \end{pmatrix}$$

A, B, and C are collinear.

$$\therefore \overrightarrow{AB} = k \overrightarrow{BC}$$

$$\therefore 9 = k \times -24$$

$$\therefore k = -\frac{3}{8}$$

$$\therefore \overrightarrow{AB} = -\frac{3}{8} \overrightarrow{BC}$$

$$\therefore -\frac{8}{3} \overrightarrow{AB} = \overrightarrow{BC}$$

$$\text{So, } -\frac{8}{3} \times -6 = a + 9$$

$$\therefore 16 = a + 9$$

$$\therefore a = 7$$

$$\text{and } -\frac{8}{3} \times 3 = b - 7$$

$$\therefore -8 = b - 7$$

$$\therefore b = -1$$

$$b \quad \overrightarrow{KL} = \begin{pmatrix} 4-1 \\ -3-(-1) \\ 7-0 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 7 \end{pmatrix}$$

$$\overrightarrow{LM} = \begin{pmatrix} a-4 \\ 2-(-3) \\ b-7 \end{pmatrix} = \begin{pmatrix} a-4 \\ 5 \\ b-7 \end{pmatrix}$$

K, L, and M are collinear.

$$\therefore \overrightarrow{KL} = k \overrightarrow{LM}$$

$$\therefore -2 = k \times 5$$

$$\therefore k = -\frac{2}{5}$$

$$\therefore \overrightarrow{KL} = -\frac{2}{5} \overrightarrow{LM}$$

$$\therefore -\frac{5}{2} \overrightarrow{KL} = \overrightarrow{LM}$$

$$\text{So, } -\frac{5}{2} \times 3 = a - 4$$

$$\therefore -\frac{15}{2} = a - 4$$

$$\therefore a = -\frac{7}{2}$$

$$\text{and } -\frac{5}{2} \times 7 = b - 7$$

$$\therefore -\frac{35}{2} = b - 7$$

$$\therefore b = -\frac{21}{2}$$

$$16 \quad a \quad \overrightarrow{CP} = \begin{pmatrix} x-X \\ y-Y \\ z-Z \end{pmatrix}$$

$$b \quad |\overrightarrow{CP}| = \left| \begin{pmatrix} x-X \\ y-Y \\ z-Z \end{pmatrix} \right|$$

$$\therefore r = \sqrt{(x-X)^2 + (y-Y)^2 + (z-Z)^2}$$

$$\therefore (x-X)^2 + (y-Y)^2 + (z-Z)^2 = r^2$$

This is a Cartesian equation for the sphere with radius  $r$  units and centre  $C(X, Y, Z)$ .

## INVESTIGATION 2

## LINEAR COMBINATIONS

$$\begin{aligned}
 \text{1 a i} \quad r \begin{pmatrix} 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} 0 \\ 1 \end{pmatrix} &= \begin{pmatrix} -4 \\ -22 \end{pmatrix} \\
 \therefore \begin{pmatrix} r \\ s \end{pmatrix} &= \begin{pmatrix} -4 \\ -22 \end{pmatrix} \\
 \therefore r &= -4, \quad s = -22
 \end{aligned}$$

$$\begin{aligned}
 \text{ii} \quad r \begin{pmatrix} 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} 0 \\ 1 \end{pmatrix} &= \begin{pmatrix} 7 \\ -19 \end{pmatrix} \\
 \therefore \begin{pmatrix} r \\ s \end{pmatrix} &= \begin{pmatrix} 7 \\ -19 \end{pmatrix} \\
 \therefore r &= 7, \quad s = -19
 \end{aligned}$$

$$\text{b Yes, } \begin{pmatrix} x \\ y \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

$$\begin{aligned}
 \text{2 a i} \quad r \begin{pmatrix} 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} 1 \\ 1 \end{pmatrix} &= \begin{pmatrix} -4 \\ -22 \end{pmatrix} \\
 \therefore \begin{pmatrix} r \\ 0 \end{pmatrix} + \begin{pmatrix} s \\ s \end{pmatrix} &= \begin{pmatrix} -4 \\ -22 \end{pmatrix} \\
 \therefore \begin{pmatrix} r+s \\ s \end{pmatrix} &= \begin{pmatrix} -4 \\ -22 \end{pmatrix} \\
 \therefore s &= -22 \quad \text{and} \quad r - 22 = -4 \\
 \therefore r &= 18
 \end{aligned}$$

$$\begin{aligned}
 \text{ii} \quad r \begin{pmatrix} 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} 1 \\ 1 \end{pmatrix} &= \begin{pmatrix} 7 \\ -19 \end{pmatrix} \\
 \therefore \begin{pmatrix} r \\ 0 \end{pmatrix} + \begin{pmatrix} s \\ s \end{pmatrix} &= \begin{pmatrix} 7 \\ -19 \end{pmatrix} \\
 \therefore \begin{pmatrix} r+s \\ s \end{pmatrix} &= \begin{pmatrix} 7 \\ -19 \end{pmatrix} \\
 \therefore s &= -19 \quad \text{and} \quad r - 19 = 7 \\
 \therefore r &= 26
 \end{aligned}$$

$$\text{b Yes, } \begin{pmatrix} x \\ y \end{pmatrix} = (x-y) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

$$\begin{aligned}
 \text{3 a i} \quad r \begin{pmatrix} 2 \\ 1 \end{pmatrix} + s \begin{pmatrix} -4 \\ -2 \end{pmatrix} &= \begin{pmatrix} 8 \\ 4 \end{pmatrix} \\
 \therefore \begin{pmatrix} 2r \\ r \end{pmatrix} + \begin{pmatrix} -4s \\ -2s \end{pmatrix} &= \begin{pmatrix} 8 \\ 4 \end{pmatrix} \\
 \therefore \begin{pmatrix} 2r-4s \\ r-2s \end{pmatrix} &= \begin{pmatrix} 8 \\ 4 \end{pmatrix}
 \end{aligned}$$

So we have  $2r-4s=8$  and  $r-2s=4$ , but these are equivalent since they are multiples of each other. There are infinitely many values of  $r$  and  $s$  which satisfy the equation.

$$\begin{aligned}
 \text{ii} \quad r \begin{pmatrix} 2 \\ 1 \end{pmatrix} + s \begin{pmatrix} -4 \\ -2 \end{pmatrix} &= \begin{pmatrix} 7 \\ -19 \end{pmatrix} \\
 \therefore \begin{pmatrix} 2r \\ r \end{pmatrix} + \begin{pmatrix} -4s \\ -2s \end{pmatrix} &= \begin{pmatrix} 7 \\ -19 \end{pmatrix} \\
 \therefore \begin{pmatrix} 2r-4s \\ r-2s \end{pmatrix} &= \begin{pmatrix} 7 \\ -19 \end{pmatrix}
 \end{aligned}$$

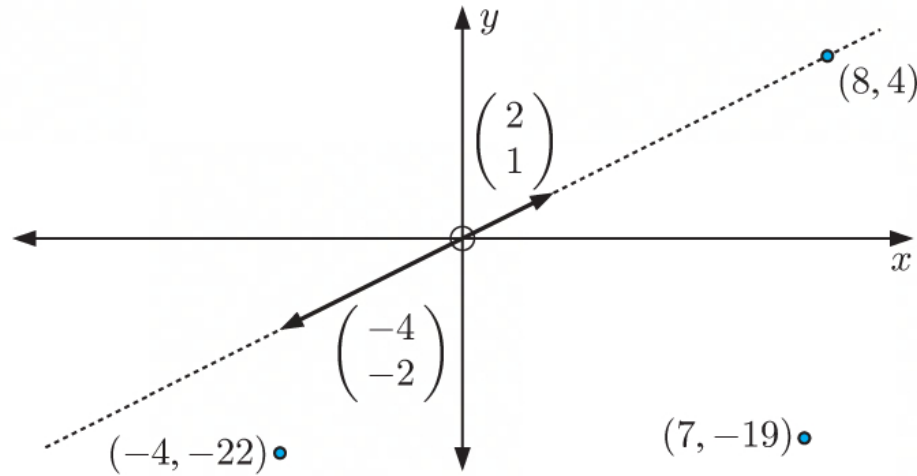
$$\therefore 2r-4s=7 \quad \text{and} \quad r-2s=-19$$

$$\therefore 2r-4s=7 \quad \text{and} \quad 2r-4s=-38$$

But  $2r-4s$  cannot be 7 and  $-38$  simultaneously, so there are no solutions.



- b**  $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$  and  $\begin{pmatrix} -4 \\ -2 \end{pmatrix}$  are multiples of each other, so every linear combination of these vectors is a vector of the form  $k \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ . These vectors are represented by the line on the diagram below.



**4 a** 
$$r \begin{pmatrix} 4 \\ 1 \end{pmatrix} + s \begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} -6 \\ 7 \end{pmatrix}$$

$$\therefore \begin{pmatrix} 4r \\ r \end{pmatrix} + \begin{pmatrix} 3s \\ 5s \end{pmatrix} = \begin{pmatrix} -6 \\ 7 \end{pmatrix}$$

$$\therefore \begin{pmatrix} 4r + 3s \\ r + 5s \end{pmatrix} = \begin{pmatrix} -6 \\ 7 \end{pmatrix}$$

$$\therefore 4r + 3s = -6 \quad \dots (1)$$

$$r + 5s = 7 \quad \dots (2)$$

Solving simultaneously,  $4r + 3s = -6 \quad \{(1)\}$   
 $4r + 20s = 28 \quad \{4 \times (2)\}$

Subtracting, 
$$\begin{array}{r} 4r + 3s = -6 \\ 4r + 20s = 28 \\ \hline -17s = -34 \end{array}$$

$$\therefore s = 2$$

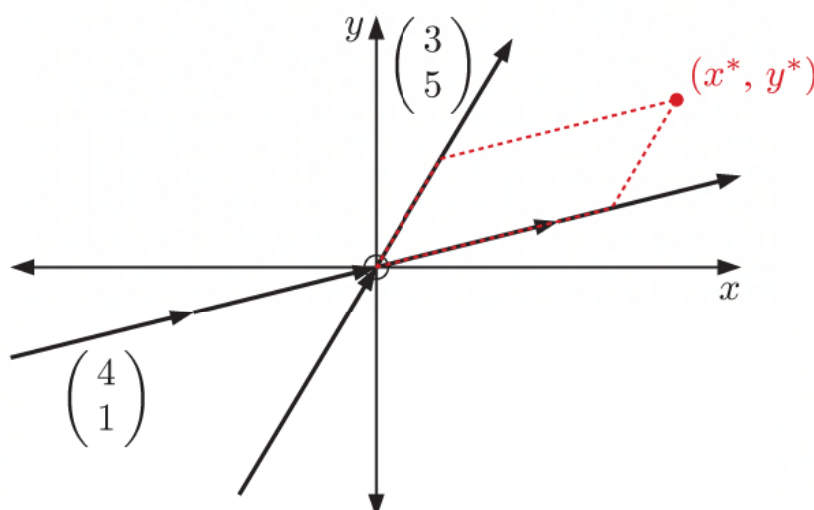
$$\therefore 4r + 3(2) = -6$$

$$\therefore 4r = -12$$

$$\therefore r = -3$$

$$\therefore r = -3, s = 2$$

- b** Yes, any vector in the plane can be written as a linear combination of  $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$  and  $\begin{pmatrix} 3 \\ 5 \end{pmatrix}$ , since  $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$  and  $\begin{pmatrix} 3 \\ 5 \end{pmatrix}$  are **not** multiples of each other.



We can arrive at any  $(x^*, y^*)$  by moving along multiples of  $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$  and  $\begin{pmatrix} 3 \\ 5 \end{pmatrix}$ .

**EXERCISE 12K**

$$\begin{aligned} 1 \quad a \quad \begin{pmatrix} 2 \\ 5 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 1 \end{pmatrix} &= 2 \times 3 + 5 \times 1 \\ &= 6 + 5 \\ &= 11 \end{aligned}$$

$$\begin{aligned} b \quad \begin{pmatrix} 5 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 4 \end{pmatrix} &= 5 \times 2 + (-1) \times 4 \\ &= 10 - 4 \\ &= 6 \end{aligned}$$

$$\begin{aligned} c \quad \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} 4 \\ -2 \\ 5 \end{pmatrix} &= 3 \times 4 + 0 \times (-2) + (-2) \times 5 \\ &= 12 + 0 - 10 \\ &= 2 \end{aligned}$$

$$\begin{aligned} d \quad \begin{pmatrix} -4 \\ -5 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 9 \\ -6 \\ -1 \end{pmatrix} &= (-4) \times 9 + (-5) \times (-6) + 3 \times (-1) \\ &= -36 + 30 - 3 \\ &= -9 \end{aligned}$$

$$\begin{aligned} 2 \quad a \quad \mathbf{q} \bullet \mathbf{p} &= \begin{pmatrix} -1 \\ 5 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 2 \end{pmatrix} \\ &= (-1) \times 3 + 5 \times 2 \\ &= -3 + 10 \\ &= 7 \end{aligned}$$

$$\begin{aligned} b \quad \mathbf{q} \bullet \mathbf{r} &= \begin{pmatrix} -1 \\ 5 \end{pmatrix} \bullet \begin{pmatrix} -2 \\ 4 \end{pmatrix} \\ &= (-1) \times (-2) + 5 \times 4 \\ &= 2 + 20 \\ &= 22 \end{aligned}$$

$$\begin{aligned} c \quad \mathbf{q} \bullet (\mathbf{p} + \mathbf{r}) &= \begin{pmatrix} -1 \\ 5 \end{pmatrix} \bullet \left[ \begin{pmatrix} 3 \\ 2 \end{pmatrix} + \begin{pmatrix} -2 \\ 4 \end{pmatrix} \right] \\ &= \begin{pmatrix} -1 \\ 5 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 6 \end{pmatrix} \\ &= (-1) \times 1 + 5 \times 6 \\ &= -1 + 30 \\ &= 29 \end{aligned}$$

$$\begin{aligned} d \quad 3\mathbf{r} \bullet \mathbf{q} &= 3 \begin{pmatrix} -2 \\ 4 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 5 \end{pmatrix} \\ &= \begin{pmatrix} -6 \\ 12 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 5 \end{pmatrix} \\ &= (-6) \times (-1) + 12 \times 5 \\ &= 6 + 60 \\ &= 66 \end{aligned}$$

$$\begin{aligned} e \quad 2\mathbf{p} \bullet 2\mathbf{p} &= 2 \begin{pmatrix} 3 \\ 2 \end{pmatrix} \bullet 2 \begin{pmatrix} 3 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} 6 \\ 4 \end{pmatrix} \bullet \begin{pmatrix} 6 \\ 4 \end{pmatrix} \\ &= 6 \times 6 + 4 \times 4 \\ &= 36 + 16 \\ &= 52 \end{aligned}$$

$$\begin{aligned} f \quad \mathbf{i} \bullet \mathbf{p} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 2 \end{pmatrix} \\ &= 1 \times 3 + 0 \times 2 \\ &= 3 + 0 \\ &= 3 \end{aligned}$$

$$\begin{aligned} g \quad \mathbf{q} \bullet \mathbf{j} &= \begin{pmatrix} -1 \\ 5 \end{pmatrix} \bullet \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= (-1) \times 0 + 5 \times 1 \\ &= 0 + 5 \\ &= 5 \end{aligned}$$

$$\begin{aligned} h \quad \mathbf{i} \bullet \mathbf{i} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= 1 \times 1 + 0 \times 0 \\ &= 1 + 0 \\ &= 1 \end{aligned}$$

$$\begin{aligned}
 \text{3 a } \mathbf{a} \bullet \mathbf{b} &= \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \\
 &= 2(-1) + 1(1) + 3(1) \\
 &= -2 + 1 + 3 \\
 &= 2
 \end{aligned}$$

$$\begin{aligned}
 \text{c } |\mathbf{a}|^2 &= \left( \sqrt{2^2 + 1^2 + 3^2} \right)^2 \\
 &= 14
 \end{aligned}$$

$$\begin{aligned}
 \text{e } \mathbf{a} \bullet (\mathbf{b} + \mathbf{c}) &= \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \bullet \left[ \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \right] \\
 &= \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} \\
 &= 2(-1) + 1(0) + 3(2) \\
 &= 4
 \end{aligned}$$

$$\begin{aligned}
 \text{4 a } \mathbf{a} \bullet \mathbf{b} &= \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \bullet \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \\
 &= a_1 b_1 + a_2 b_2 + a_3 b_3 \\
 &= b_1 a_1 + b_2 a_2 + b_3 a_3 \\
 &= \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \bullet \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \\
 &= \mathbf{b} \bullet \mathbf{a}
 \end{aligned}$$

$$\begin{aligned}
 \text{5 a } (\mathbf{i} + \mathbf{j} - \mathbf{k}) \bullet (2\mathbf{j} + \mathbf{k}) &= \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \\
 &= 1(0) + 1(2) + (-1)(1) \\
 &= 1
 \end{aligned}$$

$$\begin{aligned}
 \text{b } \mathbf{b} \bullet \mathbf{a} &= \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \\
 &= -1(2) + 1(1) + 1(3) \\
 &= -2 + 1 + 3 \\
 &= 2
 \end{aligned}$$

$$\begin{aligned}
 \text{d } \mathbf{a} \bullet \mathbf{a} &= \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \\
 &= 2(2) + 1(1) + 3(3) \\
 &= 14
 \end{aligned}$$

$$\begin{aligned}
 \text{f } \mathbf{a} \bullet \mathbf{b} + \mathbf{a} \bullet \mathbf{c} &= 2 + \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \quad \{\text{using a}\} \\
 &= 2 + 2(0) + 1(-1) + 3(1) \\
 &= 4
 \end{aligned}$$

$$\begin{aligned}
 \text{b } \mathbf{a} \bullet \mathbf{a} &= \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \bullet \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \\
 &= a_1 a_1 + a_2 a_2 + a_3 a_3 \\
 &= a_1^2 + a_2^2 + a_3^2 \\
 &= \left( \sqrt{a_1^2 + a_2^2 + a_3^2} \right)^2 \\
 &= |\mathbf{a}|^2
 \end{aligned}$$

$$\begin{aligned}
 \text{b } \mathbf{i} \bullet \mathbf{i} &= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\
 &= 1(1) + 0(0) + 0(0) \\
 &= 1
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & (3\mathbf{i} - 2\mathbf{k}) \bullet (\mathbf{i} + \mathbf{j}) \\
 &= \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \\
 &= 3(1) + 0(1) + (-2)(0) \\
 &= 3
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad & (2\mathbf{i} + \mathbf{k}) \bullet (-\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}) \\
 &= \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} \\
 &= 2(-1) + 0(3) + 1(2) \\
 &= 0
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & \mathbf{i} \bullet \mathbf{j} \\
 &= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \bullet \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\
 &= 1(0) + 0(1) + 0(0) \\
 &= 0
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & (\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}) \bullet (3\mathbf{i} - 3\mathbf{j} - 5\mathbf{k}) \\
 &= \begin{pmatrix} 1 \\ -4 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ -3 \\ -5 \end{pmatrix} \\
 &= 1(3) + (-4)(-3) + 2(-5) \\
 &= 5
 \end{aligned}$$

$$\text{6} \quad \mathbf{a} = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix}, \quad \mathbf{d} = \begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix}$$

$$\text{a} \quad \text{i} \quad \mathbf{a} \bullet \mathbf{c} = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix} = 1(3) + 0(-1) + 4(5) = 23$$

$$\text{ii} \quad \mathbf{a} \bullet \mathbf{d} = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix} \bullet \begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix} = 1(0) + 0(-2) + 4(3) = 12$$

$$\text{iii} \quad \mathbf{b} \bullet \mathbf{c} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix} = -2(3) + 1(-1) + 0(5) = -7$$

$$\text{iv} \quad \mathbf{b} \bullet \mathbf{d} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \bullet \begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix} = -2(0) + 1(-2) + 0(3) = -2$$

$$\begin{aligned}
 \text{b} \quad (\mathbf{a} + \mathbf{b}) \bullet (\mathbf{c} + \mathbf{d}) &= \left[ \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix} + \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \right] \bullet \left[ \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix} + \begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix} \right] \\
 &= \begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ -3 \\ 8 \end{pmatrix} \\
 &= -1(3) + 1(-3) + 4(8) \\
 &= 26
 \end{aligned}$$

$$\text{and} \quad \mathbf{a} \bullet \mathbf{c} + \mathbf{a} \bullet \mathbf{d} + \mathbf{b} \bullet \mathbf{c} + \mathbf{b} \bullet \mathbf{d} = 23 + 12 + (-7) + (-2) = 26$$

$$\therefore (\mathbf{a} + \mathbf{b}) \bullet (\mathbf{c} + \mathbf{d}) = \mathbf{a} \bullet \mathbf{c} + \mathbf{a} \bullet \mathbf{d} + \mathbf{b} \bullet \mathbf{c} + \mathbf{b} \bullet \mathbf{d}$$



$$\begin{aligned}
7 \quad \mathbf{a} \bullet (\mathbf{b} + \mathbf{c}) &= \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \bullet \left[ \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \right] \\
&= \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \bullet \begin{pmatrix} b_1 + c_1 \\ b_2 + c_2 \\ b_3 + c_3 \end{pmatrix} \\
&= a_1(b_1 + c_1) + a_2(b_2 + c_2) + a_3(b_3 + c_3) \\
&= a_1b_1 + a_1c_1 + a_2b_2 + a_2c_2 + a_3b_3 + a_3c_3 \\
&= (a_1b_1 + a_2b_2 + a_3b_3) + (a_1c_1 + a_2c_2 + a_3c_3) \\
&= \mathbf{a} \bullet \mathbf{b} + \mathbf{a} \bullet \mathbf{c}
\end{aligned}$$

$$\therefore \mathbf{p} \bullet (\mathbf{c} + \mathbf{d}) = \mathbf{p} \bullet \mathbf{c} + \mathbf{p} \bullet \mathbf{d}$$

$$\begin{aligned}
\text{If we let } \mathbf{p} = \mathbf{a} + \mathbf{b}, \text{ then } (\mathbf{a} + \mathbf{b}) \bullet (\mathbf{c} + \mathbf{d}) &= \mathbf{p} \bullet (\mathbf{c} + \mathbf{d}) \\
&= \mathbf{p} \bullet \mathbf{c} + \mathbf{p} \bullet \mathbf{d} \\
&= (\mathbf{a} + \mathbf{b}) \bullet \mathbf{c} + (\mathbf{a} + \mathbf{b}) \bullet \mathbf{d} \\
&= \mathbf{c} \bullet (\mathbf{a} + \mathbf{b}) + \mathbf{d} \bullet (\mathbf{a} + \mathbf{b}) \\
&= \mathbf{c} \bullet \mathbf{a} + \mathbf{c} \bullet \mathbf{b} + \mathbf{d} \bullet \mathbf{a} + \mathbf{d} \bullet \mathbf{b} \\
&= \mathbf{a} \bullet \mathbf{c} + \mathbf{a} \bullet \mathbf{d} + \mathbf{b} \bullet \mathbf{c} + \mathbf{b} \bullet \mathbf{d}
\end{aligned}$$

$$\begin{aligned}
8 \quad \mathbf{a} \quad |\mathbf{a} + \mathbf{b}|^2 &= (\mathbf{a} + \mathbf{b}) \bullet (\mathbf{a} + \mathbf{b}) \\
&= \mathbf{a} \bullet \mathbf{a} + \mathbf{a} \bullet \mathbf{b} + \mathbf{b} \bullet \mathbf{a} + \mathbf{b} \bullet \mathbf{b} \\
&= |\mathbf{a}|^2 + \mathbf{a} \bullet \mathbf{b} + \mathbf{a} \bullet \mathbf{b} + |\mathbf{b}|^2 \\
&= |\mathbf{a}|^2 + 2\mathbf{a} \bullet \mathbf{b} + |\mathbf{b}|^2
\end{aligned}$$

$$\begin{aligned}
\mathbf{b} \quad |\mathbf{a} + \mathbf{b}|^2 + |\mathbf{a} - \mathbf{b}|^2 &= (\mathbf{a} + \mathbf{b}) \bullet (\mathbf{a} + \mathbf{b}) + (\mathbf{a} - \mathbf{b}) \bullet (\mathbf{a} - \mathbf{b}) \\
&= \mathbf{a} \bullet \mathbf{a} + \cancel{\mathbf{a} \bullet \mathbf{b}} + \cancel{\mathbf{b} \bullet \mathbf{a}} + \mathbf{b} \bullet \mathbf{b} + \mathbf{a} \bullet \mathbf{a} - \cancel{\mathbf{a} \bullet \mathbf{b}} - \cancel{\mathbf{b} \bullet \mathbf{a}} + \mathbf{b} \bullet \mathbf{b} \\
&= 2\mathbf{a} \bullet \mathbf{a} + 2\mathbf{b} \bullet \mathbf{b} \\
&= 2|\mathbf{a}|^2 + 2|\mathbf{b}|^2
\end{aligned}$$

$$\begin{aligned}
\mathbf{c} \quad |\mathbf{a} + \mathbf{b}|^2 - |\mathbf{a} - \mathbf{b}|^2 &= (\mathbf{a} + \mathbf{b}) \bullet (\mathbf{a} + \mathbf{b}) - (\mathbf{a} - \mathbf{b}) \bullet (\mathbf{a} - \mathbf{b}) \\
&= \mathbf{a} \bullet \mathbf{a} + \mathbf{a} \bullet \mathbf{b} + \mathbf{b} \bullet \mathbf{a} + \mathbf{b} \bullet \mathbf{b} - (\mathbf{a} \bullet \mathbf{a} - \mathbf{a} \bullet \mathbf{b} - \mathbf{b} \bullet \mathbf{a} + \mathbf{b} \bullet \mathbf{b}) \\
&= \cancel{\mathbf{a} \bullet \mathbf{a}} + \mathbf{a} \bullet \mathbf{b} + \mathbf{b} \bullet \mathbf{a} + \cancel{\mathbf{b} \bullet \mathbf{b}} - \cancel{\mathbf{a} \bullet \mathbf{a}} + \mathbf{a} \bullet \mathbf{b} + \mathbf{b} \bullet \mathbf{a} - \cancel{\mathbf{b} \bullet \mathbf{b}} \\
&= \mathbf{a} \bullet \mathbf{b} + \mathbf{a} \bullet \mathbf{b} + \mathbf{a} \bullet \mathbf{b} + \mathbf{a} \bullet \mathbf{b} \\
&= 4\mathbf{a} \bullet \mathbf{b}
\end{aligned}$$

9  $\mathbf{a} \bullet \mathbf{b}$  is a scalar and so  $\mathbf{a} \bullet \mathbf{b} \bullet \mathbf{c}$  is the scalar product of a scalar and a vector, which is meaningless.

$$10 \quad |\mathbf{a}| = 5, \quad |\mathbf{b}| = 3$$

$$\begin{aligned}
(\mathbf{a} + \mathbf{b}) \bullet (\mathbf{a} - \mathbf{b}) &= \mathbf{a} \bullet \mathbf{a} - \mathbf{a} \bullet \mathbf{b} + \mathbf{b} \bullet \mathbf{a} - \mathbf{b} \bullet \mathbf{b} \\
&= \mathbf{a} \bullet \mathbf{a} - \mathbf{b} \bullet \mathbf{b} \\
&= |\mathbf{a}|^2 - |\mathbf{b}|^2 \\
&= 5^2 - 3^2 \\
&= 16
\end{aligned}$$

**11** If  $\mathbf{a}$  and  $\mathbf{b}$  are parallel, then  $\mathbf{b} = k\mathbf{a}$  for some scalar  $k$ .

$$\begin{aligned}
 \therefore |\mathbf{a} \bullet \mathbf{b}| &= |\mathbf{a} \bullet k\mathbf{a}| \\
 &= |k| |\mathbf{a} \bullet \mathbf{a}| \\
 &= |k| |\mathbf{a}|^2 \\
 &= |k| |\mathbf{a}| |\mathbf{a}| \\
 &= |\mathbf{a}| |k\mathbf{a}| \\
 &= |\mathbf{a}| |\mathbf{b}|
 \end{aligned}$$

## EXERCISE 12L

**1 a**  $\mathbf{a} \bullet \mathbf{b} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 5 \end{pmatrix}$

$$\begin{aligned}
 &= 3(2) + 1(5) \\
 &= 6 + 5 \\
 &= 11
 \end{aligned}$$

**2 a**  $\mathbf{r} \bullet \mathbf{s} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

$$\begin{aligned}
 &= 3(-1) + 3(2) \\
 &= -3 + 6 \\
 &= 3
 \end{aligned}$$

**b**  $\mathbf{r} \bullet \mathbf{s} = \begin{pmatrix} -1 \\ -3 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 5 \end{pmatrix}$

$$\begin{aligned}
 &= (-1)(2) + (-3)(5) \\
 &= -2 - 15 \\
 &= -17
 \end{aligned}$$

**c**  $\mathbf{r} \bullet \mathbf{s} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

$$\begin{aligned}
 &= (1)(2) + (-1)(-1) \\
 &= 2 + 1 \\
 &= 3
 \end{aligned}$$

**b**  $\cos \theta = \frac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{a}| |\mathbf{b}|}$

$$\begin{aligned}
 &= \frac{11}{\sqrt{9+1}\sqrt{4+25}} \\
 &= \frac{11}{\sqrt{290}} \\
 \therefore \theta &= \cos^{-1} \left( \frac{11}{\sqrt{290}} \right) \approx 49.8^\circ
 \end{aligned}$$

Now  $\cos \theta = \frac{\mathbf{r} \bullet \mathbf{s}}{|\mathbf{r}| |\mathbf{s}|}$

$$\begin{aligned}
 &= \frac{3}{\sqrt{3^2+3^2}\sqrt{(-1)^2+2^2}} \\
 &= \frac{3}{\sqrt{90}} \\
 \therefore \theta &= \cos^{-1} \left( \frac{3}{\sqrt{90}} \right) \approx 71.6^\circ
 \end{aligned}$$

Now  $\cos \theta = \frac{\mathbf{r} \bullet \mathbf{s}}{|\mathbf{r}| |\mathbf{s}|}$

$$\begin{aligned}
 &= \frac{-17}{\sqrt{(-1)^2+(-3)^2}\sqrt{2^2+5^2}} \\
 &= \frac{-17}{\sqrt{290}} \\
 \therefore \theta &= \cos^{-1} \left( \frac{-17}{\sqrt{290}} \right) \approx 177^\circ
 \end{aligned}$$

Now  $\cos \theta = \frac{\mathbf{r} \bullet \mathbf{s}}{|\mathbf{r}| |\mathbf{s}|}$

$$\begin{aligned}
 &= \frac{3}{\sqrt{1^2+(-1)^2}\sqrt{2^2+(-1)^2}} \\
 &= \frac{3}{\sqrt{10}} \\
 \therefore \theta &= \cos^{-1} \left( \frac{3}{\sqrt{10}} \right) \approx 18.4^\circ
 \end{aligned}$$

$$\begin{aligned}
 \text{d } \mathbf{r} \bullet \mathbf{s} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
 &= 1(1) + 0(1) \\
 &= 1 + 0 \\
 &= 1
 \end{aligned}$$

$$\begin{aligned}
 \text{Now } \cos \theta &= \frac{\mathbf{r} \bullet \mathbf{s}}{|\mathbf{r}| |\mathbf{s}|} \\
 &= \frac{1}{\sqrt{1^2 + 0^2} \sqrt{1^2 + 1^2}} \\
 &= \frac{1}{\sqrt{2}} \\
 \therefore \theta &= \cos^{-1} \left( \frac{1}{\sqrt{2}} \right) = 45^\circ
 \end{aligned}$$

$$\begin{aligned}
 \text{3 } \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} &= 1(2) + 1(3) + 5(-1) = 0 \\
 \therefore \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix} \text{ and } \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} &\text{ are perpendicular.}
 \end{aligned}$$

4  $\mathbf{a} \bullet \mathbf{b}$

$$\begin{aligned}
 &= \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \\
 &= 3(-1) + 1(1) + 2(1) \\
 &= 0
 \end{aligned}$$

$\mathbf{b} \bullet \mathbf{c}$

$$\begin{aligned}
 &= \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 5 \\ -4 \end{pmatrix} \\
 &= -1(1) + 1(5) + 1(-4) \\
 &= 0
 \end{aligned}$$

$\mathbf{a} \bullet \mathbf{c}$

$$\begin{aligned}
 &= \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 5 \\ -4 \end{pmatrix} \\
 &= 3(1) + 1(5) + 2(-4) \\
 &= 0
 \end{aligned}$$

$\therefore \mathbf{a}, \mathbf{b}, \text{ and } \mathbf{c}$  are mutually perpendicular.

$$\begin{aligned}
 \text{5 a } \cos \theta &= \frac{\begin{pmatrix} 4 \\ 0 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}}{\left| \begin{pmatrix} 4 \\ 0 \\ -2 \end{pmatrix} \right| \left| \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \right|} \\
 &= \frac{4(2) + 0(-3) + (-2)(1)}{\sqrt{4^2 + 0^2 + (-2)^2} \sqrt{2^2 + (-3)^2 + 1^2}} \\
 &= \frac{8 + 0 - 2}{\sqrt{20} \sqrt{14}} \\
 &= \frac{6}{\sqrt{280}} \\
 \therefore \theta &= \cos^{-1} \left( \frac{6}{\sqrt{280}} \right) \approx 69.0^\circ
 \end{aligned}$$

$$\begin{aligned}
 \text{b } \cos \theta &= \frac{\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}}{\left| \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} \right| \left| \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} \right|} \\
 &= \frac{3(-2) + (-1)(1) + 2(3)}{\sqrt{3^2 + (-1)^2 + 2^2} \sqrt{(-2)^2 + 1^2 + 3^2}} \\
 &= \frac{-6 - 1 + 6}{\sqrt{14} \sqrt{14}} \\
 &= -\frac{1}{14} \\
 \therefore \theta &= \cos^{-1} \left( -\frac{1}{14} \right) \approx 94.1^\circ
 \end{aligned}$$

$$\begin{aligned}
 \text{c } \cos \theta &= \frac{\begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}}{\left| \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} \right| \left| \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \right|} \\
 &= \frac{0(1) + 2(0) + (-1)(2)}{\sqrt{0^2 + 2^2 + (-1)^2} \sqrt{1^2 + 0^2 + 2^2}} \\
 &= \frac{-2}{\sqrt{5}\sqrt{5}} \\
 &= -\frac{2}{5} \\
 \therefore \theta &= \cos^{-1}\left(-\frac{2}{5}\right) \approx 114^\circ
 \end{aligned}$$

$$\begin{aligned}
 \text{d } \cos \theta &= \frac{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}}{\left| \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right| \left| \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} \right|} \\
 &= \frac{0(-1) + 1(1) + 0(3)}{\sqrt{0^2 + 1^2 + 0^2} \sqrt{(-1)^2 + 1^2 + 3^2}} \\
 &= \frac{1}{\sqrt{11}} \\
 \therefore \theta &= \cos^{-1}\left(\frac{1}{\sqrt{11}}\right) \approx 72.5^\circ
 \end{aligned}$$

6 a Since **p** and **q** are perpendicular,

$$\begin{aligned}
 \mathbf{p} \bullet \mathbf{q} &= 0 \\
 \therefore \begin{pmatrix} 3 \\ t \end{pmatrix} \bullet \begin{pmatrix} -2 \\ 1 \end{pmatrix} &= 0 \\
 \therefore 3(-2) + t(1) &= 0 \\
 \therefore -6 + t &= 0 \\
 \therefore t &= 6
 \end{aligned}$$

c Since **a** and **b** are perpendicular,

$$\begin{aligned}
 \mathbf{a} \bullet \mathbf{b} &= 0 \\
 \therefore \begin{pmatrix} t \\ t+2 \end{pmatrix} \bullet \begin{pmatrix} 2-3t \\ t \end{pmatrix} &= 0 \\
 \therefore t(2-3t) + (t+2)(t) &= 0 \\
 \therefore 2t - 3t^2 + t^2 + 2t &= 0 \\
 \therefore -2t^2 + 4t &= 0 \\
 \therefore t^2 - 2t &= 0 \\
 \therefore t(t-2) &= 0 \\
 \therefore t &= 0 \text{ or } 2
 \end{aligned}$$

e Since **p** and **q** are perpendicular,

$$\begin{aligned}
 \mathbf{p} \bullet \mathbf{q} &= 0 \\
 \therefore \begin{pmatrix} 3 \\ t \\ -2 \end{pmatrix} \bullet \begin{pmatrix} 1-t \\ -3 \\ 4 \end{pmatrix} &= 0 \\
 \therefore 3(1-t) + t(-3) + (-2)(4) &= 0 \\
 \therefore 3 - 3t - 3t - 8 &= 0 \\
 \therefore -6t &= 5 \\
 \therefore t &= -\frac{5}{6}
 \end{aligned}$$

b Since **r** and **s** are perpendicular,

$$\begin{aligned}
 \mathbf{r} \bullet \mathbf{s} &= 0 \\
 \therefore \begin{pmatrix} t \\ t+2 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ -4 \end{pmatrix} &= 0 \\
 \therefore t(3) + (t+2)(-4) &= 0 \\
 \therefore 3t - 4t - 8 &= 0 \\
 \therefore t &= -8
 \end{aligned}$$

d Since **a** and **b** are perpendicular,

$$\begin{aligned}
 \mathbf{a} \bullet \mathbf{b} &= 0 \\
 \therefore \begin{pmatrix} 3 \\ -1 \\ t \end{pmatrix} \bullet \begin{pmatrix} 2t \\ -3 \\ -4 \end{pmatrix} &= 0 \\
 \therefore 3(2t) + (-1)(-3) + t(-4) &= 0 \\
 \therefore 6t + 3 - 4t &= 0 \\
 \therefore 2t &= -3 \\
 \therefore t &= -\frac{3}{2}
 \end{aligned}$$

f Since **a** and **b** are perpendicular,

$$\begin{aligned}
 \mathbf{a} \bullet \mathbf{b} &= 0 \\
 \therefore \begin{pmatrix} 2 \\ t \\ t-2 \end{pmatrix} \bullet \begin{pmatrix} t \\ 3 \\ t \end{pmatrix} &= 0 \\
 \therefore 2t + 3t + t(t-2) &= 0 \\
 \therefore 2t + 3t + t^2 - 2t &= 0 \\
 \therefore t(t+3) &= 0 \\
 \therefore t &= 0 \text{ or } -3
 \end{aligned}$$



**7** Given that  $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ ,  $\mathbf{b} = \begin{pmatrix} 2 \\ 2 \\ r \end{pmatrix}$ , and  $\mathbf{c} = \begin{pmatrix} s \\ t \\ 1 \end{pmatrix}$  are mutually perpendicular,

$$\mathbf{a} \bullet \mathbf{b} = 0, \mathbf{b} \bullet \mathbf{c} = 0, \text{ and } \mathbf{a} \bullet \mathbf{c} = 0$$

$$\therefore \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 2 \\ r \end{pmatrix} = 0 \quad \begin{array}{l} \therefore 2 + 4 + 3r = 0 \\ \therefore 3r = -6 \\ \therefore r = -2 \end{array}$$

$$\text{and } \begin{pmatrix} 2 \\ 2 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} s \\ t \\ 1 \end{pmatrix} = 0 \quad \begin{array}{l} \therefore 2s + 2t - 2 = 0 \\ \therefore s + t = 1 \quad \dots (1) \end{array}$$

$$\text{and } \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} s \\ t \\ 1 \end{pmatrix} = 0 \quad \begin{array}{l} \therefore s + 2t + 3 = 0 \\ \therefore s + 2t = -3 \quad \dots (2) \end{array}$$

(2) - (1) gives  $t = -4$  and so  $s = 5$

$\therefore r = -2, s = 5, \text{ and } t = -4$

**8 a**  $\begin{pmatrix} 3 \\ -4 \end{pmatrix} \bullet \begin{pmatrix} 4 \\ 3 \end{pmatrix} = (3)(4) + (-4)(3) = 0$

$\therefore$  vectors of the form  $k \begin{pmatrix} 4 \\ 3 \end{pmatrix}$ ,  $k \neq 0$  are perpendicular to  $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$ .

Now  $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$  has length  $\sqrt{16 + 9} = 5$  units

$$\therefore |k| \times 5 = 10$$

$$\therefore |k| = 2$$

$$\therefore k = \pm 2$$

$\therefore$  the vectors of length 10 units which are perpendicular to  $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$  are  $\pm 2 \begin{pmatrix} 4 \\ 3 \end{pmatrix}$ , which are  $\begin{pmatrix} 8 \\ 6 \end{pmatrix}$  and  $\begin{pmatrix} -8 \\ -6 \end{pmatrix}$ .

**b**  $\begin{pmatrix} 1 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 1 \end{pmatrix} = (1)(1) + (-1)(1) = 0$

$\therefore$  vectors of the form  $k \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ ,  $k \neq 0$  are perpendicular to  $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ .

Now  $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$  has length  $\sqrt{1 + 1} = \sqrt{2}$  units

$$\therefore |k| \sqrt{2} = 3\sqrt{2}$$

$$\therefore |k| = 3$$

$$\therefore k = \pm 3$$

$\therefore$  the vectors of length  $3\sqrt{2}$  units which are perpendicular to  $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$  are  $\pm 3 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ , which are  $\begin{pmatrix} 3 \\ 3 \end{pmatrix}$  and  $\begin{pmatrix} -3 \\ -3 \end{pmatrix}$ .

$$\bullet \begin{pmatrix} -2 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 2 \end{pmatrix} = (-2)(-1) + (-1)(2) = 0$$

$\therefore$  vectors of the form  $k \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ ,  $k \neq 0$  are perpendicular to  $\begin{pmatrix} -2 \\ -1 \end{pmatrix}$ .

Now  $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$  has length  $\sqrt{1+4} = \sqrt{5}$  units

$$\therefore |k| \sqrt{5} = \sqrt{20}$$

$$\therefore |k| = \frac{2\sqrt{5}}{\sqrt{5}}$$

$$\therefore |k| = 2$$

$$\therefore k = \pm 2$$

$\therefore$  the vectors of length  $\sqrt{20}$  units which are perpendicular to  $\begin{pmatrix} -2 \\ -1 \end{pmatrix}$  are  $\pm 2 \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ ,  
which are  $\begin{pmatrix} -2 \\ 4 \end{pmatrix}$  and  $\begin{pmatrix} 2 \\ -4 \end{pmatrix}$ .

$$9 \quad \begin{pmatrix} 2 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} -3 \\ 2 \end{pmatrix} = (2)(-3) + (3)(2) = 0$$

$\therefore$  vectors of the form  $k \begin{pmatrix} -3 \\ 2 \end{pmatrix}$ ,  $k \neq 0$  are perpendicular to  $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ .

Now  $\begin{pmatrix} -3 \\ 2 \end{pmatrix}$  has length  $\sqrt{9+4} = \sqrt{13}$  units

$$\therefore |k| \sqrt{13} = 5$$

$$\therefore |k| = \frac{5}{\sqrt{13}}$$

$$\therefore k = \pm \frac{5}{\sqrt{13}}$$

$\therefore$  the vectors of length 5 units which are perpendicular to  $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$  are  $\pm \frac{5}{\sqrt{13}} \begin{pmatrix} -3 \\ 2 \end{pmatrix}$ , which are  
 $\begin{pmatrix} -\frac{15}{\sqrt{13}} \\ \frac{10}{\sqrt{13}} \end{pmatrix}$  and  $\begin{pmatrix} \frac{15}{\sqrt{13}} \\ -\frac{10}{\sqrt{13}} \end{pmatrix}$ .

$$\begin{aligned} 10 \quad \mathbf{a} \bullet \mathbf{b} &= |\mathbf{a}| |\mathbf{b}| \cos \theta \\ &= \sqrt{10} \times 3 \times \cos 71^\circ \\ &\approx 3.089 \end{aligned}$$

$$\begin{aligned} 11 \quad \mathbf{a} \quad \mathbf{p} \bullet \mathbf{q} &= |\mathbf{p}| |\mathbf{q}| \cos \theta \\ &= 2 \times 5 \times \cos 60^\circ \\ &= 5 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \mathbf{p} \bullet \mathbf{q} &= |\mathbf{p}| |\mathbf{q}| \cos \theta \\ &= 6 \times 3 \times \cos \frac{2\pi}{3} \\ &= -9 \end{aligned}$$

12 Let  $\theta$  be the angle between  $\mathbf{a}$  and  $\mathbf{b}$ .

( $\Rightarrow$ ) If  $\mathbf{a}$  and  $\mathbf{b}$  are perpendicular, then  $\theta = 90^\circ$

$$\therefore \mathbf{a} \bullet \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \times \cos 90^\circ = 0$$

( $\Leftarrow$ ) If  $\mathbf{a} \bullet \mathbf{b} = 0$ , then  $|\mathbf{a}| |\mathbf{b}| \cos \theta = 0$

$$\therefore \cos \theta = 0 \quad \{\mathbf{a}, \mathbf{b} \neq \mathbf{0}\}$$

$$\therefore \theta = 90^\circ \quad \{0^\circ \leq \theta \leq 180^\circ\}$$

$\therefore \mathbf{a}$  and  $\mathbf{b}$  are perpendicular.

$\therefore \mathbf{a}$  and  $\mathbf{b}$  are perpendicular  $\Leftrightarrow \mathbf{a} \bullet \mathbf{b} = 0$  provided  $\mathbf{a}$  and  $\mathbf{b}$  are non-zero.

**13** Let  $\theta$  be the angle between  $\mathbf{a}$  and  $\mathbf{b}$ .

( $\Leftarrow$ ) If  $\mathbf{a}$  and  $\mathbf{b}$  are parallel, then  $\theta = 0^\circ$  or  $180^\circ$

$$\therefore \mathbf{a} \bullet \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \times \cos 0^\circ \quad \text{or} \quad \mathbf{a} \bullet \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \times \cos 180^\circ$$

$$\therefore \mathbf{a} \bullet \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \quad \text{or} \quad \mathbf{a} \bullet \mathbf{b} = -|\mathbf{a}| |\mathbf{b}|$$

$$\therefore |\mathbf{a} \bullet \mathbf{b}| = |\mathbf{a}| |\mathbf{b}|$$

( $\Rightarrow$ ) If  $|\mathbf{a} \bullet \mathbf{b}| = |\mathbf{a}| |\mathbf{b}|$ , then  $\mathbf{a} \bullet \mathbf{b} = \pm |\mathbf{a}| |\mathbf{b}|$

$$\therefore \cos \theta = \frac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{a}| |\mathbf{b}|} = \pm 1, \quad \text{provided } \mathbf{a}, \mathbf{b} \neq \mathbf{0}$$

$$\therefore \theta = 0^\circ \text{ or } 180^\circ$$

$\therefore \mathbf{a}$  and  $\mathbf{b}$  are parallel.

$\therefore$  for two non-zero vectors  $\mathbf{a}$  and  $\mathbf{b}$ ,  $|\mathbf{a} \bullet \mathbf{b}| = |\mathbf{a}| |\mathbf{b}| \Leftrightarrow \mathbf{a}$  and  $\mathbf{b}$  are parallel.

**14 a**  $\mathbf{a}$  and  $\mathbf{b}$  are not perpendicular as  $\mathbf{a} \bullet \mathbf{b} \neq 0$ .

**b**  $\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \theta$

If  $\mathbf{a}$  and  $\mathbf{b}$  are parallel, then  $\theta = 0^\circ$  or  $180^\circ$

$$\therefore \cos \theta = \pm 1$$

$$\therefore -12 = |\mathbf{a}| \times 1 \times (\pm 1)$$

$$\therefore |\mathbf{a}| = \pm 12$$

$$\therefore |\mathbf{a}| = 12 \text{ units} \quad \{|\mathbf{a}| > 0\}$$

**Note:** This means that  $\cos \theta$  must be  $-1$ , so the angle between  $\mathbf{a}$  and  $\mathbf{b}$  is  $180^\circ$

$\therefore \mathbf{a}$  and  $\mathbf{b}$  are in opposite directions.

**15 a**  $\mathbf{c} \bullet \mathbf{d} = |\mathbf{c}| |\mathbf{d}| \cos \theta$

$$\therefore 5 = \sqrt{5} \sqrt{5} \cos \theta$$

$$\therefore 5 = 5 \cos \theta$$

$$\therefore \cos \theta = 1$$

$$\therefore \theta = 0^\circ$$

$$\text{So, } \mathbf{c} = \mathbf{d}$$

**b**  $\mathbf{c} \bullet \mathbf{d} = |\mathbf{c}| |\mathbf{d}| \cos \theta$

$$\therefore -5 = \sqrt{5} \sqrt{5} \cos \theta$$

$$\therefore -5 = 5 \cos \theta$$

$$\therefore \cos \theta = -1$$

$$\therefore \theta = 180^\circ$$

$$\text{So, } \mathbf{c} = -\mathbf{d}$$

**16** For example, let  $\mathbf{a} = \mathbf{i}$ ,  $\mathbf{b} = \mathbf{j}$ , and  $\mathbf{c} = \mathbf{k}$

$$\mathbf{i} \bullet \mathbf{j} = \mathbf{i} \bullet \mathbf{k} = 0 \quad \text{and} \quad \mathbf{j} \neq \mathbf{k}$$

**17** If  $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$  is perpendicular to  $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ , then  $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = 0$ .

So, to find a vector perpendicular to  $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ , we pick two non-zero integer values for  $a$  and  $b$ , then solve for  $c$ .

$$\text{For example, if } a = 1, b = 2 \text{ then } \begin{pmatrix} 1 \\ 2 \\ c \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = 0$$

$$\therefore 1 + 4 - c = 0$$

$$\therefore 5 - c = 0$$

$$\therefore c = 5$$

So,  $\begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$  is perpendicular to  $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ .

Repeating this process with a different value of  $a$  (or  $b$ ) will give another vector which is perpendicular to  $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ .

$$\begin{aligned} \text{If } a = 2, b = 2 \text{ then } \begin{pmatrix} 2 \\ 2 \\ c \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} &= 0 \\ \therefore 2 + 4 - c &= 0 \\ \therefore 6 - c &= 0 \\ \therefore c &= 6 \end{aligned}$$

So,  $\begin{pmatrix} 2 \\ 2 \\ 6 \end{pmatrix}$  is also perpendicular to  $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ , and is not parallel to  $\begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$ .

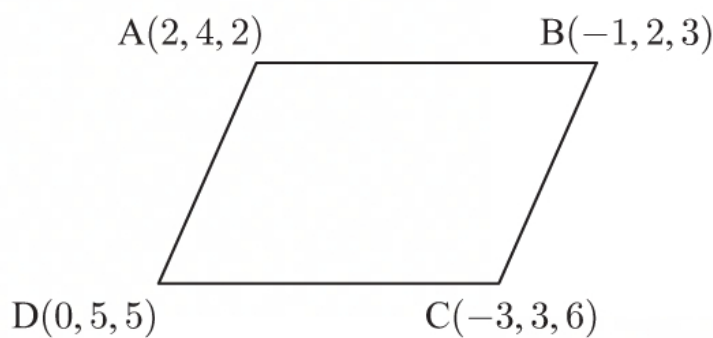
**18** A(5, 1, 2), B(6, -1, 0), C(3, 2, 0)

$$\overrightarrow{AB} = \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}, \overrightarrow{AC} = \begin{pmatrix} -2 \\ 1 \\ -2 \end{pmatrix}, \text{ and } \overrightarrow{BC} = \begin{pmatrix} -3 \\ 3 \\ 0 \end{pmatrix}$$

$$\text{Now } \overrightarrow{AB} \bullet \overrightarrow{AC} = \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} -2 \\ 1 \\ -2 \end{pmatrix} = (-2) + (-2) + 4 = 0$$

$\therefore \overrightarrow{AB}$  is perpendicular to  $\overrightarrow{AC}$  and so triangle ABC is right angled at A.

**19 a**



$$\overrightarrow{AB} = \begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix}, \overrightarrow{BC} = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$$

$$\overrightarrow{DC} = \begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix}, \overrightarrow{AD} = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$$

$\therefore \overrightarrow{AB}$  is parallel to  $\overrightarrow{DC}$  and  $\overrightarrow{BC}$  is parallel to  $\overrightarrow{AD}$ .  
 $\therefore$  ABCD is a parallelogram.

**b**  $|\overrightarrow{AB}| = \sqrt{(-3)^2 + (-2)^2 + 1^2} = \sqrt{9 + 4 + 1} = \sqrt{14}$  units

and  $|\overrightarrow{BC}| = \sqrt{(-2)^2 + 1^2 + 3^2} = \sqrt{4 + 1 + 9} = \sqrt{14}$  units

$\therefore$  ABCD is a rhombus.

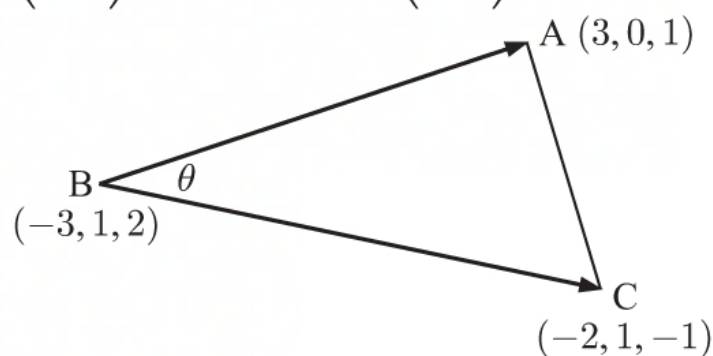
**c**  $\overrightarrow{AC} \bullet \overrightarrow{BD} = \begin{pmatrix} -5 \\ -1 \\ 4 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = (-5)(1) + (-1)(3) + 4(2) = 0$

$\therefore \overrightarrow{AC}$  is perpendicular to  $\overrightarrow{BD}$  which illustrates that the diagonals of a rhombus are perpendicular.



- 20**  $A(3, 0, 1)$ ,  $B(-3, 1, 2)$ ,  $C(-2, 1, -1)$

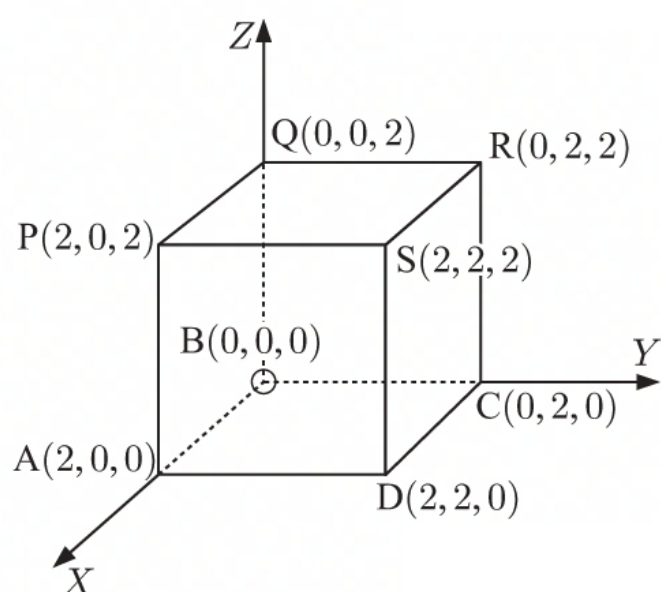
$$\overrightarrow{BC} = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} \text{ and } \overrightarrow{BA} = \begin{pmatrix} 6 \\ -1 \\ -1 \end{pmatrix}$$



$$\begin{aligned} \therefore \cos \theta &= \frac{\overrightarrow{BC} \cdot \overrightarrow{BA}}{|\overrightarrow{BC}| |\overrightarrow{BA}|} \\ &= \frac{(1)(6) + (0)(-1) + (-3)(-1)}{\sqrt{1+0+9}\sqrt{36+1+1}} \\ &= \frac{9}{\sqrt{380}} \\ \therefore \theta &= \cos^{-1}\left(\frac{9}{\sqrt{380}}\right) \\ &\approx 62.5^\circ \end{aligned}$$

If  $\overrightarrow{BA}$  and  $\overrightarrow{CB}$  are used we would find the exterior angle of the triangle at B, which is  $\approx 117.5^\circ$ .

- 21** Suppose the origin is at B.



**a**  $\overrightarrow{BA} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$  and  $\overrightarrow{BS} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$

$$\begin{aligned} \therefore \overrightarrow{BA} \cdot \overrightarrow{BS} &= \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} = 4 + 0 + 0 = 4 \\ \therefore \cos(\widehat{ABS}) &= \frac{4}{\sqrt{4+0+0}\sqrt{4+4+4}} \\ &= \frac{4}{2 \times 2\sqrt{3}} = \frac{1}{\sqrt{3}} \\ \therefore \widehat{ABS} &= \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \\ &\approx 54.7^\circ \end{aligned}$$

**b**  $\overrightarrow{BR} = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$  and  $\overrightarrow{BP} = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$

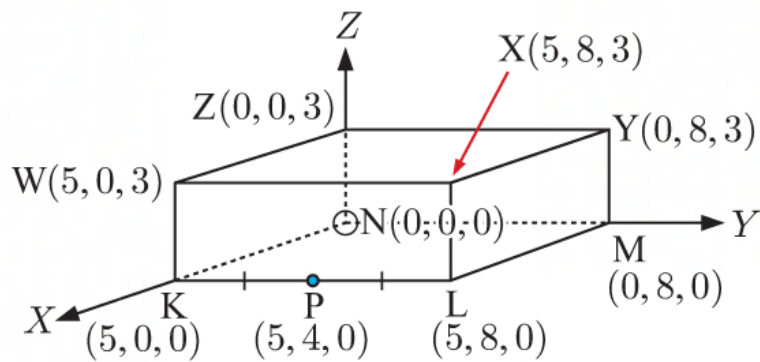
$$\begin{aligned} \therefore \overrightarrow{BR} \cdot \overrightarrow{BP} &= \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} \\ &= 0 + 0 + 4 = 4 \\ \therefore \cos(\widehat{RBP}) &= \frac{4}{\sqrt{0+4+4}\sqrt{4+0+4}} \\ &= \frac{4}{\sqrt{8} \times \sqrt{8}} = \frac{1}{2} \\ \therefore \widehat{RBP} &= 60^\circ \end{aligned}$$

**c**  $\overrightarrow{BP} = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$  and  $\overrightarrow{BS} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$

$$\begin{aligned} \therefore \overrightarrow{BP} \cdot \overrightarrow{BS} &= \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} \\ &= 4 + 0 + 4 = 8 \\ \therefore \cos(\widehat{PBS}) &= \frac{8}{\sqrt{4+0+4}\sqrt{4+4+4}} \\ &= \frac{8}{\sqrt{96}} \\ \therefore \widehat{PBS} &= \cos^{-1}\left(\frac{8}{\sqrt{96}}\right) \\ &\approx 35.3^\circ \end{aligned}$$

**22** Suppose the origin is at N.

**a**



$$\begin{aligned}\vec{NY} &= \begin{pmatrix} 0 \\ 8 \\ 3 \end{pmatrix} \text{ and } \vec{NX} = \begin{pmatrix} 5 \\ 8 \\ 3 \end{pmatrix} \\ \vec{NY} \cdot \vec{NX} &= \begin{pmatrix} 0 \\ 8 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 8 \\ 3 \end{pmatrix} = 0 + 64 + 9 = 73 \\ \therefore \cos(\widehat{YNX}) &= \frac{\vec{NY} \cdot \vec{NX}}{|\vec{NY}| |\vec{NX}|} \\ &= \frac{73}{\sqrt{0+64+9}\sqrt{25+64+9}} \\ &= \frac{73}{\sqrt{73}\sqrt{98}} = \sqrt{\frac{73}{98}} \\ \therefore \widehat{YNX} &= \cos^{-1}\left(\sqrt{\frac{73}{98}}\right) \\ &\approx 30.3^\circ\end{aligned}$$

**b**  $\vec{NY} = \begin{pmatrix} 0 \\ 8 \\ 3 \end{pmatrix}$  and  $\vec{NP} = \begin{pmatrix} 5 \\ 4 \\ 0 \end{pmatrix}$

$$\begin{aligned}\vec{NY} \cdot \vec{NP} &= \begin{pmatrix} 0 \\ 8 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 4 \\ 0 \end{pmatrix} \\ &= 0 + 32 + 0 \\ &= 32\end{aligned}$$

$$\begin{aligned}\therefore \cos(\widehat{YNP}) &= \frac{\vec{NY} \cdot \vec{NP}}{|\vec{NY}| |\vec{NP}|} \\ &= \frac{32}{\sqrt{0+64+9}\sqrt{25+16}} \\ &= \frac{32}{\sqrt{73}\sqrt{41}} \\ \therefore \widehat{YNP} &= \cos^{-1}\left(\frac{32}{\sqrt{73}\sqrt{41}}\right) \\ &\approx 54.2^\circ\end{aligned}$$

**23 a** M is the midpoint of [BC].  $\therefore$  M is at  $\left(\frac{2+1}{2}, \frac{2+3}{2}, \frac{2+1}{2}\right)$ , which is  $\left(\frac{3}{2}, \frac{5}{2}, \frac{3}{2}\right)$ .

**b** Now  $\vec{MD} = \begin{pmatrix} \frac{3}{2} \\ -\frac{1}{2} \\ -\frac{3}{2} \end{pmatrix}$  and  $\vec{MA} = \begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \\ -\frac{1}{2} \end{pmatrix}$

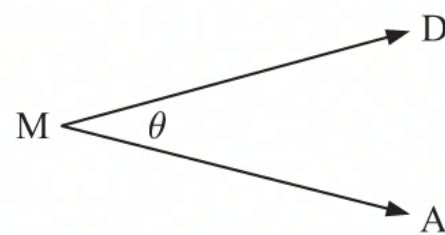
Let  $\widehat{DMA} = \theta$ .

$$\therefore \cos \theta = \frac{\vec{MD} \cdot \vec{MA}}{|\vec{MD}| |\vec{MA}|} = \frac{\begin{pmatrix} \frac{3}{2} \\ -\frac{1}{2} \\ -\frac{3}{2} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \\ -\frac{1}{2} \end{pmatrix}}{\sqrt{\frac{9}{4} + \frac{1}{4} + \frac{9}{4}} \sqrt{\frac{1}{4} + \frac{9}{4} + \frac{1}{4}}}$$

$$\therefore \cos \theta = \frac{\frac{3}{4} + \frac{3}{4} + \frac{3}{4}}{\sqrt{\frac{19}{4}} \sqrt{\frac{11}{4}}} = \frac{\frac{9}{4}}{\frac{\sqrt{209}}{4}} = \frac{9}{\sqrt{209}}$$

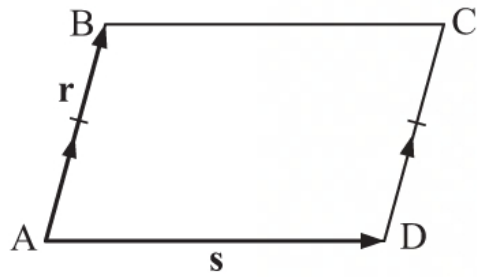
$$\therefore \theta = \cos^{-1}\left(\frac{9}{\sqrt{209}}\right)$$

$$\therefore \widehat{DMA} \approx 51.5^\circ$$



## EXERCISE 12M

1



Let  $\overrightarrow{AB} = \mathbf{r}$  and  $\overrightarrow{AD} = \mathbf{s}$ .

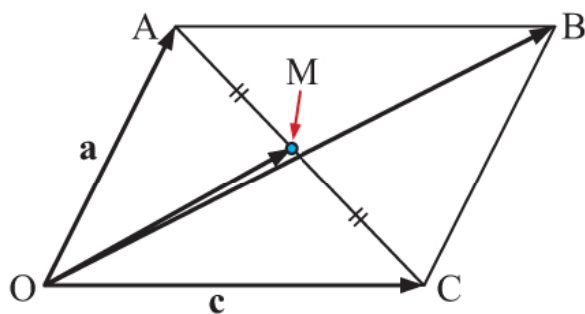
Since  $\overrightarrow{DC}$  is equal in length and parallel to  $\overrightarrow{AB}$ ,  
 $\overrightarrow{DC} = \mathbf{r}$ .

$$\begin{aligned}\text{Now, } \overrightarrow{BA} + \overrightarrow{AD} + \overrightarrow{DC} &= \overrightarrow{BC} \\ \therefore -\mathbf{r} + \mathbf{s} + \mathbf{r} &= \overrightarrow{BC} \\ \therefore \overrightarrow{BC} &= \mathbf{s}\end{aligned}$$

$\therefore \overrightarrow{BC}$  and  $\overrightarrow{AD}$  are also parallel and equal in length.

$\therefore$  ABCD is a parallelogram.

2



Let M be the midpoint of [AC],  $\overrightarrow{OA} = \mathbf{a}$ , and  $\overrightarrow{OC} = \mathbf{c}$ .

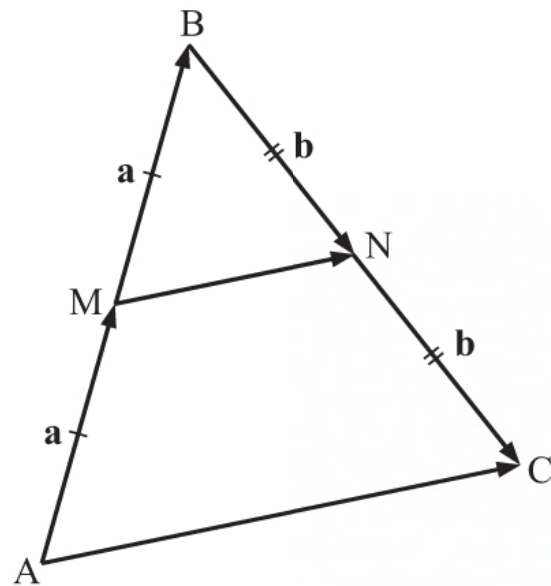
$$\begin{aligned}\text{Now, } \overrightarrow{OB} &= \overrightarrow{OA} + \overrightarrow{OC} \quad \text{and} \quad \overrightarrow{OM} = \frac{1}{2} \overrightarrow{OA} + \frac{1}{2} \overrightarrow{OC} \\ &= \mathbf{a} + \mathbf{c} \qquad \qquad \qquad = \frac{1}{2} \mathbf{a} + \frac{1}{2} \mathbf{c}\end{aligned}$$

$$\therefore \overrightarrow{OM} = \frac{1}{2} \overrightarrow{OB}$$

$\therefore$  [OB] and [AC] bisect each other

$\therefore$  the diagonals of a parallelogram bisect each other.

3



In triangle ABC, M and N are midpoints of [AB] and [BC] respectively.

Let  $\overrightarrow{AM} = \overrightarrow{MB} = \mathbf{a}$  and  $\overrightarrow{BN} = \overrightarrow{NC} = \mathbf{b}$ .

$$\text{Now } \overrightarrow{MN} = \overrightarrow{MB} + \overrightarrow{BN} = \mathbf{a} + \mathbf{b}$$

$$\text{and } \overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = 2\mathbf{a} + 2\mathbf{b}$$

$$\therefore \overrightarrow{MN} = \frac{1}{2} \overrightarrow{AC}$$

Thus  $\overrightarrow{MN}$  and  $\overrightarrow{AC}$  are parallel, and  $|\overrightarrow{MN}| = \frac{1}{2} |\overrightarrow{AC}|$

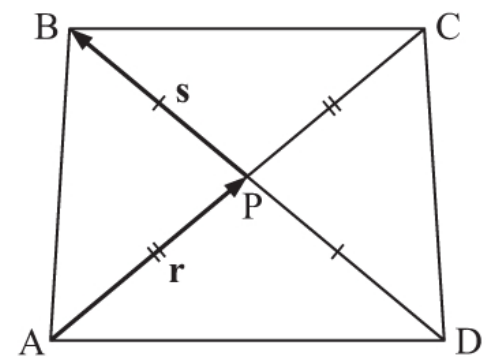
$\therefore$  the line joining the midpoints of two sides of the triangle is parallel to the third side and half its length.

4 a i  $\overrightarrow{PC} = \overrightarrow{AP} = \mathbf{r}$ ,  $\overrightarrow{DP} = \overrightarrow{PB} = \mathbf{s}$

$$\begin{aligned}\text{ii } \overrightarrow{AB} &= \overrightarrow{AP} + \overrightarrow{PB}, & \overrightarrow{DC} &= \overrightarrow{DP} + \overrightarrow{PC} \\ &= \mathbf{r} + \mathbf{s} & &= \mathbf{s} + \mathbf{r} \\ & & &= \mathbf{r} + \mathbf{s}\end{aligned}$$

b The quadrilateral ABCD has a pair of opposite sides,  $\overrightarrow{AB}$  and  $\overrightarrow{DC}$ , which are parallel and equal in length.

c If the diagonals of a quadrilateral bisect each other then the quadrilateral is a parallelogram.



5 Let  $\overrightarrow{AB} = \mathbf{a}$ ,  $\overrightarrow{BC} = \mathbf{b}$ ,  $\overrightarrow{CD} = \mathbf{c}$ , and  $\overrightarrow{DA} = \mathbf{d}$ .

$$\therefore \overrightarrow{MN} = \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b} = \frac{1}{2}(\mathbf{a} + \mathbf{b})$$

$$\overrightarrow{QP} = -\frac{1}{2}\mathbf{d} - \frac{1}{2}\mathbf{c} = \frac{1}{2}(-\mathbf{d} - \mathbf{c})$$

$$\text{But } \mathbf{a} + \mathbf{b} = -\mathbf{d} - \mathbf{c} = \overrightarrow{AC}$$

$$\therefore \overrightarrow{MN} = \overrightarrow{QP} \quad \dots (1)$$

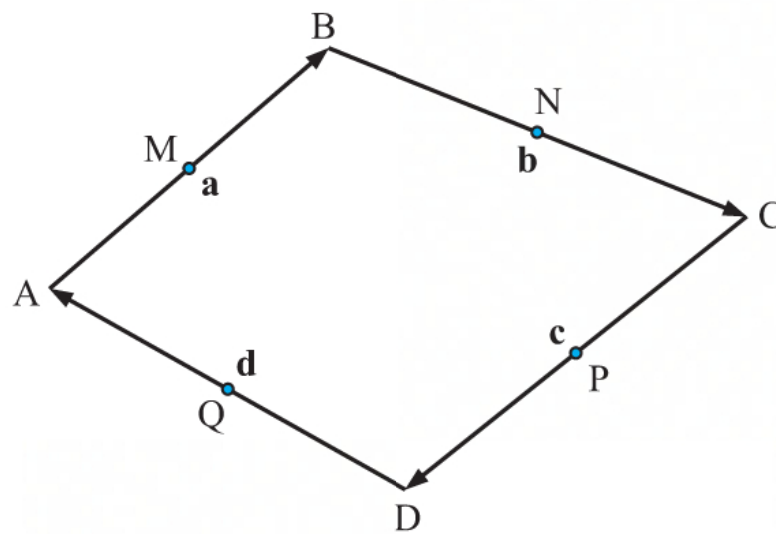
$$\text{Also, } \overrightarrow{MQ} = -\frac{1}{2}\mathbf{a} - \frac{1}{2}\mathbf{d} = \frac{1}{2}(-\mathbf{a} - \mathbf{d})$$

$$\overrightarrow{NP} = \frac{1}{2}\mathbf{b} + \frac{1}{2}\mathbf{c} = \frac{1}{2}(\mathbf{b} + \mathbf{c})$$

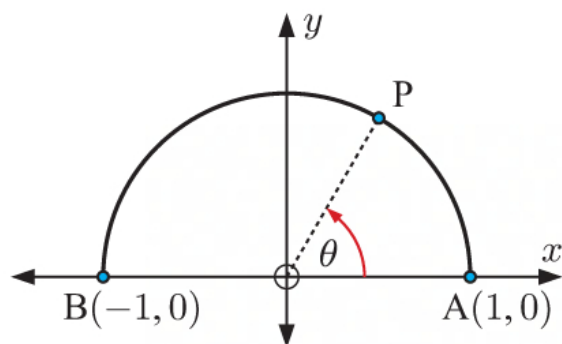
$$\text{But } -\mathbf{a} - \mathbf{d} = \mathbf{b} + \mathbf{c} = \overrightarrow{BD}$$

$$\therefore \overrightarrow{MQ} = \overrightarrow{NP} \quad \dots (2)$$

From (1) and (2), MNPQ is a parallelogram.



6



a  $P(\cos \theta, \sin \theta)$

b 
$$\overrightarrow{BP} = \begin{pmatrix} \cos \theta - (-1) \\ \sin \theta - 0 \end{pmatrix} = \begin{pmatrix} \cos \theta + 1 \\ \sin \theta \end{pmatrix}$$

$$\overrightarrow{AP} = \begin{pmatrix} \cos \theta - 1 \\ \sin \theta - 0 \end{pmatrix} = \begin{pmatrix} \cos \theta - 1 \\ \sin \theta \end{pmatrix}$$

c 
$$\begin{aligned} \overrightarrow{AP} \cdot \overrightarrow{BP} &= \begin{pmatrix} \cos \theta - 1 \\ \sin \theta \end{pmatrix} \cdot \begin{pmatrix} \cos \theta + 1 \\ \sin \theta \end{pmatrix} \\ &= (\cos \theta - 1)(\cos \theta + 1) + (\sin \theta)(\sin \theta) \\ &= \cos^2 \theta - 1 + \sin^2 \theta \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

d The angle in a semi-circle is a right angle.

7 a

If  $|\mathbf{a} + \mathbf{b}| = |\mathbf{a} - \mathbf{b}|$

then  $|\mathbf{a} + \mathbf{b}|^2 = |\mathbf{a} - \mathbf{b}|^2$

$\therefore (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) = (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b})$

$\therefore \cancel{\mathbf{a} \cdot \mathbf{a}} + \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{a} + \cancel{\mathbf{b} \cdot \mathbf{b}} = \cancel{\mathbf{a} \cdot \mathbf{a}} - \mathbf{a} \cdot \mathbf{b} - \mathbf{b} \cdot \mathbf{a} + \cancel{\mathbf{b} \cdot \mathbf{b}}$

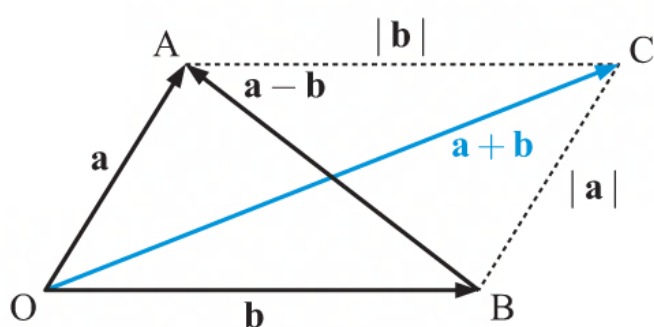
$\therefore \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{b} = 0$

$\therefore 4\mathbf{a} \cdot \mathbf{b} = 0$

$\therefore \mathbf{a} \cdot \mathbf{b} = 0$

$\therefore \mathbf{a}$  and  $\mathbf{b}$  are perpendicular.

b

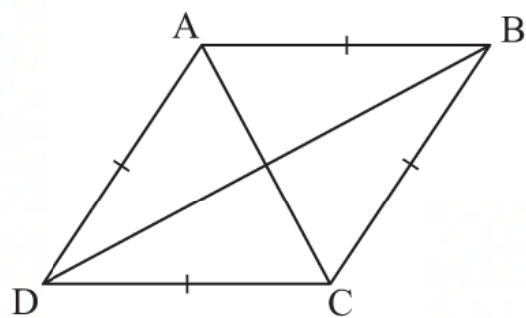


As shown,  $\overrightarrow{OC} = \mathbf{a} + \mathbf{b}$  and  $\overrightarrow{BA} = \mathbf{a} - \mathbf{b}$  are the diagonals of a parallelogram with side lengths  $|\mathbf{a}|$  and  $|\mathbf{b}|$ .

Now if  $|\mathbf{a} + \mathbf{b}| = |\mathbf{a} - \mathbf{b}|$  then the parallelogram must be a rectangle, and  $\mathbf{a}$  is perpendicular to  $\mathbf{b}$ .



8 a



$\triangle ACD$  and  $\triangle ACB$  are congruent {SSS}

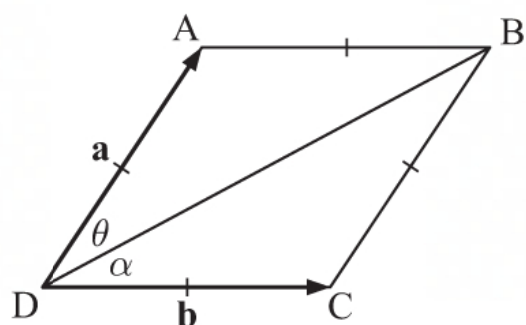
$\therefore \widehat{ACD} = \widehat{ACB}$  and  $\widehat{CAD} = \widehat{CAB}$ .

Also,  $\triangle DAB$  and  $\triangle DCB$  are congruent {SSS}

$\therefore \widehat{DBA} = \widehat{DBC}$  and  $\widehat{BDA} = \widehat{BDC}$

$\therefore$  the diagonals of a rhombus bisect the angles of the rhombus.

b



Let  $\overrightarrow{DA} = \mathbf{a}$ ,  $\overrightarrow{DC} = \mathbf{b}$ ,  $\widehat{ADB} = \theta$ , and  $\widehat{BDC} = \alpha$ .

Now  $\overrightarrow{DB} = \mathbf{a} + \mathbf{b}$

$$\therefore \cos \theta = \frac{\mathbf{a} \bullet (\mathbf{a} + \mathbf{b})}{|\mathbf{a}| |\mathbf{a} + \mathbf{b}|} = \frac{\mathbf{a} \bullet \mathbf{a} + \mathbf{a} \bullet \mathbf{b}}{|\mathbf{a}| |\mathbf{a} + \mathbf{b}|} = \frac{|\mathbf{a}|^2 + \mathbf{a} \bullet \mathbf{b}}{|\mathbf{a}| |\mathbf{a} + \mathbf{b}|}$$

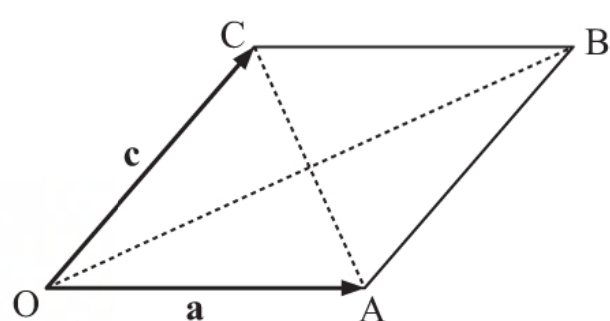
$$\text{and } \cos \alpha = \frac{\mathbf{b} \bullet (\mathbf{a} + \mathbf{b})}{|\mathbf{b}| |\mathbf{a} + \mathbf{b}|} = \frac{\mathbf{b} \bullet \mathbf{a} + \mathbf{b} \bullet \mathbf{b}}{|\mathbf{b}| |\mathbf{a} + \mathbf{b}|} = \frac{|\mathbf{b}|^2 + \mathbf{a} \bullet \mathbf{b}}{|\mathbf{b}| |\mathbf{a} + \mathbf{b}|}$$

But  $|\mathbf{a}| = |\mathbf{b}|$  since ABCD is a rhombus  $\therefore \cos \theta = \cos \alpha$

$$\therefore \theta = \alpha \quad \{0^\circ \leq \theta, \alpha \leq 180^\circ\}$$

$\therefore$  the diagonals of a rhombus bisect the angles of the rhombus.

9



Let  $\overrightarrow{OA} = \mathbf{a}$  and  $\overrightarrow{OC} = \mathbf{c}$ .

Now,  $\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \mathbf{c} - \mathbf{a}$

and  $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{OC} = \mathbf{a} + \mathbf{c}$ .

( $\Rightarrow$ ) Suppose the diagonals of parallelogram OACB meet at right angles.

$\therefore$  [OB] and [AC] are perpendicular.

$$\therefore \overrightarrow{OB} \bullet \overrightarrow{AC} = 0$$

$$\therefore (\mathbf{a} + \mathbf{c}) \bullet (\mathbf{c} - \mathbf{a}) = 0$$

$$\therefore \mathbf{a} \bullet \mathbf{c} - \mathbf{a} \bullet \mathbf{a} + \mathbf{c} \bullet \mathbf{c} - \mathbf{c} \bullet \mathbf{a} = 0$$

$$\therefore |\mathbf{c}|^2 - |\mathbf{a}|^2 = 0$$

This is satisfied if  $|\mathbf{c}| = |\mathbf{a}|$ .

$\therefore$  a pair of adjacent sides are equal in length.

$\therefore$  OACB is a rhombus.

( $\Leftarrow$ ) Suppose OACB is a rhombus.

$$\therefore |\mathbf{a}| = |\mathbf{c}|$$

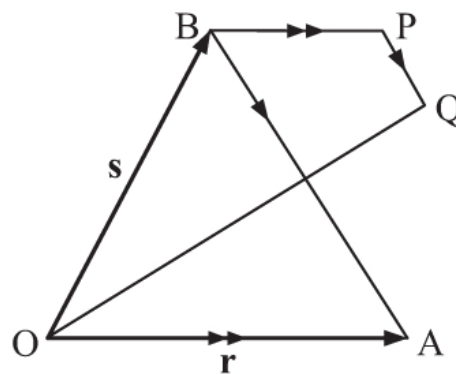
$$\therefore |\mathbf{a}|^2 = |\mathbf{c}|^2$$

$$\therefore \mathbf{a} \bullet \mathbf{a} = \mathbf{c} \bullet \mathbf{c} \quad \dots (1)$$

$$\begin{aligned} \text{Now } \overrightarrow{AC} \bullet \overrightarrow{OB} &= (\mathbf{c} - \mathbf{a}) \bullet (\mathbf{c} + \mathbf{a}) \\ &= \mathbf{c} \bullet \mathbf{c} + \mathbf{c} \bullet \mathbf{a} - \mathbf{a} \bullet \mathbf{c} - \mathbf{a} \bullet \mathbf{a} \\ &= \mathbf{c} \bullet \mathbf{c} - \mathbf{a} \bullet \mathbf{a} \\ &= 0 \quad \{\text{by (1)}\} \end{aligned}$$

$\therefore$  the diagonals of OACB meet at right angles.

$$\begin{aligned}
 \text{10 a } \overrightarrow{AB} &= -\overrightarrow{OA} + \overrightarrow{OB} = \mathbf{s} - \mathbf{r} = -\mathbf{r} + \mathbf{s} \\
 \overrightarrow{OQ} &= \overrightarrow{OB} + \overrightarrow{BP} + \overrightarrow{PQ} \\
 &= \mathbf{s} + \frac{1}{2}\overrightarrow{OA} + \frac{1}{4}\overrightarrow{BA} \\
 &= \mathbf{s} + \frac{1}{2}\mathbf{r} + \frac{1}{4}(\mathbf{r} - \mathbf{s}) \\
 &= \frac{3}{4}\mathbf{s} + \frac{3}{4}\mathbf{r} \\
 &= \frac{3}{4}\mathbf{r} + \frac{3}{4}\mathbf{s}
 \end{aligned}$$



$$\begin{aligned}
 \text{b } \overrightarrow{AB} \bullet \overrightarrow{OQ} &= (-\mathbf{r} + \mathbf{s}) \bullet \left(\frac{3}{4}\mathbf{r} + \frac{3}{4}\mathbf{s}\right) \\
 &= -\mathbf{r} \bullet \frac{3}{4}\mathbf{r} - \frac{3}{4}\mathbf{s} \bullet \mathbf{r} + \mathbf{s} \bullet \frac{3}{4}\mathbf{r} + \mathbf{s} \bullet \frac{3}{4}\mathbf{s} \\
 &= -\frac{3}{4}|\mathbf{r}|^2 + \frac{3}{4}|\mathbf{s}|^2 \\
 &= 0 \quad \{|\mathbf{r}| = |\mathbf{s}|\}
 \end{aligned}$$

$\therefore$  [AB] and [OQ] are perpendicular.

11 Suppose  $k_1 \neq l_1$ .

Now  $k_1\mathbf{a} + k_2\mathbf{b} = l_1\mathbf{a} + l_2\mathbf{b}$ , where  $\mathbf{a}$  and  $\mathbf{b}$  are non-parallel

$$\therefore k_1\mathbf{a} - l_1\mathbf{a} = l_2\mathbf{b} - k_2\mathbf{b}$$

$$\therefore (k_1 - l_1)\mathbf{a} = (l_2 - k_2)\mathbf{b}$$

$$\therefore \mathbf{a} = \left(\frac{l_2 - k_2}{k_1 - l_1}\right)\mathbf{b} \quad \{k_1 \neq l_1\}$$

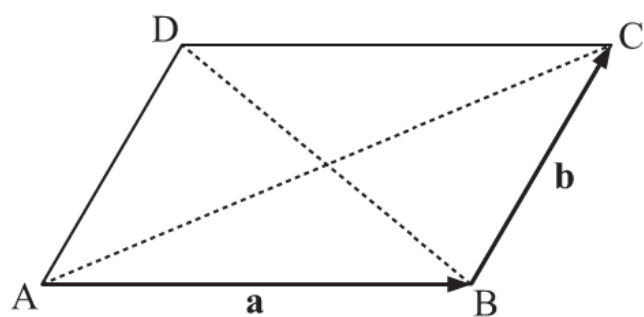
$$\therefore \mathbf{a} = r\mathbf{b} \quad \text{for some scalar } r$$

This implies that  $\mathbf{a}$  is parallel to  $\mathbf{b}$ , which is a contradiction.

So, our original assumption is false.

$$\therefore k_1 = l_1 \quad \text{and hence} \quad k_2 = l_2$$

12



Let  $\overrightarrow{AB} = \mathbf{a}$  and  $\overrightarrow{BC} = \mathbf{b}$ .

Now,  $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = \mathbf{a} + \mathbf{b}$

and  $\overrightarrow{BD} = \overrightarrow{AD} - \overrightarrow{AB} = \mathbf{b} - \mathbf{a}$ .

$$\begin{aligned}
 \text{So, } |\overrightarrow{AC}|^2 &= \overrightarrow{AC} \bullet \overrightarrow{AC} \\
 &= (\mathbf{a} + \mathbf{b}) \bullet (\mathbf{a} + \mathbf{b}) \\
 &= |\mathbf{a}|^2 + 2\mathbf{a} \bullet \mathbf{b} + |\mathbf{b}|^2
 \end{aligned}$$

$$\begin{aligned}
 \text{and } |\overrightarrow{BD}|^2 &= \overrightarrow{BD} \bullet \overrightarrow{BD} \\
 &= (\mathbf{b} - \mathbf{a}) \bullet (\mathbf{b} - \mathbf{a}) \\
 &= |\mathbf{b}|^2 - 2\mathbf{a} \bullet \mathbf{b} + |\mathbf{a}|^2
 \end{aligned}$$

$$\therefore |\overrightarrow{AC}|^2 + |\overrightarrow{BD}|^2 = 2|\mathbf{a}|^2 + 2|\mathbf{b}|^2$$

$$\begin{aligned}
 \text{And the sum of the squares of the lengths of the sides} &= |\mathbf{a}|^2 + |\mathbf{b}|^2 + |\mathbf{a}|^2 + |\mathbf{b}|^2 \\
 &= 2|\mathbf{a}|^2 + 2|\mathbf{b}|^2 \\
 &= |\overrightarrow{AC}|^2 + |\overrightarrow{BD}|^2
 \end{aligned}$$

$\therefore$  the sum of the squares of the lengths of the diagonals of a parallelogram is equal to the sum of the squares of the lengths of its sides.

$$13 \quad \mathbf{a} \quad \overrightarrow{OX} = \overrightarrow{OA} + \frac{1}{2} \overrightarrow{AB} = \mathbf{a} + \frac{1}{2}(-\mathbf{a} + \mathbf{b}) = \frac{1}{2}(\mathbf{a} + \mathbf{b})$$

$$\overrightarrow{BZ} = -\overrightarrow{OB} + \frac{1}{2} \overrightarrow{OA} = \frac{1}{2}\mathbf{a} - \mathbf{b}$$

$$\mathbf{b} \quad \overrightarrow{OP} = k \overrightarrow{OX} = \overrightarrow{OB} + h \overrightarrow{BZ} \quad \text{for some } k, h \in \mathbb{R}$$

$$\therefore \frac{k}{2}(\mathbf{a} + \mathbf{b}) = \mathbf{b} + h(\frac{1}{2}\mathbf{a} - \mathbf{b})$$

$$\therefore \frac{k}{2}\mathbf{a} + \frac{k}{2}\mathbf{b} = (1-h)\mathbf{b} + \frac{h}{2}\mathbf{a}$$

$$\text{Equating the coefficients of } \mathbf{a} \text{ and } \mathbf{b}, \quad \frac{k}{2} = \frac{h}{2} \quad \text{and} \quad \frac{k}{2} = 1-h$$

$$\therefore \frac{h}{2} = 1-h$$

$$\therefore \frac{3h}{2} = 1$$

$$\therefore h = \frac{2}{3}$$

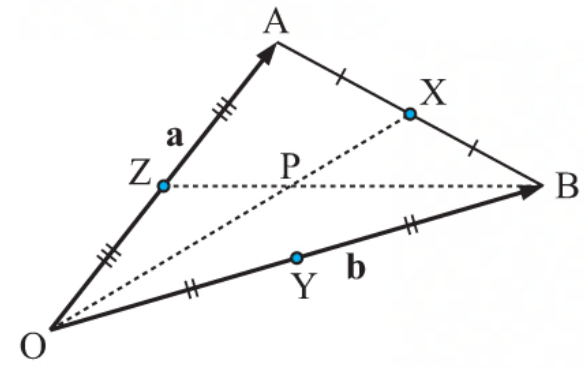
$$\begin{aligned} \therefore \overrightarrow{OP} &= \overrightarrow{OB} + \frac{2}{3} \overrightarrow{BZ} \\ &= \mathbf{b} + \frac{2}{3}(\frac{1}{2}\mathbf{a} - \mathbf{b}) \\ &= \frac{1}{3}\mathbf{a} + \frac{1}{3}\mathbf{b} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad \overrightarrow{YP} &= \overrightarrow{YO} + \overrightarrow{OP} & \text{and} \quad \overrightarrow{PA} &= -\overrightarrow{OP} + \overrightarrow{OA} \\ &= -\frac{1}{2}\overrightarrow{OB} + \overrightarrow{OP} & &= -\frac{1}{3}\mathbf{a} - \frac{1}{3}\mathbf{b} + \mathbf{a} \\ &= -\frac{1}{2}\mathbf{b} + \frac{1}{3}\mathbf{a} + \frac{1}{3}\mathbf{b} & &= \frac{2}{3}\mathbf{a} - \frac{1}{3}\mathbf{b} \\ &= \frac{1}{3}\mathbf{a} - \frac{1}{6}\mathbf{b} & &= 2(\frac{1}{3}\mathbf{a} - \frac{1}{6}\mathbf{b}) = 2\overrightarrow{YP} \end{aligned}$$

$$\mathbf{d} \quad \text{From } \mathbf{c}, \quad \overrightarrow{PA} = 2\overrightarrow{YP}$$

$\therefore$  A, P, and Y are collinear, and so the median [AY] also passes through P.

$\therefore$  the medians of the triangle meet at the common point P.



### INVESTIGATION 3

### THE VECTOR CROSS PRODUCT FORMULA

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}$$

$$= (a_2b_3 - a_3b_2) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - (a_1b_3 - a_3b_1) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + (a_1b_2 - a_2b_1) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} a_2b_3 - a_3b_2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ a_3b_1 - a_1b_3 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ a_1b_2 - a_2b_1 \end{pmatrix}$$

$$= \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}$$

$$= \mathbf{a} \times \mathbf{b} \quad \checkmark$$

**EXERCISE 12N.1**

$$\begin{aligned}
 \mathbf{1} \quad \mathbf{a} \quad \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -3 & 1 \\ 1 & 4 & -2 \end{vmatrix} \\
 &= \begin{vmatrix} -3 & 1 \\ 4 & -2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & -3 \\ 1 & 4 \end{vmatrix} \mathbf{k} \\
 &= (6 - 4)\mathbf{i} - (-4 - 1)\mathbf{j} + (8 + 3)\mathbf{k} \\
 &= 2\mathbf{i} - (-5)\mathbf{j} + 11\mathbf{k} \\
 &= \begin{pmatrix} 2 \\ 5 \\ 11 \end{pmatrix}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} \times \begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 0 & 2 \\ 3 & -1 & -2 \end{vmatrix} \\
 &= \begin{vmatrix} 0 & 2 \\ -1 & -2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -1 & 2 \\ 3 & -2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -1 & 0 \\ 3 & -1 \end{vmatrix} \mathbf{k} \\
 &= (0 + 2)\mathbf{i} - (2 - 6)\mathbf{j} + (1 - 0)\mathbf{k} \\
 &= 2\mathbf{i} - (-4)\mathbf{j} + \mathbf{k} \\
 &= \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} \times \begin{pmatrix} -3 \\ 1 \\ -5 \end{pmatrix} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -1 & 2 \\ -3 & 1 & -5 \end{vmatrix} \\
 &= \begin{vmatrix} -1 & 2 \\ 1 & -5 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 4 & 2 \\ -3 & -5 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 4 & -1 \\ -3 & 1 \end{vmatrix} \mathbf{k} \\
 &= (5 - 2)\mathbf{i} - (-20 + 6)\mathbf{j} + (4 - 3)\mathbf{k} \\
 &= 3\mathbf{i} - (-14)\mathbf{j} + \mathbf{k} \\
 &= \begin{pmatrix} 3 \\ 14 \\ 1 \end{pmatrix}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad (\mathbf{i} + \mathbf{j} - 2\mathbf{k}) \times (\mathbf{i} - \mathbf{k}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -2 \\ 1 & 0 & -1 \end{vmatrix} \\
 &= \begin{vmatrix} 1 & -2 \\ 0 & -1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & -2 \\ 1 & -1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} \mathbf{k} \\
 &= (-1 - 0)\mathbf{i} - (-1 + 2)\mathbf{j} + (0 - 1)\mathbf{k} \\
 &= -\mathbf{i} - \mathbf{j} - \mathbf{k}
 \end{aligned}$$



$$\begin{aligned}
 \text{e } (2\mathbf{i} - \mathbf{k}) \times (\mathbf{j} + 3\mathbf{k}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -1 \\ 0 & 1 & 3 \end{vmatrix} \\
 &= \begin{vmatrix} 0 & -1 \\ 1 & 3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & -1 \\ 0 & 3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} \mathbf{k} \\
 &= (0 + 1)\mathbf{i} - (6 - 0)\mathbf{j} + (2 - 0)\mathbf{k} \\
 &= \mathbf{i} - 6\mathbf{j} + 2\mathbf{k}
 \end{aligned}$$

$$\begin{aligned}
 \text{f } (\mathbf{i} - \mathbf{j} + 3\mathbf{k}) \times (-2\mathbf{i} + 3\mathbf{j} - \mathbf{k}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 3 \\ -2 & 3 & -1 \end{vmatrix} \\
 &= \begin{vmatrix} -1 & 3 \\ 3 & -1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 3 \\ -2 & -1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & -1 \\ -2 & 3 \end{vmatrix} \mathbf{k} \\
 &= (1 - 9)\mathbf{i} - (-1 + 6)\mathbf{j} + (3 - 2)\mathbf{k} \\
 &= -8\mathbf{i} - 5\mathbf{j} + \mathbf{k}
 \end{aligned}$$

$$\begin{aligned}
 \text{2 a } \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \times \begin{pmatrix} -1 \\ 3 \\ -1 \end{pmatrix} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ -1 & 3 & -1 \end{vmatrix} \\
 &= \begin{vmatrix} 2 & 3 \\ 3 & -1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 3 \\ -1 & -1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix} \mathbf{k} \\
 &= (-2 - 9)\mathbf{i} - (-1 + 3)\mathbf{j} + (3 + 2)\mathbf{k} \\
 &= -11\mathbf{i} - 2\mathbf{j} + 5\mathbf{k} \\
 &= \begin{pmatrix} -11 \\ -2 \\ 5 \end{pmatrix}
 \end{aligned}$$

$$\begin{aligned}
 \text{b } \mathbf{a} \bullet (\mathbf{a} \times \mathbf{b}) &= \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} -11 \\ -2 \\ 5 \end{pmatrix} & \mathbf{b} \bullet (\mathbf{a} \times \mathbf{b}) &= \begin{pmatrix} -1 \\ 3 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} -11 \\ -2 \\ 5 \end{pmatrix} \\
 &= -11 - 4 + 15 & &= 11 - 6 - 5 \\
 &= 0 & &= 0
 \end{aligned}$$

$$\therefore \mathbf{a} \bullet (\mathbf{a} \times \mathbf{b}) = 0 = \mathbf{b} \bullet (\mathbf{a} \times \mathbf{b})$$

**c**  $\mathbf{a} \times \mathbf{b}$  is a vector perpendicular to both  $\mathbf{a}$  and  $\mathbf{b}$ .

$$\begin{aligned}
 \text{3 a } \mathbf{i} \times \mathbf{i} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{vmatrix} \\
 &= \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} \mathbf{k} \\
 &= (0 \times 0 - 0 \times 0)\mathbf{i} - (1 \times 0 - 0 \times 1)\mathbf{j} + (1 \times 0 - 0 \times 1)\mathbf{k} \\
 &= \mathbf{0}
 \end{aligned}$$

$$\begin{aligned}
\mathbf{j} \times \mathbf{j} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{vmatrix} \\
&= \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix} \mathbf{k} \\
&= (1 \times 0 - 0 \times 1) \mathbf{i} - (0 \times 0 - 0 \times 0) \mathbf{j} + (0 \times 1 - 1 \times 0) \mathbf{k} \\
&= \mathbf{0}
\end{aligned}$$

$$\begin{aligned}
\mathbf{k} \times \mathbf{k} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{vmatrix} \\
&= \begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} \mathbf{k} \\
&= (0 \times 1 - 1 \times 0) \mathbf{i} - (0 \times 1 - 1 \times 0) \mathbf{j} + (0 \times 0 - 0 \times 0) \mathbf{k} \\
&= \mathbf{0}
\end{aligned}$$

In each case, we observe that  $\mathbf{a} \times \mathbf{a} = \mathbf{0}$ .

$$\begin{aligned}
\text{b } \mathbf{i} \times \mathbf{j} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} \\
&= \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \mathbf{k} \\
&= (0 \times 0 - 0 \times 1) \mathbf{i} - (1 \times 0 - 0 \times 0) \mathbf{j} + (1 \times 1 - 0 \times 0) \mathbf{k} \\
&= \mathbf{k}
\end{aligned}$$

$$\begin{aligned}
\mathbf{j} \times \mathbf{i} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} \\
&= \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \mathbf{k} \\
&= (1 \times 0 - 0 \times 0) \mathbf{i} - (0 \times 0 - 0 \times 1) \mathbf{j} + (0 \times 0 - 1 \times 1) \mathbf{k} \\
&= -\mathbf{k}
\end{aligned}$$

In each case, we observe that  $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$ .

$$\begin{aligned}
\text{c } \mathbf{i} \times \mathbf{k} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \\
&= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} \mathbf{k} \\
&= (1 \times 1 - 0 \times 0) \mathbf{i} - (0 \times 1 - 0 \times 0) \mathbf{j} + (0 \times 0 - 1 \times 0) \mathbf{k} \\
&= \mathbf{i}
\end{aligned}$$

$$\begin{aligned}
\mathbf{k} \times \mathbf{j} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} \\
&= \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} \mathbf{k} \\
&= (0 \times 0 - 1 \times 1) \mathbf{i} - (0 \times 0 - 1 \times 0) \mathbf{j} + (0 \times 1 - 0 \times 0) \mathbf{k} \\
&= -\mathbf{i}
\end{aligned}$$

$$\begin{aligned}
\text{ii } \mathbf{i} \times \mathbf{k} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} \\
&= \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} \mathbf{k} \\
&= (0 \times 1 - 0 \times 0) \mathbf{i} - (1 \times 1 - 0 \times 0) \mathbf{j} + (1 \times 0 - 0 \times 0) \mathbf{k} \\
&= -\mathbf{j}
\end{aligned}$$

$$\begin{aligned}
\mathbf{k} \times \mathbf{i} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} \\
&= \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} \mathbf{k} \\
&= (0 \times 0 - 1 \times 0) \mathbf{i} - (0 \times 0 - 1 \times 1) \mathbf{j} + (0 \times 0 - 0 \times 1) \mathbf{k} \\
&= \mathbf{j}
\end{aligned}$$

$$\begin{aligned}
\text{4 a } \mathbf{a} \times \mathbf{a} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \end{vmatrix} \\
&= \begin{vmatrix} a_2 & a_3 \\ a_2 & a_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ a_1 & a_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ a_1 & a_2 \end{vmatrix} \mathbf{k} \\
&= (a_2 a_3 - a_2 a_3) \mathbf{i} - (a_1 a_3 - a_1 a_3) \mathbf{j} + (a_1 a_2 - a_1 a_2) \mathbf{k} \\
&= \mathbf{0}
\end{aligned}$$

Hence  $\mathbf{a} \times \mathbf{a} = \mathbf{0}$  for all 3-dimensional vectors  $\mathbf{a}$ .

$$\begin{aligned}
\text{b } \mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\
&= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k} \\
&= (a_2 b_3 - a_3 b_2) \mathbf{i} - (a_1 b_3 - a_3 b_1) \mathbf{j} + (a_1 b_2 - a_2 b_1) \mathbf{k}
\end{aligned}$$

$$\begin{aligned}
-(\mathbf{b} \times \mathbf{a}) &= - \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix} \\
&= - \left[ \begin{vmatrix} b_2 & b_3 \\ a_2 & a_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} b_1 & b_3 \\ a_1 & a_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} b_1 & b_2 \\ a_1 & a_2 \end{vmatrix} \mathbf{k} \right] \\
&= - [(b_2 a_3 - b_3 a_2) \mathbf{i} - (b_1 a_3 - b_3 a_1) \mathbf{j} + (b_1 a_2 - b_2 a_1) \mathbf{k}] \\
&= (a_2 b_3 - a_3 b_2) \mathbf{i} - (a_1 b_3 - a_3 b_1) \mathbf{j} + (a_1 b_2 - a_2 b_1) \mathbf{k} \\
&= \mathbf{a} \times \mathbf{b}
\end{aligned}$$

$$\begin{aligned}
\text{5 a } \mathbf{b} \times \mathbf{c} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 1 \\ 0 & 1 & -2 \end{vmatrix} \\
&= \begin{vmatrix} -1 & 1 \\ 1 & -2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 1 \\ 0 & -2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} \mathbf{k} \\
&= (2 - 1) \mathbf{i} - (-4) \mathbf{j} + 2 \mathbf{k} \\
&= \mathbf{i} + 4 \mathbf{j} + 2 \mathbf{k} \\
&= \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\text{b } \mathbf{a} \bullet (\mathbf{b} \times \mathbf{c}) &= \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} \\
&= 1 + 12 + 4 \\
&= 17
\end{aligned}$$

$$\begin{aligned}
\text{6 a } \mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 2 \\ 0 & -1 & 1 \end{vmatrix} \\
&= \begin{vmatrix} 0 & 2 \\ -1 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} \mathbf{k} \\
&= 2 \mathbf{i} - \mathbf{j} - \mathbf{k}
\end{aligned}$$

$$\begin{aligned}
\text{b } \mathbf{a} \times \mathbf{c} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 2 \\ 2 & 0 & -1 \end{vmatrix} \\
&= \begin{vmatrix} 0 & 2 \\ 0 & -1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 0 \\ 2 & 0 \end{vmatrix} \mathbf{k} \\
&= -(-1 - 4) \mathbf{j} \\
&= 5 \mathbf{j}
\end{aligned}$$

$$\begin{aligned}
\text{c } (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c}) &= (2 \mathbf{i} - \mathbf{j} - \mathbf{k}) + 5 \mathbf{j} \\
&= 2 \mathbf{i} + 4 \mathbf{j} - \mathbf{k}
\end{aligned}$$

$$\begin{aligned}
\text{d } (\mathbf{b} + \mathbf{c}) &= (-\mathbf{j} + \mathbf{k}) + (2 \mathbf{i} - \mathbf{k}) \\
&= 2 \mathbf{i} - \mathbf{j}
\end{aligned}$$

$$\begin{aligned}
\therefore \mathbf{a} \times (\mathbf{b} + \mathbf{c}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 2 \\ 2 & -1 & 0 \end{vmatrix} \\
&= \begin{vmatrix} 0 & 2 \\ -1 & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} \mathbf{k} \\
&= 2 \mathbf{i} + 4 \mathbf{j} - \mathbf{k}
\end{aligned}$$



$$\begin{aligned}
7 \quad \mathbf{a} \quad \mathbf{a} \times (\mathbf{b} + \mathbf{c}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 + c_1 & b_2 + c_2 & b_3 + c_3 \end{vmatrix} \\
&= \begin{vmatrix} a_2 & a_3 \\ b_2 + c_2 & b_3 + c_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 + c_1 & b_3 + c_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 + c_1 & b_2 + c_2 \end{vmatrix} \mathbf{k} \\
&= (a_2(b_3 + c_3) - a_3(b_2 + c_2))\mathbf{i} - (a_1(b_3 + c_3) - a_3(b_1 + c_1))\mathbf{j} \\
&\quad + (a_1(b_2 + c_2) - a_2(b_1 + c_1))\mathbf{k} \\
&= (a_2b_3 + a_2c_3 - a_3b_2 - a_3c_2)\mathbf{i} - (a_1b_3 + a_1c_3 - a_3b_1 - a_3c_1)\mathbf{j} \\
&\quad + (a_1b_2 + a_1c_2 - a_2b_1 - a_2c_1)\mathbf{k}
\end{aligned}$$

$$\begin{aligned}
&\mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c} \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \\
&= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k} + \begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix} \mathbf{k} \\
&= (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k} \\
&\quad + (a_2c_3 - a_3c_2)\mathbf{i} - (a_1c_3 - a_3c_1)\mathbf{j} + (a_1c_2 - a_2c_1)\mathbf{k} \\
&= (a_2b_3 + a_2c_3 - a_3b_2 - a_3c_2)\mathbf{i} - (a_1b_3 + a_1c_3 - a_3b_1 - a_3c_1)\mathbf{j} \\
&\quad + (a_1b_2 + a_1c_2 - a_2b_1 - a_2c_1)\mathbf{k} \\
&= \mathbf{a} \times (\mathbf{b} + \mathbf{c})
\end{aligned}$$

$$\begin{aligned}
\mathbf{b} \quad (\mathbf{a} + \mathbf{b}) \times (\mathbf{c} + \mathbf{d}) &= ((\mathbf{a} + \mathbf{b}) \times \mathbf{c}) + ((\mathbf{a} + \mathbf{b}) \times \mathbf{d}) \quad \{\text{using } \mathbf{a}\} \\
&= (-(\mathbf{c} \times (\mathbf{a} + \mathbf{b})) - (\mathbf{d} \times (\mathbf{a} + \mathbf{b}))) \quad \{\text{since } \mathbf{p} \times \mathbf{q} = -\mathbf{q} \times \mathbf{p}\} \\
&= (-(\mathbf{c} \times \mathbf{a}) + (\mathbf{c} \times \mathbf{b})) - ((\mathbf{d} \times \mathbf{a}) + (\mathbf{d} \times \mathbf{b})) \\
&= ((\mathbf{a} \times \mathbf{c}) + (\mathbf{b} \times \mathbf{c})) + ((\mathbf{a} \times \mathbf{d}) + (\mathbf{b} \times \mathbf{d})) \\
&= (\mathbf{a} \times \mathbf{c}) + (\mathbf{a} \times \mathbf{d}) + (\mathbf{b} \times \mathbf{c}) + (\mathbf{b} \times \mathbf{d})
\end{aligned}$$

$$\begin{aligned}
8 \quad \mathbf{a} \quad \mathbf{a} \times (\mathbf{a} + \mathbf{b}) &= (\mathbf{a} \times \mathbf{a}) + (\mathbf{a} \times \mathbf{b}) \\
&= \mathbf{a} \times \mathbf{b} \quad \{\text{since } \mathbf{p} \times \mathbf{p} = \mathbf{0}\}
\end{aligned}$$

$$\begin{aligned}
\mathbf{b} \quad (\mathbf{a} + \mathbf{b}) \times (\mathbf{a} + \mathbf{b}) &= (\mathbf{a} \times \mathbf{a}) + (\mathbf{a} \times \mathbf{b}) + (\mathbf{b} \times \mathbf{a}) + (\mathbf{b} \times \mathbf{b}) \quad \{\text{using } \mathbf{7} \mathbf{b}\} \\
&= (\mathbf{a} \times \mathbf{b}) + (\mathbf{b} \times \mathbf{a}) \quad \{\text{since } \mathbf{p} \times \mathbf{p} = \mathbf{0}\} \\
&= (\mathbf{a} \times \mathbf{b}) - (\mathbf{a} \times \mathbf{b}) \quad \{\text{since } \mathbf{p} \times \mathbf{q} = -\mathbf{q} \times \mathbf{p}\} \\
&= \mathbf{0}
\end{aligned}$$

$$\begin{aligned}
\mathbf{c} \quad (\mathbf{a} + \mathbf{b}) \times (\mathbf{a} - \mathbf{b}) &= (\mathbf{a} \times \mathbf{a}) + (\mathbf{a} \times (-\mathbf{b})) + (\mathbf{b} \times \mathbf{a}) + (\mathbf{b} \times (-\mathbf{b})) \quad \{\text{using } \mathbf{7} \mathbf{b}\} \\
&= (\mathbf{a} \times (-\mathbf{b})) + (\mathbf{b} \times \mathbf{a}) + (\mathbf{b} \times (-\mathbf{b})) \quad \{\text{since } \mathbf{p} \times \mathbf{p} = \mathbf{0}\} \\
&= -(\mathbf{b} \times \mathbf{a}) + (\mathbf{b} \times \mathbf{a}) - (-\mathbf{b} \times \mathbf{b}) \quad \{\text{since } \mathbf{p} \times \mathbf{q} = -\mathbf{q} \times \mathbf{p}\} \\
&= (\mathbf{b} \times \mathbf{a}) + (\mathbf{b} \times \mathbf{a}) + (\mathbf{b} \times \mathbf{b}) \\
&= 2(\mathbf{b} \times \mathbf{a})
\end{aligned}$$

$$\begin{aligned}
 \text{d } 2\mathbf{a} \bullet (\mathbf{a} \times \mathbf{b}) &= \begin{pmatrix} 2a_1 \\ 2a_2 \\ 2a_3 \end{pmatrix} \bullet \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix} \\
 &= 2a_1(a_2b_3 - a_3b_2) + 2a_2(a_3b_1 - a_1b_3) + 2a_3(a_1b_2 - a_2b_1) \\
 &= \cancel{2a_1a_2b_3} - \cancel{2a_1a_3b_2} + \cancel{2a_2a_3b_1} - \cancel{2a_1a_2b_3} + \cancel{2a_1a_3b_2} - \cancel{2a_2a_3b_1} \\
 &= 0
 \end{aligned}$$

$$\begin{aligned}
 \text{9 a } \mathbf{b} \times \mathbf{c} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \\
 &= \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \mathbf{k} \\
 &= (b_2c_3 - b_3c_2)\mathbf{i} - (b_1c_3 - b_3c_1)\mathbf{j} + (b_1c_2 - b_2c_1)\mathbf{k} \\
 \therefore \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_2c_3 - b_3c_2 & b_3c_1 - b_1c_3 & b_1c_2 - b_2c_1 \end{vmatrix} \\
 &= \begin{vmatrix} a_2 & a_3 \\ b_3c_1 - b_1c_3 & b_1c_2 - b_2c_1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_2c_3 - b_3c_2 & b_1c_2 - b_2c_1 \end{vmatrix} \mathbf{j} \\
 &\quad + \begin{vmatrix} a_1 & a_2 \\ b_2c_3 - b_3c_2 & b_3c_1 - b_1c_3 \end{vmatrix} \mathbf{k} \\
 &= (a_2(b_1c_2 - b_2c_1) - a_3(b_3c_1 - b_1c_3))\mathbf{i} \\
 &\quad - (a_1(b_1c_2 - b_2c_1) - a_3(b_2c_3 - b_3c_2))\mathbf{j} \\
 &\quad + (a_1(b_3c_1 - b_1c_3) - a_2(b_2c_3 - b_3c_2))\mathbf{k} \\
 &= (a_2b_1c_2 - a_2b_2c_1 - a_3b_3c_1 + a_3b_1c_3)\mathbf{i} \\
 &\quad - (a_1b_1c_2 - a_1b_2c_1 - a_3b_2c_3 + a_3b_3c_2)\mathbf{j} \\
 &\quad + (a_1b_3c_1 - a_1b_1c_3 - a_2b_2c_3 + a_2b_3c_2)\mathbf{k} \\
 &= \begin{pmatrix} a_2b_1c_2 + a_3b_1c_3 - a_2b_2c_1 - a_3b_3c_1 \\ a_1b_2c_1 + a_3b_2c_3 - a_1b_1c_2 - a_3b_3c_2 \\ a_1b_3c_1 + a_2b_3c_2 - a_1b_1c_3 - a_2b_2c_3 \end{pmatrix}
 \end{aligned}$$

Now  $(\mathbf{a} \bullet \mathbf{c})\mathbf{b} - (\mathbf{a} \bullet \mathbf{b})\mathbf{c}$

$$\begin{aligned}
 &= (a_1c_1 + a_2c_2 + a_3c_3) \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} - (a_1b_1 + a_2b_2 + a_3b_3) \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \\
 &= \begin{pmatrix} \cancel{a_1b_1c_1} + a_2b_1c_2 + a_3b_1c_3 \\ a_1b_2c_1 + \cancel{a_2b_2c_2} + a_3b_2c_3 \\ a_1b_3c_1 + a_2b_3c_2 + \cancel{a_3b_3c_3} \end{pmatrix} - \begin{pmatrix} \cancel{a_1b_1c_1} + a_2b_2c_1 + a_3b_3c_1 \\ a_1b_1c_2 + \cancel{a_2b_2c_2} + a_3b_3c_2 \\ a_1b_1c_3 + a_2b_2c_3 + \cancel{a_3b_3c_3} \end{pmatrix} \\
 &= \begin{pmatrix} a_2b_1c_2 + a_3b_1c_3 - a_2b_2c_1 - a_3b_3c_1 \\ a_1b_2c_1 + a_3b_2c_3 - a_1b_1c_2 - a_3b_3c_2 \\ a_1b_3c_1 + a_2b_3c_2 - a_1b_1c_3 - a_2b_2c_3 \end{pmatrix} \\
 &= \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \quad \checkmark
 \end{aligned}$$

$$\begin{aligned}
\mathbf{b} \quad & \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) \\
&= (\mathbf{a} \bullet \mathbf{c})\mathbf{b} - (\mathbf{a} \bullet \mathbf{b})\mathbf{c} + (\mathbf{b} \bullet \mathbf{a})\mathbf{c} - (\mathbf{b} \bullet \mathbf{c})\mathbf{a} + (\mathbf{c} \bullet \mathbf{b})\mathbf{a} - (\mathbf{c} \bullet \mathbf{a})\mathbf{b} \quad \{\text{generalising the result in } \mathbf{a}\} \\
&= (\mathbf{a} \bullet \mathbf{c})\mathbf{b} - (\mathbf{a} \bullet \mathbf{b})\mathbf{c} + (\mathbf{a} \bullet \mathbf{b})\mathbf{c} - (\mathbf{b} \bullet \mathbf{c})\mathbf{a} + (\mathbf{b} \bullet \mathbf{c})\mathbf{a} - (\mathbf{a} \bullet \mathbf{c})\mathbf{b} \quad \{\text{since } \mathbf{p} \bullet \mathbf{q} = \mathbf{q} \bullet \mathbf{p}\} \\
&= \mathbf{0}
\end{aligned}$$

$$\begin{aligned}
10 \quad \mathbf{a} \quad & \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 3 \\ 1 & 1 & 1 \end{vmatrix} \\
&= \begin{vmatrix} -1 & 3 \\ 1 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} \mathbf{k} \\
&= (-1 - 3)\mathbf{i} - (2 - 3)\mathbf{j} + (2 + 1)\mathbf{k} \\
&= -4\mathbf{i} + \mathbf{j} + 3\mathbf{k}
\end{aligned}$$

The vectors have the form  $k \begin{pmatrix} -4 \\ 1 \\ 3 \end{pmatrix}$ ,  $k \in \mathbb{R}$ ,  $k \neq 0$ .

$$\begin{aligned}
\mathbf{b} \quad & \begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix} \times \begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 3 & 4 \\ 5 & 0 & 2 \end{vmatrix} \\
&= \begin{vmatrix} 3 & 4 \\ 0 & 2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -1 & 4 \\ 5 & 2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -1 & 3 \\ 5 & 0 \end{vmatrix} \mathbf{k} \\
&= 6\mathbf{i} - (-2 - 20)\mathbf{j} - 15\mathbf{k} \\
&= 6\mathbf{i} + 22\mathbf{j} - 15\mathbf{k}
\end{aligned}$$

The vectors have the form  $k \begin{pmatrix} 6 \\ 22 \\ -15 \end{pmatrix}$ ,  $k \in \mathbb{R}$ ,  $k \neq 0$ .

$$\begin{aligned}
\mathbf{c} \quad & (\mathbf{i} + \mathbf{j}) \times (\mathbf{i} - \mathbf{j} - \mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 1 & -1 & -1 \end{vmatrix} \\
&= \begin{vmatrix} 1 & 0 \\ -1 & -1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} \mathbf{k} \\
&= -\mathbf{i} + \mathbf{j} - 2\mathbf{k}
\end{aligned}$$

The vectors have the form  $(-\mathbf{i} + \mathbf{j} - 2\mathbf{k})n$ ,  $n \in \mathbb{R}$ ,  $n \neq 0$ .

$$\begin{aligned}
\mathbf{d} \quad & (\mathbf{i} - \mathbf{j} - \mathbf{k}) \times (2\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & -1 \\ 2 & 2 & -3 \end{vmatrix} \\
&= \begin{vmatrix} -1 & -1 \\ 2 & -3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & -1 \\ 2 & -3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & -1 \\ 2 & 2 \end{vmatrix} \mathbf{k} \\
&= (3 + 2)\mathbf{i} - (-3 + 2)\mathbf{j} + (2 + 2)\mathbf{k} \\
&= 5\mathbf{i} + \mathbf{j} + 4\mathbf{k}
\end{aligned}$$

The vectors have the form  $(5\mathbf{i} + \mathbf{j} + 4\mathbf{k})n$ ,  $n \in \mathbb{R}$ ,  $n \neq 0$ .

$$\begin{aligned}
 \mathbf{11} \quad \mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & -1 \\ 1 & -2 & 2 \end{vmatrix} \\
 &= \begin{vmatrix} 3 & -1 \\ -2 & 2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} \mathbf{k} \\
 &= (6 - 2)\mathbf{i} - (4 + 1)\mathbf{j} + (-4 - 3)\mathbf{k} \\
 &= 4\mathbf{i} - 5\mathbf{j} - 7\mathbf{k}
 \end{aligned}$$

$\therefore$  vectors perpendicular to  $\mathbf{a}$  and  $\mathbf{b}$  have the form  $k \begin{pmatrix} 4 \\ -5 \\ -7 \end{pmatrix}$ ,  $k \in \mathbb{R}$ ,  $k \neq 0$ .

$$\begin{aligned}
 \text{Now } |\mathbf{a} \times \mathbf{b}| &= \sqrt{4^2 + (-5)^2 + (-7)^2} \\
 &= \sqrt{90} \\
 &= 3\sqrt{10}
 \end{aligned}$$

$\therefore$  the two vectors of length 5 units which are perpendicular to both  $\mathbf{a}$  and  $\mathbf{b}$  are

$$\begin{aligned}
 \pm \frac{5}{3\sqrt{10}} \begin{pmatrix} 4 \\ -5 \\ -7 \end{pmatrix} &= \pm \frac{5\sqrt{10}}{30} \begin{pmatrix} 4 \\ -5 \\ -7 \end{pmatrix} \\
 &= \pm \frac{\sqrt{10}}{6} \begin{pmatrix} 4 \\ -5 \\ -7 \end{pmatrix}
 \end{aligned}$$

$$\mathbf{12} \quad \mathbf{a} \quad A(1, 3, 2), B(0, 2, -5), C(3, 1, -4)$$

$$\overrightarrow{AB} = \begin{pmatrix} 0-1 \\ 2-3 \\ -5-2 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ -7 \end{pmatrix} \quad \overrightarrow{AC} = \begin{pmatrix} 3-1 \\ 1-3 \\ -4-2 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ -6 \end{pmatrix}$$

$$\begin{aligned}
 \overrightarrow{AB} \times \overrightarrow{AC} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -1 & -7 \\ 2 & -2 & -6 \end{vmatrix} \\
 &= \begin{vmatrix} -1 & -7 \\ -2 & -6 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -1 & -7 \\ 2 & -6 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -1 & -1 \\ 2 & -2 \end{vmatrix} \mathbf{k} \\
 &= (6 - 14)\mathbf{i} - (6 + 14)\mathbf{j} + (2 + 2)\mathbf{k} \\
 &= -8\mathbf{i} - 20\mathbf{j} + 4\mathbf{k} \\
 &= -4(2\mathbf{i} + 5\mathbf{j} - \mathbf{k})
 \end{aligned}$$

$\therefore$  any non-zero multiple of  $\begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix}$  will be perpendicular to the plane.



- b**  $P(2, 0, -1), Q(0, 1, 3), R(1, -1, 1)$

$$\overrightarrow{PQ} = \begin{pmatrix} 0-2 \\ 1-0 \\ 3-(-1) \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} \quad \overrightarrow{PR} = \begin{pmatrix} 1-2 \\ -1-0 \\ 1-(-1) \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$$

$$\begin{aligned} \overrightarrow{PQ} \times \overrightarrow{PR} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 1 & 4 \\ -1 & -1 & 2 \end{vmatrix} \\ &= \begin{vmatrix} 1 & 4 \\ -1 & 2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -2 & 4 \\ -1 & 2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -2 & 1 \\ -1 & -1 \end{vmatrix} \mathbf{k} \\ &= (2+4)\mathbf{i} - (-4+4)\mathbf{j} + (2+1)\mathbf{k} \\ &= 6\mathbf{i} + 3\mathbf{k} \\ &= 3(2\mathbf{i} + \mathbf{k}) \end{aligned}$$

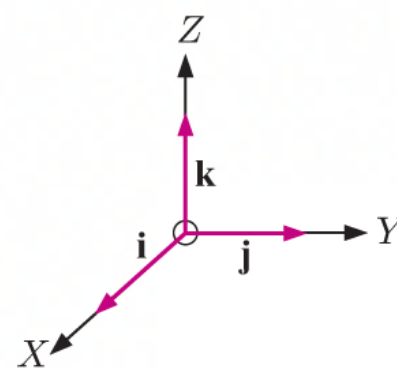
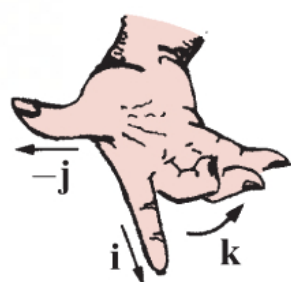
$\therefore$  any non-zero multiple of  $\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$  will be perpendicular to the plane.

## EXERCISE 12N.2

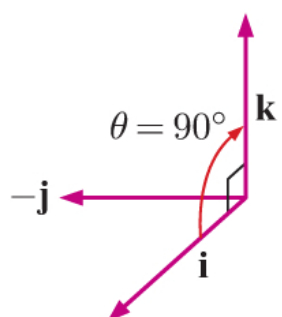
**1 a**  $\mathbf{i} \times \mathbf{k} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \times 1 - 0 \times 0 \\ 0 \times 0 - 1 \times 1 \\ 1 \times 0 - 0 \times 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} = -\mathbf{j}$

$$\mathbf{k} \times \mathbf{i} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \times 0 - 1 \times 0 \\ 1 \times 1 - 0 \times 0 \\ 0 \times 0 - 0 \times 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \mathbf{j}$$

Yes, the **right hand rule** does accurately give the direction.

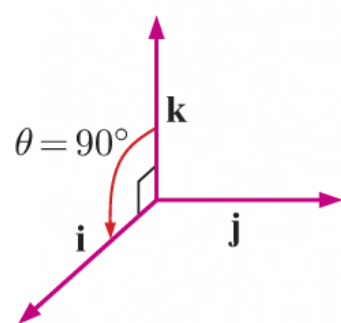


- b** If  $\mathbf{u}$  is the unit vector in the direction  $\mathbf{i} \times \mathbf{k}$ , then by the right hand rule,  $\mathbf{u} = -\mathbf{j}$ .



$$\begin{aligned} |\mathbf{i}| |\mathbf{k}| \sin \theta \mathbf{u} &= 1 \times 1 \times \sin 90^\circ \times (-\mathbf{j}) \\ &= -\mathbf{j} \end{aligned}$$

If  $\mathbf{u}$  is the unit vector in the direction  $\mathbf{k} \times \mathbf{i}$ , then by the right hand rule,  $\mathbf{u} = \mathbf{j}$ .



$$|\mathbf{k}| |\mathbf{i}| \sin \theta \mathbf{u} = 1 \times 1 \times \sin 90^\circ \times \mathbf{j} \\ = \mathbf{j}$$

$$\begin{aligned} \mathbf{a} \bullet \mathbf{b} &= \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \\ &= 2 + 0 - 3 \\ &= -1 \end{aligned}$$

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 3 \\ 1 & 0 & -1 \end{vmatrix} \\ &= \begin{vmatrix} -1 & 3 \\ 0 & -1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & -1 \\ 1 & 0 \end{vmatrix} \mathbf{k} \\ &= \mathbf{i} - (-2 - 3)\mathbf{j} + \mathbf{k} \\ &= \mathbf{i} + 5\mathbf{j} + \mathbf{k} \\ &= \begin{pmatrix} 1 \\ 5 \\ 1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} |\mathbf{a}| &= \sqrt{2^2 + (-1)^2 + 3^2} \\ &= \sqrt{4 + 1 + 9} \\ &= \sqrt{14} \end{aligned}$$

$$\begin{aligned} |\mathbf{b}| &= \sqrt{1^2 + 0^2 + (-1)^2} \\ &= \sqrt{1 + 0 + 1} \\ &= \sqrt{2} \end{aligned}$$

$$\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \theta$$

$$\begin{aligned} \therefore \cos \theta &= \frac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{a}| |\mathbf{b}|} \\ &= \frac{-1}{\sqrt{14}\sqrt{2}} \quad \{\mathbf{a} \bullet \mathbf{b} = -1 \text{ from a}\} \\ &= -\frac{1}{\sqrt{28}} \end{aligned}$$

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$\begin{aligned} \therefore \sin \theta &= \pm \sqrt{1 - \cos^2 \theta} \\ &= \pm \sqrt{1 - \left(\frac{1}{\sqrt{28}}\right)^2} \\ &= \pm \sqrt{\frac{27}{28}} \end{aligned}$$

But since  $\theta$  is the angle between two vectors,  $0^\circ \leq \theta \leq 180^\circ$ .

$$\therefore \sin \theta \geq 0$$

$$\therefore \sin \theta = \frac{\sqrt{27}}{\sqrt{28}}$$

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}| |\mathbf{b}| \sin \theta$$

$$\begin{aligned} \sin \theta &= \frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}| |\mathbf{b}|} \\ &= \frac{\left| \begin{pmatrix} 1 \\ 5 \\ 1 \end{pmatrix} \right|}{\sqrt{14}\sqrt{2}} \quad \{\text{from a and b}\} \\ &= \frac{\sqrt{1 + 25 + 1}}{\sqrt{28}} \\ &= \frac{\sqrt{27}}{\sqrt{28}} \end{aligned}$$

$$\mathbf{3} \quad (\Rightarrow) \quad \text{If } \mathbf{a} \times \mathbf{b} = \mathbf{0}, \text{ then } \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \mathbf{0}$$

$$\therefore \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k} = \mathbf{0}$$

$$\therefore (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k} = \mathbf{0}$$

$$\therefore \begin{cases} a_2b_3 - a_3b_2 = 0 & \dots (1) \\ a_1b_3 - a_3b_1 = 0 & \dots (2) \\ a_1b_2 - a_2b_1 = 0 & \dots (3) \end{cases}$$

If  $\mathbf{a}, \mathbf{b} \neq \mathbf{0}$  then at least one component of both  $\mathbf{a}$  and  $\mathbf{b}$  is non-zero.

Suppose  $a_1, b_1 \neq 0$

$$\therefore \begin{cases} a_3 = \frac{a_1}{b_1} b_3 & \{(2)\} \\ a_2 = \frac{a_1}{b_1} b_2 & \{(3)\} \\ a_1 = \frac{a_1}{b_1} b_1 \end{cases}$$

$$\therefore \mathbf{a} = \frac{a_1}{b_1} \mathbf{b}, \quad \frac{a_1}{b_1} \in \mathbb{R}, \quad \frac{a_1}{b_1} \neq 0$$

$\therefore \mathbf{a}$  and  $\mathbf{b}$  are parallel.

We can rearrange equations (1), (2), and (3) similarly for any pair of non-zero components of  $\mathbf{a}$  and  $\mathbf{b}$ .

$\therefore$  if  $\mathbf{a}$  and  $\mathbf{b}$  are non-zero vectors then  $\mathbf{a} \times \mathbf{b} = \mathbf{0} \Rightarrow \mathbf{a}$  is parallel to  $\mathbf{b}$ .

$(\Leftarrow)$  If  $\mathbf{a}$  is parallel to  $\mathbf{b}$  then  $\mathbf{a} = k\mathbf{b}$ ,  $k \in \mathbb{R}$ ,  $k \neq 0$

$$\begin{aligned} \therefore \mathbf{a} \times \mathbf{b} &= \mathbf{a} \times (k\mathbf{a}) \\ &= k(\mathbf{a} \times \mathbf{a}) \\ &= k \times \mathbf{0} \\ &= \mathbf{0} \end{aligned}$$

$\therefore \mathbf{a}$  is parallel to  $\mathbf{b} \Rightarrow \mathbf{a} \times \mathbf{b} = \mathbf{0}$

$\therefore$  if  $\mathbf{a}$  and  $\mathbf{b}$  are non-zero vectors then  $\mathbf{a} \times \mathbf{b} = \mathbf{0} \Leftrightarrow \mathbf{a}$  is parallel to  $\mathbf{b}$ .

$$\begin{aligned} \mathbf{4} \quad \mathbf{a} \quad \mathbf{p} \bullet \mathbf{q} &= \begin{pmatrix} 2 \\ 1 \\ -5 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} \\ &= 6 - 1 - 5 \\ &= 0 \end{aligned}$$

$\therefore \mathbf{p}$  and  $\mathbf{q}$  are perpendicular.

$$\begin{aligned} \mathbf{b} \quad \mathbf{p} \times \mathbf{q} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -5 \\ 3 & -1 & 1 \end{vmatrix} \\ &= \begin{vmatrix} 1 & -5 \\ -1 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & -5 \\ 3 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix} \mathbf{k} \\ &= (1 - 5)\mathbf{i} - (2 + 15)\mathbf{j} + (-2 - 3)\mathbf{k} \\ &= -4\mathbf{i} - 17\mathbf{j} - 5\mathbf{k} \\ &= \begin{pmatrix} -4 \\ -17 \\ -5 \end{pmatrix} \end{aligned}$$

$$\begin{aligned}
 \text{c } |\mathbf{p}||\mathbf{q}| &= \sqrt{2^2 + 1^2 + (-5)^2} \sqrt{3^2 + (-1)^2 + 1^2} \\
 &= \sqrt{30} \sqrt{11} \\
 &= \sqrt{330} \\
 |\mathbf{p} \times \mathbf{q}| &= \sqrt{(-4)^2 + (-17)^2 + (-5)^2} \\
 &= \sqrt{16 + 289 + 25} \\
 &= \sqrt{330} \\
 &= |\mathbf{p}||\mathbf{q}| \quad \checkmark
 \end{aligned}$$

$$\begin{aligned}
 \text{5 a } \mathbf{a} \times \mathbf{c} &= \mathbf{b} \times \mathbf{c} \\
 \therefore \mathbf{0} &= \mathbf{b} \times \mathbf{c} - \mathbf{a} \times \mathbf{c} \\
 \therefore \mathbf{0} &= (\mathbf{b} - \mathbf{a}) \times \mathbf{c} \\
 \therefore \overrightarrow{OC} &\text{ is parallel to } \overrightarrow{AB}.
 \end{aligned}$$

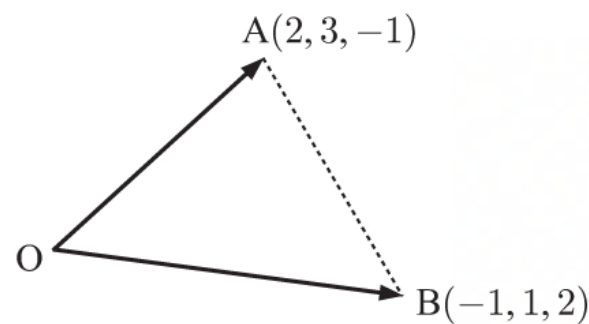
$$\begin{aligned}
 \text{b } \mathbf{a} + \mathbf{b} + \mathbf{c} &= \mathbf{0} \\
 \therefore \mathbf{b} \times (\mathbf{a} + \mathbf{b} + \mathbf{c}) &= \mathbf{0} \\
 \therefore \mathbf{b} \times \mathbf{a} + \mathbf{b} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} &= \mathbf{0} \\
 \therefore -\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} &= \mathbf{0} \\
 \{\text{since } \mathbf{b} \times \mathbf{b} = \mathbf{0}, \mathbf{b} \times \mathbf{a} = -\mathbf{a} \times \mathbf{b}\} \\
 \therefore \mathbf{b} \times \mathbf{c} &= \mathbf{a} \times \mathbf{b}
 \end{aligned}$$

$$\begin{aligned}
 \text{c } \mathbf{b} \times \mathbf{c} &= \mathbf{c} \times \mathbf{a}, \quad \mathbf{c} \neq \mathbf{0} \\
 \therefore \mathbf{b} \times \mathbf{c} - \mathbf{c} \times \mathbf{a} &= \mathbf{0} \\
 \therefore \mathbf{b} \times \mathbf{c} + \mathbf{a} \times \mathbf{c} &= \mathbf{0} \quad \{-\mathbf{c} \times \mathbf{a} = \mathbf{a} \times \mathbf{c}\} \\
 \therefore (\mathbf{b} + \mathbf{a}) \times \mathbf{c} &= \mathbf{0} \\
 \therefore \text{since } \mathbf{c} \neq \mathbf{0}, \mathbf{b} + \mathbf{a} &\text{ and } \mathbf{c} \text{ must be parallel, or } \mathbf{a} + \mathbf{b} = \mathbf{0} \\
 \therefore \mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a} &= k\mathbf{c}, \quad k \in \mathbb{R}
 \end{aligned}$$

### EXERCISE 12N.3

$$\text{1 a } \overrightarrow{OA} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \quad \text{and} \quad \overrightarrow{OB} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$$

$$\begin{aligned}
 \text{b } \overrightarrow{OA} \times \overrightarrow{OB} &= \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \times \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \\
 &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & -1 \\ -1 & 1 & 2 \end{vmatrix} \\
 &= \begin{vmatrix} 3 & -1 \\ 1 & 2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & 3 \\ -1 & 1 \end{vmatrix} \mathbf{k} \\
 &= 7\mathbf{i} - 3\mathbf{j} - 5\mathbf{k} \\
 &= \begin{pmatrix} 7 \\ -3 \\ 5 \end{pmatrix}
 \end{aligned}$$





$$\begin{aligned}
 \text{c Area of triangle OAB} &= \frac{1}{2} | \overrightarrow{\text{OA}} \times \overrightarrow{\text{OB}} | \\
 &= \frac{1}{2} \sqrt{7^2 + (-3)^2 + 5^2} \\
 &= \frac{\sqrt{83}}{2} \text{ units}^2
 \end{aligned}$$

$$2 \quad \text{a} \quad \text{A}(2, 1, 1), \text{ B}(4, 3, 0), \text{ C}(1, 3, -2)$$

$$\overrightarrow{\text{AB}} = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} \text{ and } \overrightarrow{\text{AC}} = \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix}$$

$$\begin{aligned}
 \therefore \overrightarrow{\text{AB}} \times \overrightarrow{\text{AC}} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & -1 \\ -1 & 2 & -3 \end{vmatrix} \\
 &= \begin{vmatrix} 2 & -1 \\ 2 & -3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & -1 \\ -1 & -3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & 2 \\ -1 & 2 \end{vmatrix} \mathbf{k} \\
 &= -4\mathbf{i} + 7\mathbf{j} + 6\mathbf{k}
 \end{aligned}$$

$$\begin{aligned}
 \therefore \text{area} &= \frac{1}{2} | \overrightarrow{\text{AB}} \times \overrightarrow{\text{AC}} | \\
 &= \frac{1}{2} | -4\mathbf{i} + 7\mathbf{j} + 6\mathbf{k} | \\
 &= \frac{1}{2} \sqrt{(-4)^2 + 7^2 + 6^2} \\
 &= \frac{\sqrt{101}}{2} \text{ units}^2
 \end{aligned}$$

$$b \quad \text{A}(0, 0, 0), \text{ B}(-1, 2, 3), \text{ C}(1, 2, 6)$$

$$\overrightarrow{\text{AB}} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} \text{ and } \overrightarrow{\text{AC}} = \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix}$$

$$\begin{aligned}
 \therefore \overrightarrow{\text{AB}} \times \overrightarrow{\text{AC}} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & 3 \\ 1 & 2 & 6 \end{vmatrix} \\
 &= \begin{vmatrix} 2 & 3 \\ 2 & 6 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -1 & 3 \\ 1 & 6 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -1 & 2 \\ 1 & 2 \end{vmatrix} \mathbf{k} \\
 &= 6\mathbf{i} + 9\mathbf{j} - 4\mathbf{k}
 \end{aligned}$$

$$\begin{aligned}
 \therefore \text{area} &= \frac{1}{2} | \overrightarrow{\text{AB}} \times \overrightarrow{\text{AC}} | \\
 &= \frac{1}{2} | 6\mathbf{i} + 9\mathbf{j} - 4\mathbf{k} | \\
 &= \frac{1}{2} \sqrt{6^2 + 9^2 + (-4)^2} \\
 &= \frac{\sqrt{133}}{2} \text{ units}^2
 \end{aligned}$$

**c**  $A(1, 3, 2), B(2, -1, 0), C(1, 10, 6)$

$$\overrightarrow{AB} = \begin{pmatrix} 1 \\ -4 \\ -2 \end{pmatrix} \text{ and } \overrightarrow{AC} = \begin{pmatrix} 0 \\ 7 \\ 4 \end{pmatrix}$$

$$\begin{aligned} \therefore \overrightarrow{AB} \times \overrightarrow{AC} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -4 & -2 \\ 0 & 7 & 4 \end{vmatrix} \\ &= \begin{vmatrix} -4 & -2 \\ 7 & 4 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & -2 \\ 0 & 4 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & -4 \\ 0 & 7 \end{vmatrix} \mathbf{k} \\ &= -2\mathbf{i} - 4\mathbf{j} + 7\mathbf{k} \end{aligned}$$

$$\begin{aligned} \therefore \text{area} &= \frac{1}{2} |\overrightarrow{AB} \times \overrightarrow{AC}| \\ &= \frac{1}{2} |-2\mathbf{i} - 4\mathbf{j} + 7\mathbf{k}| \\ &= \frac{1}{2} \sqrt{(-2)^2 + (-4)^2 + 7^2} \\ &= \frac{\sqrt{69}}{2} \text{ units}^2 \end{aligned}$$

**3 a**  $\text{Area} = \frac{1}{2} |\mathbf{a} \times \mathbf{b}| \text{ units}^2$

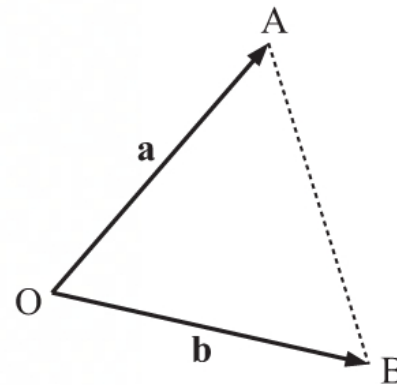
**b i**  $\begin{aligned} \overrightarrow{BO} &= -\overrightarrow{OB} \\ &= -\mathbf{b} \\ \overrightarrow{BA} &= (-\mathbf{b}) + \mathbf{a} \\ &= \mathbf{a} - \mathbf{b} \end{aligned}$

**ii**  $\begin{aligned} \text{Area} &= \frac{1}{2} |\overrightarrow{BO} \times \overrightarrow{BA}| \\ &= \frac{1}{2} |-\mathbf{b} \times (\mathbf{a} - \mathbf{b})| \text{ units}^2 \end{aligned}$

**c**  $\begin{aligned} -\mathbf{b} \times (\mathbf{a} - \mathbf{b}) &= -\mathbf{b} \times (\mathbf{a} + (-\mathbf{b})) \\ &= (-\mathbf{b} \times \mathbf{a}) + (-\mathbf{b} \times -\mathbf{b}) \\ &= \mathbf{a} \times \mathbf{b} \end{aligned}$

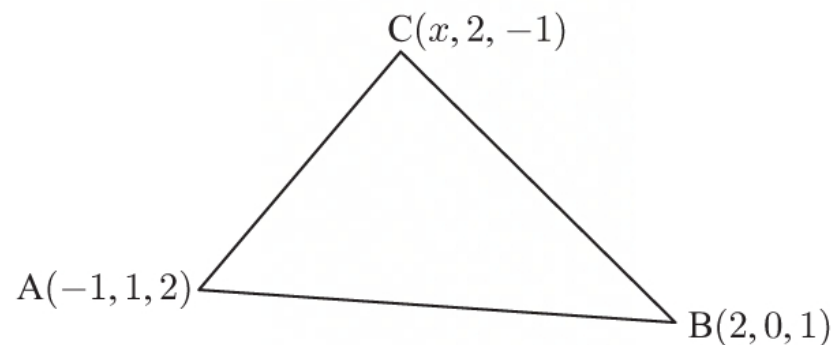
$$\therefore \frac{1}{2} |-\mathbf{b} \times (\mathbf{a} - \mathbf{b})| = \frac{1}{2} |\mathbf{a} \times \mathbf{b}|$$

$\therefore$  the areas in **a** and **b ii** are equal.



**4**  $\overrightarrow{AB} = \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix} \text{ and } \overrightarrow{AC} = \begin{pmatrix} x+1 \\ 1 \\ -3 \end{pmatrix}$

$$\begin{aligned} \therefore \overrightarrow{AB} \times \overrightarrow{AC} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & -1 \\ x+1 & 1 & -3 \end{vmatrix} \\ &= \begin{vmatrix} -1 & -1 \\ 1 & -3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 3 & -1 \\ x+1 & -3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 3 & -1 \\ x+1 & 1 \end{vmatrix} \mathbf{k} \\ &= (3+1)\mathbf{i} - (-9+(x+1))\mathbf{j} + (3+(x+1))\mathbf{k} \\ &= 4\mathbf{i} + (8-x)\mathbf{j} + (x+4)\mathbf{k} \end{aligned}$$



$$\text{Area of } \triangle ABC = \frac{1}{2} | \overrightarrow{AB} \times \overrightarrow{AC} |$$

$$\therefore \sqrt{88} = \frac{1}{2} | 4\mathbf{i} + (8-x)\mathbf{j} + (x+4)\mathbf{k} |$$

$$\therefore \sqrt{352} = \sqrt{16 + (8-x)^2 + (x+4)^2}$$

$$\therefore 352 = 16 + 64 - 16x + x^2 + x^2 + 8x + 16$$

$$\therefore 2x^2 - 8x - 256 = 0$$

$$\therefore x^2 - 4x - 128 = 0$$

$$\therefore x = \frac{4 \pm \sqrt{16 + 4(1)(128)}}{2}$$

$$= 2 \pm \sqrt{132}$$

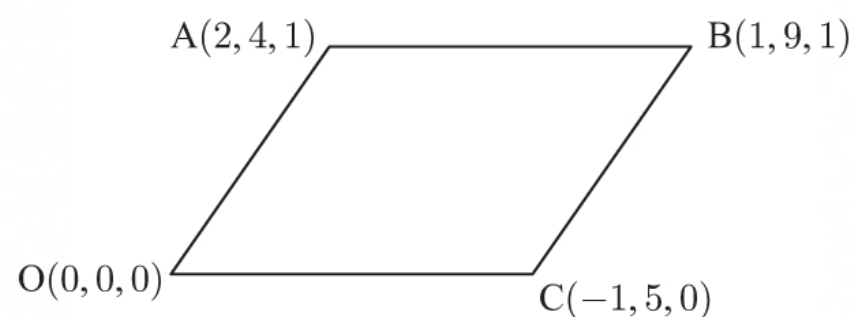
$$= 2 \pm 2\sqrt{33}$$

**5 a**  $\overrightarrow{OA} = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}$

$$\overrightarrow{CB} = \begin{pmatrix} 1 - (-1) \\ 9 - 5 \\ 1 - 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}$$

$$\overrightarrow{OC} = \begin{pmatrix} -1 \\ 5 \\ 0 \end{pmatrix}$$

$$\overrightarrow{AB} = \begin{pmatrix} 1 - 2 \\ 9 - 4 \\ 1 - 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \\ 0 \end{pmatrix}$$



Opposites are parallel and equal in length.

$\therefore$  OABC is a parallelogram.

**b** Area of OABC =  $| \overrightarrow{OA} \times \overrightarrow{OC} |$

$$= \left| \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 4 & 1 \\ -1 & 5 & 0 \end{vmatrix} \right|$$

$$= \left| \begin{vmatrix} 4 & 1 \\ 5 & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 1 \\ -1 & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & 4 \\ -1 & 5 \end{vmatrix} \mathbf{k} \right|$$

$$= | -5\mathbf{i} - \mathbf{j} + 14\mathbf{k} |$$

$$= \sqrt{(-5)^2 + (-1)^2 + 14^2}$$

$$= \sqrt{222} \text{ units}^2$$

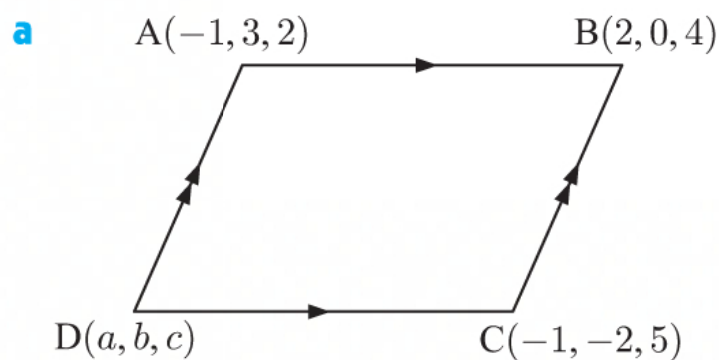
- 6  $A(-1, 2, 2)$ ,  $B(2, -1, 4)$ ,  $C(0, 1, 0)$

$$\overrightarrow{AB} = \begin{pmatrix} 3 \\ -3 \\ 2 \end{pmatrix} \text{ and } \overrightarrow{AC} = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$$

$$\begin{aligned} \therefore \overrightarrow{AB} \times \overrightarrow{AC} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -3 & 2 \\ 1 & -1 & -2 \end{vmatrix} \\ &= \begin{vmatrix} -3 & 2 \\ -1 & -2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 3 & 2 \\ 1 & -2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 3 & -3 \\ 1 & -1 \end{vmatrix} \mathbf{k} \\ &= 8\mathbf{i} + 8\mathbf{j} \end{aligned}$$

$$\begin{aligned} \therefore \text{area} &= | \overrightarrow{AB} \times \overrightarrow{AC} | \\ &= | 8\mathbf{i} + 8\mathbf{j} | \\ &= \sqrt{8^2 + 8^2} \\ &= 8\sqrt{2} \text{ units}^2 \end{aligned}$$

- 7  $A(-1, 3, 2)$ ,  $B(2, 0, 4)$ ,  $C(-1, -2, 5)$



Suppose D is  $(a, b, c)$ .

Since  $\overrightarrow{AB} = \overrightarrow{DC}$ ,

$$\begin{pmatrix} 3 \\ -3 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 - a \\ -2 - b \\ 5 - c \end{pmatrix}$$

$$\therefore -1 - a = 3, \quad -2 - b = -3, \quad \text{and } 5 - c = 2$$

$$\therefore a = -4, \quad b = 1, \quad \text{and } c = 3$$

$\therefore$  D is  $(-4, 1, 3)$ .

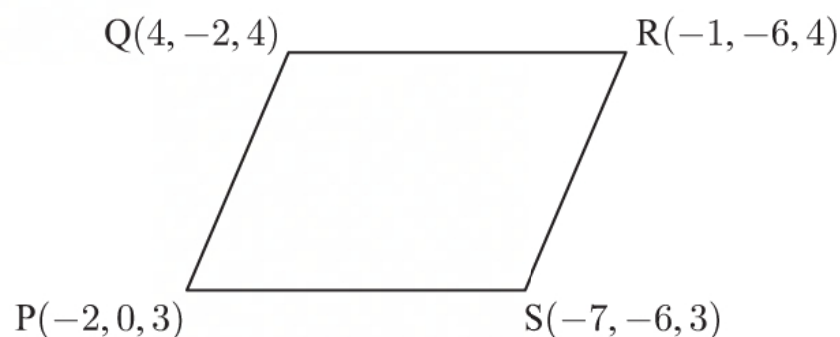
**b**

$$\overrightarrow{BC} = \begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix} \text{ and } \overrightarrow{BA} = \begin{pmatrix} -3 \\ 3 \\ -2 \end{pmatrix}$$

$$\begin{aligned} \therefore \overrightarrow{BC} \times \overrightarrow{BA} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & -2 & 1 \\ -3 & 3 & -2 \end{vmatrix} \\ &= \begin{vmatrix} -2 & 1 \\ 3 & -2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -3 & 1 \\ -3 & -2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -3 & -2 \\ -3 & 3 \end{vmatrix} \mathbf{k} \\ &= \mathbf{i} - 9\mathbf{j} - 15\mathbf{k} \end{aligned}$$

$$\begin{aligned} \therefore \text{area} &= | \overrightarrow{BC} \times \overrightarrow{BA} | \\ &= | \mathbf{i} - 9\mathbf{j} - 15\mathbf{k} | \\ &= \sqrt{1^2 + (-9)^2 + (-15)^2} \\ &= \sqrt{307} \text{ units}^2 \end{aligned}$$



**8 a**

$$\begin{aligned}\vec{PQ} &= \begin{pmatrix} 4 - (-2) \\ -2 - 0 \\ 4 - 3 \end{pmatrix} \\ &= \begin{pmatrix} 6 \\ -2 \\ 1 \end{pmatrix}\end{aligned}$$

$$\begin{aligned}\therefore |\vec{PQ}| &= \sqrt{6^2 + (-2)^2 + 1^2} \\ &= \sqrt{41} \text{ units}\end{aligned}$$

$$\begin{aligned}\vec{QR} &= \begin{pmatrix} -1 - 4 \\ -6 - (-2) \\ 4 - 4 \end{pmatrix} \\ &= \begin{pmatrix} -5 \\ -4 \\ 0 \end{pmatrix}\end{aligned}$$

$$\begin{aligned}\therefore |\vec{QR}| &= \sqrt{(-5)^2 + (-4)^2 + 0^2} \\ &= \sqrt{41} \text{ units}\end{aligned}$$

$$\begin{aligned}\vec{RS} &= \begin{pmatrix} -7 - (-1) \\ -4 - (-6) \\ 3 - 4 \end{pmatrix} \\ &= \begin{pmatrix} -6 \\ 2 \\ -1 \end{pmatrix}\end{aligned}$$

$$\begin{aligned}\therefore |\vec{RS}| &= \sqrt{(-6)^2 + 2^2 + (-1)^2} \\ &= \sqrt{41} \text{ units}\end{aligned}$$

$$\begin{aligned}\vec{SP} &= \begin{pmatrix} -2 - (-7) \\ 0 - (-4) \\ 3 - 3 \end{pmatrix} \\ &= \begin{pmatrix} 5 \\ 4 \\ 0 \end{pmatrix}\end{aligned}$$

$$\begin{aligned}\therefore |\vec{SP}| &= \sqrt{5^2 + 4^2 + 0^2} \\ &= \sqrt{41} \text{ units}\end{aligned}$$

All sides are the same length, so PQRS is a rhombus.

**b** Area of PQRS =  $|\vec{PQ} \times \vec{PS}|$

$$\begin{aligned}&= \left| \begin{pmatrix} 6 \\ -2 \\ 1 \end{pmatrix} \times \begin{pmatrix} -5 \\ -4 \\ 0 \end{pmatrix} \right| \\ &= \left| \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & -2 & 1 \\ -5 & -4 & 0 \end{vmatrix} \right| \\ &= \left| \begin{vmatrix} -2 & 1 \\ -4 & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 6 & 1 \\ -5 & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 6 & -2 \\ -5 & -4 \end{vmatrix} \mathbf{k} \right| \\ &= |4\mathbf{i} - 5\mathbf{j} - 34\mathbf{k}| \\ &= \sqrt{4^2 + (-5)^2 + (-34)^2} \\ &= \sqrt{1197} \\ &= 3\sqrt{133} \text{ units}^2\end{aligned}$$

9 a i Area of the base plane =  $|\mathbf{b} \times \mathbf{c}|$  units<sup>2</sup>

ii Perpendicular height =  $|\mathbf{a}| \sin \theta$  units

b Let  $\phi$  be the angle between  $\mathbf{a}$  and  $\mathbf{b} \times \mathbf{c}$ .

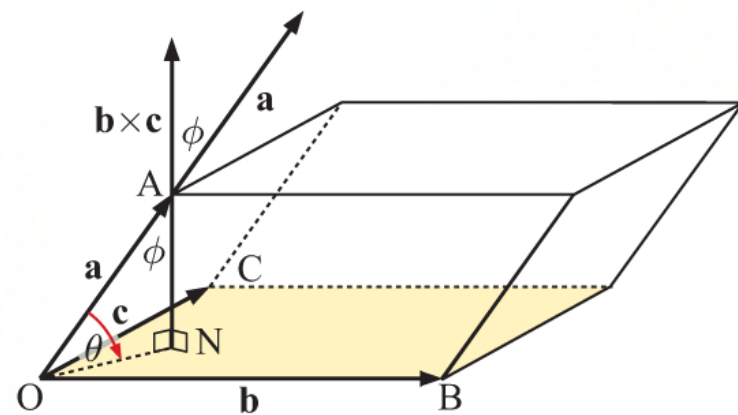
$\therefore$  perpendicular height =  $|\mathbf{a}| \cos \phi$

Volume = area of base  $\times$  perpendicular height

$$= |\mathbf{b} \times \mathbf{c}| \times |\mathbf{a}| \cos \phi$$

$$= |\mathbf{a}| |\mathbf{b} \times \mathbf{c}| \cos \phi$$

$$= |\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c})| \text{ units}^3 \quad \{\phi \text{ is the angle between } \mathbf{a} \text{ and } \mathbf{b} \times \mathbf{c}, \cos \phi \geq 0\}$$



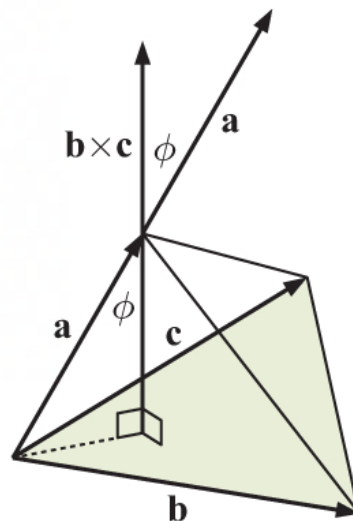
c  $\overrightarrow{OA} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}, \quad \overrightarrow{OB} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \overrightarrow{OC} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

$$\begin{aligned} \overrightarrow{OB} \times \overrightarrow{OC} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ 1 & 2 & 3 \end{vmatrix} \\ &= \begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 0 & 0 \\ 1 & 3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} \mathbf{k} \\ &= 3\mathbf{i} - \mathbf{k} \\ &= \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} \end{aligned}$$

$$\text{Volume of the parallelepiped} = |\overrightarrow{OA} \bullet (\overrightarrow{OB} \times \overrightarrow{OC})|$$

$$\begin{aligned} &= \left| \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} \right| \\ &= 9 \text{ units}^3 \end{aligned}$$

10 a Let  $\phi$  be the angle between  $\mathbf{a}$  and  $\mathbf{b} \times \mathbf{c}$ .



$$\text{Volume} = \frac{1}{3}(\text{area of base} \times \text{perpendicular height})$$

$$= \frac{1}{3} \left( \frac{1}{2} |\mathbf{b} \times \mathbf{c}| \times |\mathbf{a}| \cos \phi \right)$$

$$= \frac{1}{6} |\mathbf{a}| |\mathbf{b} \times \mathbf{c}| \cos \phi$$

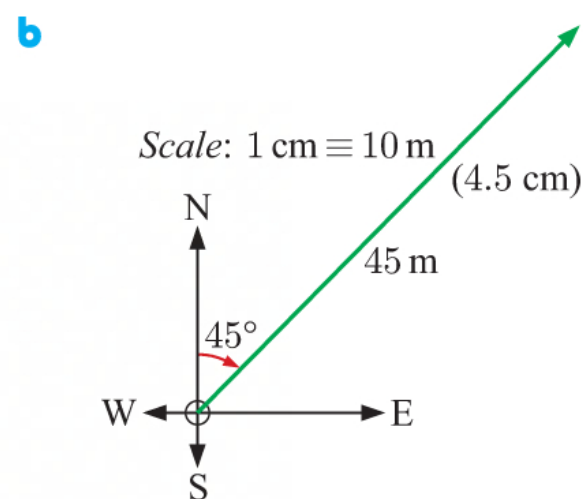
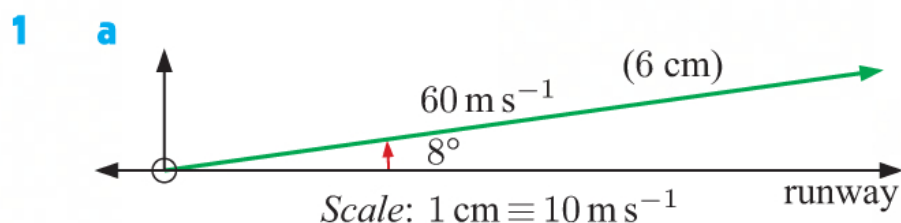
$$= \frac{1}{6} |\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c})| \text{ units}^3 \quad \{\phi \text{ is the angle between } \mathbf{a} \text{ and } \mathbf{b} \times \mathbf{c}, \cos \phi \geq 0\}$$

$$\begin{aligned} \text{b } \vec{PQ} &= \begin{pmatrix} 2-0 \\ 3-0 \\ 0-1 \end{pmatrix} & \vec{PR} &= \begin{pmatrix} -1-0 \\ 2-0 \\ 1-1 \end{pmatrix} & \vec{PS} &= \begin{pmatrix} 1-0 \\ -2-0 \\ 4-1 \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} & &= \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} & &= \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \vec{PR} \times \vec{PS} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & 0 \\ 1 & -2 & 3 \end{vmatrix} \\ &= \begin{vmatrix} 2 & 0 \\ -2 & 3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -1 & 0 \\ 1 & 3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -1 & 2 \\ 1 & -2 \end{vmatrix} \mathbf{k} \\ &= 6\mathbf{i} + 3\mathbf{j} \\ &= \begin{pmatrix} 6 \\ 3 \\ 0 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \text{Volume of tetrahedron} &= \frac{1}{6} | \vec{PQ} \bullet (\vec{PR} \times \vec{PS}) | \\ &= \frac{1}{6} \left| \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} 6 \\ 3 \\ 0 \end{pmatrix} \right| \\ &= \frac{21}{6} \\ &= 3\frac{1}{2} \text{ units}^3 \end{aligned}$$

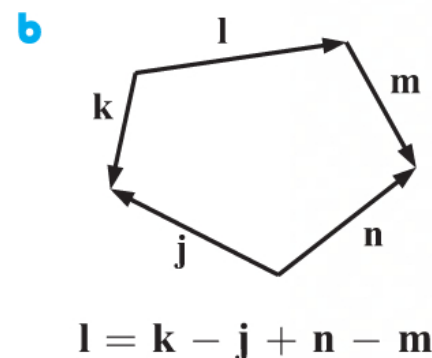
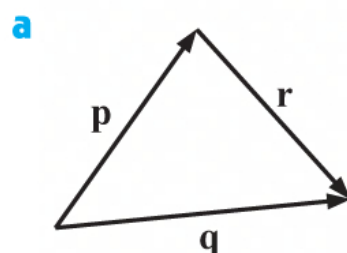
## REVIEW SET 12A

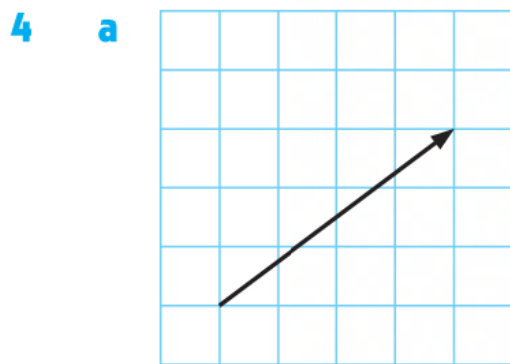


2 a  $\vec{AB} - \vec{CB} = \vec{AB} + \vec{BC} = \vec{AC}$

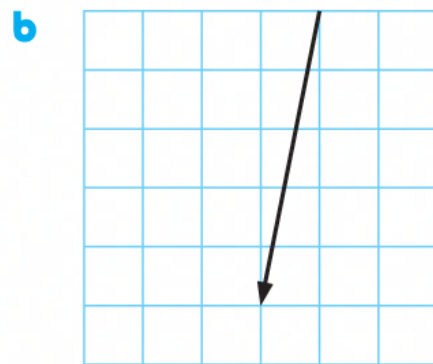
b  $\vec{AB} + \vec{BC} - \vec{DC} = \vec{AC} + \vec{CD} = \vec{AD}$

3 Note: Other answers are possible.

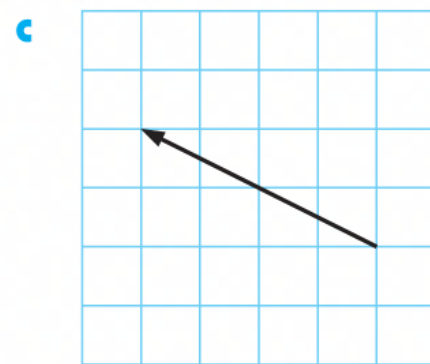




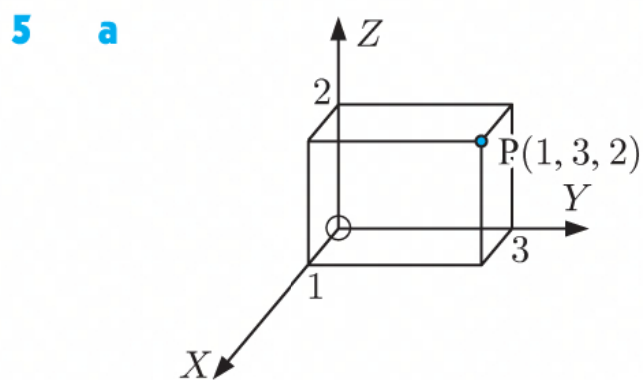
$$\begin{pmatrix} 4 \\ 3 \end{pmatrix} = 4\mathbf{i} + 3\mathbf{j}$$



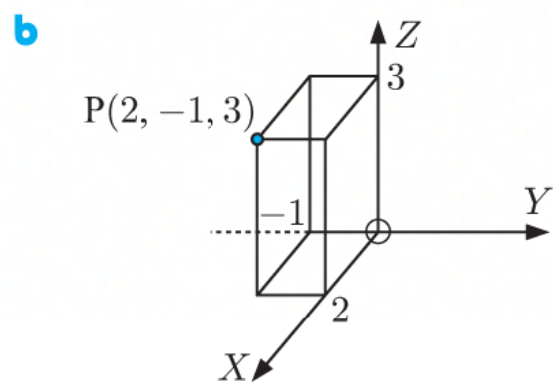
$$\begin{pmatrix} -1 \\ -5 \end{pmatrix} = -\mathbf{i} - 5\mathbf{j}$$



$$\begin{pmatrix} -4 \\ 2 \end{pmatrix} = -4\mathbf{i} + 2\mathbf{j}$$



$$\begin{aligned} OP &= \sqrt{(1-0)^2 + (3-0)^2 + (2-0)^2} \\ &= \sqrt{1+9+4} \\ &= \sqrt{14} \text{ units} \end{aligned}$$

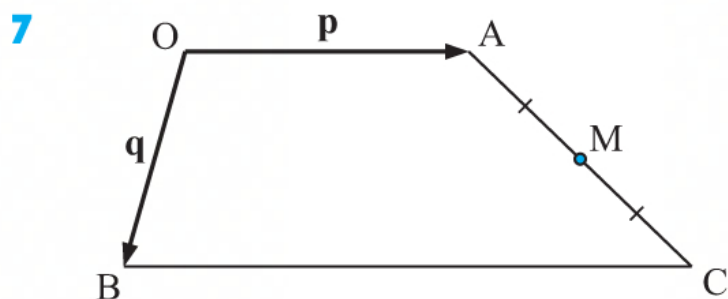


$$\begin{aligned} OP &= \sqrt{(2-0)^2 + (-1-0)^2 + (3-0)^2} \\ &= \sqrt{4+1+9} \\ &= \sqrt{14} \text{ units} \end{aligned}$$

**6**  $\vec{SP} = \vec{SR} + \vec{RQ} + \vec{QP}$

$$= -\begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix} + \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} - \begin{pmatrix} -4 \\ 1 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 4 \\ -9 \end{pmatrix}$$



**a**  $\vec{BC} = 2\vec{OA} = 2\mathbf{p}$

Now  $\vec{AC} = \vec{AO} + \vec{OB} + \vec{BC}$

$$= -\mathbf{p} + \mathbf{q} + 2\mathbf{p}$$

$$= \mathbf{p} + \mathbf{q}$$

**b**  $\vec{OM} = \vec{OA} + \vec{AM}$

$$= \mathbf{p} + \frac{1}{2}\vec{AC}$$

$$= \mathbf{p} + \frac{1}{2}(\mathbf{p} + \mathbf{q}) \quad \{\text{using a}\}$$

$$= \frac{3}{2}\mathbf{p} + \frac{1}{2}\mathbf{q}$$



$$8 \quad \mathbf{a} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 6 \\ -1 \\ -4 \end{pmatrix}$$

$$\begin{aligned} \mathbf{a} \quad \mathbf{a} - \mathbf{b} &= \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} - \begin{pmatrix} 6 \\ -1 \\ -4 \end{pmatrix} \\ &= \begin{pmatrix} 3-6 \\ -1-(-1) \\ 2-(-4) \end{pmatrix} \\ &= \begin{pmatrix} -3 \\ 0 \\ 6 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad 2\mathbf{a} + 3\mathbf{b} &= 2 \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + 3 \begin{pmatrix} 6 \\ -1 \\ -4 \end{pmatrix} \\ &= \begin{pmatrix} 6 \\ -2 \\ 4 \end{pmatrix} + \begin{pmatrix} 18 \\ -3 \\ -12 \end{pmatrix} \\ &= \begin{pmatrix} 24 \\ -5 \\ -8 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad |\mathbf{b}| \mathbf{a} &= \sqrt{6^2 + (-1)^2 + (-4)^2} \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} \\ &= \sqrt{53} \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} 3\sqrt{53} \\ -\sqrt{53} \\ 2\sqrt{53} \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad |\mathbf{a}| &= \sqrt{3^2 + (-1)^2 + 2^2} \\ &= \sqrt{14} \text{ units} \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad -\frac{1}{3}\mathbf{a} &= -\frac{1}{3} \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ \frac{1}{3} \\ -\frac{2}{3} \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad \mathbf{b} - 2\mathbf{a} &= \begin{pmatrix} 6 \\ -1 \\ -4 \end{pmatrix} - 2 \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} 6 \\ -1 \\ -4 \end{pmatrix} - \begin{pmatrix} 6 \\ -2 \\ 4 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 1 \\ -8 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \therefore |\mathbf{b} - 2\mathbf{a}| &= \sqrt{0^2 + 1^2 + (-8)^2} \\ &= \sqrt{65} \text{ units} \end{aligned}$$

$$9 \quad 3 \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} + a \begin{pmatrix} 2 \\ 0 \\ 6 \end{pmatrix} - \begin{pmatrix} b \\ -1 \\ -4 \end{pmatrix} = \begin{pmatrix} 4 \\ c \\ 7 \end{pmatrix}$$

$$\therefore \begin{pmatrix} 3 \\ -6 \\ -3 \end{pmatrix} + \begin{pmatrix} 2a \\ 0 \\ 6a \end{pmatrix} - \begin{pmatrix} b \\ -1 \\ -4 \end{pmatrix} = \begin{pmatrix} 4 \\ c \\ 7 \end{pmatrix}$$

$$\therefore \begin{pmatrix} 3 + 2a - b \\ -6 + 0 + 1 \\ -3 + 6a + 4 \end{pmatrix} = \begin{pmatrix} 4 \\ c \\ 7 \end{pmatrix}$$

$$\therefore \begin{pmatrix} 3 + 2a - b \\ -5 \\ 6a + 1 \end{pmatrix} = \begin{pmatrix} 4 \\ c \\ 7 \end{pmatrix}$$

$$\text{So, } 3 + 2a - b = 4, \quad -5 = c, \quad \text{and } 6a + 1 = 7$$

$$\therefore b = 2a - 1, \quad c = -5, \quad \text{and } a = 1$$

$$\therefore a = 1, \quad b = 1, \quad c = -5$$

$$10 \quad a \quad \overrightarrow{BC} = \begin{pmatrix} -1 - 5 \\ -4 - 0 \\ 1 - (-1) \end{pmatrix} \quad \overrightarrow{AB} = \begin{pmatrix} 5 - 2 \\ 0 - 1 \\ -1 - (-4) \end{pmatrix}$$

$$= \begin{pmatrix} -6 \\ -4 \\ 2 \end{pmatrix} \quad = \begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix}$$

$$= 2\overrightarrow{BM}$$

$$\text{Now } \overrightarrow{AM} = \overrightarrow{AB} + \overrightarrow{BM}$$

$$= \overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC}$$

$$= \begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} -6 \\ -4 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix} + \begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ -3 \\ 4 \end{pmatrix}$$

$$\therefore |\overrightarrow{AM}| = \sqrt{0^2 + (-3)^2 + 4^2}$$

$$= \sqrt{25}$$

$$= 5 \text{ units}$$

- b** Let D have position vector  $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ .

$$\text{Now } \overrightarrow{AC} = \overrightarrow{BD}$$

$$\therefore \begin{pmatrix} -1-2 \\ -4-1 \\ 1-(-4) \end{pmatrix} = \begin{pmatrix} x-5 \\ y-0 \\ z-(-1) \end{pmatrix}$$

$$\therefore \begin{pmatrix} -3 \\ -5 \\ 5 \end{pmatrix} = \begin{pmatrix} x-5 \\ y \\ z+1 \end{pmatrix}$$

$$\text{So, } x-5 = -3, \quad y = -5, \quad \text{and } z+1 = 5$$

$$\therefore x = 2$$

$$\therefore z = 4$$

$$\therefore \text{D has position vector } 2\mathbf{i} - 5\mathbf{j} + 4\mathbf{k}.$$

**11** The vectors are parallel, so  $\begin{pmatrix} -12 \\ -20 \\ 2 \end{pmatrix} = k \begin{pmatrix} 3 \\ m \\ n \end{pmatrix}$

$$\text{So, } 3k = -12, \quad km = -20, \quad kn = 2$$

$$\therefore k = -4$$

$$\therefore m = 5, \quad n = -\frac{1}{2}$$

**12**  $P(-6, 8, 2), Q(4, 6, 8), R(19, 3, 17)$

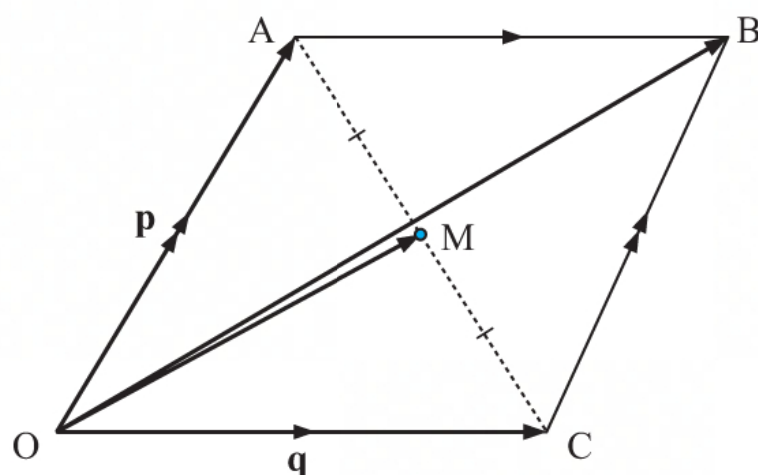
$$\overrightarrow{PQ} = \begin{pmatrix} 4-(-6) \\ 6-8 \\ 8-2 \end{pmatrix} = \begin{pmatrix} 10 \\ -2 \\ 6 \end{pmatrix} = 2 \begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix}$$

$$\overrightarrow{QR} = \begin{pmatrix} 19-4 \\ 3-6 \\ 17-8 \end{pmatrix} = \begin{pmatrix} 15 \\ -3 \\ 9 \end{pmatrix} = 3 \begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix}$$

$$\text{So, } \overrightarrow{PQ} = \frac{2}{3} \overrightarrow{QR}$$

$\therefore$  P, Q, and R are collinear.

**13**



**a i**  $\begin{aligned} \overrightarrow{OB} &= \overrightarrow{OA} + \overrightarrow{AB} \\ &= \overrightarrow{OA} + \overrightarrow{OC} \\ &= \mathbf{p} + \mathbf{q} \end{aligned}$

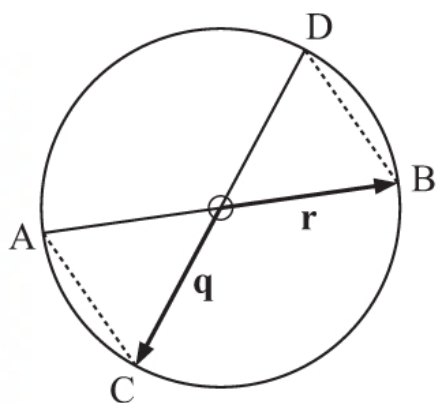
**ii**  $\begin{aligned} \overrightarrow{OM} &= \overrightarrow{OA} + \overrightarrow{AM} \\ &= \overrightarrow{OA} + \frac{1}{2} \overrightarrow{AC} \\ &= \mathbf{p} + \frac{1}{2} (\overrightarrow{AO} + \overrightarrow{OC}) \\ &= \mathbf{p} + \frac{1}{2} (-\mathbf{p} + \mathbf{q}) \\ &= \mathbf{p} - \frac{1}{2} \mathbf{p} + \frac{1}{2} \mathbf{q} \\ &= \frac{1}{2} \mathbf{p} + \frac{1}{2} \mathbf{q} \end{aligned}$

**b** We notice that  $\overrightarrow{OM} = \frac{1}{2} \overrightarrow{OB}$

$\therefore$  [OM] and [OB] are parallel, and  $OM = \frac{1}{2} OB$ .

So, O, M, and B are collinear (as O is common) and hence M is the midpoint of [OB].

14



$$\begin{array}{ll}
 \text{a i} & \overrightarrow{DB} \\
 & = \overrightarrow{DO} + \overrightarrow{OB} \\
 & = \overrightarrow{OC} + \overrightarrow{OB} \\
 & = \mathbf{q} + \mathbf{r} \\
 \text{ii} & \overrightarrow{AC} \\
 & = \overrightarrow{AO} + \overrightarrow{OC} \\
 & = \overrightarrow{OB} + \overrightarrow{OC} \\
 & = \mathbf{r} + \mathbf{q}
 \end{array}$$

b  $\overrightarrow{DB} = \overrightarrow{AC}$  {from a i and ii}  
 $\therefore$  [DB] is parallel to [AC] and equal in length.

15  $\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$  has length  $\sqrt{3^2 + 2^2 + (-1)^2} = \sqrt{14}$  units

$\therefore$  a unit vector in the opposite direction is  $-\frac{1}{\sqrt{14}} \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$

$\therefore$  a vector of length 5 units in the opposite direction is  $-\frac{5}{\sqrt{14}} \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$ .

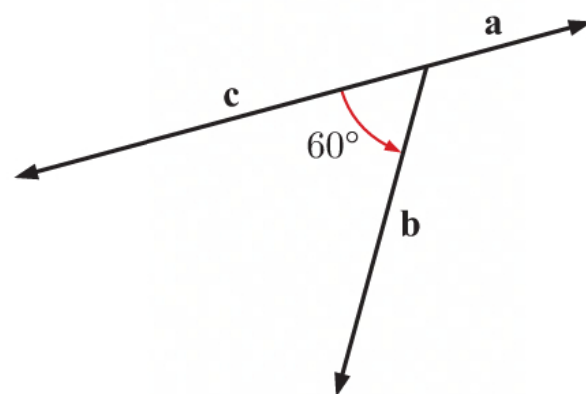
16 a  $\mathbf{p} \cdot \mathbf{q} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 5 \end{pmatrix}$   
 $= 3 \times (-1) + (-2) \times 5$   
 $= -3 - 10$   
 $= -13$

b  $\mathbf{p} - \mathbf{r} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} - \begin{pmatrix} -3 \\ 4 \end{pmatrix} = \begin{pmatrix} 6 \\ -6 \end{pmatrix}$   
 $\therefore \mathbf{q} \cdot (\mathbf{p} - \mathbf{r}) = \begin{pmatrix} -1 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 6 \\ -6 \end{pmatrix}$   
 $= (-1) \times 6 + 5 \times (-6)$   
 $= -6 - 30$   
 $= -36$

17 a  $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \theta$   
 $= 2 \times 4 \times \cos 120^\circ$   
 $= -4$

b  $\mathbf{b} \cdot \mathbf{c} = |\mathbf{b}| |\mathbf{c}| \cos \theta$   
 $= 4 \times 5 \times \cos 60^\circ$   
 $= 10$

c  $\mathbf{a} \cdot \mathbf{c} = |\mathbf{a}| |\mathbf{c}| \cos \theta$   
 $= 2 \times 5 \times \cos 180^\circ$   
 $= -10$



18 a  $\mathbf{u} \cdot \mathbf{v} = \begin{pmatrix} -4 \\ 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 3 \\ -2 \end{pmatrix}$   
 $= -4(-1) + 2(3) + 1(-2)$   
 $= 4 + 6 - 2$   
 $= 8$



**b** If  $\theta$  is the angle between  $\mathbf{u}$  and  $\mathbf{v}$ , then  $\cos \theta = \frac{\mathbf{u} \bullet \mathbf{v}}{|\mathbf{u}| |\mathbf{v}|}$

$$= \frac{4 + 6 - 2}{\sqrt{(-4)^2 + 2^2 + 1^2} \sqrt{(-1)^2 + 3^2 + (-2)^2}}$$

$$= \frac{8}{\sqrt{21} \sqrt{14}}$$

$$\therefore \theta = \cos^{-1} \left( \frac{8}{\sqrt{21} \sqrt{14}} \right)$$

$$\approx 62.2^\circ$$

**19** If  $\theta$  is the angle between  $\mathbf{a}$  and  $\mathbf{b}$ , then  $\cos \theta = \frac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{a}| |\mathbf{b}|}$

$$= \frac{3(2) + 1(5) + (-2)(1)}{\sqrt{9 + 1 + 4} \sqrt{4 + 25 + 1}}$$

$$= \frac{9}{\sqrt{14} \sqrt{30}}$$

$$\therefore \theta = \cos^{-1} \left( \frac{9}{\sqrt{14} \sqrt{30}} \right)$$

$$\approx 64.0^\circ$$

**20**  $\begin{pmatrix} 2-t \\ 3 \\ t \end{pmatrix} \bullet \begin{pmatrix} t \\ 4 \\ t+1 \end{pmatrix} = 0$

$$\therefore (2-t)t + 12 + t(t+1) = 0$$

$$\therefore 2t - t^2 + 12 + t^2 + t = 0$$

$$\therefore 3t + 12 = 0$$

$$\therefore t = -4$$

**21** Vectors parallel to  $\mathbf{i} + r\mathbf{j} + 2\mathbf{k}$  have form  $k \begin{pmatrix} 1 \\ r \\ 2 \end{pmatrix}$ ,  $k \in \mathbb{R}$ ,  $k \neq 0$ .

If these vectors are perpendicular to  $2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$  then  $k \begin{pmatrix} 1 \\ r \\ 2 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} = 0$

$$k(2 + 2r - 2) = 0$$

$$2kr = 0$$

but  $k \neq 0 \therefore r = 0$

One of these vectors is  $\mathbf{i} + 2\mathbf{k}$ , which has length  $\sqrt{1^2 + 0^2 + 2^2} = \sqrt{5}$  units.

$\therefore \frac{1}{\sqrt{5}}(\mathbf{i} + 2\mathbf{k}) = \frac{1}{\sqrt{5}}\mathbf{i} + \frac{2}{\sqrt{5}}\mathbf{k}$  is a unit vector parallel to  $\mathbf{i} + r\mathbf{j} + 2\mathbf{k}$ , and perpendicular to  $2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ .

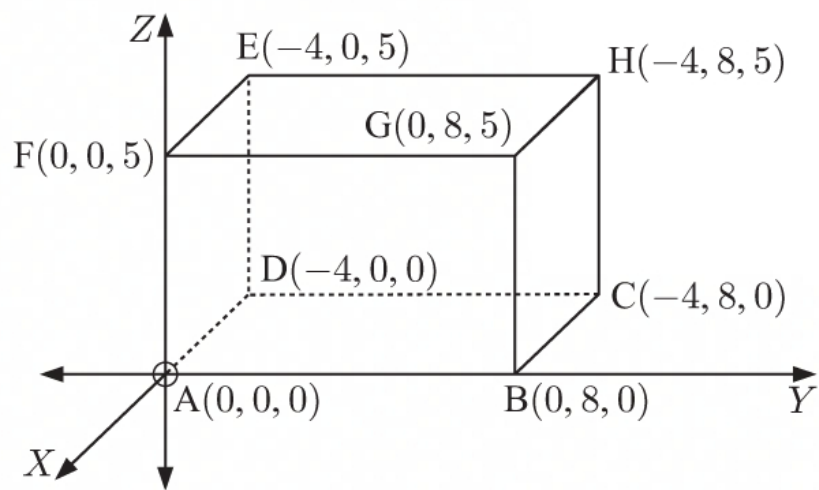
**22** Let A be the origin.

$$\overrightarrow{AG} = \begin{pmatrix} 0 \\ 8 \\ 5 \end{pmatrix}, \quad \overrightarrow{AC} = \begin{pmatrix} -4 \\ 8 \\ 0 \end{pmatrix}$$

$$\begin{aligned} \overrightarrow{AG} \bullet \overrightarrow{AC} &= \begin{pmatrix} 0 \\ 8 \\ 5 \end{pmatrix} \bullet \begin{pmatrix} -4 \\ 8 \\ 0 \end{pmatrix} \\ &= 0 \times (-4) + 8 \times 8 + 5 \times 0 \\ &= 64 \end{aligned}$$

$$\begin{aligned} |\overrightarrow{AG}| &= \sqrt{0^2 + 8^2 + 5^2} & |\overrightarrow{AC}| &= \sqrt{(-4)^2 + 8^2 + 0^2} \\ &= \sqrt{64 + 25} & &= \sqrt{16 + 64} \\ &= \sqrt{89} & &= \sqrt{80} \end{aligned}$$

$$\begin{aligned} \cos(\widehat{GAC}) &= \frac{\overrightarrow{AG} \bullet \overrightarrow{AC}}{|\overrightarrow{AG}| |\overrightarrow{AC}|} \\ \therefore \widehat{GAC} &= \cos^{-1} \left( \frac{\overrightarrow{AG} \bullet \overrightarrow{AC}}{|\overrightarrow{AG}| |\overrightarrow{AC}|} \right) \\ &= \cos^{-1} \left( \frac{64}{\sqrt{89}\sqrt{80}} \right) \\ &\approx 40.7^\circ \end{aligned}$$



**23 a**  $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 2 \\ -2 & 1 & 1 \end{vmatrix}$

$$\begin{aligned} &= \begin{vmatrix} -1 & 2 \\ 1 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 3 & 2 \\ -2 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 3 & -1 \\ -2 & 1 \end{vmatrix} \mathbf{k} \\ &= -3\mathbf{i} - 7\mathbf{j} + \mathbf{k} \\ &= \begin{pmatrix} -3 \\ -7 \\ 1 \end{pmatrix} \end{aligned}$$

**b**  $(\mathbf{a} \times \mathbf{b}) \bullet \mathbf{c} = \begin{pmatrix} -3 \\ -7 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$

$$\begin{aligned} &= -3(1) + (-7)(0) + 1(3) \\ &= 0 \end{aligned}$$

**c**  $\mathbf{b} \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 1 & 1 \\ 1 & 0 & 3 \end{vmatrix}$

$$\begin{aligned} &= \begin{vmatrix} 1 & 1 \\ 0 & 3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -2 & 1 \\ 1 & 3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -2 & 1 \\ 1 & 0 \end{vmatrix} \mathbf{k} \\ &= 3\mathbf{i} + 7\mathbf{j} + \mathbf{k} \\ &= \begin{pmatrix} 3 \\ 7 \\ -1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned}
 \mathbf{24} \quad \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 0 & 3 \\ 2 & -1 & 1 \end{vmatrix} &= \begin{vmatrix} 0 & 3 \\ -1 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -1 & 3 \\ 2 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -1 & 0 \\ 2 & -1 \end{vmatrix} \mathbf{k} \\
 &= 3\mathbf{i} + 7\mathbf{j} + \mathbf{k}
 \end{aligned}$$

$\therefore$  vectors perpendicular to  $\begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}$  and  $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$  have the form  $k \begin{pmatrix} 3 \\ 7 \\ 1 \end{pmatrix}$ ,  $k \in \mathbb{R}$ ,  $k \neq 0$ .

$$\begin{aligned}
 \text{Now } \left| \begin{pmatrix} 3 \\ 7 \\ 1 \end{pmatrix} \right| &= \sqrt{3^2 + 7^2 + 1^2} \\
 &= \sqrt{9 + 49 + 1} \\
 &= \sqrt{59}
 \end{aligned}$$

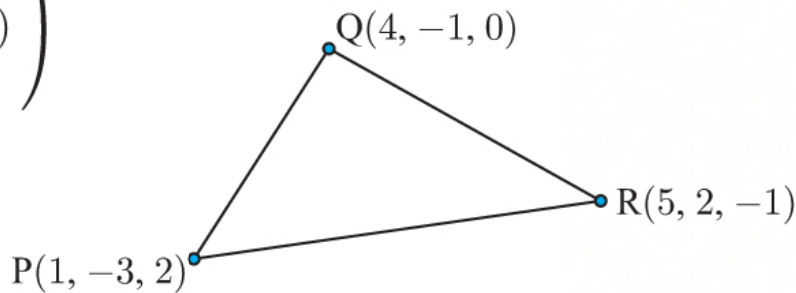
$\therefore$  the two vectors of length 3 units which are perpendicular to both  $\begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}$  and  $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$  are  $\pm \frac{3}{\sqrt{59}} \begin{pmatrix} 3 \\ 7 \\ 1 \end{pmatrix}$ .

**25**  $|\mathbf{a} \times (\mathbf{a} \times \mathbf{b})| = |\mathbf{a}| |\mathbf{a} \times \mathbf{b}| \sin \theta$ , where  $\theta$  is the angle between  $\mathbf{a}$  and  $\mathbf{a} \times \mathbf{b}$ .

Now  $\mathbf{a} \times \mathbf{b}$  is perpendicular to  $\mathbf{a}$ , so  $\theta = 90^\circ$  and  $\sin \theta = 1$ .

$$\therefore |\mathbf{a} \times (\mathbf{a} \times \mathbf{b})| = |\mathbf{a}| |\mathbf{a} \times \mathbf{b}|$$

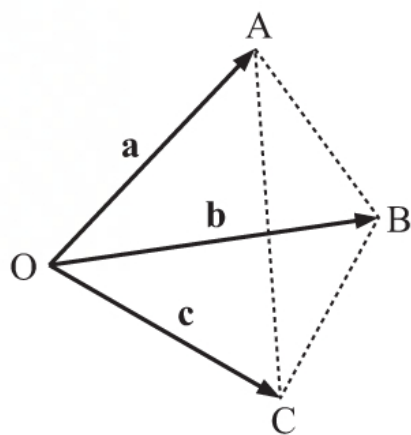
$$\begin{aligned}
 \mathbf{26} \quad \overrightarrow{PQ} &= \begin{pmatrix} 4-1 \\ -1-(-3) \\ 0-2 \end{pmatrix} \quad \text{and} \quad \overrightarrow{PR} = \begin{pmatrix} 5-1 \\ 2-(-3) \\ -1-2 \end{pmatrix} \\
 &= \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix} \quad \quad \quad = \begin{pmatrix} 4 \\ 5 \\ -3 \end{pmatrix}
 \end{aligned}$$



$$\begin{aligned}
 \therefore \overrightarrow{PQ} \times \overrightarrow{PR} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & -2 \\ 4 & 5 & -3 \end{vmatrix} \\
 &= \begin{vmatrix} 2 & -2 \\ 5 & -3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 3 & -2 \\ 4 & -3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 3 & 2 \\ 4 & 5 \end{vmatrix} \mathbf{k} \\
 &= 4\mathbf{i} + \mathbf{j} + 7\mathbf{k} \\
 &= \begin{pmatrix} 4 \\ 1 \\ 7 \end{pmatrix}
 \end{aligned}$$

$$\begin{aligned}
 \therefore \text{area of triangle PQR} &= \frac{1}{2} |\overrightarrow{PQ} \times \overrightarrow{PR}| \\
 &= \frac{1}{2} \sqrt{4^2 + 1^2 + 7^2} \\
 &= \frac{\sqrt{66}}{2} \text{ units}^2
 \end{aligned}$$

27



The total surface area  $S$  of the tetrahedron is the sum of the areas of the 4 triangular faces.

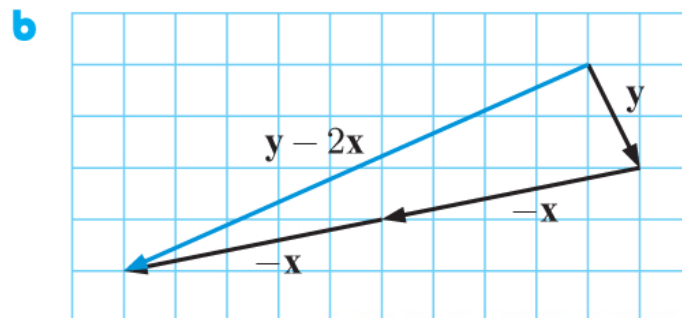
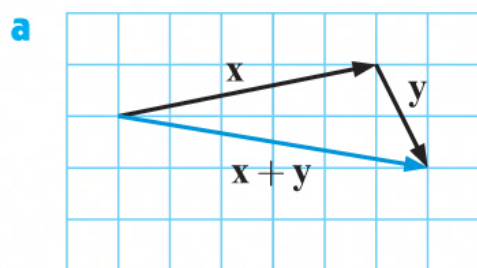
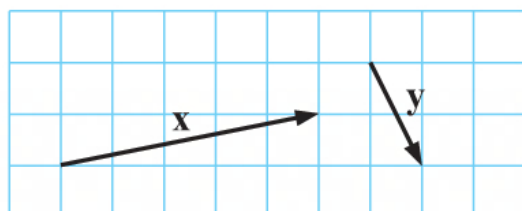
$$\text{Now } \overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -\mathbf{a} + \mathbf{b} = \mathbf{b} - \mathbf{a}$$

$$\text{and } \overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC} = -\mathbf{a} + \mathbf{c} = \mathbf{c} - \mathbf{a}$$

$$\therefore S = \frac{1}{2} [|\mathbf{a} \times \mathbf{b}| + |\mathbf{a} \times \mathbf{c}| + |\mathbf{b} \times \mathbf{c}| + |(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a})|]$$

## REVIEW SET 12B

1

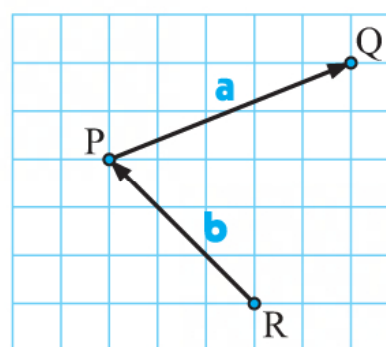


2 a  $\overrightarrow{PR} + \overrightarrow{RQ} = \overrightarrow{PQ}$

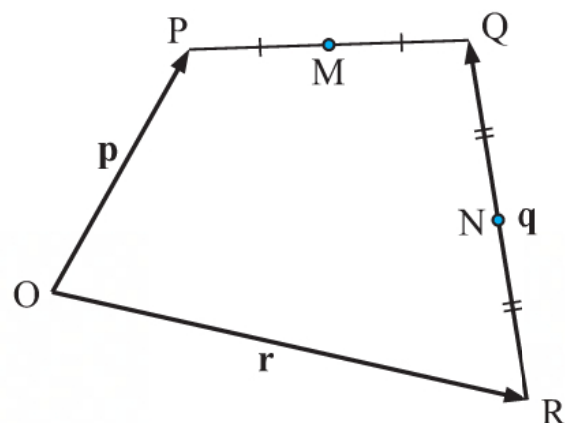
b  $\overrightarrow{PS} + \overrightarrow{SQ} + \overrightarrow{QR} = \overrightarrow{PQ} + \overrightarrow{QR}$   
 $= \overrightarrow{PR}$

3 a  $\overrightarrow{PQ} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$   
 $= 5\mathbf{i} + 2\mathbf{j}$

b  $\overrightarrow{RP} = \begin{pmatrix} -3 \\ 3 \end{pmatrix}$   
 $= -3\mathbf{i} + 3\mathbf{j}$



4



a  $\overrightarrow{OQ} = \overrightarrow{OR} + \overrightarrow{RQ} = \mathbf{r} + \mathbf{q}$

b  $\overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OR} + \overrightarrow{RQ} = -\mathbf{p} + \mathbf{r} + \mathbf{q}$

c  $\overrightarrow{ON} = \overrightarrow{OR} + \overrightarrow{RN} = \mathbf{r} + \frac{1}{2}\mathbf{q}$

d  $\overrightarrow{MN} = \overrightarrow{MQ} + \overrightarrow{QN}$   
 $= \frac{1}{2}\overrightarrow{PQ} + \frac{1}{2}\overrightarrow{QR}$   
 $= \frac{1}{2}(-\mathbf{p} + \mathbf{r} + \mathbf{q}) + \frac{1}{2}(-\mathbf{q}) \quad \{\text{from b}\}$   
 $= -\frac{1}{2}\mathbf{p} + \frac{1}{2}\mathbf{r} + \frac{1}{2}\mathbf{q} - \frac{1}{2}\mathbf{q}$   
 $= -\frac{1}{2}\mathbf{p} + \frac{1}{2}\mathbf{r}$



$$\begin{aligned} \mathbf{5} \quad \mathbf{a} \quad \mathbf{m} - \mathbf{n} + \mathbf{p} &= \begin{pmatrix} 6 \\ -3 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} + \begin{pmatrix} -1 \\ 3 \\ 6 \end{pmatrix} \\ &= \begin{pmatrix} 3 \\ -3 \\ 11 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad 2\mathbf{n} - 3\mathbf{p} &= 2 \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} - 3 \begin{pmatrix} -1 \\ 3 \\ 6 \end{pmatrix} \\ &= \begin{pmatrix} 4 \\ 6 \\ -8 \end{pmatrix} - \begin{pmatrix} -3 \\ 9 \\ 18 \end{pmatrix} \\ &= \begin{pmatrix} 7 \\ -3 \\ -26 \end{pmatrix} \end{aligned}$$

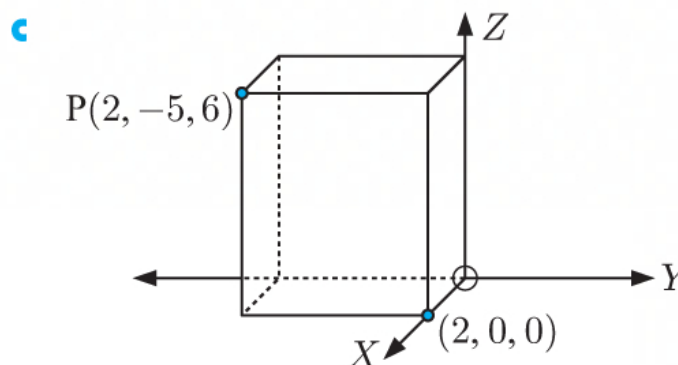
$$\begin{aligned} \mathbf{c} \quad \mathbf{m} + \mathbf{p} &= \begin{pmatrix} 6 \\ -3 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 3 \\ 6 \end{pmatrix} \\ &= \begin{pmatrix} 5 \\ 0 \\ 7 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \therefore |\mathbf{m} + \mathbf{p}| &= \sqrt{25 + 0 + 49} \\ &= \sqrt{74} \text{ units} \end{aligned}$$

$$\begin{aligned} \mathbf{6} \quad \overrightarrow{\text{CB}} &= \overrightarrow{\text{CA}} + \overrightarrow{\text{AB}} \\ &= -\overrightarrow{\text{AC}} + \overrightarrow{\text{AB}} \\ &= -\begin{pmatrix} -6 \\ 1 \\ -3 \end{pmatrix} + \begin{pmatrix} 2 \\ -7 \\ 4 \end{pmatrix} \\ &= \begin{pmatrix} 6 \\ -1 \\ 3 \end{pmatrix} + \begin{pmatrix} 2 \\ -7 \\ 4 \end{pmatrix} \\ &= \begin{pmatrix} 8 \\ -8 \\ 7 \end{pmatrix} \end{aligned}$$

$$\mathbf{7} \quad \mathbf{a} \quad \overrightarrow{\text{PQ}} = \begin{pmatrix} -1 - 2 \\ 7 - (-5) \\ 9 - 6 \end{pmatrix} = \begin{pmatrix} -3 \\ 12 \\ 3 \end{pmatrix}$$

$$\begin{aligned} \mathbf{b} \quad \text{PQ} &= \sqrt{(-3)^2 + 12^2 + 3^2} \\ &= \sqrt{9 + 144 + 9} \\ &= \sqrt{162} \\ &= 9\sqrt{2} \text{ units} \end{aligned}$$



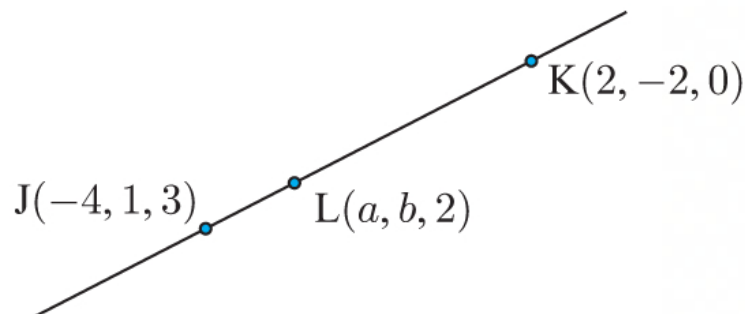
$$\begin{aligned} \therefore \text{the shortest distance from P to the X-axis} &= \sqrt{(2 - 2)^2 + (0 - (-5))^2 + (0 - 6)^2} \\ &= \sqrt{0 + 25 + 36} \\ &= \sqrt{61} \text{ units} \end{aligned}$$

$$\begin{aligned}
 8 \quad & \sqrt{k^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + (-k)^2} = 1 \\
 & \therefore k^2 + \frac{1}{2} + k^2 = 1 \\
 & \therefore 2k^2 = \frac{1}{2} \\
 & \therefore k^2 = \frac{1}{4} \\
 & \therefore k = \pm \frac{1}{2}
 \end{aligned}$$

$$9 \quad J(-4, 1, 3), \quad K(2, -2, 0), \quad L(a, b, 2)$$

$$\overrightarrow{JK} = \begin{pmatrix} 2 - (-4) \\ -2 - 1 \\ 0 - 3 \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \\ -3 \end{pmatrix}$$

$$\overrightarrow{JL} = \begin{pmatrix} a - (-4) \\ b - 1 \\ 2 - 3 \end{pmatrix} = \begin{pmatrix} a + 4 \\ b - 1 \\ -1 \end{pmatrix}$$



If J, K, and L are collinear then  $\overrightarrow{JK}$  is parallel to  $\overrightarrow{JL}$ .

$$\therefore \begin{pmatrix} 6 \\ -3 \\ -3 \end{pmatrix} = k \begin{pmatrix} a + 4 \\ b - 1 \\ -1 \end{pmatrix} \quad \text{for some } k \in \mathbb{R}, \quad k \neq 0$$

$$\therefore -3 = k(-1)$$

$$\therefore k = 3$$

$$\therefore 6 = 3(a + 4) \quad \text{and} \quad -3 = 3(b - 1)$$

$$\therefore 2 = a + 4 \quad \text{and} \quad -1 = b - 1$$

$$\therefore a = -2 \quad \text{and} \quad b = 0$$

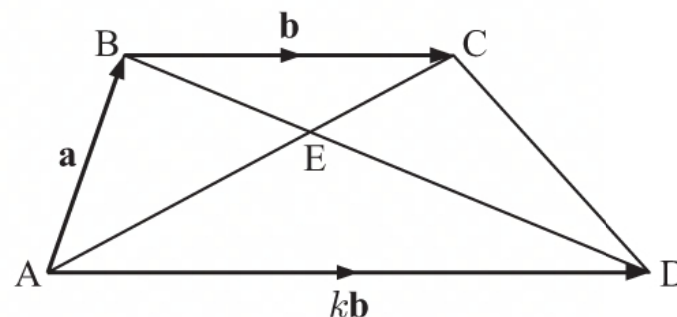
$$10 \quad |3\mathbf{i} - 2\mathbf{j} + \mathbf{k}| = \sqrt{3^2 + (-2)^2 + 1^2} = \sqrt{14}$$

$$\therefore \text{a unit vector in the direction of } 3\mathbf{i} - 2\mathbf{j} + \mathbf{k} \text{ is } \frac{1}{\sqrt{14}}(3\mathbf{i} - 2\mathbf{j} + \mathbf{k})$$

$$\therefore \text{the two vectors of length 4 units parallel to } 3\mathbf{i} - 2\mathbf{j} + \mathbf{k} \text{ are } \pm \frac{4}{\sqrt{14}}(3\mathbf{i} - 2\mathbf{j} + \mathbf{k}).$$

$$11 \quad \overrightarrow{AD} \text{ is parallel to } \overrightarrow{BC}, \text{ so } \overrightarrow{AD} = k\overrightarrow{BC} = k\mathbf{b}.$$

$$\begin{aligned}
 \text{a} \quad \overrightarrow{CD} &= \overrightarrow{CB} + \overrightarrow{BA} + \overrightarrow{AD} \\
 &= -\mathbf{b} - \mathbf{a} + k\mathbf{b} \\
 &= (k - 1)\mathbf{b} - \mathbf{a}
 \end{aligned}$$



$$\begin{aligned}\text{b } \overrightarrow{BD} &= \overrightarrow{BA} + \overrightarrow{AD} \\ &= -\mathbf{a} + k\mathbf{b}\end{aligned}$$

Let  $\overrightarrow{ED} = m\overrightarrow{BE}$  for some  $m \in \mathbb{R}$ ,  $m > 0$  {B, E, and D are collinear}

$$\therefore \overrightarrow{BD} = (m+1)\overrightarrow{BE}$$

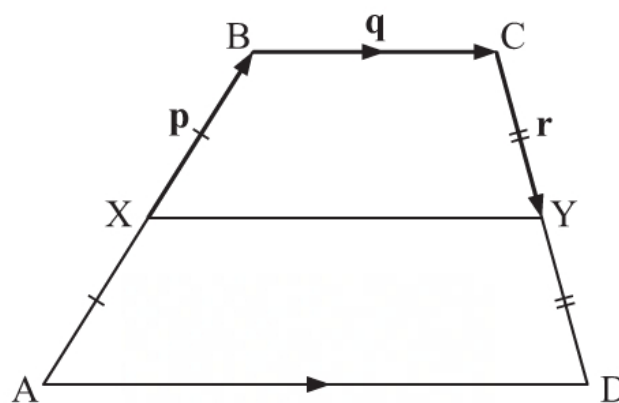
$$\therefore \overrightarrow{BE} = \frac{1}{m+1} \overrightarrow{BD} = \frac{1}{m+1} (-\mathbf{a} + k\mathbf{b})$$

$$\begin{aligned}\text{Now } \overrightarrow{AE} &= \overrightarrow{AB} + \overrightarrow{BE} \\ &= \mathbf{a} + \frac{1}{m+1} (-\mathbf{a} + k\mathbf{b}) \\ &= \frac{m}{m+1} \mathbf{a} + \frac{k}{m+1} \mathbf{b}\end{aligned}$$

$$\begin{aligned}\text{So, } \frac{m}{m+1} &= \frac{k}{m+1} \quad \{\overrightarrow{AE} \text{ is parallel to } \overrightarrow{AC} = \mathbf{a} + \mathbf{b}\} \\ \therefore m &= k \quad \{m \neq -1\} \\ \therefore \overrightarrow{ED} &= k\overrightarrow{BE}\end{aligned}$$

$$\begin{aligned}\text{12 a } \overrightarrow{AD} &= \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} \\ &= 2\mathbf{p} + \mathbf{q} + 2\mathbf{r}\end{aligned}$$

$$\text{b } \text{Since } \overrightarrow{BC} = \mathbf{q}, \overrightarrow{AD} = k\mathbf{q}.$$



$$\text{c } 2\mathbf{p} + \mathbf{q} + 2\mathbf{r} = k\mathbf{q}$$

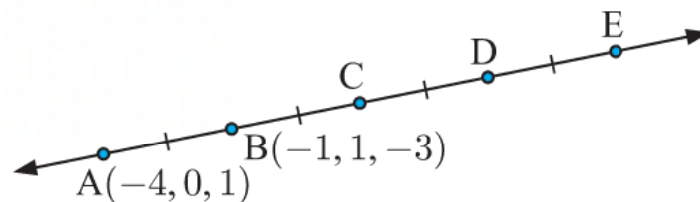
$$\therefore 2\mathbf{p} + 2\mathbf{r} = k\mathbf{q} - \mathbf{q}$$

$$\therefore \mathbf{p} + \mathbf{r} = \left(\frac{k-1}{2}\right) \mathbf{q}$$

$$\begin{aligned}\text{Then, } \overrightarrow{XY} &= \overrightarrow{XB} + \overrightarrow{BC} + \overrightarrow{CY} \\ &= \mathbf{p} + \mathbf{q} + \mathbf{r} \\ &= \left(\frac{k-1}{2}\right) \mathbf{q} + \mathbf{q} \\ &= \left(\frac{k+1}{2}\right) \mathbf{q}\end{aligned}$$

d The line joining the midpoints of the non-parallel sides of a trapezium is parallel to the other two sides.  $\left\{ \overrightarrow{XY} = \left(\frac{k+1}{2}\right) \overrightarrow{BC} \right\}$

$$\begin{aligned}\text{13 } \overrightarrow{AB} &= \begin{pmatrix} -1 - (-4) \\ 1 - 0 \\ -3 - 1 \end{pmatrix} \\ &= \begin{pmatrix} 3 \\ 1 \\ -4 \end{pmatrix}\end{aligned}$$



$$\therefore \text{C is } (-1 + 3, 1 + 1, -3 - 4), \text{ or } (2, 2, -7).$$

$$\therefore \text{D is } (2 + 3, 2 + 1, -7 - 4), \text{ or } (5, 3, -11).$$

$$\therefore \text{E is } (5 + 3, 3 + 1, -11 - 4), \text{ or } (8, 4, -15).$$

$$\begin{aligned}
 \mathbf{14} \quad |2\mathbf{i} + \mathbf{j} - 2\mathbf{k}| &= \sqrt{2^2 + 1^2 + (-2)^2} \\
 &= \sqrt{9} \\
 &= 3 \text{ units}
 \end{aligned}$$

$\therefore$  the vector in the direction of  $2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$  with length 12 units is  $4(2\mathbf{i} + \mathbf{j} - 2\mathbf{k})$   
or  $8\mathbf{i} + 4\mathbf{j} - 8\mathbf{k}$ .

$\therefore$  if X is 12 units from  $(-2, 1, -5)$  in this direction, then X is  $(-2 + 8, 1 + 4, -5 - 8)$   
or  $(6, 5, -13)$

**15** Since  $\mathbf{v}$  is parallel to  $\mathbf{w}$ , the angle  $\theta$  between  $\mathbf{v}$  and  $\mathbf{w}$  is either  $0^\circ$  or  $180^\circ$ .

$$\begin{aligned}
 \text{Now, } \mathbf{v} \bullet \mathbf{w} &= |\mathbf{v}| |\mathbf{w}| \cos \theta \\
 &= 3 \times 2 \times \cos 0^\circ \text{ or } 3 \times 2 \times \cos 180^\circ \\
 &= 6(1) \text{ or } 6(-1) \\
 &= \pm 6
 \end{aligned}$$

$$\mathbf{16} \quad \mathbf{p} \bullet (\mathbf{q} - \mathbf{r})$$

$$\begin{aligned}
 &= \begin{pmatrix} 3 \\ -2 \end{pmatrix} \bullet \left[ \begin{pmatrix} -2 \\ 5 \end{pmatrix} - \begin{pmatrix} 1 \\ -3 \end{pmatrix} \right] \\
 &= \begin{pmatrix} 3 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} -3 \\ 8 \end{pmatrix} \\
 &= 3 \times (-3) + (-2) \times 8 \\
 &= -9 - 16 \\
 &= -25
 \end{aligned}$$

$$\mathbf{p} \bullet \mathbf{q} - \mathbf{p} \bullet \mathbf{r}$$

$$\begin{aligned}
 &= \begin{pmatrix} 3 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} -2 \\ 5 \end{pmatrix} - \begin{pmatrix} 3 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -3 \end{pmatrix} \\
 &= (3 \times (-2) + (-2) \times 5) - (3 \times 1 + (-2) \times (-3)) \\
 &= (-6 - 10) - (3 + 6) \\
 &= -16 - 9 \\
 &= -25
 \end{aligned}$$

$$\therefore \mathbf{p} \bullet (\mathbf{q} - \mathbf{r}) = \mathbf{p} \bullet \mathbf{q} - \mathbf{p} \bullet \mathbf{r}$$

$$\mathbf{17} \quad \overrightarrow{\text{MP}} \bullet \overrightarrow{\text{PT}} = 0$$

$$\begin{aligned}
 \text{Letting } \overrightarrow{\text{PT}} &= \begin{pmatrix} k \\ l \end{pmatrix}, \quad \begin{pmatrix} 5 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} k \\ l \end{pmatrix} = 5k - l = 0 \\
 &\therefore l = 5k
 \end{aligned}$$

$$\therefore \overrightarrow{\text{PT}} \text{ has the form } k \begin{pmatrix} 1 \\ 5 \end{pmatrix}, \quad k \neq 0, \quad k \in \mathbb{R}.$$

$$\text{Also, } |\overrightarrow{\text{MP}}| = |\overrightarrow{\text{PT}}|$$

$$\begin{aligned}
 \therefore \sqrt{5^2 + (-1)^2} &= \sqrt{k^2 + (5k)^2} \\
 \therefore \sqrt{26} &= \sqrt{26k^2} \\
 &= |k| \sqrt{26} \\
 \therefore k &= \pm 1
 \end{aligned}$$

$$\text{So } \overrightarrow{\text{PT}} = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \text{ or } \begin{pmatrix} -1 \\ -5 \end{pmatrix}$$

$$\text{Finally, } \overrightarrow{\text{OT}} = \overrightarrow{\text{OM}} + \overrightarrow{\text{MP}} + \overrightarrow{\text{PT}}$$

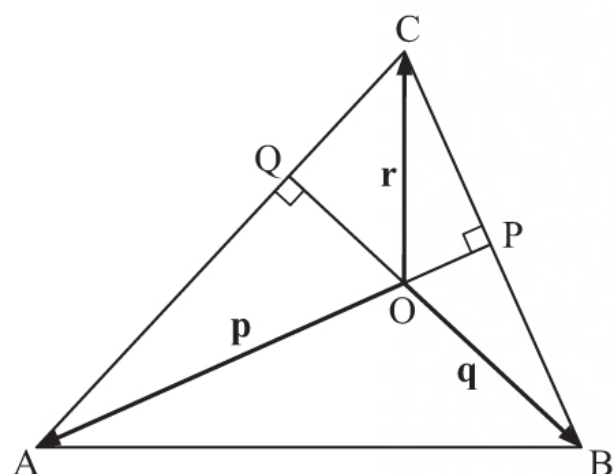
$$= \begin{pmatrix} -2 \\ 4 \end{pmatrix} + \begin{pmatrix} 5 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 5 \end{pmatrix}$$

$$\text{or } \begin{pmatrix} -2 \\ 4 \end{pmatrix} + \begin{pmatrix} 5 \\ -1 \end{pmatrix} + \begin{pmatrix} -1 \\ -5 \end{pmatrix}$$

$$\therefore \overrightarrow{\text{OT}} = \begin{pmatrix} 4 \\ 8 \end{pmatrix} \text{ or } \begin{pmatrix} 2 \\ -2 \end{pmatrix}$$



18



$$\begin{aligned} \text{a } \overrightarrow{AC} &= \overrightarrow{AO} + \overrightarrow{OC} & \overrightarrow{BC} &= \overrightarrow{BO} + \overrightarrow{OC} \\ &= -\mathbf{p} + \mathbf{r} & &= -\mathbf{q} + \mathbf{r} \end{aligned}$$

$$\begin{aligned} \text{b } \overrightarrow{BQ} &\text{ is perpendicular to } \overrightarrow{AC} \\ \therefore \overrightarrow{BO} &\text{ is perpendicular to } \overrightarrow{AC} \quad \{\text{B, O, Q collinear}\} \\ \therefore \mathbf{q} \bullet (\mathbf{r} - \mathbf{p}) &= 0 \\ \therefore \mathbf{q} \bullet \mathbf{r} - \mathbf{q} \bullet \mathbf{p} &= 0 \\ \therefore \mathbf{q} \bullet \mathbf{r} &= \mathbf{p} \bullet \mathbf{q} \end{aligned}$$

and  $\overrightarrow{AP}$  is perpendicular to  $\overrightarrow{BC}$

$$\therefore \overrightarrow{AO} \text{ is perpendicular to } \overrightarrow{BC} \quad \{\text{A, O, P collinear}\}$$

$$\begin{aligned} \therefore \mathbf{p} \bullet (\mathbf{r} - \mathbf{q}) &= 0 \\ \therefore \mathbf{p} \bullet \mathbf{r} - \mathbf{p} \bullet \mathbf{q} &= 0 \\ \therefore \mathbf{p} \bullet \mathbf{r} &= \mathbf{p} \bullet \mathbf{q} \\ \therefore \mathbf{q} \bullet \mathbf{r} &= \mathbf{p} \bullet \mathbf{q} = \mathbf{p} \bullet \mathbf{r} \\ \text{c } \overrightarrow{OC} \bullet \overrightarrow{AB} &= \mathbf{r} \bullet (-\mathbf{p} + \mathbf{q}) \\ &= -\mathbf{r} \bullet \mathbf{p} + \mathbf{r} \bullet \mathbf{q} \\ &= \mathbf{q} \bullet \mathbf{r} - \mathbf{p} \bullet \mathbf{r} \\ &= 0 \quad \{\text{using b}\} \end{aligned}$$

19

$$\begin{aligned} \text{a } \overrightarrow{PQ} &= \begin{pmatrix} 4 - (-1) \\ 0 - 2 \\ -1 - 3 \end{pmatrix} \\ &= \begin{pmatrix} 5 \\ -2 \\ -4 \end{pmatrix} \end{aligned}$$

b The vector  $\overrightarrow{OX} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$  represents the  $X$ -axis, so the angle  $\theta$  between  $\overrightarrow{OX}$  and  $\overrightarrow{PQ}$  is the angle between  $\theta$  and the  $X$ -axis.

$$\begin{aligned} \therefore \cos \theta &= \frac{\overrightarrow{PQ} \bullet \overrightarrow{OX}}{|\overrightarrow{PQ}| |\overrightarrow{OX}|} \\ &= \frac{5 + 0 + 0}{\sqrt{25 + 4 + 16} \sqrt{1 + 0 + 0}} \\ &= \frac{5}{\sqrt{45}} \\ \therefore \theta &= \cos^{-1} \left( \frac{5}{\sqrt{45}} \right) \\ &\approx 41.8^\circ \end{aligned}$$

$$\begin{aligned}
 \mathbf{p} \bullet \mathbf{q} &= \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ -4 \\ 2 \end{pmatrix} \\
 &= 2(-1) + (-1)(-4) + 4(2) \\
 &= -2 + 4 + 8 \\
 &= 10
 \end{aligned}$$

$$\begin{aligned}
 |\mathbf{p}| &= \sqrt{2^2 + (-1)^2 + 4^2} & |\mathbf{q}| &= \sqrt{(-1)^2 + (-4)^2 + 2^2} \\
 &= \sqrt{4 + 1 + 16} & &= \sqrt{1 + 16 + 4} \\
 &= \sqrt{21} & &= \sqrt{21}
 \end{aligned}$$

If  $\theta$  is the angle between  $\mathbf{p}$  and  $\mathbf{q}$ , then

$$\begin{aligned}
 \cos \theta &= \frac{\mathbf{p} \bullet \mathbf{q}}{|\mathbf{p}| |\mathbf{q}|} \\
 &= \frac{10}{\sqrt{21}\sqrt{21}} \\
 \therefore \theta &= \cos^{-1} \left( \frac{10}{\sqrt{21}\sqrt{21}} \right) \\
 &\approx 61.6^\circ
 \end{aligned}$$

**21** As the vectors are perpendicular,

$$\begin{aligned}
 \begin{pmatrix} -4 \\ t+2 \\ t \end{pmatrix} \bullet \begin{pmatrix} t \\ 1+t \\ -3 \end{pmatrix} &= 0 \\
 \therefore -4t + (t+2)(1+t) - 3t &= 0 \\
 \therefore -4t + t + t^2 + 2 + 2t - 3t &= 0 \\
 \therefore t^2 - 4t + 2 &= 0 \\
 \therefore t &= \frac{4 \pm \sqrt{16 - 4(1)(2)}}{2} \\
 \therefore t &= \frac{4 \pm \sqrt{8}}{2} = 2 \pm \sqrt{2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{u} \bullet \mathbf{v} &= \begin{pmatrix} 2 \\ -4 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} & |\mathbf{u}| &= \sqrt{2^2 + (-4)^2 + 3^2} & |\mathbf{v}| &= \sqrt{(-1)^2 + 1^2 + 3^2} \\
 &= -2 - 4 + 9 & &= \sqrt{4 + 16 + 9} & &= \sqrt{1 + 1 + 9} \\
 &= 3 & &= \sqrt{29} & &= \sqrt{11}
 \end{aligned}$$

Now

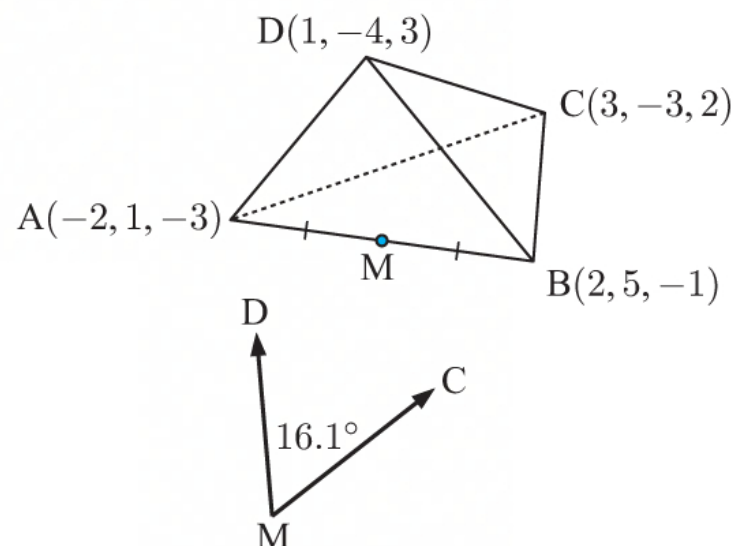
$$\begin{aligned}
 \cos \theta &= \frac{\mathbf{u} \bullet \mathbf{v}}{|\mathbf{u}| |\mathbf{v}|} \\
 &= \frac{3}{\sqrt{29}\sqrt{11}} \\
 \therefore \theta &= \cos^{-1} \left( \frac{3}{\sqrt{29}\sqrt{11}} \right) \\
 &\approx 80.3^\circ
 \end{aligned}$$

**23** M is  $\left(\frac{-2+2}{2}, \frac{1+5}{2}, \frac{-3-1}{2}\right)$  or  $(0, 3, -2)$ .

$$\therefore \overrightarrow{MD} = \begin{pmatrix} 1-0 \\ -4-3 \\ 3-(-2) \end{pmatrix} = \begin{pmatrix} 1 \\ -7 \\ 5 \end{pmatrix}, \quad \overrightarrow{MC} = \begin{pmatrix} 3-0 \\ -3-3 \\ 2-(-2) \end{pmatrix} = \begin{pmatrix} 3 \\ -6 \\ 4 \end{pmatrix}$$

$$\begin{aligned} \therefore \cos(\widehat{DMC}) &= \frac{\overrightarrow{MD} \cdot \overrightarrow{MC}}{|\overrightarrow{MD}| |\overrightarrow{MC}|} \\ &= \frac{3 + 42 + 20}{\sqrt{1+49+25} \sqrt{9+36+16}} \\ &= \frac{65}{\sqrt{75} \sqrt{61}} \end{aligned}$$

$$\begin{aligned} \therefore \widehat{DMC} &= \cos^{-1} \left( \frac{65}{\sqrt{75} \sqrt{61}} \right) \\ &\approx 16.1^\circ \end{aligned}$$



$$\begin{aligned} \mathbf{24} \quad \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & -2 \\ 1 & 1 & 2 \end{vmatrix} \\ &= \begin{vmatrix} -1 & -2 \\ 1 & 2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & -2 \\ 1 & 2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} \mathbf{k} \\ &= (-2 + 2)\mathbf{i} - (4 + 2)\mathbf{j} + (2 + 1)\mathbf{k} \\ &= -6\mathbf{j} + 3\mathbf{k} \\ &= 3(-2\mathbf{j} + \mathbf{k}) \end{aligned}$$

$\therefore$  perpendicular vectors have the form  $k \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$ ,  $k \in \mathbb{R}$ ,  $k \neq 0$ .

$$\begin{aligned} \mathbf{25} \quad |\mathbf{u} \times \mathbf{v}| &= \sqrt{1^2 + (-3)^2 + (-4)^2} \\ &= \sqrt{1 + 9 + 16} \\ &= \sqrt{26} \end{aligned}$$

$$\begin{aligned} \sin \theta &= \frac{|\mathbf{u} \times \mathbf{v}|}{|\mathbf{u}| |\mathbf{v}|} \\ &= \frac{\sqrt{26}}{3 \times 5} \\ &= \frac{\sqrt{26}}{15} \end{aligned}$$

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$\begin{aligned} \cos \theta &= \pm \sqrt{1 - \sin^2 \theta} \\ &= \pm \sqrt{1 - \frac{26}{225}} \\ &= \pm \sqrt{\frac{199}{225}} \\ &= \pm \frac{\sqrt{199}}{15} \end{aligned}$$

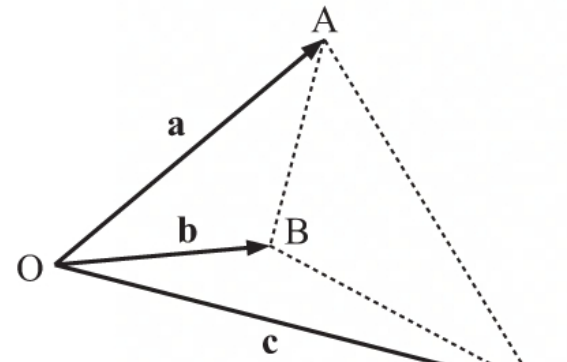
$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= |\mathbf{u}| |\mathbf{v}| \cos \theta \\ &= 3 \times 5 \times \left( \pm \frac{\sqrt{199}}{15} \right) \\ &= \pm \sqrt{199} \end{aligned}$$

So, if  $\theta$  is acute,  $\mathbf{u} \cdot \mathbf{v} = \sqrt{199}$   
and if  $\theta$  is obtuse,  $\mathbf{u} \cdot \mathbf{v} = -\sqrt{199}$

$$\mathbf{26} \quad (\Rightarrow) \quad \overrightarrow{BA} = \overrightarrow{BO} + \overrightarrow{OA} = -\mathbf{b} + \mathbf{a} = \mathbf{a} - \mathbf{b}$$

$$\text{and } \overrightarrow{BC} = \overrightarrow{BO} + \overrightarrow{OC} = -\mathbf{b} + \mathbf{c} = \mathbf{c} - \mathbf{b}$$

$$\begin{aligned} \text{Area of triangle ABC} &= \frac{1}{2} |\overrightarrow{BA} \times \overrightarrow{BC}| \\ &= \frac{1}{2} |(\mathbf{a} - \mathbf{b}) \times (\mathbf{c} - \mathbf{b})| \\ &= \frac{1}{2} |(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{b})| \end{aligned}$$



If A, B, and C are collinear, then area of triangle ABC = 0

$$\therefore (\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{b}) = \mathbf{0}$$

$$(\Leftarrow) \quad \overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = \mathbf{b} - \mathbf{a}$$

$$\text{and } \overrightarrow{BC} = \overrightarrow{BO} + \overrightarrow{OC} = \mathbf{c} - \mathbf{b}$$

If  $(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{b}) = \overrightarrow{AB} \times \overrightarrow{BC} = \mathbf{0}$  then  $\overrightarrow{AB}$  and  $\overrightarrow{BC}$  are parallel, provided they are both non-zero.

A, B, and C are distinct points, so  $\overrightarrow{AB}, \overrightarrow{BC} \neq \mathbf{0}$ .

$\therefore \overrightarrow{AB}$  and  $\overrightarrow{BC}$  are parallel.

$\therefore$  A, B, and C are collinear.

$$\mathbf{27} \quad \mathbf{a} \quad \mathbf{a} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix}$$

$$\begin{aligned} \therefore \mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 2 \\ 1 & 1 & -3 \end{vmatrix} \\ &= \begin{vmatrix} -2 & 2 \\ 1 & -3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 2 \\ 1 & -3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix} \mathbf{k} \\ &= 4\mathbf{i} + 5\mathbf{j} + 3\mathbf{k} \\ &= \begin{pmatrix} 4 \\ 5 \\ 3 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad |\mathbf{a} \times \mathbf{b}| &= \sqrt{4^2 + 5^2 + 3^2} \quad \{\text{from } \mathbf{a}\} \\ &= \sqrt{50} \\ &= 5\sqrt{2} \end{aligned}$$

$$\therefore \text{the unit vector in the direction opposite to } \mathbf{a} \times \mathbf{b} \text{ is } -\frac{1}{5\sqrt{2}} \begin{pmatrix} 4 \\ 5 \\ 3 \end{pmatrix} = \begin{pmatrix} -\frac{4}{5\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ -\frac{3}{5\sqrt{2}} \end{pmatrix}$$

$$\therefore \text{C is } 6 \left( -\frac{4}{5\sqrt{2}}, -\frac{1}{\sqrt{2}}, -\frac{3}{5\sqrt{2}} \right) \text{ which is } \left( -\frac{24}{5\sqrt{2}}, -\frac{6}{\sqrt{2}}, -\frac{18}{5\sqrt{2}} \right).$$



$$\begin{aligned}
 \text{c Area of triangle OAB} &= \frac{1}{2} | \overrightarrow{\text{OA}} \times \overrightarrow{\text{OB}} | \\
 &= \frac{1}{2} | \mathbf{a} \times \mathbf{b} | \\
 &= \frac{5\sqrt{2}}{2} \quad \{\text{from } \mathbf{b}\} \\
 &= \frac{5}{\sqrt{2}} \text{ units}^2
 \end{aligned}$$

- d Triangle OAB is the base, which has area  $\frac{5}{\sqrt{2}}$  units<sup>2</sup>.

The apex of the tetrahedron is C, which has a perpendicular height of 6 units above the base.

$$\begin{aligned}
 \text{Volume of the tetrahedron} &= \frac{1}{3} (\text{base area} \times \text{perpendicular height}) \\
 &= \frac{1}{3} \times \frac{5}{\sqrt{2}} \times 6 \\
 &= \frac{10}{\sqrt{2}} \text{ units}^3
 \end{aligned}$$

# Chapter 13

## VECTOR APPLICATIONS

### EXERCISE 13A

1 a i  $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \quad \lambda \in \mathbb{R}$  ii  $x = 1 - \lambda, \quad y = 4 + 2\lambda, \quad \lambda \in \mathbb{R}$

iii  $\lambda = 1 - x = \frac{y - 4}{2}$   
 $\therefore 2 - 2x = y - 4$   
 $\therefore 2x + y = 6$

b i If the line has direction vector  $\mathbf{b}$  perpendicular to  $\begin{pmatrix} 5 \\ 2 \end{pmatrix}$ , then  $\mathbf{b} \bullet \begin{pmatrix} 5 \\ 2 \end{pmatrix} = 0$   
 $\therefore \mathbf{b} = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$  is a reasonable choice

$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 5 \end{pmatrix}, \quad \lambda \in \mathbb{R}$

ii  $x = 5 - 2\lambda, \quad y = 2 + 5\lambda, \quad \lambda \in \mathbb{R}$

iii  $\lambda = \frac{x - 5}{-2} = \frac{y - 2}{5}$   
 $\therefore 5x - 25 = -2y + 4$   
 $\therefore 5x + 2y = 29$

c i  $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -6 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 7 \end{pmatrix}, \quad \lambda \in \mathbb{R}$

ii  $x = -6 + 3\lambda, \quad y = 7\lambda, \quad \lambda \in \mathbb{R}$

iii  $\lambda = \frac{x + 6}{3} = \frac{y}{7}$   
 $\therefore 7x + 42 = 3y$   
 $\therefore 7x - 3y = -42$

d i If the line has direction vector  $\mathbf{b}$  perpendicular to  $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$ , then  $\mathbf{b} \bullet \begin{pmatrix} 3 \\ -1 \end{pmatrix} = 0$   
 $\therefore \mathbf{b} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$  is a reasonable choice

$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \quad \lambda \in \mathbb{R}$

ii  $x = \lambda, \quad y = 2 + 3\lambda, \quad \lambda \in \mathbb{R}$

iii  $\lambda = x = \frac{y - 2}{3}$   
 $\therefore 3x = y - 2$   
 $\therefore y = 2 + 3x$

**e i** Take  $(3, 0)$  as our fixed point, so  $\mathbf{a} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$ .

The direction vector  $\mathbf{b} = \begin{pmatrix} 7-3 \\ 2-0 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \quad \lambda \in \mathbb{R}$$

**ii**  $x = 3 + 4\lambda, \quad y = 2\lambda, \quad \lambda \in \mathbb{R}$

**iii**  $\lambda = \frac{x-3}{4} = \frac{y}{2}$   
 $\therefore x-3 = 2y$   
 $\therefore x-2y = 3$

**f i** Take  $(-2, 5)$  as our fixed point, so  $\mathbf{a} = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$ .

The direction vector  $\mathbf{b} = \begin{pmatrix} 4-(-2) \\ -6-5 \end{pmatrix} = \begin{pmatrix} 6 \\ -11 \end{pmatrix}$

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 6 \\ -11 \end{pmatrix}, \quad \lambda \in \mathbb{R}$$

**ii**  $x = -2 + 6\lambda, \quad y = 5 - 11\lambda, \quad \lambda \in \mathbb{R}$

**iii**  $\lambda = \frac{x+2}{6} = \frac{y-5}{-11}$   
 $\therefore -11(x+2) = 6(y-5)$   
 $\therefore -11x - 22 = 6y - 30$   
 $\therefore 11x + 6y = 8$

**2 a**  $x = 4 - \lambda, \quad y = -3 + 2\lambda, \quad \lambda \in \mathbb{R}$

**b** When  $\lambda = 0$ ,  $x = 4 - 0 = 4$  and  $y = -3 + 2(0) = -3$   $\therefore$  the point is  $(4, -3)$ .

When  $\lambda = 1$ ,  $x = 4 - 1 = 3$  and  $y = -3 + 2(1) = -1$   $\therefore$  the point is  $(3, -1)$ .

When  $\lambda = 2$ ,  $x = 4 - 2 = 2$  and  $y = -3 + 2(2) = 1$   $\therefore$  the point is  $(2, 1)$ .

When  $\lambda = -1$ ,  $x = 4 - (-1) = 5$  and  $y = -3 + 2(-1) = -5$   
 $\therefore$  the point is  $(5, -5)$ .

When  $\lambda = -3$ ,  $x = 4 - (-3) = 7$  and  $y = -3 + 2(-3) = -9$   
 $\therefore$  the point is  $(7, -9)$ .

**3 a** When  $\lambda = 1$ ,  $\mathbf{r} = \begin{pmatrix} 1 \\ 5 \end{pmatrix} + 1 \times \begin{pmatrix} -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1-1 \\ 5+3 \end{pmatrix} = \begin{pmatrix} 0 \\ 8 \end{pmatrix}$

$\therefore$  the point is  $(0, 8)$ .

**b**  $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$  is a non-zero scalar multiple of  $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$ . It is parallel and in the opposite direction, so it could also be used to describe the direction of the line.

**c** The line passes through  $(0, 8)$  and has direction vector  $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$ .

$$\therefore \mathbf{r} = \begin{pmatrix} 0 \\ 8 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -3 \end{pmatrix}, \quad \mu \in \mathbb{R} \text{ is an alternative vector equation for line } L.$$

$$4 \quad \mathbf{a} \quad \mathbf{i} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -7 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \quad \lambda \in \mathbb{R}$$

$$\mathbf{ii} \quad x = 1 + 2\lambda, \quad y = 3 + \lambda, \quad z = -7 + 3\lambda, \quad \lambda \in \mathbb{R}$$

$$\mathbf{iii} \quad \frac{x-1}{2} = y-3 = \frac{z+7}{3}$$

$$\mathbf{b} \quad \mathbf{i} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \quad \lambda \in \mathbb{R}$$

$$\mathbf{ii} \quad x = \lambda, \quad y = 1 + \lambda, \quad z = 2 - 2\lambda, \quad \lambda \in \mathbb{R}$$

$$\mathbf{iii} \quad x = y - 1 = \frac{-z + 2}{2}$$

$$\mathbf{c} \quad \mathbf{i} \quad \text{Since the line is parallel to the } X\text{-axis, it has direction vector } \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \lambda \in \mathbb{R}$$

$$\mathbf{ii} \quad x = -2 + \lambda, \quad y = 2, \quad z = 1, \quad \lambda \in \mathbb{R}$$

$$\mathbf{iii} \quad y = 2, \quad z = 1$$

$$\mathbf{d} \quad \mathbf{i} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}, \quad \lambda \in \mathbb{R}$$

$$\mathbf{ii} \quad x = 2\lambda, \quad y = 2 - \lambda, \quad z = -1 + 3\lambda, \quad \lambda \in \mathbb{R}$$

$$\mathbf{iii} \quad \frac{x}{2} = -y + 2 = \frac{z + 1}{3}$$

$$\mathbf{e} \quad \mathbf{i} \quad \text{Since the line is perpendicular to the } XY\text{-plane, it has direction vector } \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \lambda \in \mathbb{R}$$

$$\mathbf{ii} \quad x = 3, \quad y = 2, \quad z = -1 + \lambda, \quad \lambda \in \mathbb{R}$$

$$\mathbf{iii} \quad x = 3, \quad y = 2$$

$$5 \quad \mathbf{a} \quad \overrightarrow{AB} = \begin{pmatrix} -1-1 \\ 3-2 \\ 2-1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \quad \therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}, \quad \lambda \in \mathbb{R}$$

$$\mathbf{b} \quad \overrightarrow{CD} = \begin{pmatrix} 3-0 \\ 1-1 \\ -1-3 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ -4 \end{pmatrix} \quad \therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 0 \\ -4 \end{pmatrix}, \quad \lambda \in \mathbb{R}$$

$$\mathbf{c} \quad \overrightarrow{EF} = \begin{pmatrix} 1-1 \\ -1-2 \\ 5-5 \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \\ 0 \end{pmatrix} \quad \therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ -3 \\ 0 \end{pmatrix}, \quad \lambda \in \mathbb{R}$$

$$\mathbf{d} \quad \overrightarrow{GH} = \begin{pmatrix} 5-0 \\ -1-1 \\ 3-(-1) \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 4 \end{pmatrix} \quad \therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ -2 \\ 4 \end{pmatrix}, \quad \lambda \in \mathbb{R}$$



$$6 \quad \mathbf{a} \quad \begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix} \quad \mathbf{b} \quad \begin{pmatrix} -1 \\ 1 \\ -3 \end{pmatrix}$$

$$\mathbf{c} \quad \lambda = \frac{x-2}{3} = \frac{y+1}{2} = z-1$$

$$\begin{aligned} \therefore x-2 &= 3\lambda, & y+1 &= 2\lambda, & z-1 &= \lambda \\ \therefore x &= 2+3\lambda, & y &= -1+2\lambda, & z &= 1+\lambda \end{aligned}$$

$$\therefore \text{the direction vector of the line is } \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}.$$

$$\mathbf{d} \quad \mu = \frac{1-x}{2} = \frac{y}{4} = \frac{z-3}{3}$$

$$\begin{aligned} \therefore 2\mu &= 1-x, & y &= 4\mu, & z-3 &= 3\mu \\ \therefore x &= 1-2\mu, & y &= 4\mu, & z &= 3+3\mu \end{aligned}$$

$$\therefore \text{the direction vector of the line is } \begin{pmatrix} -2 \\ 4 \\ 3 \end{pmatrix}.$$

$$7 \quad x = 1 - \lambda, \quad y = 3 + \lambda, \quad z = 3 - 2\lambda, \quad \lambda \in \mathbb{R}$$

$$\mathbf{a} \quad \text{The line meets the } YZ\text{-plane when } x = 0 \quad \therefore 1 - \lambda = 0 \\ \therefore \lambda = 1$$

$$\text{When } \lambda = 1, \quad y = 3 + 1 = 4 \quad \text{and} \quad z = 3 - 2 = 1$$

$$\therefore \text{the point is } (0, 4, 1).$$

$$\mathbf{b} \quad \text{The line meets the } XZ\text{-plane when } y = 0 \quad \therefore 3 + \lambda = 0 \\ \therefore \lambda = -3$$

$$\text{When } \lambda = -3, \quad x = 1 - (-3) = 4 \quad \text{and} \quad z = 3 - 2(-3) = 9$$

$$\therefore \text{the point is } (4, 0, 9).$$

$$\mathbf{c} \quad \text{The line meets the } XY\text{-plane when } z = 0 \quad \therefore 3 - 2\lambda = 0 \\ \therefore \lambda = \frac{3}{2}$$

$$\text{When } \lambda = \frac{3}{2}, \quad x = 1 - \frac{3}{2} = -\frac{1}{2} \quad \text{and} \quad y = 3 + \frac{3}{2} = \frac{9}{2}$$

$$\therefore \text{the point is } \left(-\frac{1}{2}, \frac{9}{2}, 0\right).$$

$$8 \quad x = x_0 + \lambda l, \quad y = y_0 + \lambda m, \quad z = z_0 + \lambda n, \quad \lambda \in \mathbb{R}$$

$$\mathbf{a} \quad \text{When } \lambda = 0, \quad x = x_0, \quad y = y_0, \quad \text{and} \quad z = z_0 \\ \therefore \text{the point is } (x_0, y_0, z_0).$$

$$\mathbf{b} \quad \begin{pmatrix} l \\ m \\ n \end{pmatrix}$$

$$\mathbf{c} \quad \text{Equating } \lambda \text{ values, the Cartesian equations are } \frac{x-x_0}{l} = \frac{y-y_0}{m} = \frac{z-z_0}{n}, \quad l, m, n \neq 0.$$

9  $x = 2 - \lambda, \quad y = 3 + 2\lambda, \quad z = 1 + \lambda, \quad \lambda \in \mathbb{R}$

The distance from the line to the point  $(1, 0, -2)$

$$= \sqrt{(2 - \lambda - 1)^2 + (3 + 2\lambda - 0)^2 + (1 + \lambda + 2)^2}$$

$$= \sqrt{(1 - \lambda)^2 + (3 + 2\lambda)^2 + (\lambda + 3)^2}$$

Now  $\sqrt{(1 - \lambda)^2 + (3 + 2\lambda)^2 + (\lambda + 3)^2} = 5\sqrt{3}$

$$\therefore (1 - \lambda)^2 + (3 + 2\lambda)^2 + (\lambda + 3)^2 = 75$$

$$\therefore 1 - 2\lambda + \lambda^2 + 9 + 12\lambda + 4\lambda^2 + \lambda^2 + 6\lambda + 9 = 75$$

$$\therefore 6\lambda^2 + 16\lambda - 56 = 0$$

$$\therefore 3\lambda^2 + 8\lambda - 28 = 0$$

$$\therefore (3\lambda + 14)(\lambda - 2) = 0$$

$$\therefore \lambda = -\frac{14}{3} \text{ or } \lambda = 2$$

When  $\lambda = 2$ , the point is  $(0, 7, 3)$ , and when  $\lambda = -\frac{14}{3}$ , the point is  $(\frac{20}{3}, -\frac{19}{3}, -\frac{11}{3})$ .

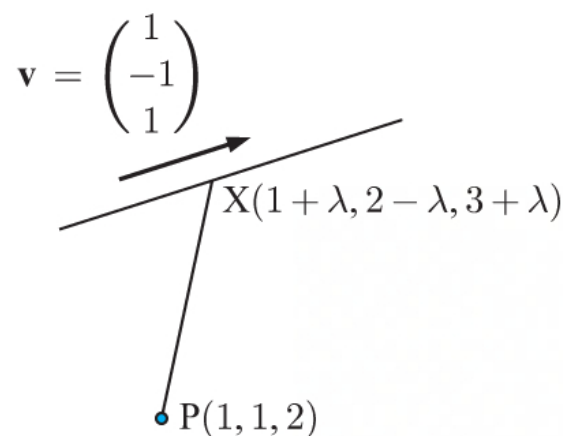
10 a Let  $X(1 + \lambda, 2 - \lambda, 3 + \lambda)$  be any point on the line.

$$\text{Then } \overrightarrow{PX} = \begin{pmatrix} 1 + \lambda - 1 \\ 2 - \lambda - 1 \\ 3 + \lambda - 2 \end{pmatrix} = \begin{pmatrix} \lambda \\ 1 - \lambda \\ 1 + \lambda \end{pmatrix}$$

$$\therefore PX = \sqrt{\lambda^2 + (1 - \lambda)^2 + (1 + \lambda)^2}$$

$$= \sqrt{\lambda^2 + (1 - 2\lambda + \lambda^2) + (1 + 2\lambda + \lambda^2)}$$

$$= \sqrt{3\lambda^2 + 2} \text{ units}$$



b  $PX$  is minimised when  $PX^2 = 3\lambda^2 + 2$  is minimised.

This occurs when  $\lambda = -\frac{b}{2a} = -\frac{0}{6} = 0$ .

c When  $PX$  is minimised,  $\lambda = 0$ .

$$\text{When } \lambda = 0, \overrightarrow{PX} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

$$\overrightarrow{PX} \bullet \mathbf{v} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$= 0 - 1 + 1$$

$$= 0$$

$\therefore \overrightarrow{PX}$  is perpendicular to the line.

**11 a** A direction vector for the line is  $\begin{pmatrix} -2-2 \\ 1-0 \\ 3-3 \end{pmatrix} = \begin{pmatrix} -4 \\ 1 \\ 0 \end{pmatrix}.$

Using  $(2, 0, 3)$  as our fixed point,  $L_2$  has equation  $\mathbf{r} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} + s \begin{pmatrix} -4 \\ 1 \\ 0 \end{pmatrix}, s \in \mathbb{R}.$

**b** Let  $L_3$  have direction vector  $\begin{pmatrix} a \\ b \\ c \end{pmatrix}.$

$L_2$  and  $L_3$  are perpendicular, so  $\begin{pmatrix} -4 \\ 1 \\ 0 \end{pmatrix} \bullet \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$   
 $\therefore -4a + b = 0$   
 $\therefore b = 4a \quad \dots (*)$

$L_1$  and  $L_3$  are perpendicular, so  $\begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix} \bullet \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$   
 $\therefore -a + b + 4c = 0$   
 $\therefore -a + 4a + 4c = 0 \quad \{\text{using } (*)\}$   
 $\therefore 3a = -4c$   
 $\therefore a = -\frac{4}{3}c$   
 $\therefore b = -\frac{16}{3}c$

$\therefore L_3$  has direction vector  $\begin{pmatrix} -\frac{4}{3}c \\ -\frac{16}{3}c \\ c \end{pmatrix}$ , or  $c \begin{pmatrix} -4 \\ -16 \\ 3 \end{pmatrix}, c \in \mathbb{R}.$

$L_3$  passes through  $(0, 0, -2)$ , so its vector equation is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} -4 \\ -16 \\ 3 \end{pmatrix}, \lambda \in \mathbb{R}.$$

$\therefore L_3$  has parametric equations  $x = -4\lambda, y = -16\lambda, z = -2 + 3\lambda$

$$\therefore \lambda = \frac{x}{-4} = \frac{y}{-16} = \frac{z+2}{3}$$

$\therefore L_3$  in Cartesian form is  $\frac{x}{-4} = \frac{y}{-16} = \frac{z+2}{3}.$

**EXERCISE 13B**

- 1  $L_1$  has direction vector  $\mathbf{b}_1 = \begin{pmatrix} 12 \\ 5 \end{pmatrix}$  and  $L_2$  has direction vector  $\mathbf{b}_2 = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$ .

$$\begin{aligned} \text{Now } \cos \theta &= \frac{|\mathbf{b}_1 \bullet \mathbf{b}_2|}{|\mathbf{b}_1| |\mathbf{b}_2|} \\ &= \frac{\left| \begin{pmatrix} 12 \\ 5 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ -4 \end{pmatrix} \right|}{\sqrt{144+25}\sqrt{9+16}} \\ &= \frac{|36 + (-20)|}{13 \times 5} \\ &= \frac{16}{65} \\ \therefore \theta &= \cos^{-1}\left(\frac{16}{65}\right) \\ \therefore \theta &\approx 75.7^\circ \end{aligned}$$

$\therefore$  the angle between  $L_1$  and  $L_2$  is about  $75.7^\circ$ .

- 2  $L_1$  has direction vector  $\begin{pmatrix} 5 \\ -3 \\ 2 \end{pmatrix}$  and  $L_2$  has direction vector  $\begin{pmatrix} 4 \\ 10 \\ 5 \end{pmatrix}$ .

$$\text{Now } \begin{pmatrix} 5 \\ -3 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} 4 \\ 10 \\ 5 \end{pmatrix} = 20 - 30 + 10 = 0$$

$\therefore L_1$  and  $L_2$  are perpendicular.

- 3  $L_1$  has direction vector  $\mathbf{b}_1 = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$  and  $L_2$  has direction vector  $\mathbf{b}_2 = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$ .

$$\begin{aligned} \text{Now } \cos \theta &= \frac{|\mathbf{b}_1 \bullet \mathbf{b}_2|}{|\mathbf{b}_1| |\mathbf{b}_2|} \\ &= \frac{\left| \begin{pmatrix} 4 \\ -3 \end{pmatrix} \bullet \begin{pmatrix} 5 \\ 4 \end{pmatrix} \right|}{\sqrt{16+9}\sqrt{25+16}} \\ &= \frac{|20 + (-12)|}{\sqrt{25}\sqrt{41}} \\ &= \frac{8}{\sqrt{25}\sqrt{41}} \\ \therefore \theta &= \cos^{-1}\left(\frac{8}{\sqrt{25}\sqrt{41}}\right) \\ \therefore \theta &\approx 75.5^\circ \end{aligned}$$

$\therefore$  the angle between  $L_1$  and  $L_2$  is about  $75.5^\circ$ .



$$\mathbf{4} \quad \mathbf{a} \quad L_1: s = \frac{x-8}{3} = \frac{9-y}{16} = \frac{z-10}{7}, \quad s \in \mathbb{R} \quad \text{and} \quad L_2: \mathbf{r}_2 = \begin{pmatrix} 15 \\ 29 \\ 5 \end{pmatrix} + t \begin{pmatrix} 3 \\ 8 \\ -5 \end{pmatrix}, \quad t \in \mathbb{R}$$

$$\begin{aligned} \text{For } L_1: \quad 3s &= x - 8, & 16s &= 9 - y, & 7s &= z - 10 \\ \therefore x &= 8 + 3s, & y &= 9 - 16s, & z &= 10 + 7s \end{aligned}$$

$$\therefore L_1 \text{ has direction vector } \mathbf{b}_1 = \begin{pmatrix} 3 \\ -16 \\ 7 \end{pmatrix} \text{ and } L_2 \text{ has direction vector } \mathbf{b}_2 = \begin{pmatrix} 3 \\ 8 \\ -5 \end{pmatrix}.$$

$$\begin{aligned} \text{Now } \cos \theta &= \frac{|\mathbf{b}_1 \bullet \mathbf{b}_2|}{|\mathbf{b}_1| |\mathbf{b}_2|} \\ &= \frac{\left| \begin{pmatrix} 3 \\ -16 \\ 7 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 8 \\ -5 \end{pmatrix} \right|}{\sqrt{9 + 256 + 49} \sqrt{9 + 64 + 25}} \\ &= \frac{|9 - 128 - 35|}{\sqrt{314} \sqrt{98}} \\ &= \frac{154}{\sqrt{314} \sqrt{98}} \end{aligned}$$

$$\begin{aligned} \therefore \theta &= \cos^{-1} \left( \frac{154}{\sqrt{314} \sqrt{98}} \right) \\ &\approx 28.6^\circ \end{aligned}$$

$\therefore$  the angle between  $L_1$  and  $L_2$  is about  $28.6^\circ$ .

$$\begin{aligned} \mathbf{b} \quad \mathbf{i} \quad \text{Since } L_3 \text{ is perpendicular to } L_1, \quad \begin{pmatrix} 0 \\ -3 \\ a \end{pmatrix} \bullet \begin{pmatrix} 3 \\ -16 \\ 7 \end{pmatrix} &= 0 \\ \therefore 48 + 7a &= 0 \\ \therefore a &= -\frac{48}{7} \end{aligned}$$

ii  $L_2$  has direction vector  $\mathbf{b}_2 = \begin{pmatrix} 3 \\ 8 \\ -5 \end{pmatrix}$  and  $L_3$  has direction vector  $\mathbf{b}_3 = \begin{pmatrix} 0 \\ -3 \\ -\frac{48}{7} \end{pmatrix}$ .

$$\begin{aligned} \text{Now } \cos \theta &= \frac{|\mathbf{b}_2 \cdot \mathbf{b}_3|}{|\mathbf{b}_2| |\mathbf{b}_3|} \\ &= \frac{\left| \begin{pmatrix} 3 \\ 8 \\ -5 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -3 \\ -\frac{48}{7} \end{pmatrix} \right|}{\sqrt{9+64+25} \sqrt{0+9+\frac{2304}{49}}} \\ &= \frac{\left| 0 - 24 + \frac{240}{7} \right|}{\sqrt{98} \sqrt{\frac{2745}{49}}} \\ &= \frac{\frac{72}{7}}{7\sqrt{2} \times \frac{\sqrt{2745}}{7}} \\ &= \frac{72}{7\sqrt{5490}} \\ \therefore \theta &= \cos^{-1} \left( \frac{72}{7\sqrt{5490}} \right) \\ &\approx 82.0^\circ \end{aligned}$$

$\therefore$  the angle between  $L_2$  and  $L_3$  is about  $82.0^\circ$ .

5  $L_1$  has direction vector  $\mathbf{b}_1 = \begin{pmatrix} 0 - (-1) \\ 2 - 5 \\ -2 - 2 \end{pmatrix}$

$$= \begin{pmatrix} 1 \\ -3 \\ -4 \end{pmatrix}$$

$L_2$  is parallel to  $\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$  and hence has direction vector  $\mathbf{b}_2 = \begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix}$ .

$$\begin{aligned} \text{Now } \cos \theta &= \frac{|\mathbf{b}_1 \cdot \mathbf{b}_2|}{|\mathbf{b}_1| |\mathbf{b}_2|} \\ &= \frac{\left| \begin{pmatrix} 1 \\ -3 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix} \right|}{\sqrt{1+9+16} \sqrt{1+9+16}} \\ &= \frac{|1+9-16|}{\sqrt{26} \sqrt{26}} \\ &= \frac{6}{26} \\ &= \frac{3}{13} \\ \therefore \theta &= \cos^{-1} \left( \frac{3}{13} \right) \\ &\approx 76.7^\circ \end{aligned}$$

$\therefore$  the angle between the lines is about  $76.7^\circ$ .

- 6  $L_1$  meets the  $X$ -axis when  $y = z = 0$ .

$$\therefore \frac{x-2}{2} = 1$$

$$\therefore x - 2 = 2$$

$$\therefore x = 4$$

$\therefore L_1$  and  $L_2$  meet on the  $X$ -axis at  $(4, 0, 0)$ .

Now,  $L_1$  has direction vector  $\mathbf{b}_1 = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$

and  $L_2$  has direction vector  $\mathbf{b}_2 = \begin{pmatrix} 5-4 \\ -3-0 \\ 1-0 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}$ .

$$\begin{aligned} \cos \theta &= \frac{|\mathbf{b}_1 \bullet \mathbf{b}_2|}{|\mathbf{b}_1| |\mathbf{b}_2|} \\ &= \frac{\left| \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} \right|}{\sqrt{4+9+1}\sqrt{1+9+1}} \\ &= \frac{|2-9-1|}{\sqrt{14}\sqrt{11}} \\ &= \frac{8}{\sqrt{14}\sqrt{11}} \end{aligned}$$

$$\begin{aligned} \therefore \theta &= \cos^{-1} \left( \frac{8}{\sqrt{14}\sqrt{11}} \right) \\ &\approx 49.9^\circ \end{aligned}$$

$\therefore$  the angle between the lines is about  $49.9^\circ$ .

- 7 a  $x - y = 3$  has gradient  $1 = \frac{1}{1}$   $\therefore$  it has direction vector  $\mathbf{b}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ .

$3x + 2y = 11$  has gradient  $-\frac{3}{2}$   $\therefore$  it has direction vector  $\mathbf{b}_2 = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ .

$$\begin{aligned} \text{Now } \cos \theta &= \frac{|\mathbf{b}_1 \bullet \mathbf{b}_2|}{|\mathbf{b}_1| |\mathbf{b}_2|} \\ &= \frac{\left| \begin{pmatrix} 1 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ -3 \end{pmatrix} \right|}{\sqrt{1+1}\sqrt{4+9}} \\ &= \frac{|2-3|}{\sqrt{2}\sqrt{13}} \\ &= \frac{1}{\sqrt{26}} \end{aligned}$$

$$\therefore \theta = \cos^{-1} \left( \frac{1}{\sqrt{26}} \right) \approx 78.7^\circ$$

$\therefore$  the angle between the lines is about  $78.7^\circ$ .

**b**  $y = x + 2$  has gradient  $1 = \frac{1}{1} \therefore$  it has direction vector  $\mathbf{b}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ .

$y = 1 - 3x$  has gradient  $-3 = \frac{-3}{1} \therefore$  it has direction vector  $\mathbf{b}_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$ .

$$\begin{aligned} \text{Now } \cos \theta &= \frac{|\mathbf{b}_1 \bullet \mathbf{b}_2|}{|\mathbf{b}_1| |\mathbf{b}_2|} \\ &= \frac{\left| \begin{pmatrix} 1 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -3 \end{pmatrix} \right|}{\sqrt{1+1}\sqrt{1+9}} \\ &= \frac{|1-3|}{\sqrt{2}\sqrt{10}} \\ &= \frac{2}{\sqrt{20}} \end{aligned}$$

$$\therefore \theta = \cos^{-1}\left(\frac{2}{\sqrt{20}}\right) \approx 63.4^\circ$$

$\therefore$  the angle between the lines is about  $63.4^\circ$ .

**c**  $y + x = 7$  has gradient  $-1 = \frac{-1}{1} \therefore$  it has direction vector  $\mathbf{b}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ .

$x - 3y + 2 = 0$  has gradient  $\frac{1}{3} \therefore$  it has direction vector  $\mathbf{b}_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ .

$$\begin{aligned} \text{Now } \cos \theta &= \frac{|\mathbf{b}_1 \bullet \mathbf{b}_2|}{|\mathbf{b}_1| |\mathbf{b}_2|} \\ &= \frac{\left| \begin{pmatrix} 1 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right|}{\sqrt{1+1}\sqrt{9+1}} \\ &= \frac{|3-1|}{\sqrt{2}\sqrt{10}} \\ &= \frac{2}{\sqrt{20}} \end{aligned}$$

$$\therefore \theta = \cos^{-1}\left(\frac{2}{\sqrt{20}}\right) \approx 63.4^\circ$$

$\therefore$  the angle between the lines is about  $63.4^\circ$ .

**d**  $y = 2 - x$  has gradient  $-1 = \frac{-1}{1} \therefore$  it has direction vector  $\mathbf{b}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ .

$x - 2y = 7$  has gradient  $\frac{1}{2} \therefore$  it has direction vector  $\mathbf{b}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ .

$$\begin{aligned} \text{Now } \cos \theta &= \frac{|\mathbf{b}_1 \bullet \mathbf{b}_2|}{|\mathbf{b}_1| |\mathbf{b}_2|} \\ &= \frac{\left| \begin{pmatrix} 1 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right|}{\sqrt{1+1}\sqrt{4+1}} \\ &= \frac{|2-1|}{\sqrt{2}\sqrt{5}} \\ &= \frac{1}{\sqrt{10}} \end{aligned}$$

$$\therefore \theta = \cos^{-1}\left(\frac{1}{\sqrt{10}}\right) \approx 71.6^\circ$$

$\therefore$  the angle between the lines is about  $71.6^\circ$ .



## EXERCISE 13C

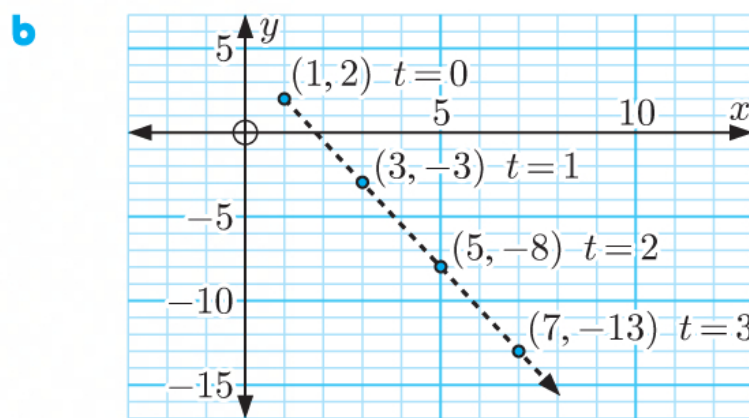
- 1 a  $x(0) = 1$  and  $y(0) = 2$ ,  
 $\therefore$  the initial position is  $(1, 2)$ .

c The velocity vector is  $\begin{pmatrix} 2 \\ -5 \end{pmatrix}$ .

d The speed is  $\left| \begin{pmatrix} 2 \\ -5 \end{pmatrix} \right| = \sqrt{2^2 + (-5)^2}$   
 $= \sqrt{29} \text{ cm s}^{-1}$

e The unit vector with direction  $\begin{pmatrix} 2 \\ -5 \end{pmatrix}$  is  $\frac{1}{\sqrt{29}} \begin{pmatrix} 2 \\ -5 \end{pmatrix}$ .

$\therefore$  the particle travelling with speed  $8 \text{ cm s}^{-1}$  in the opposite direction has velocity vector  
 $-\frac{8}{\sqrt{29}} \begin{pmatrix} 2 \\ -5 \end{pmatrix}$ .



- 2 a i When  $t = 0$ ,  $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \\ 0 \end{pmatrix}$   
 $\therefore$  the initial position of the object is  $(-4, 3, 0)$ .

ii The velocity vector of the object is  $\begin{pmatrix} 12 \\ 5 \\ 6 \end{pmatrix}$ .

iii The speed of the object  $= \sqrt{12^2 + 5^2 + 6^2} = \sqrt{205} \approx 14.3 \text{ m s}^{-1}$

- b i When  $t = 0$ ,  $x = 3$ ,  $y = 0$ ,  $z = 4$   
 $\therefore$  the initial position of the object is  $(3, 0, 4)$ .

ii The velocity vector of the object is  $\begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$ .

iii The speed of the object  $= \sqrt{2^2 + (-1)^2 + (-2)^2} = 3 \text{ m s}^{-1}$

3 a  $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + t \begin{pmatrix} 4 \\ -5 \end{pmatrix}$ ,  $t \geq 0$

b 90 minutes = 1.5 hours

When  $t = 1.5$ ,  $x = 2 + 4(1.5) = 8$  and  $y = 3 - 5(1.5) = -4.5$

$\therefore$  after 90 minutes the boat is at  $(8, -4.5)$ .

c When the boat is at  $(5, -0.75)$

$2 + 4t = 5$  and  $3 - 5t = -0.75$

$\therefore 4t = 3$   $-5t = -3.75$

$\therefore t = 0.75$   $t = 0.75$  ✓

It will take 0.75 hours = 45 minutes for the boat to reach the point  $(5, -0.75)$ .

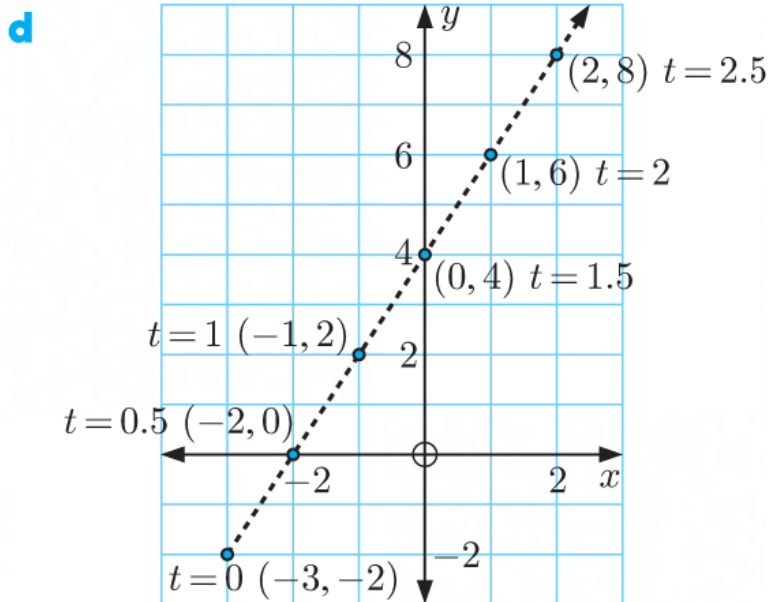
**4 a**  $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ -2 \end{pmatrix} + t \begin{pmatrix} 2 \\ 4 \end{pmatrix}, t \geq 0$

**b** When  $t = 2.5$ ,  $-3 + 2t = -3 + 5 = 2$   
and  $-2 + 4t = -2 + 10 = 8$

So, the position vector is  $\begin{pmatrix} 2 \\ 8 \end{pmatrix}$ .

**c i** The car is due north when  $x = 0$   
 $\therefore -3 + 2t = 0$   
 $\therefore t = 1.5$  seconds

**ii** The car is due west when  $y = 0$   
 $\therefore -2 + 4t = 0$   
 $\therefore t = 0.5$  seconds



**5 a**  $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$  has length  $\sqrt{4^2 + (-3)^2} = 5$   
 $\therefore 30 \begin{pmatrix} 4 \\ -3 \end{pmatrix}$  has length 150  
 $\therefore$  the velocity vector of the speed boat is  $\begin{pmatrix} 120 \\ -90 \end{pmatrix}$ .

**b**  $\begin{pmatrix} -5 \\ 12 \end{pmatrix}$  has length  $\sqrt{(-5)^2 + 12^2} = 13$   
 $\therefore \frac{3}{5} \begin{pmatrix} -5 \\ 12 \end{pmatrix}$  has length 7.8  
 $\therefore$  the velocity vector of the jogger is  $\begin{pmatrix} -3 \\ 7.2 \end{pmatrix}$ .

**c**  $6\mathbf{i} + 7\mathbf{j} = \begin{pmatrix} 6 \\ 7 \end{pmatrix}$  has length  $\sqrt{6^2 + 7^2} = \sqrt{85}$   
 $\therefore \frac{25}{\sqrt{85}} \begin{pmatrix} 6 \\ 7 \end{pmatrix}$  has length 25  
 $\therefore$  the velocity vector of the ferry is  $\begin{pmatrix} \frac{150}{\sqrt{85}} \\ \frac{175}{\sqrt{85}} \end{pmatrix} = \begin{pmatrix} \frac{150\sqrt{85}}{85} \\ \frac{175\sqrt{85}}{85} \end{pmatrix}$   
 $= \begin{pmatrix} \frac{30\sqrt{85}}{17} \\ \frac{35\sqrt{85}}{17} \end{pmatrix}$

**d**  $4\mathbf{i} + 7\mathbf{j} + \mathbf{k} = \begin{pmatrix} 4 \\ 7 \\ 1 \end{pmatrix}$  has length  $\sqrt{4^2 + 7^2 + 1^2} = \sqrt{66}$

$\therefore \frac{33}{\sqrt{66}} \begin{pmatrix} 4 \\ 7 \\ 1 \end{pmatrix} = \frac{\sqrt{66}}{2} \begin{pmatrix} 4 \\ 7 \\ 1 \end{pmatrix}$  has length 33

$\therefore$  the velocity vector of the hot air balloon is  $\begin{pmatrix} 2\sqrt{66} \\ \frac{7\sqrt{66}}{2} \\ \frac{\sqrt{66}}{2} \end{pmatrix}$ .

**e**  $-2\mathbf{i} + 5\mathbf{j} - 14\mathbf{k}$  has length  $\sqrt{(-2)^2 + 5^2 + (-14)^2} = \sqrt{225} = 15$

$\therefore 6 \begin{pmatrix} -2 \\ 5 \\ -14 \end{pmatrix}$  has length 90

$\therefore$  the velocity vector of the swooping eagle is  $\begin{pmatrix} -12 \\ 30 \\ -84 \end{pmatrix}$ .

**6** Yacht A:  $\begin{pmatrix} x_A \\ y_A \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix} + t \begin{pmatrix} 1 \\ -2 \end{pmatrix}$  Yacht B:  $\begin{pmatrix} x_B \\ y_B \end{pmatrix} = \begin{pmatrix} 1 \\ -8 \end{pmatrix} + t \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad t \geq 0$

**a** When  $t = 0$ ,  $\begin{pmatrix} x_A \\ y_A \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$   $\therefore$  A is initially at (4, 5)

and  $\begin{pmatrix} x_B \\ y_B \end{pmatrix} = \begin{pmatrix} 1 \\ -8 \end{pmatrix}$   $\therefore$  B is initially at (1, -8).

**b** The velocity vector of A is  $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ . The velocity vector of B is  $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ .

**c** Speed of A =  $\sqrt{1^2 + (-2)^2} = \sqrt{5} \text{ km h}^{-1}$ . Speed of B =  $\sqrt{2^2 + 1^2} = \sqrt{5} \text{ km h}^{-1}$ .  
The speed of each yacht does not depend on  $t$  and is therefore constant.

**d** A has direction vector  $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$  and B has direction vector  $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ .

Since  $\begin{pmatrix} 1 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 2 - 2 = 0$ , the paths of the yachts are at right angles to each other.

**7 a** P's torpedo has position  $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} -5 \\ 4 \end{pmatrix} + t \begin{pmatrix} 3 \\ -1 \end{pmatrix}$ , where  $t$  is the time in minutes after 1:34 pm and  $t \geq 0$ .

$\therefore x_1(t) = -5 + 3t, \quad y_1(t) = 4 - t.$

**b** Speed of P's torpedo =  $\sqrt{3^2 + (-1)^2} = \sqrt{10} \text{ km min}^{-1}$

- c** Q fires its torpedo after  $a$  minutes. At time  $t$ , the torpedo has travelled for  $(t - a)$  minutes.

$$\text{Q's torpedo has position } \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 15 \\ 7 \end{pmatrix} + (t - a) \begin{pmatrix} -4 \\ -3 \end{pmatrix}, \quad t \geq a$$

$$\therefore x_2(t) = 15 - 4(t - a), \quad y_2(t) = 7 - 3(t - a)$$

- d** Let  $\theta$  be the angle between the paths of the torpedoes.

$$\begin{aligned} \cos \theta &= \frac{\left| \begin{pmatrix} 3 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} -4 \\ -3 \end{pmatrix} \right|}{\left| \begin{pmatrix} 3 \\ -1 \end{pmatrix} \right| \left| \begin{pmatrix} -4 \\ -3 \end{pmatrix} \right|} \\ &= \frac{|-12 + 3|}{\sqrt{3^2 + (-1)^2} \sqrt{(-4)^2 + (-3)^2}} \\ &= \frac{9}{5\sqrt{10}} \end{aligned}$$

$$\therefore \theta = \cos^{-1} \left( \frac{9}{5\sqrt{10}} \right) \approx 55.3^\circ$$

- e** They meet when  $x_1(t) = x_2(t)$  and  $y_1(t) = y_2(t)$

$$\therefore -5 + 3t = 15 - 4(t - a) \quad \text{and} \quad 4 - t = 7 - 3(t - a)$$

$$\therefore 7t - 4a = 20 \quad \dots (1) \quad \text{and} \quad 2t - 3a = 3 \quad \dots (2)$$

$$\begin{aligned} \text{Solving simultaneously,} \quad & 21t - 12a = 60 \quad \{3 \times (1)\} \\ & -8t + 12a = -12 \quad \{-4 \times (2)\} \\ \hline & \therefore 13t = 48 \end{aligned}$$

$$\therefore t = \frac{48}{13} \quad \text{and} \quad 7\left(\frac{48}{13}\right) - 4a = 20$$

$$\therefore t \approx 3.6923 \quad \therefore 5.8462 = 4a$$

$$\therefore t \approx 3 \text{ min } 41.54 \text{ sec} \quad \therefore a \approx 1.4615 \approx 1 \text{ min } 27.7 \text{ sec}$$

At  $t = 0$ , the time is 1:34 pm.

$\therefore$  Q fired its torpedo at 1:35:28 pm and the explosion occurred at 1:37:42 pm.

- 8 a** The direction vector  $2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$  has length  $\sqrt{2^2 + (-2)^2 + 1^2} = 3$ .

The cable car moves with speed  $4.5 \text{ m s}^{-1}$ .

$$\therefore \text{the velocity vector of the cable car is } 1.5 \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ 1.5 \end{pmatrix}.$$



- b** The cable car has position vector  $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 10 \\ 3 \\ 0 \end{pmatrix} + t \begin{pmatrix} 3 \\ -3 \\ 1.5 \end{pmatrix}, \quad t \geq 0.$

$$\begin{aligned} \text{When } t = 30, \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \begin{pmatrix} 10 \\ 3 \\ 0 \end{pmatrix} + 30 \begin{pmatrix} 3 \\ -3 \\ 1.5 \end{pmatrix} \\ &= \begin{pmatrix} 10 \\ 3 \\ 0 \end{pmatrix} + \begin{pmatrix} 90 \\ -90 \\ 45 \end{pmatrix} \\ &= \begin{pmatrix} 100 \\ -87 \\ 45 \end{pmatrix} \end{aligned}$$

$\therefore$  after 30 seconds the cable car is at  $(100, -87, 45)$ .

- c** After  $t$  seconds, the  $x$ -coordinate of the cable car is  $x = 10 + 3t$ .

$$550 = 10 + 3t$$

$$\therefore 3t = 540$$

$$\therefore t = 180$$

$\therefore$  the cable car ride lasts 180 seconds.

$$\begin{aligned} \text{When } t = 180, \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \begin{pmatrix} 10 \\ 3 \\ 0 \end{pmatrix} + 180 \begin{pmatrix} 3 \\ -3 \\ 1.5 \end{pmatrix} \\ &= \begin{pmatrix} 550 \\ -537 \\ 270 \end{pmatrix} \end{aligned}$$

$\therefore$  the cable car ride ends at  $\begin{pmatrix} 550 \\ -537 \\ 270 \end{pmatrix}$  and begins at  $\begin{pmatrix} 10 \\ 3 \\ 0 \end{pmatrix}.$

$$\begin{aligned} \therefore \text{ the length of the cable car ride} &= \sqrt{(550 - 10)^2 + (-537 - 3)^2 + (270 - 0)^2} \\ &= \sqrt{540^2 + (-540)^2 + 270^2} \\ &= 810 \text{ m} \end{aligned}$$

- d** The cable car has direction vector  $\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ .

Consider the direction vector  $\begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix}$  which has the same direction in the horizontal  $XY$ -plane, but no vertical component.

$$\begin{aligned} \cos \theta &= \frac{\left| \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix} \right|}{\left| \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \right| \left| \begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix} \right|} \\ &= \frac{|4 + 4 + 0|}{\sqrt{2^2 + (-2)^2 + 1^2} \sqrt{2^2 + (-2)^2 + 0^2}} \\ &= \frac{8}{3\sqrt{8}} \end{aligned}$$

$$\therefore \theta = \cos^{-1} \left( \frac{8}{3\sqrt{8}} \right) \approx 19.5^\circ$$

$\therefore$  the cable car travels at an angle of about  $19.5^\circ$  to the horizontal.

**9 a**  $\vec{AB} = \begin{pmatrix} 3-6 \\ 10-9 \\ 2.5-3 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ -0.5 \end{pmatrix}$

**b**  $|\vec{AB}| = \sqrt{(-3)^2 + 1^2 + (-0.5)^2}$   
 $= \sqrt{10.25} \text{ km}$

The helicopter travels  $\sqrt{10.25} \text{ km}$  in 10 minutes.

$\therefore$  the helicopter's speed is  $6 \times \sqrt{10.25} \approx 19.2 \text{ km h}^{-1}$ .

**c**  $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 9 \\ 3 \end{pmatrix} + t \begin{pmatrix} -3 \\ 1 \\ -0.5 \end{pmatrix}, \quad t \in \mathbb{R}$

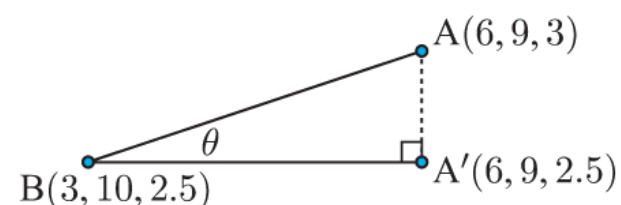
**d** If  $z = 0$ ,  $3 + (-0.5)t = 0$   
 $\therefore t = 6$

$t = 1$  represents 10 minutes, so  $t = 6$  represents 60 minutes.

$\therefore$  the helicopter lands on the helipad after 1 hour.

- e** Let  $A'$  have coordinates  $(6, 9, 2.5)$ , so  $A'$  is directly underneath  $A$  and has the same  $z$ -coordinate as  $B$ .

$\therefore \widehat{ABA'} = \theta$  is the angle the helicopter is flying at to the horizontal in the diagram alongside.



$$\begin{aligned}
 \text{Now } \overrightarrow{A'B} &= \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} \\
 \therefore \cos \theta &= \frac{\overrightarrow{AB} \bullet \overrightarrow{A'B}}{|\overrightarrow{AB}| |\overrightarrow{A'B}|} \\
 &= \frac{(-3)^2 + (1)^2 + 0}{\sqrt{(-3)^2 + 1^2 + (0.5)^2} \sqrt{(-3)^2 + 1^2}} \\
 &= \frac{2\sqrt{10}}{\sqrt{41}} \\
 \therefore \theta &= \cos^{-1} \left( \frac{2\sqrt{10}}{\sqrt{41}} \right) \\
 &\approx 8.98^\circ
 \end{aligned}$$

### EXERCISE 13D

- 1 a Let N be the point on the line closest to P.  
N has coordinates  $(2+t, 3+2t)$  for some  $t \in \mathbb{R}$ .

$$\overrightarrow{PN} \text{ is } \begin{pmatrix} 2+t-3 \\ 3+2t-2 \end{pmatrix} = \begin{pmatrix} t-1 \\ 2t+1 \end{pmatrix}.$$

The distance between P and the line is minimised when  $\overrightarrow{PN} \bullet \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 0$ .

$$\therefore \begin{pmatrix} t-1 \\ 2t+1 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 0$$

$$\therefore (t-1) + 2(2t+1) = 0$$

$$\therefore t-1+4t+2=0$$

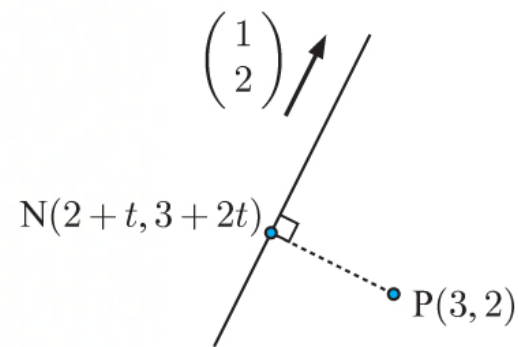
$$\therefore 5t = -1$$

$$\therefore t = -\frac{1}{5}$$

$$\begin{aligned}
 \text{Thus } \overrightarrow{PN} &= \begin{pmatrix} -\frac{1}{5} - 1 \\ -\frac{2}{5} + 1 \end{pmatrix} = \begin{pmatrix} -\frac{6}{5} \\ \frac{3}{5} \end{pmatrix} \\
 &= \frac{3}{5} \begin{pmatrix} -2 \\ 1 \end{pmatrix}
 \end{aligned}$$

$$\begin{aligned}
 \text{and } |\overrightarrow{PN}| &= \frac{3}{5} \left| \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right| = \frac{3}{5} \sqrt{(-2)^2 + 1^2} \\
 &= \frac{3\sqrt{5}}{5} \text{ units}
 \end{aligned}$$

$\therefore$  the shortest distance from P to the line is  $\frac{3\sqrt{5}}{5}$  units.



- b** Let N be the point on the line closest to Q.

N has coordinates  $(t, 1 - t)$  for some  $t \in \mathbb{R}$ .

$$\overrightarrow{QN} \text{ is } \begin{pmatrix} t - (-1) \\ 1 - t - 1 \end{pmatrix} = \begin{pmatrix} t + 1 \\ -t \end{pmatrix}.$$

The distance between Q and the line is minimised when  $\overrightarrow{QN} \bullet \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0$ .

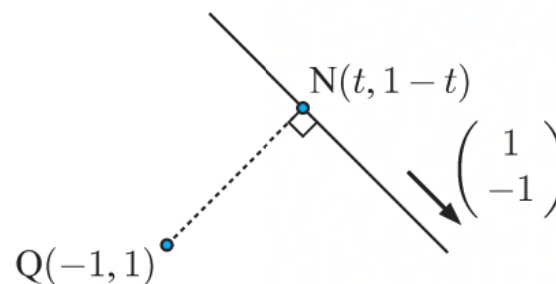
$$\therefore \begin{pmatrix} t + 1 \\ -t \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0$$

$$\therefore (t + 1) + (-1)(-t) = 0$$

$$\therefore t + 1 + t = 0$$

$$\therefore 2t = -1$$

$$\therefore t = -\frac{1}{2}$$



$$\begin{aligned} \text{Thus } \overrightarrow{QN} &= \begin{pmatrix} -\frac{1}{2} + 1 \\ -(-\frac{1}{2}) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \text{and } |\overrightarrow{QN}| &= \frac{1}{2} \left| \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right| = \frac{1}{2} \sqrt{1^2 + 1^2} \\ &= \frac{\sqrt{2}}{2} \text{ units} \end{aligned}$$

$\therefore$  the shortest distance from Q to the line is  $\frac{\sqrt{2}}{2}$  units.

- c** Let N be the point on the line closest to R.

N has coordinates  $(2 + s, 3 - s)$  for some  $s \in \mathbb{R}$ .

$$\overrightarrow{RN} \text{ is } \begin{pmatrix} 2 + s - (-3) \\ 3 - s - (-1) \end{pmatrix} = \begin{pmatrix} s + 5 \\ 4 - s \end{pmatrix}.$$

The distance from R to the line is minimised when  $\overrightarrow{RN} \bullet \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0$ .

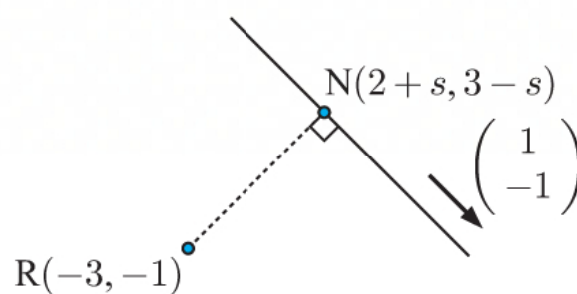
$$\therefore \begin{pmatrix} s + 5 \\ 4 - s \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0$$

$$\therefore (s + 5) + (-1)(4 - s) = 0$$

$$\therefore s + 5 - 4 + s = 0$$

$$\therefore 2s = -1$$

$$\therefore s = -\frac{1}{2}$$



$$\begin{aligned} \text{Thus } \overrightarrow{RN} &= \begin{pmatrix} -\frac{1}{2} + 5 \\ 4 - (-\frac{1}{2}) \end{pmatrix} = \begin{pmatrix} \frac{9}{2} \\ \frac{9}{2} \end{pmatrix} \\ &= \frac{9}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \text{and } |\overrightarrow{RN}| &= \frac{9}{2} \left| \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right| = \frac{9}{2} \sqrt{1^2 + 1^2} \\ &= \frac{9\sqrt{2}}{2} \text{ units} \end{aligned}$$

$\therefore$  the shortest distance from R to the line is  $\frac{9\sqrt{2}}{2}$  units.



- d** Let N be the point on the line closest to S.

N has coordinates  $(2 + 3t, 5 - 7t)$  for some  $t \in \mathbb{R}$ .

$$\overrightarrow{SN} \text{ is } \begin{pmatrix} 2 + 3t - 5 \\ 5 - 7t - (-2) \end{pmatrix} = \begin{pmatrix} 3t - 3 \\ 7 - 7t \end{pmatrix}.$$

The distance from S to the line is minimised when  $\overrightarrow{SN} \bullet \begin{pmatrix} 3 \\ -7 \end{pmatrix} = 0$ .

$$\therefore \begin{pmatrix} 3t - 3 \\ 7 - 7t \end{pmatrix} \bullet \begin{pmatrix} 3 \\ -7 \end{pmatrix} = 0$$

$$\therefore 3(3t - 3) - 7(7 - 7t) = 0$$

$$\therefore 9t - 9 - 49 + 49t = 0$$

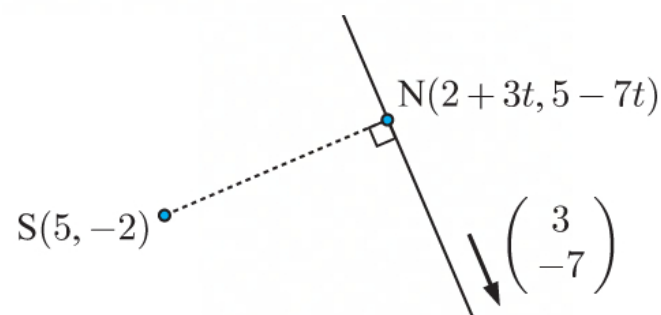
$$\therefore 58t = -58$$

$$\therefore t = -1$$

$$\text{Thus } \overrightarrow{SN} = \begin{pmatrix} 3(-1) - 3 \\ 7 - 7(-1) \end{pmatrix} = \begin{pmatrix} -6 \\ 14 \end{pmatrix}$$

$$\therefore |\overrightarrow{SN}| = 0$$

$\therefore$  S actually lies on the line, and the shortest distance is 0 units.



- 2 a**  $6\mathbf{i} - 6\mathbf{j}$

- b**  $\begin{pmatrix} -3 \\ 4 \end{pmatrix}$  has length  $\sqrt{(-3)^2 + 4^2} = \sqrt{25} = 5$  units

As the speed is  $10 \text{ km h}^{-1}$ , the liner has velocity vector  $2\begin{pmatrix} -3 \\ 4 \end{pmatrix} = \begin{pmatrix} -6 \\ 8 \end{pmatrix}$ .

$\therefore$  the liner has position vector  $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6 \\ -6 \end{pmatrix} + t\begin{pmatrix} -6 \\ 8 \end{pmatrix} = \begin{pmatrix} 6 - 6t \\ -6 + 8t \end{pmatrix}, t \geq 0$ .

- c** The liner is due east when  $y = 0$

$$\therefore -6 + 8t = 0$$

$$\therefore \text{at } t = \frac{3}{4} \text{ hours}$$

- d** The liner L is nearest the fishing boat O when  $\overrightarrow{OL} \perp \begin{pmatrix} -3 \\ 4 \end{pmatrix}$

$$\therefore \overrightarrow{OL} \bullet \begin{pmatrix} -3 \\ 4 \end{pmatrix} = 0$$

$$\therefore \begin{pmatrix} 6 - 6t \\ -6 + 8t \end{pmatrix} \bullet \begin{pmatrix} -3 \\ 4 \end{pmatrix} = 0$$

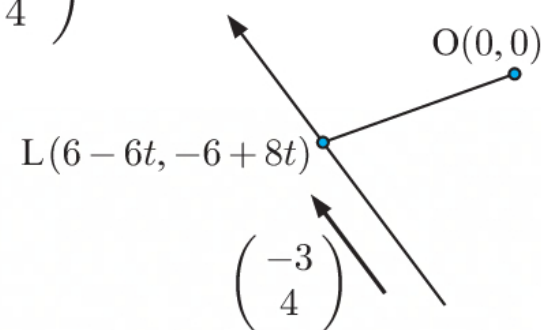
$$\therefore (-18 + 18t) + (-24 + 32t) = 0$$

$$\therefore 50t = 42$$

$$\therefore t = 0.84 \text{ hours} = 50.4 \text{ minutes}$$

$$\text{When } t = 0.84, \overrightarrow{OL} = \begin{pmatrix} 6 - 6(0.84) \\ -6 + 8(0.84) \end{pmatrix} = \begin{pmatrix} 0.96 \\ 0.72 \end{pmatrix}$$

$\therefore$  the liner is closest to the fishing boat after 0.84 hours or 50.4 minutes, when it is at  $(0.96, 0.72)$ .



**3 a**  $|\mathbf{b}| = \sqrt{(-3)^2 + (-1)^2} = \sqrt{10}$

As the speed is  $40\sqrt{10} \text{ km h}^{-1}$ , the velocity vector is  $40 \begin{pmatrix} -3 \\ -1 \end{pmatrix} = \begin{pmatrix} -120 \\ -40 \end{pmatrix}$ .

**b**  $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 200 \\ 100 \end{pmatrix} + t \begin{pmatrix} -120 \\ -40 \end{pmatrix}, t \geq 0 \quad \{t = 0 \text{ at } 12:00 \text{ noon}\}$

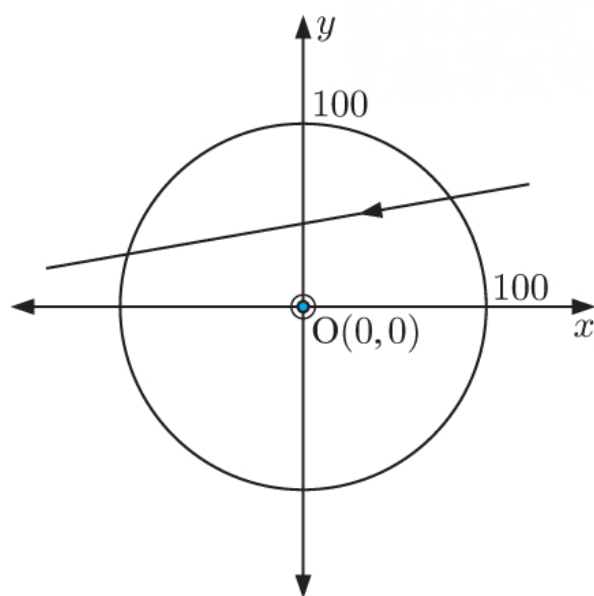
**c** At 1:00 pm,  $t = 1$  and  $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 200 - 120 \\ 100 - 40 \end{pmatrix} = \begin{pmatrix} 80 \\ 60 \end{pmatrix}$

$\therefore$  the aircraft is at (80, 60).

**d** The distance from  $O(0, 0)$  to (80, 60) is  $\left| \begin{pmatrix} 80 \\ 60 \end{pmatrix} \right| = \sqrt{80^2 + 60^2} = 100 \text{ km}$ ,

which is when it becomes visible to radar at 1:00 pm. {within 100 km of  $O(0, 0)$ }

**e**



A general point on the path is

$P(200 - 120t, 100 - 40t)$ .

Now  $\overrightarrow{OP} = \begin{pmatrix} 200 - 120t \\ 100 - 40t \end{pmatrix}$ ,

and for the closest point  $\overrightarrow{OP} \bullet \begin{pmatrix} -3 \\ -1 \end{pmatrix} = 0$

$\therefore -3(200 - 120t) - 1(100 - 40t) = 0$

$\therefore -700 + 400t = 0$

$\therefore t = \frac{7}{4} = 1\frac{3}{4} \text{ hours}$

$\therefore$  the time when the aircraft is closest is 1:45 pm, and at this time

$\overrightarrow{OP} = \begin{pmatrix} 200 - 120(\frac{7}{4}) \\ 100 - 40(\frac{7}{4}) \end{pmatrix} = \begin{pmatrix} -10 \\ 30 \end{pmatrix}$

$\therefore d_{\min} = \sqrt{(-10)^2 + 30^2}$

$= \sqrt{1000}$

$= 10\sqrt{10} \text{ km}$

**f** The aircraft disappears from radar when  $|\overrightarrow{OP}| = 100$  and  $t > 1\frac{3}{4}$

$\therefore \sqrt{(200 - 120t)^2 + (100 - 40t)^2} = 100$

$\therefore 40\,000 - 48\,000t + 14\,400t^2 + 10\,000 - 8000t + 1600t^2 = 10\,000$

$\therefore 16\,000t^2 - 56\,000t + 40\,000 = 0$

$\therefore 2t^2 - 7t + 5 = 0$

$\therefore (2t - 5)(t - 1) = 0$

$\therefore t = \frac{5}{2} \quad \{\text{as } t > 1\frac{3}{4}\}$

So, the aircraft disappears from the radar screen  $2\frac{1}{2}$  hours after noon, or at 2:30 pm.

- 4 a** The direction vector of the line is  $\mathbf{b} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$ .

Let the point  $(3, 0, -1)$  be P, and  $A(2 + 3t, -1 + 2t, 4 + t)$  be any point on the line.

$$\therefore \overrightarrow{PA} = \begin{pmatrix} 2 + 3t - 3 \\ -1 + 2t - 0 \\ 4 + t - (-1) \end{pmatrix} = \begin{pmatrix} -1 + 3t \\ -1 + 2t \\ 5 + t \end{pmatrix}$$

Now  $\overrightarrow{PA}$  and  $\mathbf{b}$  are perpendicular, so  $\overrightarrow{PA} \bullet \mathbf{b} = 0$

$$\therefore \begin{pmatrix} -1 + 3t \\ -1 + 2t \\ 5 + t \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = 0$$

$$\therefore -3 + 9t - 2 + 4t + 5 + t = 0$$

$$\therefore 14t = 0$$

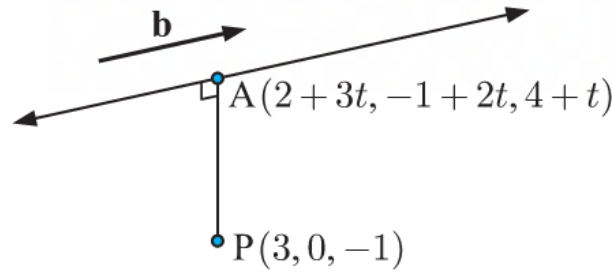
$$\therefore t = 0$$

Substituting  $t = 0$  into the parametric equations, the foot of the perpendicular is  $(2, -1, 4)$ .

- b** When  $t = 0$ ,  $\overrightarrow{PA} = \begin{pmatrix} -1 \\ -1 \\ 5 \end{pmatrix}$

$$\begin{aligned} \therefore |\overrightarrow{PA}| &= \sqrt{1 + 1 + 25} \\ &= \sqrt{27} \text{ units} \end{aligned}$$

So, the shortest distance from the point to the line is  $\sqrt{27}$  units.



- 5 a** The line has direction vector  $\mathbf{b} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$ .

Let the point  $(1, 1, 3)$  be P and  $A(1 + 2t, -1 + 3t, 2 + t)$  be any point on the line.

$$\therefore \overrightarrow{PA} = \begin{pmatrix} 1 + 2t - 1 \\ -1 + 3t - 1 \\ 2 + t - 3 \end{pmatrix} = \begin{pmatrix} 2t \\ -2 + 3t \\ -1 + t \end{pmatrix}$$

Now  $\overrightarrow{PA}$  and  $\mathbf{b}$  are perpendicular, so  $\overrightarrow{PA} \bullet \mathbf{b} = 0$

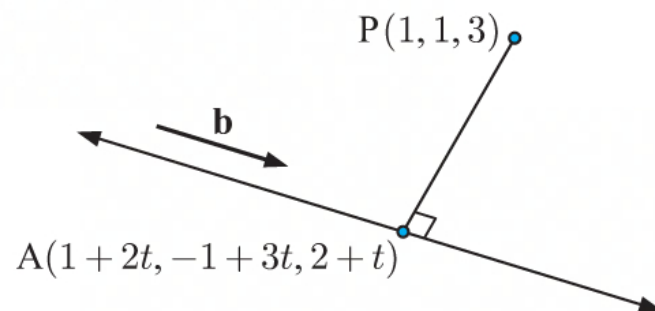
$$\therefore \begin{pmatrix} 2t \\ -2 + 3t \\ -1 + t \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = 0$$

$$\therefore 4t - 6 + 9t - 1 + t = 0$$

$$\therefore 14t = 7$$

$$\therefore t = \frac{1}{2}$$

Substituting  $t = \frac{1}{2}$  into the parametric equations, the foot of the perpendicular is  $(2, \frac{1}{2}, \frac{5}{2})$ .



**b** When  $t = \frac{1}{2}$ ,  $\overrightarrow{PA} = \begin{pmatrix} 1 \\ -2 + \frac{3}{2} \\ -1 + \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$

$$\therefore |\overrightarrow{PA}| = \sqrt{1 + \frac{1}{4} + \frac{1}{4}} = \sqrt{\frac{3}{2}} \text{ units}$$

So, the shortest distance from the point to the line is  $\sqrt{\frac{3}{2}}$  units.

**6 a** The diver swims with direction vector  $\mathbf{b} = \begin{pmatrix} 7 - 12 \\ -15 - 25 \\ 0 - (-20) \end{pmatrix} = \begin{pmatrix} -5 \\ -40 \\ 20 \end{pmatrix}$ .

$$|\mathbf{b}| = \sqrt{(-5)^2 + (-40)^2 + 20^2} = 45$$

As the speed is  $0.9 \text{ m s}^{-1}$ , the velocity vector is  $\frac{0.9}{45} \begin{pmatrix} -5 \\ -40 \\ 20 \end{pmatrix} = \begin{pmatrix} -0.1 \\ -0.8 \\ 0.4 \end{pmatrix}$ .

**b i** Let  $D(12 - 0.1t, 25 - 0.8t, -20 + 0.4t)$  be any point on the path of the diver.

If the octopus is at  $P(12, -8, -5)$ ,  $\overrightarrow{PD} = \begin{pmatrix} 12 - 0.1t - 12 \\ 25 - 0.8t - (-8) \\ -20 + 0.4t - (-5) \end{pmatrix}$

$$= \begin{pmatrix} -0.1t \\ 33 - 0.8t \\ -15 + 0.4t \end{pmatrix}$$

If D is the closest point on the line to P, then

$$\begin{pmatrix} -0.1t \\ 33 - 0.8t \\ -15 + 0.4t \end{pmatrix} \cdot \begin{pmatrix} -0.1 \\ -0.8 \\ 0.4 \end{pmatrix} = 0$$

$$\therefore -0.1(-0.1t) - 0.8(33 - 0.8t) + 0.4(-15 + 0.4t) = 0$$

$$\therefore 0.01t - 26.4 + 0.64t - 6 + 0.16t = 0$$

$$\therefore 0.81t = 32.4$$

$$\therefore t = 40$$

$\therefore$  the diver is closest to the octopus after 40 seconds.

**ii** When  $t = 40$ , the diver is at  $(12 - 0.1(40), 25 - 0.8(40), -20 + 0.4(40))$   
 $= (8, -7, -4)$

The distance between  $(8, -7, -4)$  and  $(12, -8, -5)$

$$= \sqrt{(12 - 8)^2 + (-8 - (-7))^2 + (-5 - (-4))^2}$$

$$= \sqrt{4^2 + (-1)^2 + (-1)^2}$$

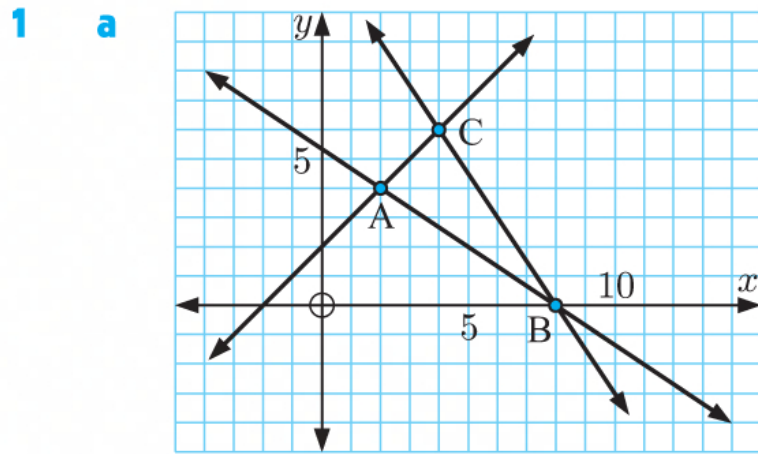
$$= \sqrt{18}$$

$$= 3\sqrt{2} \text{ m}$$

$\therefore$  the shortest distance from the diver to the octopus is  $3\sqrt{2} \text{ m}$ , which occurs at  $t = 40$  seconds.



## EXERCISE 13E



b  $A(2, 4), B(8, 0), C(4, 6)$

c  $BC = \sqrt{(4-8)^2 + (6-0)^2} = \sqrt{16+36}$   
 $= \sqrt{52} \text{ units}$

$AB = \sqrt{(8-2)^2 + (0-4)^2} = \sqrt{36+16}$   
 $= \sqrt{52} \text{ units}$

$\therefore BC = AB$  and so triangle ABC is isosceles.

d (AB) and (AC) meet at A.

$$\therefore \begin{pmatrix} -1 \\ 6 \end{pmatrix} + r \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} + s \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\therefore \begin{pmatrix} 3r - s \\ -2r - s \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$$

$$\therefore 3r - s = 1 \quad \dots (1)$$

$$-2r - s = -4 \quad \dots (2)$$

$$\therefore 3r - s = 1 \quad \{(1)\}$$

$$2r + s = 4 \quad \{- (2)\}$$

$$\therefore 5r = 5$$

$$\therefore r = 1$$

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 6 \end{pmatrix} + \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} \quad \checkmark$$

(AC) and (BC) meet at C.

$$\therefore \begin{pmatrix} 0 \\ 2 \end{pmatrix} + s \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 10 \\ -3 \end{pmatrix} + t \begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

$$\therefore \begin{pmatrix} s + 2t \\ s - 3t \end{pmatrix} = \begin{pmatrix} 10 \\ -5 \end{pmatrix}$$

$$\therefore s + 2t = 10 \quad \dots (1)$$

$$s - 3t = -5 \quad \dots (2)$$

$$\therefore s + 2t = 10 \quad \{(1)\}$$

$$-s + 3t = 5 \quad \{- (2)\}$$

$$\therefore 5t = 15$$

$$\therefore t = 3$$

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 10 \\ -3 \end{pmatrix} + 3 \begin{pmatrix} -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix} \quad \checkmark$$

(AB) and (BC) meet at B.

$$\therefore \begin{pmatrix} -1 \\ 6 \end{pmatrix} + r \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 10 \\ -3 \end{pmatrix} + t \begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

$$\therefore \begin{pmatrix} 3r + 2t \\ -2r - 3t \end{pmatrix} = \begin{pmatrix} 11 \\ -9 \end{pmatrix}$$

$$\therefore 3r + 2t = 11 \quad \dots (1)$$

$$-2r - 3t = -9 \quad \dots (2)$$

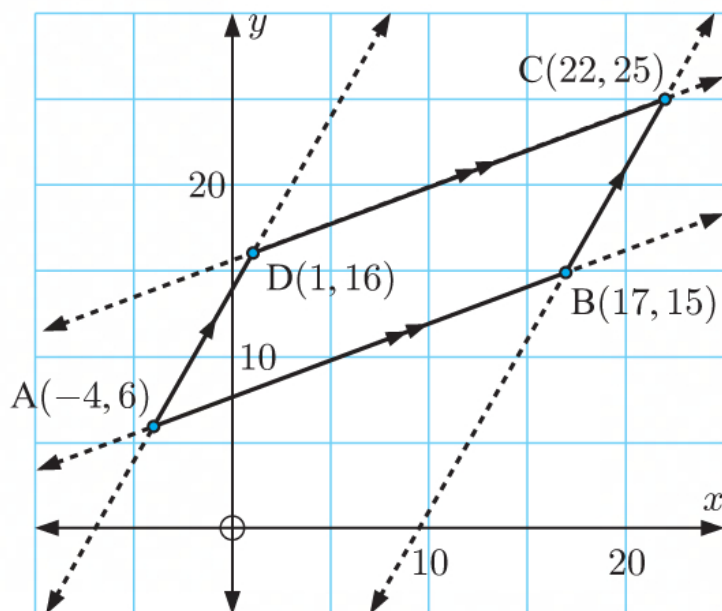
$$\therefore 9r + 6t = 33 \quad \{3 \times (1)\}$$

$$-4r - 6t = -18 \quad \{2 \times (2)\}$$

$$\therefore 5r = 15$$

$$\therefore r = 3$$

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 6 \end{pmatrix} + 3 \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 8 \\ 0 \end{pmatrix} \quad \checkmark$$

**2 a****b**  $A(-4, 6)$ ,  $B(17, 15)$ ,  $C(22, 25)$ ,  $D(1, 16)$ **c** (AB) and (AD) meet at A.

$$\therefore \begin{pmatrix} -4 \\ 6 \end{pmatrix} + r \begin{pmatrix} 7 \\ 3 \end{pmatrix} = \begin{pmatrix} -4 \\ 6 \end{pmatrix} + s \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\therefore \begin{pmatrix} 7r - s \\ 3r - 2s \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore 7r - s = 0 \quad \dots (*)$$

$$\text{and } 3r - 2s = 0$$

$$\underline{-14r + 2s = 0 \quad \{-2 \times (*)\}}$$

$$\therefore -11r = 0$$

$$\therefore r = 0$$

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -4 \\ 6 \end{pmatrix} \quad \checkmark$$

(CD) and (CB) meet at C.

$$\therefore \begin{pmatrix} 22 \\ 25 \end{pmatrix} + t \begin{pmatrix} -7 \\ -3 \end{pmatrix} = \begin{pmatrix} 22 \\ 25 \end{pmatrix} + u \begin{pmatrix} -1 \\ -2 \end{pmatrix}$$

$$\therefore \begin{pmatrix} -7t + u \\ -3t + 2u \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore -7t + u = 0 \quad \dots (*)$$

$$\text{and } -3t + 2u = 0$$

$$\underline{14t - 2u = 0 \quad \{-2 \times (*)\}}$$

$$\therefore 11t = 0$$

$$\therefore t = 0$$

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 22 \\ 25 \end{pmatrix} \quad \checkmark$$

(AB) and (CB) meet at B.

$$\therefore \begin{pmatrix} -4 \\ 6 \end{pmatrix} + r \begin{pmatrix} 7 \\ 3 \end{pmatrix} = \begin{pmatrix} 22 \\ 25 \end{pmatrix} + u \begin{pmatrix} -1 \\ -2 \end{pmatrix}$$

$$\therefore \begin{pmatrix} 7r + u \\ 3r + 2u \end{pmatrix} = \begin{pmatrix} 26 \\ 19 \end{pmatrix}$$

$$\therefore 7r + u = 26 \quad \dots (*)$$

$$\text{and } 3r + 2u = 19$$

$$\underline{-14r - 2u = -52 \quad \{-2 \times (*)\}}$$

$$\therefore -11r = -33$$

$$\therefore r = 3$$

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -4 \\ 6 \end{pmatrix} + 3 \begin{pmatrix} 7 \\ 3 \end{pmatrix} = \begin{pmatrix} 17 \\ 15 \end{pmatrix} \quad \checkmark$$

(AD) and (CD) meet at D.

$$\therefore \begin{pmatrix} -4 \\ 6 \end{pmatrix} + s \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 22 \\ 25 \end{pmatrix} + t \begin{pmatrix} -7 \\ -3 \end{pmatrix}$$

$$\therefore \begin{pmatrix} s + 7t \\ 2s + 3t \end{pmatrix} = \begin{pmatrix} 26 \\ 19 \end{pmatrix}$$

$$\therefore s + 7t = 26 \quad \dots (*)$$

$$\text{and } 2s + 3t = 19$$

$$\underline{-2s - 14t = -52 \quad \{-2 \times (*)\}}$$

$$\therefore -11t = -33$$

$$\therefore t = 3$$

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 22 \\ 25 \end{pmatrix} + 3 \begin{pmatrix} -7 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ 16 \end{pmatrix} \quad \checkmark$$

**3 a** (AB) and (AC) meet at A.

$$\therefore \begin{pmatrix} 0 \\ 2 \end{pmatrix} + r \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \end{pmatrix} + t \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\therefore \begin{pmatrix} 2r - t \\ r + t \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

$$\therefore 2r - t = 0$$

$$r + t = 3$$

$$\therefore \underline{3r = 3}$$

$$\therefore r = 1$$

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$\therefore$  A is (2, 3).

(BC) and (AC) meet at C.

$$\therefore \begin{pmatrix} 8 \\ 6 \end{pmatrix} + s \begin{pmatrix} -1 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \end{pmatrix} + t \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\therefore \begin{pmatrix} -s - t \\ -2s + t \end{pmatrix} = \begin{pmatrix} -8 \\ -1 \end{pmatrix}$$

$$\therefore -s - t = -8$$

$$-2s + t = -1$$

$$\therefore \underline{-3s = -9}$$

$$\therefore s = 3$$

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8 \\ 6 \end{pmatrix} + 3 \begin{pmatrix} -1 \\ -2 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$$

$\therefore$  C is (5, 0).

**b** A(2, 3), B(8, 6), C(5, 0)

$$AB = \sqrt{(8-2)^2 + (6-3)^2}$$

$$= \sqrt{36+9}$$

$$= \sqrt{45} \text{ units}$$

$$AC = \sqrt{(5-2)^2 + (0-3)^2}$$

$$= \sqrt{9+9}$$

$$= \sqrt{18} \text{ units}$$

$$BC = \sqrt{(5-8)^2 + (0-6)^2}$$

$$= \sqrt{9+36}$$

$$= \sqrt{45} \text{ units}$$

The two equal sides are [AB] and [BC] and they have length  $\sqrt{45}$  units. [AC] has length  $\sqrt{18}$  units.

(AB) and (BC) meet at B.

$$\therefore \begin{pmatrix} 0 \\ 2 \end{pmatrix} + r \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 6 \end{pmatrix} + s \begin{pmatrix} -1 \\ -2 \end{pmatrix}$$

$$\therefore \begin{pmatrix} 2r + s \\ r + 2s \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \end{pmatrix}$$

$$\therefore 2r + s = 8 \quad \dots (*)$$

$$r + 2s = 4$$

$$\therefore \underline{-4r - 2s = -16} \quad \{-2 \times (*)\}$$

$$\therefore \underline{-3r = -12}$$

$$\therefore r = 4$$

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} + 4 \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 6 \end{pmatrix}$$

$\therefore$  B is (8, 6).

**4 a** (QP) and (PR) meet at P.

$$\therefore \begin{pmatrix} 3 \\ -1 \end{pmatrix} + r \begin{pmatrix} 14 \\ 10 \end{pmatrix} = \begin{pmatrix} 0 \\ 18 \end{pmatrix} + t \begin{pmatrix} 5 \\ -7 \end{pmatrix}$$

$$\therefore \begin{pmatrix} 14r - 5t \\ 10r + 7t \end{pmatrix} = \begin{pmatrix} -3 \\ 19 \end{pmatrix}$$

$$\therefore 14r - 5t = -3 \quad \dots (1)$$

$$10r + 7t = 19 \quad \dots (2)$$

$$\therefore 98r - 35t = -21 \quad \{7 \times (1)\}$$

$$50r + 35t = 95 \quad \{5 \times (2)\}$$

$$\therefore \frac{148r}{74} = 74$$

$$\therefore r = \frac{1}{2}$$

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 14 \\ 10 \end{pmatrix} = \begin{pmatrix} 10 \\ 4 \end{pmatrix}$$

$\therefore$  P is (10, 4).

(QP) and (PR) meet at Q.

$$\begin{pmatrix} 3 \\ -1 \end{pmatrix} + r \begin{pmatrix} 14 \\ 10 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} + s \begin{pmatrix} 17 \\ -9 \end{pmatrix}$$

$$\therefore r \begin{pmatrix} 14 \\ 10 \end{pmatrix} = s \begin{pmatrix} 17 \\ -9 \end{pmatrix}$$

$$\therefore r = s = 0$$

$$\text{So, } \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

$\therefore$  Q is (3, -1).

**b**  $\overrightarrow{PQ} = \begin{pmatrix} 3 - 10 \\ -1 - 4 \end{pmatrix} = \begin{pmatrix} -7 \\ -5 \end{pmatrix}$

$$\overrightarrow{PR} = \begin{pmatrix} 20 - 10 \\ -10 - 4 \end{pmatrix} = \begin{pmatrix} 10 \\ -14 \end{pmatrix}$$

$$\text{and } \overrightarrow{PQ} \bullet \overrightarrow{PR} = -70 + 70 = 0$$

(QR) and (PR) meet at R.

$$\therefore \begin{pmatrix} 3 \\ -1 \end{pmatrix} + s \begin{pmatrix} 17 \\ -9 \end{pmatrix} = \begin{pmatrix} 0 \\ 18 \end{pmatrix} + t \begin{pmatrix} 5 \\ -7 \end{pmatrix}$$

$$\therefore \begin{pmatrix} 17s - 5t \\ -9s + 7t \end{pmatrix} = \begin{pmatrix} -3 \\ 19 \end{pmatrix}$$

$$\therefore 17s - 5t = -3 \quad \dots (1)$$

$$-9s + 7t = 19 \quad \dots (2)$$

$$\therefore 119s - 35t = -21 \quad \{7 \times (1)\}$$

$$-45s + 35t = 95 \quad \{5 \times (2)\}$$

$$\therefore \frac{74s}{74} = 74$$

$$\therefore s = 1$$

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} + \begin{pmatrix} 17 \\ -9 \end{pmatrix} = \begin{pmatrix} 20 \\ -10 \end{pmatrix}$$

$\therefore$  R is (20, -10).

**c**  $[PQ] \perp [PR] \therefore \widehat{QPR} = 90^\circ$

**d** Area =  $\frac{1}{2} |\overrightarrow{PQ}| |\overrightarrow{PR}|$   
 $= \frac{1}{2} \sqrt{49 + 25} \sqrt{100 + 196}$   
 $= 74 \text{ units}^2$



**5 a** (AB) and (AD) meet at A.

$$\therefore \begin{pmatrix} 2 \\ 5 \end{pmatrix} + r \begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} + u \begin{pmatrix} -3 \\ 12 \end{pmatrix}$$

$$\therefore \begin{pmatrix} 4r + 3u \\ r - 12u \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$$

$$\therefore 4r + 3u = 1$$

$$r - 12u = -4 \quad \dots (1)$$

$$\therefore 4r + 3u = 1$$

$$\begin{array}{r} -4r + 48u = 16 \quad \{-4 \times (1)\} \\ \hline \end{array}$$

$$\therefore 51u = 17$$

$$\therefore u = \frac{1}{3}$$

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \frac{1}{3} \begin{pmatrix} -3 \\ 12 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

$\therefore$  A is (2, 5).

(BC) and (CD) meet at C.

$$\therefore \begin{pmatrix} 18 \\ 9 \end{pmatrix} + s \begin{pmatrix} -8 \\ 32 \end{pmatrix} = \begin{pmatrix} 14 \\ 25 \end{pmatrix} + t \begin{pmatrix} -8 \\ -2 \end{pmatrix}$$

$$\therefore \begin{pmatrix} -8s + 8t \\ 32s + 2t \end{pmatrix} = \begin{pmatrix} -4 \\ 16 \end{pmatrix}$$

$$\therefore -8s + 8t = -4 \quad \dots (1)$$

$$32s + 2t = 16$$

$$\therefore 2s - 2t = 1 \quad \{(1) \div -4\}$$

$$32s + 2t = 16$$

$$\therefore \begin{array}{r} 34s = 17 \\ \hline \end{array}$$

$$\therefore s = \frac{1}{2}$$

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 18 \\ 9 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} -8 \\ 32 \end{pmatrix} = \begin{pmatrix} 14 \\ 25 \end{pmatrix}$$

$\therefore$  C is (14, 25).

$$\mathbf{b} \quad \overrightarrow{AC} = \begin{pmatrix} 14 - 2 \\ 25 - 5 \end{pmatrix} = \begin{pmatrix} 12 \\ 20 \end{pmatrix}$$

$$\overrightarrow{DB} = \begin{pmatrix} 18 - (-2) \\ 9 - 21 \end{pmatrix} = \begin{pmatrix} 20 \\ -12 \end{pmatrix}$$

$$\mathbf{i} \quad |\overrightarrow{AC}| = \sqrt{12^2 + 20^2} = \sqrt{544} \text{ units}$$

$$\mathbf{ii} \quad |\overrightarrow{DB}| = \sqrt{20^2 + (-12)^2} = \sqrt{544} \text{ units}$$

$$\mathbf{iii} \quad \overrightarrow{AC} \bullet \overrightarrow{DB} = 240 - 240 = 0$$

$$\mathbf{iv} \quad \text{The midpoint of [AC] is } \left( \frac{14+2}{2}, \frac{25+5}{2} \right) \text{ which is } (8, 15).$$

$$\text{The midpoint of [DB] is } \left( \frac{18-2}{2}, \frac{9+21}{2} \right) \text{ which is } (8, 15).$$

(AB) and (BC) meet at B.

$$\therefore \begin{pmatrix} 2 \\ 5 \end{pmatrix} + r \begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 18 \\ 9 \end{pmatrix} + s \begin{pmatrix} -8 \\ 32 \end{pmatrix}$$

$$\therefore \begin{pmatrix} 4r + 8s \\ r - 32s \end{pmatrix} = \begin{pmatrix} 16 \\ 4 \end{pmatrix}$$

$$\therefore 4r + 8s = 16 \quad \dots (1)$$

$$r - 32s = 4 \quad \dots (2)$$

$$\therefore r + 2s = 4 \quad \{(1) \div 4\}$$

$$\begin{array}{r} -r + 32s = -4 \quad \{-1 \times (2)\} \\ \hline \end{array}$$

$$\therefore 34s = 0$$

$$\therefore s = 0$$

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 18 \\ 9 \end{pmatrix}$$

$\therefore$  B is (18, 9).

(CD) and (AD) meet at D.

$$\therefore \begin{pmatrix} 14 \\ 25 \end{pmatrix} + t \begin{pmatrix} -8 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} + u \begin{pmatrix} -3 \\ 12 \end{pmatrix}$$

$$\therefore \begin{pmatrix} -8t + 3u \\ -2t - 12u \end{pmatrix} = \begin{pmatrix} -11 \\ -24 \end{pmatrix}$$

$$\therefore -8t + 3u = -11 \quad \dots (1)$$

$$-2t - 12u = -24 \quad \dots (2)$$

$$\therefore 16t - 6u = 22 \quad \{-2 \times (1)\}$$

$$\begin{array}{r} t + 6u = 12 \quad \{(2) \div -2\} \\ \hline \end{array}$$

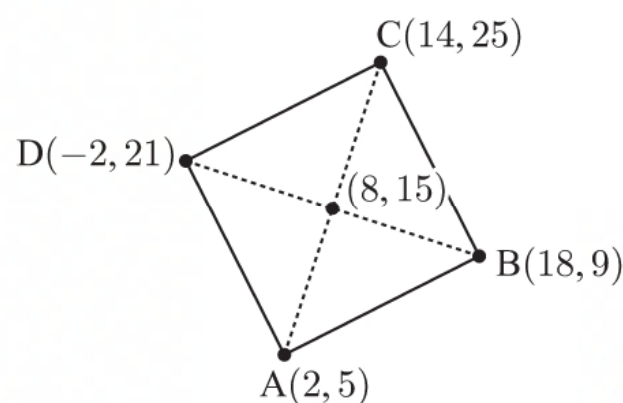
$$\therefore 17t = 34$$

$$\therefore t = 2$$

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 14 \\ 25 \end{pmatrix} + 2 \begin{pmatrix} -8 \\ -2 \end{pmatrix} = \begin{pmatrix} -2 \\ 21 \end{pmatrix}$$

$\therefore$  D is (-2, 21).

- c** The diagonals are perpendicular and equal in length, and their midpoints are both  $(8, 15)$ . So, ABCD is a square.



## EXERCISE 13F.1

- 1**
  - a** One equation is not a multiple of the other and their gradients are not the same, so the lines are intersecting.
  - b**  $x + y = 7$  can be written as  $3x + 3y = 21$  and the other line is  $3x + 3y = 1$ ,  $\therefore$  the lines are parallel.
  - c** The lines intersect at  $(2\frac{1}{2}, 2)$ .
  - d**  $x - 2y = 4$  can be written as  $2x - 4y = 8$ , so the lines are coincident.
  - e** The lines are intersecting.
  - f**  $3x - 4y = 5$  can be written as  $-3x + 4y = -5$  and the other line is  $-3x + 4y = 2$ ,  $\therefore$  the lines are parallel.

- 2** In augmented matrix form, the system is

$$\begin{aligned}
 & \left( \begin{array}{cc|c} 4 & 5 & 15 \\ 3 & -2 & 17 \end{array} \right) \\
 & \sim \left( \begin{array}{cc|c} 1 & \frac{5}{4} & \frac{15}{4} \\ 0 & -23 & 23 \end{array} \right) \quad \begin{array}{l} \frac{1}{4}R_1 \rightarrow R_1 \\ 4R_2 - 3R_1 \rightarrow R_2 \end{array} \quad \left\{ \begin{array}{ccc} 12 & -8 & 68 \\ -12 & -15 & -45 \\ \hline 0 & -23 & 23 \end{array} \right\} \\
 & \sim \left( \begin{array}{cc|c} 1 & \frac{5}{4} & \frac{15}{4} \\ 0 & 1 & -1 \end{array} \right) \quad -\frac{1}{23}R_2 \rightarrow R_2
 \end{aligned}$$

These lines are intersecting. There is exactly one solution to the equations in the augmented matrix ( $y = -1$ ,  $x = 5$ ).

- 3**
  - a**  $x + 2y = 3$  can be written as  $2x + 4y = 6$ , so the equations represent coincident lines. There are an infinite number of solutions (all the points on the line).
  - b** As the second equation is an exact multiple of the first, it will give the same solutions as the first.
  - c**
    - i** If  $x = t$ ,  $t + 2y = 3$   
 $\therefore 2y = 3 - t$   
 $\therefore y = \frac{3-t}{2}$   
 $\therefore$  the solutions are  $x = t$ ,  $y = \frac{3-t}{2}$ ,  $t \in \mathbb{R}$ .
    - ii** If  $y = s$ ,  $x + 2s = 3$   
 $\therefore x = 3 - 2s$   
 $\therefore$  the solutions are  $x = 3 - 2s$ ,  $y = s$ ,  $s \in \mathbb{R}$ .

- 4 a** In augmented matrix form, the system is:

$$\begin{pmatrix} 2 & 3 & | & 5 \\ 2 & 3 & | & 11 \end{pmatrix} \sim \begin{pmatrix} 2 & 3 & | & 5 \\ 0 & 0 & | & 6 \end{pmatrix} R_2 - R_1 \rightarrow R_2 \leftarrow \left\{ \begin{array}{ccc} 2 & 3 & 11 \\ -2 & -3 & -5 \\ \hline 0 & 0 & 6 \end{array} \right\}$$

- b**  $R_2$  shows  $0x + 0y = 6$   
 $\therefore$  there are no solutions  
 $\therefore$  the lines are parallel.

- 5 a** In augmented matrix form, the system is:

$$\begin{pmatrix} 2 & 3 & | & 5 \\ 4 & 6 & | & 10 \end{pmatrix} \sim \begin{pmatrix} 2 & 3 & | & 5 \\ 0 & 0 & | & 0 \end{pmatrix} R_2 - 2R_1 \rightarrow R_2 \leftarrow \left\{ \begin{array}{ccc} 4 & 6 & 10 \\ -4 & -6 & -10 \\ \hline 0 & 0 & 0 \end{array} \right\}$$

- b**  $R_2$  shows  $0x + 0y = 0$ , which is true for all  $x$  and  $y$ . So, the lines are coincident.  
 All solutions come from  $2x + 3y = 5$ . Let  $x = t$ ,  $y = \frac{5-2t}{3}$  for all values of  $t$   
 $\therefore$  there are infinitely many solutions of the form  $x = t$ ,  $y = \frac{5-2t}{3}$ ,  $t \in \mathbb{R}$ .

- 6 a** In augmented matrix form, the system is:

$$\begin{pmatrix} 3 & -1 & | & 2 \\ 6 & -2 & | & 4 \end{pmatrix} \sim \begin{pmatrix} 3 & -1 & | & 2 \\ 0 & 0 & | & 0 \end{pmatrix} R_2 - 2R_1 \rightarrow R_2 \leftarrow \left\{ \begin{array}{ccc} 6 & -2 & 4 \\ -6 & 2 & -4 \\ \hline 0 & 0 & 0 \end{array} \right\}$$

$R_2$  shows  $0x + 0y = 0$ , which is true for all  $x$  and  $y$ .

So, there are infinitely many solutions {the lines are coincident}.

Substitute  $x = t$  into the first equation  $3x - y = 2$

$$\therefore 3t - y = 2$$

$$\therefore y = 3t - 2$$

$\therefore$  there are infinitely many solutions of the form  $x = t$ ,  $y = 3t - 2$ ,  $t \in \mathbb{R}$ .

**b**  $3x - y = 2 \quad \dots (1)$

$6x - 2y = k \quad \dots (2)$

If  $k = 4$  then  $6x - 2y = 4$ , which is an exact multiple of equation (1).

$\therefore$  the lines are coincident and there are infinitely many solutions of the form

$$x = t, \quad y = 3t - 2, \quad t \in \mathbb{R}.$$

If  $k \neq 4$  then the equations represent parallel lines  $\therefore$  there are no solutions.



- 7 a** In augmented matrix form, the system is:

$$\begin{pmatrix} 3 & -1 & | & 8 \\ 6 & -2 & | & k \end{pmatrix} \sim \begin{pmatrix} 3 & -1 & | & 8 \\ 0 & 0 & | & k-16 \end{pmatrix} \quad R_2 - 2R_1 \rightarrow R_2 \quad \left\{ \begin{array}{ccc} 6 & -2 & k \\ -6 & 2 & -16 \\ \hline 0 & 0 & k-16 \end{array} \right\}$$

- b** If  $k = 16$  there are infinitely many solutions.

Substitute  $y = t$  into  $3x - y = 8$

$$\therefore 3x - t = 8$$

$$\therefore x = \frac{t+8}{3}$$

There are infinitely many solutions of the form  $x = \frac{t+8}{3}$ ,  $y = t$ ,  $t \in \mathbb{R}$ .

- c** The system has no solutions when  $k - 16 \neq 0$ ,  $\therefore k \neq 16$ .  
The lines are parallel but not coincident.

- 8 a** In augmented matrix form, the system is:

$$\begin{pmatrix} 4 & 8 & | & 1 \\ 2 & -a & | & 11 \end{pmatrix} \sim \begin{pmatrix} 4 & 8 & | & 1 \\ 0 & -2a-8 & | & 21 \end{pmatrix} \quad 2R_2 - R_1 \rightarrow R_2 \quad \left\{ \begin{array}{ccc} 4 & -2a & 22 \\ -4 & -8 & -1 \\ \hline 0 & -2a-8 & 21 \end{array} \right\}$$

$$\sim \begin{pmatrix} 4 & 8 & | & 1 \\ 0 & 2a+8 & | & -21 \end{pmatrix} \quad -R_2 \rightarrow R_2$$

- b** A unique solution exists provided  $2a + 8 \neq 0 \therefore a \neq -4$ .

From  $R_2$ ,  $(2a + 8)y = -21$

$$\therefore y = \frac{-21}{2a+8}$$

$$\text{and } 4x + 8y = 1$$

$$\therefore 4x + 8\left(\frac{-21}{2a+8}\right) = 1$$

$$\therefore 4x(2a+8) - 168 = 2a+8$$

$$\therefore 2x(2a+8) - 84 = a+4$$

$$\therefore 2x(2a+8) = a+88$$

$$\therefore x = \frac{a+88}{4a+16}$$

The solution is  $x = \frac{a+88}{4a+16}$ ,  $y = \frac{-21}{2a+8}$ ,  $a \neq -4$ .

- c** When  $a = -4$  there are no solutions as the lines are parallel.



- 9 a In augmented matrix form, the system is:

$$\begin{pmatrix} m & 2 & | & 6 \\ 2 & m & | & 6 \end{pmatrix} \sim \begin{pmatrix} m & 2 & | & 6 \\ 0 & m^2 - 4 & | & 6m - 12 \end{pmatrix} \quad mR_2 - 2R_1 \rightarrow R_2 \quad \leftarrow \begin{pmatrix} 2m & m^2 & 6m \\ -2m & -4 & -12 \\ 0 & m^2 - 4 & 6m - 12 \end{pmatrix}$$

A unique solution exists provided  $m^2 - 4 \neq 0$ .

So, there is a unique solution provided  $m \neq \pm 2$ .

From  $R_2$ ,  $(m^2 - 4)y = 6m - 12$

$$\therefore y = \frac{6(m-2)}{(m-2)(m+2)}$$

$$\therefore y = \frac{6}{m+2}, \quad m \neq \pm 2$$

Substituting into  $mx + 2y = 6$

$$\text{gives } mx + 2\left(\frac{6}{m+2}\right) = 6$$

$$\therefore m(m+2)x + 12 = 6(m+2)$$

$$\therefore m(m+2)x = 6m + 12 - 12$$

$$\therefore m(m+2)x = 6m$$

$$\therefore x = \frac{6}{m+2}$$

So, the unique solution is  $x = \frac{6}{m+2}$ ,  $y = \frac{6}{m+2}$  when  $m \neq \pm 2$ .

The lines intersect at a point.

- b When  $m = 2$ , the equations are  $2x + 2y = 6$  and  $2x + 2y = 6$ , so the lines are coincident.

There are infinitely many solutions of the form  $x = t$ ,  $y = \frac{6-2t}{2} = 3-t$ ,  $t \in \mathbb{R}$ .

When  $m = -2$ , the equations are  $-2x + 2y = 6$  and  $2x - 2y = 6$   
or  $-2x + 2y = -6$

$\therefore$  the lines are parallel and there are no solutions.

## EXERCISE 13F.2

- 1 a Line 1 has direction vector  $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$  and line 2 has direction vector  $\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$ .

As one vector is not a scalar multiple of the other, the lines are not parallel.

$$\begin{array}{lll} \text{Now} & 1 + 2t = -2 + 3s & 2 - t = 3 - s & 3 + t = 1 + 2s \\ \therefore & 2t - 3s = -3 \quad \dots (1) & -t + s = 1 \quad \dots (2) & t - 2s = -2 \quad \dots (3) \end{array}$$

Solving (2) and (3) simultaneously:

$$\begin{array}{r} -t + s = 1 \\ t - 2s = -2 \\ \hline \end{array}$$

$$\therefore -s = -1$$

$$\therefore s = 1 \text{ and } t = 0$$

Checking in (1):  $2t - 3s = 2(0) - 3(1) = -3$  ✓

∴ the two lines meet at  $(1, 2, 3)$  {using  $t = 0$  or  $s = 1$ }

If  $\theta$  is the acute angle between the lines, then  $\cos \theta = \frac{|6 + 1 + 2|}{\sqrt{4 + 1 + 1} \sqrt{9 + 1 + 4}} = \frac{9}{\sqrt{84}}$

$$\begin{aligned}\therefore \theta &= \cos^{-1}\left(\frac{9}{\sqrt{84}}\right) \\ &\approx 10.9^\circ\end{aligned}$$

**b** Line 1 has direction vector  $\begin{pmatrix} 2 \\ -12 \\ 12 \end{pmatrix}$  and line 2 has direction vector  $\begin{pmatrix} 4 \\ 3 \\ -1 \end{pmatrix}$ .

As one vector is not a scalar multiple of the other, the lines are not parallel.

$$\begin{aligned}\text{Now } -1 + 2\lambda &= 4\mu - 3 & 2 - 12\lambda &= 3\mu + 2 & 4 + 12\lambda &= -\mu - 1 \\ \therefore 2\lambda - 4\mu &= -2 & -12\lambda - 3\mu &= 0 & 12\lambda + \mu &= -5 \quad \dots (3) \\ \therefore \lambda - 2\mu &= -1 \quad \dots (1) & \mu &= -4\lambda \quad \dots (2)\end{aligned}$$

$$\begin{aligned}\text{Solving (1) and (2) simultaneously: } \lambda - 2(-4\lambda) &= -1 \\ \therefore 9\lambda &= -1 \\ \therefore \lambda &= -\frac{1}{9} \text{ and so } \mu = \frac{4}{9}\end{aligned}$$

Checking in (3):  $12\lambda + \mu = 12\left(-\frac{1}{9}\right) + \frac{4}{9} = -\frac{8}{9} \neq -5$  ✗

Since the system is inconsistent, the lines do not intersect, so the lines are skew.

If  $\theta$  is the acute angle between the lines, then  $\cos \theta = \frac{|8 - 36 - 12|}{\sqrt{292} \sqrt{26}} = \frac{40}{\sqrt{7592}}$

$$\begin{aligned}\therefore \theta &= \cos^{-1}\left(\frac{40}{\sqrt{7592}}\right) \\ &\approx 62.7^\circ\end{aligned}$$

**c** Line 1 has direction vector  $\begin{pmatrix} 6 \\ 8 \\ 2 \end{pmatrix}$  and line 2 has direction vector  $\begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$ .

As  $\begin{pmatrix} 6 \\ 8 \\ 2 \end{pmatrix} = 2 \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$  the two lines are parallel. Hence,  $\theta = 0^\circ$ .

To see if the lines are coincident, try to find a shared point.

When  $t = 0$ , the point on line 1 is  $(0, 3, -1)$ .

∴ the unique point on line 2 with  $z$ -coordinate  $-1$  is the point where  $1 + s = -1$   
 $\therefore s = -2$

This point is  $(-4, -8, -1)$ .

Since  $(0, 3, -1) \neq (-4, -8, -1)$ , the lines are not coincident.

∴ the lines are parallel.

- d In line 1 let  $x = 2 - y = z + 2 = t$ , so  $x = t$ ,  $y = 2 - t$ , and  $z = t - 2$ ,  $t \in \mathbb{R}$ .

Line 1 has direction vector  $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$  and line 2 has direction vector  $\begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix}$ .

As one vector is not a scalar multiple of the other, the lines are not parallel.

$$\begin{aligned} \text{Now } t = 1 + 3s \quad \dots (1) \quad & 2 - t = -2 - 2s \quad & -2 + t = 2s + \frac{1}{2} \\ \therefore -t + 2s = -4 \quad \dots (2) \quad & t - 2s = 2\frac{1}{2} \quad \dots (3) \end{aligned}$$

$$\begin{aligned} \text{Solving (1) and (2) simultaneously: } & -(1 + 3s) + 2s = -4 \\ \therefore -1 - 3s + 2s = -4 & \\ \therefore -s = -3 & \\ \therefore s = 3 \text{ and so } t = 1 + 3(3) = 10 & \end{aligned}$$

$$\text{Checking in (3): } t - 2s = 10 - 2(3) = 4 \neq 2\frac{1}{2} \quad \times$$

Since the system is inconsistent, the lines do not intersect, so the lines are skew.

$$\begin{aligned} \text{If } \theta \text{ is the acute angle between the lines, then } \cos \theta &= \frac{|3 + 2 + 2|}{\sqrt{1+1+1}\sqrt{9+4+4}} = \frac{7}{\sqrt{3}\sqrt{17}} \\ \therefore \theta &= \cos^{-1}\left(\frac{7}{\sqrt{3}\sqrt{17}}\right) \\ &\approx 11.4^\circ \end{aligned}$$

- e Line 1 has direction vector  $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$  and line 2 has direction vector  $\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$ .

As one vector is not a scalar multiple of the other, the lines are not parallel.

$$\begin{aligned} 1 + \lambda = 2 + 3\mu \quad & 2 - \lambda = 3 - 2\mu \quad & 3 + 2\lambda = \mu - 5 \\ \therefore \lambda - 3\mu = 1 \quad \dots (1) \quad & -\lambda + 2\mu = 1 \quad \dots (2) \quad & 2\lambda - \mu = -8 \quad \dots (3) \end{aligned}$$

$$\begin{aligned} \text{Solving (1) and (2) simultaneously: } & \lambda - 3\mu = 1 \\ & -\lambda + 2\mu = 1 \\ \hline & \therefore -\mu = 2 \\ & \therefore \mu = -2 \text{ and } \lambda - 3(-2) = 1 \\ & \therefore \lambda = -5 \end{aligned}$$

$$\text{Checking in (3): } 2\lambda - \mu = 2(-5) - (-2) = -8 \quad \checkmark$$

The two lines meet at  $(-4, 7, -7)$ . {using  $\mu = -2$  or  $\lambda = -5$ }

$$\begin{aligned} \text{If } \theta \text{ is the acute angle between the lines, then } \cos \theta &= \frac{|3 + 2 + 2|}{\sqrt{1+1+4}\sqrt{9+4+1}} = \frac{7}{\sqrt{84}} \\ \therefore \theta &= \cos^{-1}\left(\frac{7}{\sqrt{84}}\right) \\ &\approx 40.2^\circ \end{aligned}$$



**f** Line 1 has direction vector  $\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$  and line 2 has direction vector  $\begin{pmatrix} 4 \\ -2 \\ 0 \end{pmatrix}$ .

Now  $\begin{pmatrix} 4 \\ -2 \\ 0 \end{pmatrix} = -2 \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$ , so the lines are parallel and hence  $\theta = 0^\circ$ .

All points on line 1 have  $z$ -coordinate 5 and all points on line 2 have  $z$ -coordinate 3.  
 $\therefore$  the lines are parallel.

**g** Line 1 has direction vector  $\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$  and line 2 has direction vector  $\begin{pmatrix} -4 \\ 2 \\ -6 \end{pmatrix}$ .

As  $\begin{pmatrix} -4 \\ 2 \\ -6 \end{pmatrix} = -2 \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ , the two lines are parallel. Hence  $\theta = 0^\circ$ .

When  $\lambda = 1$ , the point on line 1 is  $(3, -1, 4)$ .

The unique point on line 2 with  $x$ -coordinate 3 is the point where  $3 - 4\mu = 3$   
 $\therefore \mu = 0$

This point is  $(3, -1, 4)$ .

Lines 1 and 2 are parallel and share the point  $(3, -1, 4)$ .

$\therefore$  the lines are coincident.

**2** Line 1 is  $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$  with direction vector  $\mathbf{a} = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$ .

Line 2 is  $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 4 \\ 2 \end{pmatrix}$  with direction vector  $\mathbf{b} = \begin{pmatrix} -2 \\ 4 \\ 2 \end{pmatrix}$ .

Line 3 is  $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$  with direction vector  $\mathbf{c} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ .

Line 1 and line 2:

Since  $\begin{pmatrix} -2 \\ 4 \\ 2 \end{pmatrix} = -2 \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$ , line 1 and line 2 are parallel.

When  $\lambda = 0$ , the point on line 1 is  $(3, -1, 2)$ .

The unique point on line 2 with  $y$ -coordinate  $-1$  is the point where  $4\mu = -1$   
 $\therefore \mu = -\frac{1}{4}$

This point is  $(\frac{3}{2}, -1, -\frac{3}{2})$ .

Since  $(\frac{3}{2}, -1, -\frac{3}{2}) \neq (3, -1, 2)$ , line 1 and line 2 are not coincident.



Line 1 and line 3:

Equating  $x$ ,  $y$ , and  $z$  values in lines 1 and 3 gives

$$\begin{array}{lll} 3 + \lambda = t & -1 - 2\lambda = 1 + 2t & 2 - \lambda = 1 + t \\ \therefore t = 3 + \lambda & \therefore 2t = -2 - 2\lambda & \therefore \lambda + t = 1 \dots (1) \\ & \therefore t = -1 - \lambda & \end{array}$$

Solving these we get  $3 + \lambda = -1 - \lambda$

$$\therefore 2\lambda = -4$$

$$\therefore \lambda = -2 \text{ and so } t = 3 - 2 \therefore t = 1$$

Checking in (1):  $\lambda + t = -2 + 1 = -1 \neq 1$  ✗

So, there is no simultaneous solution to all 3 equations.

$\therefore$  the lines do not intersect.

$$\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} \neq k \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \text{ for any } k \in \mathbb{R}.$$

$\therefore$  lines 1 and 3 are not parallel.

Since they do not intersect and are not parallel, they are skew.

If  $\theta$  is the acute angle between  $\mathbf{a}$  and  $\mathbf{c}$ , then

$$\cos \theta = \frac{|\mathbf{a} \cdot \mathbf{c}|}{|\mathbf{a}| |\mathbf{c}|} = \frac{|1 - 4 - 1|}{\sqrt{1+4+1}\sqrt{1+4+1}} = \frac{4}{\sqrt{6}\sqrt{6}} = \frac{2}{3}$$

$$\therefore \theta = \cos^{-1}\left(\frac{2}{3}\right) \approx 48.2^\circ$$

So, line 1 and line 3 are skew with an angle of about  $48.2^\circ$  between them.

Line 2 and line 3:

Equating  $x$ ,  $y$ , and  $z$  values in lines 2 and 3 gives

$$\begin{array}{lll} 1 - 2\mu = t & 4\mu = 1 + 2t & -1 + 2\mu = 1 + t \\ \therefore t = 1 - 2\mu & \therefore 2t = -1 + 4\mu & \therefore 2\mu - t = 2 \dots (2) \\ \therefore 2t = 2 - 4\mu & & \end{array}$$

Solving these we get  $2 - 4\mu = -1 + 4\mu$

$$\therefore 8\mu = 3$$

$$\therefore \mu = \frac{3}{8} \text{ and so } t = 1 - 2\left(\frac{3}{8}\right) \therefore t = \frac{1}{4}$$

Checking in (2):  $2\mu - t = 2\left(\frac{3}{8}\right) - \frac{1}{4} = \frac{6}{8} - \frac{2}{8} = \frac{4}{8} = \frac{1}{2} \neq 2$  ✗

So, there is no simultaneous solution to all 3 equations.

$\therefore$  the lines do not intersect.

$$\begin{pmatrix} -2 \\ 4 \\ 2 \end{pmatrix} \neq k \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \text{ for any } k \in \mathbb{R}.$$

$\therefore$  lines 2 and 3 are not parallel.

Since they do not intersect and are not parallel, they are skew.

If  $\phi$  is the acute angle between  $\mathbf{b}$  and  $\mathbf{c}$ , then

$$\cos \phi = \frac{|\mathbf{b} \cdot \mathbf{c}|}{|\mathbf{b}| |\mathbf{c}|} = \frac{|-2 + 8 + 2|}{\sqrt{4 + 16 + 4} \sqrt{1 + 4 + 1}} = \frac{8}{\sqrt{144}} = \frac{2}{3}$$

$$\therefore \phi = \cos^{-1}\left(\frac{2}{3}\right)$$

$$\approx 48.2^\circ$$

So, line 2 and line 3 are skew with an angle of about  $48.2^\circ$  between them.

## INVESTIGATION 1

## MOTION OF PARTICLES IN SPACE

**1**  $L_1$  has direction vector  $\begin{pmatrix} -2 \\ -3 \\ 4 \end{pmatrix}$  and  $L_2$  has direction vector  $\begin{pmatrix} 0 \\ -2 \\ 2 \end{pmatrix}$ .

As one vector is not a scalar multiple of the other, the lines are not parallel.

$$\begin{array}{lll} 5 - 2t = 1 & 4 - 3t = -4 - 2u & 4 + 4t = 14 + 2u \\ \therefore 2t = 4 & \therefore 4 - 3(2) = -4 - 2u & \therefore 4 + 4(2) = 14 + 2(-1) \\ \therefore t = 2 & \therefore 4 - 6 = -4 - 2u & \therefore 12 = 12 \quad \checkmark \\ & \therefore 2u = -2 & \\ & \therefore u = -1 & \end{array}$$

The two lines meet at  $(1, -2, 12)$ . {using  $t = 2$  or  $u = -1$ }

**2 a**  $x_A(t) = 5 - 2t$ ,  $y_A(t) = 4 - 3t$ ,  $z_A(t) = 4 + 4t$

$$x_A(0) = 5, \quad y_A(0) = 4, \quad z_A(0) = 4$$

$\therefore$  particle A is initially at  $(5, 4, 4)$ .

$$x_B(t) = 1, \quad y_B(t) = -4 - 2t, \quad z_B(t) = 14 + 2t$$

$$x_B(0) = 1, \quad y_B(0) = -4, \quad z_B(0) = 14$$

$\therefore$  particle B is initially at  $(1, -4, 14)$ .

**b** Particle A has velocity vector  $\begin{pmatrix} -2 \\ -3 \\ 4 \end{pmatrix}$  and particle B has velocity vector  $\begin{pmatrix} 0 \\ -2 \\ 2 \end{pmatrix}$ .

**c** The position vectors of particles A and B at time  $t \geq 0$  are given by the lines  $L_1$  and  $L_2$  respectively. {from **1**}

So, we know that their paths intersect only at the point  $(1, -2, 12)$ .

Particle A is at this point when  $t = 2$ , and particle B is never at this point as  $t \geq 0$ .

$\therefore$  the particles will not collide.

- 3 a** At time  $t$ , particle A has position  $(5 - 2t, 4 - 3t, 4 + 4t)$   
and particle B has position  $(1, -4 - 2t, 14 + 2t)$ .

The distance between particles A and B at time  $t$  is

$$\begin{aligned} D &= \sqrt{(1 - (5 - 2t))^2 + (-4 - 2t - (4 - 3t))^2 + (14 + 2t - (4 + 4t))^2} \\ &= \sqrt{(2t - 4)^2 + (t - 8)^2 + (10 - 2t)^2} \\ &= \sqrt{4t^2 - 16t + 16 + t^2 - 16t + 64 + 100 - 40t + 4t^2} \end{aligned}$$

$$\therefore D = \sqrt{9t^2 - 72t + 180}$$

**b**  $D^2 = 9t^2 - 72t + 180$

$D$  is minimised when  $D^2$  is minimised.

$$\begin{aligned} D^2 \text{ is a quadratic, so it is a minimum when } t &= -\frac{b}{2a} \\ &= -\frac{(-72)}{2(9)} \\ &= \frac{72}{18} \\ &= 4 \end{aligned}$$

$$\begin{aligned} \text{When } t = 4, \quad D &= \sqrt{9(4)^2 - 72(4) + 180} \\ &= \sqrt{144 - 288 + 180} \\ &= \sqrt{36} \\ &= 6 \text{ m} \end{aligned}$$

$\therefore$  the minimum distance between the particles is 6 m, at time  $t = 4$  seconds.

### EXERCISE 13F.3

- 1 a** Line 1:  $x = 1 + 2t, y = -t, z = 2 + 3t, t \in \mathbb{R}$   
Line 2:  $x = y = z = s, s \in \mathbb{R}$

The lines have direction vectors  $\mathbf{v} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$  and  $\mathbf{w} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$$\begin{aligned} \text{so } \mathbf{v} \times \mathbf{w} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 3 \\ 1 & 1 & 1 \end{vmatrix} \\ &= \begin{vmatrix} -1 & 3 \\ 1 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} \mathbf{k} \\ &= -4\mathbf{i} + \mathbf{j} + 3\mathbf{k} \\ &= \begin{pmatrix} -4 \\ 1 \\ 3 \end{pmatrix} \end{aligned}$$

Let A and B be points on the skew lines such that  $|\overrightarrow{AB}|$  is the shortest distance between them.

$\therefore$  A is  $(1 + 2t, -t, 2 + 3t)$  and B is  $(s, s, s)$  for some  $s, t \in \mathbb{R}$ .

$$\begin{aligned} \text{Now } \overrightarrow{AB} \parallel \mathbf{v} \times \mathbf{w}, \text{ so } \begin{pmatrix} s-1-2t \\ s+t \\ s-2-3t \end{pmatrix} &= k \begin{pmatrix} -4 \\ 1 \\ 3 \end{pmatrix} \text{ for some } k \in \mathbb{R} \\ \therefore s-2t-1 &= -4k \quad \dots (1) \\ s+t &= k \quad \dots (2) \\ s-3t-2 &= 3k \quad \dots (3) \end{aligned}$$

Substituting (2) into (1) and (3) gives

$$\begin{aligned} s-2t-1 &= -4s-4t \quad \text{and} \quad s-3t-2 = 3s+3t \\ \therefore 5s+2t &= 1 \quad \dots (4) \quad \text{and} \quad -2s-6t = 2 \\ &\quad \quad \quad \frac{15s+6t = 3}{\therefore 13s = 5} \quad \{3 \times (4)\} \\ &\quad \quad \quad \therefore s = \frac{5}{13} \end{aligned}$$

$$\begin{aligned} \text{Using (4), } \frac{25}{13} + 2t &= 1 \\ \therefore 2t &= -\frac{12}{13} \\ \therefore t &= -\frac{6}{13} \end{aligned}$$

$$\therefore \overrightarrow{AB} = \begin{pmatrix} \frac{5}{13} - 1 + \frac{12}{13} \\ \frac{5}{13} - \frac{6}{13} \\ \frac{5}{13} - 2 + \frac{18}{13} \end{pmatrix} = \begin{pmatrix} \frac{4}{13} \\ -\frac{1}{13} \\ -\frac{3}{13} \end{pmatrix}$$

$$\begin{aligned} \text{and the shortest distance between the lines } |\overrightarrow{AB}| &= \sqrt{\left(\frac{4}{13}\right)^2 + \left(-\frac{1}{13}\right)^2 + \left(-\frac{3}{13}\right)^2} \\ &= \frac{2}{\sqrt{26}} \text{ units.} \end{aligned}$$

- b** Line 1:  $x = 1 - t, y = 1 + t, z = 3 - t, t \in \mathbb{R}$   
 Line 2:  $x = 2 + s, y = 1 - 2s, z = s, s \in \mathbb{R}$

$$\text{The lines have direction vectors } \mathbf{v} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} \text{ and } \mathbf{w} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$$\begin{aligned} \text{so } \mathbf{v} \times \mathbf{w} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & -1 \\ 1 & -2 & 1 \end{vmatrix} \\ &= \begin{vmatrix} 1 & -1 \\ -2 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -1 & -1 \\ 1 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -1 & 1 \\ 1 & -2 \end{vmatrix} \mathbf{k} \\ &= -\mathbf{i} + 0\mathbf{j} + \mathbf{k} \\ &= \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \end{aligned}$$

Let A and B be points on the skew lines such that  $|\overrightarrow{AB}|$  is the shortest distance between them.

$\therefore$  A is  $(1 - t, 1 + t, 3 - t)$  and B is  $(2 + s, 1 - 2s, s)$  for some  $s, t \in \mathbb{R}$ .



Now  $\overrightarrow{AB} \parallel \mathbf{v} \times \mathbf{w}$ , so  $\begin{pmatrix} 2+s-1+t \\ 1-2s-1-t \\ s-3+t \end{pmatrix} = k \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$  for some  $k \in \mathbb{R}$

$$\therefore 1+s+t = -k \quad \dots (1)$$

$$-2s-t = 0 \quad \dots (2)$$

$$-3+s+t = k \quad \dots (3)$$

Substituting (3) into (1) gives  $4 = -2k$

$$\therefore k = -2$$

$$\therefore \overrightarrow{AB} = -2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}$$

and the shortest distance between the lines  $|\overrightarrow{AB}| = \sqrt{2^2 + 0^2 + (-2)^2}$   
 $= 2\sqrt{2}$  units.

**2 a** Since the lines intersect, the shortest distance between these lines is 0 units.

**b** Line 1:  $x = -1 + 2\lambda$ ,  $y = 2 - 12\lambda$ ,  $z = 4 + 12\lambda$ ,  $\lambda \in \mathbb{R}$

Line 2:  $x = 4\mu - 3$ ,  $y = 3\mu + 2$ ,  $z = -\mu - 1$ ,  $\mu \in \mathbb{R}$

The lines have direction vectors  $\mathbf{v} = \begin{pmatrix} 2 \\ -12 \\ 12 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ -6 \\ 6 \end{pmatrix}$  and  $\mathbf{w} = \begin{pmatrix} 4 \\ 3 \\ -1 \end{pmatrix}$

$$\begin{aligned} \text{so } \mathbf{v} \times \mathbf{w} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -6 & 6 \\ 4 & 3 & -1 \end{vmatrix} \\ &= \begin{vmatrix} -6 & 6 \\ 3 & -1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 6 \\ 4 & -1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & -6 \\ 4 & 3 \end{vmatrix} \mathbf{k} \\ &= -12\mathbf{i} + 25\mathbf{j} + 27\mathbf{k} \\ &= \begin{pmatrix} -12 \\ 25 \\ 27 \end{pmatrix} \end{aligned}$$

Let A and B be points on the skew lines such that  $|\overrightarrow{AB}|$  is the shortest distance between them.

$\therefore$  A is  $(-1 + 2\lambda, 2 - 12\lambda, 4 + 12\lambda)$  and B is  $(4\mu - 3, 3\mu + 2, -\mu - 1)$   
 for some  $\lambda, \mu \in \mathbb{R}$ .

Now  $\overrightarrow{AB} \parallel \mathbf{v} \times \mathbf{w}$ , so  $\begin{pmatrix} 4\mu - 3 + 1 - 2\lambda \\ 3\mu + 2 - 2 + 12\lambda \\ -\mu - 1 - 4 - 12\lambda \end{pmatrix} = k \begin{pmatrix} -12 \\ 25 \\ 27 \end{pmatrix}$  for some  $k \in \mathbb{R}$

$$\therefore -2 - 2\lambda + 4\mu = -12k \quad \dots (1)$$

$$12\lambda + 3\mu = 25k \quad \dots (2)$$

$$-5 - 12\lambda - \mu = 27k \quad \dots (3)$$

Adding (2) and (3) gives  $-5 + 2\mu = 52k \quad \dots (4)$

$$\begin{array}{rcl}
 -12 - 12\lambda + 24\mu = -72k & \{6 \times (1)\} \\
 12\lambda + 3\mu = 25k & \{(2)\} \\
 \hline
 \text{Adding, } -12 & + 27\mu = -47k & \dots (5) \\
 135 - 54\mu = -1404k & \{-27 \times (4)\} \\
 -24 + 54\mu = -94k & \{2 \times (5)\} \\
 \hline
 \text{Adding, } 111 & = -1498k \\
 \therefore k = -\frac{111}{1498}
 \end{array}$$

$$\therefore \vec{AB} = -\frac{111}{1498} \begin{pmatrix} -12 \\ 25 \\ 27 \end{pmatrix} = \begin{pmatrix} \frac{666}{749} \\ -\frac{2775}{1498} \\ -\frac{2997}{1498} \end{pmatrix}$$

and the shortest distance between the lines  $|\vec{AB}|$

$$\begin{aligned}
 &= \sqrt{\left(\frac{666}{749}\right)^2 + \left(-\frac{2775}{1498}\right)^2 + \left(-\frac{2997}{1498}\right)^2} \\
 &= \frac{111}{\sqrt{1498}} \approx 2.87 \text{ units}
 \end{aligned}$$

• Line 1:  $x = 6t, y = 3 + 8t, z = -1 + 2t, t \in \mathbb{R}$

Line 2:  $x = 2 + 3s, y = 4s, z = 1 + s, s \in \mathbb{R}$

Since the lines are parallel, we find the shortest distance between the lines using the shortest distance from a point to a line.

Let P be a point on line 1 and N be a point on line 2.

If P is a point where  $t = 0$ , then P is  $(0, 3, -1)$ .

Now N has coordinates  $(2 + 3s, 4s, 1 + s)$  for some  $s$ ,

$$\text{and } \vec{PN} \text{ is } \begin{pmatrix} 2 + 3s - 0 \\ 4s - 3 \\ 1 + s + 1 \end{pmatrix} = \begin{pmatrix} 3s + 2 \\ 4s - 3 \\ s + 2 \end{pmatrix}.$$

The distance between P and line 2 is minimised when  $\vec{PN} \bullet \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} = 0$

$$\therefore \begin{pmatrix} 3s + 2 \\ 4s - 3 \\ s + 2 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} = 0$$

$$\therefore 3(3s + 2) + 4(4s - 3) + (s + 2) = 0$$

$$\therefore 9s + 6 + 16s - 12 + s + 2 = 0$$

$$\therefore 26s = 4$$

$$\therefore s = \frac{4}{26} = \frac{2}{13}$$

$$\begin{aligned}
 \text{Thus } \vec{PN} &= \begin{pmatrix} \frac{6}{13} + 2 \\ \frac{8}{13} - 3 \\ \frac{2}{13} + 2 \end{pmatrix} = \begin{pmatrix} \frac{32}{13} \\ -\frac{31}{13} \\ \frac{28}{13} \end{pmatrix} \quad \text{and} \quad |\vec{PN}| = \sqrt{\left(\frac{32}{13}\right)^2 + \left(-\frac{31}{13}\right)^2 + \left(\frac{28}{13}\right)^2} \\
 &= \frac{\sqrt{2769}}{13} \approx 4.05 \text{ units}
 \end{aligned}$$

**d** Line 1:  $x = 2 - y = z + 2$ , which can be re-written as

$$x = t, \quad y = 2 - t, \quad z = t - 2, \quad t \in \mathbb{R}$$

Line 2:  $x = 1 + 3s, \quad y = -2 - 2s, \quad z = 2s + \frac{1}{2}, \quad s \in \mathbb{R}$

The lines have direction vectors  $\mathbf{v} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$  and  $\mathbf{w} = \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix}$

$$\begin{aligned} \text{so } \mathbf{v} \times \mathbf{w} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ 3 & -2 & 2 \end{vmatrix} \\ &= \begin{vmatrix} -1 & 1 \\ -2 & 2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & -1 \\ 3 & -2 \end{vmatrix} \mathbf{k} \\ &= 0\mathbf{i} + \mathbf{j} + \mathbf{k} \\ &= \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \end{aligned}$$

Let A and B be points on the skew lines such that  $|\overrightarrow{AB}|$  is the shortest distance between them.

$\therefore$  A is  $(t, 2 - t, t - 2)$  and B is  $(1 + 3s, -2 - 2s, 2s + \frac{1}{2})$  for some  $s, t \in \mathbb{R}$ .

$$\begin{aligned} \text{Now } \overrightarrow{AB} \parallel \mathbf{v} \times \mathbf{w}, \text{ so } \begin{pmatrix} 1 + 3s - t \\ -2 - 2s - 2 + t \\ 2s + \frac{1}{2} - t + 2 \end{pmatrix} &= k \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \text{ for some } k \in \mathbb{R} \\ \therefore 1 + 3s - t &= 0 \quad \dots (1) \\ -4 - 2s + t &= k \quad \dots (2) \\ \frac{5}{2} + 2s - t &= k \quad \dots (3) \end{aligned}$$

$$\begin{aligned} \text{Adding (2) and (3) gives } -\frac{3}{2} &= 2k \\ \therefore k &= -\frac{3}{4} \end{aligned}$$

$$\therefore \overrightarrow{AB} = -\frac{3}{4} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{3}{4} \\ -\frac{3}{4} \end{pmatrix}$$

$$\begin{aligned} \text{and the shortest distance between the lines } |\overrightarrow{AB}| &= \sqrt{\left(-\frac{3}{4}\right)^2 + \left(-\frac{3}{4}\right)^2} \\ &= \frac{3\sqrt{2}}{4} \text{ units} \end{aligned}$$

**e** Since the lines intersect, the shortest distance between these lines is 0 units.

**f** Line 1:  $x = 1 - 2t, \quad y = 8 + t, \quad z = 5, \quad t \in \mathbb{R}$

Line 2:  $x = 2 + 4s, \quad y = -1 - 2s, \quad z = 3, \quad s \in \mathbb{R}$

Let P be a point on line 1 and N be a point on line 2.

If P is a point where  $t = 0$ , then P is  $(1, 8, 5)$ .

Now N has coordinates  $(2 + 4s, -1 - 2s, 3)$  for some  $s$ ,

$$\text{and } \overrightarrow{PN} \text{ is } \begin{pmatrix} 2 + 4s - 1 \\ -1 - 2s - 8 \\ 3 - 5 \end{pmatrix} = \begin{pmatrix} 1 + 4s \\ -9 - 2s \\ -2 \end{pmatrix}.$$

The distance between P and line 2 is minimised when  $\overrightarrow{PN} \bullet \begin{pmatrix} 4 \\ -2 \\ 0 \end{pmatrix} = 0$

$$\therefore \begin{pmatrix} 1+4s \\ -9-2s \\ -2 \end{pmatrix} \bullet \begin{pmatrix} 4 \\ -2 \\ 0 \end{pmatrix} = 0$$

$$\therefore 4(1+4s) - 2(-9-2s) + 0(-2) = 0$$

$$\therefore 4 + 16s + 18 + 4s = 0$$

$$\therefore 20s = -22$$

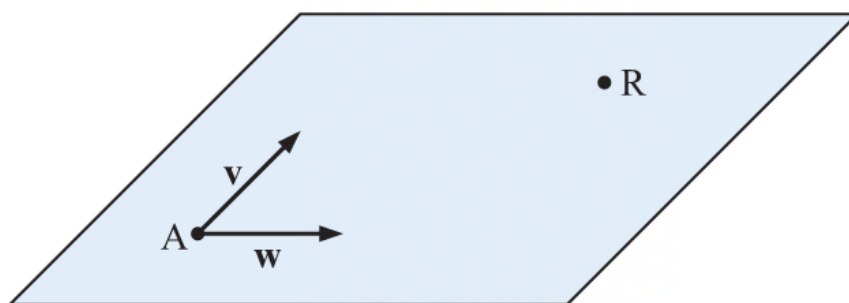
$$\therefore s = -\frac{22}{20} = -\frac{11}{10}$$

$$\text{Thus } \overrightarrow{PN} = \begin{pmatrix} 1 - \frac{22}{5} \\ -9 + \frac{11}{5} \\ -2 \end{pmatrix} = \begin{pmatrix} -\frac{17}{5} \\ -\frac{34}{5} \\ -2 \end{pmatrix} \text{ and } |\overrightarrow{PN}| = \sqrt{\left(-\frac{17}{5}\right)^2 + \left(-\frac{34}{5}\right)^2 + (-2)^2} \\ = \frac{\sqrt{6180}}{10} \approx 7.86 \text{ units}$$

- 9 Since the lines are coincident, the shortest distance between these lines is 0 units.

## INVESTIGATION 2

## LINEAR COMBINATIONS



- 1 a The vector  $s\mathbf{v}$  is parallel to  $\mathbf{v}$  for  $s \in \mathbb{R}$ .

$\therefore s\mathbf{v}$  is parallel to the plane.

$\therefore$  the vector  $\mathbf{a} + s\mathbf{v}$  travels from O to A, and then parallel to the plane that A is on.

$\therefore$  the position vector  $\mathbf{a} + s\mathbf{v}$  lies on the plane for any  $s \in \mathbb{R}$ .

- b Suppose B is a point on the plane with position vector  $\mathbf{a} + s\mathbf{v}$ ,  $s \in \mathbb{R}$ . {using a}

Now,  $\mathbf{w}$  is parallel to the plane

$\therefore t\mathbf{w}$  is parallel to the plane.

$\therefore$  the vector  $\mathbf{a} + s\mathbf{v} + t\mathbf{w}$  travels from O to B, and then parallel to the plane that B is on.

$\therefore$  the position vector  $\mathbf{a} + s\mathbf{v} + t\mathbf{w}$  lies on the plane for any  $s, t \in \mathbb{R}$ .

- c If  $\mathbf{v}$  and  $\mathbf{w}$  are parallel, then  $\mathbf{w} = k\mathbf{v}$  for some  $k \in \mathbb{R}$ .

Then, the position vector of R is  $\mathbf{r} = \mathbf{a} + s\mathbf{v} + t\mathbf{w}$

$$= \mathbf{a} + s\mathbf{v} + t(k\mathbf{v})$$

$$= \mathbf{a} + (s + tk)\mathbf{v} \text{ for some } s, t \in \mathbb{R}$$

So, R lies on the line which passes through A in the direction  $\mathbf{v}$ .

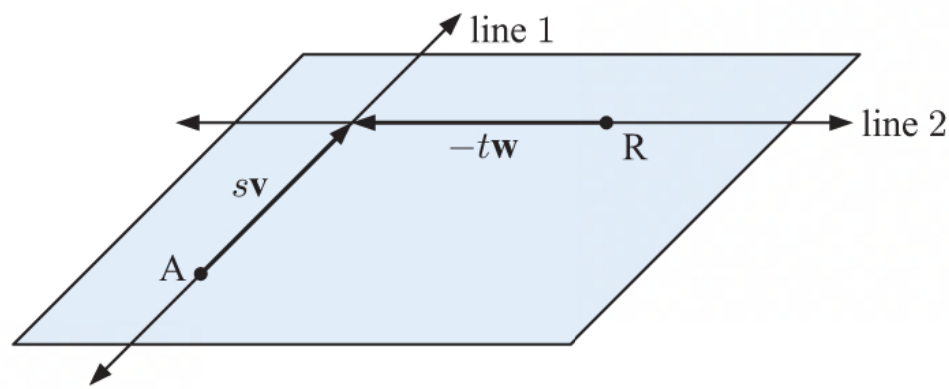
However, there are many points on the plane which do not lie on this line.

Thus, to travel from A to *any* point R using a linear combination of  $\mathbf{v}$  and  $\mathbf{w}$ , it is important that  $\mathbf{v}$  and  $\mathbf{w}$  are non-parallel.



- 2** Line 1:  $\mathbf{a} + s\mathbf{v}$ ,  $s \in \mathbb{R}$   
 Line 2:  $\mathbf{r} - t\mathbf{w}$ ,  $t \in \mathbb{R}$

**a**



**b** The lines meet where  $\mathbf{a} + s\mathbf{v} = \mathbf{r} - t\mathbf{w}$

**c** The vector equation in **b** is  $\mathbf{a} + s\mathbf{v} = \mathbf{r} - t\mathbf{w}$   
 which is  $\mathbf{r} = \mathbf{a} + s\mathbf{v} + t\mathbf{w}$

Solving this equation for  $s$  and  $t$  gives us a way to move from A to R using a linear combination of  $\mathbf{v}$  and  $\mathbf{w}$ .

Geometrically, this describes travelling from A along line 1 to the point where the lines meet, then along line 2 to R.

**d** Since line 1 and line 2 both lie on the plane, and are not parallel, as  $\mathbf{v}$  and  $\mathbf{w}$  are non-parallel, the lines must intersect at a unique point.

$\therefore$  solving the vector equation in **b** for  $s$  and  $t$  will give us a unique solution.

- 3 a** Yes. Given any point R on the plane, and any two vectors  $\mathbf{v}$  and  $\mathbf{w}$  which are parallel to the plane but not each other, we can solve the equation in **2 c** to find a linear combination of  $\mathbf{v}$  and  $\mathbf{w}$  which goes from A to R.
- b** No. If A is on the plane and  $\mathbf{v}$  and  $\mathbf{w}$  are parallel to the plane, then you will always remain on the plane when travelling from A along a linear combination of  $\mathbf{v}$  and  $\mathbf{w}$ . {using **1 b**}

## EXERCISE 13G

- 1 a**  $\mathbf{n} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ , and the point  $(-1, 2, 4)$  lies on the plane.

$\therefore$  the equation is  $2x - y + 3z = 2(-1) - 2 + 3(4)$  which is  $2x - y + 3z = 8$ .

- b**  $\overrightarrow{AB} = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$  is a vector normal to the plane, and  $(2, 3, 1)$  lies on the plane.

$\therefore$  the equation is  $3x + 4y + z = 3(2) + 4(3) + 1$

$$\therefore 3x + 4y + z = 19$$

- 2 a**  $2x + 3y - z = 8$  has  $\mathbf{n} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$

- b**  $3x - y + 0z = 11$  has  $\mathbf{n} = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$

- c**  $0x + 0y + z = 2$  has  $\mathbf{n} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

- d**  $1x + 0y + 0z = 0$  has  $\mathbf{n} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

- 3 a** The  $y$ -axis is perpendicular to the  $XZ$ -plane  $\therefore$  a normal vector is  $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$   
 $\therefore$  since the origin lies on the plane, it has equation  $y = 0$ .

- b** Since the plane is perpendicular to the  $Z$ -axis, it has normal vector  $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$   
 $\therefore$  since  $(2, -1, 4)$  lies on the plane, it has equation  $z = 4$ .

- 4 a**  $A(0, 2, 6)$ ,  $B(1, 3, 2)$ ,  $C(-1, 2, 4)$

- i**  $\overrightarrow{AB} = \begin{pmatrix} 1 \\ 1 \\ -4 \end{pmatrix}$  and  $\overrightarrow{CB} = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$  are two non-parallel vectors in the plane.

Using  $C$  as the known (fixed) point on the plane,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix} + s \begin{pmatrix} 1 \\ 1 \\ -4 \end{pmatrix} + t \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}, \quad s, t \in \mathbb{R}.$$

- ii** If  $\mathbf{n}$  is the normal vector, then

$$\begin{aligned} \mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -4 \\ -1 & 0 & -2 \end{vmatrix} \\ &= \begin{vmatrix} 1 & -4 \\ 0 & -2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & -4 \\ -1 & -2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 1 \\ -1 & 0 \end{vmatrix} \mathbf{k} \\ &= -2\mathbf{i} + 6\mathbf{j} + \mathbf{k} \end{aligned}$$

$\therefore$  since  $A(0, 2, 6)$  lies on the plane, it has equation

$$-2x + 6y + z = -2(0) + 6(2) + 6$$

$$\therefore -2x + 6y + z = 18$$

- b**  $A(3, 1, 2)$ ,  $B(0, 4, 0)$ ,  $C(0, 0, 1)$

- i**  $\overrightarrow{AB} = \begin{pmatrix} -3 \\ 3 \\ -2 \end{pmatrix}$  and  $\overrightarrow{CB} = \begin{pmatrix} 0 \\ 4 \\ -1 \end{pmatrix}$  are two non-parallel vectors in the plane.

Using  $C$  as the known (fixed) point on the plane,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + s \begin{pmatrix} -3 \\ 3 \\ -2 \end{pmatrix} + t \begin{pmatrix} 0 \\ 4 \\ -1 \end{pmatrix}, \quad s, t \in \mathbb{R}.$$

ii If  $\mathbf{n}$  is the normal vector, then

$$\begin{aligned}\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 3 & -2 \\ -3 & -1 & -1 \end{vmatrix} \\ &= \begin{vmatrix} 3 & -2 \\ -1 & -1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -3 & -2 \\ -3 & -1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -3 & 3 \\ -3 & -1 \end{vmatrix} \mathbf{k} \\ &= -5\mathbf{i} + 3\mathbf{j} + 12\mathbf{k}\end{aligned}$$

$\therefore$  since  $C(0, 0, 1)$  lies on the plane, it has equation

$$-5x + 3y + 12z = -5(0) + 3(0) + 12(1)$$

$$\therefore -5x + 3y + 12z = 12$$

c  $A(2, 0, 3)$ ,  $B(0, -1, 2)$ ,  $C(4, -3, 0)$

i  $\overrightarrow{AB} = \begin{pmatrix} -2 \\ -1 \\ -1 \end{pmatrix}$  and  $\overrightarrow{CB} = \begin{pmatrix} -4 \\ 2 \\ 2 \end{pmatrix}$  are two non-parallel vectors in the plane.

Using  $C$  as the known (fixed) point on the plane,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \\ 0 \end{pmatrix} + s \begin{pmatrix} -2 \\ -1 \\ -1 \end{pmatrix} + t \begin{pmatrix} -4 \\ 2 \\ 2 \end{pmatrix} \quad s, t \in \mathbb{R}.$$

ii If  $\mathbf{n}$  is the normal vector, then

$$\begin{aligned}\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & -1 & -1 \\ 2 & -3 & -3 \end{vmatrix} \\ &= \begin{vmatrix} -1 & -1 \\ -3 & -3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -2 & -1 \\ 2 & -3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -2 & -1 \\ 2 & -3 \end{vmatrix} \mathbf{k} \\ &= -8\mathbf{j} + 8\mathbf{k} \quad \text{or} \quad -8(\mathbf{j} - \mathbf{k})\end{aligned}$$

$\therefore$  since  $A(2, 0, 3)$  lies on the plane, it has equation

$$y - z = 0 - 3$$

$$\therefore -y + z = 3$$

5 The normal vector  $\mathbf{n} = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} \times \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}$

$$\begin{aligned}\therefore \mathbf{n} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -3 \\ -2 & 1 & -1 \end{vmatrix} \\ &= \begin{vmatrix} 0 & -3 \\ 1 & -1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & -3 \\ -2 & -1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 0 \\ -2 & 1 \end{vmatrix} \mathbf{k} \\ &= 3\mathbf{i} + 7\mathbf{j} + \mathbf{k}\end{aligned}$$

$\therefore$  since  $(3, -1, 1)$  lies on the plane, it has equation  $3x + 7y + z = 3(3) + 7(-1) + 1$

$$\therefore 3x + 7y + z = 3$$

- 6 The line  $x = 1 + t$ ,  $y = 2 - t$ ,  $z = 3 + 2t$  has direction vector  $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ .

Also, letting  $t = 0$ , the point  $(1, 2, 3)$  lies on the plane and we call this point B.

$$\therefore \overrightarrow{AB} = \begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix} \text{ and if } \mathbf{n} \text{ is the normal vector to the plane, then } \mathbf{n} = \overrightarrow{AB} \times \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

$$\begin{aligned} \therefore \mathbf{n} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 0 & 2 \\ 1 & -1 & 2 \end{vmatrix} \\ &= \begin{vmatrix} 0 & 2 \\ -1 & 2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -2 & 2 \\ 1 & 2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -2 & 0 \\ 1 & -1 \end{vmatrix} \mathbf{k} \\ &= 2\mathbf{i} + 6\mathbf{j} + 2\mathbf{k} \text{ or } 2(\mathbf{i} + 3\mathbf{j} + \mathbf{k}) \end{aligned}$$

$$\begin{aligned} \therefore \text{ since } A(3, 2, 1) \text{ lies on the plane, it has equation } x + 3y + z &= 3 + 3(2) + 1 \\ \therefore x + 3y + z &= 10 \end{aligned}$$

7  $\overrightarrow{AB} = \begin{pmatrix} -1 \\ -3 \\ -1 \end{pmatrix}$

- a The normal  $\mathbf{n}$  is perpendicular to both the  $X$ -axis and  $\overrightarrow{AB}$ .

$$\begin{aligned} \text{Since the } X\text{-axis has direction vector } \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \\ \mathbf{n} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ -1 & -3 & -1 \end{vmatrix} \\ &= \begin{vmatrix} 0 & 0 \\ -3 & -1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 0 \\ -1 & -1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 0 \\ -1 & -3 \end{vmatrix} \mathbf{k} \\ &= \mathbf{j} - 3\mathbf{k} \end{aligned}$$

$$\begin{aligned} \text{Since } A(1, 2, 3) \text{ is in the plane, the plane has equation } y - 3z &= 1(2) - 3(3) \\ \therefore y - 3z &= -7 \end{aligned}$$

- b The normal  $\mathbf{n}$  is perpendicular to both the  $Y$ -axis and  $\overrightarrow{AB}$ .

$$\begin{aligned} \text{Since the } Y\text{-axis has direction vector } \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \\ \mathbf{n} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ -1 & -3 & -1 \end{vmatrix} \\ &= \begin{vmatrix} 1 & 0 \\ -3 & -1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 0 & 0 \\ -1 & -1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 0 & 1 \\ -1 & -3 \end{vmatrix} \mathbf{k} \\ &= -\mathbf{i} + \mathbf{k} \end{aligned}$$

$$\begin{aligned} \text{Since } A(1, 2, 3) \text{ is in the plane, the plane has equation } -x + z &= -1(1) + 1(3) = 2 \\ \therefore x - z &= -2 \end{aligned}$$



- c** The normal  $\mathbf{n}$  is perpendicular to both the  $Z$ -axis and  $\overrightarrow{AB}$ .

Since the  $Z$ -axis has direction vector  $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ ,

$$\begin{aligned}\mathbf{n} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ -1 & -3 & -1 \end{vmatrix} \\ &= \begin{vmatrix} 0 & 1 \\ -3 & -1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 0 & 1 \\ -1 & -1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 0 & 0 \\ -1 & -3 \end{vmatrix} \mathbf{k} \\ &= 3\mathbf{i} - \mathbf{j}\end{aligned}$$

Since  $A(1, 2, 3)$  is in the plane, the plane has equation  $3x - y = 3(1) - 1(2)$   
 $\therefore 3x - y = 1$

- 8 a** The normal to  $x - 3y + 4z = 8$  is  $\begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix}$ , and this is the direction vector of the line.

$\therefore$  since the line passes through  $(1, -2, 0)$ , it has equation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix}, \lambda \in \mathbb{R}.$$

- b** The normal to  $x - y - 2z = 11$  is  $\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$ .

$\therefore$  since the line passes through  $(3, 4, -1)$ , it has equation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}, \lambda \in \mathbb{R}.$$

- 9** The plane  $x + 4y - z = -2$  has normal  $\mathbf{n} = \begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix}$  which passes through  $(3, 4, -1)$ .

$\therefore$  the normal has parametric equations  $x = 3 + t$ ,  $y = 4 + 4t$ ,  $z = -1 - t$ ,  $t \in \mathbb{R}$ . It will meet any of the coordinate axes if any two of the values of  $x$ ,  $y$ , and  $z$  are zero at the same time.

Now  $x = 0$  when  $t = -3$  and  $y = z = 0$  when  $t = -1$ .

$\therefore$  the normal meets the  $X$ -axis when  $t = -1$ , at the point  $(2, 0, 0)$ .

**10** Now  $x - 1 = \frac{y-2}{2} = z + 3$  has direction vector  $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

and  $x + 1 = y - 3 = 2z + 5$  has direction vector  $\begin{pmatrix} 1 \\ 1 \\ \frac{1}{2} \end{pmatrix}$   $\left\{ \text{since } 2z + 5 = \frac{z + \frac{5}{2}}{\frac{1}{2}} \right\}$

$$\begin{aligned} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ \frac{1}{2} \end{pmatrix} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ 1 & 1 & \frac{1}{2} \end{vmatrix} \\ &= \begin{vmatrix} 2 & 1 \\ 1 & \frac{1}{2} \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 1 \\ 1 & \frac{1}{2} \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} \mathbf{k} \\ &= 0\mathbf{i} + \frac{1}{2}\mathbf{j} - \mathbf{k} \end{aligned}$$

$\therefore \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$  is perpendicular to both lines.

A plane with normal  $\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$  has equation  $y - 2z = c$  for some  $c$ .

Now for line 1,  $\frac{y-2}{2} = z + 3$  and for line 2,  $y - 3 = 2z + 5$   
 $\therefore y - 2 = 2z + 6$   $\therefore y - 2z = 8$   
 $\therefore y - 2z = 8$

$\therefore y - 2z = 8$  is a plane containing both lines, so the lines are coplanar.

**11 a**  $L_1$  and  $L_2$  meet where  $-4 + 3\lambda = \frac{2 + \lambda - 5}{2} = \frac{-(-1 + 2\lambda) - 1}{2}$   
 $\therefore -4 + 3\lambda = \frac{\lambda - 3}{2}$   
 $\therefore -8 + 6\lambda = \lambda - 3$   
 $\therefore 5\lambda = 5$   
 $\therefore \lambda = 1$

Check:  $\frac{\lambda - 3}{2} = \frac{1 - 2\lambda - 1}{2}$   
 $\therefore \lambda - 3 = -2\lambda$   
 $\therefore 3\lambda = 3$   
 $\therefore \lambda = 1$  ✓

$$\begin{aligned} \text{When } \lambda = 1, \mathbf{r} &= \begin{pmatrix} -4 \\ 2 \\ -1 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} \end{aligned}$$

$\therefore L_1$  and  $L_2$  meet at  $(-1, 3, 1)$ .

**b**  $L_1$  has direction vector  $\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$  and  $L_2$  has direction vector  $\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$ .

$$\begin{aligned} \text{If } \mathbf{n} \text{ is the normal vector, then } \mathbf{n} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & 2 \\ 1 & 2 & -2 \end{vmatrix} \\ &= \begin{vmatrix} 1 & 2 \\ 2 & -2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 3 & 2 \\ 1 & -2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} \mathbf{k} \\ &= -6\mathbf{i} + 8\mathbf{j} + 5\mathbf{k} \end{aligned}$$

$\therefore$  since  $(-1, 3, 1)$  lies on the plane, it has equation

$$-6x + 8y + 5z = -6(-1) + 8(3) + 5(1)$$

$$\therefore -6x + 8y + 5z = 35$$

$\therefore 6x - 8y - 5z = -35$  is the equation of the plane containing  $L_1$  and  $L_2$ .

**12 a** Since  $A(1, 2, k)$  lies on  $x + 2y - 2z = 8$ ,  $1 + 2(2) - 2k = 8$

$$\therefore 1 + 4 - 2k = 8$$

$$\therefore -2k = 3$$

$$\therefore k = -\frac{3}{2}$$

**b** Since  $x + 2y - 2z = 8$ , the plane has normal vector  $\mathbf{n} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$ .

$\therefore$  the normal through A has parametric equations

$$x = 1 + t, \quad y = 2 + 2t, \quad z = -\frac{3}{2} - 2t, \quad t \in \mathbb{R}.$$

$\therefore$  for a point on the normal 6 units from A,

$$\sqrt{(1+t-1)^2 + (2+2t-2)^2 + (-\frac{3}{2}-2t+\frac{3}{2})^2} = 6$$

$$\therefore \sqrt{t^2 + 4t^2 + 4t^2} = 6$$

$$\therefore 9t^2 = 36$$

$$\therefore t^2 = 4$$

$$\therefore t = \pm 2$$

$\therefore$  B is  $(3, 6, -\frac{11}{2})$  or  $(-1, -2, \frac{5}{2})$ .

**13**  $\overrightarrow{AB} = \begin{pmatrix} -1 \\ 3 \\ -3 \end{pmatrix}$  so (AB) has parametric equations  
 $x = 2 - t, \quad y = -1 + 3t, \quad z = 3 - 3t, \quad t \in \mathbb{R} \quad \dots (*)$

This line meets  $x + 2y - z = 21$  when

$$(2 - t) + 2(-1 + 3t) - (3 - 3t) = 21$$

$$\therefore 2 - t - 2 + 6t - 3 + 3t = 21$$

$$\therefore 8t = 24$$

$$\therefore t = 3$$

Substituting  $t = 3$  into  $(*)$ , the line meets the plane at  $(-1, 8, -6)$ .

**14 a**  $\overrightarrow{PQ} = \begin{pmatrix} 1 \\ 2 \\ -5 \end{pmatrix}$  so (PQ) has parametric equations  
 $x = 1 + t, y = -2 + 2t, z = 4 - 5t, t \in \mathbb{R}.$

**b i** The line meets the  $YZ$ -plane when  $x = 0$ , or when  $t = -1$ .  
 This corresponds to the point  $(0, -4, 9)$ .

**ii** The line meets  $y + z = 2$  when  $-2 + 2t + 4 - 5t = 2$   
 $\therefore -3t = 0$   
 $\therefore t = 0$

This corresponds to the point  $(1, -2, 4)$ .

**iii** The line meets  $\frac{x-3}{2} = \frac{y+2}{3} = \frac{z-30}{-1}$

when  $\frac{1+t-3}{2} = \frac{-2+2t+2}{3} = \frac{4-5t-30}{-1}$

$$\therefore \frac{t-2}{2} = \frac{2t}{3} = 5t+26$$

$$\therefore 3t-6 = 4t = 30t+156$$

$$\therefore 3t-6 = 4t \quad \text{and} \quad 4t = 30t+156$$

$$\therefore t = -6 \quad \text{and} \quad -26t = 156$$

$$\therefore t = -6$$

$\therefore$  the lines meet at the point corresponding to  $t = -6$ , which is  $(-5, -14, 34)$ .

**15 a** The normal to the plane  $P$  is  $\mathbf{n} = \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix}$ , which is the direction vector of the line  $L$ .

$$\therefore L = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix} + t \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix}, \quad t \in \mathbb{R}$$

$$\therefore x = 1 + 2t, \quad y = 5 - 5t, \quad z = 3 + t, \quad t \in \mathbb{R}$$

**b**  $L$  meets  $P$  where  $2(1+2t) - 5(5-5t) + 3+t = 10$

$$\therefore 2 + 4t - 25 + 25t + 3 + t = 10$$

$$\therefore 30t = 30$$

$$\therefore t = 1$$

$$\text{When } t = 1, \quad x = 1 + 2(1) = 3$$

$$y = 5 - 5(1) = 0$$

$$z = 3 + 1 = 4$$

$\therefore L$  meets  $P$  at  $(3, 0, 4)$ .



**16 a**  $2x + y - 2z = -11$  has  $\mathbf{n} = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$

$\therefore$  the parametric equations of (AN) are  $x = 1 + 2t$ ,  $y = 0 + t$ ,  $z = 2 - 2t$ ,  $t \in \mathbb{R}$ .

This line meets the plane when  $2(1 + 2t) + t - 2(2 - 2t) = -11$

$$\therefore 2 + 4t + t - 4 + 4t = -11$$

$$\therefore 9t = -9$$

$$\therefore t = -1$$

$\therefore$  N is  $(-1, -1, 4)$  and  $\overrightarrow{AN} = \begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix}$ .

The shortest distance,  $|\overrightarrow{AN}| = \sqrt{(-2)^2 + (-1)^2 + 2^2}$   
 $= \sqrt{9} = 3$  units.

**b**  $x - y + 3z = -10$  has  $\mathbf{n} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$

$\therefore$  the parametric equations of (AN) are  $x = 2 + t$ ,  $y = -1 - t$ ,  $z = 3 + 3t$ ,  $t \in \mathbb{R}$ .

This line meets the plane when  $(2 + t) - (-1 - t) + 3(3 + 3t) = -10$

$$\therefore 2 + t + 1 + t + 9 + 9t = -10$$

$$\therefore 11t = -22$$

$$\therefore t = -2$$

$\therefore$  N is  $(0, 1, -3)$  and  $\overrightarrow{AN} = \begin{pmatrix} -2 \\ 2 \\ -6 \end{pmatrix}$ .

The shortest distance,  $|\overrightarrow{AN}| = \sqrt{(-2)^2 + 2^2 + (-6)^2}$   
 $= \sqrt{44} = 2\sqrt{11}$  units.

**c**  $4x - y - 2z = 8$  has  $\mathbf{n} = \begin{pmatrix} 4 \\ -1 \\ -2 \end{pmatrix}$

$\therefore$  the parametric equations of (AN) are

$$x = 1 + 4t, \quad y = -4 - t, \quad z = -3 - 2t, \quad t \in \mathbb{R}.$$

This line meets the plane when  $4(1 + 4t) - (-4 - t) - 2(-3 - 2t) = 8$

$$\therefore 4 + 16t + 4 + t + 6 + 4t = 8$$

$$\therefore 21t = -6$$

$$\therefore t = -\frac{2}{7}$$

$\therefore$  N is  $(-\frac{1}{7}, -\frac{26}{7}, -\frac{17}{7})$  and  $\overrightarrow{AN} = \begin{pmatrix} -\frac{8}{7} \\ \frac{2}{7} \\ \frac{4}{7} \end{pmatrix}$ .

The shortest distance,  $|\overrightarrow{AN}| = \sqrt{\left(-\frac{8}{7}\right)^2 + \left(\frac{2}{7}\right)^2 + \left(\frac{4}{7}\right)^2}$   
 $= \sqrt{\frac{84}{49}} = 2\sqrt{\frac{3}{7}}$  units.

- 17** Let N be the foot of the normal from A(3, 1, 2) to the plane  $x + 2y + z = 1$ .

$x + 2y + z = 1$  has normal vector  $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ .

$\therefore$  the parametric equations of (AN) are  $x = 3 + t$ ,  $y = 1 + 2t$ ,  $z = 2 + t$ ,  $t \in \mathbb{R}$ .

The line meets the plane  $x + 2y + z = 1$  where

$$3 + t + 2(1 + 2t) + 2 + t = 1$$

$$\therefore 3 + t + 2 + 4t + 2 + t = 1$$

$$\therefore 7 + 6t = 1$$

$$\therefore 6t = -6$$

$$\therefore t = -1$$

$\therefore$  N has coordinates (2, -1, 1).

Now let  $A'(a, b, c)$  be the mirror image of A when reflected in the plane.

$$\therefore \overrightarrow{NA'} = \overrightarrow{AN}$$

$$\therefore \begin{pmatrix} a - 2 \\ b + 1 \\ c - 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix} \quad \therefore \begin{matrix} a - 2 = -1, & b + 1 = -2, & \text{and} & c - 1 = -1 \\ \therefore a = 1, & b = -3, & \text{and} & c = 0 \end{matrix}$$

$\therefore$  A' has coordinates (1, -3, 0).

- 18 a** A vector normal to the plane is

$$\begin{aligned} (2\mathbf{i} - \mathbf{j} + \mathbf{k}) \times (\mathbf{i} + \mathbf{j} - 2\mathbf{k}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 1 \\ 1 & 1 & -2 \end{vmatrix} \\ &= \begin{vmatrix} -1 & 1 \\ 1 & -2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} \mathbf{k} \\ &= \mathbf{i} + 5\mathbf{j} + 3\mathbf{k} \end{aligned}$$

From the vector equation of the plane, the point (2, 0, -4) lies in the plane.

$\therefore$  the Cartesian equation of the plane is  $x + 5y + 3z = 2 + 5(0) + 3(-4)$

$$\therefore x + 5y + 3z = -10 \quad \dots (*)$$

- b** Parametric equations for the normal through A(5, 8, 5) are

$$x = 5 + \lambda, \quad y = 8 + 5\lambda, \quad z = 5 + 3\lambda, \quad \lambda \in \mathbb{R} \quad \{\text{using } (*)\}$$

so N has coordinates (5 +  $\lambda$ , 8 + 5 $\lambda$ , 5 + 3 $\lambda$ ) for some  $\lambda$ .

Substituting the coordinates into (\*) gives:  $(5 + \lambda) + 5(8 + 5\lambda) + 3(5 + 3\lambda) = -10$

$$\therefore 5 + \lambda + 40 + 25\lambda + 15 + 9\lambda = -10$$

$$\therefore 35\lambda = -70$$

$$\therefore \lambda = -2$$

$\therefore$  N has coordinates (3, -2, -1).

**19 a** A vector normal to the plane is

$$\begin{aligned}
 (2\mathbf{i} + \mathbf{j} + \mathbf{k}) \times (4\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 1 \\ 4 & 2 & -2 \end{vmatrix} \\
 &= \begin{vmatrix} 1 & 1 \\ 2 & -2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 1 \\ 4 & -2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} \mathbf{k} \\
 &= -4\mathbf{i} + 8\mathbf{j}
 \end{aligned}$$

From the vector equation of the plane, the point  $(3, 1, 2)$  lies in the plane.

$\therefore$  the Cartesian equation of the plane is  $-4x + 8y = -4(3) + 8(1)$

$$\therefore -4x + 8y = -4 \quad \dots (*)$$

Parametric equations for the normal through A are

$$x = 3 - 4t, \quad y = 2 + 8t, \quad z = 1, \quad t \in \mathbb{R}$$

so the foot of the normal from A is  $N(3 - 4t, 2 + 8t, 1)$  for some  $t$ .

Substituting the coordinates into  $(*)$  gives:  $-4(3 - 4t) + 8(2 + 8t) = -4$

$$\therefore -12 + 16t + 16 + 64t = -4$$

$$\therefore 80t = -8$$

$$\therefore t = -\frac{1}{10}$$

$\therefore$  N has coordinates  $(\frac{17}{5}, \frac{6}{5}, 1)$ .

$$\begin{aligned}
 \text{The shortest distance, } |\overrightarrow{AN}| &= \sqrt{(3.4 - 3)^2 + (1.2 - 2)^2 + (1 - 1)^2} \\
 &= \sqrt{0.8} \\
 &= \frac{2}{\sqrt{5}} \text{ units.}
 \end{aligned}$$

**b** A vector normal to the plane is

$$\begin{aligned}
 (3\mathbf{i} - \mathbf{j} + 2\mathbf{k}) \times (-\mathbf{i} + \mathbf{j} - \mathbf{k}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 2 \\ -1 & 1 & -1 \end{vmatrix} \\
 &= \begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 3 & 2 \\ -1 & -1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 3 & -1 \\ -1 & 1 \end{vmatrix} \mathbf{k} \\
 &= -\mathbf{i} + \mathbf{j} + 2\mathbf{k}
 \end{aligned}$$

From the vector equation of the plane, the point  $(1, -1, 1)$  lies in the plane.

$\therefore$  the Cartesian equation of the plane is  $-x + y + 2z = -(1) + (-1) + 2(1)$

$$\therefore -x + y + 2z = 0 \quad \dots (*)$$

Parametric equations for the normal through A are

$$x = 1 - t, \quad y = t, \quad z = -2 + 2t, \quad t \in \mathbb{R}$$

so the foot of the normal from A is  $N(1 - t, t, -2 + 2t)$  for some  $t$ .

Substituting the coordinates into  $(*)$  gives:  $-(1 - t) + t + 2(-2 + 2t) = 0$

$$\therefore -1 + t + t - 4 + 4t = 0$$

$$\therefore 6t = 5$$

$$\therefore t = \frac{5}{6}$$

$\therefore$  N has coordinates  $(\frac{1}{6}, \frac{5}{6}, -\frac{1}{3})$ .

$$\begin{aligned}
 \text{The shortest distance, } |\overrightarrow{AN}| &= \sqrt{\left(\frac{1}{6} - 1\right)^2 + \left(\frac{5}{6} - 0\right)^2 + \left(-\frac{1}{3} + 2\right)^2} \\
 &= \sqrt{\frac{25}{6}} \\
 &= \frac{5}{\sqrt{6}} \text{ units.}
 \end{aligned}$$

- 20**  $\mathbf{v}$  and  $\mathbf{w}$  are non-parallel vectors in the plane. The normal vector to the plane is  $\mathbf{n} = \mathbf{v} \times \mathbf{w}$ . We want to show that  $\mathbf{n} \perp (s\mathbf{v} + t\mathbf{w})$  for all  $s, t \in \mathbb{R}$  except when  $s = t = 0$ .

$$\begin{aligned}
 \mathbf{n} \cdot (s\mathbf{v} + t\mathbf{w}) &= (\mathbf{v} \times \mathbf{w}) \cdot (s\mathbf{v} + t\mathbf{w}) & \{\mathbf{n} = \mathbf{v} \times \mathbf{w}\} \\
 &= (\mathbf{v} \times \mathbf{w}) \cdot s\mathbf{v} + (\mathbf{v} \times \mathbf{w}) \cdot t\mathbf{w} & \{\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}\} \\
 &= 0 + 0 & \{(\mathbf{v} \times \mathbf{w}) \perp \mathbf{v} \therefore (\mathbf{v} \times \mathbf{w}) \perp \text{to any non-zero scalar multiple of } \mathbf{v}. \text{ Similarly for } \mathbf{w}. \} \\
 &= 0 \text{ for all } s, t \in \mathbb{R}
 \end{aligned}$$

$\mathbf{v}$  and  $\mathbf{w}$  are non-parallel, so  $\mathbf{n}$  is non-zero, and  $s\mathbf{v} + t\mathbf{w} = \mathbf{0}$  only when  $s = t = 0$ .

$\therefore \mathbf{n} \perp (s\mathbf{v} + t\mathbf{w})$  for all  $s, t \neq 0$   $\{\mathbf{a} \cdot \mathbf{b} = 0 \Leftrightarrow \mathbf{a} \perp \mathbf{b} \text{ provided } \mathbf{a} \text{ and } \mathbf{b} \text{ are non-zero}\}$

- 21 a** If  $N$  is the point on the plane such that  $(NP)$  is normal to the plane, then  $\triangle NPQ$  is right angled at  $N$ .

Draw a line parallel to  $\mathbf{n}$  through  $Q$ .

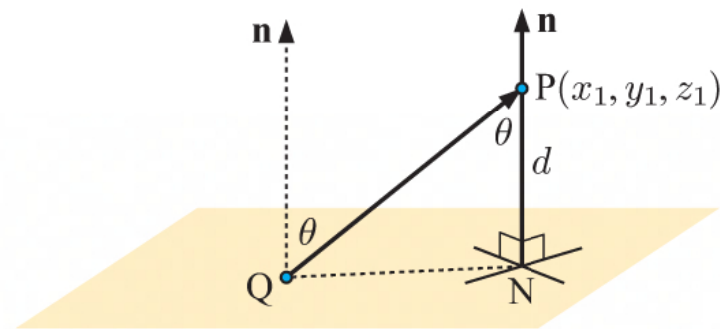
Let  $\theta$  be the angle between  $\mathbf{n}$  and  $\overrightarrow{QP}$ .

$$\therefore \cos \theta = \frac{|\overrightarrow{QP} \cdot \mathbf{n}|}{|\overrightarrow{QP}| |\mathbf{n}|} \quad \dots (1)$$

But  $\widehat{QPN} = \theta$   $\{\text{alternate angles}\}$

$$\therefore \cos \theta = \frac{d}{|\overrightarrow{QP}|} \quad \dots (2)$$

$$\begin{aligned}
 \text{Equating (1) and (2), } \frac{d}{|\overrightarrow{QP}|} &= \frac{|\overrightarrow{QP} \cdot \mathbf{n}|}{|\overrightarrow{QP}| |\mathbf{n}|} \\
 \therefore d &= \frac{|\overrightarrow{QP} \cdot \mathbf{n}|}{|\mathbf{n}|}
 \end{aligned}$$



- b** Since  $Q$  is any point on the plane, it has coordinates  $(x, y, z)$  such that  $Ax + By + Cz + D = 0$ .

The normal vector to the plane is  $\mathbf{n} = \begin{pmatrix} A \\ B \\ C \end{pmatrix}$ .

$$\begin{aligned}
 \text{Using a, } d &= \frac{|\overrightarrow{QP} \cdot \mathbf{n}|}{|\mathbf{n}|} = \frac{\left| \begin{pmatrix} x_1 - x \\ y_1 - y \\ z_1 - z \end{pmatrix} \cdot \begin{pmatrix} A \\ B \\ C \end{pmatrix} \right|}{\sqrt{A^2 + B^2 + C^2}} \\
 &= \frac{|Ax_1 - Ax + By_1 - By + Cz_1 - Cz|}{\sqrt{A^2 + B^2 + C^2}} \\
 &= \frac{|Ax_1 + By_1 + Cz_1 - (Ax + By + Cz)|}{\sqrt{A^2 + B^2 + C^2}} \\
 &= \frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}
 \end{aligned}$$



- c 16 a** Given  $A(1, 0, 2)$  and the plane  $2x + y - 2z + 11 = 0$ ,

$$d = \frac{|2x_1 + y_1 - 2z_1 + 11|}{\sqrt{2^2 + 1^2 + (-2)^2}} = \frac{|2(1) + 1(0) - 2(2) + 11|}{\sqrt{9}} = \frac{9}{3} = 3 \text{ units}$$

- 16 b** Given  $A(2, -1, 3)$  and the plane  $x - y + 3z = -10$ ,

$$d = \frac{|x_1 - y_1 + 3z_1 + 10|}{\sqrt{1^2 + (-1)^2 + 3^2}} = \frac{|2 - (-1) + 3(3) + 10|}{\sqrt{11}} = \frac{22}{\sqrt{11}} = 2\sqrt{11} \text{ units}$$

- 16 c** Given  $A(1, -4, -3)$  and the plane  $4x - y - 2z = 8$ ,

$$d = \frac{|4x_1 - y_1 - 2z_1 - 8|}{\sqrt{4^2 + (-1)^2 + (-2)^2}} = \frac{|4 - (-4) - 2(-3) - 8|}{\sqrt{21}} = \frac{6}{\sqrt{21}} \text{ units or } 2\sqrt{\frac{3}{7}} \text{ units}$$

- d** Using the formula derived in **b**,

$$\begin{aligned} \text{i} \quad d &= \frac{|x_1 + 2y_1 - z_1 - 10|}{\sqrt{1^2 + 2^2 + (-1)^2}} \\ &= \frac{|0 + 2(0) - 0 - 10|}{\sqrt{6}} \\ &= \frac{10}{\sqrt{6}} \text{ units} \end{aligned}$$

$$\begin{aligned} \text{ii} \quad d &= \frac{|x_1 + y_1 - z_1 - 2|}{\sqrt{1^2 + 1^2 + (-1)^2}} \\ &= \frac{|1 + (-3) - 2 - 2|}{\sqrt{3}} \\ &= \frac{|-6|}{\sqrt{3}} \\ &= \frac{6}{\sqrt{3}} = 2\sqrt{3} \text{ units} \end{aligned}$$

- 22 a** First choose a point on the first plane  $x + y + 2z = 4$ , for example,  $(0, 0, 2)$ .

Using the formula from **21 b** to calculate the distance from this point to the second plane,

$$\begin{aligned} d &= \frac{|2x_1 + 2y_1 + 4z_1 + 11|}{\sqrt{2^2 + 2^2 + 4^2}} \\ &= \frac{|2(0) + 2(0) + 4(2) + 11|}{\sqrt{24}} \\ &= \frac{19}{\sqrt{24}} = \frac{19}{2\sqrt{6}} \text{ units.} \end{aligned}$$

- b** Choose a point on the plane  $ax + by + cz + d_1 = 0$ , for example,  $\left(0, 0, -\frac{d_1}{c}\right)$ .

Using the formula from **21 b** to calculate the distance from this point to the second plane,

$$\begin{aligned} d &= \frac{|ax_1 + by_1 + cz_1 + d_2|}{\sqrt{a^2 + b^2 + c^2}} \\ &= \frac{\left|a(0) + b(0) + c\left(-\frac{d_1}{c}\right) + d_2\right|}{\sqrt{a^2 + b^2 + c^2}} \\ &= \frac{|d_2 - d_1|}{\sqrt{a^2 + b^2 + c^2}} \text{ units.} \end{aligned}$$

**23** The line  $x = 2 + t$ ,  $y = -1 + 2t$ ,  $z = -3t$  has direction vector  $\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$ ,

and  $\begin{pmatrix} 11 \\ -4 \\ 1 \end{pmatrix}$  is a vector normal to the plane  $11x - 4y + z = 0$ .

But  $\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 11 \\ -4 \\ 1 \end{pmatrix} = 11 - 8 - 3 = 0$

$\therefore$  these vectors are perpendicular and so the line is parallel to the plane.

Choose any point on the line, for example,  $(2, -1, 0)$ .

The distance from the line to the plane  $d = \frac{|11x_1 - 4y_1 + z_1|}{\sqrt{11^2 + (-4)^2 + 1^2}}$   
 $= \frac{|11(2) - 4(-1) + 0|}{\sqrt{138}}$   
 $= \frac{26}{\sqrt{138}}$  units.

**24** Since the planes are parallel to  $2x - y + 2z = 5$ , they have equation  $2x - y + 2z = a$  for some  $a$ .  
 Choose any point on  $2x - y + 2z = 5$ , for example,  $(0, -5, 0)$ .

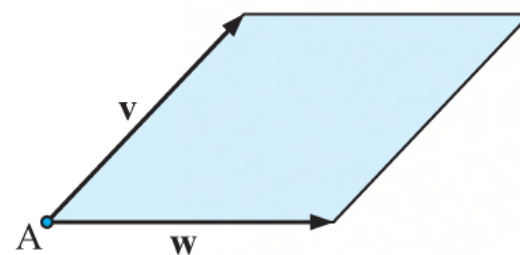
The distance from this point to the plane  $2x - y + 2z = a$ ,  $d = \frac{|2x_1 - y_1 + 2z_1 - a|}{\sqrt{2^2 + (-1)^2 + 2^2}}$   
 $\therefore 2 = \frac{|2(0) - (-5) + 2(0) - a|}{3}$   
 $\therefore 6 = |5 - a|$   
 $\therefore 5 - a = \pm 6$   
 $\therefore a = 5 \pm 6$   
 $\therefore a = -1 \text{ or } a = 11$

$\therefore$  the planes are  $2x - y + 2z = -1$  and  $2x - y + 2z = 11$ .

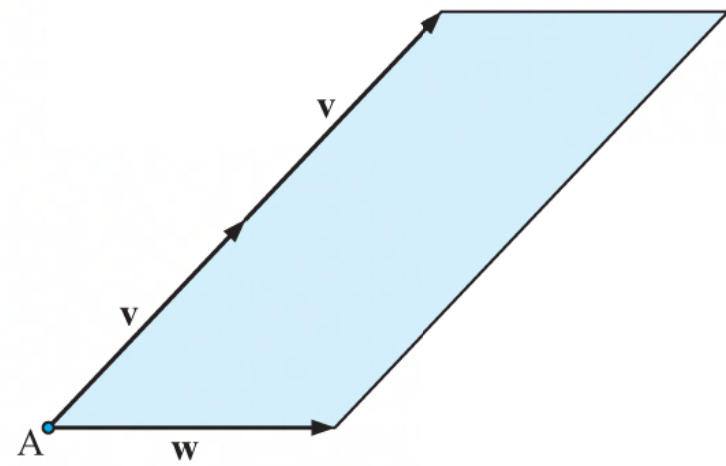
## INVESTIGATION 3

## REGIONS OF A PLANE

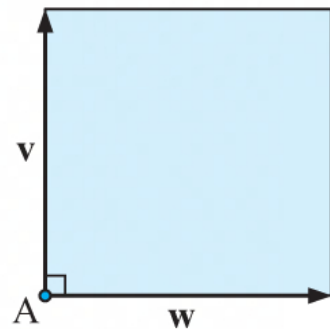
- 1 a** The shaded region includes all points  
 $\mathbf{r} = \mathbf{a} + s\mathbf{v} + t\mathbf{w}$ ,  $0 \leq s \leq 1$ ,  $0 \leq t \leq 1$ .
- b** The region is a parallelogram with side lengths  
 $|\mathbf{v}|$  and  $|\mathbf{w}|$ .



- c The region would still be a parallelogram, with side lengths  $2|\mathbf{v}|$  and  $|\mathbf{w}|$ .

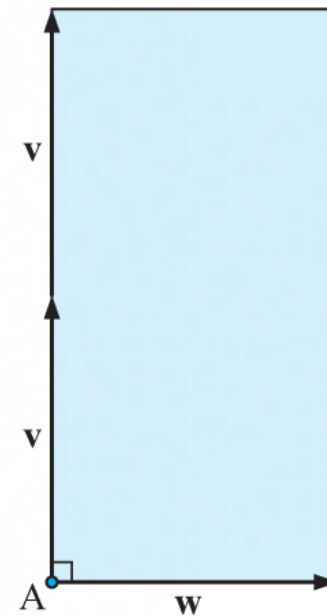


2 a



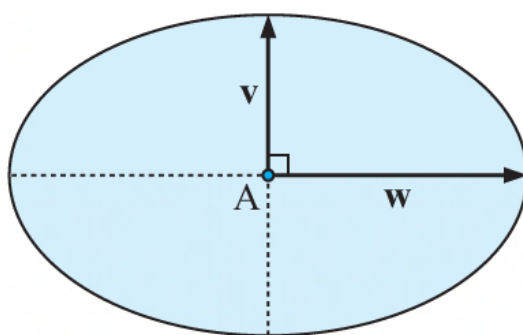
The region formed on the plane  $\mathbf{r} = \mathbf{a} + s\mathbf{v} + t\mathbf{w}$  is a square with side length  $|\mathbf{v}| = |\mathbf{w}|$ .

b



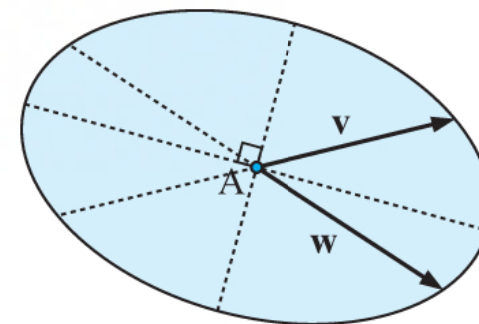
The region formed on the plane  $\mathbf{r} = \mathbf{a} + s\mathbf{v} + t\mathbf{w}$  is a rectangle with length  $2|\mathbf{v}| = 2|\mathbf{w}|$  and width  $|\mathbf{w}| = |\mathbf{v}|$ .

3 a



The region formed on the plane  $\mathbf{r} = \mathbf{a} + s\mathbf{v} + t\mathbf{w}$  is an ellipse with semi-minor and semi-major axes  $|\mathbf{v}|$  and  $|\mathbf{w}|$ .

b

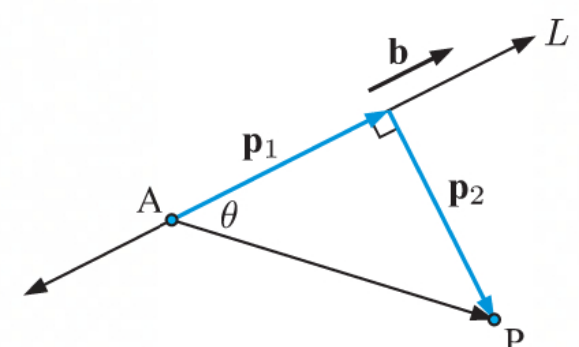


The region formed on the plane  $\mathbf{r} = \mathbf{a} + s\mathbf{v} + t\mathbf{w}$  is also an ellipse.

## INVESTIGATION 4

## SHORTEST DISTANCE FORMULAE

- 1 a  $\mathbf{p}_2$  is perpendicular to  $L$ , and covers the exact distance between  $L$  and  $P$ .  
 $\therefore$  the shortest distance from  $P$  to  $L$  is given by  $|\mathbf{p}_2|$ .

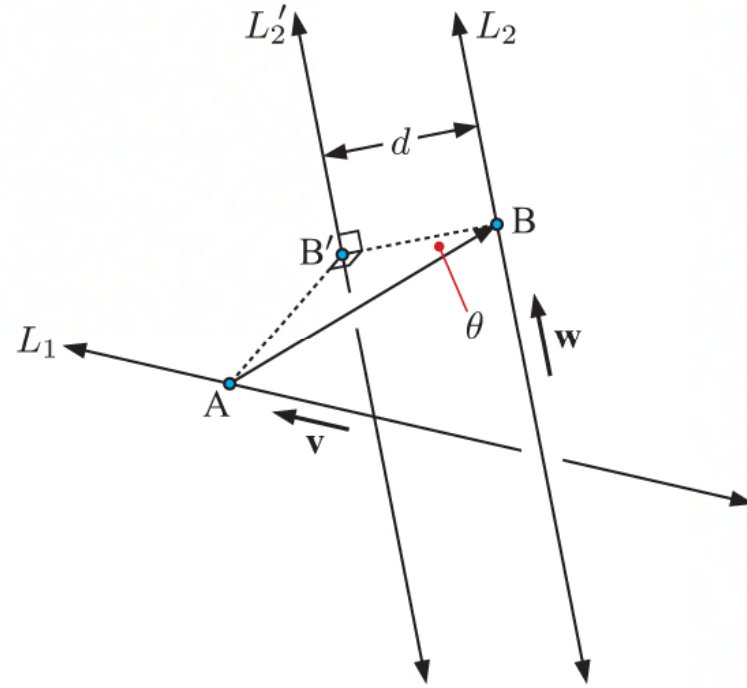


**b** From the diagram,  $\sin \theta = \frac{|\mathbf{p}_2|}{|\overrightarrow{AP}|}$

$$\begin{aligned} \text{So, } \frac{|\overrightarrow{AP} \times \mathbf{b}|}{|\mathbf{b}|} &= \frac{|\overrightarrow{AP}| |\mathbf{b}| \sin \theta}{|\mathbf{b}|} \\ &= |\overrightarrow{AP}| \frac{|\mathbf{p}_2|}{|\overrightarrow{AP}|} \\ &= |\mathbf{p}_2| \quad \text{which is the shortest distance from } P \text{ to } L \text{ as in } \mathbf{a}. \end{aligned}$$

**c** The expression in **b** is only useful in 3-dimensional space as the vector cross product is only defined for 3-dimensional vectors.

- 2 a**  $\overrightarrow{B'B}$  is perpendicular to the plane in which  $L_1$  and  $L_2'$  lie.  
 $L_2'$  is a translation of  $L_2$ , so has the same direction vector  $\mathbf{w}$ .  
 $L_1$  has direction vector  $\mathbf{v}$ , so  $\mathbf{v} \times \mathbf{w}$  is also perpendicular to the plane in which  $L_1$  and  $L_2'$  lie.  
 $\therefore \overrightarrow{B'B}$  is a scalar multiple of  $\mathbf{v} \times \mathbf{w}$ .



**b** From **a**,  $\overrightarrow{B'B}$  is a scalar multiple of  $\mathbf{v} \times \mathbf{w}$ , so  $\overrightarrow{B'B} = k(\mathbf{v} \times \mathbf{w})$  for some  $k \in \mathbb{R}$ ,  $k \neq 0$ .

$$\text{From the diagram, } \cos \theta = \frac{|\overrightarrow{B'B}|}{|\overrightarrow{AB}|}$$

$$\begin{aligned} \therefore |\overrightarrow{AB} \bullet \overrightarrow{B'B}| &= |\overrightarrow{AB}| |\overrightarrow{B'B}| \frac{|\overrightarrow{B'B}|}{|\overrightarrow{AB}|} \\ &= |\overrightarrow{AB}| |k(\mathbf{v} \times \mathbf{w})| \frac{|\overrightarrow{B'B}|}{|\overrightarrow{AB}|} \end{aligned}$$

$$\begin{aligned} \text{So, } \frac{|\overrightarrow{AB} \bullet (\mathbf{v} \times \mathbf{w})|}{|\mathbf{v} \times \mathbf{w}|} &= \frac{|\overrightarrow{AB} \bullet \frac{1}{k} \overrightarrow{B'B}|}{|\mathbf{v} \times \mathbf{w}|} \\ &= \frac{|\overrightarrow{AB} \bullet \overrightarrow{B'B}|}{|k| |\mathbf{v} \times \mathbf{w}|} \\ &= \frac{|k| |\mathbf{v} \times \mathbf{w}| |\overrightarrow{B'B}|}{|k| |\mathbf{v} \times \mathbf{w}|} \\ &= |\overrightarrow{B'B}| \\ &= d \end{aligned}$$

which is the shortest distance between  $L_2$  and  $L_2'$ .

$B'$  is on the plane in which  $L_1$  and  $L_2'$  lie, so  $\overrightarrow{B'B}$  is also perpendicular to  $L_1$ .

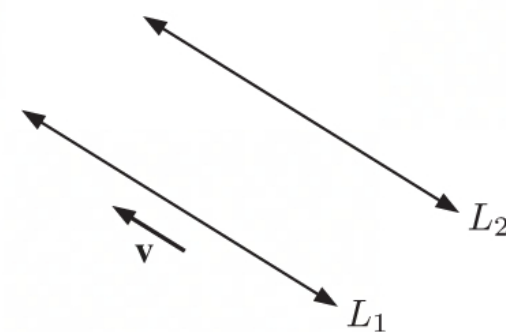
$\therefore |\overrightarrow{B'B}| = d$  is the shortest distance between  $L_1$  and  $L_2$ .



- c** If  $L_1$  and  $L_2$  are non-coincident parallel lines, then they both share the same direction vector  $\mathbf{v}$ .

Since  $\mathbf{v} \times \mathbf{v} = \mathbf{0}$ , the expression in **b** cannot be used to find the distance between  $L_1$  and  $L_2$ .

In the following verifications we use a vector  $\overrightarrow{AB}$  which is parallel to  $\mathbf{v} \times \mathbf{w}$ , but *any* vector between a point on each line would produce the same result.



**d** In **1 a**,  $\mathbf{v} \times \mathbf{w} = \begin{pmatrix} -4 \\ 1 \\ 3 \end{pmatrix}$

and  $\overrightarrow{AB} = \begin{pmatrix} \frac{4}{13} \\ -\frac{1}{13} \\ -\frac{3}{13} \end{pmatrix}$

$$\begin{aligned} d &= \frac{|\overrightarrow{AB} \bullet (\mathbf{v} \times \mathbf{w})|}{|\mathbf{v} \times \mathbf{w}|} \\ &= \frac{\left| \begin{pmatrix} \frac{4}{13} \\ -\frac{1}{13} \\ -\frac{3}{13} \end{pmatrix} \bullet \begin{pmatrix} -4 \\ 1 \\ 3 \end{pmatrix} \right|}{\left| \begin{pmatrix} -4 \\ 1 \\ 3 \end{pmatrix} \right|} \\ &= \frac{\left| -\frac{16}{3} - \frac{1}{13} - \frac{9}{13} \right|}{\sqrt{(-4)^2 + 1^2 + 3^2}} \\ &= \frac{2}{\sqrt{26}} \text{ units } \checkmark \end{aligned}$$

In **2 a**, the lines intersect at a point.

$\therefore \overrightarrow{AB}$  is on the plane perpendicular to  $\mathbf{v} \times \mathbf{w}$ .

$$\therefore \overrightarrow{AB} \bullet (\mathbf{v} \times \mathbf{w}) = 0$$

$$\begin{aligned} \therefore d &= \frac{|\overrightarrow{AB} \bullet (\mathbf{v} \times \mathbf{w})|}{|\mathbf{v} \times \mathbf{w}|} \\ &= 0 \text{ units } \checkmark \end{aligned}$$

In **1 b**,  $\mathbf{v} \times \mathbf{w} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

and  $\overrightarrow{AB} = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}$

$$\begin{aligned} d &= \frac{|\overrightarrow{AB} \bullet (\mathbf{v} \times \mathbf{w})|}{|\mathbf{v} \times \mathbf{w}|} \\ &= \frac{\left| \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right|}{\left| \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right|} \\ &= \frac{|-2 + 0 - 2|}{\sqrt{(-1)^2 + 0^2 + 1^2}} \\ &= \frac{4}{\sqrt{2}} \\ &= 2\sqrt{2} \text{ units } \checkmark \end{aligned}$$

In **2 b**,  $\mathbf{v} \times \mathbf{w} = \begin{pmatrix} -12 \\ 25 \\ 27 \end{pmatrix}$

and  $\overrightarrow{AB} = \begin{pmatrix} \frac{666}{749} \\ -\frac{2775}{1498} \\ -\frac{2997}{1498} \end{pmatrix}$

$$\begin{aligned} d &= \frac{|\overrightarrow{AB} \bullet (\mathbf{v} \times \mathbf{w})|}{|\mathbf{v} \times \mathbf{w}|} \\ &= \frac{\left| \begin{pmatrix} \frac{666}{749} \\ -\frac{2775}{1498} \\ -\frac{2997}{1498} \end{pmatrix} \bullet \begin{pmatrix} -12 \\ 25 \\ 27 \end{pmatrix} \right|}{\left| \begin{pmatrix} -12 \\ 25 \\ 27 \end{pmatrix} \right|} \\ &= \frac{\left| -\frac{7992}{749} - \frac{69375}{1498} - \frac{80919}{1498} \right|}{\sqrt{(-12)^2 + 25^2 + 27^2}} \\ &= \frac{111}{\sqrt{1498}} \text{ units } \checkmark \end{aligned}$$

In **2 c**, the lines are non-coincident and parallel, so we cannot use the expression to verify our answer.

$$\text{In } \mathbf{2 d}, \mathbf{v} \times \mathbf{w} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$\text{and } \overrightarrow{AB} = \begin{pmatrix} 0 \\ -\frac{3}{4} \\ -\frac{3}{4} \end{pmatrix}$$

$$\begin{aligned} d &= \frac{|\overrightarrow{AB} \bullet (\mathbf{v} \times \mathbf{w})|}{|\mathbf{v} \times \mathbf{w}|} \\ &= \frac{\left| \begin{pmatrix} 0 \\ -\frac{3}{4} \\ -\frac{3}{4} \end{pmatrix} \bullet \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right|}{\left| \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right|} \\ &= \frac{\left| 0 - \frac{3}{4} - \frac{3}{4} \right|}{\sqrt{0^2 + 1^2 + 1^2}} \\ &= \frac{\frac{6}{4}}{\sqrt{2}} \\ &= \frac{3}{2\sqrt{2}} \\ &= \frac{3\sqrt{2}}{4} \text{ units } \checkmark \end{aligned}$$

In **2 e**, the lines intersect at a point.

$\therefore \overrightarrow{AB}$  is on the plane perpendicular to  $\mathbf{v} \times \mathbf{w}$ .

$$\therefore \overrightarrow{AB} \bullet (\mathbf{v} \times \mathbf{w}) = 0$$

$$\begin{aligned} \therefore d &= \frac{|\overrightarrow{AB} \bullet (\mathbf{v} \times \mathbf{w})|}{|\mathbf{v} \times \mathbf{w}|} \\ &= 0 \text{ units } \checkmark \end{aligned}$$

In **2 f**, the lines are non-coincident and parallel, so we cannot use the expression to verify our answer.

In **2 g**,  $\mathbf{v} \times \mathbf{w} = \mathbf{0}$  since the lines are coincident, so we cannot use the expression to verify our answer.

**EXERCISE 13H**

**1 a**  $\mathbf{n} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$  and  $\mathbf{d} = \begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix}$

If  $\phi$  is the angle between the plane and the line, then

$$\begin{aligned} \sin \phi &= \frac{|\mathbf{n} \bullet \mathbf{d}|}{|\mathbf{n}| |\mathbf{d}|} \\ &= \frac{|4 - 3 + 1|}{\sqrt{3}\sqrt{26}} \\ &= \frac{2}{\sqrt{78}} \\ \therefore \phi &= \sin^{-1}\left(\frac{2}{\sqrt{78}}\right) \\ &\approx 13.1^\circ \end{aligned}$$

**b**  $\mathbf{n} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$  and  $\mathbf{d} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$

If  $\phi$  is the angle between the plane and the line, then

$$\begin{aligned} \sin \phi &= \frac{|\mathbf{n} \bullet \mathbf{d}|}{|\mathbf{n}| |\mathbf{d}|} \\ &= \frac{|2 - 3 + 1|}{\sqrt{6}\sqrt{11}} \\ &= 0 \\ \therefore \phi &= 0^\circ \end{aligned}$$

So, the line and plane are parallel.

**c**  $\mathbf{n} = \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}$  and  $\mathbf{d} = \begin{pmatrix} 1 \\ -1 \\ \frac{1}{2} \end{pmatrix}$

If  $\phi$  is the angle between the plane and the line, then

$$\begin{aligned} \sin \phi &= \frac{|\mathbf{n} \bullet \mathbf{d}|}{|\mathbf{n}| |\mathbf{d}|} \\ &= \frac{|3 + (-4) + (-\frac{1}{2})|}{\sqrt{26}\sqrt{\frac{9}{4}}} \\ &= \frac{\left|-\frac{3}{2}\right|}{\frac{3}{2}\sqrt{26}} \\ &= \frac{1}{\sqrt{26}} \\ \therefore \phi &= \sin^{-1}\left(\frac{1}{\sqrt{26}}\right) \\ &\approx 11.3^\circ \end{aligned}$$

$$\mathbf{d} \quad \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -4 & -1 \\ 1 & 1 & -2 \end{vmatrix} = \begin{vmatrix} -4 & -1 \\ 1 & -2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 3 & -1 \\ 1 & -2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 3 & -4 \\ 1 & 1 \end{vmatrix} \mathbf{k} = 9\mathbf{i} + 5\mathbf{j} + 7\mathbf{k}$$

and  $\mathbf{d} = \mathbf{i} - \mathbf{j} + \mathbf{k}$ .

If  $\phi$  is the angle between the plane and the line, then

$$\begin{aligned} \sin \phi &= \frac{|\mathbf{n} \cdot \mathbf{d}|}{|\mathbf{n}| |\mathbf{d}|} \\ &= \frac{|9 - 5 + 7|}{\sqrt{81 + 25 + 49} \sqrt{1 + 1 + 1}} \\ &= \frac{11}{\sqrt{155} \sqrt{3}} \\ \therefore \phi &= \sin^{-1} \left( \frac{11}{\sqrt{155} \sqrt{3}} \right) \\ &\approx 30.7^\circ \end{aligned}$$

$$2 \quad \mathbf{a} \quad \mathbf{n} = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} \quad \text{and} \quad \mathbf{d} = \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$$

If  $\phi$  is the angle between the plane and the line, then

$$\begin{aligned} \sin \phi &= \frac{|\mathbf{n} \cdot \mathbf{d}|}{|\mathbf{n}| |\mathbf{d}|} \\ &= \frac{|6 - 2 - 4|}{\sqrt{9} \sqrt{26}} \\ &= 0 \\ \therefore \phi &= 0^\circ \end{aligned}$$

So, the line and plane are parallel.

**b** Letting  $t = 0$ ,  $(1, 4, 2)$  lies on the line.

The distance from this point to the plane  $2x + 2y - z = 5$  is

$$\begin{aligned} d &= \frac{|2x_1 + 2y_1 - z_1 - 5|}{\sqrt{2^2 + 2^2 + (-1)^2}} \\ &= \frac{|2(1) + 2(4) - 2 - 5|}{3} \\ &= \frac{3}{3} \\ &= 1 \text{ unit} \end{aligned}$$

$$3 \quad \mathbf{a} \quad \mathbf{n} = \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix} \quad \text{and the } X\text{-axis has direction vector } \mathbf{d} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

If  $\phi$  is the angle between the plane and the  $X$ -axis, then

$$\begin{aligned} \sin \phi &= \frac{|\mathbf{n} \cdot \mathbf{d}|}{|\mathbf{n}| |\mathbf{d}|} \\ &= \frac{|3 + 0 + 0|}{\sqrt{26} \sqrt{1}} \\ &= \frac{3}{\sqrt{26}} \\ \therefore \phi &= \sin^{-1} \left( \frac{3}{\sqrt{26}} \right) \\ &\approx 36.0^\circ \end{aligned}$$



**b**  $\mathbf{n} = \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$  and the  $Y$ -axis has direction vector  $\mathbf{d} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ .

If  $\phi$  is the angle between the plane and the  $Y$ -axis, then

$$\begin{aligned} \sin \phi &= \frac{|\mathbf{n} \cdot \mathbf{d}|}{|\mathbf{n}| |\mathbf{d}|} \\ &= \frac{|0 - 1 + 0|}{\sqrt{26}\sqrt{1}} \\ &= \frac{1}{\sqrt{26}} \\ \therefore \phi &= \sin^{-1}\left(\frac{1}{\sqrt{26}}\right) \\ &\approx 11.3^\circ \end{aligned}$$

**c**  $\mathbf{n} = \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$  and the  $Z$ -axis has direction vector  $\mathbf{d} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ .

If  $\phi$  is the angle between the plane and the  $Z$ -axis, then

$$\begin{aligned} \sin \phi &= \frac{|\mathbf{n} \cdot \mathbf{d}|}{|\mathbf{n}| |\mathbf{d}|} \\ &= \frac{|0 + 0 + 4|}{\sqrt{26}\sqrt{1}} \\ &= \frac{4}{\sqrt{26}} \\ \therefore \phi &= \sin^{-1}\left(\frac{4}{\sqrt{26}}\right) \\ &\approx 51.7^\circ \end{aligned}$$

**4 a** The line  $L$  has direction vector  $\overrightarrow{AB} = \begin{pmatrix} 3 \\ 0 \\ -6 \end{pmatrix}$ , and the plane  $\pi_1$  has normal  $\mathbf{n} = \begin{pmatrix} 4 \\ -2 \\ 5 \end{pmatrix}$ .

If  $\phi$  is the angle between  $\pi_1$  and  $L$ , then

$$\begin{aligned} \sin \phi &= \frac{|\mathbf{n} \cdot \overrightarrow{AB}|}{|\mathbf{n}| |\overrightarrow{AB}|} \\ &= \frac{|12 + 0 - 30|}{\sqrt{45}\sqrt{45}} \\ &= \frac{18}{45} \\ \therefore \phi &= \sin^{-1}\left(\frac{18}{45}\right) \\ &\approx 23.6^\circ \end{aligned}$$

**b** When  $\pi_1$  cuts the  $Y$ -axis, the  $x$  and  $z$ -coordinates are both 0.

$$\begin{aligned} \therefore 4(0) - 2y + 5(0) &= 16 \\ \therefore -2y &= 16 \\ \therefore y &= -8 \end{aligned}$$

$\therefore \pi_1$  cuts the  $Y$ -axis at  $C(0, -8, 0)$ .

**c i**  $\vec{AC} = \begin{pmatrix} 3 \\ -10 \\ -5 \end{pmatrix}$  which is perpendicular to  $\pi_2$ .

$\therefore$  the normal of  $\pi_2$  is  $\mathbf{n} = \begin{pmatrix} 3 \\ -10 \\ -5 \end{pmatrix}$ .

Using the point  $B(0, 2, -1)$ , the equation of  $\pi_2$  is

$$3x - 10y - 5z = 3(0) - 10(2) - 5(-1)$$

$$\therefore 3x - 10y - 5z = -15$$

**ii** The line  $L$  has direction vector  $\vec{AB} = \begin{pmatrix} 3 \\ 0 \\ -6 \end{pmatrix}$  and the plane  $\pi_2$  has normal  $\mathbf{n} = \begin{pmatrix} 3 \\ -10 \\ -5 \end{pmatrix}$ .

If  $\phi$  is the angle between  $\pi_2$  and  $L$ , then

$$\begin{aligned} \sin \phi &= \frac{|\mathbf{n} \bullet \vec{AB}|}{|\mathbf{n}| |\vec{AB}|} \\ &= \frac{|9 + 0 + 30|}{\sqrt{45}\sqrt{34}} \\ &= \frac{39}{\sqrt{45}\sqrt{134}} \\ \therefore \phi &= \sin^{-1}\left(\frac{39}{\sqrt{45}\sqrt{134}}\right) \\ &\approx 30.1^\circ \end{aligned}$$

**5 a**  $\mathbf{n}_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$  and  $\mathbf{n}_2 = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$

If  $\theta$  is the acute angle between the

planes, then

$$\begin{aligned} \cos \theta &= \frac{|\mathbf{n}_1 \bullet \mathbf{n}_2|}{|\mathbf{n}_1| |\mathbf{n}_2|} \\ &= \frac{|2 - 3 + 2|}{\sqrt{6}\sqrt{14}} \\ &= \frac{1}{\sqrt{84}} \\ \therefore \theta &= \cos^{-1}\left(\frac{1}{\sqrt{84}}\right) \\ &\approx 83.7^\circ \end{aligned}$$

**b**  $\mathbf{n}_1 = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$  and  $\mathbf{n}_2 = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$

If  $\theta$  is the acute angle between the

planes, then

$$\begin{aligned} \cos \theta &= \frac{|\mathbf{n}_1 \bullet \mathbf{n}_2|}{|\mathbf{n}_1| |\mathbf{n}_2|} \\ &= \frac{|3 - 1 - 3|}{\sqrt{11}\sqrt{11}} \\ &= \frac{1}{11} \\ \therefore \theta &= \cos^{-1}\left(\frac{1}{11}\right) \\ &\approx 84.8^\circ \end{aligned}$$

$$\text{c } \mathbf{n}_1 = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} \text{ and } \mathbf{n}_2 = \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix}$$

If  $\theta$  is the acute angle between the planes, then

$$\begin{aligned} \cos \theta &= \frac{|\mathbf{n}_1 \bullet \mathbf{n}_2|}{|\mathbf{n}_1| |\mathbf{n}_2|} \\ &= \frac{|6 - 4 - 1|}{\sqrt{11}\sqrt{21}} \\ &= \frac{1}{\sqrt{231}} \\ \therefore \theta &= \cos^{-1}\left(\frac{1}{\sqrt{231}}\right) \\ &\approx 86.2^\circ \end{aligned}$$

$$\begin{aligned} \text{d } \mathbf{n}_1 &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & -1 \\ 2 & -4 & 3 \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ -4 & 3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -1 & -1 \\ 2 & 3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -1 & 1 \\ 2 & -4 \end{vmatrix} \mathbf{k} \\ &= -\mathbf{i} + \mathbf{j} + 2\mathbf{k} \\ &= \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \mathbf{n}_2 &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & -1 & -1 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} -1 & -1 \\ 1 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -2 & -1 \\ 1 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -2 & -1 \\ 1 & 1 \end{vmatrix} \mathbf{k} \\ &= \mathbf{j} - \mathbf{k} \\ &= \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \end{aligned}$$

If  $\theta$  is the acute angle between the planes, then

$$\begin{aligned} \cos \theta &= \frac{|\mathbf{n}_1 \bullet \mathbf{n}_2|}{|\mathbf{n}_1| |\mathbf{n}_2|} \\ &= \frac{|0 + 1 - 2|}{\sqrt{6}\sqrt{2}} \\ &= \frac{1}{\sqrt{12}} \\ \therefore \theta &= \cos^{-1}\left(\frac{1}{\sqrt{12}}\right) \\ &\approx 73.2^\circ \end{aligned}$$

$$\begin{aligned}
 \text{e } \mathbf{n}_1 &= \begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix} \text{ and } \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 0 \\ 2 & 1 & 1 \end{vmatrix} \\
 &= \begin{vmatrix} -1 & 0 \\ 1 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 3 & 0 \\ 2 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 3 & -1 \\ 2 & 1 \end{vmatrix} \mathbf{k} \\
 &= -\mathbf{i} - 3\mathbf{j} + 5\mathbf{k} \\
 &= \begin{pmatrix} -1 \\ -3 \\ 5 \end{pmatrix}
 \end{aligned}$$

$$\begin{aligned}
 \text{If } \theta \text{ is the acute angle between the planes, then } \cos \theta &= \frac{|\mathbf{n}_1 \bullet \mathbf{n}_2|}{|\mathbf{n}_1| |\mathbf{n}_2|} \\
 &= \frac{|-3 + 12 + 5|}{\sqrt{26}\sqrt{35}} \\
 \therefore \theta &= \cos^{-1} \left( \frac{14}{\sqrt{26}\sqrt{35}} \right) \\
 &\approx 62.3^\circ
 \end{aligned}$$

$$\text{6 } x + 5y - 3z = 8 \text{ has normal } \mathbf{n}_1 = \begin{pmatrix} 1 \\ 5 \\ -3 \end{pmatrix}.$$

$$2x + 2y + kz = 9 \text{ has normal } \mathbf{n}_2 = \begin{pmatrix} 2 \\ 2 \\ k \end{pmatrix}.$$

The planes are perpendicular, so  $\mathbf{n}_1 \bullet \mathbf{n}_2 = 0$

$$\begin{aligned}
 \therefore \begin{pmatrix} 1 \\ 5 \\ -3 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 2 \\ k \end{pmatrix} &= 0 \\
 \therefore 2 + 10 - 3k &= 0 \\
 \therefore 3k &= 12 \\
 \therefore k &= 4
 \end{aligned}$$

$$\begin{aligned}
 \text{7 a } L_1: \mathbf{r} &= \begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} t \\ 3 \\ -1 + 4t \end{pmatrix} \\
 L_2: \mathbf{r} &= \begin{pmatrix} 8 \\ -3 \\ k \end{pmatrix} + s \begin{pmatrix} -2 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 8 - 2s \\ -3 + 2s \\ k + 3s \end{pmatrix}
 \end{aligned}$$

$$\begin{aligned}
 \text{The lines intersect where } t = 8 - 2s \quad \dots (1) \quad 3 = -3 + 2s, \quad -1 + 4t = k + 3s \quad \dots (2) \\
 \therefore 2s = 6 \\
 \therefore s = 3
 \end{aligned}$$

$$\begin{aligned}
 \text{In (1): } t &= 8 - 2(3) & \text{In (2): } -1 + 4(2) &= k + 3(3) \\
 \therefore t &= 8 - 6 & \therefore -1 + 8 &= k + 9 \\
 \therefore t &= 2 & \therefore k &= -2
 \end{aligned}$$

$\therefore k = -2$ , and the point of intersection is  $(2, 3, 7)$  (substituting  $t = 2$  or  $s = 3$ ).



- b**  $\pi_1$  contains lines  $L_1$  and  $L_2$  with direction vectors  $\begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}$  and  $\begin{pmatrix} -2 \\ 2 \\ 3 \end{pmatrix}$  respectively.

$$\begin{aligned}
 \therefore \pi_1 \text{ has normal } \mathbf{n}_1 &= \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix} \times \begin{pmatrix} -2 \\ 2 \\ 3 \end{pmatrix} \\
 &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 4 \\ -2 & 2 & 3 \end{vmatrix} \\
 &= \begin{vmatrix} 0 & 4 \\ 2 & 3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 4 \\ -2 & 3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 0 \\ -2 & 2 \end{vmatrix} \mathbf{k} \\
 &= -8\mathbf{i} - 11\mathbf{j} + 2\mathbf{k} \\
 &= \begin{pmatrix} -8 \\ -11 \\ 2 \end{pmatrix}
 \end{aligned}$$

Using the point  $(2, 3, 7)$  on the plane, the equation of  $\pi_1$  is

$$\begin{aligned}
 -8x - 11y + 2z &= -8(2) - 11(3) + 2(7) \\
 \therefore -8x - 11y + 2z &= -35 \\
 \therefore 8x + 11y - 2z &= 35
 \end{aligned}$$

- c** Plane  $\pi_2$  has normal  $\mathbf{n}_2 = \begin{pmatrix} 0 \\ 2 \\ 11 \end{pmatrix}$ .

$$\begin{aligned}
 \mathbf{n}_1 \bullet \mathbf{n}_2 &= \begin{pmatrix} -8 \\ -11 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} 0 \\ 2 \\ 11 \end{pmatrix} \\
 &= 0 - 22 + 22 \\
 &= 0
 \end{aligned}$$

$\therefore \mathbf{n}_1$  and  $\mathbf{n}_2$  are perpendicular to each other, and since  $\mathbf{n}_1$  and  $\mathbf{n}_2$  are perpendicular to  $\pi_1$  and  $\pi_2$  respectively,  $\pi_1$  and  $\pi_2$  are also perpendicular.

- d i**  $L_1$  has direction vector  $\mathbf{d}_1 = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}$  and  $L_2$  has direction vector  $\mathbf{d}_2 = \begin{pmatrix} -2 \\ 2 \\ 3 \end{pmatrix}$ .

$$\begin{aligned}
 \text{If } \theta \text{ is the angle between } L_1 \text{ and } L_2, \text{ then } \cos \theta &= \frac{|\mathbf{d}_1 \bullet \mathbf{d}_2|}{|\mathbf{d}_1| |\mathbf{d}_2|} \\
 &= \frac{|-2 + 0 + 12|}{\sqrt{17}\sqrt{17}} \\
 &= \frac{10}{17} \\
 \therefore \theta &= \cos^{-1}\left(\frac{10}{17}\right) \\
 &\approx 54.0^\circ
 \end{aligned}$$

ii  $L_1$  has direction vector  $\mathbf{d}_1 = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}$  and  $\pi_2$  has normal  $\mathbf{n}_2 = \begin{pmatrix} 0 \\ 2 \\ 11 \end{pmatrix}$ .

If  $\phi$  is the angle between  $L_1$  and  $\pi_2$ , then

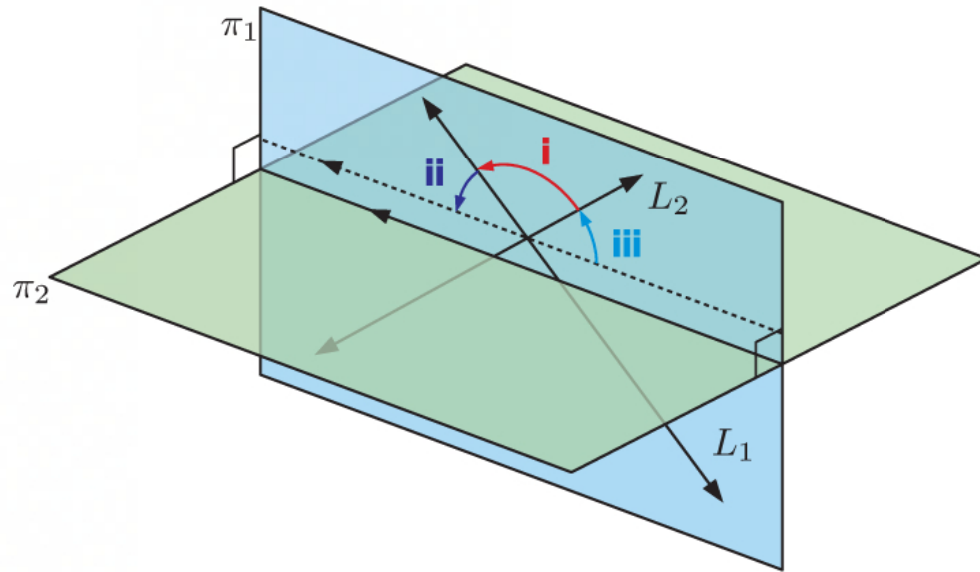
$$\begin{aligned} \sin \phi &= \frac{|\mathbf{d}_1 \bullet \mathbf{n}_2|}{|\mathbf{d}_1| |\mathbf{n}_2|} \\ &= \frac{|0 + 0 + 44|}{\sqrt{17}\sqrt{125}} \\ &= \frac{44}{\sqrt{17}\sqrt{125}} \\ \therefore \phi &= \sin^{-1}\left(\frac{44}{\sqrt{17}\sqrt{125}}\right) \\ &\approx 72.6^\circ \end{aligned}$$

iii  $L_2$  has direction vector  $\mathbf{d}_2 = \begin{pmatrix} -2 \\ 2 \\ 3 \end{pmatrix}$  and  $\pi_2$  has normal  $\mathbf{n}_2 = \begin{pmatrix} 0 \\ 2 \\ 11 \end{pmatrix}$ .

If  $\phi$  is the angle between  $L_2$  and  $\pi_2$ , then

$$\begin{aligned} \sin \phi &= \frac{|\mathbf{d}_2 \bullet \mathbf{n}_2|}{|\mathbf{d}_2| |\mathbf{n}_2|} \\ &= \frac{|0 + 4 + 33|}{\sqrt{17}\sqrt{125}} \\ &= \frac{37}{\sqrt{17}\sqrt{125}} \\ \therefore \phi &= \sin^{-1}\left(\frac{37}{\sqrt{17}\sqrt{125}}\right) \\ &\approx 53.4^\circ \end{aligned}$$

e  $54.0^\circ + 72.6^\circ + 53.4^\circ = 180^\circ$

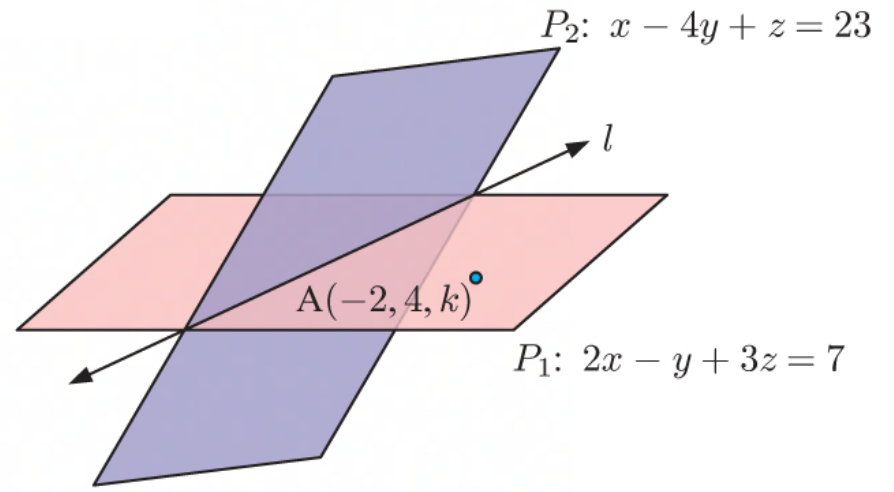


**8 a**  $P_1$  has normal vector  $\mathbf{n}_1 = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ .

$P_2$  has normal vector  $\mathbf{n}_2 = \begin{pmatrix} 1 \\ -4 \\ 1 \end{pmatrix}$ .

The acute angle between the planes is

$$\begin{aligned} \theta &= \cos^{-1} \left( \frac{|\mathbf{n}_1 \bullet \mathbf{n}_2|}{|\mathbf{n}_1| |\mathbf{n}_2|} \right) \\ &= \cos^{-1} \left( \frac{|2 + 4 + 3|}{\sqrt{14} \sqrt{18}} \right) \\ &= \cos^{-1} \left( \frac{9}{\sqrt{14} \sqrt{18}} \right) \\ &\approx 55.5^\circ \end{aligned}$$



**b**  $(-2, 4, k)$  lies on  $2x - y + 3z = 7$   
 $\therefore 2(-2) - 4 + 3k = 7$   
 $\therefore -4 - 4 + 3k = 7$   
 $\therefore 3k = 15$   
 $\therefore k = 5$

**c**  $P_1$  has normal vector  $\mathbf{n}_1 = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ .

$\therefore$  the line through  $A(-2, 4, 5)$  in the direction  $\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$  has vector equation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \\ 5 \end{pmatrix} + t \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$

$$\therefore x = -2 + 2t, \quad y = 4 - t, \quad z = 5 + 3t, \quad t \in \mathbb{R}$$

The line intersects  $P_2$  when  $-2 + 2t - 4(4 - t) + 5 + 3t = 23$

$$\therefore -2 + 2t - 16 + 4t + 5 + 3t = 23$$

$$\therefore 9t = 36$$

$$\therefore t = 4$$

When  $t = 4$ ,  $x = -2 + 2(4) = 6$

$$y = 4 - 4 = 0$$

$$z = 5 + 3(4) = 17$$

$\therefore$  the line through A, normal to  $P_1$ , cuts  $P_2$  at  $B(6, 0, 17)$ .

**d i**  $2(-4) - (-6) + 3(3) = -8 + 6 + 9$   
 $= 7 \quad \checkmark$

$\therefore (-4, -6, 3)$  lies on  $P_1$ .

$\therefore (-4, -6, 3)$  lies on  $l$ .

$$(-4) - 4(-6) + 3 = -4 + 24 + 3$$

$$= 23 \quad \checkmark$$

$\therefore (-4, -6, 3)$  also lies on  $P_2$ .

ii  $l$  is parallel to both  $P_1$  and  $P_2$ , so  $l$  must be perpendicular to the normals of each plane.

$$\therefore l \text{ has direction } \mathbf{n}_1 \times \mathbf{n}_2 = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \times \begin{pmatrix} 1 \\ -4 \\ 1 \end{pmatrix}.$$

$$\begin{aligned} \text{e } l \text{ has direction vector } \mathbf{d}_1 &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 3 \\ 1 & -4 & 1 \end{vmatrix} \\ &= \begin{vmatrix} -1 & 3 \\ -4 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & -1 \\ 1 & -4 \end{vmatrix} \mathbf{k} \\ &= 11\mathbf{i} + \mathbf{j} - 7\mathbf{k} \\ &= \begin{pmatrix} 11 \\ 1 \\ -7 \end{pmatrix} \end{aligned}$$

The midpoint of  $[AB]$  is  $M\left(\frac{-2+6}{2}, \frac{4+0}{2}, \frac{5+17}{2}\right)$  or  $M(2, 2, 11)$ .

The line through  $(-4, -6, 3)$  and  $(2, 2, 11)$  has direction vector  $\mathbf{d}_2 = \begin{pmatrix} 6 \\ 8 \\ 8 \end{pmatrix}$  or  $\begin{pmatrix} 3 \\ 4 \\ 4 \end{pmatrix}$ .

$P_3$  has normal vector  $\mathbf{n}_3 = \mathbf{d}_1 \times \mathbf{d}_2$

$$\begin{aligned} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 11 & 1 & -7 \\ 3 & 4 & 4 \end{vmatrix} \\ &= \begin{vmatrix} 1 & -7 \\ 4 & 4 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 11 & -7 \\ 3 & 4 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 11 & 1 \\ 3 & 4 \end{vmatrix} \mathbf{k} \\ &= 32\mathbf{i} - 65\mathbf{j} + 41\mathbf{k} \\ &= \begin{pmatrix} 32 \\ -65 \\ 41 \end{pmatrix} \end{aligned}$$

Using  $(2, 2, 11)$  on the plane, the equation of  $P_3$  is

$$32x - 65y + 41z = 32(2) - 65(2) + 41(11)$$

$$\therefore 32x - 65y + 41z = 385$$



**f i**  $P_1$  has normal vector  $\mathbf{n}_1 = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ .

$P_3$  has normal vector  $\mathbf{n}_3 = \begin{pmatrix} 32 \\ -65 \\ 41 \end{pmatrix}$ .

The acute angle between the planes is

$$\begin{aligned} \theta &= \cos^{-1} \left( \frac{|\mathbf{n}_1 \bullet \mathbf{n}_3|}{|\mathbf{n}_1| |\mathbf{n}_3|} \right) \\ &= \cos^{-1} \left( \frac{|64 + 65 + 123|}{\sqrt{14}\sqrt{6930}} \right) \\ &= \cos^{-1} \left( \frac{252}{\sqrt{14}\sqrt{6930}} \right) \\ &\approx 36.0^\circ \end{aligned}$$

**ii**  $P_2$  has normal vector  $\mathbf{n}_2 = \begin{pmatrix} 1 \\ -4 \\ 1 \end{pmatrix}$ .

$P_3$  has normal vector  $\mathbf{n}_3 = \begin{pmatrix} 32 \\ -65 \\ 41 \end{pmatrix}$ .

The acute angle between the planes is

$$\begin{aligned} \theta &= \cos^{-1} \left( \frac{|\mathbf{n}_2 \bullet \mathbf{n}_3|}{|\mathbf{n}_2| |\mathbf{n}_3|} \right) \\ &= \cos^{-1} \left( \frac{|32 + 260 + 41|}{\sqrt{18}\sqrt{6930}} \right) \\ &= \cos^{-1} \left( \frac{333}{\sqrt{18}\sqrt{6930}} \right) \\ &\approx 19.5^\circ \end{aligned}$$

**g**  $\text{sum} \approx 36.0^\circ + 19.5^\circ \approx 55.5^\circ$ ; this is equal to the angle between  $P_1$  and  $P_2$ .

## EXERCISE 13I

**1 a** The planes are either intersecting, parallel, or coincident. So, there are 3 geometric interpretations.

**b i** They are parallel if

$$a_1 = ka_2$$

$$b_1 = kb_2$$

$$c_1 = kc_2$$

and  $d_1 \neq kd_2$  for some  $k \in \mathbb{R}$ .

**ii** They are coincident if

$$a_1 = ka_2$$

$$b_1 = kb_2$$

$$c_1 = kc_2$$

and  $d_1 = kd_2$  for some  $k \in \mathbb{R}$ .

**c i**  $\begin{pmatrix} 1 & -3 & 2 & | & 8 \\ 3 & -9 & 2 & | & 4 \end{pmatrix}$

$$\sim \begin{pmatrix} 1 & -3 & 2 & | & 8 \\ 0 & 0 & -4 & | & -20 \end{pmatrix} \quad R_2 - 3R_1 \rightarrow R_2 \quad \left\{ \begin{array}{ccc|c} 3 & -9 & 2 & 4 \\ -3 & 9 & -6 & -24 \\ \hline 0 & 0 & -4 & -20 \end{array} \right\}$$

$$\therefore -4z = -20$$

$$\therefore z = 5$$

$$\text{Now } x - 3y + 2z = 8 \quad \text{or} \quad x = 3y - 2z + 8$$

$$\text{and if we let } y = t, \text{ then } x = 3t - 2(5) + 8 = -2 + 3t$$

$$\therefore \text{ the planes meet in the line } x = -2 + 3t, y = t, z = 5, t \in \mathbb{R}$$

ii 
$$\begin{pmatrix} 2 & 1 & 1 & | & 5 \\ 1 & -1 & 1 & | & 3 \end{pmatrix} \sim \begin{pmatrix} 2 & 1 & 1 & | & 5 \\ 0 & -3 & 1 & | & 1 \end{pmatrix} \begin{matrix} 2R_2 - R_1 \rightarrow R_2 \end{matrix} \leftarrow \left\{ \begin{array}{cccc} 2 & -2 & 2 & 6 \\ -2 & -1 & -1 & -5 \\ 0 & -3 & 1 & 1 \end{array} \right\}$$

$\therefore -3y + z = 1$  and  $2x + y + z = 5$

$\therefore$  if we let  $z = t$ , then  $-3y + z = 1$  and  $2x + y + z = 5$

$\therefore -3y + t = 1$   $\therefore 2x + \frac{t-1}{3} + z = 5$

$\therefore -3y = 1 - t$   $\therefore 6x + t - 1 + 3t = 15$

$\therefore y = \frac{t-1}{3}$   $\therefore 6x = 16 - 4t$

$\therefore x = \frac{8-2t}{3}$

$\therefore$  the planes meet in the line  $x = \frac{8-2t}{3}, y = \frac{t-1}{3}, z = t, t \in \mathbb{R}$

iii 
$$\begin{pmatrix} 1 & 2 & -3 & | & 6 \\ 3 & 6 & -9 & | & 18 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -3 & | & 6 \\ 0 & 0 & 0 & | & 0 \end{pmatrix} \begin{matrix} R_2 - 3R_1 \rightarrow R_2 \end{matrix} \leftarrow \left\{ \begin{array}{cccc} 3 & 6 & -9 & 18 \\ -3 & -6 & 9 & -18 \\ 0 & 0 & 0 & 0 \end{array} \right\}$$

$\therefore$  there are infinitely many solutions, as the planes are coincident.

Let  $y = s$  and  $z = t$  in  $x + 2y - 3z = 6, s, t \in \mathbb{R}$

$\therefore x = 6 - 2s + 3t$

$\therefore x = 6 - 2s + 3t, y = s, z = t, s, t \in \mathbb{R}$  is the general solution of the plane.

2 a 
$$\begin{pmatrix} 1 & 2 & -1 & | & 6 \\ 2 & 4 & k & | & 12 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -1 & | & 6 \\ 0 & 0 & k+2 & | & 0 \end{pmatrix} \begin{matrix} R_2 - 2R_1 \rightarrow R_2 \end{matrix}$$

If  $k = -2$ , the two planes are coincident.  
 $\therefore$  infinitely many solutions.

If  $k \neq -2$ , the two planes meet in a line.  
 $\therefore$  infinitely many solutions.

b 
$$\begin{pmatrix} 1 & -1 & 3 & | & 8 \\ 2 & -2 & 6 & | & k \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 3 & | & 8 \\ 0 & 0 & 0 & | & k-16 \end{pmatrix} \begin{matrix} R_2 - 2R_1 \rightarrow R_2 \end{matrix}$$

If  $k = 16$ , the planes are coincident.  
 $\therefore$  infinitely many solutions.

If  $k \neq 16$ , the planes are parallel but not coincident.  
 $\therefore$  no solutions exist.

- 3 (1)  $P_1 = P_2 = P_3$ : infinitely many solutions where  $x, y$ , and  $z$  are in terms of two parameters, say  $s$  and  $t$ . The solution is a plane.
- (2)  $P_1 = P_2$  are coincident and cut by  $P_3$ : infinitely many solutions where  $x, y$ , and  $z$  are in terms of one parameter, say  $t$ . The solution is a line.
- (3)  $P_1 = P_2$  with  $P_3$  parallel but not coincident: no solutions exist.
- (4)  $P_1$  and  $P_2$  are parallel but not coincident, and  $P_3$  cuts both planes: no solutions exist.
- (5)  $P_1, P_2$ , and  $P_3$  are all parallel but not coincident: no solutions exist.
- (6)  $P_1, P_2$ , and  $P_3$  meet in a unique point  $(a, b, c)$ , so that  $x = a, y = b, z = c$ .
- (7)  $P_1, P_2$ , and  $P_3$  meet in a common line: infinitely many solutions where  $x, y$ , and  $z$  are in terms of one parameter, say  $t$ .

- (8)  $P_1$ ,  $P_2$ , and  $P_3$  are such that the line of intersection between any two is parallel to the third plane, but not coincident: no solutions exist.

**4 a** In augmented matrix form, the system is:

$$\begin{aligned} & \left( \begin{array}{ccc|c} 1 & 1 & -1 & -5 \\ 1 & -1 & 2 & 11 \\ 4 & 1 & -5 & -18 \end{array} \right) \\ & \sim \left( \begin{array}{ccc|c} 1 & 1 & -1 & -5 \\ 0 & -2 & 3 & 16 \\ 0 & -3 & -1 & 2 \end{array} \right) \begin{array}{l} R_2 - R_1 \rightarrow R_2 \\ R_3 - 4R_1 \rightarrow R_3 \end{array} \\ & \sim \left( \begin{array}{ccc|c} 1 & 1 & -1 & -5 \\ 0 & -2 & 3 & 16 \\ 0 & 0 & -11 & -44 \end{array} \right) 2R_3 - 3R_2 \rightarrow R_3 \end{aligned}$$

$\therefore$  the planes meet at the unique point  $(1, -2, 4)$ .

$$\text{Now } -11z = -44$$

$$\therefore z = 4$$

$$\text{and } -2y + 3z = 16$$

$$\therefore -2y + 12 = 16$$

$$\therefore -2y = 4$$

$$\therefore y = -2$$

$$\text{and } x + y - z = -5$$

$$\therefore x = -5 - (-2) + 4$$

$$\therefore x = 1$$

**b** In augmented matrix form, the system is:

$$\begin{aligned} & \left( \begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 2 & 1 & -1 & 8 \\ 5 & -2 & 5 & 11 \end{array} \right) \\ & \sim \left( \begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 3 & -5 & 6 \\ 0 & 3 & -5 & 6 \end{array} \right) \begin{array}{l} R_2 - 2R_1 \rightarrow R_2 \\ R_3 - 5R_1 \rightarrow R_3 \end{array} \\ & \sim \left( \begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 3 & -5 & 6 \\ 0 & 0 & 0 & 0 \end{array} \right) R_3 - R_2 \rightarrow R_3 \end{aligned}$$

$\therefore$  the three planes meet in a common line

$$x = \frac{9-t}{3}, \quad y = \frac{6+5t}{3}, \quad z = t, \quad t \in \mathbb{R}.$$

$$\text{Let } z = t$$

$$\text{Now } 3y - 5z = 6$$

$$\therefore 3y = 6 + 5t$$

$$\therefore y = \frac{6+5t}{3}$$

$$\text{and } x - y + 2z = 1$$

$$\therefore x = 1 + \frac{6+5t}{3} - 2t$$

$$\therefore x = \frac{3+6+5t-6t}{3}$$

$$\therefore x = \frac{9-t}{3}$$

**c** In augmented matrix form, the system is:

$$\begin{aligned} & \left( \begin{array}{ccc|c} 1 & 2 & -1 & 8 \\ 2 & -1 & -1 & 5 \\ 3 & -4 & -1 & 2 \end{array} \right) \\ & \sim \left( \begin{array}{ccc|c} 1 & 2 & -1 & 8 \\ 0 & -5 & 1 & -11 \\ 0 & -10 & 2 & -22 \end{array} \right) \begin{array}{l} R_2 - 2R_1 \rightarrow R_2 \\ R_3 - 3R_1 \rightarrow R_3 \end{array} \\ & \sim \left( \begin{array}{ccc|c} 1 & 2 & -1 & 8 \\ 0 & -5 & 1 & -11 \\ 0 & 0 & 0 & 0 \end{array} \right) R_3 - 2R_2 \rightarrow R_3 \end{aligned}$$

$$\text{Let } y = t$$

$$\text{Now } -5y + z = -11$$

$$\therefore z = -11 + 5t$$

$$\text{and } x + 2y - z = 8$$

$$\therefore x = 8 - 2y + z$$

$$\therefore x = 8 - 2t - 11 + 5t$$

$$\therefore x = -3 + 3t$$

$\therefore$  the three planes meet in a common line  $x = 3t - 3$ ,  $y = t$ ,  $z = 5t - 11$ ,  $t \in \mathbb{R}$



**d** In augmented matrix form, the system is:

$$\begin{pmatrix} 1 & -1 & 1 & | & 8 \\ 2 & -2 & 2 & | & 11 \\ 1 & 3 & -1 & | & -2 \end{pmatrix} \\ \sim \begin{pmatrix} 1 & -1 & 1 & | & 8 \\ 0 & 0 & 0 & | & -5 \\ 0 & 4 & -2 & | & -10 \end{pmatrix} \begin{array}{l} R_2 - 2R_1 \rightarrow R_2 \\ R_3 - R_1 \rightarrow R_3 \end{array}$$

The equations are inconsistent and there are no solutions.

The first two planes are parallel and are cut by the third plane.

**e** In augmented matrix form, the system is:

$$\begin{pmatrix} 1 & 1 & -2 & | & 1 \\ 1 & -1 & 1 & | & 4 \\ 3 & 3 & -6 & | & 3 \end{pmatrix} \\ \sim \begin{pmatrix} 1 & 1 & -2 & | & 1 \\ 0 & -2 & 3 & | & 3 \\ 0 & 0 & 0 & | & 0 \end{pmatrix} \begin{array}{l} R_2 - R_1 \rightarrow R_2 \\ R_3 - 3R_1 \rightarrow R_3 \end{array}$$

There are two coincident planes cut by a third plane.

$\therefore$  there are infinitely many solutions in the line

$$x = \frac{t+5}{2}, \quad y = \frac{3t-3}{2}, \quad z = t, \quad t \in \mathbb{R}$$

$$\text{Let } z = t$$

$$\text{Now } -2y + 3z = 3$$

$$\therefore 2y = 3z - 3$$

$$\therefore y = \frac{3t-3}{2}$$

$$\text{and } x + y - 2z = 1$$

$$\therefore x = 1 - y + 2z$$

$$\therefore x = 1 - \frac{3t-3}{2} + 2t$$

$$\therefore x = \frac{2-3t+3+4t}{2}$$

$$\therefore x = \frac{t+5}{2}$$

**f** In augmented matrix form, the system is:

$$\begin{pmatrix} 1 & -1 & -1 & | & 5 \\ 1 & 1 & 1 & | & 1 \\ 5 & -1 & 2 & | & 17 \end{pmatrix} \\ \sim \begin{pmatrix} 1 & -1 & -1 & | & 5 \\ 0 & 2 & 2 & | & -4 \\ 0 & 4 & 7 & | & -8 \end{pmatrix} \begin{array}{l} R_2 - R_1 \rightarrow R_2 \\ R_3 - 5R_1 \rightarrow R_3 \end{array} \\ \sim \begin{pmatrix} 1 & -1 & -1 & | & 5 \\ 0 & 2 & 2 & | & -4 \\ 0 & 0 & 3 & | & 0 \end{pmatrix} R_3 - 2R_2 \rightarrow R_3$$

$$\text{Now } 3z = 0$$

$$\therefore z = 0$$

$$\text{and } 2y + 2z = -4$$

$$\therefore 2y = -4$$

$$\therefore y = -2$$

$$\text{and } x - y - z = 5$$

$$\therefore x = 5 + (-2) + 0$$

$$\therefore x = 3$$

$\therefore$  the planes meet at the unique point  $(3, -2, 0)$ .

**5 a** In augmented matrix form, the system is:

$$\begin{pmatrix} 2 & -1 & 4 & | & 5 \\ 1 & 1 & -2 & | & -3 \\ -1 & -4 & 10 & | & 6 \end{pmatrix} \\ \sim \begin{pmatrix} 2 & -1 & 4 & | & 5 \\ 0 & 3 & -8 & | & -11 \\ 0 & -9 & 24 & | & 17 \end{pmatrix} \begin{array}{l} 2R_2 - R_1 \rightarrow R_2 \\ 2R_3 + R_1 \rightarrow R_3 \end{array} \\ \sim \begin{pmatrix} 2 & -1 & 4 & | & 5 \\ 0 & 3 & -8 & | & -11 \\ 0 & 0 & 0 & | & -16 \end{pmatrix} R_3 + 3R_2 \rightarrow R_3$$

The equations are inconsistent and there are no solutions.

The line of intersection of any two planes is parallel to the third plane.



**b** In augmented matrix form, the system is:

$$\begin{pmatrix} 2 & -1 & 4 & | & 5 \\ 1 & 1 & -2 & | & -3 \end{pmatrix} \\ \sim \begin{pmatrix} 2 & -1 & 4 & | & 5 \\ 0 & 3 & -8 & | & -11 \end{pmatrix} \quad 2R_2 - R_1 \rightarrow R_2$$

If we let  $z = t$  in row 2, then  $3y - 8t = -11$

$$\therefore 3y = 8t - 11$$

$$\therefore y = \frac{8t - 11}{3}$$

Thus in row 1,  $2x - \frac{8t - 11}{3} + 4t = 5$

$$\therefore 6x - 8t + 11 + 12t = 15$$

$$\therefore 6x = 4 - 4t$$

$$\therefore x = \frac{2 - 2t}{3}$$

$\therefore$  the solutions have the form  $x = \frac{2 - 2t}{3}$ ,  $y = \frac{8t - 11}{3}$ ,  $z = t$ ,  $t \in \mathbb{R}$ .

**c** The plane  $-x - 4y + 10z = 6$  has normal vector  $\mathbf{n} = \begin{pmatrix} -1 \\ -4 \\ 10 \end{pmatrix}$ .

The line  $l$  has direction vector  $\mathbf{d} = \begin{pmatrix} -\frac{2}{3} \\ \frac{8}{3} \\ 1 \end{pmatrix}$ .

$$\begin{aligned} \mathbf{n} \bullet \mathbf{d} &= \begin{pmatrix} -1 \\ -4 \\ 10 \end{pmatrix} \bullet \begin{pmatrix} -\frac{2}{3} \\ \frac{8}{3} \\ 1 \end{pmatrix} \\ &= \frac{2}{3} - \frac{32}{3} + 10 \\ &= 0 \end{aligned}$$

So  $l$  is perpendicular to the normal of the plane  $-x - 4y + 10z = 6$ .

$\therefore l$  is parallel to the plane  $-x - 4y + 10z = 6$ .

6 In augmented matrix form, the system is:

$$\begin{aligned}
 & \left( \begin{array}{ccc|c} 1 & -1 & 3 & 1 \\ 2 & -3 & -1 & 3 \\ 3 & -5 & -5 & k \end{array} \right) & \left\{ \begin{array}{cccc} 2 & -3 & -1 & 3 \\ -2 & 2 & -6 & -2 \\ \hline 0 & -1 & -7 & 1 \end{array} \right\} \\
 & \sim \left( \begin{array}{ccc|c} 1 & -1 & 3 & -1 \\ 0 & -1 & -7 & 1 \\ 0 & -2 & -14 & k-3 \end{array} \right) \begin{array}{l} R_2 - 2R_1 \rightarrow R_2 \\ R_3 - 3R_1 \rightarrow R_3 \end{array} & \left\{ \begin{array}{cccc} 3 & -5 & -5 & k \\ -3 & 3 & -9 & -3 \\ \hline 0 & -2 & -14 & k-3 \end{array} \right\} \\
 & \sim \left( \begin{array}{ccc|c} 1 & -1 & 3 & -1 \\ 0 & -1 & -7 & 1 \\ 0 & 0 & 0 & k-5 \end{array} \right) R_3 - 2R_2 \rightarrow R_3 & \left\{ \begin{array}{cccc} 0 & -2 & -14 & k-3 \\ 0 & 2 & 14 & -2 \\ \hline 0 & 0 & 0 & k-5 \end{array} \right\}
 \end{aligned}$$

Case 1: If  $k = 5$ , the planes meet in a common line.

$$\text{Let } z = t$$

$$\text{Now } -y - 7z = 1$$

$$\therefore y = -1 - 7t$$

$$\text{and } x - y + 3z = 1$$

$$\therefore x = 1 + y - 3z$$

$$\therefore x = 1 - 1 - 7t - 3t$$

$$\therefore x = -10t$$

$$\therefore x = -10t, y = -1 - 7t, z = t, t \in \mathbb{R}$$

Case 2: If  $k \neq 5$ , there are no solutions.

Since no two planes are parallel, the line of intersection of any two planes is parallel to the third plane.

7 In augmented matrix form, the system is:

$$\begin{aligned}
 & \left( \begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 3 & 1 & 2 & 0 \\ 5 & 3 & a & 2 \end{array} \right) & \left\{ \begin{array}{cccc} 3 & 1 & 2 & 0 \\ -3 & 3 & -6 & -3 \\ \hline 0 & 4 & -4 & -3 \end{array} \right\} \\
 & \sim \left( \begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 4 & -4 & -3 \\ 0 & 8 & a-10 & -3 \end{array} \right) \begin{array}{l} R_2 - 3R_1 \rightarrow R_2 \\ R_3 - 5R_1 \rightarrow R_3 \end{array} & \left\{ \begin{array}{cccc} 5 & 3 & a & 2 \\ -5 & 5 & -10 & -5 \\ \hline 0 & 8 & a-10 & -3 \end{array} \right\} \\
 & \sim \left( \begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 4 & -4 & -3 \\ 0 & 0 & a-2 & 3 \end{array} \right) R_3 - 2R_2 \rightarrow R_3 & \left\{ \begin{array}{cccc} 0 & 8 & a-10 & -3 \\ 0 & -8 & 8 & 6 \\ \hline 0 & 0 & a-2 & 3 \end{array} \right\}
 \end{aligned}$$

Case 1: If  $a = 2$ , there are no solutions.

Since no two planes are parallel, the line of intersection of any two planes is parallel to the third plane.

Case 2: If  $a \neq 2$ , there is a unique solution.

$$(a-2)z = 3$$

$$\therefore z = \frac{3}{a-2}$$

$$\therefore 4y - 4\left(\frac{3}{a-2}\right) = -3$$

$$\therefore 4y - \frac{12}{a-2} = -3$$

$$\therefore 4y = \frac{-3(a-2)}{a-2} + \frac{12}{a-2}$$

$$\therefore 4y = \frac{18-3a}{a-2}$$

$$\therefore y = \frac{18-3a}{4(a-2)}$$

$$\therefore x - \frac{18-3a}{4(a-2)} + 2\left(\frac{3}{a-2}\right) = 1$$

$$\therefore x - \frac{18-3a}{4(a-2)} + \frac{24}{4(a-2)} = 1$$

$$\therefore x = \frac{4(a-2)}{4(a-2)} + \frac{18-3a}{4(a-2)} - \frac{24}{4(a-2)}$$

$$\therefore x = \frac{a-14}{4(a-2)}$$

If  $a \neq 2$  there is a unique solution  $x = \frac{a-14}{4(a-2)}$ ,  $y = \frac{18-3a}{4(a-2)}$ ,  $z = \frac{3}{a-2}$ .

The planes meet at a point.

**8 a** In augmented matrix form, the system is:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ -1 & -2 & 2 & 1 \\ 3 & 6 & k & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 5 & 5 \\ 0 & 0 & 9-k & 7 \end{pmatrix} \begin{matrix} R_2 + R_1 \rightarrow R_2 \\ 3R_1 - R_3 \rightarrow R_3 \end{matrix}$$

$$\begin{Bmatrix} -1 & -2 & 2 & 1 \\ 1 & 2 & 3 & 4 \\ 0 & 0 & 5 & 5 \end{Bmatrix}$$

$$\begin{Bmatrix} 3 & 6 & 9 & 12 \\ -3 & -6 & -k & -5 \\ 0 & 0 & 9-k & 7 \end{Bmatrix}$$

**b** When  $k \neq 2$ ,  $R_2$  and  $R_3$  of the reduced matrix in **a** are inconsistent with one another. Consider the original augmented matrix.

If  $k = 9$ , the coefficients of  $R_1$  and  $R_3$  are multiples of each other.

If  $k = -6$ , the coefficients of  $R_2$  and  $R_3$  are multiples of each other.

$\therefore$  two of the planes are parallel if  $k = 9$  or  $k = -6$ . For the remaining values of  $k$  for which the system is inconsistent, the line of intersection of any two planes is parallel to the third plane.

**c** The system has infinitely many solutions when  $k = 2$ . The augmented matrix is equivalent to  $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 5 & 5 \end{pmatrix}$ .

$$\therefore 5z = 5$$

$$\text{Let } y = t$$

$$\therefore z = 1$$

$$\therefore x + 2t + 3 = 4$$

$$\therefore x = 1 - 2t$$

$\therefore$  if  $k = 2$ , the planes intersect in the line  $x = 1 - 2t$ ,  $y = t$ ,  $z = 1$ ,  $t \in \mathbb{R}$ .





$$\begin{aligned}
\text{Substituting into row 1 gives } x + \frac{6(m-2)}{m+5} + \frac{-7m}{m+5} &= -1 \\
\therefore x(m+5) + 6(m-2) - 7m &= -(m+5) \\
\therefore x(m+5) + 6m - 12 - 7m &= -m - 5 \\
\therefore x(m+5) &= 7 \\
\therefore x &= \frac{7}{m+5}
\end{aligned}$$

$\therefore$  for all  $m$  except  $m = -5$  and  $m = -1$ , the system has the unique solution

$$x = \frac{7}{m+5}, \quad y = \frac{3(m-2)}{m+5}, \quad z = \frac{-7}{m+5}.$$

**10 a** In augmented matrix form, the system is:

$$\begin{aligned}
&\left( \begin{array}{ccc|c} 2 & -1 & 3 & 4 \\ 2 & 1 & a+3 & 10-a \\ 4 & 6 & a^2+6 & a^2 \end{array} \right) \\
&\sim \left( \begin{array}{ccc|c} 1 & -\frac{1}{2} & \frac{3}{2} & 2 \\ 0 & 2 & a & 6-a \\ 0 & 8 & a^2 & a^2-8 \end{array} \right) \begin{array}{l} \frac{1}{2}R_1 \rightarrow R_1 \\ R_2 - R_1 \rightarrow R_2 \\ R_3 - 2R_2 \rightarrow R_3 \end{array} \\
&\sim \left( \begin{array}{ccc|c} 1 & -\frac{1}{2} & \frac{3}{2} & 2 \\ 0 & 2 & a & 6-a \\ 0 & 0 & a^2-4a & a^2+4a-32 \end{array} \right) R_3 - 4R_2 \rightarrow R_3 \\
&\sim \left( \begin{array}{ccc|c} 1 & -\frac{1}{2} & \frac{3}{2} & 2 \\ 0 & 2 & a & 6-a \\ 0 & 0 & a(a-4) & (a+8)(a-4) \end{array} \right)
\end{aligned}$$

**b** If  $a = 0$ , row 3 becomes  $0x + 0y + 0z = -32$ .

$\therefore$  the system is inconsistent and there are no solutions.

In this case we have three planes with no common point of intersection. No two planes are coincident or parallel. So, the line of intersection of any two planes is parallel to the third.

**c** If  $a = 4$ , row 3 is a row of zeros.

$\therefore$  there are infinitely many solutions.

If  $z = t$ , then  $2y + 4t = 6 - 4$

$$\therefore 2y = 2 - 4t$$

$$\therefore y = 1 - 2t$$

and  $2x - (1 - 2t) + 3t = 4$

$$\therefore 2x - 1 + 2t + 3t = 4$$

$$\therefore 2x = 5 - 5t$$

$$\therefore x = \frac{5-5t}{2}$$

$\therefore$  if  $a = 4$ , the planes intersect in the line  $x = \frac{5-5t}{2}$ ,  $y = 1 - 2t$ ,  $z = t$ ,  $t \in \mathbb{R}$ .

**d** If  $a \neq 0$  or  $4$ , then row 3 becomes  $a(a-4)z = (a+8)(a-4)$

$$\therefore az = a + 8$$

$$\therefore z = \frac{a+8}{a}$$

Substituting into row 2 gives  $2y + a\left(\frac{a+8}{a}\right) = 6 - a$

$$\therefore 2y + a + 8 = 6 - a$$

$$\therefore 2y = -2 - 2a$$

$$\therefore y = -1 - a$$

Substituting into row 1 gives  $2x - (-1 - a) + 3\left(\frac{a+8}{a}\right) = 4$

$$\therefore 2x + 1 + a + \frac{3a+24}{a} = 4$$

$$\therefore 2ax + a + a^2 + 3a + 24 = 4a$$

$$\therefore 2ax = -a^2 - 24$$

$$\therefore x = -\frac{a}{2} - \frac{12}{a}$$

$\therefore$  for all  $a$  except  $a = 0$  and  $a = 4$ , the system has unique solution

$$x = -\frac{a}{2} - \frac{12}{a}, \quad y = -1 - a, \quad z = \frac{a+8}{a}.$$

The planes meet at a point.

If  $a = 2$ ,  $x = -\frac{2}{2} - \frac{12}{2} = -7$

$$y = -1 - 2 = -3$$

$$z = \frac{2+8}{2} = 5$$

$\therefore$  if  $a = 2$ , the planes meet at  $(-7, -3, 5)$ .

**11 a** In augmented matrix form, the system is:

$$\begin{aligned} & \left( \begin{array}{ccc|c} 1 & 3 & 3 & a-1 \\ 2 & -1 & 1 & 7 \\ 3 & -5 & a & 16 \end{array} \right) & \left\{ \begin{array}{ccc|c} 2 & -1 & 1 & 7 \\ -2 & -6 & -6 & -2a+2 \\ 0 & -7 & -5 & 9-2a \end{array} \right\} \\ \sim & \left( \begin{array}{ccc|c} 1 & 3 & 3 & a-1 \\ 0 & -7 & -5 & 9-2a \\ 0 & -14 & a-9 & 19-3a \end{array} \right) & \begin{array}{l} R_2 - 2R_1 \rightarrow R_2 \\ R_3 - 3R_1 \rightarrow R_3 \end{array} \left\{ \begin{array}{ccc|c} 3 & -5 & a & 16 \\ -3 & -9 & -9 & -3a+3 \\ 0 & -14 & a-9 & 19-3a \end{array} \right\} \\ \sim & \left( \begin{array}{ccc|c} 1 & 3 & 3 & a-1 \\ 0 & -7 & -5 & 9-2a \\ 0 & 0 & a+1 & a+1 \end{array} \right) & \begin{array}{l} R_3 - 2R_2 \rightarrow R_3 \end{array} \left\{ \begin{array}{ccc|c} 0 & -14 & a-9 & 19-3a \\ 0 & 14 & 10 & 4a-18 \\ 0 & 0 & a+1 & a+1 \end{array} \right\} \end{aligned}$$

- b** If  $a = -1$ , row 3 is a row of zeros.  
 $\therefore$  there are infinitely many solutions.

If we let  $z = t$  in row 2, then  $-7y - 5t = 9 - 2(-1)$

$$\therefore -7y - 5t = 11$$

$$\therefore y = \frac{-5t - 11}{7}$$

and substituting into row 1 gives  $x + 3\left(\frac{-5t - 11}{7}\right) + 3t = (-1) - 1$

$$\therefore x - \frac{15t}{7} - \frac{33}{7} + 3t = -2$$

$$\therefore x = \frac{19}{7} - \frac{6t}{7}$$

$\therefore$  there are infinitely many solutions of form  $x = \frac{19 - 6t}{7}$ ,  $y = \frac{-5t - 11}{7}$ ,  $z = t$ ,  $t \in \mathbb{R}$ .

In this case we have three planes which meet in a line.

- c** If  $a \neq -1$ , then  $(a + 1)z = a + 1$   
 $\therefore z = 1$

Using row 2,  $-7y - 5(1) = 9 - 2a$

$$\therefore -7y = -2a + 14$$

$$y = \frac{2a}{7} - 2$$

and substituting into row 1 gives  $x + 3\left(\frac{2a}{7} - 2\right) + 3(1) = a - 1$

$$\therefore x + \frac{6a}{7} - 6 + 3 = a - 1$$

$$\therefore x = \frac{a}{7} + 2$$

$\therefore$  the unique solution is  $x = \frac{1}{7}a + 2$ ,  $y = \frac{2}{7}a - 2$ ,  $z = 1$ .

In this case we have three planes which meet at a point.

**12**  $P_1$  meets  $P_2$  where  $\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix} + r \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} + s \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$$\therefore \begin{cases} 2 + 3\lambda + \mu = 3 + 2r + s \\ -1 + \mu = -1 + s \\ \lambda - \mu = 3 - r \end{cases} \quad \text{which gives} \quad \begin{cases} 3\lambda + \mu = 2r + s + 1 \\ \mu = s \\ \lambda - \mu = 3 - r \end{cases}$$

If  $\mu = a$  say, then  $s = a$ ,  $3\lambda + a = 2r + a + 1$ , and  $\lambda - a = 3 - r$

$$\therefore r = 3 - \lambda + a$$

$$\therefore 3\lambda + a = 6 - 2\lambda + 2a + a + 1$$

$$\therefore 5\lambda = 2a + 7$$

$$\therefore \lambda = \frac{2a + 7}{5} \quad \text{and} \quad r = 3 + a - \frac{2a + 7}{5}$$

$$\therefore r = \frac{3a + 8}{5}$$

$$\therefore \text{ if } \mu = a, \lambda = \frac{2a + 7}{5}, r = \frac{3a + 8}{5}, s = a \quad \dots (1)$$

$$P_2 \text{ meets } P_3 \text{ where } \begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix} + r \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} + s \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} + t \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} - u \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$$

$$\therefore \begin{cases} 3 + 2r + s = 2 + t \\ -1 + s = -1 - t + u \\ 3 - r = 2 - 2u \end{cases} \text{ which gives } \begin{cases} 2r + s + 1 = t \\ s = u - t \\ 2u - r = -1 \end{cases}$$

So, if  $u = b$  say, then  $r = 2b + 1$  and  $4b + 2 + b - t + 1 = t$

$$\therefore 5b + 3 = 2t \text{ and so } t = \frac{5b + 3}{2}$$

$$\text{and } s = u - t = b - \frac{5b + 3}{2} = \frac{-3b - 3}{2}$$

So, if  $u = b$ ,  $r = 2b + 1$ ,  $t = \frac{5b + 3}{2}$ ,  $s = \frac{-3b - 3}{2}$  .... (2)

From (1) and (2),  $\frac{3a + 8}{5} = 2b + 1$  and  $a = \frac{-3b - 3}{2}$

$$\therefore 3a + 8 = 10b + 5 \text{ and } 2a = -3b - 3$$

$$\therefore \begin{cases} 3a - 10b = -3 \\ 2a + 3b = -3 \end{cases} \text{ which has solutions } a = -\frac{39}{29}, b = -\frac{3}{29}$$

In (2),  $u = -\frac{3}{29}$ ,  $t = \frac{5(-\frac{3}{29}) + 3}{2} = \frac{36}{29}$

$$\therefore \mathbf{r}_3 = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} + \frac{36}{29} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \frac{3}{29} \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3\frac{7}{29} \\ -2\frac{10}{29} \\ 2\frac{6}{29} \end{pmatrix} = \begin{pmatrix} \frac{94}{29} \\ -\frac{68}{29} \\ \frac{64}{29} \end{pmatrix}$$

$\therefore$  all 3 planes meet at  $(\frac{94}{29}, -\frac{68}{29}, \frac{64}{29})$ .

## REVIEW SET 13A

**1 a** The vector equation is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -6 \\ 3 \end{pmatrix} + t \begin{pmatrix} 4 \\ -3 \end{pmatrix}, t \in \mathbb{R}$$

$$\mathbf{c} \quad \frac{x + 6}{4} = \frac{y - 3}{-3} = t$$

$$\therefore -3x - 18 = 4y - 12$$

So, the Cartesian equation is  $3x + 4y = -6$ .

**b** The parametric equations are

$$x = -6 + 4t, y = 3 - 3t, t \in \mathbb{R}$$

**2**  $(-3, m)$  lies on the line, so  $\begin{pmatrix} -3 \\ m \end{pmatrix} = \begin{pmatrix} 18 \\ -2 \end{pmatrix} + \begin{pmatrix} -7t \\ 4t \end{pmatrix}$

$$\therefore -3 = 18 - 7t \text{ and } m = -2 + 4t$$

$$\therefore 7t = 21$$

$$\therefore t = 3 \text{ and so } m = -2 + 4(3) = 10$$



**3 a**  $\vec{AB} = \begin{pmatrix} -2 \\ 2 \\ -4 \end{pmatrix}$

$\therefore$  since A lies on the line, the line has equation  $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 2 \\ -4 \end{pmatrix}, \lambda \in \mathbb{R}$

**b** If C lies on (AB) and is 2 units from A, then C has coordinates  $(2 + \lambda, -1 - \lambda, 3 + 2\lambda)$  and  $\vec{AC}$  has length  $\sqrt{(\lambda)^2 + (-\lambda)^2 + (2\lambda)^2} = 2$

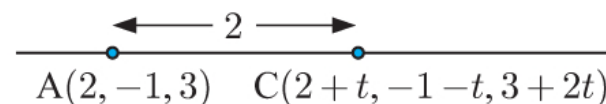
$$\therefore \sqrt{6\lambda^2} = 2$$

$$\therefore 6\lambda^2 = 4$$

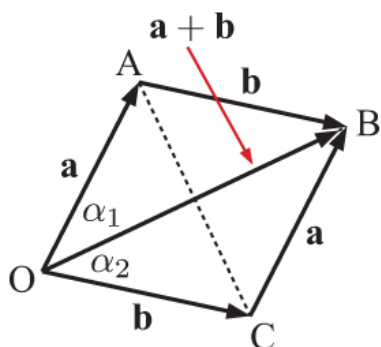
$$\therefore \lambda^2 = \frac{4}{6}$$

$$\therefore \lambda = \pm \frac{2}{\sqrt{6}}$$

$\therefore$  C has coordinates  $\left(2 + \frac{2}{\sqrt{6}}, -1 - \frac{2}{\sqrt{6}}, 3 + \frac{4}{\sqrt{6}}\right)$  or  $\left(2 - \frac{2}{\sqrt{6}}, -1 + \frac{2}{\sqrt{6}}, 3 - \frac{4}{\sqrt{6}}\right)$ .



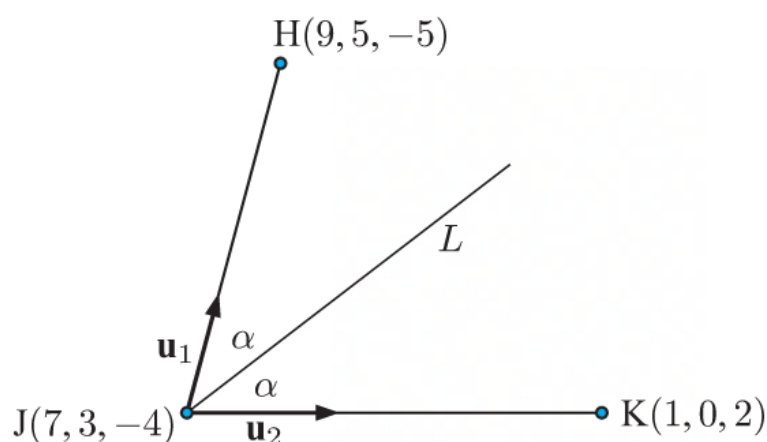
**4 a**



As **a** and **b** are unit vectors, OACB is a rhombus.

But the angles of a rhombus are bisected by its diagonals, so **a + b** bisects the angle between vector **a** and vector **b**.

**b**



$$\vec{JH} = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}, \quad \vec{JK} = \begin{pmatrix} -6 \\ -3 \\ 6 \end{pmatrix}$$

We create unit vectors **u**<sub>1</sub> and **u**<sub>2</sub> in the directions of  $\vec{JH}$  and  $\vec{JK}$  respectively.

$$\therefore \mathbf{u}_1 = \frac{1}{\sqrt{4+4+1}} \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \\ -\frac{1}{3} \end{pmatrix} \quad \text{and} \quad \mathbf{u}_2 = \frac{1}{\sqrt{36+9+36}} \begin{pmatrix} -6 \\ -3 \\ 6 \end{pmatrix} = \begin{pmatrix} -\frac{2}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$$

Using **a**, the line *L* has direction vector  $\mathbf{u}_1 + \mathbf{u}_2 = \begin{pmatrix} 0 \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}$ , or  $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ .

$\therefore$  the vector equation of the line *L* is  $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R}$

$$\text{c } \overrightarrow{HK} = \begin{pmatrix} 1-9 \\ 0-5 \\ 2-5 \end{pmatrix} = \begin{pmatrix} -8 \\ -5 \\ 7 \end{pmatrix} \text{ so (HK) has equation } \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + s \begin{pmatrix} -8 \\ -5 \\ 7 \end{pmatrix}, \quad s \in \mathbb{R}.$$

This line meets  $L$  where

$$\begin{aligned} 7 &= 1 - 8s, & 3 + \lambda &= -5s, & \text{and} & -4 + \lambda &= 2 + 7s \quad \dots (*) \\ \therefore 8s &= -6 & \text{and so } 3 + \lambda &= \frac{15}{4} \\ \therefore s &= -\frac{3}{4} & \therefore \lambda &= \frac{3}{4} \end{aligned}$$

Substituting  $s = -\frac{3}{4}$  and  $\lambda = \frac{3}{4}$  into  $(*)$ ,  $-4 + \frac{3}{4} = 2 + 7(-\frac{3}{4}) = -\frac{13}{4}$  ✓

$$\text{So, } \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} - \frac{3}{4} \begin{pmatrix} -8 \\ -5 \\ 7 \end{pmatrix} = \begin{pmatrix} 1+6 \\ 0+\frac{15}{4} \\ 2-\frac{21}{4} \end{pmatrix} = \begin{pmatrix} 7 \\ 3\frac{3}{4} \\ -3\frac{1}{4} \end{pmatrix}$$

$\therefore L$  meets (HK) at  $(7, 3\frac{3}{4}, -3\frac{1}{4})$ .

**5 a** Lines (AB) and (AC) meet at A.

$$\begin{aligned} \therefore \begin{pmatrix} 4 \\ -1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 3 \end{pmatrix} &= \begin{pmatrix} -1 \\ 0 \end{pmatrix} + u \begin{pmatrix} 3 \\ 1 \end{pmatrix} \\ \therefore \begin{pmatrix} 4+t \\ -1+3t \end{pmatrix} &= \begin{pmatrix} -1+3u \\ u \end{pmatrix} \\ \therefore t-3u &= -5 \quad \dots (*) \\ 3t-u &= 1 \\ \therefore -3t+9u &= 15 \quad \{-3 \times (*)\} \\ \underline{3t-u} &= 1 \\ \therefore 8u &= 16 \\ \therefore u &= 2 \end{aligned}$$

$$\text{So, } \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

$\therefore A$  is at  $(5, 2)$ .

Lines (BC) and (AC) meet at C.

$$\begin{aligned} \therefore \begin{pmatrix} 7 \\ 4 \end{pmatrix} + s \begin{pmatrix} 1 \\ -1 \end{pmatrix} &= \begin{pmatrix} -1 \\ 0 \end{pmatrix} + u \begin{pmatrix} 3 \\ 1 \end{pmatrix} \\ \therefore \begin{pmatrix} 7+s \\ 4-s \end{pmatrix} &= \begin{pmatrix} -1+3u \\ u \end{pmatrix} \\ \therefore s-3u &= -8 \\ -s-u &= -4 \\ \underline{-s-u} &= -4 \\ \therefore -4u &= -12 \\ \therefore u &= 3 \end{aligned}$$

$$\text{So, } \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} + 3 \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 3 \end{pmatrix}$$

$\therefore C$  is at  $(8, 3)$ .

Lines (AB) and (BC) meet at B.

$$\begin{aligned} \therefore \begin{pmatrix} 4 \\ -1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 3 \end{pmatrix} &= \begin{pmatrix} 7 \\ 4 \end{pmatrix} + s \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ \therefore \begin{pmatrix} 4+t \\ -1+3t \end{pmatrix} &= \begin{pmatrix} 7+s \\ 4-s \end{pmatrix} \\ \therefore t-s &= 3 \\ 3t+s &= 5 \\ \underline{3t+s} &= 5 \\ \therefore 4t &= 8 \\ \therefore t &= 2 \end{aligned}$$

$$\text{So, } \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ 5 \end{pmatrix}$$

$\therefore B$  is at  $(6, 5)$ .

$$\begin{aligned} \text{b } \vec{AB} &= \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \text{ so } |\vec{AB}| = \sqrt{1+9} = \sqrt{10} \text{ units} & \vec{BC} &= \begin{pmatrix} 2 \\ -2 \end{pmatrix}, \text{ so } |\vec{BC}| = \sqrt{4+4} = \sqrt{8} \text{ units} \\ \vec{AC} &= \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \text{ so } |\vec{AC}| = \sqrt{9+1} = \sqrt{10} \text{ units} \end{aligned}$$

c Triangle ABC is isosceles.

6 a i The yacht is initially at  $(-6, 10)$ , so its initial position vector is  $-6\mathbf{i} + 10\mathbf{j}$ .

ii  $-\mathbf{i} - 3\mathbf{j}$  has length  $\sqrt{(-1)^2 + (-3)^2} = \sqrt{10}$

$\therefore 5(-\mathbf{i} - 3\mathbf{j})$  has length  $5\sqrt{10}$

$\therefore$  the velocity vector of the yacht is  $-5\mathbf{i} - 15\mathbf{j}$

$$\text{iii } \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -6 \\ 10 \end{pmatrix} + t \begin{pmatrix} -5 \\ -15 \end{pmatrix}$$

$\therefore$  the position vector of the yacht after  $t$  hours is

$$\begin{aligned} & -6\mathbf{i} + 10\mathbf{j} + t(-5\mathbf{i} - 15\mathbf{j}) \\ & = (-6 - 5t)\mathbf{i} + (10 - 15t)\mathbf{j}, \quad t \geq 0 \end{aligned}$$

b Let P be the point on the yacht's path when it is closest to the beacon.

$$\text{Then } \vec{OP} = \begin{pmatrix} -6 - 5t \\ 10 - 15t \end{pmatrix} \text{ and}$$

$$\vec{OP} \bullet \begin{pmatrix} -1 \\ -3 \end{pmatrix} = 0$$

$$\therefore -1(-6 - 5t) - 3(10 - 15t) = 0$$

$$\therefore 6 + 5t - 30 + 45t = 0$$

$$\therefore 50t = 24$$

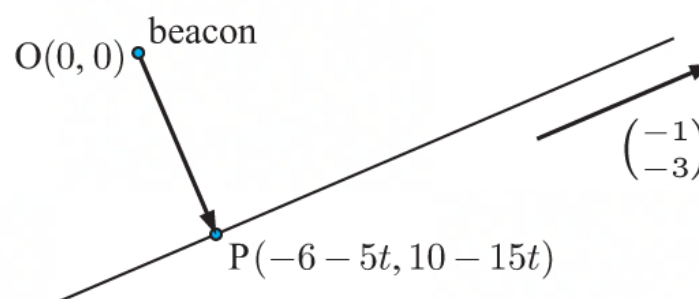
$$\therefore t = 0.48 \text{ hours (or 28.8 minutes)}$$

$$\text{c When } t = 0.48, \vec{OP} = \begin{pmatrix} -6 - 5(0.48) \\ 10 - 15(0.48) \end{pmatrix} = \begin{pmatrix} -8.4 \\ 2.8 \end{pmatrix}$$

$$\text{and } |\vec{OP}| = \sqrt{(-8.4)^2 + (2.8)^2} \approx 8.85 \text{ km}$$

The closest that the yacht gets to the beacon is about  $8.85 \text{ km} > 8 \text{ km}$ .

$\therefore$  the yacht will not hit the reef.



$$\text{7 a } \begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} + t \begin{pmatrix} 1 \\ -3 \end{pmatrix} \text{ where } t \geq 0. \text{ When } t = 0, \text{ the time is } 2:17 \text{ pm.}$$

$$\therefore x_1(t) = 2 + t, \quad y_1(t) = 4 - 3t, \quad t \geq 0$$

b After  $t$  minutes have passed, submarine Y18's torpedo has been moving for  $(t - 2)$  minutes.

$$\therefore x_2(t) = 11 - (t - 2), \quad y_2(t) = 3 + a(t - 2)$$

$$\therefore x_2(t) = 13 - t, \quad y_2(t) = (3 - 2a) + at, \quad t \geq 2$$

- c The interception occurs when  $2 + t = 13 - t$  and  $4 - 3t = (3 - 2a) + at$   
 $\therefore 2t = 11$   
 $\therefore t = \frac{11}{2}$

$\therefore$  the interception occurs  $5\frac{1}{2}$  minutes after 2:17 pm = 2:22:30 pm.

- d When  $t = \frac{11}{2}$ ,  $4 - 3(\frac{11}{2}) = (3 - 2a) + a(\frac{11}{2})$   
 $\therefore -\frac{25}{2} = 3 + \frac{7a}{2}$   
 $\therefore -25 = 6 + 7a$   
 $\therefore 7a = -31$   
 $\therefore a = -\frac{31}{7}$

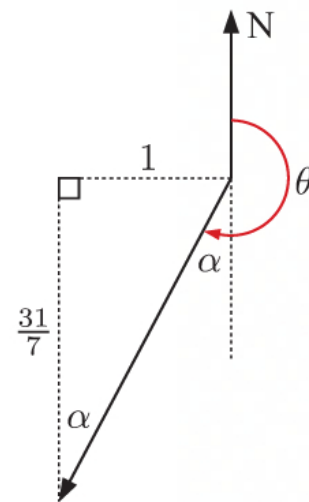
Y18's torpedo has velocity vector  $\begin{pmatrix} -1 \\ -\frac{31}{7} \end{pmatrix}$

with speed  $= \sqrt{(-1)^2 + \left(-\frac{31}{7}\right)^2}$   
 $\approx 4.54$  km per minute

$$\tan \alpha = \frac{1}{\frac{31}{7}} = \frac{7}{31}$$

$$\therefore \alpha = \tan^{-1}\left(\frac{7}{31}\right) \approx 12.7^\circ$$

So, the torpedo has a speed of about  $4.54 \text{ km min}^{-1}$ , travelling on the bearing  $\approx 192.7^\circ$ .



- e If  $\theta$  is the acute angle between the paths of the torpedoes, then
- $$\cos \theta = \frac{\left| \begin{pmatrix} 1 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -\frac{31}{7} \end{pmatrix} \right|}{\left| \begin{pmatrix} 1 \\ -3 \end{pmatrix} \right| \left| \begin{pmatrix} -1 \\ -\frac{31}{7} \end{pmatrix} \right|}$$
- $$= \frac{\left| -1 + \frac{93}{7} \right|}{\sqrt{10} \frac{\sqrt{1010}}{7}}$$
- $$= \frac{86}{\sqrt{10}\sqrt{1010}}$$
- $$\therefore \theta = \cos^{-1}\left(\frac{86}{\sqrt{10}\sqrt{1010}}\right)$$
- $$\approx 31.2^\circ$$

- 8 Since  $2 - x = y - 3 = -\frac{1}{2}z$ ,  $\frac{x-2}{-1} = \frac{y-3}{1} = \frac{z}{-2}$

$\therefore$  the line has direction vector  $\mathbf{u} = \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}$ , and passes through  $(2, 3, 0)$

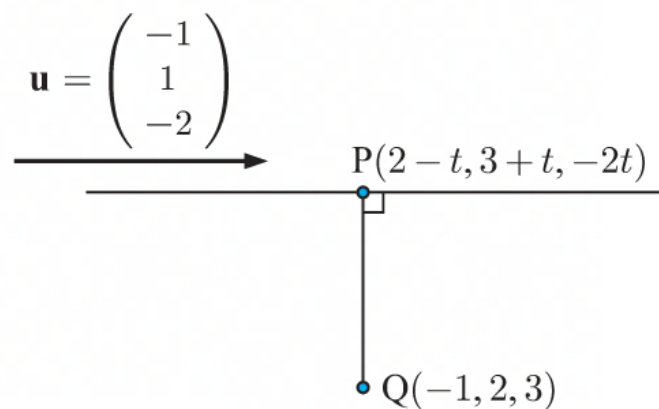
$\therefore$  if P is a point on the line with coordinates  $(2 - t, 3 + t, -2t)$ , then

$$\overrightarrow{\text{QP}} = \begin{pmatrix} 2 - t - (-1) \\ 3 + t - 2 \\ -2t - 3 \end{pmatrix} = \begin{pmatrix} 3 - t \\ 1 + t \\ -2t - 3 \end{pmatrix}$$



If P is chosen such that  $\overrightarrow{QP}$  is perpendicular to the line, then  $\mathbf{u} \bullet \overrightarrow{QP} = 0$

$$\begin{aligned}\therefore \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} 3-t \\ 1+t \\ -2t-3 \end{pmatrix} &= 0 \\ \therefore -(3-t) + 1(1+t) - 2(-2t-3) &= 0 \\ \therefore -3+t+1+t+4t+6 &= 0 \\ \therefore 6t &= -4 \\ \therefore t &= -\frac{2}{3}\end{aligned}$$



$\therefore$  P is at  $(2 + \frac{2}{3}, 3 - \frac{2}{3}, 2(\frac{2}{3}))$ , so the foot of the perpendicular is at  $(\frac{8}{3}, \frac{7}{3}, \frac{4}{3})$ .

- 9 A(-1, 0, 2), B(0, -1, 1), C(1, 2, -1)

a  $\overrightarrow{AB} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$  and  $\overrightarrow{AC} = \begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix}$

$$\begin{aligned}\therefore \mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & -1 \\ 2 & 2 & -3 \end{vmatrix} \\ &= \begin{vmatrix} -1 & -1 \\ 2 & -3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & -1 \\ 2 & -3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & -1 \\ 2 & 2 \end{vmatrix} \mathbf{k} \\ &= 5\mathbf{i} + \mathbf{j} + 4\mathbf{k} \\ &= \begin{pmatrix} 5 \\ 1 \\ 4 \end{pmatrix}\end{aligned}$$

$\therefore$  since A lies on the plane it has equation  $5x + y + 4z = 5(-1) + 0 + 4(2)$

$$\therefore 5x + y + 4z = 3$$

- b Since the normal has direction  $\begin{pmatrix} 5 \\ 1 \\ 4 \end{pmatrix}$  and passes through (0, 0, 0), it has equation

$$\begin{aligned}x &= 0 + 5t, & y &= 0 + t, & z &= 0 + 4t \\ \therefore x &= 5t, & y &= t, & z &= 4t, & t \in \mathbb{R}\end{aligned}$$

- c The line meets the plane when  $5(5t) + t + 4(4t) = 3$

$$\therefore 25t + t + 16t = 3$$

$$\therefore 42t = 3$$

$$\therefore t = \frac{1}{14}$$

So, the line meets the plane at  $(\frac{5}{14}, \frac{1}{14}, \frac{2}{7})$ .

**10 a**  $\frac{x-8}{3} = \frac{y+9}{-16} = \frac{z-10}{7}$  has direction vector  $\mathbf{v}_1 = \begin{pmatrix} 3 \\ -16 \\ 7 \end{pmatrix}$

$x = 15 + 3t, y = 29 + 8t, z = 5 - 5t$  has direction vector  $\mathbf{v}_2 = \begin{pmatrix} 3 \\ 8 \\ -5 \end{pmatrix}$

$\therefore$  since the direction vectors are not scalar multiples of each other, the lines are not parallel.

If they intersect, then  $\frac{15+3t-8}{3} = \frac{29+8t+9}{-16} = \frac{5-5t-10}{7}$

$\therefore t + \frac{7}{3} = -\frac{1}{2}t - \frac{19}{8} = -\frac{5}{7}t - \frac{5}{7}$

Now  $t + \frac{7}{3} = -\frac{1}{2}t - \frac{19}{8}$  requires  $\frac{3}{2}t = -\frac{19}{8} - \frac{7}{3} = -\frac{113}{24} \therefore t = -\frac{113}{36}$

and  $t + \frac{7}{3} = -\frac{5}{7}t - \frac{5}{7}$  requires  $\frac{12}{7}t = -\frac{5}{7} - \frac{7}{3} = -\frac{64}{21} \therefore t = -\frac{16}{9}$

Hence the lines do not intersect, and since they are not parallel, they are skew.

**b** If  $\theta$  is the acute angle between  $L_1$  and  $L_2$ ,

$$\begin{aligned} \text{then } \cos \theta &= \frac{|\mathbf{v}_1 \bullet \mathbf{v}_2|}{|\mathbf{v}_1||\mathbf{v}_2|} = \frac{|3(3) + (-16)(8) + 7(-5)|}{\sqrt{3^2 + (-16)^2 + 7^2}\sqrt{3^2 + 8^2 + (-5)^2}} \\ &= \frac{154}{\sqrt{314}\sqrt{98}} \\ \therefore \theta &= \cos^{-1}\left(\frac{154}{\sqrt{314}\sqrt{98}}\right) \\ &\approx 28.6^\circ \end{aligned}$$

**c** Let  $L_3$  be a line which is parallel to  $L_1$ , but intersects with  $L_2$ .

$\therefore L_3$  has direction vector  $\mathbf{v}_1 = \begin{pmatrix} 3 \\ -16 \\ 7 \end{pmatrix}$ .

Consider the plane  $P$  that contains  $L_2$  and  $L_3$ .

The normal vector of  $P$  is:

$$\begin{aligned} \mathbf{v}_1 \times \mathbf{v}_2 &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -16 & 7 \\ 3 & 8 & -5 \end{vmatrix} \\ &= \begin{vmatrix} -16 & 7 \\ 8 & -5 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 3 & 7 \\ 3 & -5 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 3 & -16 \\ 3 & 8 \end{vmatrix} \mathbf{k} \\ &= 24\mathbf{i} + 36\mathbf{j} + 72\mathbf{k} \end{aligned}$$

A(8, -9, 10) lies on  $L_1$ , and B(15, 29, 5) lies on  $L_2$ .

From **Investigation 4** question **2 b**, the shortest distance between  $L_1$  and  $L_2$  is

$$\begin{aligned} d &= \frac{|\overrightarrow{AB} \bullet (\mathbf{v}_1 \times \mathbf{v}_2)|}{|\mathbf{v}_1 \times \mathbf{v}_2|} \\ &= \frac{|7(24) + 38(36) - 5(72)|}{\sqrt{24^2 + 36^2 + 72^2}} \\ &= \frac{1176}{\sqrt{7056}} = 14 \text{ units} \end{aligned}$$

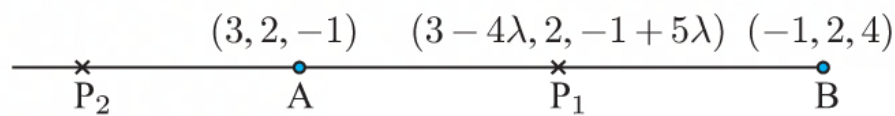
**11 a**  $\overrightarrow{AB} = \begin{pmatrix} -4 \\ 0 \\ 5 \end{pmatrix}$

$\therefore$  the line through A and B is  $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} -4 \\ 0 \\ 5 \end{pmatrix}, \lambda \in \mathbb{R}$

**b** The equation of the plane is

$$-4x + 0y + 5z = -4(-1) + 5(4)$$

$$\therefore -4x + 5z = 24$$



**c** The distance from a point on the line to A is  $d = \sqrt{(-4\lambda)^2 + 0^2 + (5\lambda)^2} = \sqrt{41\lambda^2}$

$$\therefore \text{since } d = 2\sqrt{41} \text{ units, } \sqrt{41\lambda^2} = 2\sqrt{41}$$

$$\therefore \lambda^2 = 4$$

$$\therefore \lambda = \pm 2$$

$\therefore$  the points are  $(-5, 2, 9)$  and  $(11, 2, -11)$ .

**12**  $\overrightarrow{CD} = \begin{pmatrix} 3 \\ -1 \\ -3 \end{pmatrix}$

$\therefore$  the line passing through C and D has parametric equations

$$x = -3 + 3t, y = 2 - t, z = -1 - 3t, t \in \mathbb{R}.$$

The line meets  $2x - y + z = 3$  when  $2(-3 + 3t) - (2 - t) + (-1 - 3t) = 3$

$$\therefore -6 + 6t - 2 + t - 1 - 3t = 3$$

$$\therefore 4t = 12$$

$$\therefore t = 3$$

$\therefore$  they meet at  $(6, -1, -10)$ .

**13 a**  $\overrightarrow{LM} = \begin{pmatrix} -2 \\ 2 \\ -2 \end{pmatrix} = -2 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

$\therefore$  since L lies on the line, it has parametric equations

$$x = 1 + t, y = 0 - t, z = 1 + t, t \in \mathbb{R}$$

The line meets  $x - 2y - 3z = 14$  when

$$(1 + t) - 2(-t) - 3(1 + t) = 14$$

$$\therefore 1 + t + 2t - 3 - 3t = 14$$

$$\therefore -2 = 14 \text{ which is absurd}$$

$\therefore$  the line and plane do not meet, and hence must be parallel.

**b** From Exercise 13G question 21 b,

$$\text{the distance } d = \frac{|x_1 - 2y_1 - 3z_1 - 14|}{\sqrt{1 + 4 + 9}} = \frac{|1 - 2(0) - 3(1) - 14|}{\sqrt{14}} = \frac{16}{\sqrt{14}} \text{ units.}$$

**14 a** From Exercise 13G question 21 b, the distance of  $A(-1, 3, 2)$  from  $2x - y + 2z = 8$  is

$$\begin{aligned} d &= \frac{|2x_1 - y_1 + 2z_1 - 8|}{\sqrt{2^2 + (-1)^2 + 2^2}} = \frac{|2(-1) - 3 + 2(2) - 8|}{3} \\ &= \frac{|-9|}{3} = 3 \text{ units} \end{aligned}$$

- b** The point on the plane nearest A is the foot of the normal to the plane that passes through A.

Since the normal has direction vector  $\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$  and passes through  $(-1, 3, 2)$ ,

it has equation  $x = -1 + 2t$ ,  $y = 3 - t$ ,  $z = 2 + 2t$ ,  $t \in \mathbb{R}$ .

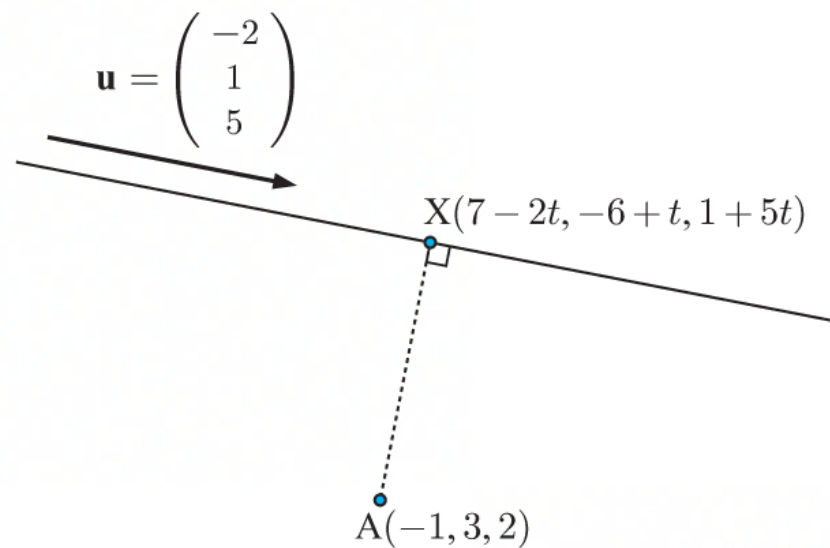
$$\begin{aligned} \text{The normal meets the plane when } 2(-1 + 2t) - (3 - t) + 2(2 + 2t) &= 8 \\ \therefore -2 + 4t - 3 + t + 4 + 4t &= 8 \\ \therefore 9t &= 9 \\ \therefore t &= 1 \end{aligned}$$

$\therefore$  the point is  $(1, 2, 4)$ .

- c** Suppose N is the foot of the perpendicular from A to the line, so N has coordinates  $(7 - 2t, -6 + t, 1 + 5t)$  for some  $t \in \mathbb{R}$ . The shortest distance from A to the line is AN.

$$\overrightarrow{AN} = \begin{pmatrix} 8 - 2t \\ t - 9 \\ -1 + 5t \end{pmatrix} \text{ and the line has direction vector } \mathbf{u} = \begin{pmatrix} -2 \\ 1 \\ 5 \end{pmatrix}.$$

$$\begin{aligned} \text{Now } \mathbf{u} \bullet \overrightarrow{AN} &= 0 \\ \therefore \begin{pmatrix} -2 \\ 1 \\ 5 \end{pmatrix} \bullet \begin{pmatrix} 8 - 2t \\ t - 9 \\ -1 + 5t \end{pmatrix} &= 0 \\ \therefore -16 + 4t + t - 9 - 5 + 25t &= 0 \\ \therefore 30t &= 30 \\ \therefore t &= 1 \\ \therefore |\overrightarrow{AN}| &= \sqrt{6^2 + (-8)^2 + 4^2} \\ &= \sqrt{36 + 64 + 16} = \sqrt{116} \text{ units} \end{aligned}$$



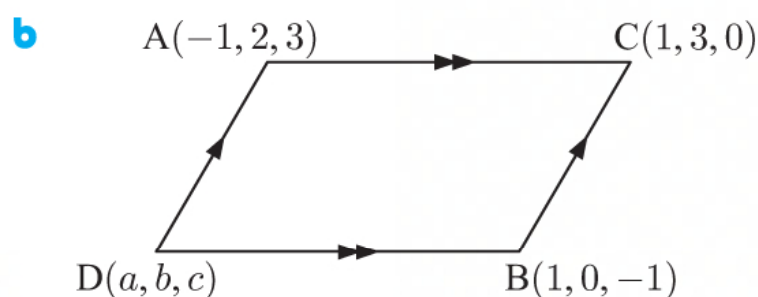
- 15 a**  $A(-1, 2, 3)$ ,  $B(1, 0, -1)$ ,  $C(1, 3, 0)$

$$\overrightarrow{AB} = \begin{pmatrix} 2 \\ -2 \\ -4 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \text{ and } \overrightarrow{AC} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$$

$\therefore$  a normal to the plane containing A, B, and C is

$$\begin{aligned} \mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & -2 \\ 2 & 1 & -3 \end{vmatrix} \\ &= \begin{vmatrix} -1 & -2 \\ 1 & -3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & -2 \\ 2 & -3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} \mathbf{k} \\ &= 5\mathbf{i} - \mathbf{j} + 3\mathbf{k} \\ &= \begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix} \end{aligned}$$





Suppose D has coordinates  $(a, b, c)$

$$\text{Since } \overrightarrow{AD} = \overrightarrow{CB}, \quad \begin{pmatrix} a+1 \\ b-2 \\ c-3 \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \\ -1 \end{pmatrix}$$

$$\therefore a = -1, \quad b = -1, \quad \text{and} \quad c = 2$$

$\therefore$  D is at  $(-1, -1, 2)$ .

**c** From **a**,  $\overrightarrow{AC} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$  and from **b**,  $\overrightarrow{AD} = \begin{pmatrix} 0 \\ -3 \\ -1 \end{pmatrix}$

$$\begin{aligned} \overrightarrow{AC} \times \overrightarrow{AD} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -3 \\ 0 & -3 & -1 \end{vmatrix} \\ &= \begin{vmatrix} 1 & -3 \\ -3 & -1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & -3 \\ 0 & -1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & 1 \\ 0 & -3 \end{vmatrix} \mathbf{k} \\ &= -10\mathbf{i} + 2\mathbf{j} - 6\mathbf{k} \end{aligned}$$

$$\begin{aligned} \therefore \text{area of parallelogram} &= |-10\mathbf{i} + 2\mathbf{j} - 6\mathbf{k}| \\ &= \sqrt{100 + 4 + 36} \\ &= \sqrt{140} \approx 11.8 \text{ units}^2 \end{aligned}$$

**d** From **a**,  $\overrightarrow{AB}$  has direction vector  $\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$

$\therefore$  the line through A and B has parametric equations

$$x = 1 + t, \quad y = 0 - t, \quad z = -1 - 2t, \quad t \in \mathbb{R}.$$

If  $P(1 + t, -t, -1 - 2t)$  is the foot of the perpendicular,

then  $\overrightarrow{CP} = \begin{pmatrix} t \\ -t-3 \\ -1-2t \end{pmatrix}$  and  $\overrightarrow{CP} \cdot \overrightarrow{AB} = 0$

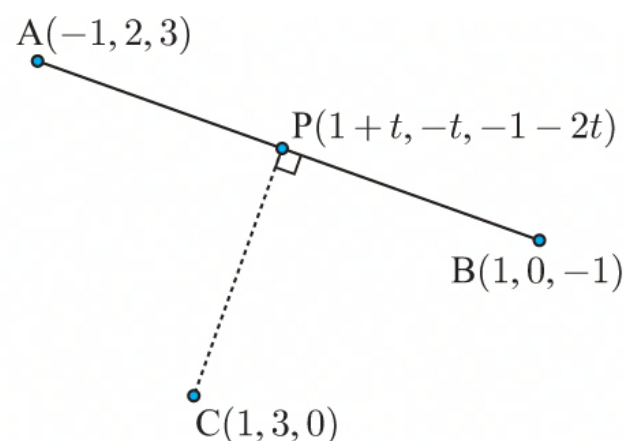
$$\therefore \begin{pmatrix} t \\ -t-3 \\ -1-2t \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} = 0$$

$$\therefore t + t + 3 + 2 + 4t = 0$$

$$\therefore 6t = -5$$

$$\therefore t = -\frac{5}{6}$$

$$\therefore P \text{ is } \left(1 - \frac{5}{6}, \frac{5}{6}, -1 + \frac{10}{6}\right) \text{ or } \left(\frac{1}{6}, \frac{5}{6}, \frac{2}{3}\right).$$



**16 a** The lines meet where  $\begin{pmatrix} 3 \\ -2 \\ -2 \end{pmatrix} + s \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} + t \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$

$$\therefore \begin{pmatrix} -s \\ s \\ 2s \end{pmatrix} = \begin{pmatrix} -t \\ -t+2 \\ t+1 \end{pmatrix}$$

$$\therefore \begin{cases} -s = -t & \Rightarrow s = t & \dots (1) \\ s = -t + 2 & \dots (2) \\ 2s = t + 1 & \dots (3) \end{cases}$$

Substituting (1) into (2) gives  $t = -t + 2$

$$\therefore 2t = 2$$

$$\therefore t = 1 \text{ and } s = 1$$

Checking in (3):  $2(1) = 1 + 1$  ✓

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ -2 \end{pmatrix} + 1 \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$$

$\therefore$  A is  $(2, -1, 0)$ .

**b**  $B(0, -3, 2)$  lies on  $L_2$  if  $\begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} + t \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \\ 2 \end{pmatrix}$  for some  $t \in \mathbb{R}$

$$\therefore \begin{cases} 3 - t = 0 \\ -t = -3 \\ -1 + t = 2 \end{cases}$$

$t = 3$  satisfies all these equations

$\therefore B(0, -3, 2)$  lies on  $L_2$ .

**c**  $\overrightarrow{BC} = \begin{pmatrix} 3 \\ 1 \\ -4 \end{pmatrix}$

$\therefore$  the vector equation of the line (BC) is  $\mathbf{r} = \begin{pmatrix} 0 \\ -3 \\ 2 \end{pmatrix} + u \begin{pmatrix} 3 \\ 1 \\ -4 \end{pmatrix}, u \in \mathbb{R}$ .

**d**  $\overrightarrow{AB} = \begin{pmatrix} -2 \\ -2 \\ 2 \end{pmatrix}, \overrightarrow{AC} = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$

$\therefore$  the plane containing A, B, and C has normal vector

$$\begin{aligned} \mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & -2 & 2 \\ 1 & -1 & -2 \end{vmatrix} \\ &= \begin{vmatrix} -2 & 2 \\ -1 & -2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -2 & 2 \\ 1 & -2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -2 & -2 \\ 1 & -1 \end{vmatrix} \mathbf{k} \\ &= 6\mathbf{i} - 2\mathbf{j} + 4\mathbf{k} \end{aligned}$$

$\therefore$  the equation is  $6x - 2y + 4z = 6(2) - 2(-1) + 4(0) = 14$  {using point A}

$$\therefore 3x - y + 2z = 7$$

$$\begin{aligned}
 \text{e Area of triangle ABC} &= \frac{1}{2} | \overrightarrow{AB} \times \overrightarrow{AC} | \\
 &= \frac{1}{2} | 6\mathbf{i} - 2\mathbf{j} + 4\mathbf{k} | \\
 &= \frac{1}{2} \sqrt{6^2 + (-2)^2 + 4^2} \\
 &= \frac{1}{2} \sqrt{56} \\
 &= \sqrt{14} \text{ units}^2
 \end{aligned}$$

$$\text{f The normal to the plane has direction vector } \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}.$$

$\therefore$  the normal to the plane passing through  $C(3, -2, -2)$  is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}, \lambda \in \mathbb{R}$$

$$D(9, -4, 2) \text{ lies on this line if } \begin{pmatrix} 3 \\ -2 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 9 \\ -4 \\ 2 \end{pmatrix} \text{ for some } \lambda \in \mathbb{R}$$

$$\therefore \begin{cases} 3 + 3\lambda = 9 \\ -2 - \lambda = -4 \\ -2 + 2\lambda = 2 \end{cases}$$

$\lambda = 2$  satisfies all three equations

$\therefore D$  lies on this line.

$$\text{17 } A(-1, 2, 3), B(2, 0, -1), C(-3, 2, -4)$$

$$\text{a } \overrightarrow{AB} = \begin{pmatrix} 3 \\ -2 \\ -4 \end{pmatrix}, \overrightarrow{AC} = \begin{pmatrix} -2 \\ 0 \\ -7 \end{pmatrix}$$

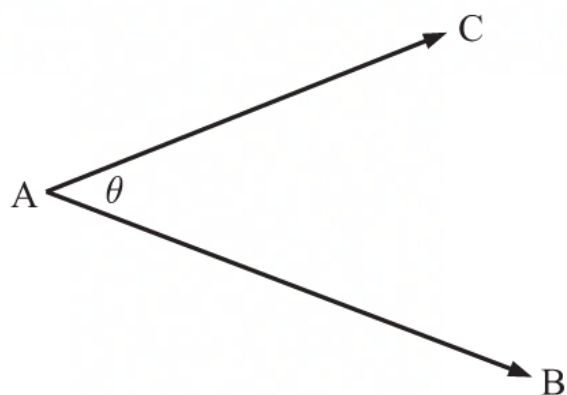
$\therefore$  a normal vector to the plane is

$$\begin{aligned}
 \mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 0 & -7 \\ 3 & -2 & -4 \end{vmatrix} \\
 &= \begin{vmatrix} 0 & -7 \\ -2 & -4 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -2 & -7 \\ 3 & -4 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -2 & 0 \\ 3 & -2 \end{vmatrix} \mathbf{k} \\
 &= -14\mathbf{i} - 29\mathbf{j} + 4\mathbf{k}
 \end{aligned}$$

$\therefore$  since  $B$  lies on the plane, it has equation  $14x + 29y - 4z = 14(2) + 29(0) - 4(-1)$

$$\therefore 14x + 29y - 4z = 32$$

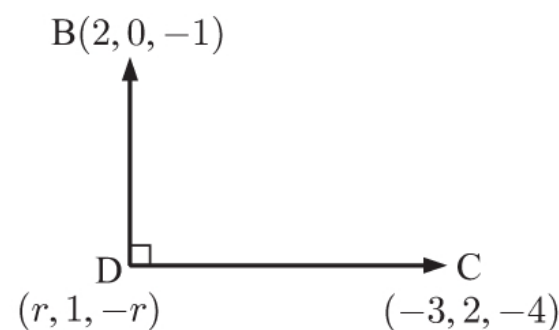
b



$$\begin{aligned}
 \cos(\widehat{CAB}) &= \frac{|\overrightarrow{AB} \cdot \overrightarrow{AC}|}{|\overrightarrow{AB}| |\overrightarrow{AC}|} \\
 &= \frac{|-6 + 0 + 28|}{\sqrt{9 + 4 + 16} \sqrt{4 + 0 + 49}} \\
 &= \frac{22}{\sqrt{29}\sqrt{53}}
 \end{aligned}$$

$$\therefore \widehat{CAB} = \cos^{-1} \left( \frac{22}{\sqrt{29}\sqrt{53}} \right) \approx 55.9^\circ$$

c If D is at  $(r, 1, -r)$  then  $\overrightarrow{DB} = \begin{pmatrix} 2-r \\ -1 \\ -1+r \end{pmatrix}$   
 and  $\overrightarrow{DC} = \begin{pmatrix} -3-r \\ 1 \\ -4+r \end{pmatrix}$



Now  $\widehat{BDC}$  is a right angle, so  $\overrightarrow{DB} \bullet \overrightarrow{DC} = 0$   
 $\therefore (2-r)(-3-r) + (-1) + (-1+r)(-4+r) = 0$   
 $\therefore -6 - 2r + 3r + r^2 - 1 + 4 - r - 4r + r^2 = 0$   
 $\therefore 2r^2 - 4r - 3 = 0$

$$\therefore r = \frac{4 \pm \sqrt{16 + 24}}{4}$$

$$\therefore r = \frac{2 \pm \sqrt{10}}{2}$$

18 a  $P_1$  has normal  $\mathbf{n}_1 = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$  and  
 $L$  has direction vector  $\mathbf{l} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ .

If  $\phi$  is the acute angle between  $L$  and  $P_1$ ,

then  $\sin \phi = \frac{|2 - 2 + 2|}{\sqrt{9}\sqrt{6}}$   
 $= \frac{2}{3\sqrt{6}}$   
 $\therefore \phi = \sin^{-1}\left(\frac{2}{3\sqrt{6}}\right) \approx 15.8^\circ$

b  $P_1$  and  $P_2$  have normals  $\mathbf{n}_1 = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$   
 and  $\mathbf{n}_2 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$  respectively.

If  $\theta$  is the acute angle between  $P_1$  and  $P_2$ ,

then  $\cos \theta = \frac{|2 - 1 - 4|}{\sqrt{9}\sqrt{6}}$   
 $= \frac{3}{3\sqrt{6}}$   
 $= \frac{1}{\sqrt{6}}$   
 $\therefore \theta = \cos^{-1}\left(\frac{1}{\sqrt{6}}\right) \approx 65.9^\circ$

19 a  $P_1, P_2$ , and  $P_3$  are mutually perpendicular, and have normal vectors  $\mathbf{n}_1 = \begin{pmatrix} a \\ 3 \\ -1 \end{pmatrix}$ ,

$\mathbf{n}_2 = \begin{pmatrix} b \\ 0 \\ 2 \end{pmatrix}$ , and  $\mathbf{n}_3 = \begin{pmatrix} c \\ -5 \\ -3 \end{pmatrix}$  respectively.

$$\mathbf{n}_1 \bullet \mathbf{n}_2 = 0$$

$$\therefore \begin{pmatrix} a \\ 3 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} b \\ 0 \\ 2 \end{pmatrix} = 0$$

$$\therefore ab + 0 - 2 = 0$$

$$\therefore ab = 2$$

$$\therefore b = \frac{2}{a} \quad \dots (1)$$

$$\mathbf{n}_1 \bullet \mathbf{n}_3 = 0$$

$$\therefore \begin{pmatrix} a \\ 3 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} c \\ -5 \\ -3 \end{pmatrix} = 0$$

$$\therefore ac - 15 + 3 = 0$$

$$\therefore ac = 12$$

$$\therefore c = \frac{12}{a} \quad \dots (2)$$



$$\begin{aligned}
 & \mathbf{n}_2 \bullet \mathbf{n}_3 = 0 \\
 \therefore & \begin{pmatrix} b \\ 0 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} c \\ -5 \\ -3 \end{pmatrix} = 0 \\
 & \therefore bc + 0 - 6 = 0 \\
 & \therefore bc = 6 \\
 \therefore & \frac{2}{a} \times \frac{12}{a} = 6 \quad \{\text{using (1) and (2)}\} \\
 & \therefore \frac{24}{a^2} = 6 \\
 & \therefore a^2 = 4 \\
 & \therefore a = 2 \quad \{a, b, c \text{ positive}\} \\
 & \therefore b = 1, \quad c = 6 \quad \{\text{using (1) and (2)}\}
 \end{aligned}$$

- b** The equations of the planes  $P_1$ ,  $P_2$ , and  $P_3$ , in augmented matrix form, are:

$$\begin{aligned}
 & \left( \begin{array}{ccc|c} 2 & 3 & -1 & 2 \\ 1 & 0 & 2 & 5 \\ 6 & -5 & -3 & 20 \end{array} \right) \quad \left\{ \begin{array}{cccc} 2 & 0 & 4 & 10 \\ -2 & -3 & 1 & -2 \\ \hline 0 & -3 & 5 & 8 \end{array} \right\} \\
 \sim & \left( \begin{array}{ccc|c} 2 & 3 & -1 & 2 \\ 0 & -3 & 5 & 8 \\ 0 & -14 & 0 & 14 \end{array} \right) \begin{array}{l} 2R_2 - R_1 \rightarrow R_2 \\ R_3 - 3R_1 \rightarrow R_3 \end{array} \quad \left\{ \begin{array}{cccc} 6 & -5 & -3 & 20 \\ -6 & -9 & 3 & -6 \\ \hline 0 & -14 & 0 & 14 \end{array} \right\}
 \end{aligned}$$

$$\begin{aligned}
 \therefore -14y &= 14 \\
 \therefore y &= -1
 \end{aligned}$$

$$\begin{aligned}
 \text{Substituting into row 2 gives } & -3(-1) + 5z = 8 \\
 \therefore & 3 + 5z = 8 \\
 \therefore & 5z = 5 \\
 \therefore & z = 1
 \end{aligned}$$

$$\begin{aligned}
 \text{Substituting into row 1 gives } & 2x + 3(-1) - 1 = 2 \\
 \therefore & 2x - 3 - 1 = 2 \\
 \therefore & 2x = 6 \\
 \therefore & x = 3
 \end{aligned}$$

$\therefore$  the planes intersect at  $(3, -1, 1)$ .

- c** For planes  $P_1$  and  $P_2$  we know that  $\left( \begin{array}{ccc|c} 2 & 3 & -1 & 2 \\ 1 & 0 & 2 & 5 \end{array} \right) \sim \left( \begin{array}{ccc|c} 2 & 3 & -1 & 2 \\ 0 & -3 & 5 & 8 \end{array} \right)$ .

$$\begin{aligned}
 \text{Let } z = t \quad \therefore & -3y + 5t = 8 \\
 \therefore & 3y = 5t - 8 \\
 \therefore & y = \frac{5}{3}t - \frac{8}{3}
 \end{aligned}$$

$$\begin{aligned}
 \text{Substituting into row 1 gives } & 2x + 3\left(\frac{5}{3}t - \frac{8}{3}\right) - t = 2 \\
 \therefore & 2x + 5t - 8 - t = 2 \\
 \therefore & 2x = 10 - 4t \\
 \therefore & x = 5 - 2t
 \end{aligned}$$

So the line  $L$  has direction vector  $\begin{pmatrix} -2 \\ \frac{5}{3} \\ 1 \end{pmatrix}$  or  $\begin{pmatrix} -6 \\ 5 \\ 3 \end{pmatrix}$ .

Using the point of intersection  $(3, -1, 1)$ ,  $L_1$  has vector equation

$$\mathbf{r} = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} + t \begin{pmatrix} -6 \\ 5 \\ 3 \end{pmatrix}, \quad t \in \mathbb{R}$$

**d**  $P_3$  has equation  $6x - 5y - 3z = 20$

$$\begin{aligned} 6(2) - 5(2) - 3(-6) &= 12 - 10 + 18 \\ &= 20 \quad \checkmark \end{aligned}$$

$\therefore P(2, 2, -6)$  lies on  $P_3$ .

**e**  $(3, -1, 1)$  is a point on  $L$  and  $P$  is the point  $(2, 2, -6)$ .

The line between these points has direction vector  $\begin{pmatrix} 2-3 \\ 2-(-1) \\ -6-1 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ -7 \end{pmatrix}$ .

The line  $L$  has direction vector  $\begin{pmatrix} -6 \\ 5 \\ 3 \end{pmatrix}$ .

Both of these lines lie on the plane  $P_4$ .

$$\begin{aligned} \therefore P_4 \text{ has normal vector } \mathbf{n}_4 &= \begin{pmatrix} -1 \\ 3 \\ -7 \end{pmatrix} \times \begin{pmatrix} -6 \\ 5 \\ 3 \end{pmatrix} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 3 & -7 \\ -6 & 5 & 3 \end{vmatrix} \\ &= \begin{vmatrix} 3 & -7 \\ 5 & 3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -1 & -7 \\ -6 & 3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -1 & 3 \\ -6 & 5 \end{vmatrix} \mathbf{k} \\ &= 44\mathbf{i} + 45\mathbf{j} + 13\mathbf{k} \end{aligned}$$

Using the point  $P(2, 2, -6)$  on the plane, the equation of  $P_4$  is

$$44x + 45y + 13z = 44(2) + 45(2) + 13(-6)$$

$$\therefore 44x + 45y + 13z = 100$$

**f**  $P_1$  has normal vector  $\mathbf{n}_1 = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$

$P_4$  has normal vector  $\mathbf{n}_4 = \begin{pmatrix} 44 \\ 45 \\ 13 \end{pmatrix}$

The acute angle between the planes is  $\theta = \cos^{-1} \left( \frac{|\mathbf{n}_1 \bullet \mathbf{n}_4|}{|\mathbf{n}_1| |\mathbf{n}_4|} \right)$

$$= \cos^{-1} \left( \frac{|88 + 135 - 13|}{\sqrt{14}\sqrt{4130}} \right)$$

$$= \cos^{-1} \left( \frac{210}{\sqrt{14}\sqrt{4130}} \right)$$

$$\approx 29.2^\circ$$

**20 a**  $\mathbf{p} \times \mathbf{q} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 2 \\ 2 & 3 & -1 \end{vmatrix}$

$$= \begin{vmatrix} -1 & 2 \\ 3 & -1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} \mathbf{k}$$

$$= -5\mathbf{i} + 5\mathbf{j} + 5\mathbf{k}$$

$$= 5 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

**b**  $L$  has direction vector  $\begin{pmatrix} 2 \\ 1 \\ m \end{pmatrix}$ , and  $\mathbf{p} \times \mathbf{q}$  is parallel to  $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ .

$\therefore \mathbf{p} \times \mathbf{q}$  is perpendicular to  $L$  if  $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 1 \\ m \end{pmatrix} = 0$

$$\therefore -2 + 1 + m = 0$$

$$\therefore m = 1$$

**c**  $P$  has normal vector  $\mathbf{n} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ , and  $(1, -2, 3)$  lies on the plane {letting  $\lambda = 0$ }

$\therefore P$  has equation  $-x + y + z = -1 + (-2) + 3$

$$\therefore x - y - z = 0$$

**d**  $A$  lies on the plane if  $4 - t - 2 = 0$

$$\therefore t = 2$$

**e**  $\overrightarrow{AB} = \begin{pmatrix} 2 \\ -5 \\ 3 \end{pmatrix}$  and  $P$  has normal vector  $\mathbf{n} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

$$\begin{aligned} \therefore \text{ if } \theta \text{ is the angle between } (AB) \text{ and } P, \text{ then } \sin \theta &= \frac{\left| \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -5 \\ 3 \end{pmatrix} \right|}{\sqrt{(-1)^2 + 1^2 + 1^2} \sqrt{2^2 + (-5)^2 + 3^2}} \\ &= \frac{|-2 - 5 + 3|}{\sqrt{3}\sqrt{38}} \\ &= \frac{4}{\sqrt{114}} \\ \therefore \theta &= \sin^{-1}\left(\frac{4}{\sqrt{114}}\right) \approx 22.0^\circ \end{aligned}$$

**21** In augmented matrix form, the system is:

$$\begin{aligned} &\left( \begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 2 & 1 & -1 & -1 \\ 7 & 2 & k & -k \end{array} \right) && \left\{ \begin{array}{ccc|c} 2 & 1 & -1 & -1 \\ -2 & 2 & -2 & -10 \\ \hline 0 & 3 & -3 & -11 \end{array} \right\} \\ \sim &\left( \begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 3 & -3 & -11 \\ 0 & 9 & k-7 & -k-35 \end{array} \right) && \begin{array}{l} R_2 - 2R_1 \rightarrow R_2 \\ R_3 - 7R_1 \rightarrow R_3 \end{array} \left\{ \begin{array}{ccc|c} 7 & 2 & k & -k \\ -7 & 7 & -7 & -35 \\ \hline 0 & 9 & k-7 & -k-35 \end{array} \right\} \\ \sim &\left( \begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 3 & -3 & -11 \\ 0 & 0 & k+2 & -k-2 \end{array} \right) && \begin{array}{l} R_3 - 3R_2 \rightarrow R_3 \end{array} \left\{ \begin{array}{ccc|c} 0 & 9 & k-7 & -k-35 \\ 0 & -9 & 9 & 33 \\ \hline 0 & 0 & k+2 & -k-2 \end{array} \right\} \end{aligned}$$

Thus  $(k+2)z = -(k+2)$

If  $k \neq -2$  then  $z = -1$ , and as  $3y - 3z = -11$ ,

$$\text{then } 3y = -14$$

$$\therefore y = -\frac{14}{3}$$

$$\text{and } x - y + z = 5,$$

$$\therefore x = 5 - \frac{14}{3} + 1 = \frac{4}{3}$$

$\therefore$  we have three planes that meet at the unique point  $\left(\frac{4}{3}, -\frac{14}{3}, -1\right)$ .

If  $k = -2$ , then the 3 planes meet in a common line and hence there are an infinite number of solutions.

In this case, let  $z = t$ ,  $t \in \mathbb{R}$ .

$$\text{Now } 3y - 3z = -11, \quad \text{and} \quad x - y + z = 5$$

$$\therefore 3y = -11 + 3t$$

$$\therefore y = -\frac{11}{3} + t$$

$$\therefore x = 5 + y - z$$

$$\therefore x = 5 - \frac{11}{3} + t - t$$

$$\therefore x = \frac{4}{3}$$

$$\therefore x = \frac{4}{3}, \quad y = -\frac{11}{3} + t, \quad z = t, \quad t \in \mathbb{R}$$



**22 a**  $2(-2t + 2) + t + (3t + 1) = -4t + 4 + t + 3t + 1 = 5$  ✓

∴ the plane contains the line.

**b** If  $x + ky + z = 3$  contains  $L_1$ , then  $(-2t + 2) + k(t) + (3t + 1) = 3$

$$\therefore t(-2 + k + 3) + 2 + 1 = 3$$

$$\therefore t(k + 1) = 0 \quad \dots (*)$$

For  $(*)$  to be true for all  $t \in \mathbb{R}$ ,  $k + 1 = 0$

$$\therefore k = -1$$

**c** From **a** and **b**, both  $2x + y + z = 5$  and  $x - y + z = 3$  contain  $L_1$ .

These planes are not coincident, as their normal vectors are not scalar multiples of each other.

∴ the system only has infinite solutions if  $L_1$  is the intersection of all 3 planes.

So, substituting  $L_1$  into row 3 gives

$$-2(-2t + 2) + pt + 2(3t + 1) = q \quad \text{for all } t \in \mathbb{R}$$

$$\therefore 4t - 4 + pt + 6t + 2 = q$$

$$\therefore t(10 + p) = q + 2$$

This equation has infinitely many solutions for  $t$

when  $10 + p = 0$  and  $q + 2 = 0$  {equating coefficients}

$$\therefore p = -10 \quad \text{and} \quad q = -2$$

## REVIEW SET 13B

**1 a** When  $t = 1$ ,  $\mathbf{r} = \begin{pmatrix} 3 \\ -3 \end{pmatrix} + 1 \begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 + 2 \\ -3 + 5 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$

∴ the point is  $(5, 2)$ .

**b**  $\begin{pmatrix} 4 \\ 10 \end{pmatrix}$  is a non-zero scalar multiple of  $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$ , so it could also be used to describe the direction of the line.

**c** The line passes through point  $(5, 2)$  and has direction vector  $\begin{pmatrix} 4 \\ 10 \end{pmatrix}$ .

∴  $\mathbf{r} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} + s \begin{pmatrix} 4 \\ 10 \end{pmatrix}$ ,  $s \in \mathbb{R}$  is an alternative vector equation for the line.

**2 a i**  $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix}$ ,  $\lambda \in \mathbb{R}$

**ii**  $x = 2 + 4\lambda$ ,  $y = -3 + 2\lambda$ ,  $z = 1 - \lambda$ ,  $\lambda \in \mathbb{R}$

**b i** The line has direction vector  $\begin{pmatrix} 5 - (-1) \\ -2 - 6 \\ 0 - 3 \end{pmatrix} = \begin{pmatrix} 6 \\ -8 \\ -3 \end{pmatrix}$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 6 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 6 \\ -8 \\ -3 \end{pmatrix}, \quad \lambda \in \mathbb{R}$$

**ii**  $x = -1 + 6\lambda$ ,  $y = 6 - 8\lambda$ ,  $z = 3 - 3\lambda$ ,  $\lambda \in \mathbb{R}$

**3**  $L_1$  has direction vector  $\mathbf{b}_1 = \begin{pmatrix} 5-0 \\ -2-3 \end{pmatrix} = \begin{pmatrix} 5 \\ -5 \end{pmatrix}$

$L_2$  has direction vector  $\mathbf{b}_2 = \begin{pmatrix} -6-(-2) \\ 7-4 \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$

If  $\theta$  is the angle between  $L_1$  and  $L_2$ , then  $\cos \theta = \frac{|\mathbf{b}_1 \bullet \mathbf{b}_2|}{|\mathbf{b}_1| |\mathbf{b}_2|}$

$$= \frac{|-20 - 15|}{\sqrt{25 + 25} \sqrt{16 + 9}}$$

$$= \frac{35}{5\sqrt{50}} = \frac{7}{\sqrt{50}}$$

$$\therefore \theta = \cos^{-1}\left(\frac{7}{\sqrt{50}}\right) \approx 8.13^\circ$$

**4 a** (KL) has direction vector  $\begin{pmatrix} 5 \\ -2 \end{pmatrix}$  and (MN) has direction vector  $\begin{pmatrix} -5 \\ 2 \end{pmatrix}$ .

Now  $\begin{pmatrix} 5 \\ -2 \end{pmatrix} = -\begin{pmatrix} -5 \\ 2 \end{pmatrix}$ , so (KL)  $\parallel$  (MN).

**b**  $\overrightarrow{KL} = a \begin{pmatrix} 5 \\ -2 \end{pmatrix}$ ,  $\overrightarrow{NK} = b \begin{pmatrix} 4 \\ 10 \end{pmatrix}$ ,  $\overrightarrow{MN} = c \begin{pmatrix} -5 \\ 2 \end{pmatrix}$  {for some constants  $a, b, c$ }

$\therefore \overrightarrow{KL} \bullet \overrightarrow{NK} = ab(20 - 20) = 0$  and  $\overrightarrow{NK} \bullet \overrightarrow{MN} = bc(-20 + 20) = 0$

$\therefore$  (NK) is perpendicular to both (KL) and (MN).

**c** (KL) and (NK) meet at K.

$$\therefore \begin{pmatrix} 2 \\ 19 \end{pmatrix} + p \begin{pmatrix} 5 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \end{pmatrix} + r \begin{pmatrix} 4 \\ 10 \end{pmatrix}$$

$$\therefore \begin{pmatrix} 5p - 4r \\ -2p - 10r \end{pmatrix} = \begin{pmatrix} 1 \\ -12 \end{pmatrix}$$

$$\therefore 5p - 4r = 1 \quad \dots (1)$$

$$2p + 10r = 12 \quad \dots (2)$$

$$\therefore 25p - 20r = 5 \quad \{5 \times (1)\}$$

$$4p + 20r = 24 \quad \{2 \times (2)\}$$

$$\therefore \frac{29p}{29} = \frac{29}{29}$$

$$\therefore p = 1$$

and  $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 19 \end{pmatrix} + \begin{pmatrix} 5 \\ -2 \end{pmatrix} = \begin{pmatrix} 7 \\ 17 \end{pmatrix}$

$\therefore$  K is (7, 17).

(KL) and (ML) meet at L.

$$\therefore \begin{pmatrix} 2 \\ 19 \end{pmatrix} + p \begin{pmatrix} 5 \\ -2 \end{pmatrix} = \begin{pmatrix} 33 \\ -5 \end{pmatrix} + q \begin{pmatrix} -11 \\ 16 \end{pmatrix}$$

$$\therefore \begin{pmatrix} 5p + 11q \\ -2p - 16q \end{pmatrix} = \begin{pmatrix} 31 \\ -24 \end{pmatrix}$$

$$\therefore 5p + 11q = 31 \quad \dots (1)$$

$$-2p - 16q = -24 \quad \dots (2)$$

$$\therefore 10p + 22q = 62 \quad \{2 \times (1)\}$$

$$-10p - 80q = -120 \quad \{5 \times (2)\}$$

$$\therefore \frac{-58q}{-58} = \frac{-58}{-58}$$

$$\therefore q = 1$$

and  $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 33 \\ -5 \end{pmatrix} + \begin{pmatrix} -11 \\ 16 \end{pmatrix} = \begin{pmatrix} 22 \\ 11 \end{pmatrix}$

$\therefore$  L is (22, 11).

(ML) and (MN) meet at M.

$$\therefore \begin{pmatrix} 33 \\ -5 \end{pmatrix} + q \begin{pmatrix} -11 \\ 16 \end{pmatrix} = \begin{pmatrix} 43 \\ -9 \end{pmatrix} + s \begin{pmatrix} -5 \\ 2 \end{pmatrix}$$

$$\therefore \begin{pmatrix} -11q + 5s \\ 16q - 2s \end{pmatrix} = \begin{pmatrix} 10 \\ -4 \end{pmatrix}$$

$$\therefore -11q + 5s = 10 \quad \dots (1)$$

$$16q - 2s = -4 \quad \dots (2)$$

$$\therefore -22q + 10s = 20 \quad \{2 \times (1)\}$$

$$80q - 10s = -20 \quad \{5 \times (2)\}$$

$$\therefore 58q = 0$$

$$\therefore q = 0 \text{ and so } \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 33 \\ -5 \end{pmatrix}$$

 $\therefore$  M is (33, -5).

(NK) and (MN) meet at N.

$$\therefore \begin{pmatrix} 3 \\ 7 \end{pmatrix} + r \begin{pmatrix} 4 \\ 10 \end{pmatrix} = \begin{pmatrix} 43 \\ -9 \end{pmatrix} + s \begin{pmatrix} -5 \\ 2 \end{pmatrix}$$

$$\therefore \begin{pmatrix} 4r + 5s \\ 10r - 2s \end{pmatrix} = \begin{pmatrix} 40 \\ -16 \end{pmatrix}$$

$$\therefore 4r + 5s = 40 \quad \dots (1)$$

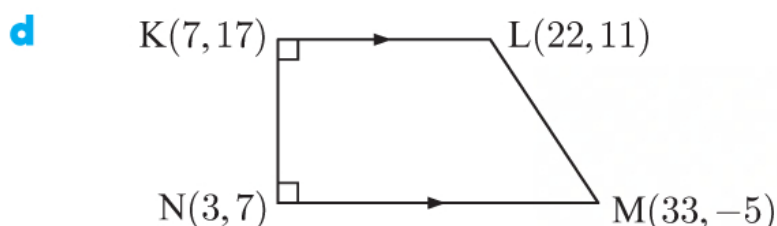
$$10r - 2s = -16 \quad \dots (2)$$

$$\therefore 8r + 10s = 80 \quad \{2 \times (1)\}$$

$$50r - 10s = -80 \quad \{5 \times (2)\}$$

$$\therefore 58r = 0$$

$$\therefore r = 0 \text{ and so } \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$$

 $\therefore$  N is (3, 7).

$$\begin{aligned} NM &= \sqrt{(33-3)^2 + (-5-7)^2} \\ &= \sqrt{900 + 144} \\ &= \sqrt{1044} \text{ units} \end{aligned}$$

$$\begin{aligned} \therefore \text{area} &= \left( \frac{\sqrt{261} + \sqrt{1044}}{2} \right) \times \sqrt{116} \\ &= 261 \text{ units}^2 \end{aligned}$$

$$\begin{aligned} KL &= \sqrt{(22-7)^2 + (11-17)^2} \\ &= \sqrt{225 + 36} \\ &= \sqrt{261} \text{ units} \end{aligned}$$

$$\begin{aligned} KN &= \sqrt{(7-3)^2 + (17-7)^2} \\ &= \sqrt{16 + 100} \\ &= \sqrt{116} \text{ units} \end{aligned}$$

**5**  $L_1$  has direction vector  $\mathbf{b}_1 = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$ ,  $L_2$  has direction vector  $\mathbf{b}_2 = \begin{pmatrix} 5 \\ -12 \end{pmatrix}$

$$\begin{aligned} \text{If } \theta \text{ is the angle between } L_1 \text{ and } L_2, \text{ then } \cos \theta &= \frac{|\mathbf{b}_1 \bullet \mathbf{b}_2|}{|\mathbf{b}_1| |\mathbf{b}_2|} \\ &= \frac{|-20 - 36|}{\sqrt{16+9} \sqrt{25+144}} \\ &= \frac{56}{65} \\ \therefore \theta &= \cos^{-1} \left( \frac{56}{65} \right) \\ &\approx 30.5^\circ \end{aligned}$$

**6 a**

$$\overrightarrow{AB} = \begin{pmatrix} 0-3 \\ 2-(-1) \\ -2-1 \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \\ -3 \end{pmatrix}$$

$$\therefore |\overrightarrow{AB}| = \sqrt{9+9+9} = \sqrt{27} = 3\sqrt{3} \text{ units}$$

$$\mathbf{b} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix} + t \begin{pmatrix} -3 \\ 3 \\ -3 \end{pmatrix}, \quad t \in \mathbb{R}$$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}, \quad \text{where } \lambda = 3t$$

$$\therefore \mathbf{r} = 2\mathbf{j} - 2\mathbf{k} + \lambda(-\mathbf{i} + \mathbf{j} - \mathbf{k}), \quad \text{where } \lambda \in \mathbb{R}$$

A lies on the line  $\mathbf{r}$  when  $\lambda = -3$  and B lies on  $\mathbf{r}$  when  $\lambda = 0$ .

$\therefore$  the line between A and B can be described by  $\mathbf{r}$ .

$$\mathbf{c} \quad \text{The line with equation } t(\mathbf{i} + \mathbf{j} + \mathbf{k}) \text{ has direction vector } \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

$$\begin{aligned} \text{If } \theta \text{ is the angle between } \overrightarrow{\text{AB}} \text{ and the line, then } \cos \theta &= \frac{\left| \overrightarrow{\text{AB}} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right|}{\left| \overrightarrow{\text{AB}} \right| \left| \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right|} \\ &= \frac{|-3 + 3 - 3|}{3\sqrt{3} \times \sqrt{1+1+1}} \\ &= \frac{|-3|}{3\sqrt{3} \times \sqrt{3}} \\ &= \frac{3}{9} \\ &= \frac{1}{3} \\ \therefore \theta &= \cos^{-1}\left(\frac{1}{3}\right) \\ &\approx 70.5^\circ \end{aligned}$$

$$\mathbf{7} \quad \text{The direction vector is } \begin{pmatrix} 3 \\ -1 \end{pmatrix} \text{ which has length } \sqrt{3^2 + (-1)^2} = \sqrt{10} \text{ units}$$

$$\therefore 2\sqrt{10} \begin{pmatrix} 3 \\ -1 \end{pmatrix} \text{ has length 20 units. So, the velocity vector is } \begin{pmatrix} 6\sqrt{10} \\ -2\sqrt{10} \end{pmatrix} \text{ or } 2\sqrt{10}(3\mathbf{i} - \mathbf{j}).$$

$$\mathbf{8} \quad \mathbf{a} \quad x(0) = -4, \quad y(0) = 3, \quad \text{and } z(0) = 1, \quad \text{so the initial position is } (-4, 3, 1).$$

$$\mathbf{b} \quad x(4) = -4 + 8(4) = 28, \quad y(4) = 3 + 6(4) = 27, \quad \text{and } z(4) = 1 - 2(4) = -7, \quad \text{so at } t = 4, \text{ the position is } (28, 27, -7).$$

$$\mathbf{c} \quad \text{The velocity vector is } \begin{pmatrix} 8 \\ 6 \\ -2 \end{pmatrix}.$$

$$\mathbf{d} \quad \text{The speed is } \sqrt{8^2 + 6^2 + (-2)^2} = \sqrt{104} \approx 10.2 \text{ m s}^{-1}.$$



$$9 \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -10 \\ 5 \\ 12 \end{pmatrix} + t \begin{pmatrix} 6 \\ 14 \\ -0.4 \end{pmatrix}$$

$$a \quad \text{When } t = 0, \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -10 \\ 5 \\ 12 \end{pmatrix}$$

$\therefore$  the initial position of the zip-liner is  $(-10, 5, 12)$ .

$$b \quad \text{The velocity vector is } \begin{pmatrix} 6 \\ 14 \\ -0.4 \end{pmatrix}.$$

$$c \quad \text{The speed of the zip-liner is } \left| \begin{pmatrix} 6 \\ 14 \\ -0.4 \end{pmatrix} \right| = \sqrt{6^2 + 14^2 + (-0.4)^2} \\ \approx 15.2 \text{ m s}^{-1}$$

$$d \quad z = 12 - 0.4t$$

$$\text{When } z = 0, \quad 12 - 0.4t = 0$$

$$\therefore 0.4t = 12$$

$$\therefore t = 30$$

$\therefore$  it takes 30 seconds to reach the end of the line.

$$e \quad \text{When } t = 30, \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -10 \\ 5 \\ 12 \end{pmatrix} + 30 \begin{pmatrix} 6 \\ 14 \\ -0.4 \end{pmatrix} \\ = \begin{pmatrix} -10 \\ 5 \\ 12 \end{pmatrix} + \begin{pmatrix} 180 \\ 420 \\ -12 \end{pmatrix} \\ = \begin{pmatrix} 170 \\ 425 \\ 0 \end{pmatrix}$$

$\therefore$  the endpoint of the line is  $(170, 425, 0)$ .

$$f \quad \text{The zip-line has direction vector } \mathbf{b} = \begin{pmatrix} 6 \\ 14 \\ -0.4 \end{pmatrix}.$$

The zip-liner has position  $Z(-10 + 6t, 5 + 14t, 12 - 0.4t)$  at time  $t$ .

The friend is watching from  $F(52, 144, 3)$ .

$$\therefore \overrightarrow{FZ} = \begin{pmatrix} -10 + 6t - 52 \\ 5 + 14t - 144 \\ 12 - 0.4t - 3 \end{pmatrix} = \begin{pmatrix} 6t - 62 \\ 14t - 139 \\ 9 - 0.4t \end{pmatrix}$$

If  $Z$  is the closest point on the line to  $F$ , then  $\overrightarrow{FZ} \perp \mathbf{b}$ .

$$\therefore \overrightarrow{FZ} \bullet \mathbf{b} = 0$$

$$\therefore \begin{pmatrix} 6t - 62 \\ 14t - 139 \\ 9 - 0.4t \end{pmatrix} \bullet \begin{pmatrix} 6 \\ 14 \\ -0.4 \end{pmatrix} = 0$$

$$\therefore 6(6t - 62) + 14(14t - 139) - 0.4(9 - 0.4t) = 0$$

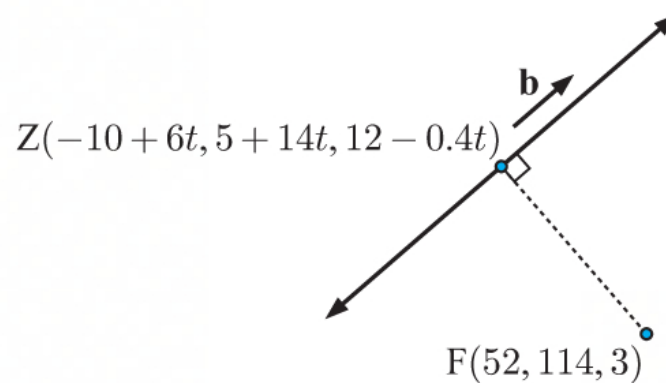
$$\therefore 36t - 372 + 196t - 1946 - 3.6 + 0.16t = 0$$

$$\therefore 232.16t = 2321.6$$

$$\therefore t = 10$$

Substituting  $t = 10$  into the parametric equations, the foot of the perpendicular is  $(50, 145, 8)$ .

$\therefore$  the position of the zip-liner when he is closest to his friend is  $(50, 145, 8)$ .



**10** Boat A:  $x_A(t) = 3 - t$ ,  $y_A(t) = 2t - 4$

Boat B:  $x_B(t) = 4 - 3t$ ,  $y_B(t) = 3 - 2t$

**a**  $x_A(0) = 3$ ,  $y_A(0) = -4$  and  $x_B(0) = 4$ ,  $y_B(0) = 3$

$\therefore$  the initial position of A is  $(3, -4)$  and the initial position of B is  $(4, 3)$ .

**b** The velocity vector of A is  $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$  and the velocity vector of B is  $\begin{pmatrix} -3 \\ -2 \end{pmatrix}$ .

**c** If the angle  $\theta$  is between the paths of the boats, then

$$\cos \theta = \frac{\left| \begin{pmatrix} -1 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} -3 \\ -2 \end{pmatrix} \right|}{\sqrt{1+4}\sqrt{9+4}}$$

$$= \frac{|3 - 4|}{\sqrt{5}\sqrt{13}}$$

$$= \frac{1}{\sqrt{65}}$$

$$\therefore \theta = \cos^{-1}\left(\frac{1}{\sqrt{65}}\right)$$

$$\approx 82.9^\circ$$

**d** If  $D$  is the distance between the boats

$$\begin{aligned} D &= \sqrt{[(4 - 3t) - (3 - t)]^2 + [(3 - 2t) - (2t - 4)]^2} \\ &= \sqrt{[1 - 2t]^2 + [7 - 4t]^2} \\ &= \sqrt{1 - 4t + 4t^2 + 49 - 56t + 16t^2} \\ &= \sqrt{20t^2 - 60t + 50} \end{aligned}$$

Under the square root we have a quadratic in  $t$ , so  $D$  is minimised when

$$t = -\frac{b}{2a} = \frac{60}{40} = 1.5 \text{ hours}$$

$\therefore$  the boats are closest after 1.5 hours.

- 11 a** Line 1 has direction vector  $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$  and line 2 has direction vector  $\begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix}$ .

As one vector is not a scalar multiple of the other, the lines are not parallel.

$$\text{Now, } 2 + t = -8 + 4s \quad \dots (1) \quad -1 + 2t = s \quad \dots (2) \quad 3 - t = 7 - 2s \quad \dots (3)$$

Substituting (2) into (1),  $2 + t = -8 + 4(-1 + 2t)$

$$\therefore 2 + t = -8 - 4 + 8t$$

$$\therefore 7t = 14$$

$$\therefore t = 2$$

$$\therefore s = -1 + 2(2) = 3$$

Checking in (3): LHS =  $3 - 2 = 1$  RHS =  $7 - 2(3) = 1$  ✓

$\therefore s = 3, t = 2$  satisfies all three equations.

$\therefore$  the lines meet at  $(4, 3, 1)$  {substituting  $t = 2$  into line 1}

$$\begin{aligned} \text{If } \theta \text{ is the angle between the lines, then } \cos \theta &= \frac{\left| \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix} \right|}{\sqrt{1+4+1}\sqrt{16+1+4}} \\ &= \frac{|4+2+2|}{\sqrt{6}\sqrt{21}} \\ &= \frac{8}{3\sqrt{14}} \\ \therefore \theta &= \cos^{-1}\left(\frac{8}{3\sqrt{14}}\right) \\ &\approx 44.5^\circ \end{aligned}$$

- b** Line 1 has direction vector  $\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$  and line 2 has direction vector  $\begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$ .

As one vector is not a scalar multiple of the other, the lines are not parallel.

$$\begin{aligned} \text{Now, } 3 + t &= 2 - s, & 5 - 2t &= 1 + 3s, & -1 + 3t &= 4 + s \\ \therefore t + s &= -1 \quad \dots (1) & 2t + 3s &= 4 \quad \dots (2) & 3t - s &= 5 \quad \dots (3) \end{aligned}$$

Solving (1) and (3) simultaneously:  $t + s = -1$

$$3t - s = 5$$

$$\therefore \frac{4t}{4} = 4$$

$$\therefore t = 1 \quad \therefore s = -2$$

Checking in (2):  $2(1) + 3(-2) = -4$  ✗

$\therefore$  the system of equations is inconsistent and so the lines are skew.

If  $\theta$  is the angle between the lines, then

$$\begin{aligned}\cos \theta &= \frac{\left| \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} \right|}{\sqrt{1+4+9}\sqrt{1+9+1}} \\ &= \frac{|-1-6+3|}{\sqrt{14}\sqrt{11}} \\ &= \frac{4}{\sqrt{154}} \\ \therefore \theta &= \cos^{-1}\left(\frac{4}{\sqrt{154}}\right) \\ &\approx 71.2^\circ\end{aligned}$$

**c**  $\frac{x-3}{2} = \frac{y-4}{1} = \frac{z+1}{-2}$  has direction vector  $\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$

while  $x = -1 + 3t$ ,  $y = 2 + 2t$ ,  $z = 3 - t$  has direction vector  $\mathbf{v}_2 = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$

As one vector is not a scalar multiple of the other, the lines are not parallel.

If the lines intersect, then  $\frac{-1+3t-3}{2} = \frac{2+2t-4}{1} = \frac{3-t+1}{-2}$

$$\therefore \frac{3}{2}t - 2 = 2t - 2 = \frac{t}{2} - 2$$

Now  $t = 0$  satisfies this relation, so the lines intersect at  $(-1, 2, 3)$ .

If  $\theta$  is the acute angle between the lines, then

$$\begin{aligned}\cos \theta &= \frac{|\mathbf{v}_1 \cdot \mathbf{v}_2|}{|\mathbf{v}_1||\mathbf{v}_2|} \\ &= \frac{|2 \times 3 + 1 \times 2 + -2 \times -1|}{\sqrt{9}\sqrt{14}} \\ &= \frac{|6+2+2|}{3\sqrt{14}} \\ &= \frac{10}{3\sqrt{14}} \\ \therefore \theta &= \cos^{-1}\left(\frac{10}{3\sqrt{14}}\right) \\ &\approx 27.0^\circ\end{aligned}$$

**12 a**  $A(-1, 2, 3)$ ,  $B(1, 0, -1)$ ,  $C(0, -1, 5)$

$$\overrightarrow{AB} = \begin{pmatrix} 2 \\ -2 \\ -4 \end{pmatrix} \text{ and } \overrightarrow{AC} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$$

$$\begin{aligned}\therefore \text{ a normal to the plane is } \mathbf{n} &= \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -2 & -4 \\ 1 & -3 & 2 \end{vmatrix} \\ &= \begin{vmatrix} -2 & -4 \\ -3 & 2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & -4 \\ 1 & 2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & -2 \\ 1 & -3 \end{vmatrix} \mathbf{k} \\ &= -16\mathbf{i} - 8\mathbf{j} - 4\mathbf{k} \text{ or } -4(4\mathbf{i} + 2\mathbf{j} + \mathbf{k})\end{aligned}$$

$\therefore$  the plane has equation  $4x + 2y + z = 4(1) + 2(0) + 1(-1)$

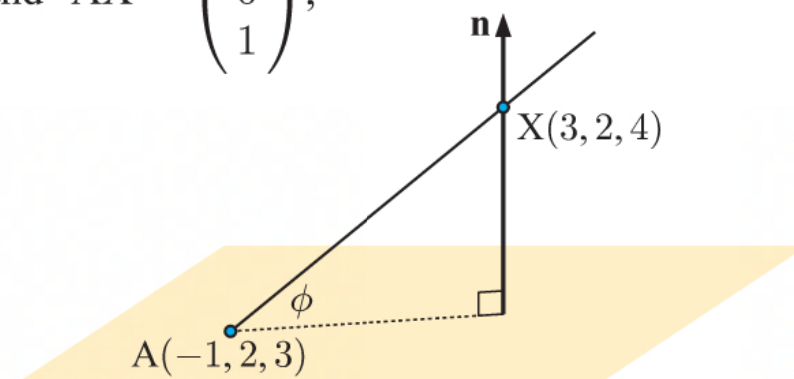
$$\therefore 4x + 2y + z = 3$$



**b** Given the plane has normal  $\mathbf{n} = \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$  and  $\overrightarrow{AX} = \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}$ ,

$$\begin{aligned} \sin \phi &= \frac{|\mathbf{n} \cdot \overrightarrow{AX}|}{|\mathbf{n}| |\overrightarrow{AX}|} \\ &= \frac{|4 \times 4 + 2 \times 0 + 1 \times 1|}{\sqrt{21}\sqrt{17}} \\ &= \frac{17}{\sqrt{21}\sqrt{17}} \end{aligned}$$

$$\therefore \phi = \sin^{-1}\left(\frac{17}{\sqrt{21}\sqrt{17}}\right) \approx 64.1^\circ$$



**13** A vector normal to the plane is

$$\begin{aligned} (3\mathbf{i} + \mathbf{j} - \mathbf{k}) \times (-\mathbf{i} + 2\mathbf{j} + \mathbf{k}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & -1 \\ -1 & 2 & 1 \end{vmatrix} \\ &= \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 3 & -1 \\ -1 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 3 & 1 \\ -1 & 2 \end{vmatrix} \mathbf{k} \\ &= 3\mathbf{i} - 2\mathbf{j} + 7\mathbf{k} \end{aligned}$$

From the vector equation of the plane, the point  $(-5, -1, -5)$  lies in the plane.

$$\begin{aligned} \therefore \text{the Cartesian equation of the plane is } 3x - 2y + 7z &= 3(-5) - 2(-1) + 7(-5) \\ \therefore 3x - 2y + 7z &= -48 \quad \dots (*) \end{aligned}$$

Parametric equations for the normal through A are

$$x = 1 + 3\lambda, \quad y = 5 - 2\lambda, \quad z = 3 + 7\lambda, \quad \lambda \in \mathbb{R}$$

so N has coordinates  $(1 + 3\lambda, 5 - 2\lambda, 3 + 7\lambda)$  for some  $\lambda$ .

Substituting the coordinates into (\*) gives:

$$\begin{aligned} 3(1 + 3\lambda) - 2(5 - 2\lambda) + 7(3 + 7\lambda) &= -48 \\ \therefore 3 + 9\lambda - 10 + 4\lambda + 21 + 49\lambda &= -48 \\ \therefore 62\lambda &= -62 \\ \therefore \lambda &= -1 \end{aligned}$$

$\therefore$  N has coordinates  $(-2, 7, -4)$ .

**14** From **Exercise 13G** question **21 b**, the distance of  $X(-1, 1, 3)$  from  $x - 2y - 2z = 8$

$$\text{is } d = \frac{|x_1 - 2y_1 - 2z_1 - 8|}{\sqrt{1^2 + (-2)^2 + (-2)^2}} = \frac{|-1 - 2(1) - 2(3) - 8|}{3} = \frac{|-17|}{3} = \frac{17}{3} \text{ units}$$

**15 a**  $L_1$  meets  $2x + y - z = 2$  when  $2(3t - 4) + (t + 2) - (2t - 1) = 2$

$$\therefore 6t - 8 + t + 2 - 2t + 1 = 2$$

$$\therefore 5t = 7$$

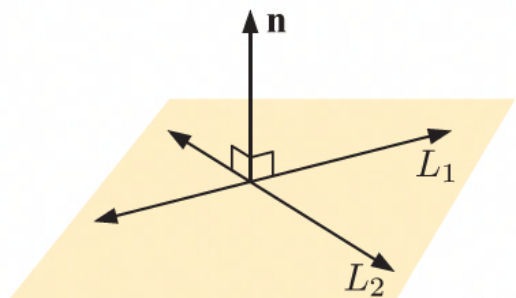
$$\therefore t = \frac{7}{5}$$

$\therefore$  the lines meet at  $(3(\frac{7}{5}) - 4, \frac{7}{5} + 2, 2(\frac{7}{5}) - 1)$

which is  $(\frac{1}{5}, \frac{17}{5}, \frac{9}{5})$

**b**  $L_1$  meets  $L_2$  when  $3t - 4 = \frac{t + 2 - 5}{2} = \frac{-(2t - 1) - 1}{2}$   
 $\therefore 6t - 8 = t - 3 = -2t$   
 $\therefore 5t = 5$  and  $3t = 3$   
 $\therefore t = 1$

So,  $L_1$  and  $L_2$  meet at  $(-1, 3, 1)$ .

**c**

$$\begin{aligned} \mathbf{n} &= \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & 2 \\ 1 & 2 & -2 \end{vmatrix} \\ &= \begin{vmatrix} 1 & 2 \\ 2 & -2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 3 & 2 \\ 1 & -2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} \mathbf{k} \\ &= -6\mathbf{i} + 8\mathbf{j} + 5\mathbf{k} \end{aligned}$$

$\therefore$  using the point of intersection  $(-1, 3, 1)$ , the plane has equation

$$\begin{aligned} -6x + 8y + 5z &= -6(-1) + 8(3) + 5(1) \\ \therefore -6x + 8y + 5z &= 35 \\ \therefore 6x - 8y - 5z &= -35 \end{aligned}$$

**16**  $x - 1 = \frac{y + 2}{2} = \frac{z - 3}{4}$  has direction vector  $\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$  and  $6x + 7y - 5z = 8$  has  $\mathbf{n} = \begin{pmatrix} 6 \\ 7 \\ -5 \end{pmatrix}$ .

Now  $\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 6 \\ 7 \\ -5 \end{pmatrix} = 6 + 14 - 20 = 0$

$\therefore$  since these two vectors are perpendicular, the line is parallel to the plane.

Choose any point on the line, for example,  $(1, -2, 3)$ .

From **Exercise 13G** question **21 b**, the distance from the line to the plane is

$$\begin{aligned} d &= \frac{|6x_1 + 7y_1 - 5z_1 - 8|}{\sqrt{6^2 + 7^2 + (-5)^2}} \\ &= \frac{|6(1) + 7(-2) - 5(3) - 8|}{\sqrt{110}} \\ &= \frac{31}{\sqrt{110}} \text{ units} \end{aligned}$$

**17**  $P(2, 0, 1)$ ,  $Q(3, 4, -2)$ ,  $R(-1, 3, 2)$

**a**  $\overrightarrow{PQ} = \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}$

$$|\overrightarrow{PQ}| = \sqrt{1 + 16 + 9} = \sqrt{26} \text{ units}$$

and  $\overrightarrow{QR} = \begin{pmatrix} -4 \\ -1 \\ 4 \end{pmatrix}$

**b** Since  $\overrightarrow{PQ} = \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}$  and P is at  $(2, 0, 1)$ ,

(PQ) has parametric equations  $x = 2 + \lambda$ ,  $y = 0 + 4\lambda$ ,  $z = 1 - 3\lambda$ ,  $\lambda \in \mathbb{R}$ .

$$\therefore x = 2 + \lambda, \quad y = 4\lambda, \quad z = 1 - 3\lambda, \quad \lambda \in \mathbb{R}$$

**c** A vector equation of the plane is  $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} -4 \\ -1 \\ 4 \end{pmatrix}, \quad \lambda, \mu \in \mathbb{R}.$

**18 a**  $L$  has parametric equations  $x = 3 + 6t$ ,  $y = -1 + t$ ,  $z = 4 - t$ ,  $t \in \mathbb{R}$ .  
Substituting these into the equation of  $\pi_1$  gives:

$$\begin{aligned} (3 + 6t) - 2(-1 + t) + 4(4 - t) &= 3 + 6t + 2 - 2t + 16 - 4t \\ &= 21 \quad \text{for all } t \in \mathbb{R} \end{aligned}$$

$\therefore \pi_1$  contains  $L$ .

**b** The line  $L$  intersects the plane  $5x - y + 3z = 2$   
when  $5(3 + 6t) - (-1 + t) + 3(4 - t) = 2$   
 $\therefore 15 + 30t + 1 - t + 12 - 3t = 2$   
 $\therefore 26t = -26$   
 $\therefore t = -1$

Substituting  $t = -1$  into the equations for  $L$ ,  $x = 3 + 6(-1) = -3$   
 $y = -1 + (-1) = -2$   
 $z = 4 - (-1) = 5$

$\therefore L$  intersects  $\pi_2$  at the point  $(-3, -2, 5)$ .

**c i** The line  $L$  has direction vector  $\mathbf{l} = \begin{pmatrix} 6 \\ 1 \\ -1 \end{pmatrix}$ .

The plane  $\pi_2$  has normal  $\mathbf{n}_2 = \begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix}$ .

The angle between  $L$  and  $\pi_2$  is  $\phi = \sin^{-1} \left( \frac{|\mathbf{l} \bullet \mathbf{n}_2|}{|\mathbf{l}| |\mathbf{n}_2|} \right)$   
 $= \sin^{-1} \left( \frac{|30 - 1 - 3|}{\sqrt{36 + 1 + 1} \sqrt{25 + 1 + 9}} \right)$   
 $= \sin^{-1} \left( \frac{26}{\sqrt{38} \sqrt{35}} \right)$   
 $\approx 45.47^\circ$

ii The plane  $\pi_1$  has normal  $\mathbf{n}_1 = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$ .

The plane  $\pi_2$  has normal  $\mathbf{n}_2 = \begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix}$ .

$$\begin{aligned} \text{The angle between } \pi_1 \text{ and } \pi_2 \text{ is } \theta &= \cos^{-1} \left( \frac{|\mathbf{n}_1 \bullet \mathbf{n}_2|}{|\mathbf{n}_1| |\mathbf{n}_2|} \right) \\ &= \cos^{-1} \left( \frac{|5 + 2 + 12|}{\sqrt{1 + 4 + 16} \sqrt{25 + 1 + 9}} \right) \\ &= \cos^{-1} \left( \frac{19}{\sqrt{21} \sqrt{35}} \right) \\ &\approx 45.51^\circ \end{aligned}$$

19 a Line 1 can be written as  $x = 2 - y = z + 2$

The lines meet when  $1 + 3\lambda = 2 - (-2 - 2\lambda) = (6 + 2\lambda) + 2$  for some  $\lambda \in \mathbb{R}$

$$\therefore 1 + 3\lambda = 4 + 2\lambda = 8 + 2\lambda$$

$$\therefore 1 + 3\lambda = 4 + 2\lambda \quad \text{and} \quad 4 + 2\lambda = 8 + 2\lambda$$

$$\therefore \lambda = 3 \quad \text{and} \quad 4 = 8 \quad \times$$

$\therefore$  no value of  $\lambda$  satisfies both equations.

$\therefore$  the lines do not meet.

The lines have direction vectors  $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$  and  $\begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix}$   $\therefore$  they are not parallel.

$\therefore$  lines 1 and 2 are skew.

b The lines have direction vectors  $\mathbf{v} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$  and  $\mathbf{w} = \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix}$

$$\begin{aligned} \text{so } \mathbf{v} \times \mathbf{w} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ 3 & -2 & 2 \end{vmatrix} \\ &= \begin{vmatrix} -1 & 1 \\ -2 & 2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & -1 \\ 3 & -2 \end{vmatrix} \mathbf{k} \\ &= 0\mathbf{i} + \mathbf{j} + \mathbf{k} \\ &= \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \end{aligned}$$

Let A and B be points on the skew lines such that  $|\overrightarrow{AB}|$  is the shortest distance between them.

$\therefore$  A is  $(s, 2 - s, -2 + s)$  and B is  $(1 + 3t, -2 - 2t, 6 + 2t)$  for some  $s, t \in \mathbb{R}$ .



Now  $\overrightarrow{AB} \parallel \mathbf{v} \times \mathbf{w}$ , so  $\begin{pmatrix} 1+3t-s \\ -2-2t-2+s \\ 6+2t+2-s \end{pmatrix} = k \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$  for some  $k \in \mathbb{R}$

$$\therefore 1-s+3t=0 \quad \dots (1)$$

$$-4+s-2t=k \quad \dots (2)$$

$$8-s+2t=k \quad \dots (3)$$

Adding (2) and (3) gives  $4=2k$

$$\therefore k=2$$

$$\therefore \overrightarrow{AB} = 2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$$

and the shortest distance between the lines  $|\overrightarrow{AB}| = \sqrt{2^2 + 2^2} = 2\sqrt{2}$  units.

**c** Let the common perpendicular meet lines 1 and 2 at A and B respectively.

Subtracting (3) from (1) in **b** gives  $-7+t=-2$

$$\therefore t=5$$

Substituting  $t=5$  into (1) gives  $1-s+3(5)=0$

$$\therefore s=16$$

$\therefore$  A is  $(16, -14, 14)$  and B is  $(16, -12, 16)$ .

$\therefore$  the common perpendicular meets lines 1 and 2 at  $A(16, -14, 14)$  and  $B(16, -12, 16)$ .

**20**  $\mathbf{n} = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$  and  $\mathbf{d} = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$

The angle between the plane and the line is  $\phi = \sin^{-1} \left( \frac{|\mathbf{n} \bullet \mathbf{d}|}{|\mathbf{n}| |\mathbf{d}|} \right)$

$$= \sin^{-1} \left( \frac{|2-4+1|}{\sqrt{4+4+1}\sqrt{1+4+1}} \right)$$

$$= \sin^{-1} \left( \frac{1}{\sqrt{54}} \right)$$

$$\approx 7.82^\circ$$

**21 a** In augmented matrix form, the system is:

$$\begin{pmatrix} 1 & -2 & 3 & | & 1 \\ 1 & p & 2 & | & 0 \\ -2 & p^2 & -4 & | & q \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 3 & | & 1 \\ 0 & p+2 & -1 & | & -1 \\ 0 & p^2-4 & 2 & | & q+2 \end{pmatrix} \begin{matrix} R_2 - R_1 \rightarrow R_2 \\ R_3 + 2R_1 \rightarrow R_3 \end{matrix}$$

$$\sim \begin{pmatrix} 1 & -2 & 3 & | & 1 \\ 0 & p+2 & -1 & | & -1 \\ 0 & 0 & p & | & p+q \end{pmatrix} \begin{matrix} \\ \\ R_3 - 3R_2 \rightarrow R_3 \end{matrix}$$

$$\begin{pmatrix} 1 & p & 2 & | & 0 \\ -1 & 2 & -3 & | & -1 \\ 0 & p+2 & -1 & | & -1 \end{pmatrix} \begin{matrix} \\ \\ \\ \end{matrix}$$

$$\begin{pmatrix} -2 & p^2 & -4 & | & q \\ 2 & -4 & 6 & | & 2 \\ 0 & p^2-4 & 2 & | & q+2 \end{pmatrix} \begin{matrix} \\ \\ \\ \end{matrix}$$

$$\begin{pmatrix} 0 & p^2-4 & 2 & | & q+2 \\ 0 & -(p^2-4) & p-2 & | & p-2 \\ 0 & 0 & p & | & p+q \end{pmatrix} \begin{matrix} \\ \\ \\ \end{matrix}$$

- b**
- i** If  $p \neq 0$  and  $p \neq -2$ , then there is a unique solution. The three planes meet at a point.
  - ii** If  $p = 0$  or  $-2$  and  $q \neq 0$ , then there are no solutions. Planes 2 and 3 are parallel but not coincident, and each intersect with plane 1 along a different line.
  - iii** If  $p = 0$  or  $-2$  and  $q = 0$ , then there are infinitely many solutions. Planes 2 and 3 are coincident, and intersect with plane 1 along a line.

- c** If  $p = -2$  and  $q = 0$ , then row 2 is  $0x + 0y - z = -1$   
 $\therefore z = 1$

Letting  $y = t$  in row 1,  $x - 2t + 3 = 1$   
 $\therefore x = 2t - 2$

$\therefore x = 2t - 2, y = t, z = 1, t \in \mathbb{R}$

If  $p = 0$  and  $q = 0$ , then row 3 is  $0x + 0y + 0z = 0$

Letting  $z = t$  in row 2,  $2y - t = -1$   
 $\therefore 2y = t - 1$   
 $\therefore y = \frac{t-1}{2}$

In row 1,  $x - 2\left(\frac{t-1}{2}\right) + 3t = 1$   
 $\therefore x - (t-1) + 3t = 1$   
 $\therefore x - t + 1 + 3t = 1$   
 $\therefore x = -2t$

$\therefore x = -2t, y = \frac{t-1}{2}, z = t, t \in \mathbb{R}$

- 22 a** Planes  $A$  and  $B$  have augmented matrix

$$\begin{pmatrix} 1 & 3 & 2 & | & 5 \\ 2 & 1 & 9 & | & 20 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 2 & | & 5 \\ 0 & -5 & 5 & | & 10 \end{pmatrix} \xrightarrow{R_2 - 2R_1 \rightarrow R_2} \begin{pmatrix} 1 & 3 & 2 & | & 5 \\ 0 & -5 & 5 & | & 10 \end{pmatrix} \xrightarrow{\frac{1}{5}R_2 \rightarrow R_2} \begin{pmatrix} 1 & 3 & 2 & | & 5 \\ 0 & -1 & 1 & | & 2 \end{pmatrix}$$

$\therefore -y + z = 2$  and  $x + 3y + 2z = 5$

$\therefore$  if we let  $y = s$ , then  $z = 2 + s$  and  $x = 5 - 3y - 2z$   
 $= 5 - 3s - 2(2 + s)$   
 $= 5 - 3s - 4 - 2s$   
 $\therefore x = 1 - 5s$

$\therefore$  planes  $A$  and  $B$  intersect in the line  $L_1$ :  $x = 1 - 5s, y = s, z = 2 + s, s \in \mathbb{R}$ .

**b** Planes  $C$  and  $B$  have augmented matrix

$$\begin{pmatrix} 1 & -1 & 6 & 8 \\ 2 & 1 & 9 & 20 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 6 & 8 \\ 0 & 3 & -3 & 4 \end{pmatrix} \begin{matrix} R_2 - 2R_1 \rightarrow R_2 \end{matrix} \leftarrow \left\{ \begin{array}{cccc} 2 & 1 & 9 & 20 \\ -2 & 2 & -12 & -16 \\ \hline 0 & 3 & -3 & 4 \end{array} \right\}$$

$$\therefore 3y - 3z = 4 \quad \text{and} \quad x - y + 6z = 8$$

$$\therefore \text{if we let } y = t, \text{ then } z = -\frac{4}{3} + y = -\frac{4}{3} + t \quad \text{and} \quad x = 8 + y - 6z$$

$$= 8 + t - 6\left(-\frac{4}{3} + t\right)$$

$$= 8 + t + 8 - 6t$$

$$\therefore x = 16 - 5t$$

$\therefore$  planes  $B$  and  $C$  intersect in the line  $L_2$ :  $x = 16 - 5t$ ,  $y = t$ ,  $z = -\frac{4}{3} + t$ ,  $t \in \mathbb{R}$ .

**c** Planes  $A$  and  $C$  have augmented matrix

$$\begin{pmatrix} 1 & 3 & 2 & 5 \\ 1 & -1 & 6 & 8 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 2 & 5 \\ 0 & -4 & 4 & 3 \end{pmatrix} \begin{matrix} R_2 - R_1 \rightarrow R_2 \end{matrix} \leftarrow \left\{ \begin{array}{cccc} 1 & -1 & 6 & 8 \\ -1 & -3 & -2 & -5 \\ \hline 0 & -4 & 4 & 3 \end{array} \right\}$$

$$\therefore -4y + 4z = 3 \quad \text{and} \quad x + 3y + 2z = 5$$

$$\therefore \text{if we let } y = u, \text{ then } z = \frac{3}{4} + y = \frac{3}{4} + u \quad \text{and} \quad x = 5 - 3y - 2z$$

$$= 5 - 3u - 2\left(\frac{3}{4} + u\right)$$

$$= 5 - 3u - \frac{3}{2} - 2u$$

$$\therefore x = \frac{7}{2} - 5u$$

$\therefore$  planes  $A$  and  $C$  intersect in the line  $L_3$ :  $x = \frac{7}{2} - 5u$ ,  $y = u$ ,  $z = \frac{3}{4} + u$ ,  $u \in \mathbb{R}$ .

**d**  $L_1$  is  $\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + s \begin{pmatrix} -5 \\ 1 \\ 1 \end{pmatrix}$  with direction vector  $\mathbf{a} = \begin{pmatrix} -5 \\ 1 \\ 1 \end{pmatrix}$ .

$L_2$  is  $\begin{pmatrix} 16 \\ 0 \\ -\frac{4}{3} \end{pmatrix} + t \begin{pmatrix} -5 \\ 1 \\ 1 \end{pmatrix}$  with direction vector  $\mathbf{b} = \begin{pmatrix} -5 \\ 1 \\ 1 \end{pmatrix}$ .

$L_3$  is  $\begin{pmatrix} \frac{7}{2} \\ 0 \\ \frac{3}{4} \end{pmatrix} + u \begin{pmatrix} -5 \\ 1 \\ 1 \end{pmatrix}$  with direction vector  $\mathbf{c} = \begin{pmatrix} -5 \\ 1 \\ 1 \end{pmatrix}$ .

Since  $\mathbf{a} = \mathbf{b} = \mathbf{c}$ ,  $L_1$ ,  $L_2$ , and  $L_3$  are parallel.

When  $s = 0$ , the point on  $L_1$  is  $(1, 0, 2)$ .

For  $L_2$ ,  $y = t$ , so the unique point on  $L_2$  with  $y$ -coordinate 0 is the point where  $t = 0$ .

This point is  $(16, 0, -\frac{4}{3})$ .

For  $L_3$ ,  $y = u$ , so the unique point on  $L_3$  with  $y$ -coordinate 0 is the point where  $u = 0$ .

This point is  $(\frac{7}{2}, 0, \frac{3}{4})$ .

$$(1, 0, 2) \neq (16, 0, -\frac{4}{3}), \quad (16, 0, -\frac{4}{3}) \neq (\frac{7}{2}, 0, \frac{3}{4}), \quad \text{and} \quad (1, 0, 2) \neq (\frac{7}{2}, 0, \frac{3}{4}).$$

$\therefore$  none of the lines are coincident.

- e** The three planes have no common point of intersection. The line of intersection of any two planes is parallel to the third plane.

**23 a** In augmented matrix form, the system is:

$$\begin{pmatrix} 1 & -3 & 2 & | & -5 \\ 3 & 1 & (2-k) & | & 10 \\ -2 & 6 & k & | & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & -3 & 2 & | & -5 \\ 0 & 10 & -4-k & | & 25 \\ 0 & 0 & k+4 & | & -5 \end{pmatrix} \quad \begin{matrix} R_2 - 3R_1 \rightarrow R_2 \\ R_3 + 2R_1 \rightarrow R_3 \end{matrix}$$

$$\begin{matrix} \left\{ \begin{array}{cccc} 3 & 1 & 2-k & 10 \\ -3 & 9 & -6 & 15 \\ \hline 0 & 10 & -4-k & 25 \end{array} \right\} \\ \left\{ \begin{array}{cccc} -2 & 6 & k & 5 \\ 2 & -6 & 4 & -10 \\ \hline 0 & 0 & k+4 & -5 \end{array} \right\} \end{matrix}$$

- b** If  $k = -4$ , row 3 becomes  $0x + 0y + 0z = -5$   
 $\therefore$  the system is inconsistent and there are no solutions.

When  $k = -4$  the system becomes

$$\begin{cases} x - 3y + 2z = -5 \\ 3x + y + 6z = 10 \\ -2x + 6y - 4z = 5 \end{cases} \quad \begin{matrix} \text{Planes 1 and 3 are parallel but not coincident.} \\ \text{They are each intersected by plane 2.} \end{matrix}$$

- c i** There is a unique solution when  $k \neq -4$ .

- ii** If  $k \neq -4$ , then row 3 becomes  $(k+4)z = -5$

$$\therefore z = \frac{-5}{k+4}$$

and substituting into row 2 gives  $10y - (4+k) \left( \frac{-5}{k+4} \right) = 25$

$$\therefore 10y + 5 = 25$$

$$\therefore 10y = 20$$

$$\therefore y = 2$$

and substituting into row 1 gives  $x - 3(2) + 2 \left( \frac{-5}{k+4} \right) = -5$

$$\therefore x = 1 + \frac{10}{k+4}$$

$\therefore$  the unique solution is  $x = 1 + \frac{10}{k+4}$ ,  $y = 2$ ,  $z = \frac{-5}{k+4}$ .

In this case we have three planes which meet at a point.

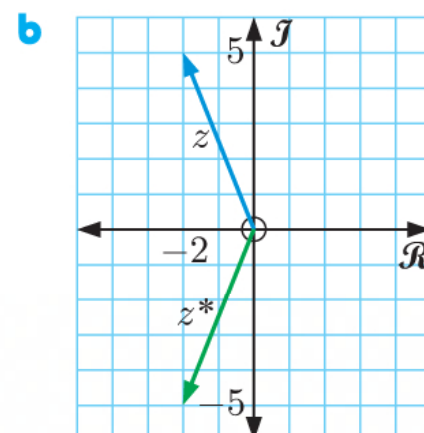
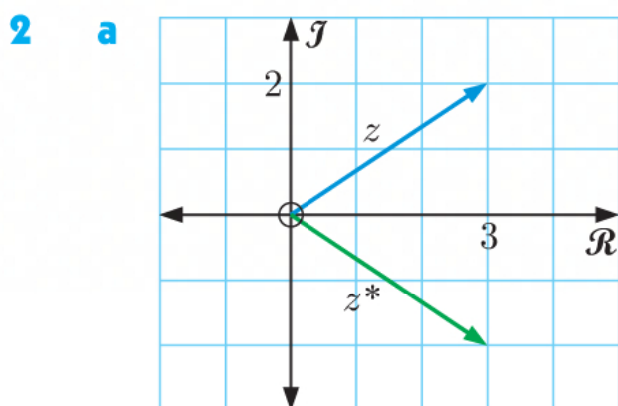
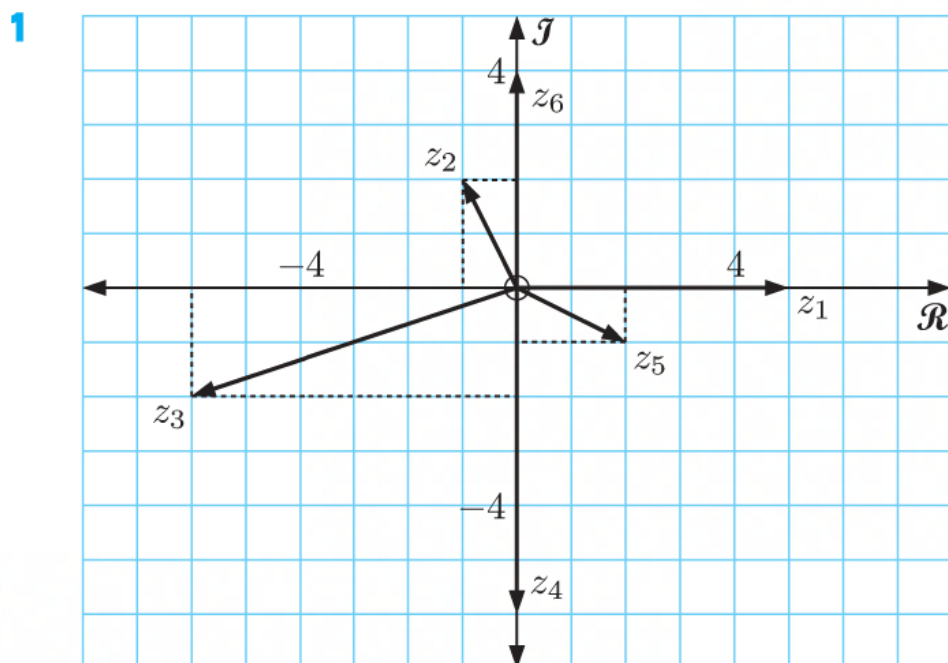
- iii** When  $k = 1$ ,  $x = 1 + \frac{10}{1+4} = 3$ ,  $y = 2$ , and  $z = \frac{-5}{1+4} = -1$



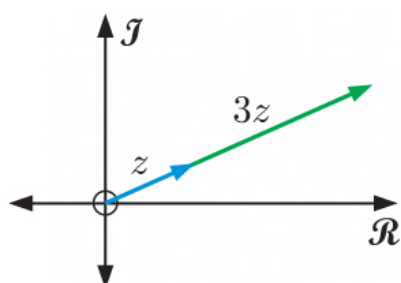
# Chapter 14

## COMPLEX NUMBERS

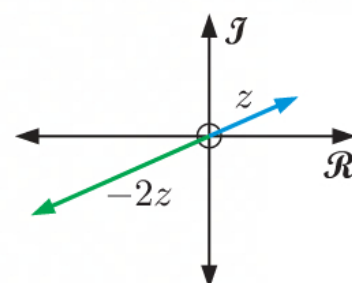
### EXERCISE 14A



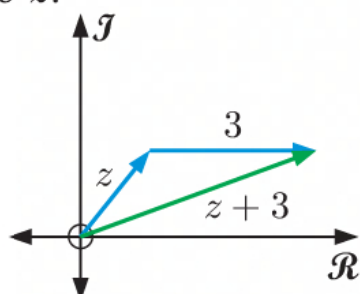
3 a  $3z$  is parallel to  $z$  and 3 times its length.



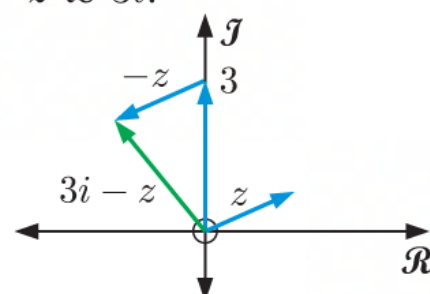
b  $-2z$  is parallel to  $z$ , in the opposite direction, and twice its length.



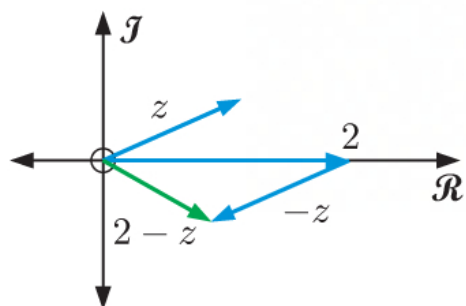
c Add 3 to  $z$ .



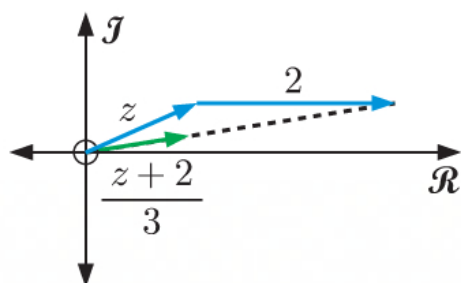
d Add  $-z$  to  $3i$ .



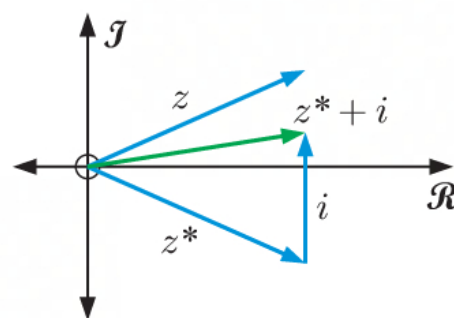
- e Add  $-z$  to 2.



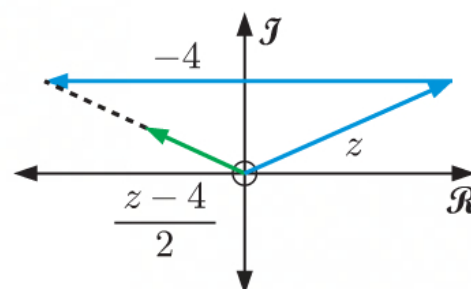
- g Add  $z$  and 2 and find the vector  $\frac{1}{3}$  of the length of the result.



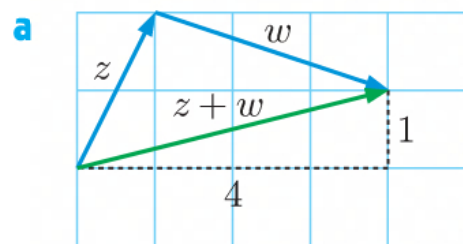
- f Reflect  $z$  in a horizontal line through the start of  $z$  and then add  $i$ .



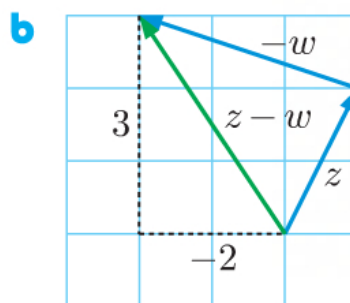
- h Add  $z$  and  $-4$  and find the vector  $\frac{1}{2}$  of the length of the result.



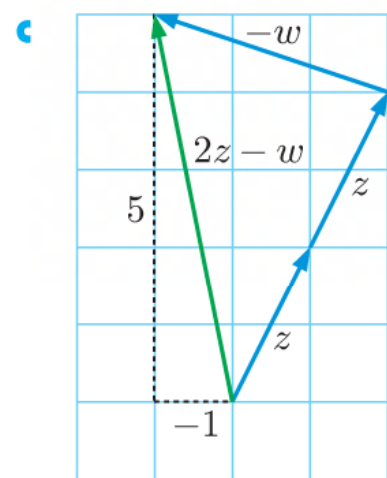
4  $z = 1 + 2i$ ,  $w = 3 - i$



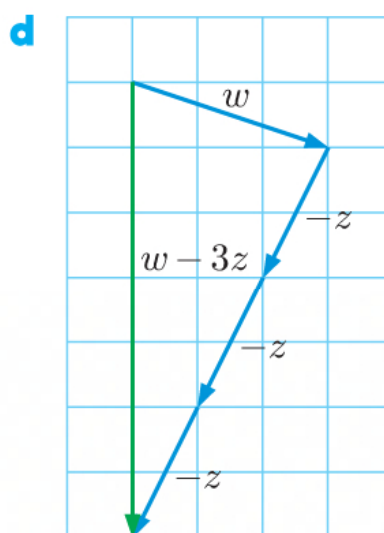
$$\begin{aligned} z + w &= (1 + 2i) + (3 - i) \\ &= 4 + i \end{aligned}$$



$$\begin{aligned} z - w &= (1 + 2i) - (3 - i) \\ &= 1 + 2i - 3 + i \\ &= -2 + 3i \end{aligned}$$

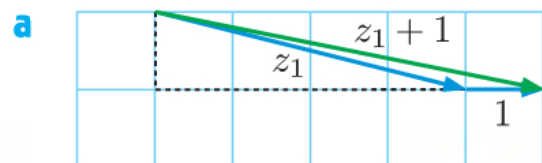


$$\begin{aligned} 2z - w &= 2(1 + 2i) - (3 - i) \\ &= 2 + 4i - 3 + i \\ &= -1 + 5i \end{aligned}$$

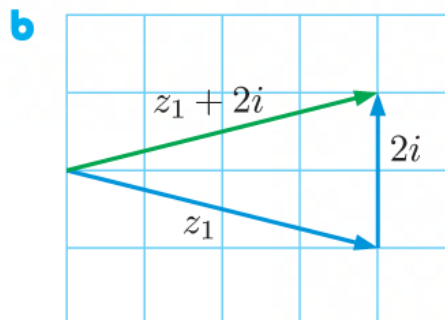


$$\begin{aligned} w - 3z &= (3 - i) - 3(1 + 2i) \\ &= 3 - i - 3 - 6i \\ &= -7i \end{aligned}$$

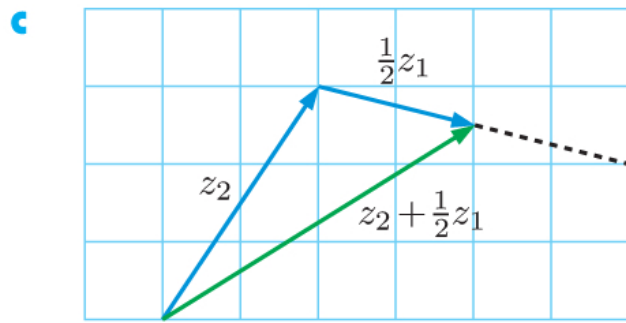
5  $z_1 = 4 - i$ ,  $z_2 = 2 + 3i$



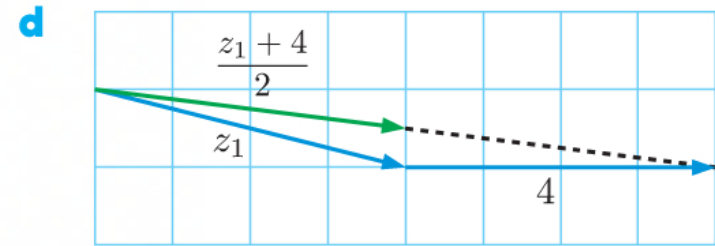
$$\begin{aligned} z_1 + 1 &= 4 - i + 1 \\ &= 5 - i \end{aligned}$$



$$\begin{aligned} z_1 + 2i &= 4 - i + 2i \\ &= 4 + i \end{aligned}$$

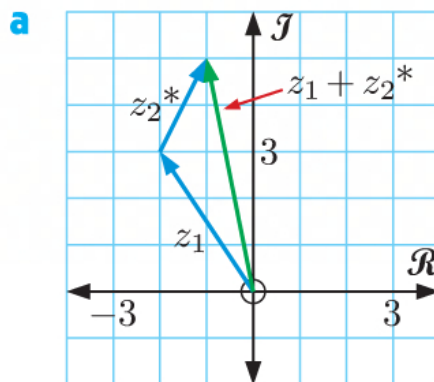


$$\begin{aligned} z_2 + \frac{1}{2}z_1 &= (2 + 3i) + \frac{1}{2}(4 - i) \\ &= 2 + 3i + 2 - \frac{1}{2}i \\ &= 4 + \frac{5}{2}i \end{aligned}$$

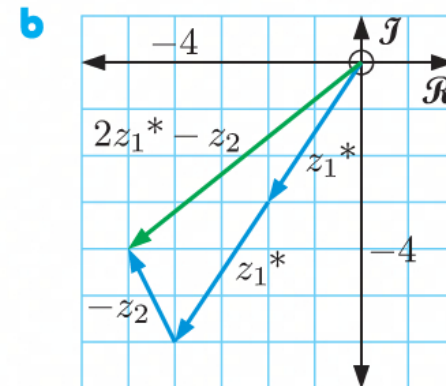


$$\begin{aligned} \frac{z_1 + 4}{2} &= \frac{4 - i + 4}{2} \\ &= \frac{8 - i}{2} \\ &= 4 - \frac{1}{2}i \end{aligned}$$

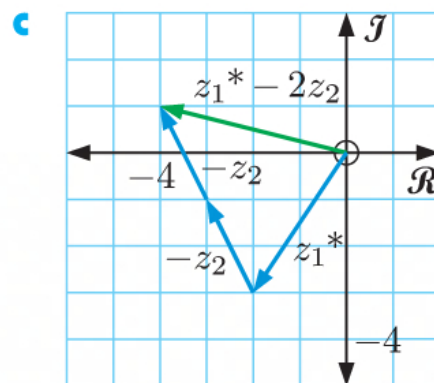
**6**  $z_1 = -2 + 3i, \quad z_2 = 1 - 2i$



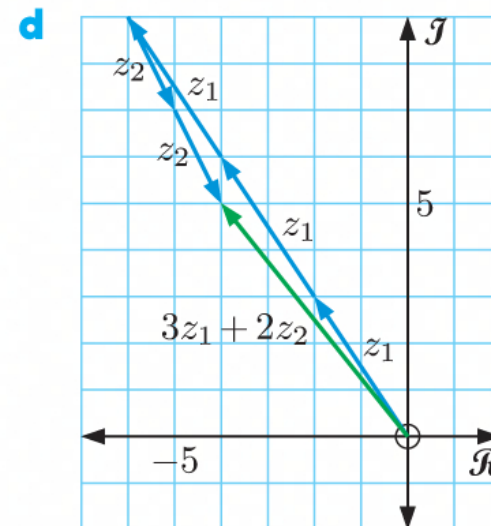
$$\begin{aligned} z_1 + z_2^* &= (-2 + 3i) + (1 + 2i) \\ &= -1 + 5i \end{aligned}$$



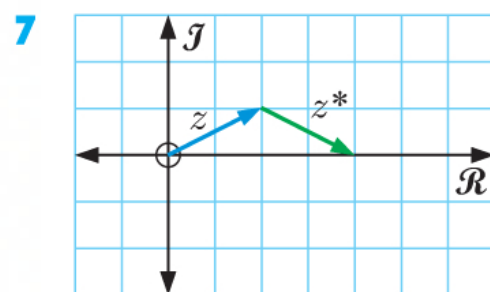
$$\begin{aligned} 2z_1^* - z_2 &= 2(-2 - 3i) - (1 - 2i) \\ &= -4 - 6i - 1 + 2i \\ &= -5 - 4i \end{aligned}$$



$$\begin{aligned} z_1^* - 2z_2 &= (-2 - 3i) - 2(1 - 2i) \\ &= -2 - 3i - 2 + 4i \\ &= -4 + i \end{aligned}$$



$$\begin{aligned} 3z_1 + 2z_2 &= 3(-2 + 3i) + 2(1 - 2i) \\ &= -6 + 9i + 2 - 4i \\ &= -4 + 5i \end{aligned}$$



Suppose  $z = a + bi$ , where  $a, b \in \mathbb{R}$

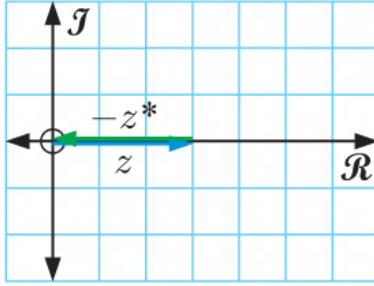
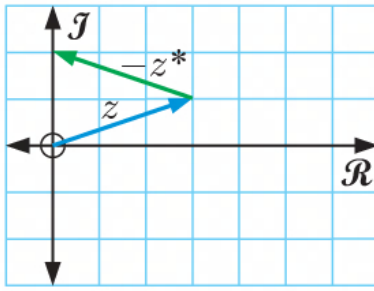
$$\therefore z^* = a - bi$$

and  $z + z^* = a + bi + a - bi$

$$= 2a, \text{ which is real (since } a \text{ is real)}$$

$\therefore z + z^*$  is always real for any complex number  $z$ .

8



Suppose  $z = a + bi$ , where  $a, b \in \mathbb{R}$

$$\therefore z^* = a - bi$$

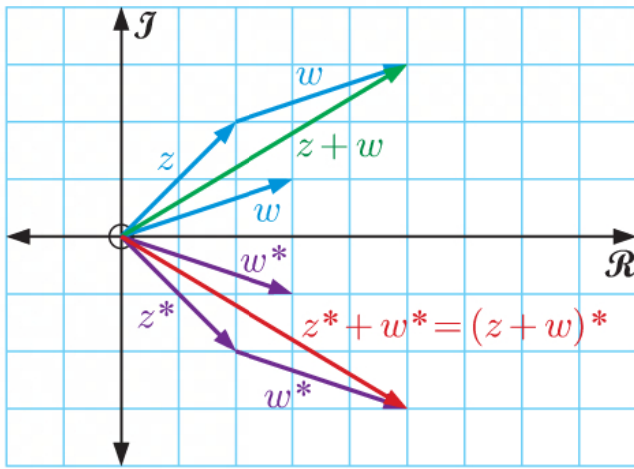
$$\begin{aligned} \text{and } z - z^* &= (a + bi) - (a - bi) \\ &= a + bi - a + bi \\ &= 2bi \end{aligned}$$

Since  $b$  is real,  $z - z^*$  is purely imaginary for  $b \neq 0$ .

If  $b = 0$  then  $z - z^* = 0$ .

$\therefore z - z^*$  is purely imaginary, unless  $z$  is real,  
then  $z - z^* = 0$ .

9 a

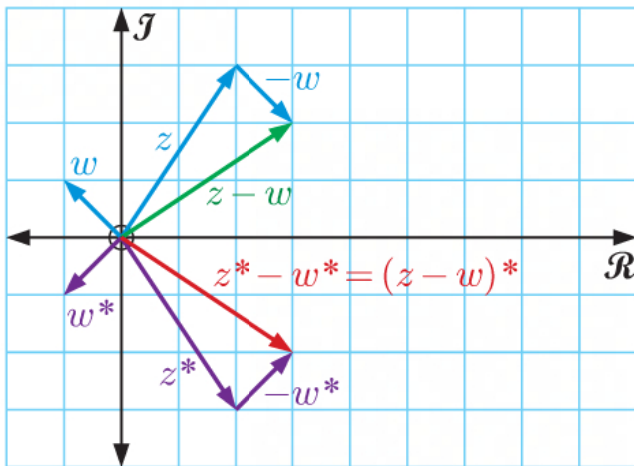


Let  $z = a + bi$  and  $w = c + di$

$$\begin{aligned} \therefore (z + w)^* &= (a + bi + c + di)^* \\ &= ((a + c) + (b + d)i)^* \\ &= (a + c) - (b + d)i \\ &= a + c - bi - di \\ &= (a - bi) + (c - di) \\ &= z^* + w^* \end{aligned}$$

$\therefore (z + w)^* = z^* + w^*$  for all complex  $z, w$

b

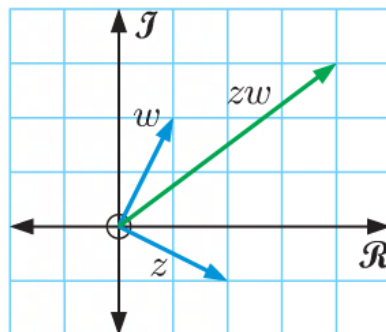


Let  $z = a + bi$  and  $w = c + di$

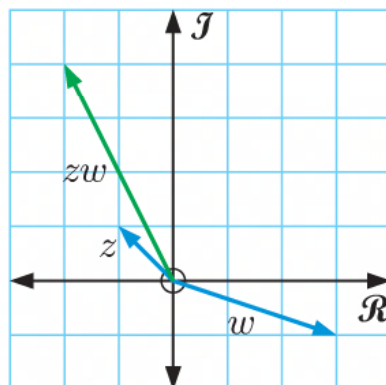
$$\begin{aligned} \therefore (z - w)^* &= ((a + bi) - (c + di))^* \\ &= (a + bi - c - di)^* \\ &= ((a - c) + (b - d)i)^* \\ &= (a - c) - (b - d)i \\ &= a - c - bi + di \\ &= (a - bi) - (c - di) \\ &= z^* - w^* \end{aligned}$$

$\therefore (z - w)^* = z^* - w^*$  for all complex  $z, w$

10 a  $zw = (2 - i)(1 + 2i)$   
 $= 2 + 4i - i + 2$   
 $= 4 + 3i$

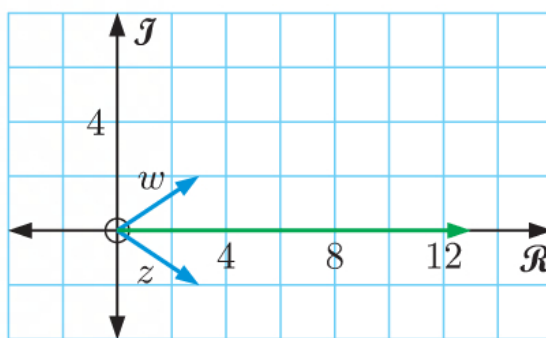


b  $zw = (-1 + i)(3 - i)$   
 $= -3 + i + 3i + 1$   
 $= -2 + 4i$

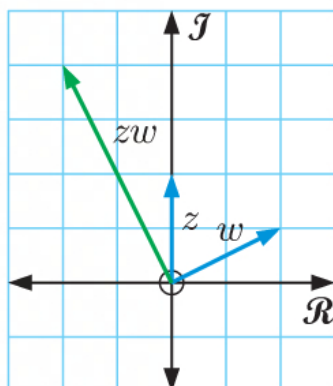




$$\begin{aligned}
 \text{c } zw &= (3 - 2i)(3 + 2i) \\
 &= 9 + 6i - 6i + 4 \\
 &= 13
 \end{aligned}$$



$$\begin{aligned}
 \text{d } zw &= (2i)(2 + i) \\
 &= 4i - 2 \\
 &= -2 + 4i
 \end{aligned}$$



## EXERCISE 14B

$$\begin{aligned}
 \text{1 a } |3 - 4i| &= \sqrt{3^2 + (-4)^2} \\
 &= \sqrt{9 + 16} \\
 &= \sqrt{25} \\
 &= 5
 \end{aligned}$$

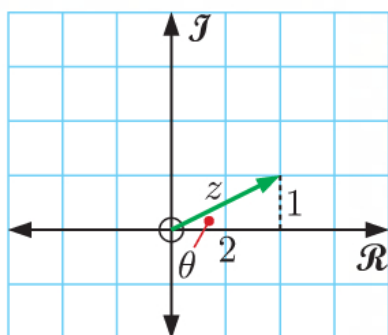
$$\begin{aligned}
 \text{b } |5 + 12i| &= \sqrt{5^2 + 12^2} \\
 &= \sqrt{25 + 144} \\
 &= \sqrt{169} \\
 &= 13
 \end{aligned}$$

$$\begin{aligned}
 \text{c } |-8 + 2i| &= \sqrt{(-8)^2 + 2^2} \\
 &= \sqrt{64 + 4} \\
 &= \sqrt{68} \\
 &= 2\sqrt{17}
 \end{aligned}$$

$$\begin{aligned}
 \text{d } |3i| &= \sqrt{0^2 + 3^2} \\
 &= \sqrt{9} \\
 &= 3
 \end{aligned}$$

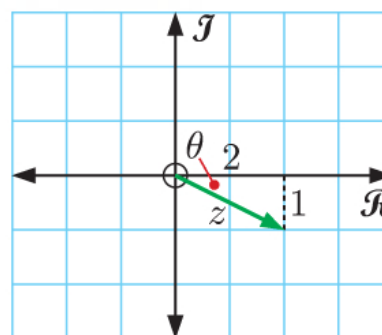
$$\begin{aligned}
 \text{e } |-4| &= \sqrt{(-4)^2 + 0^2} \\
 &= \sqrt{16} \\
 &= 4
 \end{aligned}$$

$$\begin{aligned}
 \text{2 a } |z| &= \sqrt{2^2 + 1^2} \\
 &= \sqrt{5}
 \end{aligned}$$



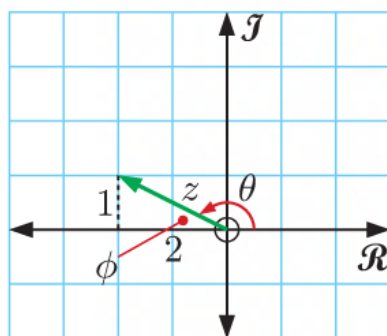
$$\begin{aligned}
 \tan \theta &= \frac{1}{2} \\
 \therefore \theta &\approx 0.464 \\
 \therefore \arg z &\approx 0.464
 \end{aligned}$$

$$\begin{aligned}
 \text{b } |z| &= \sqrt{2^2 + (-1)^2} \\
 &= \sqrt{5}
 \end{aligned}$$



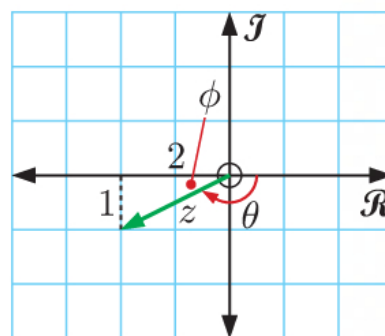
$$\begin{aligned}
 \tan \theta &= \frac{1}{2} \\
 \therefore \theta &\approx 0.464 \\
 \therefore \arg z &\approx -0.464
 \end{aligned}$$

$$\begin{aligned} \text{c } |z| &= \sqrt{(-2)^2 + 1^2} \\ &= \sqrt{5} \end{aligned}$$



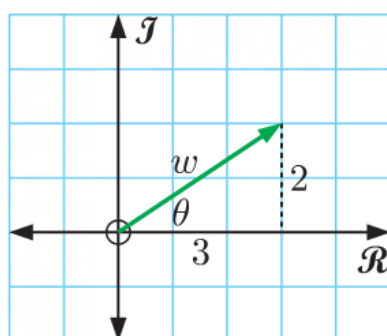
$$\begin{aligned} \tan \phi &= \frac{1}{2} \\ \therefore \phi &\approx 0.464 \\ \text{But } \theta &= \pi - \phi \\ \therefore \arg z &\approx 2.68 \end{aligned}$$

$$\begin{aligned} \text{d } |z| &= \sqrt{(-2)^2 + (-1)^2} \\ &= \sqrt{5} \end{aligned}$$



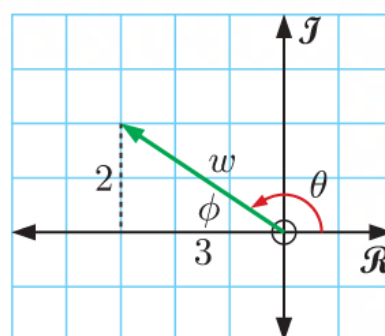
$$\begin{aligned} \tan \phi &= \frac{1}{2} \\ \therefore \phi &\approx 0.464 \\ \text{But } \theta &= -\pi + \phi \\ \therefore \arg z &\approx -2.68 \end{aligned}$$

$$\begin{aligned} \text{3 a } |w| &= \sqrt{3^2 + 2^2} \\ &= \sqrt{13} \end{aligned}$$



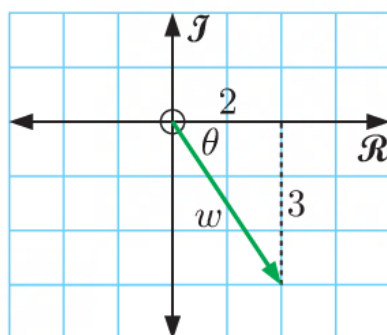
$$\begin{aligned} \tan \theta &= \frac{2}{3} \\ \therefore \theta &\approx 0.588 \\ \therefore \arg w &\approx 0.588 \end{aligned}$$

$$\begin{aligned} \text{b } |w| &= \sqrt{(-3)^2 + 2^2} \\ &= \sqrt{13} \end{aligned}$$



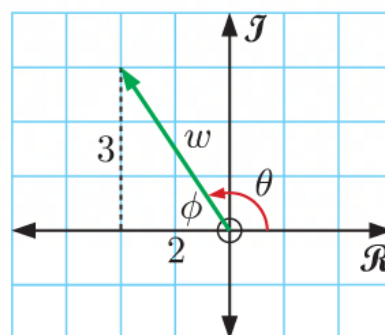
$$\begin{aligned} \tan \phi &= \frac{2}{3} \\ \therefore \phi &\approx 0.588 \\ \text{But } \theta &= \pi - \phi \\ \therefore \arg w &\approx 2.55 \end{aligned}$$

$$\begin{aligned} \text{c } |w| &= \sqrt{2^2 + (-3)^2} \\ &= \sqrt{13} \end{aligned}$$



$$\begin{aligned} \tan \theta &= \frac{3}{2} \\ \therefore \theta &\approx 0.983 \\ \therefore \arg w &\approx -0.983 \end{aligned}$$

$$\begin{aligned} \text{d } |w| &= \sqrt{(-2)^2 + 3^2} \\ &= \sqrt{13} \end{aligned}$$



$$\begin{aligned} \tan \phi &= \frac{3}{2} \\ \therefore \phi &\approx 0.983 \\ \text{But } \theta &= \pi - \phi \\ \therefore \arg w &\approx 2.16 \end{aligned}$$

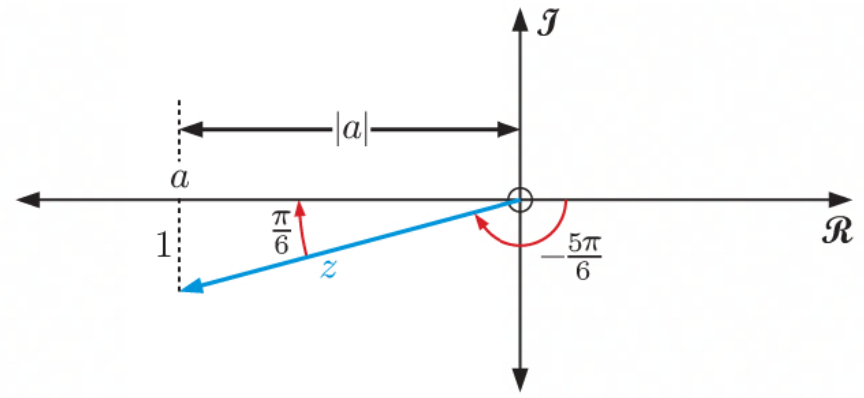
$$4 \quad \tan\left(\frac{\pi}{6}\right) = \frac{1}{|a|}$$

$$\therefore |a| = \frac{1}{\tan(\frac{\pi}{6})}$$

$$\therefore a = \pm\sqrt{3}$$

but from the Argand diagram,  $a$  is negative

$$\therefore a = -\sqrt{3}$$



$$5 \quad a \quad |z|$$

$$= |2 + i|$$

$$= \sqrt{2^2 + 1^2}$$

$$= \sqrt{5}$$

$$d \quad zz^*$$

$$= (2 + i)(2 - i)$$

$$= 4 - 2i + 2i - i^2$$

$$= 4 + 1$$

$$= 5$$

$$b \quad |z^*|$$

$$= |2 - i|$$

$$= \sqrt{2^2 + (-1)^2}$$

$$= \sqrt{5}$$

$$e \quad |zw|$$

$$= |(2 + i)(-1 + 3i)|$$

$$= |-2 + 6i - i + 3i^2|$$

$$= |-5 + 5i|$$

$$= \sqrt{(-5)^2 + 5^2}$$

$$= \sqrt{50}$$

$$= 5\sqrt{2}$$

$$c \quad |z^*|^2$$

$$= (\sqrt{5})^2 \quad \{\text{from } b\}$$

$$= 5$$

$$f \quad |z||w|$$

$$= |2 + i||-1 + 3i|$$

$$= \sqrt{2^2 + 1^2}\sqrt{(-1)^2 + 3^2}$$

$$= \sqrt{5} \times \sqrt{10}$$

$$= \sqrt{50}$$

$$= 5\sqrt{2}$$

$$g \quad \left| \frac{z}{w} \right|$$

$$= \left| \frac{2 + i}{-1 + 3i} \right|$$

$$= \left| \left( \frac{2 + i}{-1 + 3i} \right) \times \left( \frac{-1 - 3i}{-1 - 3i} \right) \right|$$

$$= \left| \frac{-2 - 6i - i - 3i^2}{(-1)^2 - (3i)^2} \right|$$

$$= \left| \frac{-2 + 3 - 7i}{10} \right|$$

$$= \left| \frac{1}{10} - \frac{7}{10}i \right|$$

$$= \sqrt{\left(\frac{1}{10}\right)^2 + \left(\frac{-7}{10}\right)^2}$$

$$= \sqrt{\frac{1+49}{100}}$$

$$= \sqrt{\frac{50}{100}}$$

$$= \frac{1}{\sqrt{2}}$$

$$h \quad \frac{|z|}{|w|}$$

$$= \frac{|2 + i|}{|-1 + 3i|}$$

$$= \frac{\sqrt{2^2 + 1^2}}{\sqrt{(-1)^2 + 3^2}}$$

$$= \frac{\sqrt{5}}{\sqrt{10}}$$

$$= \sqrt{\frac{5}{10}}$$

$$= \frac{1}{\sqrt{2}}$$

$$i \quad z^2 = (2 + i)^2$$

$$= 4 + 4i + i^2$$

$$= 3 + 4i$$

$$\therefore |z^2| = \sqrt{3^2 + 4^2}$$

$$= \sqrt{25}$$

$$= 5$$

$$j \quad |z|^2$$

$$= (\sqrt{5})^2 \quad \{\text{from } a\}$$

$$= 5$$

$$\begin{aligned}
 \mathbf{k} \quad z^3 &= z^2 \times z \\
 &= (3 + 4i)(2 + i) \quad \{\text{from } \mathbf{i}\} \\
 &= 6 + 3i + 8i + 4i^2 \\
 &= 2 + 11i \\
 \therefore |z^3| &= \sqrt{2^2 + 11^2} \\
 &= \sqrt{4 + 121} \\
 &= \sqrt{125} \\
 &= 5\sqrt{5}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{l} \quad |z|^3 &= (\sqrt{5})^3 \quad \{\text{from } \mathbf{a}\} \\
 &= \sqrt{125} \\
 &= 5\sqrt{5}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{6} \quad \mathbf{a} \quad \text{Let } z &= a + bi \text{ where } a, b \in \mathbb{R} \\
 \therefore |z^*| &= |a - bi| \\
 &= \sqrt{a^2 + (-b)^2} \\
 &= \sqrt{a^2 + b^2} \\
 &= |a + bi| \\
 &= |z|
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad \text{Let } z &= a + bi \text{ where } a, b \in \mathbb{R} \\
 \therefore zz^* &= (a + bi)(a - bi) \\
 &= a^2 - \cancel{abi} + \cancel{abi} - b^2i^2 \\
 &= a^2 + b^2 \\
 &= (\sqrt{a^2 + b^2})^2 \\
 &= |z|^2
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{7} \quad \mathbf{a} \quad z &= \cos \theta + i \sin \theta \\
 \therefore |z| &= \sqrt{\cos^2 \theta + \sin^2 \theta} \\
 &= \sqrt{1} \\
 &= 1
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad z &= r(\cos \theta + i \sin \theta), \quad r \in \mathbb{R} \\
 &= r \cos \theta + ri \sin \theta \\
 \therefore |z| &= \sqrt{r^2 \cos^2 \theta + r^2 \sin^2 \theta} \\
 &= \sqrt{r^2(\cos^2 \theta + \sin^2 \theta)} \\
 &= \sqrt{r^2} \\
 &= |r|
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{8} \quad \left| \frac{z}{w} \right| \times |w| &= \left| \frac{z}{w} \times w \right| \quad \{\text{using } |z_1| |z_2| = |z_1 z_2|\} \\
 \therefore \left| \frac{z}{w} \right| \times |w| &= |z| \\
 \therefore \left| \frac{z}{w} \right| &= \frac{|z|}{|w|} \quad \text{provided } w \neq 0 \quad \{\text{dividing both sides by } |w|\}
 \end{aligned}$$

$$\mathbf{9} \quad \mathbf{a} \quad P_n \text{ is: } |z_1 z_2 \dots z_n| = |z_1| |z_2| \dots |z_n| \text{ for all complex numbers } z_1, z_2, \dots, z_n \text{ and } n \in \mathbb{Z}^+.$$

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$  then  $\text{LHS} = |z_1|$ ,  $\text{RHS} = |z_1|$   $\therefore P_1$  is true.

(2) If  $P_k$  is true then  $|z_1 z_2 \dots z_k| = |z_1| |z_2| \dots |z_k|$

$$\begin{aligned}
 \therefore |z_1 z_2 \dots z_k z_{k+1}| &= |(z_1 z_2 \dots z_k) z_{k+1}| \\
 &= |z_1 z_2 \dots z_k| |z_{k+1}| \quad \{\text{as } |z| |w| = |zw|\} \\
 &= |z_1| |z_2| \dots |z_k| |z_{k+1}| \quad \{\text{using } P_k\}
 \end{aligned}$$

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

$$\mathbf{b} \quad |z_1 z_2 \dots z_n| = |z_1| |z_2| \dots |z_n| \quad \{\text{from } \mathbf{a}\}$$

Letting  $z_1 = z_2 = \dots = z_n = z$  we have  $|zz \dots z| = |z| |z| \dots |z|$

$$\therefore |z^n| = |z|^n$$



$$\begin{aligned}
 \text{c If } z = \frac{3}{4} - i \text{ then } |z| &= \sqrt{\left(\frac{3}{4}\right)^2 + (-1)^2} \\
 &= \sqrt{\frac{25}{16}} \\
 &= \frac{5}{4} \\
 \therefore |z^{30}| &= |z|^{30} \quad \{\text{using b with } n = 30\} \\
 &= \left(\frac{5}{4}\right)^{30} \\
 &\approx 808
 \end{aligned}$$

$$\begin{aligned}
 10 \text{ a } |3z| &= |3||z| \\
 &= 3 \times 2 \\
 &= 6
 \end{aligned}$$

$$\begin{aligned}
 \text{b } |-2z| &= |-2||z| \\
 &= 2 \times 2 \\
 &= 4
 \end{aligned}$$

$$\begin{aligned}
 \text{c } |(2-i)z| &= |2-i||z| \\
 &= \sqrt{4+1} \times 2 \\
 &= 2\sqrt{5}
 \end{aligned}$$

$$\begin{aligned}
 \text{d } |iz| &= |i||z| \\
 &= 1 \times 2 \\
 &= 2
 \end{aligned}$$

$$\begin{aligned}
 \text{e } \left| \frac{1}{z} \right| &= \frac{|1|}{|z|} \\
 &= \frac{1}{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{f } \left| \frac{z^2}{3i} \right| &= \frac{|z^2|}{|3i|} \\
 &= \frac{|z|^2}{|3||i|} \\
 &= \frac{2^2}{3} \\
 &= \frac{4}{3}
 \end{aligned}$$

$$11 \text{ a } w = \frac{z+i}{z-i}, \text{ where } z = a+bi, \ a, b \in \mathbb{R}$$

$$\begin{aligned}
 \therefore w &= \frac{a+bi+i}{a+bi-i} \\
 &= \frac{a+(b+1)i}{a+(b-1)i} \\
 &= \left( \frac{a+(b+1)i}{a+(b-1)i} \right) \times \left( \frac{a-(b-1)i}{a-(b-1)i} \right) \\
 &= \frac{a^2 - a(b-1)i + a(b+1)i - (b+1)(b-1)i^2}{a^2 - (b-1)^2 i^2} \\
 &= \frac{a^2 - \cancel{abi} + ai + \cancel{abi} + ai + b^2 - 1}{a^2 + (b-1)^2} \\
 &= \left( \frac{a^2 + b^2 - 1}{a^2 + (b-1)^2} \right) + \left( \frac{2a}{a^2 + (b-1)^2} \right) i
 \end{aligned}$$

$$\text{b } \operatorname{Re}(w) = \frac{a^2 + b^2 - 1}{a^2 + (b-1)^2} = \frac{a^2 + b^2 - 1}{a^2 + b^2 - 2b + 1}$$

$$\text{Since } |z| = 1, \sqrt{a^2 + b^2} = 1 \quad \therefore a^2 + b^2 = 1$$

$$\therefore \operatorname{Re}(w) = \frac{1-1}{1-2b+1} = 0 \text{ provided } b \neq 1$$

If  $b = 1$ , then  $\operatorname{Re}(w)$  is undefined.

**12**

$$\frac{50}{z^*} - \frac{10}{z} = 2 + 9i \quad \text{where } z = a + bi, \quad a, b \in \mathbb{R}$$

$$\therefore 50z - 10z^* = (2 + 9i)(|z|^2) \quad \{\text{multiply both sides by } zz^* = |z|^2\}$$

$$\therefore 50(a + bi) - 10(a - bi) = (2 + 9i)(40) \quad \{|z| = 2\sqrt{10} \quad \therefore |z|^2 = 40\}$$

$$\therefore 50a + 50bi - 10a + 10bi = 80 + 360i$$

$$\therefore 40a + 60bi = 80 + 360i$$

Equating real and imaginary parts,  $40a = 80$  and  $60b = 360$

$$\therefore a = 2 \quad \text{and} \quad b = 6$$

$$\therefore z = 2 + 6i$$

**13 a**

$$2|z + 4| = |z + 16|$$

$$\therefore 4|z + 4|^2 = |z + 16|^2$$

$$\therefore 4(z + 4)(z + 4)^* = (z + 16)(z + 16)^* \quad \{\text{as } |z|^2 = zz^*\}$$

$$\therefore 4(z + 4)(z^* + 4) = (z + 16)(z^* + 16) \quad \{\text{as } (z + w)^* = z^* + w^*\}$$

$$\therefore 4zz^* + \cancel{16z} + \cancel{16z^*} + 64 = zz^* + \cancel{16z} + \cancel{16z^*} + 256$$

$$\therefore 3zz^* = 192$$

$$\therefore zz^* = 64$$

$$\therefore |z|^2 = 64$$

$$\therefore |z| = 8 \quad \{\text{as } |z| > 0\}$$

**b**

$$|z - 18| = 3|z - 2|$$

$$\therefore |z - 18|^2 = 9|z - 2|^2$$

$$\therefore (z - 18)(z - 18)^* = 9(z - 2)(z - 2)^* \quad \{\text{as } |z|^2 = zz^*\}$$

$$\therefore (z - 18)(z^* - 18) = 9(z - 2)(z^* - 2) \quad \{\text{as } (z + w)^* = z^* + w^*\}$$

$$\therefore zz^* - \cancel{18z} - \cancel{18z^*} + 324 = 9zz^* - \cancel{18z} - \cancel{18z^*} + 36$$

$$\therefore 8zz^* = 288$$

$$\therefore zz^* = 36$$

$$\therefore |z|^2 = 36$$

$$\therefore |z| = 6 \quad \{\text{as } |z| > 0\}$$

**c**

$$\left| \frac{z + 20}{z + 5} \right| = 2$$

$$\therefore \frac{|z + 20|}{|z + 5|} = 2 \quad \left\{ \text{as } \left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|} \right\}$$

$$\therefore |z + 20| = 2|z + 5|$$

$$\therefore |z + 20|^2 = 4|z + 5|^2$$

$$\therefore (z + 20)(z + 20)^* = 4(z + 5)(z + 5)^* \quad \{\text{as } |z|^2 = zz^*\}$$

$$\therefore (z + 20)(z^* + 20) = 4(z + 5)(z^* + 5) \quad \{\text{as } (z + w)^* = z^* + w^*\}$$

$$\therefore zz^* + \cancel{20z} + \cancel{20z^*} + 400 = 4zz^* + \cancel{20z} + \cancel{20z^*} + 100$$

$$\therefore 3zz^* = 300$$

$$\therefore zz^* = 100$$

$$\therefore |z|^2 = 100$$

$$\therefore |z| = 10 \quad \{\text{as } |z| > 0\}$$

**d**

$$\left| \frac{z-50}{z-2} \right| = 5$$

$$\therefore \frac{|z-50|}{|z-2|} = 5$$

$$\therefore |z-50| = 5|z-2|$$

$$\therefore |z-50|^2 = 25|z-2|^2$$

$$\therefore (z-50)(z-50)^* = 25(z-2)(z-2)^*$$

$$\therefore (z-50)(z^*-50) = 25(z-2)(z^*-2)$$

$$\therefore zz^* - \cancel{50z} - \cancel{50z^*} + 2500 = 25zz^* - \cancel{50z} - \cancel{50z^*} + 100$$

$$\therefore 24zz^* = 2400$$

$$\therefore zz^* = 100$$

$$\therefore |z|^2 = 100$$

$$\therefore |z| = 10$$

$$\left\{ \text{as } \left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|} \right\}$$

$$\{ \text{as } |z|^2 = zz^* \}$$

$$\{ \text{as } (z+w)^* = z^* + w^* \}$$

$$\{ \text{as } |z| > 0 \}$$

**14 a**

$$3|z-5| = |z-45|$$

$$\therefore 9|z-5|^2 = |z-45|^2$$

$$\therefore 9(z-5)(z-5)^* = (z-45)(z-45)^*$$

$$\therefore 9(z-5)(z^*-5) = (z-45)(z^*-45)$$

$$\therefore 9zz^* - \cancel{45z} - \cancel{45z^*} + 225 = zz^* - \cancel{45z} - \cancel{45z^*} + 2025$$

$$\therefore 8zz^* = 1800$$

$$\therefore zz^* = 225$$

$$\therefore |z|^2 = 225$$

$$\therefore |z| = 15$$

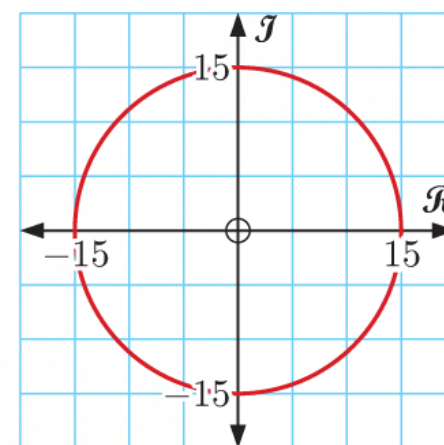
$$\{ \text{as } |z|^2 = zz^* \}$$

$$\{ \text{as } (z+w)^* = z^* + w^* \}$$

$$\{ \text{as } |z| > 0 \}$$

**b** From **a**, the set of complex numbers  $z$  that satisfy  $3|z-5| = |z-45|$  is equivalent to the set of all  $z$  such that  $|z| = 15$ .

This corresponds to a circle centred at the origin with radius 15 units.

**15**

$$|z+w| = |z-w|$$

$$\therefore |z+w|^2 = |z-w|^2$$

$$\therefore (z+w)(z+w)^* = (z-w)(z-w)^*$$

$$\therefore (z+w)(z^*+w^*) = (z-w)(z^*-w^*)$$

$$\therefore \cancel{zz^*} + zw^* + z^*w + \cancel{ww^*} = \cancel{zz^*} - zw^* - z^*w + \cancel{ww^*}$$

$$\therefore 2zw^* = -2z^*w$$

$$\therefore zw^* = -z^*w$$

$$\therefore \frac{z}{z^*} = -\frac{w}{w^*} \quad \{z \neq 0 \text{ and } w \neq 0 \text{ so } z^* \neq 0 \text{ and } w^* \neq 0\}$$

$$\{ \text{as } |z|^2 = zz^* \}$$

$$\{ \text{as } (z+w)^* = z^* + w^* \}$$

**16**  $z = 1 + 3i, \quad w = 3 + 2i$

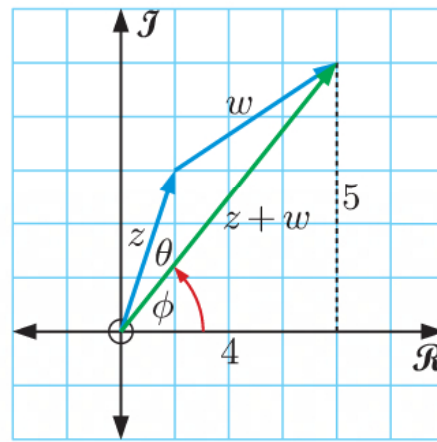
**a**  $z + w = (1 + 3i) + (3 + 2i)$   
 $= 4 + 5i$

**b**  $|z + w| = \sqrt{4^2 + 5^2}$   
 $= \sqrt{41}$

$$\tan \phi = \frac{5}{4}$$

$$\therefore \phi \approx 0.896$$

$$\therefore \arg(z + w) \approx 0.896$$



**c**  $|z| = \sqrt{1^2 + 3^2} = \sqrt{10}, \quad |w| = \sqrt{3^2 + 2^2} = \sqrt{13}, \quad \text{and} \quad |z + w| = \sqrt{41}$

Now  $|w|^2 = |z|^2 + |z + w|^2 - 2|z||z + w|\cos \theta$  {cosine rule}

$$\therefore (\sqrt{13})^2 = (\sqrt{10})^2 + (\sqrt{41})^2 - 2\sqrt{10}\sqrt{41}\cos \theta$$

$$\therefore 13 = 10 + 41 - 2\sqrt{10}\sqrt{41}\cos \theta$$

$$\therefore \cos \theta = \frac{38}{2\sqrt{410}}$$

$$\therefore \theta \approx 0.353$$

**d**  $\arg z = \tan^{-1}\left(\frac{3}{1}\right) \approx 1.249$

$$\arg(z + w) + \theta \approx 0.896 + 0.353$$

$$\approx 1.249 \quad \checkmark$$

**17 a**  $z^* = -iz$

$$\therefore x - iy = -i(x + iy)$$

$$\therefore x - iy = -ix + y$$

$$\therefore x - iy = y - ix$$

Equating real and imaginary parts,  $x = y$  and  $-y = -x$

$$\therefore y = x$$

**b**  $\arg(z - i) = \frac{\pi}{6}$

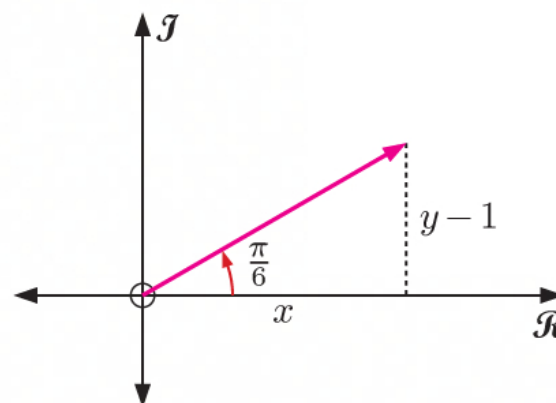
$$\therefore \arg(x + iy - i) = \frac{\pi}{6}$$

$$\therefore \arg(x + (y - 1)i) = \frac{\pi}{6}$$

$$\therefore \tan \frac{\pi}{6} = \frac{1}{\sqrt{3}} = \frac{y - 1}{x}$$

$$\therefore y - 1 = \frac{x}{\sqrt{3}}$$

$$\therefore y = \frac{1}{\sqrt{3}}x + 1, \quad x > 0$$





$$\begin{aligned}
\text{c} \quad & |z+3| + |z-3| = 8 \\
& \therefore |z+3| = 8 - |z-3| \\
& \therefore |z+3|^2 = (8 - |z-3|)^2 \\
& \therefore |z+3|^2 = 64 - 16|z-3| + |z-3|^2 \\
& \therefore (z+3)(z+3)^* = 64 - 16|z-3| + (z-3)(z-3)^* \quad \{\text{as } |z|^2 = zz^*\} \\
& \therefore (z+3)(z^*+3) = 64 - 16|z-3| + (z-3)(z^*-3) \quad \{\text{as } (z+w)^* = z^*+w^*\} \\
& \therefore \cancel{zz^*} + 3z + 3z^* + \cancel{9} = 64 - 16|(x-3) + yi| + \cancel{zz^*} - 3z - 3z^* + \cancel{9} \\
& \therefore 6(z+z^*) = 64 - 16\sqrt{(x-3)^2 + y^2} \\
& \therefore 6(x+yi+x-yi) = 64 - 16\sqrt{(x-3)^2 + y^2} \\
& \therefore 12x - 64 = -16\sqrt{(x-3)^2 + y^2} \\
& \therefore 3x - 16 = -4\sqrt{(x-3)^2 + y^2} \\
& \therefore (3x-16)^2 = 16[(x-3)^2 + y^2] \\
& \therefore 9x^2 - 96x + 256 = 16(x^2 - 6x + 9 + y^2) \\
& \therefore 9x^2 - \cancel{96x} + 256 = 16x^2 - \cancel{96x} + 144 + 16y^2 \\
& \therefore 112 = 7x^2 + 16y^2 \\
& \therefore 7x^2 + 16y^2 = 112
\end{aligned}$$

## EXERCISE 14C

1 a A(3, 6), B(-1, 2),  $z = 3 + 6i$ ,  $w = -1 + 2i$

$$\begin{aligned}
\text{i} \quad & z - w = (3 + 6i) - (-1 + 2i) \\
& = 4 + 4i \\
& \therefore |z - w| = \sqrt{4^2 + 4^2} \\
& = \sqrt{32} \\
& = 4\sqrt{2} \\
& \therefore AB = 4\sqrt{2} \text{ units}
\end{aligned}$$

$$\begin{aligned}
\text{ii} \quad & \frac{z+w}{2} = \frac{(3+6i) + (-1+2i)}{2} \\
& = \frac{2+8i}{2} \\
& = 1+4i \\
& \therefore \text{the midpoint of [AB] is } (1, 4).
\end{aligned}$$

b A(-4, 7), B(1, -3),  $z = -4 + 7i$ ,  $w = 1 - 3i$

$$\begin{aligned}
\text{i} \quad & z - w = (-4 + 7i) - (1 - 3i) \\
& = -5 + 10i \\
& \therefore |z - w| = \sqrt{(-5)^2 + 10^2} \\
& = \sqrt{125} \\
& = 5\sqrt{5} \\
& \therefore AB = 5\sqrt{5} \text{ units}
\end{aligned}$$

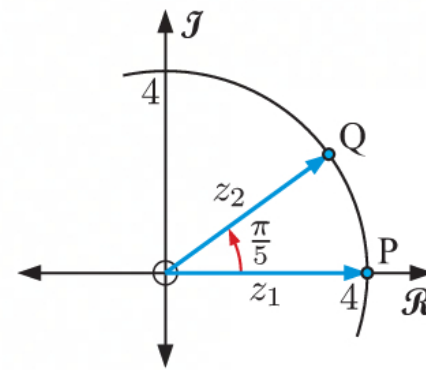
$$\begin{aligned}
\text{ii} \quad & \frac{z+w}{2} = \frac{(-4+7i) + (1-3i)}{2} \\
& = \frac{-3+4i}{2} \\
& = -\frac{3}{2} + 2i \\
& \therefore \text{the midpoint of [AB] is } \left(-\frac{3}{2}, 2\right).
\end{aligned}$$

- 2 a** P and Q both lie on a circle of radius 4 units, centred at the origin O.

P lies on the horizontal axis, so P is (4, 0).

Q makes an angle of  $\frac{\pi}{5}$  with the real axis, so Q is  $(4 \cos \frac{\pi}{5}, 4 \sin \frac{\pi}{5})$ .

$z_1 \equiv \overrightarrow{OP}$ , and  $z_2 \equiv \overrightarrow{OQ}$ , so  $|z_1 - z_2|$  is the distance between P and Q.



$$\begin{aligned} PQ &= \sqrt{(4 \cos \frac{\pi}{5} - 4)^2 + (4 \sin \frac{\pi}{5} - 0)^2} \\ &= \sqrt{16 \cos^2(\frac{\pi}{5}) - 32 \cos \frac{\pi}{5} + 16 + 16 \sin^2(\frac{\pi}{5})} \\ &= \sqrt{16 - 32 \cos \frac{\pi}{5} + 16} \quad \{\cos^2 \theta + \sin^2 \theta = 1\} \\ &= \sqrt{32 - 32 \cos \frac{\pi}{5}} \end{aligned}$$

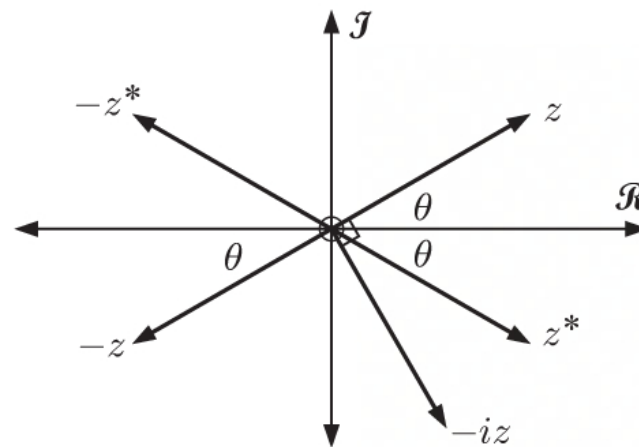
$$\therefore |z_1 - z_2| = \sqrt{32 - 32 \cos \frac{\pi}{5}}$$

- b** Perimeter of triangle OPQ = OP + OQ + PQ

$$\begin{aligned} &= |z_1| + |z_2| + |z_1 - z_2| \\ &= 4 + 4 + \sqrt{32 - 32 \cos \frac{\pi}{5}} \quad \{\text{from a}\} \\ &= 8 + \sqrt{32 - 32 \cos \frac{\pi}{5}} \text{ units} \end{aligned}$$

$$\begin{aligned} \text{Area of triangle OPQ} &= \frac{1}{2} \times OP \times OQ \times \sin \widehat{POQ} \\ &= \frac{1}{2} \times 4 \times 4 \times \sin \frac{\pi}{5} \\ &= 8 \sin \frac{\pi}{5} \text{ units}^2 \end{aligned}$$

- 3 a**  $z \mapsto z^*$  Reflection in the real axis.  
**b**  $z \mapsto -z$  Rotation of  $\pi$  about O.  
**c**  $z \mapsto -z^*$  Reflection in the imaginary axis.  
**d**  $z \mapsto -iz$  Clockwise rotation of  $\frac{\pi}{2}$  about O.



## ACTIVITY 1

## THE TRIANGLE INEQUALITY

$$\begin{aligned}
 1 \quad a \quad |z_1 + z_2| &= |3 + 4i + 6i| \\
 &= |3 + 10i| \\
 &= \sqrt{3^2 + 10^2} \\
 &= \sqrt{109}
 \end{aligned}$$

$$\begin{aligned}
 |z_1| + |z_2| &= |3 + 4i| + |6i| \\
 &= \sqrt{3^2 + 4^2} + \sqrt{0^2 + 6^2} \\
 &= \sqrt{9 + 16} + \sqrt{36} \\
 &= \sqrt{25} + \sqrt{36} \\
 &= 5 + 6 \\
 &= 11
 \end{aligned}$$

$$11 = \sqrt{121} > \sqrt{109}$$

$$\therefore |z_1 + z_2| \leq |z_1| + |z_2| \quad \checkmark$$

$$\begin{aligned}
 b \quad |z_1 + z_2| &= |2 - 3i + 4 + 7i| \\
 &= |6 + 4i| \\
 &= \sqrt{6^2 + 4^2} \\
 &= \sqrt{36 + 16} \\
 &= \sqrt{52}
 \end{aligned}$$

$$\begin{aligned}
 |z_1| + |z_2| &= |2 - 3i| + |4 + 7i| \\
 &= \sqrt{2^2 + (-3)^2} + \sqrt{4^2 + 7^2} \\
 &= \sqrt{4 + 9} + \sqrt{16 + 49} \\
 &= \sqrt{13} + \sqrt{65}
 \end{aligned}$$

$$\sqrt{52} < \sqrt{65}$$

$$\therefore \sqrt{52} < \sqrt{65} + \sqrt{13}$$

$$\therefore |z_1 + z_2| \leq |z_1| + |z_2| \quad \checkmark$$

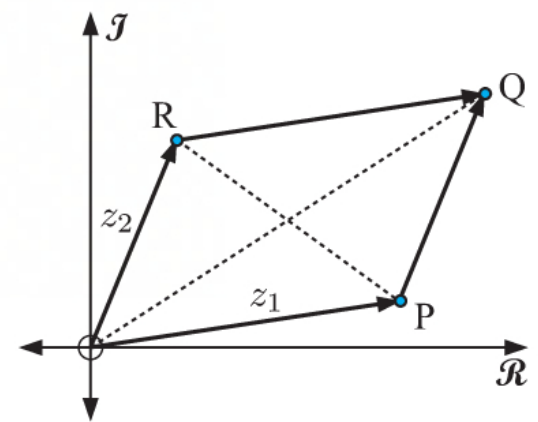
- 2 a Opposite sides in a parallelogram are parallel and have the same length, and so the vectors representing opposite sides are equal.

$$\therefore \overrightarrow{RQ} = \overrightarrow{OP}, \quad \overrightarrow{PQ} = \overrightarrow{OR}$$

So,  $\overrightarrow{RQ}$  represents  $z_1$  and  $\overrightarrow{PQ}$  represents  $z_2$ .

i  $\overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{PQ}$   
which represents  
 $z_1 + z_2$

ii  $\overrightarrow{PR} = \overrightarrow{PO} + \overrightarrow{OR}$   
which represents  
 $-z_1 + z_2$  or  
 $z_2 - z_1$



- b The sum of the lengths of two sides of a triangle is greater than the length of the remaining side.

So, in triangle OPQ,  $OP + PQ > OQ$

$$\therefore |z_1| + |z_2| > |z_1 + z_2| \quad \{\text{from a i}\}$$

The case of equality  $|z_1| + |z_2| = |z_1 + z_2|$  occurs if P lies on [OQ].

In this case,  $\overrightarrow{OP} \parallel \overrightarrow{OQ}$  and  $OP = k OQ$  for some  $k \geq 0$ . So,  $z_1$  and  $z_2$  must satisfy  $z_1 = kz_2$  for some  $k \geq 0$ .

**Note:** If O lies on [PQ] and P and Q are not O, then

$$OQ < PQ \quad \text{and so} \quad |z_1 + z_2| < |z_2|$$

$$\therefore |z_1 + z_2| < |z_1| + |z_2|$$

So,  $|z_1 + z_2| = |z_1| + |z_2|$  is impossible.

**c** In triangle OPR,  $PR + OP > OR$

$$\therefore |z_2 - z_1| + |z_1| > |z_2| \quad \{\text{from a ii}\}$$

$$\therefore |z_2 - z_1| > |z_2| - |z_1|$$

The case of equality  $|z_2 - z_1| = |z_2| - |z_1|$  occurs if P lies on [OR]. In this case,  $\vec{OP} \parallel \vec{OR}$  and  $OP = k OR$  for some  $0 \leq k \leq 1$ . So,  $z_1$  and  $z_2$  must satisfy  $z_1 = kz_2$  for some  $0 \leq k \leq 1$ .

**Note:** If R lies on [OP] and R is not O or R, then

$$OR < OP \quad \text{and so} \quad |z_2| < |z_1|$$

$$\therefore |z_2| - |z_1| < 0$$

But  $|z_2 - z_1| > 0$ , so  $|z_2 - z_1| = |z_2| - |z_1|$  is impossible.

A similar situation occurs if O lies on [PR] and P and R are not O.

**3**  $P_k$  is  $|z_1 + z_2 + \dots + z_k| \leq |z_1| + |z_2| + \dots + |z_k|$  for any integer  $k \geq 1$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $k = 1$  then  $\text{LHS} = |z_1|$ ,  $\text{RHS} = |z_1|$ , so  $P_1$  is true.

(2) If  $P_n$  is true then  $|z_1 + z_2 + \dots + z_n| \leq |z_1| + |z_2| + \dots + |z_n|$

$$\therefore |(z_1 + z_2 + \dots + z_n) + z_{n+1}| \leq |z_1 + z_2 + \dots + z_n| + |z_{n+1}|$$

$$\{\text{as } |z + w| \leq |z| + |w|\}$$

$$\leq |z_1| + |z_2| + \dots + |z_n| + |z_{n+1}| \quad \{\text{using } P_n\}$$

$$\therefore |z_1 + z_2 + \dots + z_n + z_{n+1}| \leq |z_1| + |z_2| + \dots + |z_n| + |z_{n+1}|$$

Since  $P_1$  is true, and  $P_{n+1}$  is true whenever  $P_n$  is true,

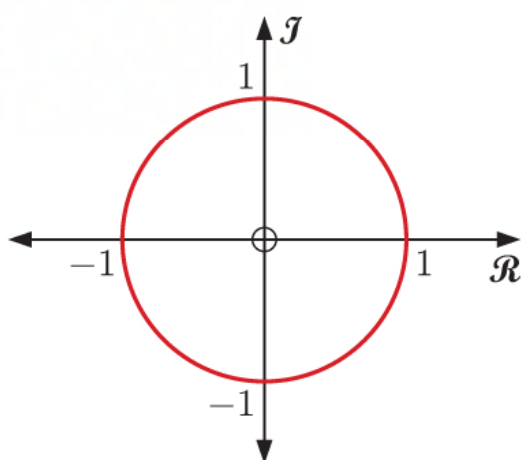
$P_k$  is true for every integer  $k \geq 1$ . {principle of mathematical induction}

The case of equality occurs when  $z_1, z_2, \dots, z_k$  are non-negative scalar multiples of each other.

## ACTIVITY 2

## LOCUS

**1 a**



**b**

$$z = x + yi, \quad |z| = 1$$

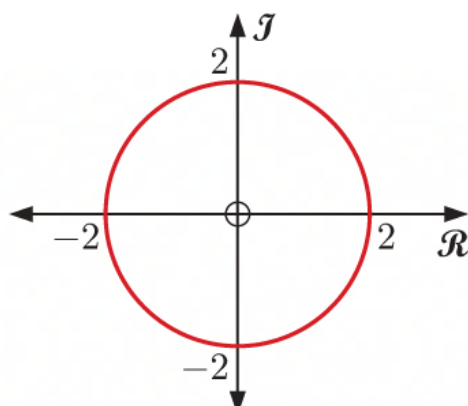
$$\therefore |z| = |x + yi|$$

$$= \sqrt{x^2 + y^2}$$

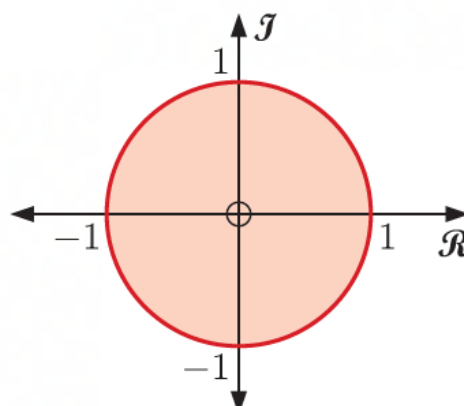
$$\therefore \sqrt{x^2 + y^2} = 1$$

or  $x^2 + y^2 = 1$  which is the equation of the unit circle.

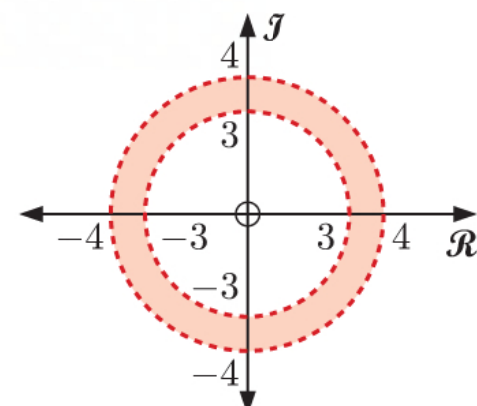
**2 a**  $|z| = 2$



**b**  $|z| \leq 1$

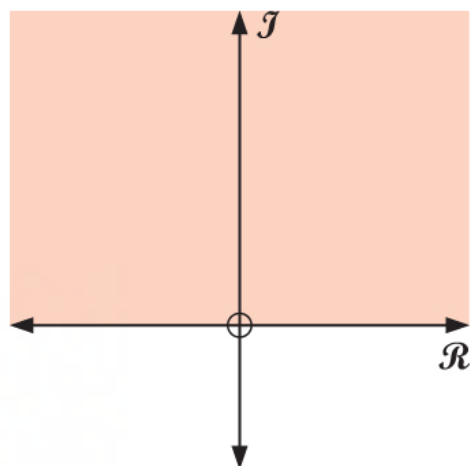


**c**  $3 < |z| < 4$

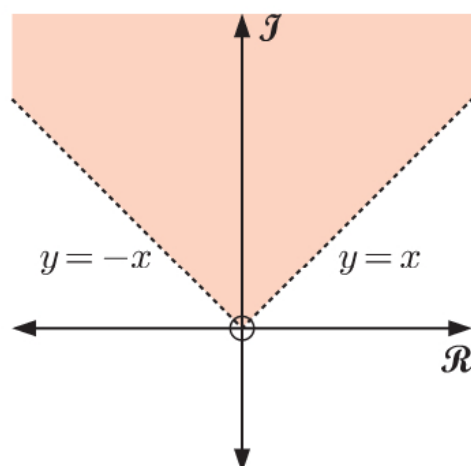




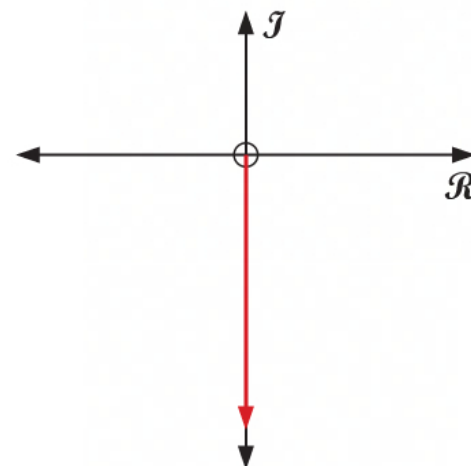
**d**  $0 \leq \arg z \leq \pi$



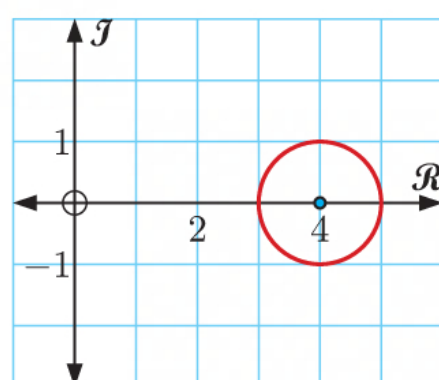
**e**  $\frac{\pi}{4} < \arg z < \frac{3\pi}{4}$



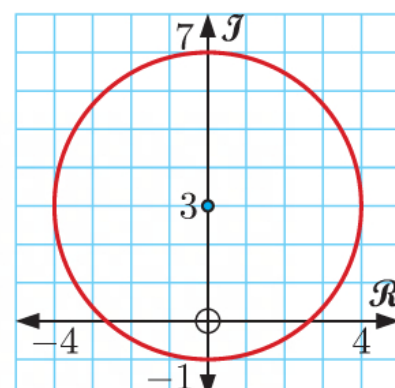
**f**  $\arg z = -\frac{\pi}{2}$



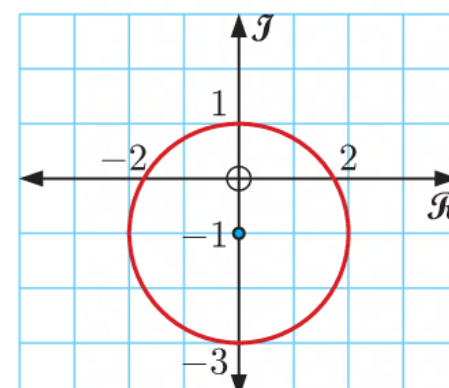
**3 a**  $|z - 4| = 1$



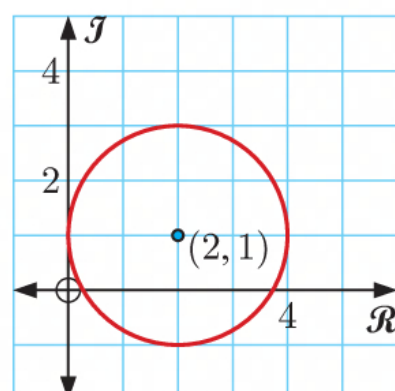
**b**  $|z - 3i| = 4$



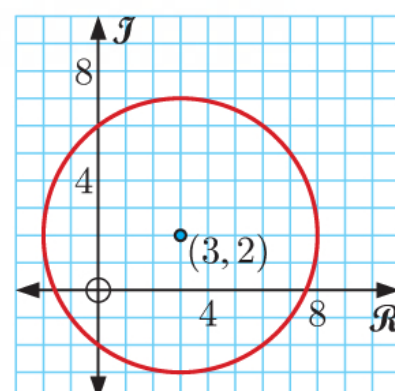
**c**  $|z + i| = 2$



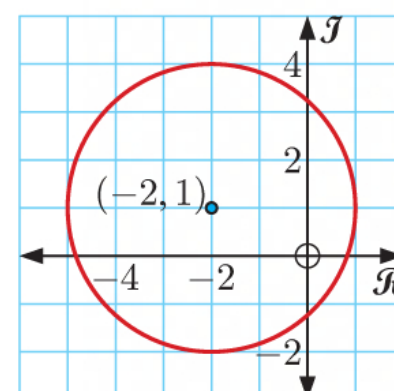
**d**  $|z - (2 + i)| = 2$



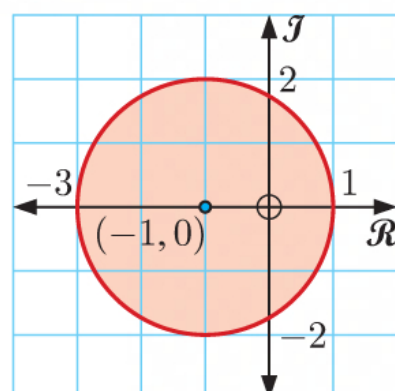
**e**  $|z - 3 - 2i| = 5$



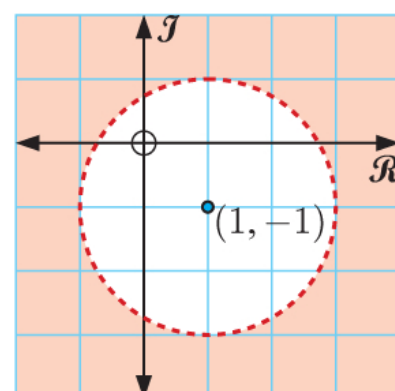
**f**  $|z + 2 - i| = 3$



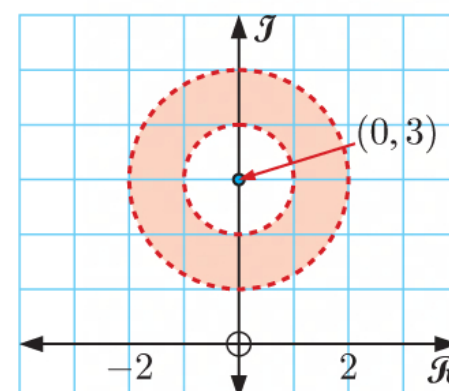
**g**  $|z + 1| \leq 2$



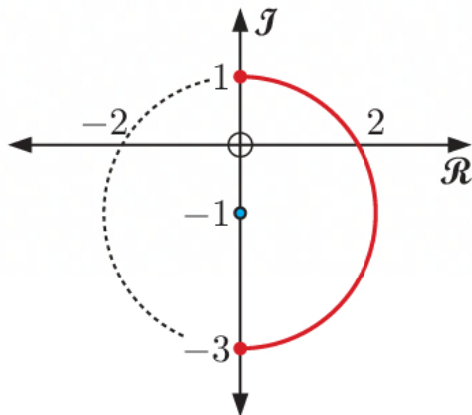
**h**  $|z - 1 + i| > 2$



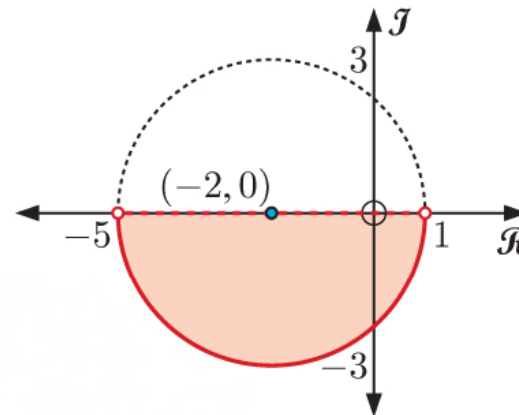
**i**  $1 < |z - 3i| < 2$



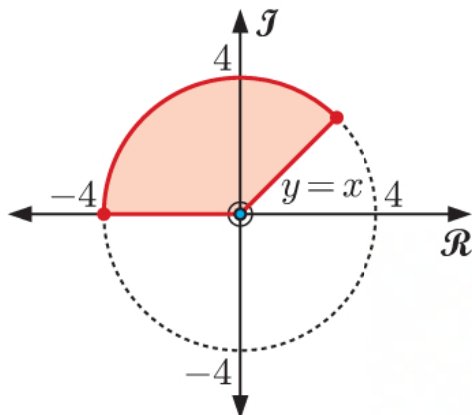
**4 a**  $\{z \mid |z + i| = 2 \text{ and } \operatorname{Re}(z) \geq 0\}$



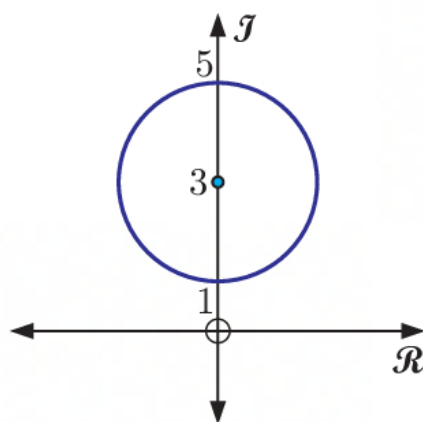
**b**  $\{z \mid |z + 2| \leq 3 \text{ and } \operatorname{Im}(z) < 0\}$



**c**  $\{z \mid |z| \leq 4 \text{ and } \frac{\pi}{4} \leq \arg z \leq \pi\}$

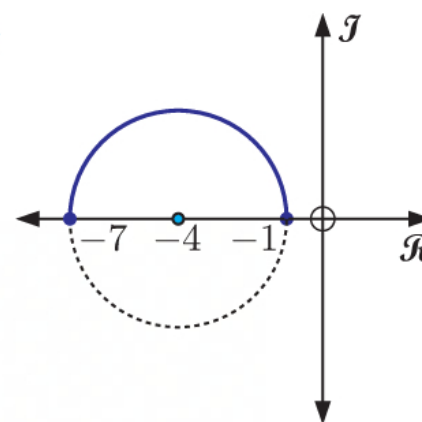


**5 a**



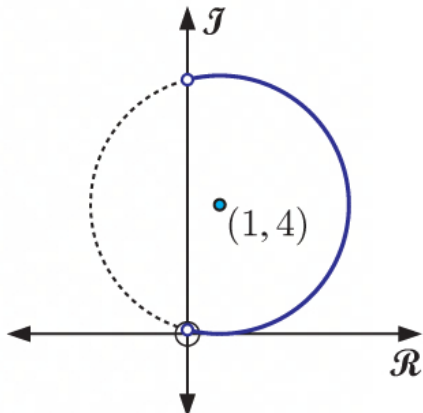
$\{z \mid |z - 3i| = 2\}$

**b**



$\{z \mid |z + 4| = 3 \text{ and } \operatorname{Im}(z) \geq 0\}$

**c**

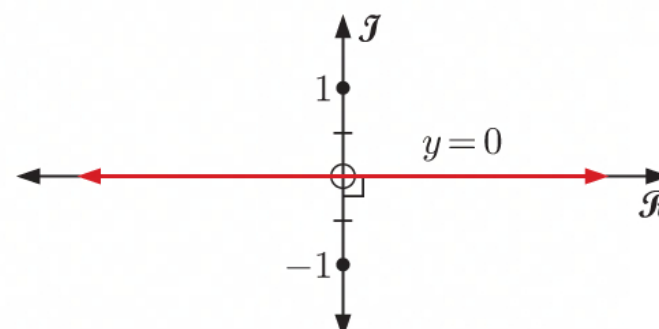


$\{z \mid |z - (1 + 4i)| = 4 \text{ and } \operatorname{Re}(z) > 0\}$

**6 a**  $|z + i| = |z - i|$

The point representing  $z$  is equidistant from  $A(0, 1)$  and  $B(0, -1)$ , so  $z$  lies on the perpendicular bisector of  $[AB]$ .

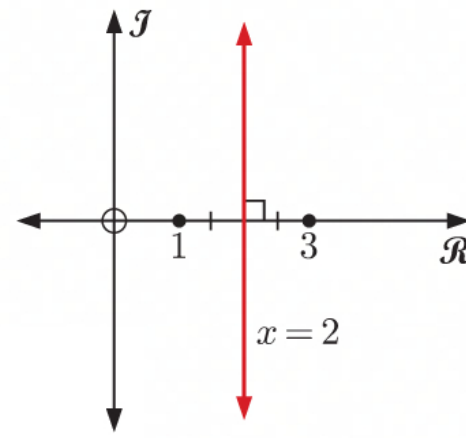
This is the line  $y = 0$ .



**b**  $|z - 1| = |z - 3|$

The point representing  $z$  is equidistant from  $A(1, 0)$  and  $B(3, 0)$ , so  $z$  lies on the perpendicular bisector of  $[AB]$ .

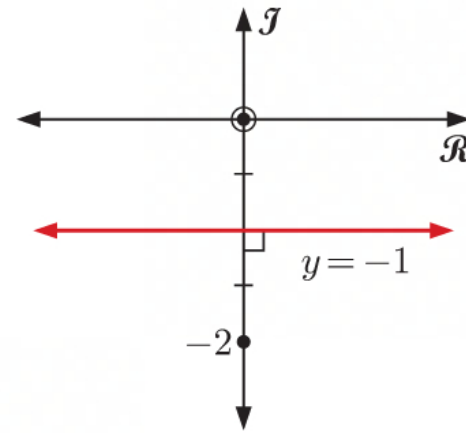
This is the line  $x = 2$ .



**c**  $|z + 2i| = |z|$

The point representing  $z$  is equidistant from  $A(0, -2)$  and  $B(0, 0)$ , so  $z$  lies on the perpendicular bisector of  $[AB]$ .

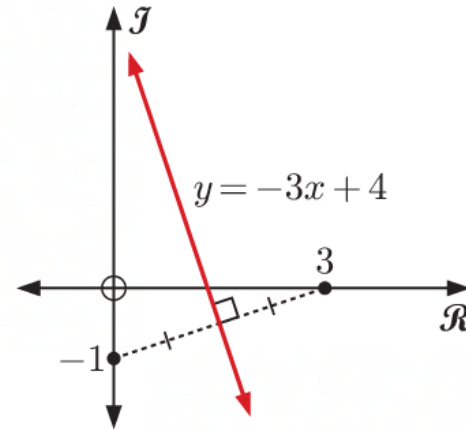
This is the line  $y = -1$ .



**d**  $|z - 3| = |z + i|$

The point representing  $z$  is equidistant from  $A(3, 0)$  and  $B(0, -1)$ , so  $z$  lies on the perpendicular bisector of  $[AB]$ .

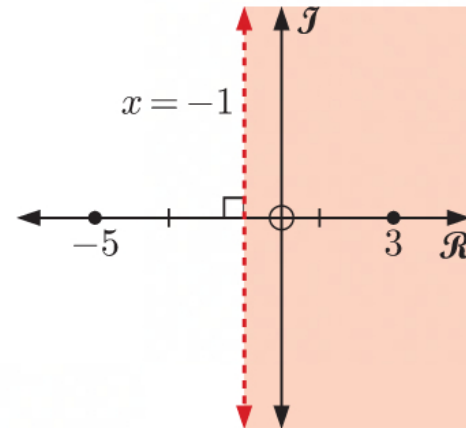
This is the line  $y = -3x + 4$ .



**e**  $|z - 3| < |z + 5|$

The point representing  $z$  is closer to  $A(3, 0)$  than  $B(-5, 0)$ , so  $z$  lies to the right of the perpendicular bisector of  $[AB]$ .

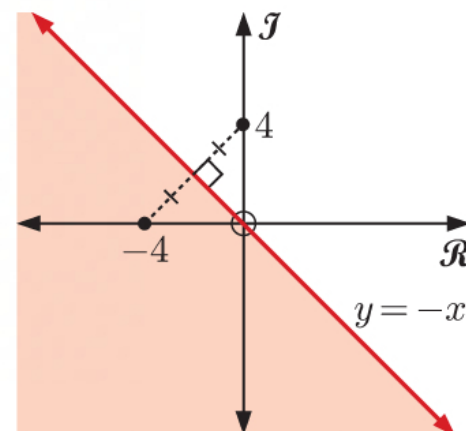
This is the region  $x > -1$ .



**f**  $|z - 4i| \geq |z + 4|$

The point representing  $z$  is either equidistant from  $A(0, 4)$  and  $B(-4, 0)$ , or closer to  $B$  than  $A$ . So  $z$  lies on or below the perpendicular bisector of  $[AB]$ .

This is the region  $y \leq -x$ .



**7 a**  $\{z \mid |z + 2i| = 2\}$

- b i** Defining points C and D as shown, we construct the right angled triangle CDP where  $CD = 4$ ,  $CP = 2$ , and  $DP = \sqrt{4^2 - 2^2} = 2\sqrt{3}$ .

$$\begin{aligned}\cos(\widehat{CDP}) &= \frac{(2\sqrt{3})^2 + 4^2 - 2^2}{2 \times 2\sqrt{3} \times 4} \quad \{\text{cosine rule}\} \\ &= \frac{\sqrt{3}}{2}\end{aligned}$$

Using the triangle ODP, we have that

$$\begin{aligned}OP^2 &= 6^2 + (2\sqrt{3})^2 - 2 \times 6 \times 2\sqrt{3} \times \cos(\widehat{ODP}) \quad \{\text{cosine rule}\} \\ &= 36 + 12 - 36 \\ &= 12\end{aligned}$$

$$\therefore OP = 2\sqrt{3} \quad \{\text{as } OP > 0\}$$

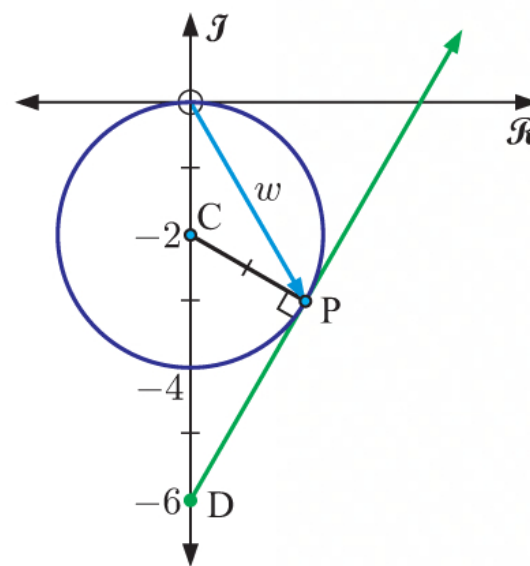
$$\therefore |w| = 2\sqrt{3} \quad \{\text{as } w \equiv \overrightarrow{OP}\}$$

- ii** Using the results from **i**,  $\triangle OPD$  is equilateral, so  $\widehat{DOP} = \widehat{ODP}$

$$\therefore \widehat{ODP} = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$$

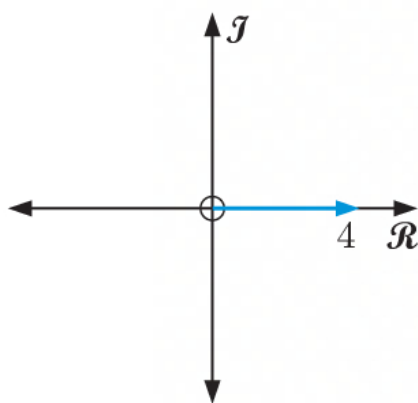
$$\therefore \arg w = -\frac{\pi}{2} + \frac{\pi}{6} = -\frac{\pi}{3}$$

**iii**  $|w| = 2\sqrt{3}$  and  $\arg w = -\frac{5\pi}{6}$ , so  $w = 2\sqrt{3} \cos(-\frac{\pi}{3}) + i \times 2\sqrt{3} \sin(-\frac{\pi}{3})$   
 $= \sqrt{3} - 3i$



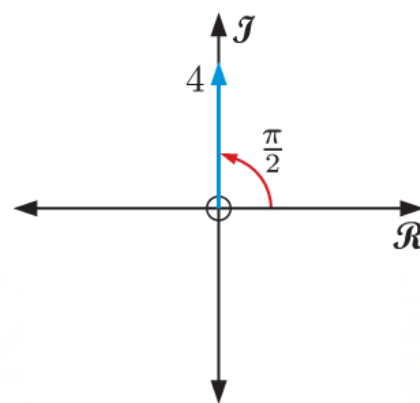
## EXERCISE 14D.1

**1 a**



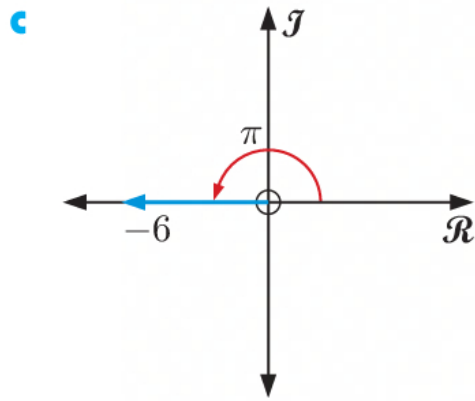
$$\begin{aligned}|4| &= 4 \\ \arg(4) &= 0 \\ \therefore 4 &= 4 \operatorname{cis} 0\end{aligned}$$

**b**



$$\begin{aligned}|4i| &= 4 \\ \arg(4i) &= \frac{\pi}{2} \\ \therefore 4i &= 4 \operatorname{cis} \frac{\pi}{2}\end{aligned}$$

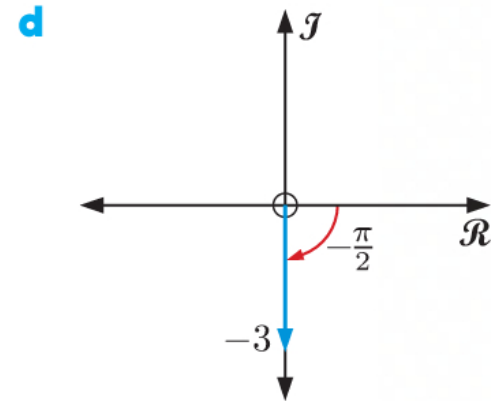




$$|-6| = 6$$

$$\arg(-6) = \pi$$

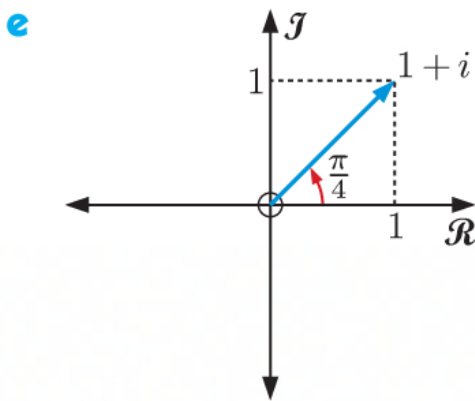
$$\therefore -6 = 6 \operatorname{cis} \pi$$



$$|-3i| = 3$$

$$\arg(-3i) = -\frac{\pi}{2}$$

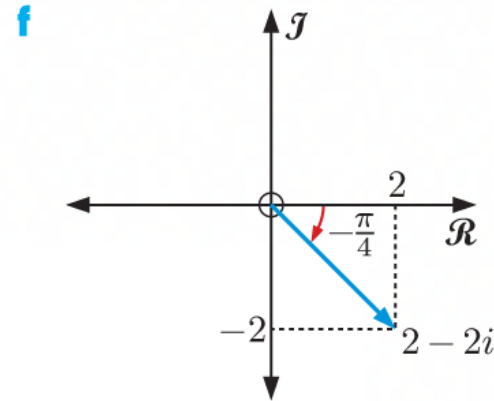
$$\therefore -3i = 3 \operatorname{cis} \left(-\frac{\pi}{2}\right)$$



$$|1 + i| = \sqrt{1 + 1} = \sqrt{2}$$

$$\arg(1 + i) = \frac{\pi}{4}$$

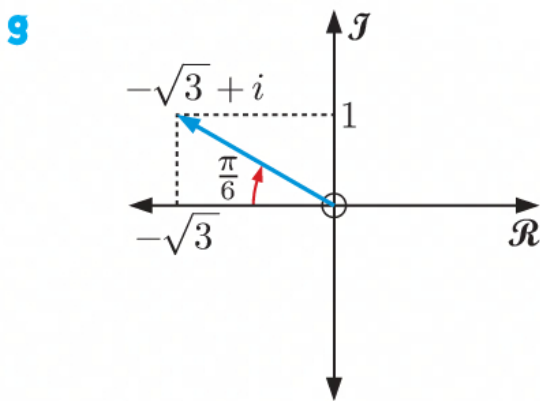
$$\therefore 1 + i = \sqrt{2} \operatorname{cis} \frac{\pi}{4}$$



$$|2 - 2i| = \sqrt{4 + 4} = \sqrt{8} = 2\sqrt{2}$$

$$\arg(2 - 2i) = -\frac{\pi}{4}$$

$$\therefore 2 - 2i = 2\sqrt{2} \operatorname{cis} \left(-\frac{\pi}{4}\right)$$



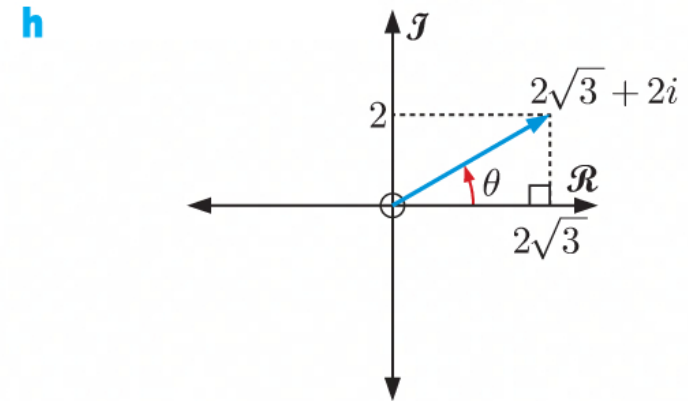
$$|-\sqrt{3} + i| = \sqrt{(\sqrt{3})^2 + 1^2}$$

$$= \sqrt{4} = 2$$

$$\arg(-\sqrt{3} + i) = \pi - \frac{\pi}{6}$$

$$= \frac{5\pi}{6}$$

$$\therefore -\sqrt{3} + i = 2 \operatorname{cis} \frac{5\pi}{6}$$



$$|2\sqrt{3} + 2i| = \sqrt{(2\sqrt{3})^2 + 2^2}$$

$$= \sqrt{12 + 4} = \sqrt{16} = 4$$

$$\tan \theta = \frac{2}{2\sqrt{3}}$$

$$\therefore \tan \theta = \frac{1}{\sqrt{3}}$$

$$\therefore \theta = \frac{\pi}{6}$$

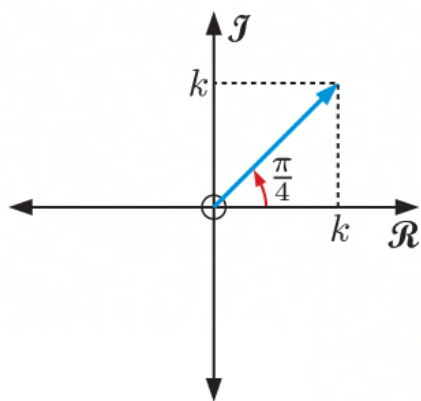
$$\arg(2\sqrt{3} + 2i) = \frac{\pi}{6}$$

$$\therefore 2\sqrt{3} + 2i = 4 \operatorname{cis} \frac{\pi}{6}$$

- 2**  $z = 0 = 0 + 0i$  cannot be written in polar form. The vector representing  $z$  has length zero, and an argument is not defined (no angle can be formed with the positive  $x$ -axis).

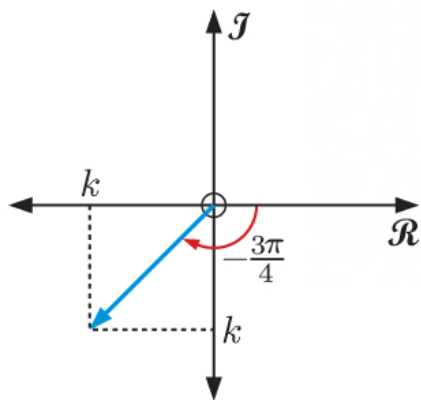
**3** If  $k = 0$  it is not possible. {from **2**}

$$\begin{aligned}\text{If } k > 0, \quad |z| &= \sqrt{k^2 + k^2} \\ &= \sqrt{2k^2} \\ &= k\sqrt{2} \\ \arg z &= \frac{\pi}{4} \\ \therefore z &= k\sqrt{2} \operatorname{cis} \frac{\pi}{4}\end{aligned}$$



$$\begin{aligned}\text{If } k < 0, \quad |z| &= \sqrt{k^2 + k^2} \\ &= \sqrt{2k^2} \\ &= |k|\sqrt{2}\end{aligned}$$

$$\begin{aligned}\text{Since } k < 0, \quad |z| &= -k\sqrt{2} \\ \arg z &= -\frac{3\pi}{4} \\ \therefore z &= -k\sqrt{2} \operatorname{cis} \left(-\frac{3\pi}{4}\right)\end{aligned}$$



**4 a**

$$\begin{aligned}2 \operatorname{cis} \frac{\pi}{2} \\ &= 2 \left( \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \right) \\ &= 2(0 + i \times 1) \\ &= 2i\end{aligned}$$

**c**

$$\begin{aligned}4 \operatorname{cis} \frac{\pi}{6} \\ &= 4 \left( \cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right) \\ &= 4 \left( \frac{\sqrt{3}}{2} + i \times \frac{1}{2} \right) \\ &= 2\sqrt{3} + 2i\end{aligned}$$

**e**

$$\begin{aligned}\sqrt{3} \operatorname{cis} \frac{2\pi}{3} \\ &= \sqrt{3} \left( \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \right) \\ &= \sqrt{3} \left( -\frac{1}{2} + i \times \frac{\sqrt{3}}{2} \right) \\ &= -\frac{\sqrt{3}}{2} + \frac{3}{2}i\end{aligned}$$

**5 a**

$$\begin{aligned}\operatorname{cis} \frac{3\pi}{4} \\ &= \cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \\ &= -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\end{aligned}$$

**c**

$$\begin{aligned}|\operatorname{cis} \theta| \\ &= |\cos \theta + i \sin \theta| \\ &= \sqrt{\cos^2 \theta + \sin^2 \theta} \\ &= \sqrt{1} = 1\end{aligned}$$

**b**

$$\begin{aligned}8 \operatorname{cis} \frac{\pi}{4} \\ &= 8 \left( \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) \\ &= 8 \left( \frac{1}{\sqrt{2}} + i \times \frac{1}{\sqrt{2}} \right) \\ &= 4\sqrt{2} + 4\sqrt{2}i\end{aligned}$$

**d**

$$\begin{aligned}\sqrt{2} \operatorname{cis} \left(-\frac{\pi}{4}\right) \\ &= \sqrt{2} \left[ \cos \left(-\frac{\pi}{4}\right) + i \sin \left(-\frac{\pi}{4}\right) \right] \\ &= \sqrt{2} \left[ \frac{1}{\sqrt{2}} + i \times -\frac{1}{\sqrt{2}} \right] \\ &= 1 - i\end{aligned}$$

**f**

$$\begin{aligned}5 \operatorname{cis} \pi \\ &= 5(\cos \pi + i \sin \pi) \\ &= 5(-1 + i \times 0) \\ &= -5\end{aligned}$$

**b**

$$\begin{aligned}\operatorname{cis} 0 \\ &= \cos 0 + i \sin 0 \\ &= 1\end{aligned}$$

**6** Using technology:

$$\begin{aligned}
 \text{a} \quad & \sqrt{3} \operatorname{cis}(2.5187) \\
 &= \sqrt{3} [\cos(2.5187) + i \sin(2.5187)] \\
 &\approx -1.41 + 1.01i
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & 2.83649 \operatorname{cis}(-2.68432) \\
 &= 2.83649 [\cos(-2.68432) + i \sin(-2.68432)] \\
 &\approx -2.55 - 1.25i
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & \sqrt{11} \operatorname{cis}\left(-\frac{3\pi}{8}\right) \\
 &= \sqrt{11} \left[ \cos\left(-\frac{3\pi}{8}\right) + i \sin\left(-\frac{3\pi}{8}\right) \right] \\
 &\approx 1.27 - 3.06i
 \end{aligned}$$

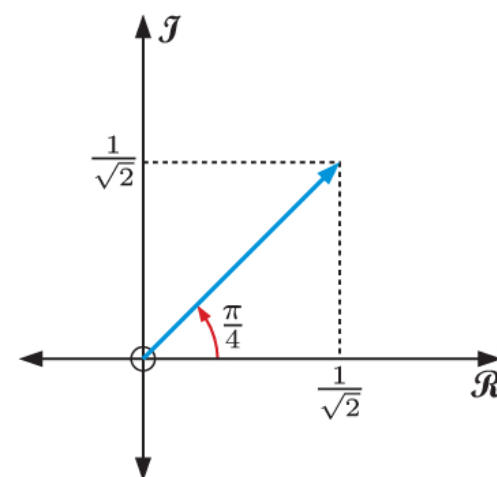
**7** Using technology:

$$\begin{aligned}
 \text{a} \quad & |3 - 4i| = 5 \\
 & \arg(3 - 4i) \approx -0.927 \\
 & \therefore 3 - 4i \approx 5 \operatorname{cis}(-0.927)
 \end{aligned}$$

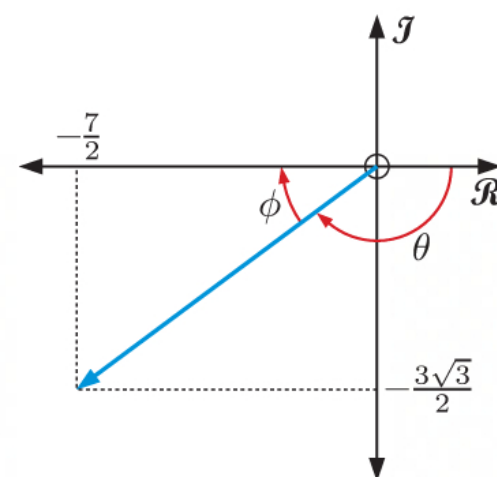
$$\begin{aligned}
 \text{b} \quad & |-5 - 12i| = 13 \\
 & \arg(-5 - 12i) \approx -1.97 \\
 & \therefore -5 - 12i \approx 13 \operatorname{cis}(-1.97)
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & |-11.6814 + 13.2697i| \approx 17.7 \\
 & \arg(-11.6814 + 13.2697i) \approx 2.29 \\
 & \therefore -11.6814 + 13.2697i \approx 17.7 \operatorname{cis}(2.29)
 \end{aligned}$$

$$\begin{aligned}
 \text{8 a} \quad & 3 \operatorname{cis} \frac{\pi}{4} + \operatorname{cis} \left(-\frac{3\pi}{4}\right) \\
 &= 3 \left[ \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right] + \left[ \cos \left(-\frac{3\pi}{4}\right) + i \sin \left(-\frac{3\pi}{4}\right) \right] \\
 &= 3 \left[ \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i \right] + \left[ -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i \right] \\
 &= \frac{3}{\sqrt{2}} + \frac{3}{\sqrt{2}} i - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i \\
 &= \frac{2}{\sqrt{2}} + \frac{2}{\sqrt{2}} i \\
 &= 2 \left( \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i \right) \\
 &= 2 \operatorname{cis} \frac{\pi}{4}
 \end{aligned}$$



$$\begin{aligned}
 \text{b} \quad & 2 \operatorname{cis} \frac{2\pi}{3} + 5 \operatorname{cis} \left(-\frac{2\pi}{3}\right) \\
 &= 2 \left[ \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \right] + 5 \left[ \cos \left(-\frac{2\pi}{3}\right) + i \sin \left(-\frac{2\pi}{3}\right) \right] \\
 &= 2 \left[ -\frac{1}{2} + i \frac{\sqrt{3}}{2} \right] + 5 \left[ -\frac{1}{2} - i \frac{\sqrt{3}}{2} \right] \\
 &= -1 + \sqrt{3}i - \frac{5}{2} - \frac{5\sqrt{3}}{2} i \\
 &= -\frac{7}{2} - \frac{3\sqrt{3}}{2} i
 \end{aligned}$$



$$\begin{aligned}
 \left| -\frac{7}{2} - \frac{3\sqrt{3}}{2} i \right| &= \sqrt{\left(-\frac{7}{2}\right)^2 + \left(\frac{3\sqrt{3}}{2}\right)^2} \\
 &= \sqrt{\frac{49}{4} + \frac{27}{4}} \\
 &= \sqrt{\frac{76}{4}} \\
 &= \sqrt{19}
 \end{aligned}$$

$$\therefore \sin \phi = \frac{\frac{3\sqrt{3}}{2}}{\sqrt{19}} = \frac{3\sqrt{3}}{2\sqrt{19}}$$

$$\therefore \phi = \sin^{-1} \left( \frac{3\sqrt{3}}{2\sqrt{19}} \right) \approx 0.639$$

$$\begin{aligned} \therefore \arg\left(-\frac{7}{2} - \frac{3\sqrt{3}}{2}i\right) &= -\pi + \phi \\ &\approx -2.50 \end{aligned}$$

$$\therefore 2 \operatorname{cis} \frac{2\pi}{3} + 5 \operatorname{cis} \left(-\frac{2\pi}{3}\right) \approx \sqrt{19} \operatorname{cis}(-2.50)$$

$$\begin{aligned} 9 \quad \operatorname{cis} \alpha \operatorname{cis} \beta &= (\cos \alpha + i \sin \alpha)(\cos \beta + i \sin \beta) \\ &= \cos \alpha \cos \beta + i \cos \alpha \sin \beta + i \sin \alpha \cos \beta + i^2 \sin \alpha \sin \beta \\ &= [\cos \alpha \cos \beta - \sin \alpha \sin \beta] + i [\sin \alpha \cos \beta + \sin \beta \cos \alpha] \\ &= \cos(\alpha + \beta) + i \sin(\alpha + \beta) \\ &= \operatorname{cis}(\alpha + \beta) \end{aligned}$$

## EXERCISE 14D.2

$$\begin{aligned} 1 \quad a \quad &\operatorname{cis} \theta \operatorname{cis} 2\theta \\ &= \operatorname{cis}(\theta + 2\theta) \\ &= \operatorname{cis} 3\theta \end{aligned}$$

$$\begin{aligned} c \quad &(\operatorname{cis} \theta)^3 \\ &= (\operatorname{cis} \theta)(\operatorname{cis} \theta)(\operatorname{cis} \theta) \\ &= (\operatorname{cis} 2\theta)(\operatorname{cis} \theta) \\ &= \operatorname{cis} 3\theta \end{aligned}$$

$$\begin{aligned} e \quad &2 \operatorname{cis} \frac{\pi}{12} \operatorname{cis} \frac{\pi}{6} \\ &= 2 \operatorname{cis} \left( \frac{\pi}{12} + \frac{\pi}{6} \right) \\ &= 2 \operatorname{cis} \frac{3\pi}{12} \\ &= 2 \operatorname{cis} \frac{\pi}{4} \\ &= 2 \left( \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) \\ &= 2 \left( \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i \right) \\ &= \sqrt{2} + i\sqrt{2} \end{aligned}$$

$$\begin{aligned} b \quad &\frac{\operatorname{cis} 3\theta}{\operatorname{cis} \theta} \\ &= \operatorname{cis}(3\theta - \theta) \\ &= \operatorname{cis} 2\theta \end{aligned}$$

$$\begin{aligned} d \quad &\operatorname{cis} \frac{\pi}{18} \operatorname{cis} \frac{\pi}{9} \\ &= \operatorname{cis} \left( \frac{\pi}{18} + \frac{\pi}{9} \right) \\ &= \operatorname{cis} \frac{3\pi}{18} \\ &= \operatorname{cis} \frac{\pi}{6} \\ &= \cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \\ &= \frac{\sqrt{3}}{2} + \frac{1}{2}i \end{aligned}$$

$$\begin{aligned} f \quad &2 \operatorname{cis} \frac{2\pi}{5} \times 4 \operatorname{cis} \frac{8\pi}{5} \\ &= 8 \operatorname{cis} \left( \frac{2\pi}{5} + \frac{8\pi}{5} \right) \\ &= 8 \operatorname{cis} \frac{10\pi}{5} \\ &= 8 \operatorname{cis} 2\pi \\ &= 8(\cos 2\pi + i \sin 2\pi) \\ &= 8(1) \\ &= 8 \end{aligned}$$



$$\begin{aligned}
 \text{g} \quad & \frac{4 \operatorname{cis} \frac{\pi}{12}}{2 \operatorname{cis} \frac{7\pi}{12}} \\
 &= 2 \operatorname{cis} \left( \frac{\pi}{12} - \frac{7\pi}{12} \right) \\
 &= 2 \operatorname{cis} \left( -\frac{6\pi}{12} \right) \\
 &= 2 \operatorname{cis} \left( -\frac{\pi}{2} \right) \\
 &= 2 \left[ \cos \left( -\frac{\pi}{2} \right) + i \sin \left( -\frac{\pi}{2} \right) \right] \\
 &= 2(-i) \\
 &= -2i
 \end{aligned}$$

$$\begin{aligned}
 \text{i} \quad & \left( \sqrt{2} \operatorname{cis} \frac{\pi}{8} \right)^4 \\
 &= \sqrt{2} \operatorname{cis} \frac{\pi}{8} \times \sqrt{2} \operatorname{cis} \frac{\pi}{8} \times \sqrt{2} \operatorname{cis} \frac{\pi}{8} \times \sqrt{2} \operatorname{cis} \frac{\pi}{8} \\
 &= (\sqrt{2})^4 \operatorname{cis} \left( \frac{\pi}{8} + \frac{\pi}{8} + \frac{\pi}{8} + \frac{\pi}{8} \right) \\
 &= 4 \operatorname{cis} \frac{\pi}{2} \\
 &= 4 \left( \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \right) \\
 &= 4i
 \end{aligned}$$

$$\begin{aligned}
 \text{h} \quad & \frac{\sqrt{32} \operatorname{cis} \frac{\pi}{8}}{\sqrt{2} \operatorname{cis} \left( -\frac{7\pi}{8} \right)} \\
 &= \frac{\sqrt{32}}{\sqrt{2}} \operatorname{cis} \left[ \frac{\pi}{8} - \left( -\frac{7\pi}{8} \right) \right] \\
 &= \sqrt{16} \operatorname{cis} \frac{8\pi}{8} \\
 &= 4 \operatorname{cis} \pi \\
 &= 4(\cos \pi + i \sin \pi) \\
 &= 4(-1) \\
 &= -4
 \end{aligned}$$

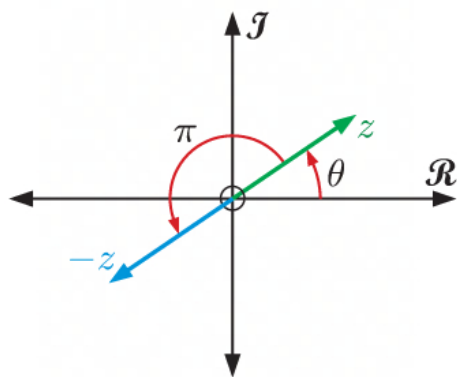
$$\begin{aligned}
 \text{2 a} \quad & \operatorname{cis} 17\pi \\
 &= \operatorname{cis} (\pi + 8 \times 2\pi) \\
 &= \operatorname{cis} \pi \\
 &= -1
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & \operatorname{cis}(-37\pi) \\
 &= \operatorname{cis} (\pi - 19 \times 2\pi) \\
 &= \operatorname{cis} \pi \\
 &= -1
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & \operatorname{cis} \frac{91\pi}{3} \\
 &= \operatorname{cis} \left( \frac{\pi}{3} + 15 \times 2\pi \right) \\
 &= \operatorname{cis} \frac{\pi}{3} \\
 &= \frac{1}{2} + \frac{\sqrt{3}}{2}i
 \end{aligned}$$

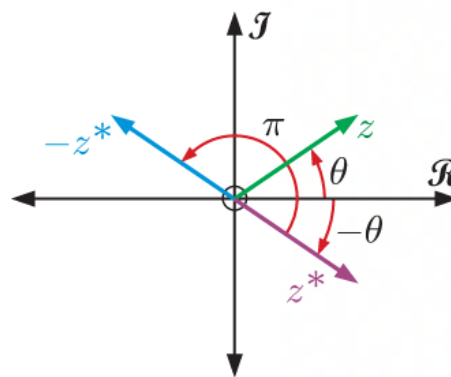
$$\begin{aligned}
 \text{3 a} \quad & z = 2 \operatorname{cis} \theta \\
 & |z| = 2 \\
 & \arg z = \theta
 \end{aligned}$$

$$\text{c} \quad -z = 2 \operatorname{cis} (\theta + \pi)$$



$$\text{b} \quad z^* = 2 \operatorname{cis} (-\theta)$$

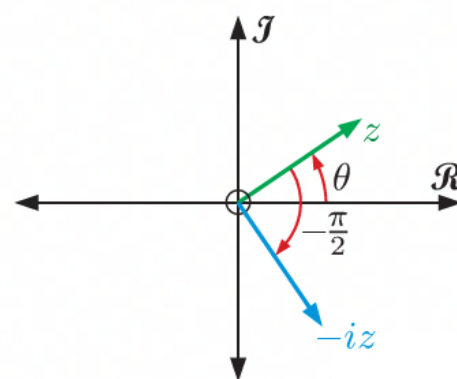
$$\text{d} \quad -z^* = 2 \operatorname{cis} (\pi - \theta)$$



$$\begin{aligned}
 \text{4 a} \quad & -i = 1 \operatorname{cis} \left( -\frac{\pi}{2} \right) \\
 &= \operatorname{cis} \left( -\frac{\pi}{2} \right)
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & -iz = \operatorname{cis} \left( -\frac{\pi}{2} \right) \times r \operatorname{cis} \theta \\
 &= r \operatorname{cis} \left( \theta - \frac{\pi}{2} \right)
 \end{aligned}$$

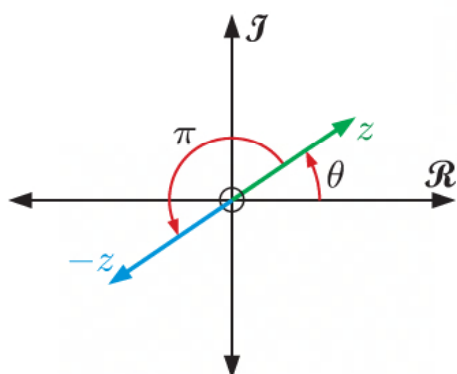
- c**  $z$  is rotated clockwise through  $\frac{\pi}{2}$  about the origin.



**5 a**  $-1 = \text{cis } \pi$

**b**  $-z = \text{cis } \pi \times r \text{cis } \theta$   
 $= r \text{cis}(\theta + \pi)$

- c**  $z$  is rotated anticlockwise through  $\pi$  about the origin.



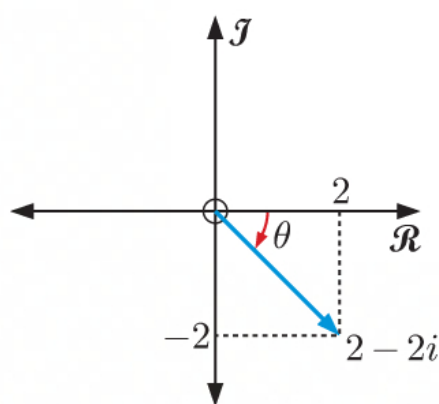
- 6**  $z^*$  is the reflection of  $z$  in the real axis.

$$\therefore |z^*| = |z| = r \quad \text{and} \quad \arg(z^*) = -\arg z = -\theta$$

$$\therefore z^* = r \text{cis}(-\theta)$$

### EXERCISE 14D.3

**1 a**



If  $z = 2 - 2i$ , then  $|z| = \sqrt{2^2 + (-2)^2} = \sqrt{8} = 2\sqrt{2}$

Now  $\tan \theta = \frac{-2}{2} = -1$

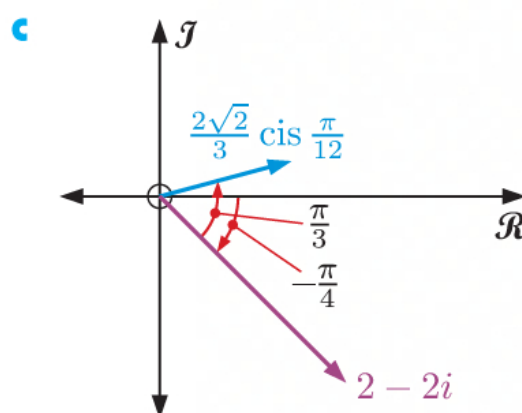
$$\therefore \theta = -\frac{\pi}{4}$$

$$\therefore \arg z = -\frac{\pi}{4}$$

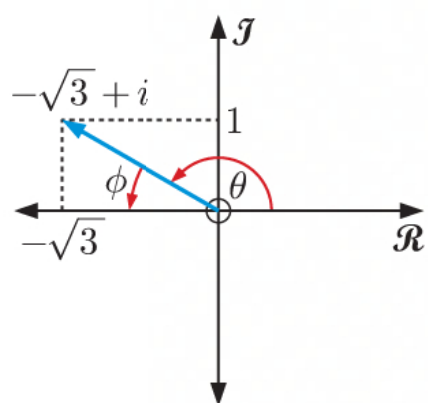
$$\therefore z = 2\sqrt{2} \text{cis} \left(-\frac{\pi}{4}\right)$$

**b**

$$\begin{aligned} & z \times \frac{1}{3} \text{cis } \frac{\pi}{3} \\ &= 2\sqrt{2} \text{cis} \left(-\frac{\pi}{4}\right) \times \frac{1}{3} \text{cis } \frac{\pi}{3} \\ &= \frac{2\sqrt{2}}{3} \text{cis} \left(-\frac{\pi}{4} + \frac{\pi}{3}\right) \\ &= \frac{2\sqrt{2}}{3} \text{cis } \frac{\pi}{12} \end{aligned}$$



- d** When  $z$  is multiplied by  $\frac{1}{3} \text{cis } \frac{\pi}{3}$ , it is dilated with scale factor  $\frac{1}{3}$ , then rotated anticlockwise through  $\frac{\pi}{3}$  about the origin.

**2 a**

If  $z = -\sqrt{3} + i$ , then  $|z| = \sqrt{(-\sqrt{3})^2 + 1^2} = \sqrt{4} = 2$

Now  $\tan \phi = \frac{1}{\sqrt{3}}$

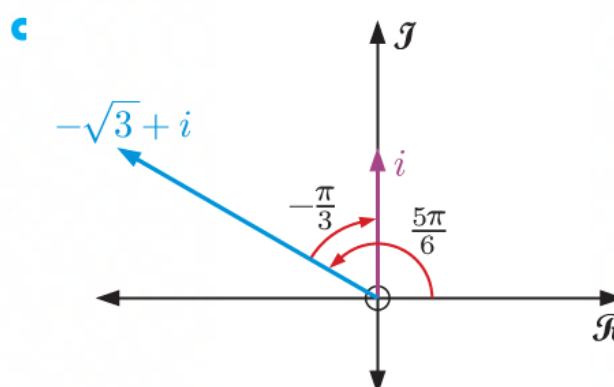
$$\therefore \phi = \frac{\pi}{6}$$

But  $\theta = \pi - \phi$

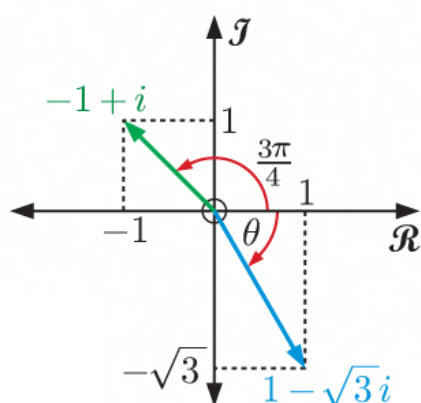
$$\therefore \arg z = \frac{5\pi}{6}$$

$$\therefore z = 2 \operatorname{cis} \frac{5\pi}{6}$$

$$\begin{aligned} \text{b} \quad & z \times \frac{1}{2} \operatorname{cis} \left(-\frac{\pi}{3}\right) \\ &= 2 \operatorname{cis} \frac{5\pi}{6} \times \frac{1}{2} \operatorname{cis} \left(-\frac{\pi}{3}\right) \\ &= \operatorname{cis} \left(\frac{5\pi}{6} - \frac{\pi}{3}\right) \\ &= \operatorname{cis} \frac{\pi}{2} \\ &= i \end{aligned}$$



**d** When  $z$  is multiplied by  $\frac{1}{2} \operatorname{cis} \left(-\frac{\pi}{3}\right)$ , it is dilated with scale factor  $\frac{1}{2}$ , then rotated clockwise through  $\frac{\pi}{3}$  about the origin.

**3 a**

If  $z = -1 + i$ , then  $|z| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$

$$\arg z = \frac{3\pi}{4}$$

$$\therefore z = \sqrt{2} \operatorname{cis} \frac{3\pi}{4}$$

If  $w = 1 - \sqrt{3}i$ , then  $|w| = \sqrt{1^2 + (-\sqrt{3})^2} = \sqrt{4} = 2$

Now  $\tan \theta = \frac{\sqrt{3}}{1} = \sqrt{3}$

$$\therefore \theta = \frac{\pi}{3}$$

$$\therefore \arg w = -\frac{\pi}{3} \quad \{\operatorname{Re}(z) > 0, \operatorname{Im}(z) < 0\}$$

$$\therefore w = 2 \operatorname{cis} \left(-\frac{\pi}{3}\right)$$

$$\begin{aligned} \text{b} \quad & zw = \sqrt{2} \operatorname{cis} \frac{3\pi}{4} \times 2 \operatorname{cis} \left(-\frac{\pi}{3}\right) \\ &= 2\sqrt{2} \operatorname{cis} \left(\frac{3\pi}{4} - \frac{\pi}{3}\right) \\ &= 2\sqrt{2} \operatorname{cis} \frac{5\pi}{12} \end{aligned}$$

**c** When  $z$  is multiplied by  $w$ , it is dilated with scale factor 2, then rotated clockwise through  $\frac{\pi}{3}$  about the origin.

$$\begin{aligned}
4 \quad a \quad & \cos \frac{\pi}{12} + i \sin \frac{\pi}{12} \\
&= \operatorname{cis} \frac{\pi}{12} \\
&= \operatorname{cis} \left( \frac{\pi}{3} - \frac{\pi}{4} \right) \\
&= \operatorname{cis} \frac{\pi}{3} \times \operatorname{cis} \left( -\frac{\pi}{4} \right) \quad \{ \operatorname{cis}(\theta + \phi) = \operatorname{cis} \theta \times \operatorname{cis} \phi \} \\
&= \left[ \cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right] \times \left[ \cos \left( -\frac{\pi}{4} \right) + i \sin \left( -\frac{\pi}{4} \right) \right] \\
&= \left( \frac{1}{2} + \frac{\sqrt{3}}{2} i \right) \left( \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i \right) \\
&= \frac{1}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} i + \frac{\sqrt{3}}{2\sqrt{2}} i - \frac{\sqrt{3}}{2\sqrt{2}} i^2 \\
&= \left( \frac{1}{2\sqrt{2}} + \frac{\sqrt{3}}{2\sqrt{2}} \right) + i \left( \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} \right)
\end{aligned}$$

$$\text{Equating real parts:} \quad \cos \frac{\pi}{12} = \left( \frac{1 + \sqrt{3}}{2\sqrt{2}} \right) \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2} + \sqrt{6}}{4}$$

$$\text{Equating imaginary parts:} \quad \sin \frac{\pi}{12} = \left( \frac{\sqrt{3} - 1}{2\sqrt{2}} \right) \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{6} - \sqrt{2}}{4}$$

$$\begin{aligned}
b \quad & \cos \frac{11\pi}{12} + i \sin \frac{11\pi}{12} \\
&= \operatorname{cis} \frac{11\pi}{12} \\
&= \operatorname{cis} \left( \frac{3\pi}{12} + \frac{8\pi}{12} \right) \\
&= \operatorname{cis} \left( \frac{\pi}{4} + \frac{2\pi}{3} \right) \\
&= \operatorname{cis} \frac{\pi}{4} \times \operatorname{cis} \frac{2\pi}{3} \quad \{ \operatorname{cis}(\theta + \phi) = \operatorname{cis} \theta \times \operatorname{cis} \phi \} \\
&= \left( \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) \times \left( \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \right) \\
&= \left( \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i \right) \left( -\frac{1}{2} + \frac{\sqrt{3}}{2} i \right) \\
&= -\frac{1}{2\sqrt{2}} + \frac{\sqrt{3}}{2\sqrt{2}} i - \frac{1}{2\sqrt{2}} i + \frac{\sqrt{3}}{2\sqrt{2}} i^2 \\
&= \left( -\frac{1}{2\sqrt{2}} - \frac{\sqrt{3}}{2\sqrt{2}} \right) + i \left( \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} \right)
\end{aligned}$$

$$\text{Equating real parts:} \quad \cos \frac{11\pi}{12} = \left( \frac{-1 - \sqrt{3}}{2\sqrt{2}} \right) \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{-\sqrt{2} - \sqrt{6}}{4}$$

$$\text{Equating imaginary parts:} \quad \sin \frac{11\pi}{12} = \left( \frac{\sqrt{3} - 1}{2\sqrt{2}} \right) \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{6} - \sqrt{2}}{4}$$

5  $P_n$  is:  $\arg(z^n) = n \arg z$  for all  $n \in \mathbb{Z}^+$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ ,  $\arg(z^1) = \arg z$

$\therefore P_1$  is true.

(2) If  $P_k$  is true, then  $\arg(z^k) = k \arg z$

Now  $\arg(z^{k+1}) = \arg(z^k z)$

$$= \arg(z^k) + \arg z \quad \{ \text{as } \arg(zw) = \arg z + \arg w \}$$

$$= k \arg z + \arg z \quad \{ \text{using } P_k \}$$

$$= (k + 1) \arg z$$

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}



6 Let  $z = R \operatorname{cis} \theta$  and  $w = r \operatorname{cis} \phi$ ,  $w \neq 0$

$$\frac{z}{w} = \frac{R \operatorname{cis} \theta}{r \operatorname{cis} \phi} = \frac{R}{r} \operatorname{cis} (\theta - \phi)$$

$$\therefore \left| \frac{z}{w} \right| = \frac{R}{r} = \frac{|z|}{|w|}$$

and  $\arg \left( \frac{z}{w} \right) = \theta - \phi = \arg z - \arg w$  if  $w \neq 0$

7 a  $z = 3 \operatorname{cis} \theta$

$$\begin{aligned} \therefore -z &= (-1) \times 3 \operatorname{cis} \theta \\ &= \operatorname{cis} \pi \times 3 \operatorname{cis} \theta \quad \text{or} \quad \operatorname{cis}(-\pi) \times 3 \operatorname{cis} \theta \\ &= 3 \operatorname{cis}(\theta \pm \pi) \end{aligned}$$

$$\therefore |-z| = 3$$

$$\therefore \arg(-z) = \theta \pm \pi$$

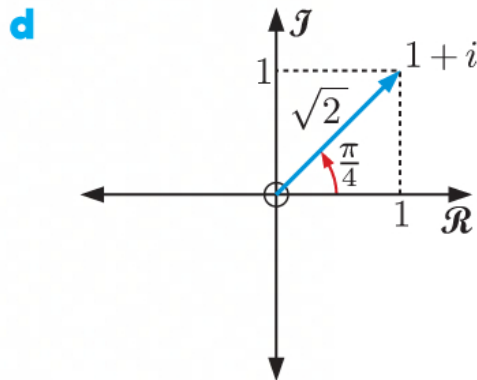
b  $z^* = 3 \operatorname{cis}(-\theta)$

$$\therefore |z^*| = 3 \quad \text{and} \quad \arg(z^*) = -\theta$$

c  $iz = i \times 3 \operatorname{cis} \theta$

$$\begin{aligned} &= \operatorname{cis} \frac{\pi}{2} \times 3 \operatorname{cis} \theta \\ &= 3 \operatorname{cis} \left( \theta + \frac{\pi}{2} \right) \end{aligned}$$

$$\therefore |iz| = 3 \quad \text{and} \quad \arg(iz) = \theta + \frac{\pi}{2} \quad \{\text{since } \theta \text{ is acute, } -\pi < \theta + \frac{\pi}{2} \leq \pi\}$$



$$1 + i = \sqrt{2} \operatorname{cis} \frac{\pi}{4}$$

$$\begin{aligned} \therefore (1+i)z &= \sqrt{2} \operatorname{cis} \frac{\pi}{4} \times 3 \operatorname{cis} \theta \\ &= 3\sqrt{2} \operatorname{cis} \left( \theta + \frac{\pi}{4} \right) \end{aligned}$$

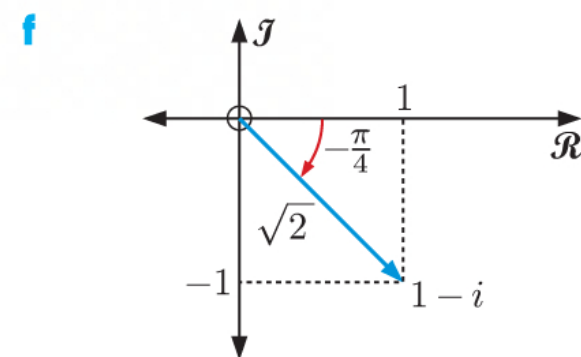
$$\therefore |(1+i)z| = 3\sqrt{2} \quad \text{and} \quad \arg[(1+i)z] = \theta + \frac{\pi}{4}$$

{since  $\theta$  is acute,  $-\pi < \theta + \frac{\pi}{2} \leq \pi$ }

e  $\frac{z}{i} = \frac{3 \operatorname{cis} \theta}{\operatorname{cis} \frac{\pi}{2}}$

$$= 3 \operatorname{cis} \left( \theta - \frac{\pi}{2} \right)$$

$$\therefore \left| \frac{z}{i} \right| = 3 \quad \text{and} \quad \arg \left( \frac{z}{i} \right) = \theta - \frac{\pi}{2} \quad \{\text{since } \theta \text{ is acute, } -\pi < \theta - \frac{\pi}{2} \leq \pi\}$$



$$1 - i = \sqrt{2} \operatorname{cis} \left( -\frac{\pi}{4} \right)$$

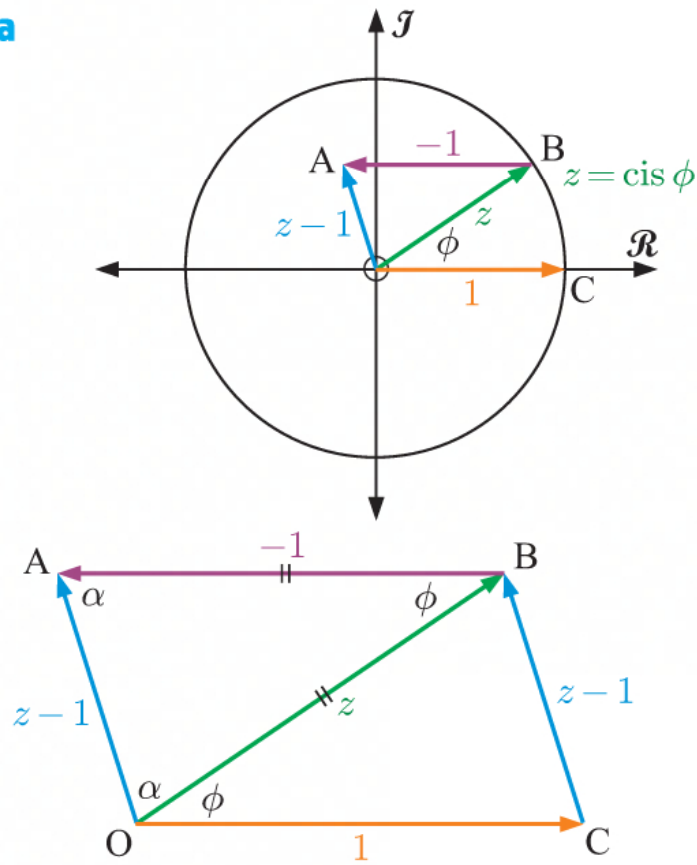
$$\begin{aligned} \therefore \frac{z}{1-i} &= \frac{3 \operatorname{cis} \theta}{\sqrt{2} \operatorname{cis} \left( -\frac{\pi}{4} \right)} \\ &= \frac{3}{\sqrt{2}} \operatorname{cis} \left( \theta + \frac{\pi}{4} \right) \end{aligned}$$

$$\therefore \left| \frac{z}{1-i} \right| = \frac{3}{\sqrt{2}}$$

$$\text{and} \quad \arg \left( \frac{z}{1-i} \right) = \theta + \frac{\pi}{4}$$

$$\{\text{since } \theta \text{ is acute, } -\pi < \theta + \frac{\pi}{4} \leq \pi\}$$

8 a



$|z| = 1$ , so  $z$  ends on the unit circle.  
 $z - 1$  is the vector  $\overrightarrow{OA}$  shown.

Now OACB is a parallelogram.

$\widehat{ABO} = \phi$  {alternate angles}

$OB = AB = 1$

$\therefore \triangle ABO$  is isosceles.

$\therefore \widehat{BAO} = \widehat{AOB} = \alpha$

$\therefore 2\alpha + \phi = \pi$

$\therefore \alpha = \frac{\pi - \phi}{2}$

$\therefore \arg(z - 1) = \frac{\pi - \phi}{2} + \phi$

$= \frac{\pi}{2} - \frac{\phi}{2} + \phi$

$= \frac{\pi}{2} + \frac{\phi}{2} \dots (1)$

{since  $\phi$  is acute,  $-\pi < \frac{\pi}{2} + \frac{\phi}{2} \leq \pi$ }

Using the cosine rule in  $\triangle ABO$ :

$OA^2 = 1^2 + 1^2 - 2(1)(1)\cos\phi$

$\therefore OA^2 = 2 - 2\cos\phi$

$\therefore OA^2 = 2 - 2\left(1 - 2\sin^2\left(\frac{\phi}{2}\right)\right)$

$\therefore OA^2 = 2 - 2 + 4\sin^2\left(\frac{\phi}{2}\right)$

$\therefore OA^2 = 4\sin^2\left(\frac{\phi}{2}\right)$

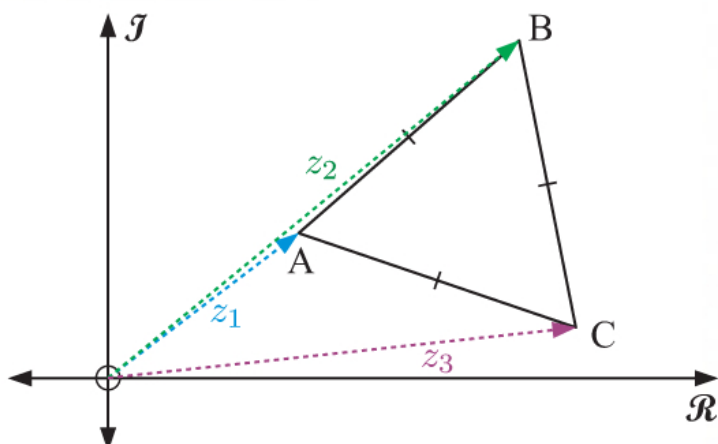
$\therefore OA = 2\sin\frac{\phi}{2}$  {as  $OA > 0$ }

$\therefore |z - 1| = 2\sin\frac{\phi}{2} \dots (2)$

b  $z - 1 = 2\sin\frac{\phi}{2} \operatorname{cis}\left(\frac{\pi}{2} + \frac{\phi}{2}\right)$  {using (1) and (2) in a}

$(z - 1)^* = 2\sin\frac{\phi}{2} \operatorname{cis}\left(-\frac{\pi}{2} - \frac{\phi}{2}\right)$

9 a



$z_2 - z_1 = \overrightarrow{AB}$

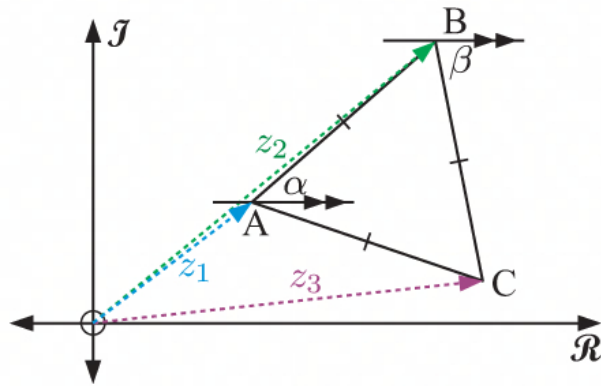
$z_3 - z_2 = \overrightarrow{BC}$

$$\begin{aligned} \text{b} \quad \left| \frac{z_2 - z_1}{z_3 - z_2} \right| &= \frac{|z_2 - z_1|}{|z_3 - z_2|} \\ &= \frac{|\overrightarrow{AB}|}{|\overrightarrow{BC}|} \end{aligned}$$

But  $\triangle ABC$  is equilateral

$$\therefore |\overrightarrow{AB}| = |\overrightarrow{BC}|$$

$$\therefore \left| \frac{z_2 - z_1}{z_3 - z_2} \right| = 1$$



$$\text{Let } \arg(z_2 - z_1) = \alpha$$

$$\text{and } \arg(z_3 - z_2) = -\beta \text{ as shown}$$

$$\begin{aligned} \therefore \arg\left(\frac{z_2 - z_1}{z_3 - z_2}\right) &= \arg(z_2 - z_1) - \arg(z_3 - z_2) \\ &= \alpha - (-\beta) \\ &= \alpha + \beta \end{aligned}$$

But  $\widehat{ABC} = \frac{\pi}{3}$  since the triangle is equilateral

$$\therefore \alpha + \beta + \frac{\pi}{3} = \pi \quad \{\text{co-interior angles}\}$$

$$\therefore \alpha + \beta = \frac{2\pi}{3}$$

$$\therefore \arg\left(\frac{z_2 - z_1}{z_3 - z_2}\right) = \frac{2\pi}{3}$$

$$\text{c From b, } \frac{z_2 - z_1}{z_3 - z_2} = 1 \operatorname{cis} \frac{2\pi}{3}$$

$$\begin{aligned} \therefore \left(\frac{z_2 - z_1}{z_3 - z_2}\right)^3 &= \left(\operatorname{cis} \frac{2\pi}{3}\right)^3 \\ &= \operatorname{cis} \frac{2\pi}{3} \times \operatorname{cis} \frac{2\pi}{3} \times \operatorname{cis} \frac{2\pi}{3} \\ &= \operatorname{cis} \left(\frac{2\pi}{3} + \frac{2\pi}{3} + \frac{2\pi}{3}\right) \\ &= \operatorname{cis} 2\pi \\ &= 1 \end{aligned}$$

$$\begin{aligned} \text{10 a Sum of roots} &= 2 \operatorname{cis} \frac{2\pi}{3} + 2 \operatorname{cis} \frac{4\pi}{3} \\ &= 2 \left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}\right) + 2 \left(\cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3}\right) \\ &= 2\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) + 2\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) \\ &= -1 + \sqrt{3}i - 1 - \sqrt{3}i \\ &= -2 \end{aligned}$$

$$\begin{aligned} \text{Product of roots} &= 2 \operatorname{cis} \frac{2\pi}{3} \times 2 \operatorname{cis} \frac{4\pi}{3} \\ &= 4 \operatorname{cis} \left(\frac{2\pi}{3} + \frac{4\pi}{3}\right) \\ &= 4 \operatorname{cis} 2\pi \\ &= 4(1 + 0i) \\ &= 4 \end{aligned}$$

$$\therefore \text{the equations are } a(x^2 - (-2)x + 4) = 0$$

$$\therefore a(x^2 + 2x + 4) = 0, \quad a \neq 0, \quad a \in \mathbb{R}.$$

$$\begin{aligned}
 \text{b Sum of roots} &= \sqrt{2} \operatorname{cis} \frac{\pi}{4} + \sqrt{2} \operatorname{cis} \left(-\frac{\pi}{4}\right) \\
 &= \sqrt{2} \left[ \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right] + \sqrt{2} \left[ \cos \left(-\frac{\pi}{4}\right) + i \sin \left(-\frac{\pi}{4}\right) \right] \\
 &= \sqrt{2} \left( \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i \right) + \sqrt{2} \left( \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i \right) \\
 &= 1 + i + 1 - i \\
 &= 2
 \end{aligned}$$

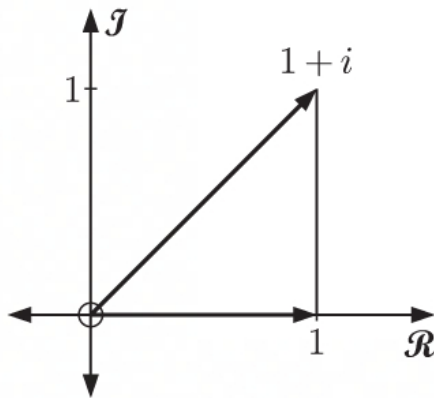
$$\begin{aligned}
 \text{Product of roots} &= \sqrt{2} \operatorname{cis} \frac{\pi}{4} \times \sqrt{2} \operatorname{cis} \left(-\frac{\pi}{4}\right) \\
 &= 2 \operatorname{cis} \left( \frac{\pi}{4} - \frac{\pi}{4} \right) \\
 &= 2 \operatorname{cis} 0 \\
 &= 2(1 + 0i) \\
 &= 2
 \end{aligned}$$

$\therefore$  the equations are  $a(x^2 - 2x + 2) = 0$ ,  $a \neq 0$ ,  $a \in \mathbb{R}$ .

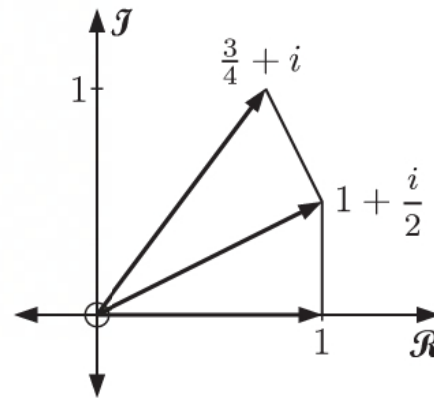
## INVESTIGATION

## EULER'S FORM

**1 a i** Sequence:  $1, \left(1 + \frac{i}{1}\right)$   
 $= 1, 1 + i$

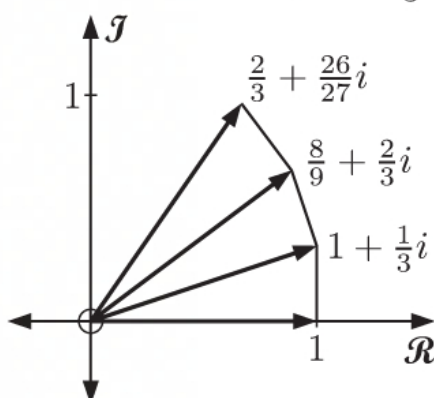


**ii** Sequence:  $1, \left(1 + \frac{i}{2}\right), \left(1 + \frac{i}{2}\right)^2$   
 $= 1, 1 + \frac{i}{2}, \frac{3}{4} + i$



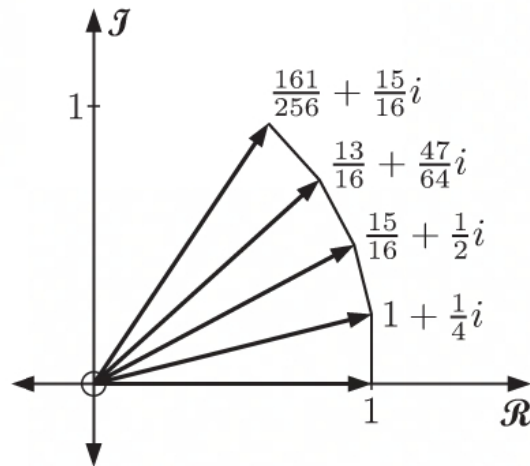
For the following parts, we have calculated the real and imaginary parts using the binomial expansion  $\left(1 + \frac{i}{n}\right)^n = \sum_{r=0}^n \binom{n}{r} \left(\frac{1}{n}\right)^r i^r$ , and the fact that  $i^0 = 1$ ,  $i^1 = i$ ,  $i^2 = -1$ ,  $i^3 = -i$ , and  $i^4 = 1$ .

**iii** Sequence:  $1, \left(1 + \frac{i}{3}\right), \left(1 + \frac{i}{3}\right)^2, \left(1 + \frac{i}{3}\right)^3$   
 $= 1, 1 + \frac{1}{3}i, \frac{8}{9} + \frac{2}{3}i, \frac{2}{3} + \frac{26}{27}i$

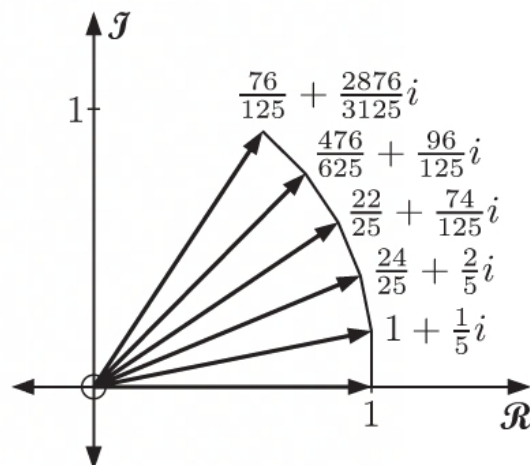




**iv** Sequence:  $1, \left(1 + \frac{i}{4}\right), \left(1 + \frac{i}{4}\right)^2, \left(1 + \frac{i}{4}\right)^3, \left(1 + \frac{i}{4}\right)^4$   
 $= 1, 1 + \frac{1}{4}i, \frac{15}{16} + \frac{1}{2}i, \frac{13}{16} + \frac{47}{64}i, \frac{161}{256} + \frac{15}{16}i$



**v** Sequence:  $1, \left(1 + \frac{i}{5}\right), \left(1 + \frac{i}{5}\right)^2, \left(1 + \frac{i}{5}\right)^3, \left(1 + \frac{i}{5}\right)^4, \left(1 + \frac{i}{5}\right)^5$   
 $= 1, 1 + \frac{1}{5}i, \frac{24}{25} + \frac{2}{5}i, \frac{22}{25} + \frac{74}{125}i, \frac{476}{625} + \frac{96}{125}i,$



**b i** As  $n$  gets large, the line segments joining successive complex numbers begin to form an arc of the unit circle.

**ii**  $\cos 1^\circ \approx 0.54030$  and  $\sin 1^\circ \approx 0.84147$  {using technology}

**iii** From the software,  $\left(1 + \frac{i}{500}\right)^{500} \approx 0.5408 + 0.8423i$

**iv** As  $n \rightarrow \infty$ ,  $\left(1 + \frac{i}{n}\right)^n \rightarrow (\cos 1^\circ) + (\sin 1^\circ)i = e^i$  {Euler}

**2 a i** As  $n$  gets large, the line segments joining successive complex numbers begin to form an arc of the unit circle, about twice as large as the arc in question 1.

**ii**  $\cos 2^\circ \approx -0.41615$  and  $\sin 2^\circ \approx 0.90930$  {using technology}

**iii** As  $n \rightarrow \infty$ ,  $\left(1 + \frac{2i}{n}\right)^n \rightarrow (\cos 2^\circ) + (\sin 2^\circ)i = e^{2i}$  {Euler}

**b i** As  $n \rightarrow \infty$ ,  $\left(1 + \frac{i \frac{\pi}{2}}{n}\right)^n \rightarrow 0 + 1 \times i = e^{i \frac{\pi}{2}}$  {Euler}

**ii** Multiplying 1 by  $e^{i \frac{\pi}{2}}$  rotates it anticlockwise about the origin by  $\frac{\pi}{2}$ , similarly to how multiplying  $z$  by  $\text{cis } \frac{\pi}{2}$  rotates it anticlockwise about the origin by  $\frac{\pi}{2}$ .

**3 a** The effect of *real* compound growth is to increase the size or modulus of a quantity. So, when we multiply a complex number  $z$  by  $e^r$ , the vector representing  $z$  is *enlarged* with scale factor  $e^r$ . This means the *modulus* of  $z$  is multiplied by  $e^r$ .

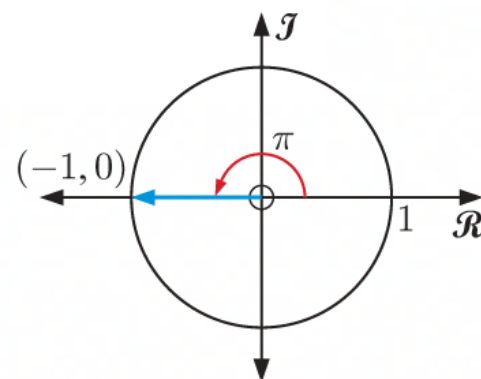
- b** The effect of *imaginary* compound growth is to rotate the quantity in the complex plane. So, when we multiply a complex number  $z$  by  $\text{cis } \theta = e^{i\theta}$ ,  $z$  is *rotated* anticlockwise through angle  $\theta$  about the origin. This means the *argument* of  $z$  is increased by  $\theta$ .
- 4** When  $z$  is multiplied by  $w$ , the vector representing  $z$  is enlarged with scale factor  $|w|$ , and is rotated anticlockwise through angle  $\phi$  about the origin.

## EXERCISE 14E

**Note:** We assume that all arguments are in  $]-\pi, \pi]$ .

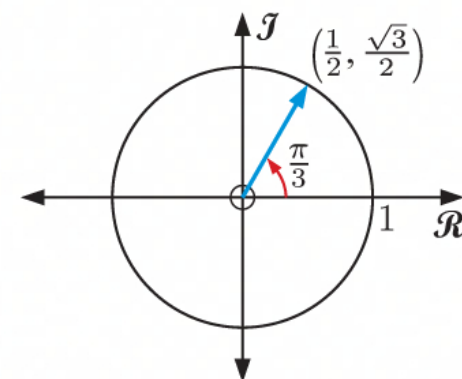
**1 a** 
$$\begin{aligned} e^{i\pi} &= \cos \pi + i \sin \pi \\ &= -1 \end{aligned}$$

We expand  $\text{cis } \pi$  using a unit circle diagram.



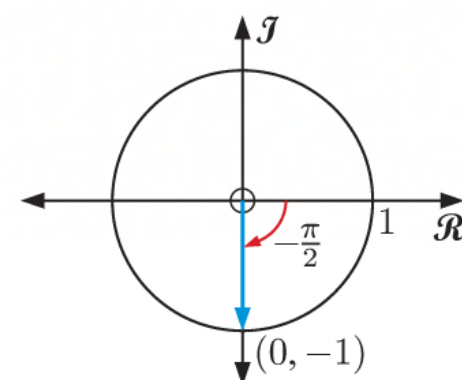
**b** 
$$\begin{aligned} e^{i\frac{\pi}{3}} &= \cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \\ &= \frac{1}{2} + \frac{\sqrt{3}}{2}i \end{aligned}$$

We expand  $\text{cis } \frac{\pi}{3}$  using a unit circle diagram.



**c** 
$$\begin{aligned} e^{-i\frac{\pi}{2}} &= \cos \left(-\frac{\pi}{2}\right) + i \sin \left(-\frac{\pi}{2}\right) \\ &= -i \end{aligned}$$

We expand  $\text{cis } \left(-\frac{\pi}{2}\right)$  using a unit circle diagram.



**2 a** 
$$\begin{aligned} |-1 + i| &= \sqrt{2}, \quad \arg(-1 + i) = \frac{3\pi}{4} \\ \therefore -1 + i &= \sqrt{2} \text{cis } \frac{3\pi}{4} \quad (\text{polar form}) \\ &= \sqrt{2}e^{i\frac{3\pi}{4}} \quad (\text{Euler form}) \end{aligned}$$

**b** 
$$\begin{aligned} 3 \text{cis } \left(-\frac{\pi}{6}\right) &= 3 \cos \left(-\frac{\pi}{6}\right) + 3i \sin \left(-\frac{\pi}{6}\right) \\ &= \frac{3\sqrt{3}}{2} - \frac{3}{2}i \quad (\text{Cartesian form}) \\ &= 3e^{-i\frac{\pi}{6}} \quad (\text{Euler form}) \end{aligned}$$

**c** 
$$\begin{aligned} 2e^{i\frac{2\pi}{3}} &= 2 \cos \frac{2\pi}{3} + 2i \sin \frac{2\pi}{3} \\ &= -1 + i\sqrt{3} \quad (\text{Cartesian form}) \\ &= 2 \text{cis } \frac{2\pi}{3} \quad (\text{polar form}) \end{aligned}$$

**3 a** 
$$\begin{aligned} \text{cis } \theta \text{cis } \phi &= e^{i\theta}e^{i\phi} \\ &= e^{i(\theta+\phi)} \\ &= \text{cis}(\theta + \phi) \end{aligned}$$

**b** 
$$\begin{aligned} \frac{\text{cis } \theta}{\text{cis } \phi} &= \frac{e^{i\theta}}{e^{i\phi}} \\ &= e^{i(\theta-\phi)} \\ &= \text{cis}(\theta - \phi) \end{aligned}$$

**4**  $z = \text{cis } \theta$

**a**  $\sqrt{z} = (\text{cis } \theta)^{\frac{1}{2}}$

$$= (e^{i\theta})^{\frac{1}{2}}$$

$$= e^{\frac{i\theta}{2}}$$

$$= \text{cis } \frac{\theta}{2}$$

$$\therefore \arg(\sqrt{z}) = \frac{\theta}{2}$$

**c**  $-iz^{\frac{2}{5}} = \text{cis } \left(-\frac{\pi}{2}\right) \times (\text{cis } \theta)^{\frac{2}{5}}$

$$= e^{-i\frac{\pi}{2}} \times (e^{i\theta})^{\frac{2}{5}}$$

$$= e^{-i\frac{\pi}{2}} \times e^{i\frac{2\theta}{5}}$$

$$= e^{i(\frac{2\theta}{5} - \frac{\pi}{2})}$$

$$= \text{cis } \left(\frac{2\theta}{5} - \frac{\pi}{2}\right)$$

$$\therefore \arg(-iz^{\frac{2}{5}}) = \frac{2\theta}{5} - \frac{\pi}{2}$$

**b**  $iz^4 = \text{cis } \frac{\pi}{2} \times (\text{cis } \theta)^4$

$$= e^{i\frac{\pi}{2}} \times (e^{i\theta})^4$$

$$= e^{i\frac{\pi}{2}} \times e^{4i\theta}$$

$$= e^{i(4\theta + \frac{\pi}{2})}$$

$$= \text{cis } \left(4\theta + \frac{\pi}{2}\right)$$

$$\therefore \arg(iz^4) = 4\theta + \frac{\pi}{2}$$

**d**  $\frac{i}{\sqrt[3]{z}} = iz^{-\frac{1}{3}}$

$$= \text{cis } \frac{\pi}{2} \times (\text{cis } \theta)^{-\frac{1}{3}}$$

$$= e^{i\frac{\pi}{2}} \times (e^{i\theta})^{-\frac{1}{3}}$$

$$= e^{i\frac{\pi}{2}} \times e^{-i\frac{\theta}{3}}$$

$$= e^{i(\frac{\pi}{2} - \frac{\theta}{3})}$$

$$= \text{cis } \left(\frac{\pi}{2} - \frac{1}{3}\theta\right)$$

$$\therefore \arg\left(\frac{i}{\sqrt[3]{z}}\right) = \frac{\pi}{2} - \frac{1}{3}\theta$$

**5**  $z = 2e^{i\theta}, \quad w = 3e^{i\phi}$

**a**  $\sqrt{w} = (3e^{i\phi})^{\frac{1}{2}}$

$$= \sqrt{3}e^{i\frac{\phi}{2}}$$

$$= \sqrt{3} \text{cis } \frac{\phi}{2}$$

$$\therefore |\sqrt{w}| = \sqrt{3}, \quad \arg(\sqrt{w}) = \frac{\phi}{2}$$

**c**  $w^3 \times \sqrt{z} = (3e^{i\phi})^3 \times (2e^{i\theta})^{\frac{1}{2}}$

$$= 27e^{3i\phi} \times \sqrt{2}e^{i\frac{\theta}{2}}$$

$$= 27\sqrt{2} \times e^{i(3\phi + \frac{\theta}{2})}$$

$$= 27\sqrt{2} \text{cis}(3\phi + \frac{\theta}{2})$$

$$\therefore |w^3 \times \sqrt{z}| = 27\sqrt{2}, \quad \arg(w^3 \times \sqrt{z}) = 3\phi + \frac{\theta}{2}$$

**6 a**  $e^i = \cos(1) + i \sin(1) \quad \{\theta = 1\}$

$$\approx 0.540 + 0.841i$$

**b**  $3^i = (e^{\ln 3})^i$

$$= e^{i \ln 3}$$

$$= \cos(\ln 3) + i \sin(\ln 3)$$

$$\approx 0.455 + 0.891i$$

$$\begin{aligned}
 \text{c} \quad i &= 0 + 1i \\
 &= \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \\
 &= e^{i\frac{\pi}{2}} \\
 \therefore i^i &= (e^{i\frac{\pi}{2}})^i \\
 &= e^{i^2\frac{\pi}{2}} \\
 &= e^{-\frac{\pi}{2}}
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad i^i &= e^{-\frac{\pi}{2}} \quad \{\text{from c}\} \\
 \therefore (i^i)^i &= (e^{-\frac{\pi}{2}})^i \\
 &= e^{-i\frac{\pi}{2}} \\
 \therefore ((i^i)^i)^2 &= (e^{-i\frac{\pi}{2}})^2 \\
 &= e^{-i\pi} \\
 &= \cos(-\pi) + i \sin(-\pi) \\
 &= -1 + i(0) \\
 &= -1
 \end{aligned}$$

$$\begin{aligned}
 7 \quad \text{a} \quad \frac{e^{i\theta} + e^{-i\theta}}{2} &= \frac{\cos \theta + i \sin \theta + \cos(-\theta) + i \sin(-\theta)}{2} \\
 &= \frac{\cos \theta + i \sin \theta + \cos \theta - i \sin \theta}{2} \quad \{\cos(-\theta) = \cos \theta, \quad \sin(-\theta) = -\sin \theta\} \\
 &= \frac{2 \cos \theta}{2} = \cos \theta \quad \text{as required}
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad \frac{e^{i\theta} - e^{-i\theta}}{2i} &= \frac{\cos \theta + i \sin \theta - (\cos(-\theta) + i \sin(-\theta))}{2} \\
 &= \frac{\cos \theta + i \sin \theta - (\cos \theta - i \sin \theta)}{2i} \quad \{\cos(-\theta) = \cos \theta, \quad \sin(-\theta) = -\sin \theta\} \\
 &= \frac{2i \sin \theta}{2i} = \sin \theta \quad \text{as required}
 \end{aligned}$$

## EXERCISE 14F

**Note:** We assume that all arguments are in  $]-\pi, \pi]$ .

$$\begin{aligned}
 1 \quad (|z| \operatorname{cis} \theta)^n &= (|z| e^{i\theta})^n \quad \{\text{Euler's form}\} \\
 &= |z|^n (e^{i\theta})^n \\
 &= |z|^n e^{in\theta} \\
 &= |z|^n \operatorname{cis} n\theta \quad \{\text{polar form}\}
 \end{aligned}$$

$$\begin{aligned}
 2 \quad \text{a} \quad & \left( \sqrt{2} \operatorname{cis} \frac{\pi}{5} \right)^{10} \\
 &= (\sqrt{2})^{10} \operatorname{cis} \frac{10\pi}{5} \\
 &= 2^5 \operatorname{cis} 2\pi \\
 &= 2^5 \\
 &= 32
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & \left( \sqrt{2} \operatorname{cis} \frac{\pi}{8} \right)^{12} \\
 &= (\sqrt{2})^{12} \operatorname{cis} \frac{12\pi}{8} \\
 &= 2^6 \operatorname{cis} \frac{3\pi}{2} \\
 &= 64(-i) \\
 &= -64i
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & \left( \operatorname{cis} \frac{\pi}{12} \right)^{36} \\
 &= \operatorname{cis} \frac{36\pi}{12} \\
 &= \operatorname{cis} 3\pi \\
 &= -1
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad & \sqrt{5} \operatorname{cis} \frac{\pi}{7} \\
 &= \left( 5 \operatorname{cis} \frac{\pi}{7} \right)^{\frac{1}{2}} \\
 &= \sqrt{5} \operatorname{cis} \left( \frac{1}{2} \times \frac{\pi}{7} \right) \\
 &= \sqrt{5} \operatorname{cis} \frac{\pi}{14} \\
 & \text{(or } \approx 2.18 + 0.498i)
 \end{aligned}$$



$$\begin{aligned}
 \text{e} \quad & \sqrt[3]{8 \operatorname{cis} \frac{\pi}{2}} \\
 &= \left(8 \operatorname{cis} \frac{\pi}{2}\right)^{\frac{1}{3}} \\
 &= 2 \operatorname{cis} \frac{\pi}{6} \\
 &= 2 \left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) \\
 &= \sqrt{3} + i
 \end{aligned}$$

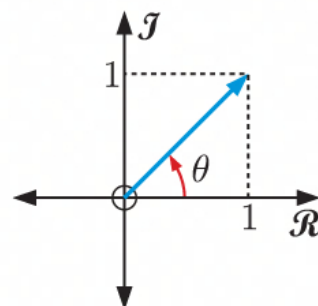
$$\begin{aligned}
 \text{f} \quad & \left(8 \operatorname{cis} \frac{\pi}{5}\right)^{\frac{5}{3}} \\
 &= 8^{\frac{5}{3}} \operatorname{cis} \left(\frac{5}{3} \times \frac{\pi}{5}\right) \\
 &= 2^5 \operatorname{cis} \frac{\pi}{3} \\
 &= 32 \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) \\
 &= 16 + 16\sqrt{3}i
 \end{aligned}$$

**3 a**  $1 + i$  has modulus  $\sqrt{1^2 + 1^2} = \sqrt{2}$

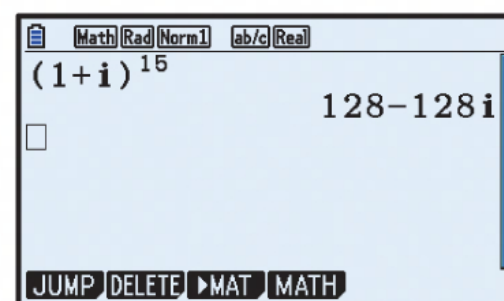
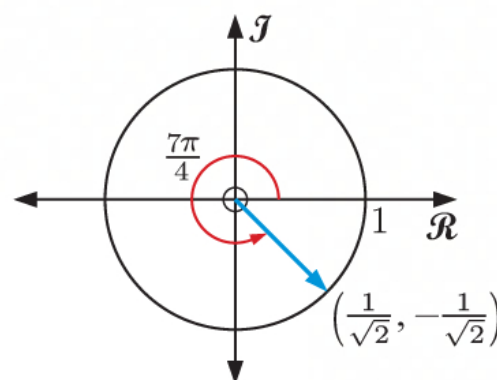
Now  $\tan \theta = 1$

$\therefore \arg(1 + i) = \frac{\pi}{4}$

$\therefore 1 + i = \sqrt{2} \operatorname{cis} \frac{\pi}{4}$



$$\begin{aligned}
 \therefore (1 + i)^{15} &= \left(\sqrt{2} \operatorname{cis} \frac{\pi}{4}\right)^{15} \\
 &= (\sqrt{2})^{15} \operatorname{cis} \frac{15\pi}{4} \\
 &= (\sqrt{2})^{15} \operatorname{cis} \frac{7\pi}{4} \\
 &= (\sqrt{2})^{15} \left(\cos \frac{7\pi}{4} + i \sin \frac{7\pi}{4}\right) \\
 &= (\sqrt{2})^{15} \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) \\
 &= 128\sqrt{2} \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) \\
 &= 128 - 128i
 \end{aligned}$$

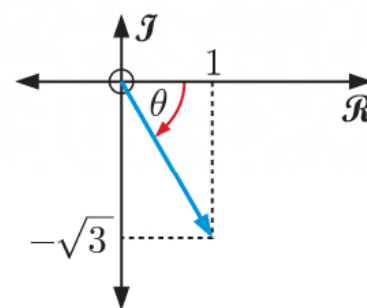


**b**  $1 - i\sqrt{3}$  has modulus  $\sqrt{1^2 + (-\sqrt{3})^2} = \sqrt{4} = 2$

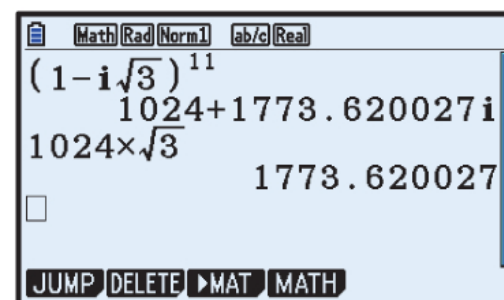
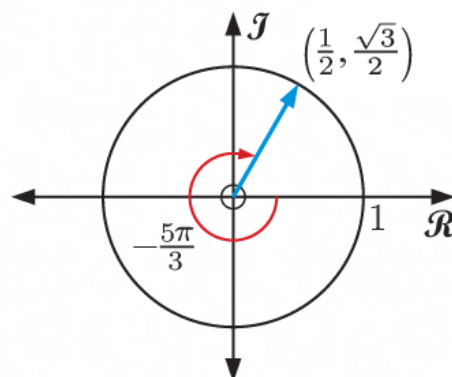
Now  $\tan \theta = \frac{\sqrt{3}}{1} = \sqrt{3}$

$\therefore \arg(1 - i\sqrt{3}) = -\frac{\pi}{3}$

$\therefore 1 - i\sqrt{3} = 2 \operatorname{cis} \left(-\frac{\pi}{3}\right)$



$$\begin{aligned}
 \therefore (1 - i\sqrt{3})^{11} &= \left(2 \operatorname{cis} \left(-\frac{\pi}{3}\right)\right)^{11} \\
 &= 2^{11} \operatorname{cis} \left(-\frac{11\pi}{3}\right) \\
 &= 2^{11} \operatorname{cis} \left(-\frac{5\pi}{3}\right) \\
 &= 2^{11} \left[\cos \left(-\frac{5\pi}{3}\right) + i \sin \left(-\frac{5\pi}{3}\right)\right] \\
 &= 2^{11} \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) \\
 &= 2048 \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) \\
 &= 1024 + 1024\sqrt{3}i
 \end{aligned}$$

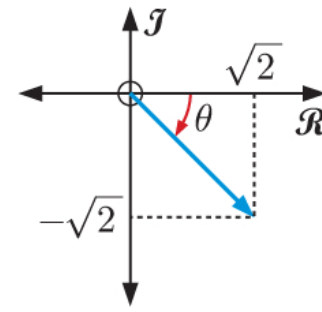


**c**  $\sqrt{2} - i\sqrt{2}$  has modulus  $\sqrt{(\sqrt{2})^2 + (-\sqrt{2})^2} = \sqrt{4} = 2$

Now  $\tan \theta = \frac{\sqrt{2}}{\sqrt{2}} = 1$

$\therefore \arg(\sqrt{2} - i\sqrt{2}) = -\frac{\pi}{4}$

$\therefore \sqrt{2} - i\sqrt{2} = 2 \operatorname{cis} \left(-\frac{\pi}{4}\right)$



$\therefore (\sqrt{2} - i\sqrt{2})^{-19}$

$= (2 \operatorname{cis} (-\frac{\pi}{4}))^{-19}$

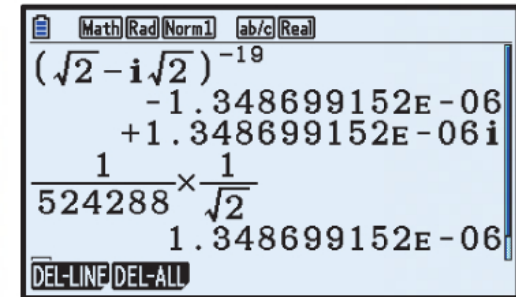
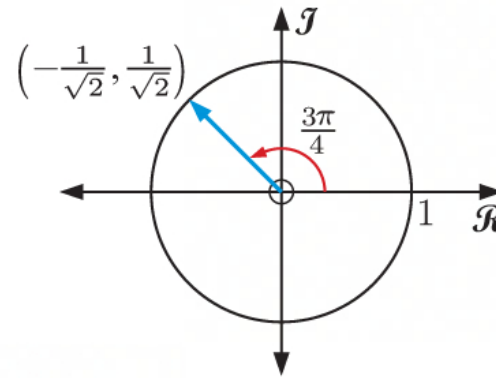
$= 2^{-19} \operatorname{cis} \frac{19\pi}{4}$

$= 2^{-19} \operatorname{cis} \frac{3\pi}{4}$

$= 2^{-19} \left[ \cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right]$

$= 2^{-19} \left( -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i \right)$

$= \frac{1}{524288} \left( -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i \right)$



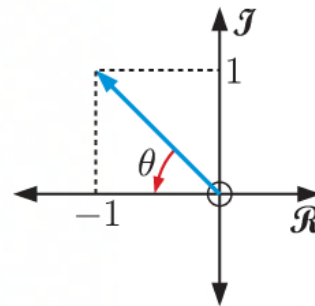
**d**  $-1 + i$  has modulus  $\sqrt{(-1)^2 + 1^2} = \sqrt{2}$

Now  $\tan \theta = \frac{1}{-1} = -1$

$\therefore \theta = \frac{\pi}{4}$

$\therefore \arg(-1 + i) = \pi - \frac{\pi}{4}$   
 $= \frac{3\pi}{4}$

$\therefore -1 + i = \sqrt{2} \operatorname{cis} \frac{3\pi}{4}$



$\therefore (-1 + i)^{-11}$

$= (\sqrt{2} \operatorname{cis} \frac{3\pi}{4})^{-11}$

$= (\sqrt{2})^{-11} \operatorname{cis} \left(-\frac{33\pi}{4}\right)$

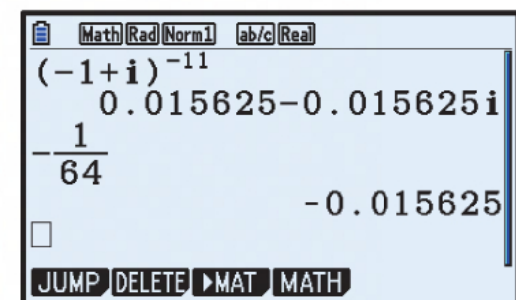
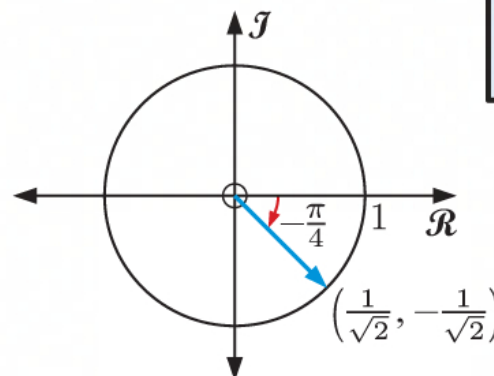
$= (\sqrt{2})^{-11} \operatorname{cis} \left(-\frac{\pi}{4}\right)$

$= (\sqrt{2})^{-11} \left[ \cos \left(-\frac{\pi}{4}\right) + i \sin \left(-\frac{\pi}{4}\right) \right]$

$= (\sqrt{2})^{-11} \left( \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i \right)$

$= (\sqrt{2})^{-12} (1 - i)$

$= \frac{1}{64} (1 - i)$

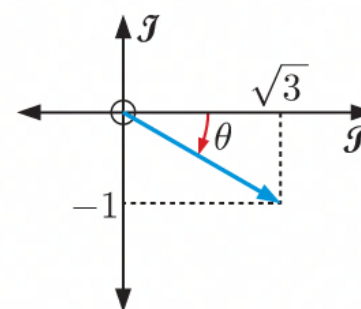


**e**  $\sqrt{3} - i$  has modulus  $\sqrt{(\sqrt{3})^2 + (-1)^2} = \sqrt{4} = 2$

Now  $\tan \theta = \frac{-1}{\sqrt{3}}$

$\therefore \arg(\sqrt{3} - i) = -\frac{\pi}{6}$

$\therefore \sqrt{3} - i = 2 \operatorname{cis} \left(-\frac{\pi}{6}\right)$



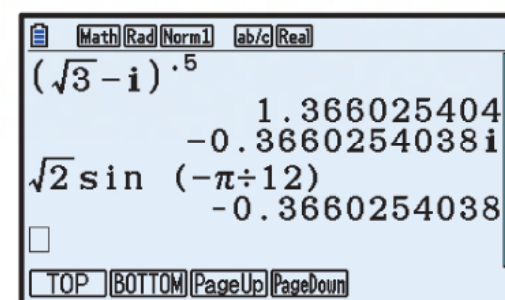
$\therefore (\sqrt{3} - i)^{\frac{1}{2}}$

$= (2 \operatorname{cis} \left(-\frac{\pi}{6}\right))^{\frac{1}{2}}$

$= 2^{\frac{1}{2}} \operatorname{cis} \left(-\frac{\pi}{12}\right)$

$= 2^{\frac{1}{2}} \left[ \cos \left(-\frac{\pi}{12}\right) + i \sin \left(-\frac{\pi}{12}\right) \right]$

$= \sqrt{2} \cos \left(-\frac{\pi}{12}\right) + i \sqrt{2} \sin \left(-\frac{\pi}{12}\right)$

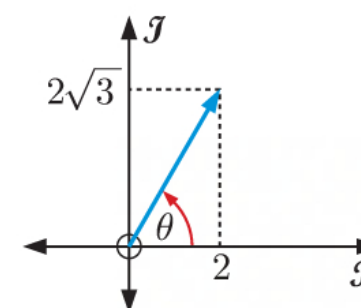


**f**  $2 + 2i\sqrt{3}$  has modulus  $\sqrt{2^2 + (2\sqrt{3})^2} = \sqrt{16} = 4$

Now  $\tan \theta = \frac{2\sqrt{3}}{2} = \sqrt{3}$

$\therefore \arg(2 + 2i\sqrt{3}) = \frac{\pi}{3}$

$\therefore 2 + 2i\sqrt{3} = 4 \operatorname{cis} \frac{\pi}{3}$



$\therefore (2 + 2i\sqrt{3})^{-\frac{5}{2}}$

$= (4 \operatorname{cis} \frac{\pi}{3})^{-\frac{5}{2}}$

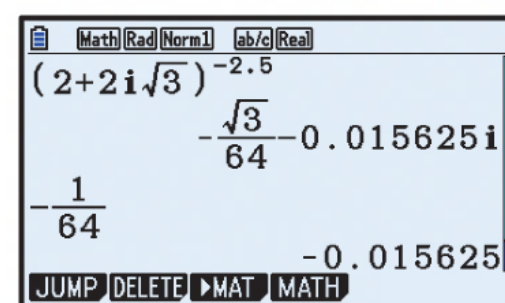
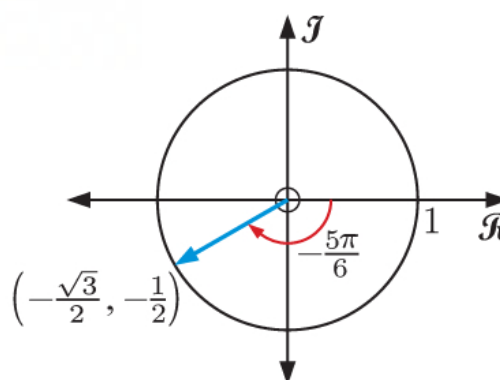
$= 4^{-\frac{5}{2}} \operatorname{cis} \left(-\frac{5\pi}{6}\right)$

$= 2^{-\frac{5}{2} \times 2} \operatorname{cis} \left(-\frac{5\pi}{6}\right)$

$= 2^{-5} \left[ \cos \left(-\frac{5\pi}{6}\right) + i \sin \left(-\frac{5\pi}{6}\right) \right]$

$= \frac{1}{32} \left( -\frac{\sqrt{3}}{2} - \frac{1}{2}i \right)$

$= \frac{1}{64} (-\sqrt{3} - i)$



**4 a**  $z = |z| \operatorname{cis} \theta$

$\therefore \sqrt{z} = (|z| \operatorname{cis} \theta)^{\frac{1}{2}}$

$\therefore \sqrt{z} = |z|^{\frac{1}{2}} \operatorname{cis} \frac{\theta}{2} \quad \{\text{De Moivre's theorem}\}$

**b**  $-\frac{\pi}{2} < \phi \leq \frac{\pi}{2}$

**c** True:  $\cos \phi \geq 0$  for all  $-\frac{\pi}{2} < \phi \leq \frac{\pi}{2}$

**5**  $\operatorname{cis}(-\theta) = \cos(-\theta) + i \sin(-\theta)$

$= \cos \theta - i \sin \theta$

$\therefore (\cos \theta - i \sin \theta)^{-3} = [\operatorname{cis}(-\theta)]^{-3}$

$= \operatorname{cis} 3\theta \quad \{\text{De Moivre's theorem}\}$

$$6 \quad a \quad z = 1 + i = \sqrt{2} \operatorname{cis} \frac{\pi}{4}$$

$$\begin{aligned} \therefore z^n &= (\sqrt{2})^n \operatorname{cis} \frac{n\pi}{4} \quad \{\text{De Moivre's theorem}\} \\ &= 2^{\frac{n}{2}} \operatorname{cis} \frac{n\pi}{4} \end{aligned}$$

$$\begin{aligned} b \quad i \quad &\text{If } z^n \text{ is real then } \sin \frac{n\pi}{4} = 0 \\ &\therefore \frac{n\pi}{4} = 0 + k\pi, \quad k \in \mathbb{Z} \\ &\therefore n = 4k, \quad k \in \mathbb{Z} \end{aligned}$$

$$\begin{aligned} ii \quad &\text{If } z^n \text{ is purely imaginary then } \cos \frac{n\pi}{4} = 0 \\ &\therefore \frac{n\pi}{4} = \frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z} \\ &\therefore n = 2 + 4k, \quad k \in \mathbb{Z} \end{aligned}$$

$$\begin{aligned} 7 \quad a \quad &z = 2 \operatorname{cis} \theta \\ &\therefore z^3 = 2^3 \operatorname{cis} 3\theta \\ &\therefore |z^3| = 8 \\ \text{and } \arg(z^3) &= 3\theta \end{aligned}$$

$$\begin{aligned} b \quad &z = 2 \operatorname{cis} \theta \\ &\therefore iz^2 = i(2 \operatorname{cis} \theta)^2 \\ &= \operatorname{cis} \frac{\pi}{2} \times 4 \operatorname{cis} 2\theta \\ &= 4 \operatorname{cis} \left( \frac{\pi}{2} + 2\theta \right) \\ &\therefore |iz^2| = 4 \\ \text{and } \arg(iz^2) &= \frac{\pi}{2} + 2\theta \end{aligned}$$

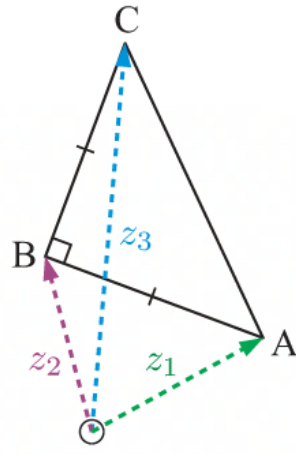
$$\begin{aligned} c \quad &z = 2 \operatorname{cis} \theta \\ &\therefore \frac{1}{z} = (2 \operatorname{cis} \theta)^{-1} \\ &= \frac{1}{2} \operatorname{cis} (-\theta) \\ &\therefore \left| \frac{1}{z} \right| = \frac{1}{2} \\ \text{and } \arg\left(\frac{1}{z}\right) &= -\theta \end{aligned}$$

$$\begin{aligned} d \quad &z = 2 \operatorname{cis} \theta \\ &\therefore -\frac{i}{z^2} = -i \times z^{-2} \\ &= \operatorname{cis} \left( -\frac{\pi}{2} \right) \times (2 \operatorname{cis} \theta)^{-2} \\ &= 2^{-2} \operatorname{cis} \left( -\frac{\pi}{2} \right) \operatorname{cis} (-2\theta) \\ &= \frac{1}{4} \operatorname{cis} \left( -\frac{\pi}{2} - 2\theta \right) \\ &\therefore \left| -\frac{i}{z^2} \right| = \frac{1}{4} \quad \text{and} \quad \arg\left(-\frac{i}{z^2}\right) = -\frac{\pi}{2} - 2\theta \end{aligned}$$

$$\begin{aligned} 8 \quad &\text{If } z = \operatorname{cis} \theta, \quad \text{then } \frac{z^2 - 1}{z^2 + 1} = \frac{(\operatorname{cis} \theta)^2 - 1}{(\operatorname{cis} \theta)^2 + 1} \\ &= \frac{\operatorname{cis} 2\theta - 1}{\operatorname{cis} 2\theta + 1} \quad \{\text{De Moivre's theorem}\} \\ &= \frac{\cos 2\theta + i \sin 2\theta - 1}{\cos 2\theta + i \sin 2\theta + 1} \\ &= \frac{(\cancel{1} - 2 \sin^2 \theta) + i \sin 2\theta - \cancel{1}}{(2 \cos^2 \theta - \cancel{1}) + i \sin 2\theta + \cancel{1}} \\ &= \frac{-2 \sin^2 \theta + 2i \cos \theta \sin \theta}{2 \cos^2 \theta + 2i \cos \theta \sin \theta} \\ &= \frac{2 \sin \theta (i \cos \theta + i^2 \sin \theta)}{2 \cos \theta (\cos \theta + i \sin \theta)} \\ &= \frac{i \sin \theta \operatorname{cis} \theta}{\cos \theta \operatorname{cis} \theta} \\ &= i \tan \theta \end{aligned}$$



9 a



$$\overrightarrow{BC} = \overrightarrow{BO} + \overrightarrow{OC} = -z_2 + z_3 = z_3 - z_2$$

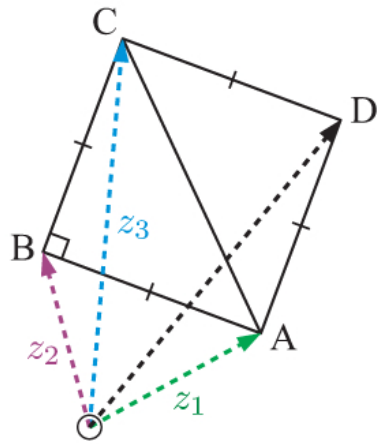
$$\overrightarrow{BA} = \overrightarrow{BO} + \overrightarrow{OA} = -z_2 + z_1 = z_1 - z_2$$

Suppose  $\overrightarrow{BA}$  has length  $r$  and argument  $\theta$ .

$$\therefore \overrightarrow{BA} = r \operatorname{cis} \theta \quad \text{and} \quad \overrightarrow{BC} = r \operatorname{cis}(\theta + \frac{\pi}{2})$$

$$\begin{aligned} \therefore -(z_3 - z_2)^2 &= \operatorname{cis} \pi \times r^2 \operatorname{cis}(2\theta + \pi) \\ &= r^2 \operatorname{cis}(2\theta + 2\pi) \\ &= r^2 \operatorname{cis} 2\theta \\ &= (r \operatorname{cis} \theta)^2 \\ &= (z_1 - z_2)^2 \end{aligned}$$

b



$$\overrightarrow{CD} = \overrightarrow{BA} = z_1 - z_2$$

$$\text{Now } \overrightarrow{OD} = \overrightarrow{OC} + \overrightarrow{CD}$$

$$\therefore \overrightarrow{OD} = z_3 + z_1 - z_2$$

$$\therefore z_3 + z_1 - z_2 \text{ represents } D$$

10 a  $\cos 3\theta + i \sin 3\theta = \operatorname{cis} 3\theta$

$$= (\operatorname{cis} \theta)^3 \quad \{\text{De Moivre's theorem}\}$$

$$= (\cos \theta + i \sin \theta)^3$$

$$= \cos^3 \theta + 3 \cos^2 \theta (i \sin \theta) + 3 \cos \theta (i \sin \theta)^2 + (i \sin \theta)^3$$

$$= [\cos^3 \theta - 3 \cos \theta \sin^2 \theta] + i [3 \cos^2 \theta \sin \theta - \sin^3 \theta]$$

i Equating real parts:

$$\cos 3\theta = \cos^3 \theta - 3 \cos \theta \sin^2 \theta$$

$$= \cos^3 \theta - 3 \cos \theta (1 - \cos^2 \theta)$$

$$= \cos^3 \theta - 3 \cos \theta + 3 \cos^3 \theta$$

$$= 4 \cos^3 \theta - 3 \cos \theta$$

ii Equating imaginary parts:

$$\sin 3\theta = 3 \cos^2 \theta \sin \theta - \sin^3 \theta$$

$$= 3(1 - \sin^2 \theta) \sin \theta - \sin^3 \theta$$

$$= 3 \sin \theta - 3 \sin^3 \theta - \sin^3 \theta$$

$$= 3 \sin \theta - 4 \sin^3 \theta$$

$$\begin{aligned}
 \text{b } \tan 3\theta &= \frac{\sin 3\theta}{\cos 3\theta} = \frac{3\sin\theta - 4\sin^3\theta}{4\cos^3\theta - 3\cos\theta} \quad \{\text{using a}\} \\
 &= \frac{\sin\theta(3 - 4\sin^2\theta)}{\cos\theta(4\cos^2\theta - 3)} \\
 &= \frac{\sin\theta [3(\cos^2\theta + \sin^2\theta) - 4\sin^2\theta]}{\cos\theta [4\cos^2\theta - 3(\cos^2\theta + \sin^2\theta)]} \quad \{\cos^2\theta + \sin^2\theta = 1\} \\
 &= \frac{\sin\theta(3\cos^2\theta - \sin^2\theta)}{\cos\theta(\cos^2\theta - 3\sin^2\theta)} \\
 &= \left( \frac{3\sin\theta\cos^2\theta - \sin^3\theta}{\cos^3\theta - 3\sin^2\theta\cos\theta} \right) \div \left( \frac{\cos^3\theta}{\cos^3\theta} \right) \\
 &= \frac{3\frac{\sin\theta}{\cos\theta} - \frac{\sin^3\theta}{\cos^3\theta}}{1 - 3\frac{\sin^2\theta}{\cos^2\theta}} \\
 &= \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}
 \end{aligned}$$

c i

$$4x^3 - 3x = -\frac{1}{\sqrt{2}}$$

$$\text{Let } x = \cos\theta$$

$$\therefore 4\cos^3\theta - 3\cos\theta = -\frac{1}{\sqrt{2}}$$

$$\therefore \cos 3\theta = -\frac{1}{\sqrt{2}}$$

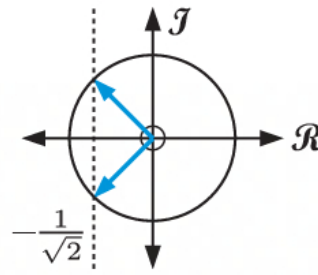
$$\therefore 3\theta = \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{11\pi}{4}, \frac{13\pi}{4}, \frac{19\pi}{4}, \frac{21\pi}{4} \quad \{0 \leq 3\theta \leq 6\pi\}$$

$$\therefore \theta = \frac{\pi}{4}, \frac{5\pi}{12}, \frac{11\pi}{12}, \frac{13\pi}{12}, \frac{19\pi}{12}, \frac{7\pi}{4} \quad \{0 \leq \theta \leq 2\pi\}$$

$$\therefore x = \cos\theta = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} \quad \{= \cos\frac{7\pi}{4}\}$$

$$\text{or } \cos\frac{5\pi}{12} \quad \{= \cos\frac{19\pi}{12}\}$$

$$\text{or } \cos\frac{11\pi}{12} \quad \{= \cos\frac{13\pi}{12}\}$$



$$\text{ii } x^3 - 3\sqrt{3}x^2 - 3x + \sqrt{3} = 0$$

$$\therefore \sqrt{3} - 3\sqrt{3}x^2 = 3x - x^3$$

$$\therefore \sqrt{3}(1 - 3x^2) = 3x - x^3$$

$$\therefore \sqrt{3} = \frac{3x - x^3}{1 - 3x^2}$$

$$\text{Let } x = \tan\theta$$

$$\therefore \sqrt{3} = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$$

$$\therefore \tan 3\theta = \sqrt{3}$$

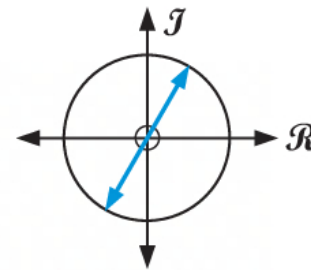
$$\therefore 3\theta = \frac{\pi}{3}, \frac{4\pi}{3}, \frac{7\pi}{3}, \frac{10\pi}{3}, \frac{13\pi}{3}, \frac{16\pi}{3} \quad \{0 \leq 3\theta \leq 6\pi\}$$

$$\therefore \theta = \frac{\pi}{9}, \frac{4\pi}{9}, \frac{7\pi}{9}, \frac{10\pi}{9}, \frac{13\pi}{9}, \frac{16\pi}{9} \quad \{0 \leq \theta \leq 2\pi\}$$

$$\therefore x = \tan\theta = \tan\frac{\pi}{9} \quad \{= \tan\frac{10\pi}{9}\}$$

$$\text{or } \tan\frac{4\pi}{9} \quad \{= \tan\frac{13\pi}{9}\}$$

$$\text{or } \tan\frac{7\pi}{9} \quad \{= \tan\frac{16\pi}{9}\}$$



$$11 \quad \cos 4\theta + i \sin 4\theta = \text{cis } 4\theta$$

$$\begin{aligned} &= (\text{cis } \theta)^4 \quad \{\text{De Moivre's theorem}\} \\ &= (\cos \theta + i \sin \theta)^4 \\ &= \cos^4 \theta + 4 \cos^3 \theta (i \sin \theta) + 6 \cos^2 \theta (i \sin \theta)^2 + 4 \cos \theta (i \sin \theta)^3 + (i \sin \theta)^4 \\ &= \cos^4 \theta + [4 \cos^3 \theta \sin \theta]i - 6 \cos^2 \theta \sin^2 \theta - [4 \cos \theta \sin^3 \theta]i + \sin^4 \theta \\ &= [\cos^4 \theta - 6 \cos^2 \theta \sin^2 \theta + \sin^4 \theta] + [4 \cos^3 \theta \sin \theta - 4 \cos \theta \sin^3 \theta]i \end{aligned}$$

$$\begin{aligned} \text{a Equating real parts: } \cos 4\theta &= \cos^4 \theta - 6 \cos^2 \theta (1 - \cos^2 \theta) + (1 - \cos^2 \theta)^2 \\ &= \cos^4 \theta - 6 \cos^2 \theta + 6 \cos^4 \theta + 1 - 2 \cos^2 \theta + \cos^4 \theta \\ &= 8 \cos^4 \theta - 8 \cos^2 \theta + 1 \end{aligned}$$

$$\text{b Equating imaginary parts: } \sin 4\theta = 4 \cos^3 \theta \sin \theta - 4 \cos \theta \sin^3 \theta$$

$$\begin{aligned} 12 \quad \text{a i If } z = \text{cis } \theta, \text{ then } z^n + \frac{1}{z^n} &= z^n + z^{-n} \\ &= (\text{cis } \theta)^n + (\text{cis } \theta)^{-n} \\ &= \text{cis } (n\theta) + \text{cis } (-n\theta) \quad \{\text{De Moivre's theorem}\} \\ &= (\cos n\theta + i \sin n\theta) + (\cos(-n\theta) + i \sin(-n\theta)) \\ &= \cos n\theta + i \sin n\theta + \cos n\theta - i \sin n\theta \\ &= 2 \cos n\theta \end{aligned}$$

$$\text{ii In a i, if we let } n = 1, \text{ we get } z + \frac{1}{z} = 2 \cos \theta.$$

$$\begin{aligned} \text{iii } \left(z + \frac{1}{z}\right)^3 &= z^3 + 3z^2 \left(\frac{1}{z}\right) + 3z \left(\frac{1}{z}\right)^2 + \left(\frac{1}{z}\right)^3 \\ &= z^3 + 3z + \frac{3}{z} + \frac{1}{z^3} \end{aligned}$$

$$\text{iv From a iii, } \left(z + \frac{1}{z}\right)^3 = \left(z^3 + \frac{1}{z^3}\right) + 3 \left(z + \frac{1}{z}\right)$$

$$\text{Using a i and a ii, } (2 \cos \theta)^3 = 2 \cos 3\theta + 3(2 \cos \theta)$$

$$\therefore 8 \cos^3 \theta = 2 \cos 3\theta + 6 \cos \theta$$

$$\therefore \cos^3 \theta = \frac{1}{4} \cos 3\theta + \frac{3}{4} \cos \theta$$

$$\text{v If we let } \theta = \frac{13\pi}{12} \text{ in a iv, we get}$$

$$\begin{aligned} \cos^3 \left(\frac{13\pi}{12}\right) &= \frac{1}{4} \cos \frac{39\pi}{12} + \frac{3}{4} \cos \frac{13\pi}{12} \\ &= \frac{1}{4} \cos \frac{13\pi}{4} + \frac{3}{4} \cos \left(\frac{3\pi}{4} + \frac{\pi}{3}\right) \\ &= \frac{1}{4} \cos \frac{5\pi}{4} + \frac{3}{4} \left(\cos \frac{3\pi}{4} \cos \frac{\pi}{3} - \sin \frac{3\pi}{4} \sin \frac{\pi}{3}\right) \\ &= \frac{1}{4} \left(-\frac{1}{\sqrt{2}}\right) + \frac{3}{4} \left[\left(-\frac{1}{\sqrt{2}}\right) \left(\frac{1}{2}\right) - \left(\frac{1}{\sqrt{2}}\right) \left(\frac{\sqrt{3}}{2}\right)\right] \\ &= -\frac{1}{4\sqrt{2}} - \frac{3}{8\sqrt{2}} - \frac{3\sqrt{3}}{8\sqrt{2}} \\ &= \frac{-2-3-3\sqrt{3}}{8\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} \\ &= \frac{-5\sqrt{2}-3\sqrt{6}}{16} \end{aligned}$$

$$\begin{aligned}
\text{b If } z = \operatorname{cis} \theta, \text{ then } z^n - \frac{1}{z^n} &= z^n - z^{-n} \\
&= (\operatorname{cis} \theta)^n - (\operatorname{cis} \theta)^{-n} \\
&= \operatorname{cis} n\theta - \operatorname{cis} (-n\theta) \quad \{\text{De Moivre's theorem}\} \\
&= \cos n\theta + i \sin n\theta - [\cos(-n\theta) + i \sin(-n\theta)] \\
&= \cos n\theta + i \sin n\theta - \cos(-n\theta) - i \sin(-n\theta) \\
&= \cos n\theta + i \sin n\theta - \cos n\theta + i \sin n\theta \\
&= 2i \sin n\theta \quad \dots (*)
\end{aligned}$$

If we let  $n = 1$ ,  $z - \frac{1}{z} = 2i \sin \theta$

$$\begin{aligned}
\therefore (2i \sin \theta)^3 &= \left(z - \frac{1}{z}\right)^3 \\
\therefore 8i^3 \sin^3 \theta &= z^3 + 3z^2 \left(-\frac{1}{z}\right) + 3z \left(-\frac{1}{z}\right)^2 + \left(-\frac{1}{z}\right)^3 \\
&= z^3 - 3z + \frac{3}{z} - \frac{1}{z^3} \\
&= z^3 - \frac{1}{z^3} - 3 \left(z - \frac{1}{z}\right) \\
\therefore 8i^3 \sin^3 \theta &= 2i \sin 3\theta - 3 \times 2i \sin \theta \quad \{\text{using } (*)\} \\
\therefore -8i \sin^3 \theta &= 2i \sin 3\theta - 6i \sin \theta \\
\therefore \sin^3 \theta &= -\frac{1}{4} \sin 3\theta + \frac{3}{4} \sin \theta \\
\therefore \sin^3 \theta &= \frac{3}{4} \sin \theta - \frac{1}{4} \sin 3\theta
\end{aligned}$$

$$\begin{aligned}
\text{c } \sin^3 \theta \cos^3 \theta &= \left(\frac{3}{4} \sin \theta - \frac{1}{4} \sin 3\theta\right) \left(\frac{1}{4} \cos 3\theta + \frac{3}{4} \cos \theta\right) \quad \{\text{using 12 a iv and b}\} \\
&= \frac{3}{16} \sin \theta \cos 3\theta + \frac{9}{16} \sin \theta \cos \theta - \frac{1}{16} \sin 3\theta \cos 3\theta - \frac{3}{16} \sin 3\theta \cos \theta \\
&= \frac{3}{16} (\sin \theta \cos 3\theta - \sin 3\theta \cos \theta) + \frac{9}{32} (2 \sin \theta \cos \theta) - \frac{1}{32} (2 \sin 3\theta \cos 3\theta) \\
&= \frac{3}{16} (\sin(\theta - 3\theta)) + \frac{9}{32} \sin 2\theta - \frac{1}{32} \sin 6\theta \\
&= -\frac{3}{16} \sin 2\theta + \frac{9}{32} \sin 2\theta - \frac{1}{32} \sin 6\theta \\
&= \frac{3}{32} \sin 2\theta - \frac{1}{32} \sin 6\theta \\
&= \frac{1}{32} (3 \sin 2\theta - \sin 6\theta)
\end{aligned}$$

$$\begin{aligned}
\text{13 a } z &= r \operatorname{cis} \theta \\
\therefore z \times z^2 \times z^3 &= r \operatorname{cis} \theta \times r^2 \operatorname{cis} 2\theta \times r^3 \operatorname{cis} 3\theta \\
&= r^6 \operatorname{cis} 6\theta
\end{aligned}$$

$$\begin{aligned}
\text{b } z \times z^2 \times z^3 \times \dots \times z^k &= r \operatorname{cis} \theta \times r^2 \operatorname{cis} 2\theta \times r^3 \operatorname{cis} 3\theta \times \dots \times r^k \operatorname{cis} k\theta \\
&= r^{(1+2+3+\dots+k)} \operatorname{cis} ((1+2+3+\dots+k)\theta)
\end{aligned}$$

$1 + 2 + 3 + \dots + k$  is an arithmetic series with sum  $\frac{k(k+1)}{2}$ .

$$\therefore z \times z^2 \times z^3 \times \dots \times z^k = r^{\frac{k(k+1)}{2}} \operatorname{cis} \left(\frac{k(k+1)\theta}{2}\right)$$



**c i**  $z = 2 \operatorname{cis} \frac{\pi}{7}$

$$\begin{aligned} z \times z^2 \times z^3 \times \dots \times z^k &= 2^{\frac{k(k+1)}{2}} \operatorname{cis} \left[ \frac{k(k+1)\pi}{14} \right] \quad \{\text{using b}\} \\ &= 2^{\frac{k(k+1)}{2}} \left[ \cos \left( \frac{k(k+1)\pi}{14} \right) + i \sin \left( \frac{k(k+1)\pi}{14} \right) \right] \end{aligned}$$

which is real when  $\sin \left( \frac{k(k+1)\pi}{14} \right) = 0$

$$\therefore \frac{k(k+1)\pi}{14} = n\pi, \quad n \in \mathbb{Z}$$

$$\therefore k(k+1) = 14n$$

which has smallest integer solution  $k = 6, n = 3$

$$\begin{aligned} \therefore |z^1 \times z^2 \times z^3 \times \dots \times z^6| &= 2^{\frac{6(7)}{2}} \\ &= 2^{21} \end{aligned}$$

**ii**  $z \times z^2 \times z^3 \times \dots \times z^k = 2^{\frac{k(k+1)}{2}} \left[ \cos \left( \frac{k(k+1)\pi}{14} \right) + i \sin \left( \frac{k(k+1)\pi}{14} \right) \right]$

which is purely imaginary when  $\cos \left( \frac{k(k+1)\pi}{14} \right) = 0$

$$\therefore \frac{k(k+1)\pi}{14} = (2n-1) \frac{\pi}{2}, \quad n \in \mathbb{Z}^+$$

$$\therefore \frac{k(k+1)}{14} = \frac{2n-1}{2}$$

$$\therefore k(k+1) = 7(2n-1) \quad \dots (*)$$

But  $k(k+1)$  is even for all  $k \in \mathbb{Z}^+$  and  $7(2n-1)$  is odd for all  $n \in \mathbb{Z}^+$ .

So,  $(*)$  is never true.

$\therefore z^1 \times z^2 \times z^3 \times \dots \times z^k$  is never purely imaginary.

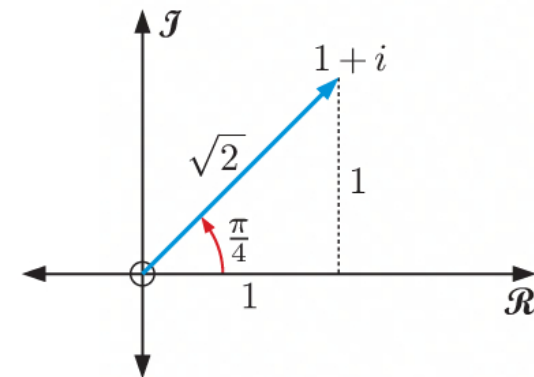
**14**  $(1+i)^{2n} = \binom{2n}{0} 1^{2n} i^0 + \binom{2n}{1} 1^{2n-1} i^1 + \binom{2n}{2} 1^{2n-2} i^2 + \binom{2n}{3} 1^{2n-3} i^3 + \dots + \binom{2n}{2n} 1^0 i^{2n}$   
{binomial theorem}

$$= \binom{2n}{0} + \binom{2n}{1} i - \binom{2n}{2} - \binom{2n}{3} i + \dots + \binom{2n}{2n} (-1)^n$$

But  $(1+i)^{2n} = \left( \sqrt{2} \operatorname{cis} \frac{\pi}{4} \right)^{2n}$

$$= 2^n \operatorname{cis} \left( \frac{n\pi}{2} \right) \quad \{\text{De Moivre's theorem}\}$$

$$= 2^n \left[ \cos \left( \frac{n\pi}{2} \right) + i \sin \left( \frac{n\pi}{2} \right) \right]$$



So,  $\binom{2n}{0} + \binom{2n}{1} i - \binom{2n}{2} - \binom{2n}{3} i + \dots + \binom{2n}{2n} (-1)^n = 2^n \left[ \cos \left( \frac{n\pi}{2} \right) + i \sin \left( \frac{n\pi}{2} \right) \right]$

Equating real parts:

$$\binom{2n}{0} - \binom{2n}{2} + \binom{2n}{4} - \binom{2n}{6} + \dots + (-1)^n \binom{2n}{2n} = 2^n \cos \left( \frac{n\pi}{2} \right), \quad n \in \mathbb{Z}^+$$

$$\begin{aligned}
15 \quad & 1 + \operatorname{cis} \theta + \operatorname{cis} 2\theta + \operatorname{cis} 3\theta + \dots + \operatorname{cis} n\theta \\
&= 1 + (\cos \theta + i \sin \theta) + (\cos 2\theta + i \sin 2\theta) + (\cos 3\theta + i \sin 3\theta) + \dots + (\cos n\theta + i \sin n\theta) \\
&= (1 + \cos \theta + \cos 2\theta + \cos 3\theta + \dots + \cos n\theta) + i(\sin \theta + \sin 2\theta + \sin 3\theta + \dots + \sin n\theta) \\
&= \sum_{r=0}^n \cos r\theta + i \sum_{r=1}^n \sin r\theta
\end{aligned}$$

$$\therefore \operatorname{Re}(1 + \operatorname{cis} \theta + \operatorname{cis} 2\theta + \operatorname{cis} 3\theta + \dots + \operatorname{cis} n\theta) = \sum_{r=0}^n \cos r\theta \quad \dots (1)$$

Now  $1 + \operatorname{cis} \theta + \operatorname{cis} 2\theta + \operatorname{cis} 3\theta + \dots + \operatorname{cis} n\theta = 1 + \operatorname{cis} \theta + (\operatorname{cis} \theta)^2 + (\operatorname{cis} \theta)^3 + \dots + (\operatorname{cis} \theta)^n$ , which is a geometric series with  $u_1 = 1$ ,  $r = \operatorname{cis} \theta$

$$\therefore \text{it has sum } S_n = \frac{u_1(r^n - 1)}{r - 1}$$

$$\begin{aligned}
\underbrace{1 + \operatorname{cis} \theta + \operatorname{cis} 2\theta + \dots + \operatorname{cis} n\theta}_{n+1 \text{ terms}} &= S_{n+1} \\
&= \frac{1((\operatorname{cis} \theta)^{n+1} - 1)}{\operatorname{cis} \theta - 1} \\
&= \frac{\operatorname{cis} [(n+1)\theta] - 1}{\operatorname{cis} \theta - 1} \\
&= \frac{\cos [(n+1)\theta] + i \sin [(n+1)\theta] - 1}{\cos \theta + i \sin \theta - 1} \\
&= \left( \frac{\cos [(n+1)\theta] + i \sin [(n+1)\theta] - 1}{\cos \theta - 1 + i \sin \theta} \right) \left( \frac{\cos \theta - 1 - i \sin \theta}{\cos \theta - 1 - i \sin \theta} \right)
\end{aligned}$$

$$\begin{aligned}
\therefore \operatorname{Re}(1 + \operatorname{cis} \theta + \operatorname{cis} 2\theta + \dots + \operatorname{cis} n\theta) &= \frac{\cos [(n+1)\theta] \cos \theta - \cos [(n+1)\theta] + \sin [(n+1)\theta] \sin \theta - \cos \theta + 1}{(\cos \theta - 1)^2 + \sin^2 \theta} \\
&= \frac{(\cos [(n+1)\theta] \cos \theta + \sin [(n+1)\theta] \sin \theta) - \cos [(n+1)\theta] - \cos \theta + 1}{\cos^2 \theta - 2 \cos \theta + 1 + \sin^2 \theta} \\
&= \frac{\cos n\theta - \cos \theta - \cos [(n+1)\theta] + 1}{2 - 2 \cos \theta} \quad \dots (2) \quad \{ \cos A \cos B + \sin A \sin B = \cos(A - B) \}
\end{aligned}$$

$$\text{Equating (1) and (2) gives } \sum_{r=0}^n \cos r\theta = \frac{\cos n\theta - \cos \theta - \cos [(n+1)\theta] + 1}{2 - 2 \cos \theta}$$

$$\begin{aligned}
16 \quad a \quad & 2 \cos \frac{\theta}{2} \operatorname{cis} \frac{\theta}{2} \\
&= 2 \cos \frac{\theta}{2} \left( \cos \frac{\theta}{2} + i \sin \frac{\theta}{2} \right) \\
&= 2 \cos^2 \left( \frac{\theta}{2} \right) + 2i \cos \frac{\theta}{2} \sin \frac{\theta}{2} \\
&= \cos \theta + 1 + i \left( 2 \cos \frac{\theta}{2} \sin \frac{\theta}{2} \right) \quad \left\{ \begin{array}{l} \cos 2X = 2 \cos^2 X - 1 \\ \therefore 2 \cos^2 X = \cos 2X + 1 \end{array} \right\} \\
&= \cos \theta + 1 + i \sin \theta \quad \{\text{double angle formulae}\} \\
&= 1 + \operatorname{cis} \theta
\end{aligned}$$

$$\begin{aligned}
b \quad \text{Consider } (1 + \operatorname{cis} \theta)^n &= \binom{n}{0} 1^n (\operatorname{cis} \theta)^0 + \binom{n}{1} 1^{n-1} (\operatorname{cis} \theta)^1 + \binom{n}{2} 1^{n-2} (\operatorname{cis} \theta)^2 + \dots \\
&\quad + \binom{n}{n} 1^0 (\operatorname{cis} \theta)^n \quad \{\text{binomial theorem}\} \\
&= \binom{n}{0} + \binom{n}{1} \operatorname{cis} \theta + \binom{n}{2} \operatorname{cis} 2\theta + \dots + \binom{n}{n} \operatorname{cis} n\theta \\
\therefore \operatorname{Re} [(1 + \operatorname{cis} \theta)^n] &= \binom{n}{0} + \binom{n}{1} \cos \theta + \binom{n}{2} \cos 2\theta + \dots + \binom{n}{n} \cos n\theta \\
&= \sum_{r=0}^n \binom{n}{r} \cos r\theta
\end{aligned}$$

$$\begin{aligned}
\text{So } \sum_{r=0}^n \binom{n}{r} \cos r\theta &= \Re[(1 + \operatorname{cis} \theta)^n] \\
&= \Re \left[ \left( 2 \cos \frac{\theta}{2} \operatorname{cis} \frac{\theta}{2} \right)^n \right] \quad \{\text{using a}\} \\
&= \Re \left[ 2^n \cos^n \left( \frac{\theta}{2} \right) \operatorname{cis} \left( \frac{n\theta}{2} \right) \right] \\
&= 2^n \cos^n \left( \frac{\theta}{2} \right) \cos \left( \frac{n\theta}{2} \right)
\end{aligned}$$

## EXERCISE 14G.1

**1** The cube roots of 1 are the 3 solutions of  $z^3 = 1$ .

**a** By factorisation,  $z^3 = 1$

$$\therefore z^3 - 1 = 0$$

$$\therefore (z - 1)(z^2 + z + 1) = 0 \quad \{z = 1 \text{ is a solution, so } (z - 1) \text{ is a factor}\}$$

$$\therefore z = 1 \text{ or } \frac{-1 \pm \sqrt{1-4}}{2}$$

$$\therefore z = 1 \text{ or } -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

**b** By the “ $n$ th roots method”,

$$z^3 = 1$$

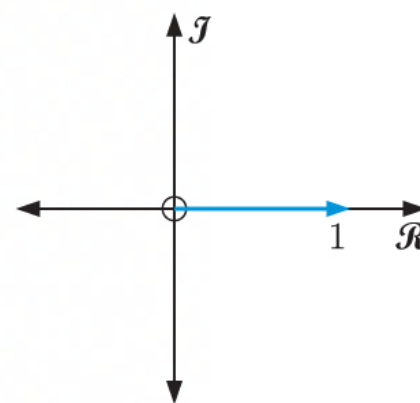
$$\therefore z^3 = \operatorname{cis}(0 + k2\pi) \quad \text{where } k \in \mathbb{Z} \quad \{\text{polar form}\}$$

$$\therefore z = (\operatorname{cis} k2\pi)^{\frac{1}{3}}$$

$$\therefore z = \operatorname{cis} \frac{k2\pi}{3} \quad \{\text{De Moivre}\}$$

$$\therefore z = \operatorname{cis} 0, \operatorname{cis} \frac{2\pi}{3}, \operatorname{cis} \frac{4\pi}{3} \quad \{\text{letting } k = 0, 1, 2\}$$

$$\therefore z = 1, -\frac{1}{2} + \frac{\sqrt{3}}{2}i, -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$



**2 a**  $z^3 = -8i$

$$\therefore z^3 = 8 \operatorname{cis} \left( -\frac{\pi}{2} + k2\pi \right) \quad \text{where } k \in \mathbb{Z} \quad \{\text{polar form}\}$$

$$\therefore z = \left( 8 \operatorname{cis} \left( -\frac{\pi}{2} + k2\pi \right) \right)^{\frac{1}{3}}$$

$$\therefore z = 8^{\frac{1}{3}} \operatorname{cis} \left( -\frac{\pi}{6} + \frac{k2\pi}{3} \right) \quad \{\text{De Moivre}\}$$

$$\therefore z = 2 \operatorname{cis} \left( -\frac{\pi}{6} \right), 2 \operatorname{cis} \frac{3\pi}{6}, 2 \operatorname{cis} \frac{7\pi}{6} \quad \{\text{letting } k = 0, 1, 2\}$$

$$\therefore z = 2 \left( \frac{\sqrt{3}}{2} - \frac{1}{2}i \right), 2 \operatorname{cis} \frac{\pi}{2}, 2 \left( -\frac{\sqrt{3}}{2} - \frac{1}{2}i \right)$$

$$\therefore z = \sqrt{3} - i, 2i, -\sqrt{3} - i$$

**b**  $z^3 = -27i$

$\therefore z^3 = 27 \operatorname{cis} \left( -\frac{\pi}{2} + k2\pi \right) \quad \text{where } k \in \mathbb{Z} \quad \{\text{polar form}\}$

$\therefore z = \left( 27 \operatorname{cis} \left( -\frac{\pi}{2} + k2\pi \right) \right)^{\frac{1}{3}}$

$\therefore z = 27^{\frac{1}{3}} \operatorname{cis} \left( -\frac{\pi}{6} + \frac{k4\pi}{6} \right) \quad \{\text{De Moivre}\}$

$\therefore z = 3 \operatorname{cis} \left( -\frac{\pi}{6} \right), 3 \operatorname{cis} \frac{3\pi}{6}, 3 \operatorname{cis} \frac{7\pi}{6} \quad \{\text{letting } k = 0, 1, 2\}$

$\therefore z = 3 \left( \frac{\sqrt{3}}{2} - \frac{1}{2}i \right), 3 \operatorname{cis} \frac{\pi}{2}, 3 \left( -\frac{\sqrt{3}}{2} - \frac{1}{2}i \right)$

$\therefore z = \frac{3\sqrt{3}}{2} - \frac{3}{2}i, 3i, -\frac{3\sqrt{3}}{2} - \frac{3}{2}i$

**3**  $z^3 = -1$

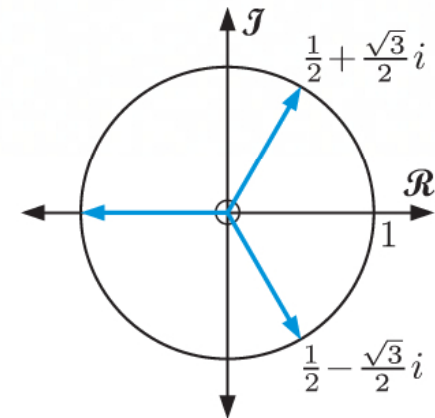
$\therefore z^3 = 1 \operatorname{cis} (\pi + k2\pi) \quad \text{where } k \in \mathbb{Z} \quad \{\text{polar form}\}$

$\therefore z = \left( \operatorname{cis} (\pi + k2\pi) \right)^{\frac{1}{3}}$

$\therefore z = \operatorname{cis} \left( \frac{\pi}{3} + \frac{k2\pi}{3} \right) \quad \{\text{De Moivre}\}$

$\therefore z = \operatorname{cis} \frac{\pi}{3}, \operatorname{cis} \pi, \operatorname{cis} \frac{5\pi}{3} \quad \{\text{letting } k = 0, 1, 2\}$

$\therefore z = \frac{1}{2} + i\frac{\sqrt{3}}{2}, -1, \frac{1}{2} - i\frac{\sqrt{3}}{2}$



**4 a**  $z^4 = 16$

$\therefore z^4 = 16 \operatorname{cis} (0 + k2\pi) \quad \text{where } k \in \mathbb{Z} \quad \{\text{polar form}\}$

$\therefore z = \left( 16 \operatorname{cis} k2\pi \right)^{\frac{1}{4}}$

$\therefore z = 16^{\frac{1}{4}} \operatorname{cis} \frac{k\pi}{2} \quad \{\text{De Moivre}\}$

$\therefore z = 2 \operatorname{cis} 0, 2 \operatorname{cis} \frac{\pi}{2}, 2 \operatorname{cis} \pi, 2 \operatorname{cis} \frac{3\pi}{2} \quad \{\text{letting } k = 0, 1, 2, 3\}$

$\therefore z = 2, 2i, -2, -2i$

$\therefore z = \pm 2 \text{ or } \pm 2i$

**b**  $z^4 = -16$

$\therefore z^4 = 16 \operatorname{cis} (\pi + k2\pi) \quad \text{where } k \in \mathbb{Z} \quad \{\text{polar form}\}$

$\therefore z = \left( 16 \operatorname{cis} (\pi + k2\pi) \right)^{\frac{1}{4}}$

$\therefore z = 16^{\frac{1}{4}} \operatorname{cis} \left( \frac{\pi}{4} + \frac{k2\pi}{4} \right) \quad \{\text{De Moivre}\}$

$\therefore z = 2 \operatorname{cis} \frac{\pi}{4}, 2 \operatorname{cis} \frac{3\pi}{4}, 2 \operatorname{cis} \frac{5\pi}{4}, 2 \operatorname{cis} \frac{7\pi}{4} \quad \{\text{letting } k = 0, 1, 2, 3\}$

$\therefore z = 2 \left( \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}} \right), 2 \left( -\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}} \right), 2 \left( -\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}} \right), 2 \left( \frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}} \right)$

$\therefore z = \sqrt{2} \pm i\sqrt{2}, -\sqrt{2} \pm i\sqrt{2}$



**5**  $z^4 = -i$

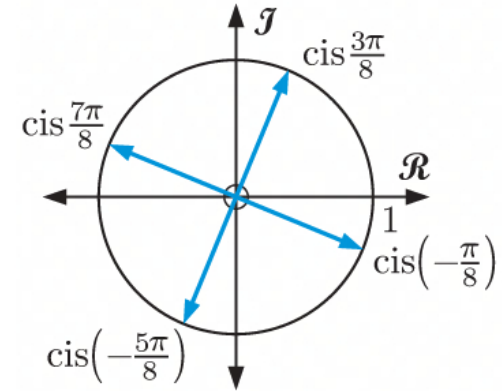
$\therefore z^4 = \text{cis}\left(-\frac{\pi}{2} + k2\pi\right)$  where  $k \in \mathbb{Z}$  {polar form}

$\therefore z = \left(\text{cis}\left(-\frac{\pi}{2} + k2\pi\right)\right)^{\frac{1}{4}}$

$\therefore z = \text{cis}\left(-\frac{\pi}{8} + \frac{k\pi}{2}\right)$  {De Moivre}

$\therefore z = \text{cis}\left(-\frac{\pi}{8} + \frac{k4\pi}{8}\right)$

$\therefore z = \text{cis}\left(-\frac{5\pi}{8}\right), \text{cis}\left(-\frac{\pi}{8}\right), \text{cis}\frac{3\pi}{8}, \text{cis}\frac{7\pi}{8}$   
{letting  $k = -1, 0, 1, 2$ }



**6 a**  $z^3 = 2 + 2i$

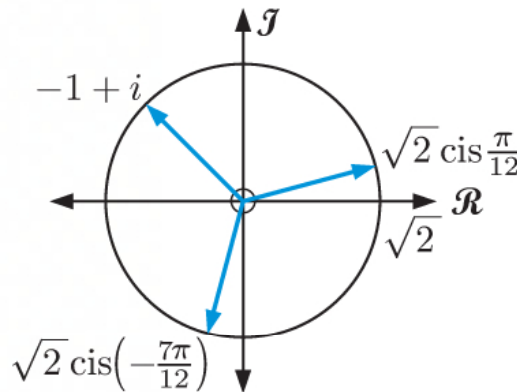
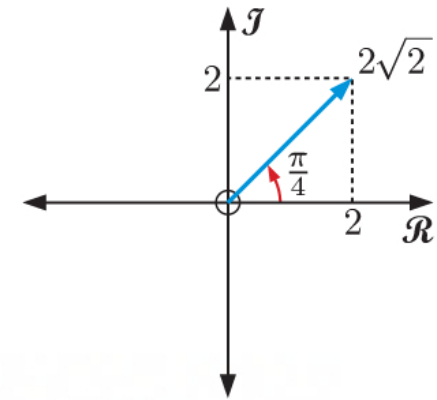
$\therefore z^3 = 2\sqrt{2} \text{cis}\left(\frac{\pi}{4} + k2\pi\right)$  where  $k \in \mathbb{Z}$  {polar form}

$\therefore z = \left[2\sqrt{2} \text{cis}\left(\frac{\pi}{4} + k2\pi\right)\right]^{\frac{1}{3}}$

$\therefore z = \left(2\sqrt{2}\right)^{\frac{1}{3}} \text{cis}\left(\frac{\pi}{12} + \frac{k2\pi}{3}\right)$  {De Moivre}

$\therefore z = \sqrt{2} \text{cis}\left(\frac{\pi}{12} + \frac{k8\pi}{12}\right)$

$\therefore z = \sqrt{2} \text{cis}\left(-\frac{7\pi}{12}\right), \sqrt{2} \text{cis}\frac{\pi}{12}, \sqrt{2} \text{cis}\frac{3\pi}{4} = -1 + i$  {letting  $k = -1, 0, 1$ }



**b**  $z^3 = -2 + 2i$

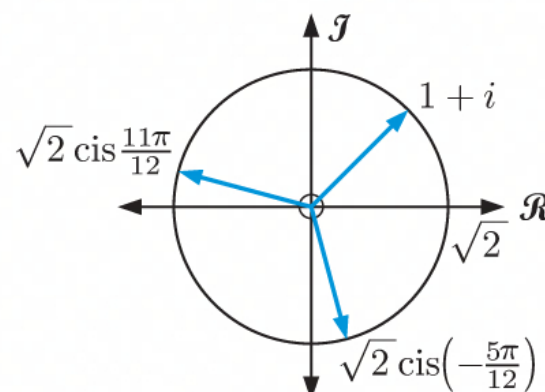
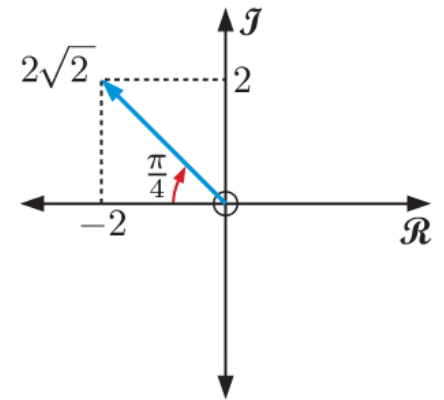
$\therefore z^3 = 2\sqrt{2} \text{cis}\left(\frac{3\pi}{4} + k2\pi\right)$  where  $k \in \mathbb{Z}$  {polar form}

$\therefore z = \left[2\sqrt{2} \text{cis}\left(\frac{3\pi}{4} + k2\pi\right)\right]^{\frac{1}{3}}$

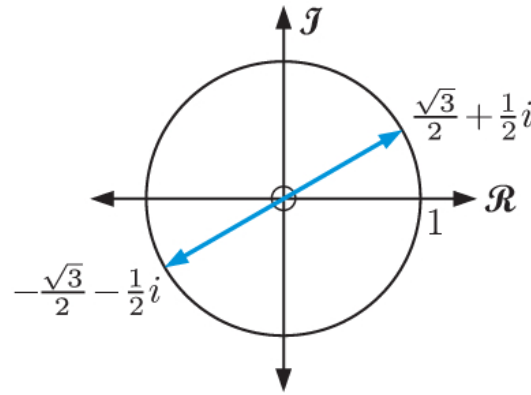
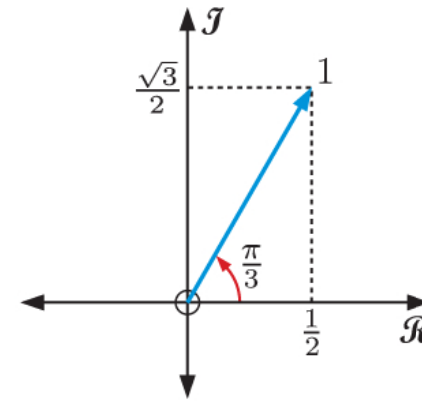
$\therefore z = \left(2\sqrt{2}\right)^{\frac{1}{3}} \text{cis}\left(\frac{\pi}{4} + \frac{k2\pi}{3}\right)$  {De Moivre}

$\therefore z = \sqrt{2} \text{cis}\left(\frac{3\pi}{12} + \frac{k8\pi}{12}\right)$

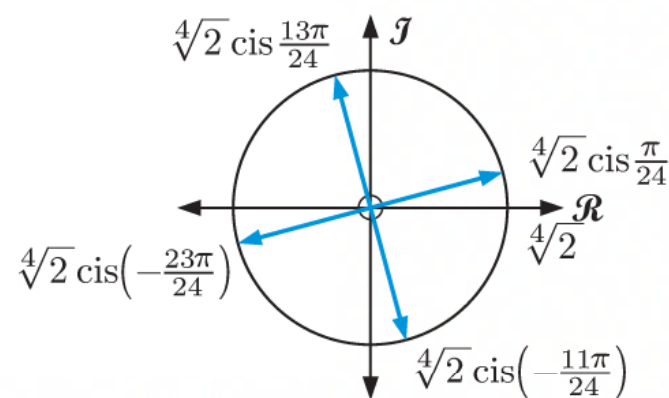
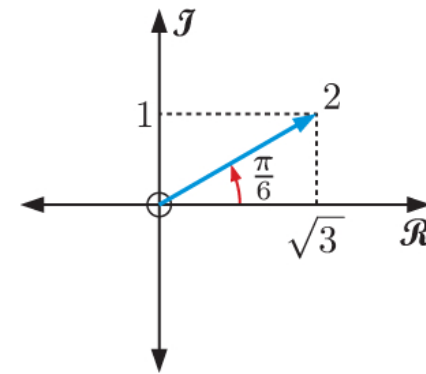
$\therefore z = \sqrt{2} \text{cis}\left(-\frac{5\pi}{12}\right), \sqrt{2} \text{cis}\frac{\pi}{4} = 1 + i, \sqrt{2} \text{cis}\frac{11\pi}{12}$  {letting  $k = -1, 0, 1$ }



$$\begin{aligned}
 \text{c} \quad z^2 &= \frac{1}{2} + \frac{\sqrt{3}}{2}i \\
 \therefore z^2 &= \text{cis} \left( \frac{\pi}{3} + k2\pi \right) \quad \text{where } k \in \mathbb{Z} \quad \{\text{polar form}\} \\
 \therefore z &= \left[ \text{cis} \left( \frac{\pi}{3} + k2\pi \right) \right]^{\frac{1}{2}} \\
 \therefore z &= \text{cis} \left( \frac{\pi}{6} + k\pi \right) \quad \{\text{De Moivre}\} \\
 \therefore z &= \text{cis} \left( \frac{\pi}{6} + \frac{k6\pi}{6} \right) \\
 \therefore z &= \text{cis} \left( -\frac{5\pi}{6} \right) = -\frac{\sqrt{3}}{2} - \frac{1}{2}i, \quad \text{cis} \frac{\pi}{6} = \frac{\sqrt{3}}{2} + \frac{1}{2}i \quad \{\text{letting } k = -1, 0\}
 \end{aligned}$$



$$\begin{aligned}
 \text{d} \quad z^4 &= \sqrt{3} + i \\
 \therefore z^4 &= 2 \text{cis} \left( \frac{\pi}{6} + k2\pi \right) \quad \text{where } k \in \mathbb{Z} \quad \{\text{polar form}\} \\
 \therefore z &= \left[ 2 \text{cis} \left( \frac{\pi}{6} + k2\pi \right) \right]^{\frac{1}{4}} \\
 \therefore z &= 2^{\frac{1}{4}} \text{cis} \left( \frac{\pi}{24} + \frac{k\pi}{2} \right) \quad \{\text{De Moivre}\} \\
 \therefore z &= \sqrt[4]{2} \text{cis} \left( \frac{\pi}{24} + \frac{k12\pi}{24} \right) \\
 \therefore z &= \sqrt[4]{2} \text{cis} \left( -\frac{23\pi}{24} \right), \sqrt[4]{2} \text{cis} \left( -\frac{11\pi}{24} \right), \sqrt[4]{2} \text{cis} \frac{\pi}{24}, \sqrt[4]{2} \text{cis} \frac{13\pi}{24} \\
 &\quad \{\text{letting } k = -2, -1, 0, 1\}
 \end{aligned}$$



**e**  $z^5 = -4 - 4i$

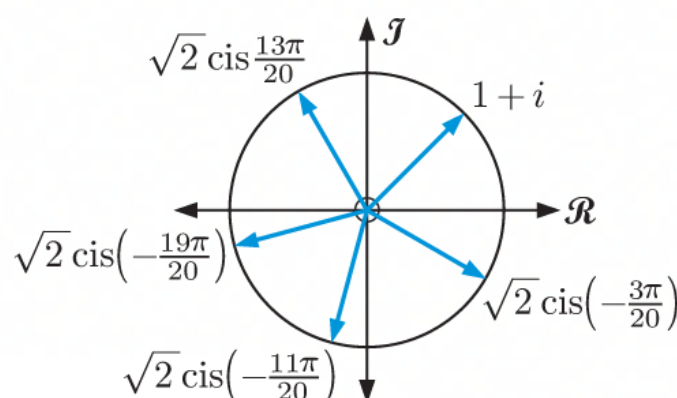
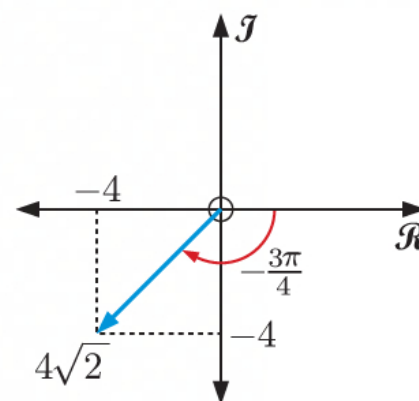
$\therefore z^5 = 4\sqrt{2} \operatorname{cis}\left(-\frac{3\pi}{4} + k2\pi\right)$  where  $k \in \mathbb{Z}$  {polar form}

$\therefore z = \left[4\sqrt{2} \operatorname{cis}\left(-\frac{3\pi}{4} + k2\pi\right)\right]^{\frac{1}{5}}$

$\therefore z = \left(4\sqrt{2}\right)^{\frac{1}{5}} \operatorname{cis}\left(-\frac{3\pi}{20} + \frac{k2\pi}{5}\right)$  {De Moivre}

$\therefore z = \sqrt{2} \operatorname{cis}\left(-\frac{3\pi}{20} + \frac{k8\pi}{20}\right)$

$\therefore z = \sqrt{2} \operatorname{cis}\left(-\frac{19\pi}{20}\right), \sqrt{2} \operatorname{cis}\left(-\frac{11\pi}{20}\right), \sqrt{2} \operatorname{cis}\left(-\frac{3\pi}{20}\right), \sqrt{2} \operatorname{cis}\frac{\pi}{4} = 1 + i,$   
 $\sqrt{2} \operatorname{cis}\frac{13\pi}{20}$  {letting  $k = -2, -1, 0, 1, 2$ }



**f**  $z^3 = -2\sqrt{3} - 2i$

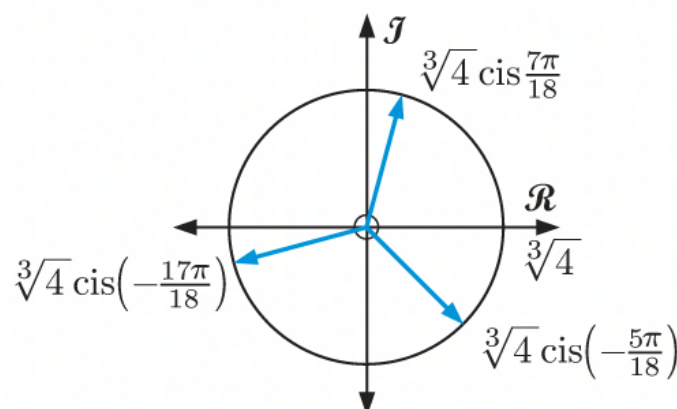
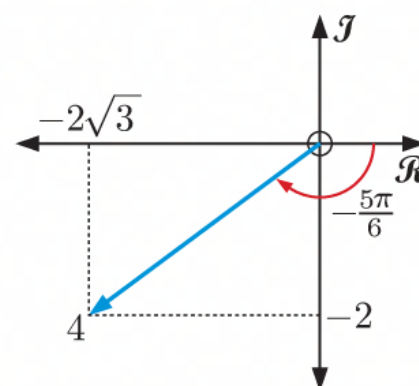
$\therefore z^3 = 4 \operatorname{cis}\left(-\frac{5\pi}{6} + k2\pi\right)$  where  $k \in \mathbb{Z}$  {polar form}

$\therefore z = \left[4 \operatorname{cis}\left(-\frac{5\pi}{6} + k2\pi\right)\right]^{\frac{1}{3}}$

$\therefore z = 4^{\frac{1}{3}} \operatorname{cis}\left(-\frac{5\pi}{18} + \frac{k2\pi}{3}\right)$  {De Moivre}

$\therefore z = \sqrt[3]{4} \operatorname{cis}\left(-\frac{5\pi}{18} + \frac{k12\pi}{18}\right)$

$\therefore z = \sqrt[3]{4} \operatorname{cis}\left(-\frac{17\pi}{18}\right), \sqrt[3]{4} \operatorname{cis}\left(-\frac{5\pi}{18}\right), \sqrt[3]{4} \operatorname{cis}\frac{7\pi}{18}$  {letting  $k = -1, 0, 1$ }



**7 a**  $z^4 + 1 = 0$

$\therefore z^4 = -1$

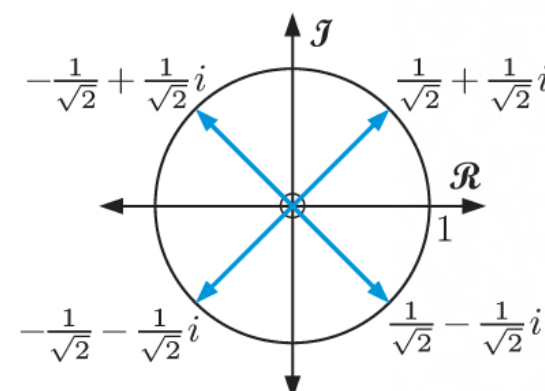
$\therefore z^4 = \operatorname{cis}(\pi + k2\pi)$  where  $k \in \mathbb{Z}$  {polar form}

$\therefore z = [\operatorname{cis}(\pi + k2\pi)]^{\frac{1}{4}}$

$\therefore z = \operatorname{cis}\left(\frac{\pi}{4} + \frac{k2\pi}{4}\right)$  {De Moivre}

$\therefore z = \operatorname{cis}\left(-\frac{3\pi}{4}\right), \operatorname{cis}\left(-\frac{\pi}{4}\right), \operatorname{cis}\frac{\pi}{4}, \operatorname{cis}\frac{3\pi}{4}$   
{letting  $k = -2, -1, 0, 1$ }

$\therefore z = \frac{1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}}i, -\frac{1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}}i$



**b** The roots  $\frac{1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}}i$  have sum  $= \sqrt{2}$

$$\text{and product} = \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) = 1$$

$\therefore$  they come from the quadratic factor  $z^2 - \sqrt{2}z + 1$ .

The roots  $-\frac{1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}}i$  have sum  $= -\sqrt{2}$

$$\text{and product} = \left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) = 1$$

$\therefore$  they come from the quadratic factor  $z^2 + \sqrt{2}z + 1$ .

$$\text{Thus } z^4 + 1 = (z^2 - \sqrt{2}z + 1)(z^2 + \sqrt{2}z + 1)$$

**8 a**

$$\begin{aligned} z &= \frac{\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)^2}{\left(\cos \frac{\pi}{10} - i \sin \frac{\pi}{10}\right)^5 \left(\cos \frac{\pi}{30} + i \sin \frac{\pi}{30}\right)^{25}} \\ &= \frac{\text{cis}\left(-\frac{\pi}{6}\right)^2}{\text{cis}\left(-\frac{\pi}{10}\right)^5 \left(\text{cis} \frac{\pi}{30}\right)^{25}} \\ &= \frac{\text{cis}\left(-\frac{\pi}{3}\right)}{\text{cis}\left(-\frac{\pi}{2}\right) \text{cis} \frac{5\pi}{6}} \quad \{\text{De Moivre}\} \\ &= \frac{\text{cis}\left(-\frac{\pi}{3}\right)}{\text{cis} \frac{\pi}{3}} \\ &= \text{cis}\left(-\frac{2\pi}{3}\right) \end{aligned}$$

$$\therefore |z| = 1, \quad \arg z = -\frac{2\pi}{3}$$

$$\begin{aligned} \text{b } z^3 &= \left[\text{cis}\left(-\frac{2\pi}{3}\right)\right]^3 \\ &= \text{cis}(-2\pi) \quad \{\text{De Moivre}\} \\ &= 1 \end{aligned}$$

$\therefore z$  is a cube root of 1.

$$\begin{aligned} \text{c } (1 - 2z)(2z^2 - 1) &= 2z^2 - 1 - 4z^3 + 2z \\ &= 2z^2 - 1 - 4(1) + 2z \quad \{z^3 = 1\} \\ &= 2z^2 + 2z - 5 \end{aligned}$$

$$\begin{aligned} \text{Now } z^3 &= 1 \quad \therefore z^2 = z^{-1}, \quad z \neq 0 \\ &= \left[\text{cis}\left(-\frac{2\pi}{3}\right)\right]^{-1} \\ &= \text{cis} \frac{2\pi}{3} \quad \{\text{De Moivre}\} \\ &= z^* \end{aligned}$$

$$\begin{aligned} \therefore 2z^2 + 2z - 5 &= 2z^* + 2z - 5 \\ &= 2(z + z^*) - 5 \end{aligned}$$

which is real as  $z + z^*$  is always real.



**9 a**  $-16i = 16 \operatorname{cis} \left(-\frac{\pi}{2}\right)$

**b i**  $z^4 = -16i$

$$\therefore z^4 = 16 \operatorname{cis} \left(-\frac{\pi}{2} + k2\pi\right) \quad \text{where } k \in \mathbb{Z} \quad \{\text{polar form}\}$$

$$\therefore z = \left[16 \operatorname{cis} \left(-\frac{\pi}{2} + k2\pi\right)\right]^{\frac{1}{4}}$$

$$\therefore z = 16^{\frac{1}{4}} \operatorname{cis} \left(-\frac{\pi}{8} + \frac{k\pi}{2}\right) \quad \{\text{De Moivre}\}$$

$$\therefore z = 2 \operatorname{cis} \left(-\frac{\pi}{8} + \frac{k4\pi}{8}\right)$$

$$\therefore z = 2 \operatorname{cis} \left(-\frac{5\pi}{8}\right), 2 \operatorname{cis} \left(-\frac{\pi}{8}\right), 2 \operatorname{cis} \frac{3\pi}{8}, 2 \operatorname{cis} \frac{7\pi}{8}$$

{letting  $k = -1, 0, 1, 2$ }

The 4th root in the second quadrant is  $z = 2 \operatorname{cis} \frac{7\pi}{8} \quad \left\{\frac{\pi}{2} < \frac{7\pi}{8} < \pi\right\}$

**ii** In Cartesian form,  $z = 2 \left(\cos \frac{7\pi}{8} + i \sin \frac{7\pi}{8}\right)$   
 $= 2 \cos \frac{7\pi}{8} + \left(2 \sin \frac{7\pi}{8}\right) i$

## EXERCISE 14G.2

**1 a i**  $(z + 3)^3 = 1$

$$\therefore z + 3 = 1, w, \text{ or } w^2 \quad \text{where } w = \operatorname{cis} \frac{2\pi}{3}$$

$$\therefore z + 3 = w^n \quad \text{where } n = 0, 1, 2$$

$$\therefore z = w^n - 3 \quad \text{where } n = 0, 1, 2 \quad \text{and } w = \operatorname{cis} \frac{2\pi}{3}$$

**ii**  $(z - 1)^3 = 8$

$$\therefore \left(\frac{z - 1}{2}\right)^3 = 1$$

$$\therefore \frac{z - 1}{2} = 1, w, \text{ or } w^2 \quad \text{where } w = \operatorname{cis} \frac{2\pi}{3}$$

$$\therefore \frac{z - 1}{2} = w^n \quad \text{where } n = 0, 1, 2$$

$$\therefore z = 2w^n + 1 \quad \text{where } n = 0, 1, 2 \quad \text{and } w = \operatorname{cis} \frac{2\pi}{3}$$

**iii**  $(2z - 1)^3 = -1$

$$\therefore (2z - 1)^3 = (-1)^3$$

$$\therefore \left(\frac{2z - 1}{-1}\right)^3 = 1$$

$$\therefore (1 - 2z)^3 = 1$$

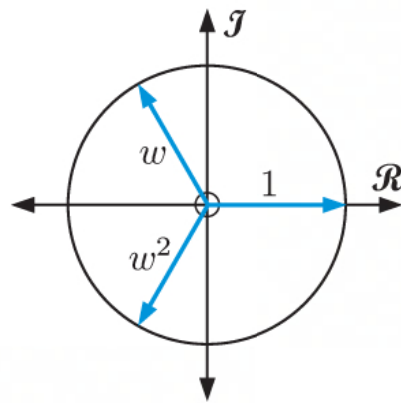
$$\therefore 1 - 2z = 1, w, w^2 \quad \text{where } w = \operatorname{cis} \frac{2\pi}{3}$$

$$\therefore 1 - 2z = w^n \quad \text{where } n = 0, 1, 2$$

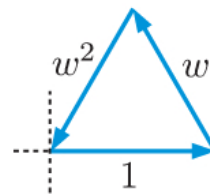
$$\therefore 2z = 1 - w^n$$

$$\therefore z = \frac{1 - w^n}{2} \quad \text{where } n = 0, 1, 2 \quad \text{and } w = \operatorname{cis} \frac{2\pi}{3}$$

- b** The following represents the cube roots of unity:



Adding these vectorially



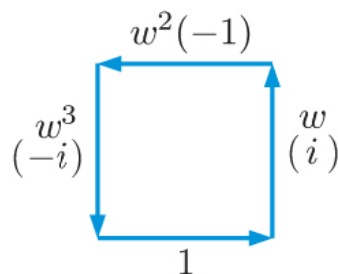
the resultant vector is **0**

$$\therefore 1 + w + w^2 = 0$$

- 2 a** If  $w = \text{cis } \frac{\pi}{2}$ ,  $w^2 = \text{cis } \pi$  and  $w^3 = \text{cis } \frac{3\pi}{2}$  {De Moivre}  
 $\therefore w = i$ ,  $w^2 = -1$ , and  $w^3 = -i$   
 $\therefore 1, i, -1, -i$  can be written as  $1, w, w^2, w^3$ , where  $w = \text{cis } \frac{\pi}{2}$

- b**  $1 + w + w^2 + w^3 = 1 + (i) + (-1) + (-i)$   
 $= 1 + i - 1 - i$   
 $= 0$

- c** Adding these vectorially:



The resultant vector is **0**.

- 3 a** The fifth roots of unity are the solutions of  $z^5 = 1$ .

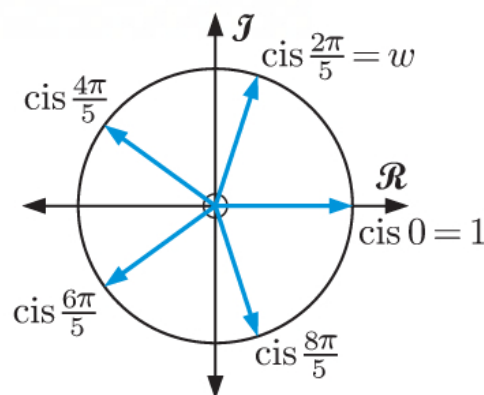
Now  $1 = \text{cis}(0 + k2\pi)$  where  $k \in \mathbb{Z}$  {polar form}

$$\therefore z^5 = \text{cis } k2\pi$$

$$\therefore z = (\text{cis } k2\pi)^{\frac{1}{5}}$$

$$\therefore z = \text{cis } \frac{k2\pi}{5} \quad \{\text{De Moivre}\}$$

$$\therefore z = \text{cis } 0 = 1, \text{cis } \frac{2\pi}{5}, \text{cis } \frac{4\pi}{5}, \text{cis } \frac{6\pi}{5}, \text{cis } \frac{8\pi}{5} \quad \{\text{letting } k = 0, 1, 2, 3, 4\}$$



- b** Letting  $w = \text{cis } \frac{2\pi}{5}$ , then  $w^2 = (\text{cis } \frac{2\pi}{5})^2 = \text{cis } \frac{4\pi}{5}$   
 $w^3 = (\text{cis } \frac{2\pi}{5})^3 = \text{cis } \frac{6\pi}{5}$   
 $w^4 = (\text{cis } \frac{2\pi}{5})^4 = \text{cis } \frac{8\pi}{5}$

$\therefore$  the roots are  $1, w, w^2, w^3$ , and  $w^4$ .

$$\begin{aligned}
 & (1 + w + w^2 + w^3 + w^4)(1 - w) \\
 &= 1 + w + w^2 + w^3 + w^4 - w - w^2 - w^3 - w^4 - w^5 \\
 &= 1 - w^5
 \end{aligned}$$

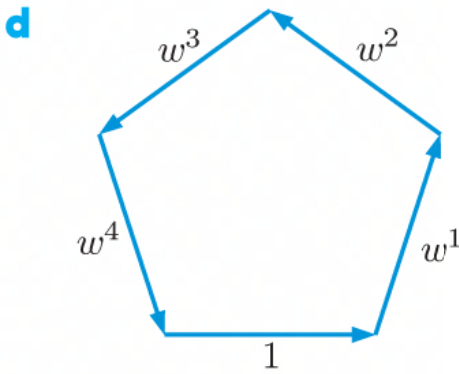
$$w = \operatorname{cis} \frac{2\pi}{5}$$

$$\begin{aligned}
 \therefore w^5 &= \left(\operatorname{cis} \frac{2\pi}{5}\right)^5 \\
 &= \operatorname{cis} 2\pi = 1 \quad \{\text{De Moivre}\}
 \end{aligned}$$

$$\therefore 1 - w^5 = 0$$

$$\therefore (1 + w + w^2 + w^3 + w^4)(1 - w) = 0$$

$$\text{But } w \neq 1, \text{ so } 1 + w + w^2 + w^3 + w^4 = 0$$



**4 a** The  $n$ th roots of unity are the solutions to  $z^n = 1$

$$z^n = 1$$

$$\therefore z^n = \operatorname{cis}(0 + k2\pi) \quad \text{where } k \in \mathbb{Z} \quad \{\text{polar form}\}$$

$$\therefore z^n = \operatorname{cis} k2\pi$$

$$\therefore z = (\operatorname{cis} k2\pi)^{\frac{1}{n}}$$

$$\therefore z = \operatorname{cis} \frac{k2\pi}{n} \quad \{\text{De Moivre}\}$$

The  $n$ th root of unity with smallest positive argument is  $w = \operatorname{cis} \frac{2\pi}{n}$ , when  $k = 1$ .

**b i** The  $n$ th roots of unity are

$$\operatorname{cis} 0 = 1 \quad \text{or}$$

$$\operatorname{cis} \frac{2\pi}{n} = w \quad \text{or}$$

$$\operatorname{cis} \frac{4\pi}{n} = \left(\operatorname{cis} \frac{2\pi}{n}\right)^2 = w^2 \quad \text{or}$$

$$\vdots$$

$$\operatorname{cis} \left[\frac{2\pi}{n}(n-1)\right] = \left(\operatorname{cis} \frac{2\pi}{n}\right)^{n-1} = w^{n-1} \quad \{\text{letting } k = 0, 1, 2, 3, \dots, n-1\}$$

$$\therefore \text{ the } n \text{ roots of } z^n = 1 \text{ are } 1, w, w^2, w^3, \dots, w^{n-1} \text{ where } w = \operatorname{cis} \frac{2\pi}{n}$$

ii  $1 + w + w^2 + \dots + w^{n-1}$  is a geometric series with  $u_1 = 1$  and  $r = w = \text{cis } \frac{2\pi}{n}$ .

$$\begin{aligned} \therefore \text{ it has sum } S_n &= \frac{u_1(r^n - 1)}{r - 1} \\ &= \frac{1(w^n - 1)}{w - 1} \\ &= \frac{\left(\text{cis } \frac{2\pi}{n}\right)^n - 1}{\text{cis } \frac{2\pi}{n} - 1} \\ &= \frac{\text{cis } 2\pi - 1}{\text{cis } \frac{2\pi}{n} - 1} \quad \{\text{De Moivre}\} \\ &= \frac{1 - 1}{\text{cis } \frac{2\pi}{n} - 1} \\ &= 0 \end{aligned}$$

$$\text{So, } 1 + w + w^2 + \dots + w^{n-1} = 0$$

5 Let  $\alpha = r \text{cis } \theta$

$$\therefore z^n = r \text{cis } (\theta + k2\pi) \quad \text{where } k \in \mathbb{Z}$$

$$\therefore z = [r \text{cis } (\theta + k2\pi)]^{\frac{1}{n}}$$

$$\therefore z = r^{\frac{1}{n}} \text{cis } \left(\frac{\theta + k2\pi}{n}\right) \quad \{\text{De Moivre}\}$$

$$\therefore z = r^{\frac{1}{n}} \text{cis } \frac{\theta}{n} \text{cis } \frac{k2\pi}{n}$$

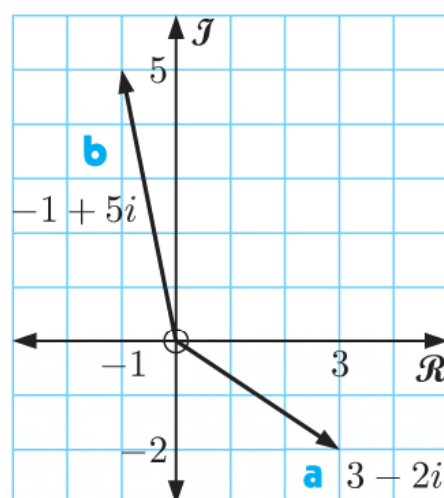
$$\begin{aligned} \therefore \text{ the } n \text{ roots of } z^n = \alpha \text{ are } &r^{\frac{1}{n}} \text{cis } \frac{\theta}{n}, r^{\frac{1}{n}} \text{cis } \frac{\theta}{n} \text{cis } \frac{2\pi}{n}, r^{\frac{1}{n}} \text{cis } \frac{\theta}{n} \text{cis } \frac{4\pi}{n}, \dots, \\ &r^{\frac{1}{n}} \text{cis } \frac{\theta}{n} \text{cis } \left[\frac{2\pi}{n}(n-1)\right] \quad \{\text{letting } k = 0, 1, 2, \dots, n-1\} \end{aligned}$$

$\therefore$  the sum of the  $n$  roots of  $z^n = \alpha$  is

$$\begin{aligned} &r^{\frac{1}{n}} \text{cis } \frac{\theta}{n} + r^{\frac{1}{n}} \text{cis } \frac{\theta}{n} \text{cis } \frac{2\pi}{n} + r^{\frac{1}{n}} \text{cis } \frac{\theta}{n} \text{cis } \frac{4\pi}{n} + \dots + r^{\frac{1}{n}} \text{cis } \frac{\theta}{n} \text{cis } \left(\frac{2\pi}{n}(n-1)\right) \\ &= r^{\frac{1}{n}} \text{cis } \frac{\theta}{n} \underbrace{\left[1 + \text{cis } \frac{2\pi}{n} + \text{cis } \frac{4\pi}{n} + \dots + \text{cis } \left(\frac{2\pi}{n}(n-1)\right)\right]}_{\text{these are the } n\text{th roots of unity, whose sum} = 0 \quad \{\text{using 4}\}} \\ &= 0 \end{aligned}$$

## REVIEW SET 14A

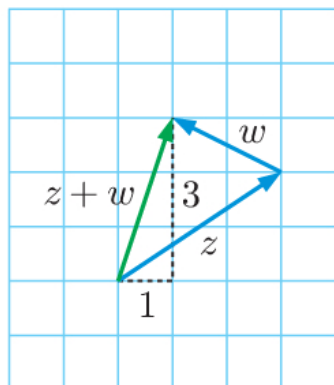
1



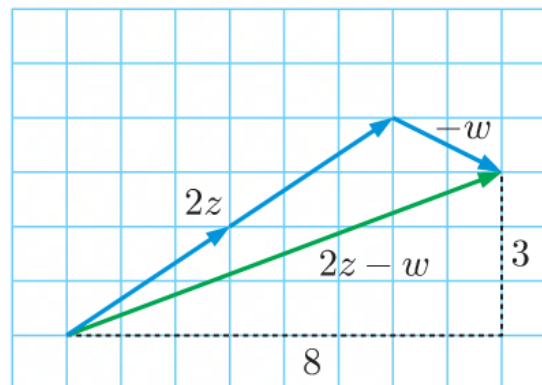


**2**  $z = 3 + 2i$ ,  $w = -2 + i$

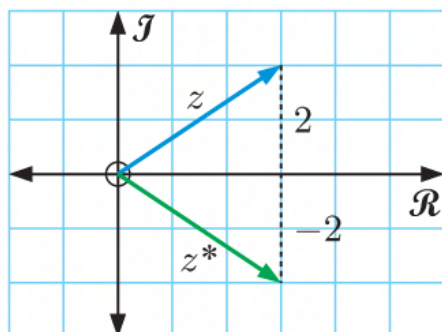
**a**  $z + w$   
 $= 3 + 2i + (-2 + i)$   
 $= 3 + 2i - 2 + i$   
 $= 1 + 3i$



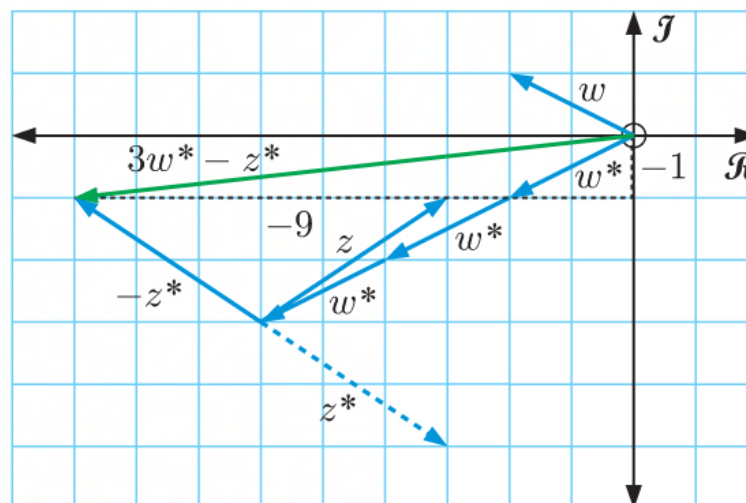
**b**  $2z - w$   
 $= 2(3 + 2i) - (-2 + i)$   
 $= 6 + 4i + 2 - i$   
 $= 8 + 3i$



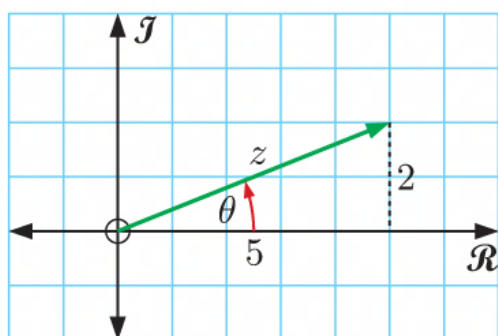
**c**  $z^*$   
 $= (3 + 2i)^*$   
 $= 3 - 2i$



**d**  $3w^* - z^*$   
 $= 3(-2 - i) - (3 - 2i)$   
 $= -6 - 3i - 3 + 2i$   
 $= -9 - i$

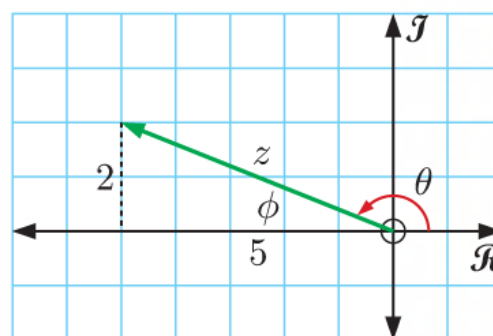


**3 a**  $|z| = \sqrt{5^2 + 2^2}$   
 $= \sqrt{29}$



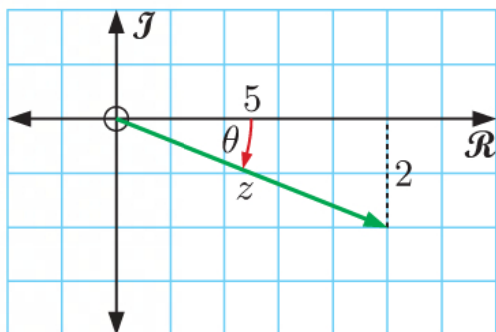
$\tan \theta = \frac{2}{5}$   
 $\therefore \arg z \approx 0.381$

**b**  $|z| = \sqrt{(-5)^2 + 2^2}$   
 $= \sqrt{29}$



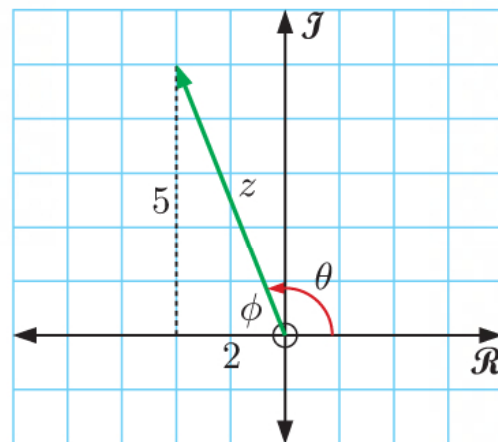
$\tan \phi = \frac{2}{5}$   
 $\therefore \phi \approx 0.381$   
 But  $\theta = \pi - \phi$   
 $\approx 2.76$   
 $\therefore \arg z \approx 2.76$

$$\begin{aligned} \text{c } |z| &= \sqrt{5^2 + (-2)^2} \\ &= \sqrt{29} \end{aligned}$$



$$\begin{aligned} \tan \theta &= \frac{2}{5} \\ \therefore \theta &\approx 0.381 \\ \therefore \arg z &\approx -0.381 \end{aligned}$$

$$\begin{aligned} \text{d } |z| &= \sqrt{(-2)^2 + 5^2} \\ &= \sqrt{29} \end{aligned}$$



$$\begin{aligned} \tan \phi &= \frac{5}{2} \\ \therefore \phi &\approx 1.19 \\ \text{But } \theta &= \pi - \phi \\ \therefore \arg z &\approx 1.95 \end{aligned}$$

$$\begin{aligned} \text{4 a } |5z| &= |5| |z| \\ &= 5 \times 5 \\ &= 25 \end{aligned}$$

$$\begin{aligned} \text{b } |-4z^*| &= |-4| |z^*| \\ &= 4 |z| \\ &= 4 \times 5 \\ &= 20 \end{aligned}$$

$$\begin{aligned} \text{c } |(3+i)z| &= |3+i| |z| \\ &= \sqrt{3^2 + 1^2} \times 5 \\ &= 5\sqrt{10} \end{aligned}$$

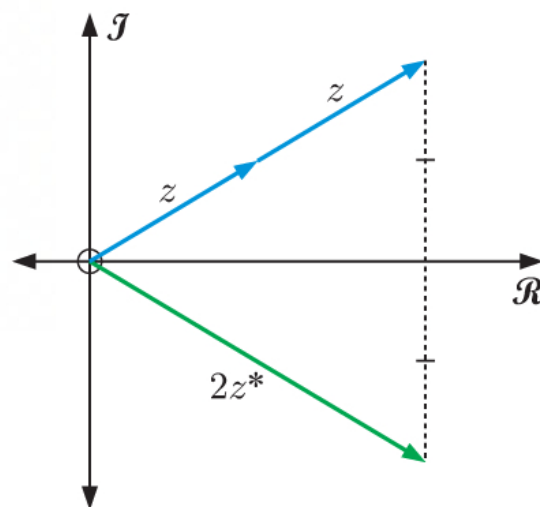
$$\begin{aligned} \text{d } \left| \frac{i}{z} \right| &= \frac{|i|}{|z|} \\ &= \frac{1}{5} \end{aligned}$$

$$\begin{aligned} \text{e } \left| \frac{3}{z^2} \right| &= \frac{|3|}{|z^2|} \\ &= \frac{3}{|z|^2} \\ &= \frac{3}{5^2} \\ &= \frac{3}{25} \end{aligned}$$

$$\begin{aligned} \text{f } \left| \frac{3+4i}{z} \right| &= \frac{|3+4i|}{|z|} \\ &= \frac{\sqrt{3^2 + 4^2}}{5} \\ &= \frac{5}{5} \\ &= 1 \end{aligned}$$

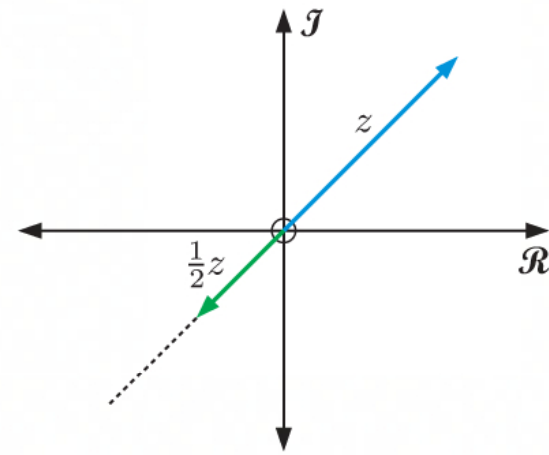
$$\text{5 a } z \mapsto z^* \mapsto 2z^*$$

A reflection in the real axis, followed by a stretch with scale factor 2.



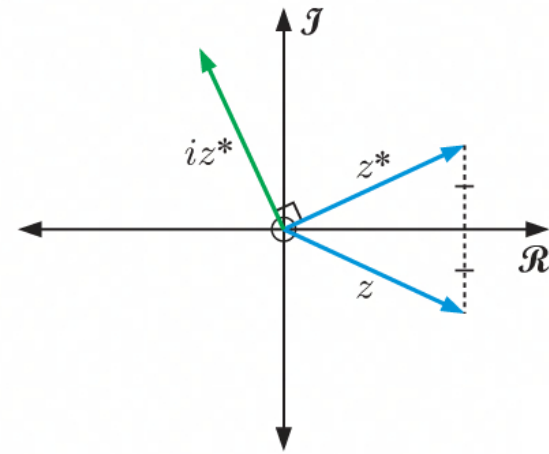
**b**  $z \mapsto -z \mapsto -\frac{1}{2}z$

A rotation of  $\pi$  about O, followed by a stretch with scale factor  $\frac{1}{2}$ .



**c**  $z \mapsto z^* \mapsto iz^*$

A reflection in the real axis, followed by an anticlockwise rotation of  $\frac{\pi}{2}$  about O.



**6**

$$\begin{aligned}(x + iy)^n &= X + Yi \\ \therefore |(x + yi)^n| &= |X + Yi| \\ \therefore |x + iy|^n &= |X + Yi| \\ \therefore \left(\sqrt{x^2 + y^2}\right)^n &= \sqrt{X^2 + Y^2}\end{aligned}$$

Squaring both sides,  $X^2 + Y^2 = (x^2 + y^2)^n$

**7**  $i - \sqrt{3}$  has modulus  $\sqrt{(-\sqrt{3})^2 + 1^2} = \sqrt{4} = 2$

Now  $\tan \theta = \frac{1}{\sqrt{3}}$

$\therefore \theta = \frac{\pi}{6}$

$\therefore \arg(i - \sqrt{3}) = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

$\therefore i - \sqrt{3} = 2 \operatorname{cis} \frac{5\pi}{6}$

$\therefore (i - \sqrt{3})^5 = 2^5 \operatorname{cis} \frac{25\pi}{6}$  {De Moivre's theorem}

$= 32 \operatorname{cis} \frac{\pi}{6}$

$= 32 \left( \cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right)$

$= 32 \left( \frac{\sqrt{3}}{2} + \frac{1}{2}i \right)$

$= 16\sqrt{3} + 16i$

$\therefore$  the real part of  $(i - \sqrt{3})^5$  is  $16\sqrt{3}$  and the imaginary part is 16.

$$\begin{aligned}
 8 \quad w &= \frac{1+z}{1+z^*} \\
 &= \left( \frac{1+\operatorname{cis} \phi}{1+\operatorname{cis}(-\phi)} \right) \times \frac{\operatorname{cis} \phi}{\operatorname{cis} \phi} \\
 &= \frac{(1+\operatorname{cis} \phi) \operatorname{cis} \phi}{\operatorname{cis} \phi + \operatorname{cis} 0} \\
 &= \frac{(1+\operatorname{cis} \phi) \operatorname{cis} \phi}{1+\operatorname{cis} \phi} \\
 &= \operatorname{cis} \phi
 \end{aligned}$$

9 The cube roots of  $-64i$  are the 3 solutions of  $z^3 = -64i$

$$\therefore z^3 = 64 \operatorname{cis}\left(-\frac{\pi}{2} + k2\pi\right) \quad \text{where } k \in \mathbb{Z} \quad \{\text{polar form}\}$$

$$\therefore z = [64 \operatorname{cis}\left(-\frac{\pi}{2} + k2\pi\right)]^{\frac{1}{3}}$$

$$\therefore z = 64^{\frac{1}{3}} \operatorname{cis}\left(-\frac{\pi}{6} + \frac{k2\pi}{3}\right) \quad \{\text{De Moivre's theorem}\}$$

$$\therefore z = 4 \operatorname{cis}\left(-\frac{\pi}{6} + \frac{k4\pi}{6}\right)$$

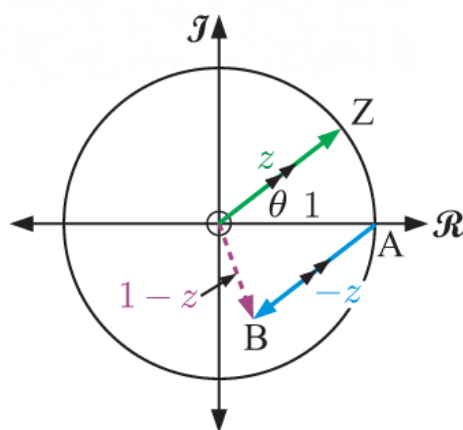
$$\therefore z = 4 \operatorname{cis}\left(-\frac{5\pi}{6}\right), 4 \operatorname{cis}\left(-\frac{\pi}{6}\right), 4 \operatorname{cis} \frac{\pi}{2} \quad \{\text{letting } k = -1, 0, 1\}$$

$$\therefore z = 4\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right), 4\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right), 4i$$

$$\therefore z = -2\sqrt{3} - 2i, 2\sqrt{3} - 2i, 4i$$

$$\begin{aligned}
 10 \quad a \quad (2z)^{-1} &= (2 \operatorname{cis} \theta)^{-1} \\
 &= 2^{-1} \operatorname{cis}(-\theta) \quad \{\text{De Moivre's theorem}\} \\
 \therefore |(2z)^{-1}| &= \frac{1}{2} \quad \text{and} \quad \arg[(2z)^{-1}] = -\theta
 \end{aligned}$$

$$\begin{aligned}
 b \quad 1 - z &= 1 - \operatorname{cis} \theta \\
 &= (1 - \cos \theta) - i \sin \theta \\
 \therefore |1 - z| &= \sqrt{(1 - \cos \theta)^2 + \sin^2 \theta} \\
 &= \sqrt{1 - 2 \cos \theta + \cos^2 \theta + \sin^2 \theta} \\
 &= \sqrt{2 - 2 \cos \theta} \\
 &= \sqrt{2 - 2(1 - 2 \sin^2(\frac{\theta}{2}))} \\
 &= \sqrt{4 \sin^2(\frac{\theta}{2})} \\
 &= 2 \sin \frac{\theta}{2}
 \end{aligned}$$



$\triangle OAB$  is isosceles since  $|z| = 1$ ,

so we let  $\widehat{AOB} = \widehat{ABO} = \phi$

Since  $[OZ] \parallel [AB]$ ,  $\widehat{OAB} = \theta$  {alternate angles}

$$\therefore \phi + \phi + \theta = \pi$$

$$\therefore 2\phi = \pi - \theta$$

$$\phi = \frac{\pi}{2} - \frac{\theta}{2}$$

But  $\arg(1 - z) = -\phi$ ,

$$\text{so } \arg(1 - z) = -\left(\frac{\pi}{2} - \frac{\theta}{2}\right) = \frac{\theta}{2} - \frac{\pi}{2}$$



**11**  $-1 + i\sqrt{3}$  has modulus  $\sqrt{(-1)^2 + (\sqrt{3})^2} = \sqrt{4} = 2$

$$\text{Now } \tan \theta = \frac{\sqrt{3}}{-1} = -\sqrt{3}$$

$$\therefore \theta = \frac{2\pi}{3}$$

$$\therefore \arg(-1 + i\sqrt{3}) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

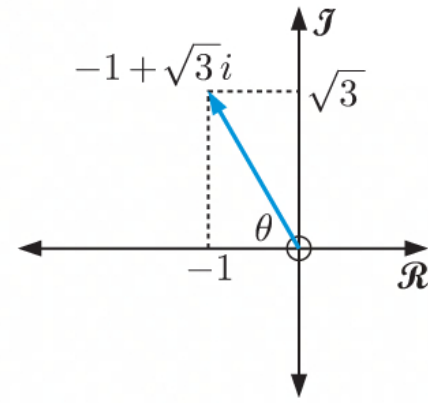
$$\therefore -1 + i\sqrt{3} = 2 \operatorname{cis} \frac{2\pi}{3}$$

$$\begin{aligned} \therefore (-1 + i\sqrt{3})^m &= 2^m \operatorname{cis} \frac{m2\pi}{3} \quad \{\text{De Moivre's theorem}\} \\ &= 2^m \left( \cos \frac{m2\pi}{3} + i \sin \frac{m2\pi}{3} \right) \end{aligned}$$

This is real provided  $\sin \frac{m2\pi}{3} = 0$

$$\therefore \frac{m2\pi}{3} = 0 + k\pi \quad \text{where } k \in \mathbb{Z}$$

$$\therefore m = \frac{3k}{2}, \quad k \in \mathbb{Z}$$



**12**  $z_1 = \operatorname{cis} \frac{\pi}{6}$  and  $z_2 = \operatorname{cis} \frac{\pi}{4}$

$$\begin{aligned} \therefore \left( \frac{z_1}{z_2} \right)^3 &= \left( \frac{\operatorname{cis} \frac{\pi}{6}}{\operatorname{cis} \frac{\pi}{4}} \right)^3 \\ &= \left[ \operatorname{cis} \left( \frac{\pi}{6} - \frac{\pi}{4} \right) \right]^3 \\ &= \left[ \operatorname{cis} \left( -\frac{\pi}{12} \right) \right]^3 \\ &= \operatorname{cis} \left( -\frac{3\pi}{12} \right) \quad \{\text{De Moivre's theorem}\} \\ &= \operatorname{cis} \left( -\frac{\pi}{4} \right) \\ &= \cos \left( -\frac{\pi}{4} \right) + i \sin \left( -\frac{\pi}{4} \right) \\ &= \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i \end{aligned}$$

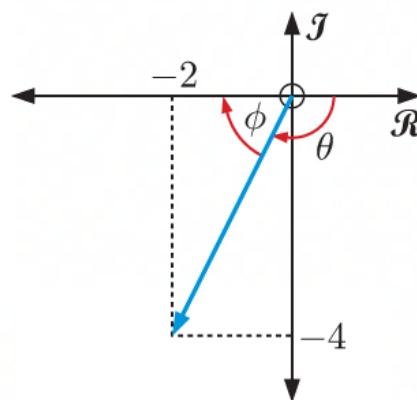
**13**  $z = 4 + i$ ,  $w = 2 - 3i$

$$\begin{aligned} \text{a} \quad 2w^* - iz &= 2(2 + 3i) - i(4 + i) \\ &= 4 + 6i - 4i - i^2 \\ &= 5 + 2i \end{aligned}$$

$$\begin{aligned} \text{b} \quad |w - z^*| &= |(2 - 3i) - (4 - i)| \\ &= |2 - 3i - 4 + i| \\ &= |-2 - 2i| \\ &= \sqrt{(-2)^2 + (-2)^2} \\ &= \sqrt{8} \\ &= 2\sqrt{2} \end{aligned}$$

$$\begin{aligned}
 \text{c } |z^{10}| &= |z|^{10} \\
 &= |4+i|^{10} \\
 &= (\sqrt{16+1})^{10} \\
 &= (\sqrt{17})^{10} \\
 &= 17^5
 \end{aligned}$$

$$\begin{aligned}
 \text{d } \arg(w-z) &= \arg[(2-3i) - (4+i)] \\
 &= \arg(-2-4i)
 \end{aligned}$$



$$\tan \phi = \frac{4}{2} = 2$$

$$\therefore \phi \approx 1.11$$

$$\begin{aligned}
 \text{But } \theta &= \pi - \phi \\
 &\approx 2.03
 \end{aligned}$$

$$\therefore \arg(w-z) \approx -2.03$$

**14 a** If  $\arg z = \frac{\pi}{2}$ , then we have a ray vertically upwards beginning at the origin.

If  $\arg(z-i) = \frac{\pi}{2}$ , the graph is translated  $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ , and we have a ray vertically upwards beginning at  $i$ .

$\therefore x = 0$ , and geometrically we require  $y > 1$ .

**b**

$$\left| \frac{z+2}{z-2} \right| = 2$$

$$\therefore \frac{|z+2|}{|z-2|} = 2$$

$$\therefore |z+2| = 2|z-2|$$

$$\text{If } z = x+iy, \text{ then } \sqrt{(x+2)^2 + y^2} = 2\sqrt{(x-2)^2 + y^2}$$

$$\therefore (x+2)^2 + y^2 = 4(x-2)^2 + 4y^2$$

$$\therefore x^2 + 4x + 4 + y^2 = 4x^2 - 16x + 16 + 4y^2$$

$$\therefore 3x^2 + 3y^2 - 20x + 12 = 0, \text{ which is a circle}$$

$$\text{15 } 2 - 2\sqrt{3}i \text{ has modulus } \sqrt{2^2 + (-2\sqrt{3})^2} = \sqrt{16} = 4$$

$$\text{Now } \tan \theta = \frac{2\sqrt{3}}{2}$$

$$= \sqrt{3}$$

$$\therefore \theta = \frac{\pi}{3}$$

$$\therefore \arg(2 - 2\sqrt{3}i) = -\frac{\pi}{3}$$

$$\therefore 2 - 2\sqrt{3}i = 4 \operatorname{cis} \left( -\frac{\pi}{3} \right)$$

$$\therefore (2 - 2\sqrt{3}i)^n = 4^n \operatorname{cis} \left( -\frac{n\pi}{3} \right)$$

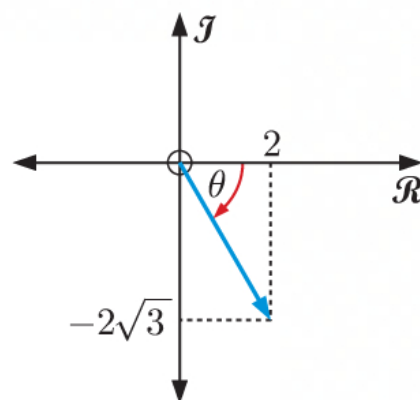
{De Moivre's theorem}

which is purely imaginary if  $\cos \left( -\frac{n\pi}{3} \right) = 0$

$$\therefore \cos \frac{n\pi}{3} = 0$$

$$\therefore \frac{n\pi}{3} = \frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z}$$

$$\therefore n = 3\left(k + \frac{1}{2}\right), \quad k \in \mathbb{Z}$$



**16** The cube roots of  $-27$  are the 3 solutions of  $z^3 = -27$ .

$$\therefore z^3 = 27 \operatorname{cis}(\pi + k2\pi) \quad \text{where } k \in \mathbb{Z} \quad \{\text{polar form}\}$$

$$\therefore z = [27 \operatorname{cis}(\pi + k2\pi)]^{\frac{1}{3}}$$

$$\therefore z = 27^{\frac{1}{3}} \operatorname{cis}\left(\frac{\pi + k2\pi}{3}\right) \quad \{\text{De Moivre's theorem}\}$$

$$\therefore z = 3 \operatorname{cis}\left(\frac{\pi + k2\pi}{3}\right)$$

$$\therefore z = 3 \operatorname{cis}\left(-\frac{\pi}{3}\right), 3 \operatorname{cis}\frac{\pi}{3}, 3 \operatorname{cis}\pi \quad \{\text{letting } k = -1, 0, 1\}$$

$$\therefore z = 3\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right), 3\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right), -3$$

$$\therefore z = \frac{3}{2} \pm \frac{3\sqrt{3}}{2}i \quad \text{or} \quad -3$$

**17 a**  $z = 4 \operatorname{cis} \theta$

$$\therefore z^3 = (4 \operatorname{cis} \theta)^3$$

$$= 4^3 \operatorname{cis} 3\theta \quad \{\text{De Moivre's theorem}\}$$

$$\therefore |z^3| = 64 \quad \text{and} \quad \arg(z^3) = 3\theta$$

**b**  $\frac{1}{z} = z^{-1}$

$$= (4 \operatorname{cis} \theta)^{-1}$$

$$= 4^{-1} \operatorname{cis}(-\theta) \quad \{\text{De Moivre's theorem}\}$$

$$\therefore \left|\frac{1}{z}\right| = \frac{1}{4} \quad \text{and} \quad \arg\left(\frac{1}{z}\right) = -\theta$$

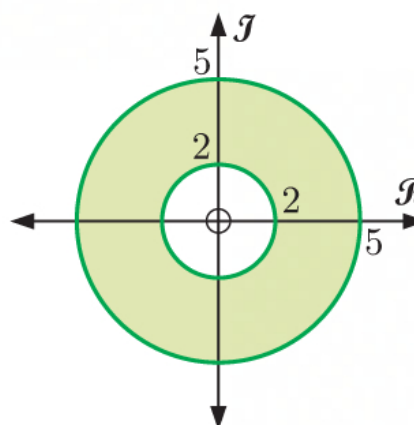
**c**  $z = 4 \operatorname{cis} \theta$

$$\therefore iz^* = (\operatorname{cis} \frac{\pi}{2})(4 \operatorname{cis}(-\theta))$$

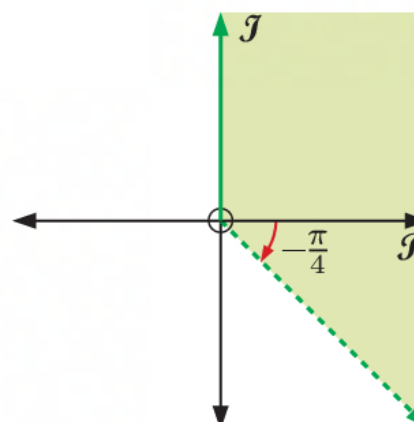
$$= 4 \operatorname{cis}\left(\frac{\pi}{2} - \theta\right)$$

$$\therefore |iz^*| = 4 \quad \text{and} \quad \arg(iz^*) = \frac{\pi}{2} - \theta$$

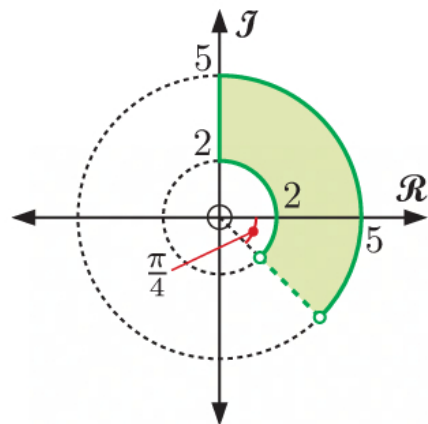
**18** The region defined by  $\{z \mid 2 \leq |z| \leq 5\}$  is:



The region defined by  $\{z \mid -\frac{\pi}{4} < \arg z \leq \frac{\pi}{2}\}$  is:



So the region defined by  $\{z \mid 2 \leq |z| \leq 5 \text{ and } -\frac{\pi}{4} < \arg z \leq \frac{\pi}{2}\}$  is:



**19** Let  $|z| = r$  and  $\arg z = \theta$ , so  $z = r \operatorname{cis} \theta$

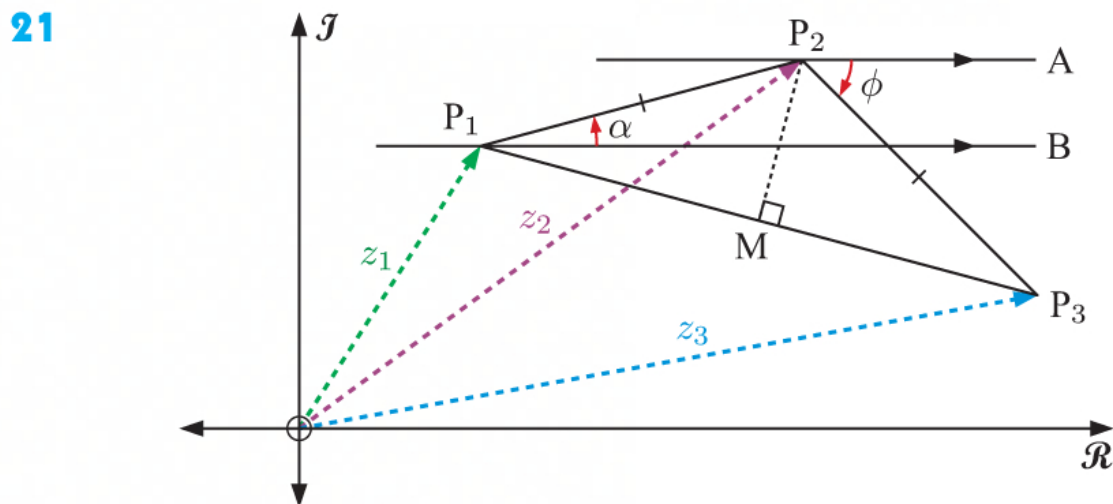
$$\begin{aligned} \text{Now } \frac{1}{z} &= (r \operatorname{cis} \theta)^{-1} = r^{-1} \operatorname{cis}(-\theta) \quad \{\text{De Moivre's theorem}\} \\ &= \frac{1}{r} \operatorname{cis}(-\theta) \end{aligned}$$

$$\therefore \left| \frac{1}{z} \right| = \frac{1}{r} = \frac{1}{|z|} \quad \{z \neq 0\}, \quad \text{and} \quad \arg\left(\frac{1}{z}\right) = -\theta = -\arg z$$

**20**  $z = \operatorname{cis} \alpha$

$$\begin{aligned} \therefore 1 + z &= 1 + \operatorname{cis} \alpha \\ &= 1 + \cos \alpha + i \sin \alpha \\ &= \left(1 + 2 \cos^2\left(\frac{\alpha}{2}\right) - 1\right) + i \left(2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2}\right) \\ &= 2 \cos^2\left(\frac{\alpha}{2}\right) + i \left(2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2}\right) \\ &= 2 \cos \frac{\alpha}{2} \left(\cos \frac{\alpha}{2} + i \sin \frac{\alpha}{2}\right) \\ &= 2 \cos \frac{\alpha}{2} \operatorname{cis} \frac{\alpha}{2} \end{aligned}$$

$$\therefore |1 + z| = 2 \cos \frac{\alpha}{2} \quad \text{and} \quad \arg(1 + z) = \frac{\alpha}{2}$$



**a**  $\arg(z_2 - z_1) = \alpha$ , and suppose  $\arg(z_3 - z_2) = -\phi$  as shown.

Let M be the midpoint of  $P_1$  and  $P_3$ , so the distance  $P_1M = \frac{\sqrt{3}P_1P_2}{2}$ .

$\widehat{P_1MP_2}$  is a right angle as  $\triangle P_1P_2P_3$  is isosceles,

$$\begin{aligned} \text{and } P_2M &= \sqrt{(P_1P_2)^2 - \left(\frac{\sqrt{3}}{2}P_1P_2\right)^2} \quad \{\text{Pythagoras}\} \\ &= \frac{1}{2}P_1P_2 \end{aligned}$$



$$\therefore \tan(\widehat{P_1 P_2 M}) = \frac{\frac{\sqrt{3}}{2} P_1 P_2}{\frac{1}{2} P_1 P_2} = \sqrt{3}, \quad \text{so} \quad \widehat{P_1 P_2 M} = \frac{\pi}{3} \quad \text{and} \quad \widehat{P_1 P_2 P_3} = \frac{2\pi}{3}$$

$$\therefore \alpha + \frac{2\pi}{3} + \phi = \pi \quad \{[P_1 B] \parallel [P_2 A], \text{ co-interior angles}\}$$

$$\therefore -\phi = \alpha - \frac{\pi}{3}$$

$$\therefore \arg(z_3 - z_2) = \alpha - \frac{\pi}{3} \quad \text{as required.}$$

**b**  $(z_2 - z_1) \equiv \overrightarrow{P_1 P_2}$  and  $(z_3 - z_2) \equiv \overrightarrow{P_2 P_3}$

$$\begin{aligned} \left| \frac{z_2 - z_1}{z_3 - z_2} \right| &= \frac{|z_2 - z_1|}{|z_3 - z_2|} \\ &= \frac{|\overrightarrow{P_1 P_2}|}{|\overrightarrow{P_2 P_3}|} \\ &= 1 \quad \{\triangle P_1 P_2 P_3 \text{ is isosceles with } P_1 P_2 = P_2 P_3\} \end{aligned}$$

$$\begin{aligned} \text{and} \quad \arg\left(\frac{z_2 - z_1}{z_3 - z_2}\right) &= \arg(z_2 - z_1) - \arg(z_3 - z_2) \\ &= \alpha - \left(\alpha - \frac{\pi}{3}\right) \quad \{\text{from a}\} \\ &= \frac{\pi}{3} \end{aligned}$$

**22** The fifth roots of unity are  $1, w, w^2, w^3$ , and  $w^4$ , where  $w = \text{cis } \frac{2\pi}{5}$ .

**a**  $(2z - 1)^5 = 32$

$$\therefore \frac{(2z - 1)^5}{2^5} = 1$$

$$\therefore \left(\frac{2z - 1}{2}\right)^5 = 1$$

$$\therefore \left(z - \frac{1}{2}\right)^5 = 1$$

$$\therefore z - \frac{1}{2} = 1, w, w^2, w^3, \text{ or } w^4$$

$$\therefore z = \frac{3}{2}, w + \frac{1}{2}, w^2 + \frac{1}{2}, w^3 + \frac{1}{2}, \text{ or } w^4 + \frac{1}{2} \quad \text{where } w = \text{cis } \frac{2\pi}{5}$$

**b**  $z^5 + 5z^4 + 10z^3 + 10z^2 + 5z = 0$

$$\therefore z^5 + 5z^4 + 10z^3 + 10z^2 + 5z + 1 = 1$$

$$\therefore (z + 1)^5 = 1$$

$$\therefore z + 1 = 1, w, w^2, w^3, w^4$$

$$\therefore z = 0, w - 1, w^2 - 1, w^3 - 1, \text{ or } w^4 - 1$$

$$\text{where } w = \text{cis } \frac{2\pi}{5}$$

**c**  $(z + 1)^5 = (z - 1)^5$

$$\therefore \frac{(z + 1)^5}{(z - 1)^5} = 1, \quad z \neq 1$$

$$\therefore \left(\frac{z + 1}{z - 1}\right)^5 = 1$$

$$\therefore \frac{z + 1}{z - 1} = 1, w, w^2, w^3, \text{ or } w^4$$

$$\frac{z + 1}{z - 1} = 1 \quad \text{has no solutions as } z + 1 \neq z - 1 \quad \text{for any } z$$

$$\text{If } \frac{z+1}{z-1} = w^k, \quad k = 1, 2, 3, 4$$

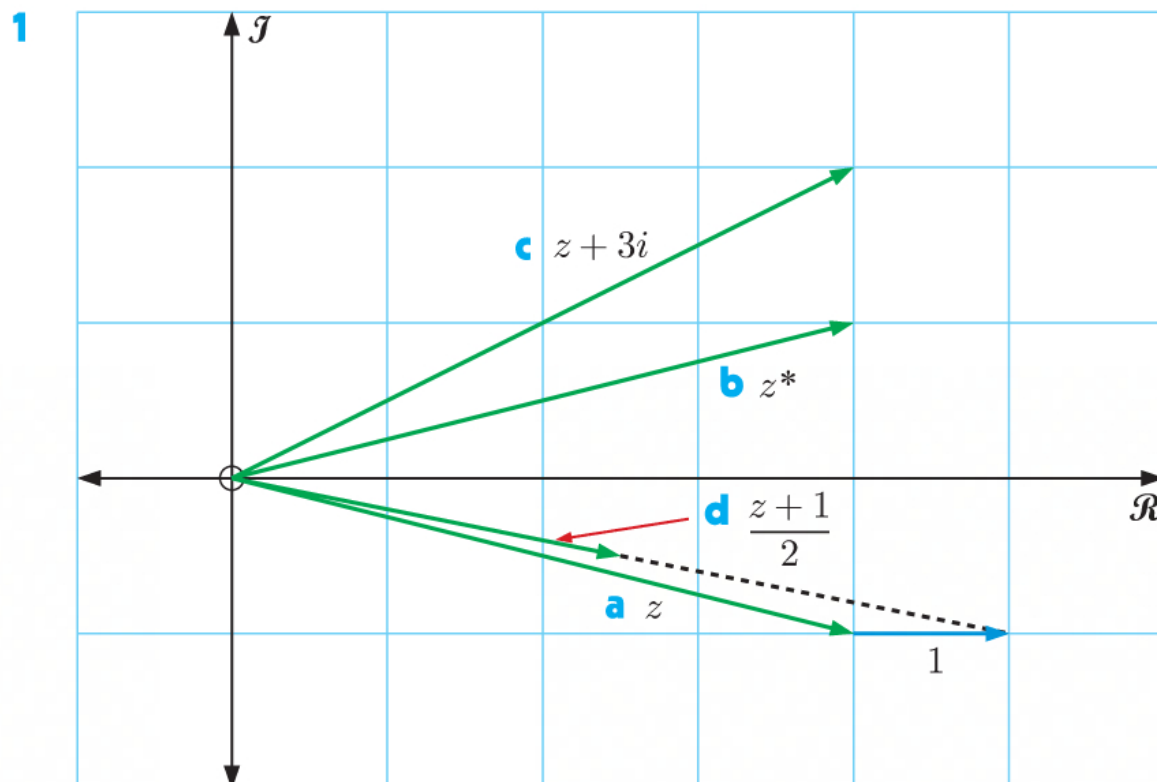
$$z+1 = w^k z - w^k$$

$$\therefore z(1 - w^k) = -w^k - 1$$

$$\therefore z = \frac{-w^k - 1}{1 - w^k} = \frac{w^k + 1}{w^k - 1}$$

$$\therefore z = \frac{w+1}{w-1}, \frac{w^2+1}{w^2-1}, \frac{w^3+1}{w^3-1}, \frac{w^4+1}{w^4-1} \quad \text{where } w = \text{cis } \frac{2\pi}{5}$$

## REVIEW SET 14B



**2 a**

$$\begin{aligned} |3z| &= |3| |z| \\ &= 3 \times 4 \\ &= 12 \end{aligned}$$

**b**

$$\begin{aligned} |2iz| &= |2i| |z| \\ &= 2 \times 4 \\ &= 8 \end{aligned}$$

**c**

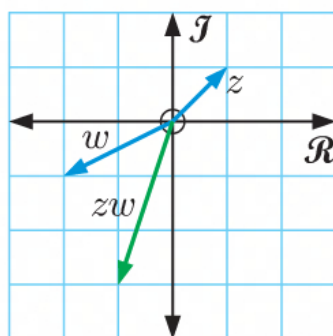
$$\begin{aligned} \left| \frac{i+1}{z} \right| &= \frac{|i+1|}{|z|} \\ &= \frac{\sqrt{1^2 + 1^2}}{4} \\ &= \frac{\sqrt{2}}{4} \end{aligned}$$

**d**

$$\begin{aligned} |(3-i)z| &= |3-i| |z| \\ &= 4 \times \sqrt{3^2 + (-1)^2} \\ &= 4\sqrt{10} \end{aligned}$$

**3**

$$\begin{aligned} zw &= (1+i)(-2-i) \\ &= -2-i-2i+1 \\ &= -1-3i \end{aligned}$$



$$4 \quad z = -1 + 3i, \quad w = 3 + i$$

$$a \quad |z| = \sqrt{(-1)^2 + 3^2} \\ = \sqrt{10}$$

$$b \quad |w| = \sqrt{3^2 + 1^2} \\ = \sqrt{10}$$

$$c \quad |z^*|^2 = |z|^2 \\ = (\sqrt{10})^2 \\ = 10$$

$$d \quad zz^* = |z|^2 \\ = (\sqrt{10})^2 \\ = 10$$

$$e \quad |zw| = |z||w| \\ = \sqrt{10} \times \sqrt{10} \\ = 10$$

$$f \quad \left| \frac{z}{w} \right| = \frac{|z|}{|w|} \\ = \frac{\sqrt{10}}{\sqrt{10}} \\ = 1$$

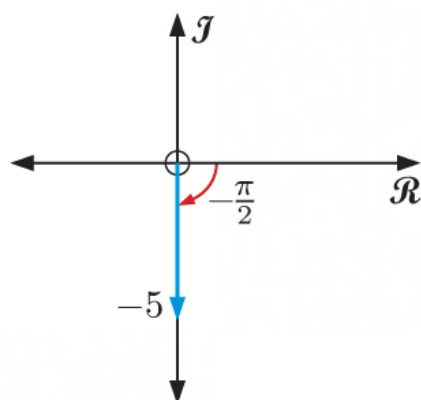
$$g \quad |z^2| = |z|^2 \\ = (\sqrt{10})^2 \\ = 10$$

$$h \quad |z^3| = |z|^3 \\ = (\sqrt{10})^3 \\ = 10\sqrt{10}$$

5

$$\begin{aligned} 3|z+2| &= |z-6| \\ \therefore 9|z+2|^2 &= |z-6|^2 \\ \therefore 9(z+2)(z+2)^* &= (z-6)(z-6)^* \quad \{\text{as } |z|^2 = zz^*\} \\ \therefore 9(z+2)(z^*+2) &= (z-6)(z^*-6) \quad \{\text{as } (z+w)^* = z^*+w^*\} \\ \therefore 9zz^* + 18z + 18z^* + 36 &= zz^* - 6z - 6z^* + 36 \\ \therefore 8zz^* + 24z + 24z^* &= 0 \\ \therefore 8|z|^2 + 24z + 24z^* &= 0 \\ \therefore 8(\sqrt{x^2+y^2})^2 + 24(x+iy) + 24(x-iy) &= 0 \\ \therefore 8x^2 + 8y^2 + 24x + \cancel{24iy} + 24x - \cancel{24iy} &= 0 \\ \therefore 8x^2 + 8y^2 + 48x &= 0 \\ \therefore x^2 + y^2 + 6x &= 0 \end{aligned}$$

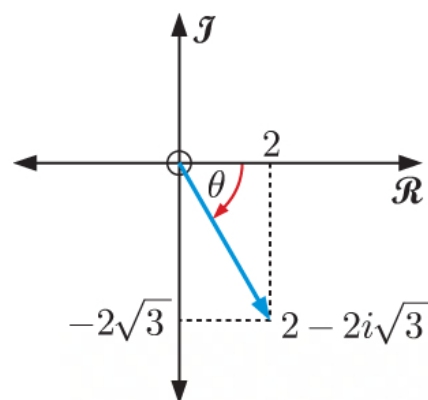
$$\begin{aligned} 6 \quad |z-w|^2 + |z+w|^2 &= (z-w)(z-w)^* + (z+w)(z+w)^* \\ &= (z-w)(z^*-w^*) + (z+w)(z^*+w^*) \\ &= zz^* - \cancel{zw^*} - \cancel{wz^*} + ww^* + zz^* + \cancel{zw^*} + \cancel{wz^*} + ww^* \\ &= 2zz^* + 2ww^* \\ &= 2|z|^2 + 2|w|^2 \\ &= 2(|z|^2 + |w|^2) \end{aligned}$$

**7 a**

$$|-5i| = 5$$

$$\arg(-5i) = -\frac{\pi}{2}$$

$$\therefore -5i = 5 \operatorname{cis}\left(-\frac{\pi}{2}\right)$$

**b**

$$|2 - 2i\sqrt{3}| = \sqrt{2^2 + (-2\sqrt{3})^2}$$

$$= \sqrt{16}$$

$$= 4$$

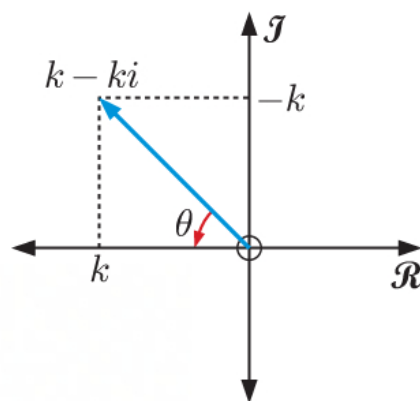
$$\tan \theta = \frac{2\sqrt{3}}{2}$$

$$= \sqrt{3}$$

$$\therefore \theta = \frac{\pi}{3}$$

$$\therefore \arg(2 - 2i\sqrt{3}) = -\frac{\pi}{3}$$

$$\therefore 2 - 2i\sqrt{3} = 4 \operatorname{cis}\left(-\frac{\pi}{3}\right)$$

**c**

$$|k - ki| = \sqrt{k^2 + (-k)^2}$$

$$= \sqrt{2k^2}$$

$$= |k| \sqrt{2}$$

Since  $k < 0$ ,  $|k - ki| = -k\sqrt{2}$

$$\tan \theta = \frac{k}{k}$$

$$= 1 \quad \{k \neq 0\}$$

$$\therefore \theta = \frac{\pi}{4}$$

$$\therefore \arg(k - ki) = \pi - \frac{\pi}{4}$$

$$= \frac{3\pi}{4}$$

$$\therefore k - ki = -k\sqrt{2} \operatorname{cis} \frac{3\pi}{4}$$

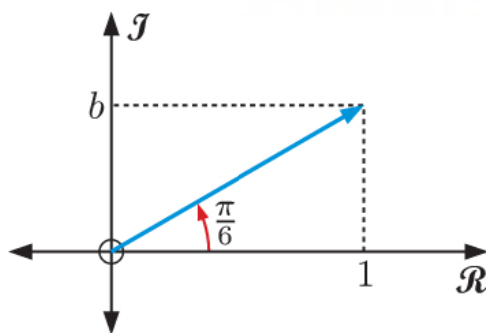
which is in polar form since  $k < 0$

**8**  $z = (1 + bi)^2$  has argument  $\frac{\pi}{3}$

$\therefore 1 + bi$  has argument  $\frac{\pi}{6}$   $\{b > 0\}$

$$\therefore \tan \frac{\pi}{6} = \frac{b}{1}$$

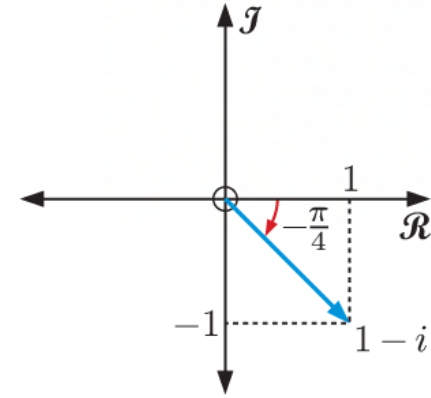
$$\therefore b = \frac{1}{\sqrt{3}}$$





**9 a**  $\text{cis } \theta \times \text{cis } \phi = (\cos \theta + i \sin \theta)(\cos \phi + i \sin \phi)$   
 $= (\cos \theta \cos \phi - \sin \theta \sin \phi) + i(\sin \theta \cos \phi + \cos \theta \sin \phi)$   
 $= \cos(\theta + \phi) + i \sin(\theta + \phi)$   
 $= \text{cis}(\theta + \phi)$

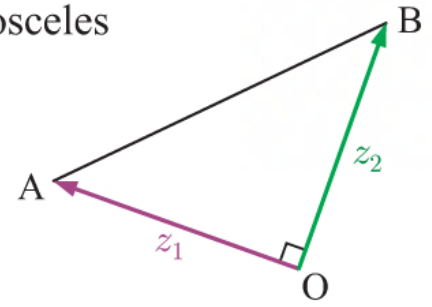
**b**  $1 - i = \sqrt{2} \text{cis} \left(-\frac{\pi}{4}\right)$   
 $\therefore (1 - i)z = \sqrt{2} \text{cis} \left(-\frac{\pi}{4}\right) \times 2\sqrt{2} \text{cis } \alpha$   
 $= 4 \text{cis} \left(\alpha - \frac{\pi}{4}\right) \quad \{\text{using a}\}$   
 $\therefore \arg[(1 - i)z] = \alpha - \frac{\pi}{4}$



**10 a**  $\left| \frac{z_1^2}{z_2^2} \right| = \frac{|z_1|^2}{|z_2|^2}$  But  $|z_1| = |z_2|$  since the triangle is isosceles

$\therefore \left| \frac{z_1^2}{z_2^2} \right| = 1$

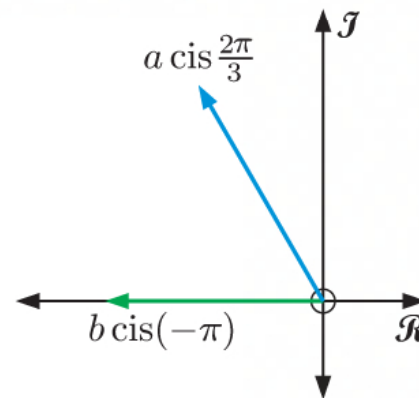
Also,  $\arg \left( \frac{z_1^2}{z_2^2} \right) = \arg(z_1^2) - \arg(z_2^2)$   
 $= 2 \arg z_1 - 2 \arg z_2$   
 $= 2(\arg z_1 - \arg z_2)$   
 $= 2 \times \frac{\pi}{2}$  since  $z_1$  and  $z_2$  are perpendicular, and  $\arg z_1 > \arg z_2$   
 $= \pi$



**b**  $\frac{z_1^2}{z_2^2} = \text{cis } \pi = -1$   
 $\therefore z_1^2 = -z_2^2$   
 $\therefore z_1^2 + z_2^2 = 0$

**11**  $z = \sqrt[4]{a} \left( \cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right) = \sqrt[4]{a} \text{cis } \frac{\pi}{6}$   
 $w = \sqrt[4]{b} \left( \cos \frac{\pi}{4} - i \sin \frac{\pi}{4} \right) = \sqrt[4]{b} \left( \cos \left(-\frac{\pi}{4}\right) + i \sin \left(-\frac{\pi}{4}\right) \right) = \sqrt[4]{b} \text{cis} \left(-\frac{\pi}{4}\right)$

$\therefore \left( \frac{z}{w} \right)^4 = \frac{z^4}{w^4} = \frac{(\sqrt[4]{a} \text{cis } \frac{\pi}{6})^4}{(\sqrt[4]{b} \text{cis} (-\frac{\pi}{4}))^4}$   
 $= \frac{a \text{cis } \frac{2\pi}{3}}{b \text{cis}(-\pi)} \quad \{\text{De Moivre}\}$   
 $= \frac{a \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)}{b(-1)}$   
 $= \frac{a}{2b} - \frac{a\sqrt{3}}{2b}i$



$\therefore \text{Re} \left( \left( \frac{z}{w} \right)^4 \right) = \frac{a}{2b}, \quad \text{Im} \left( \left( \frac{z}{w} \right)^4 \right) = -\frac{a\sqrt{3}}{2b}$

**12 a** The 5th roots of unity are the solutions of  $z^5 = 1$ .

$$\therefore z^5 = \text{cis}(0 + k2\pi) \quad \text{where } k \in \mathbb{Z} \quad \{\text{polar form}\}$$

$$\therefore z^5 = \text{cis } k2\pi$$

$$\therefore z = (\text{cis } k2\pi)^{\frac{1}{5}}$$

$$\therefore z = \text{cis } \frac{k2\pi}{5} \quad \{\text{De Moivre's theorem}\}$$

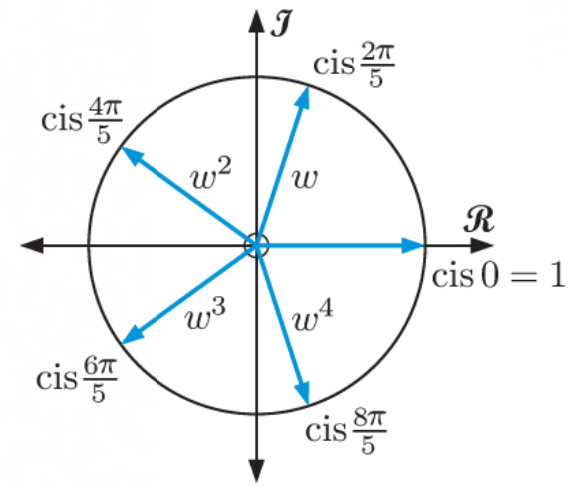
$$\therefore z = \text{cis } 0 = 1 \quad \text{or}$$

$$\text{cis } \frac{2\pi}{5} = w \quad \text{or}$$

$$\text{cis } \frac{4\pi}{5} = (\text{cis } \frac{2\pi}{5})^2 = w^2 \quad \text{or}$$

$$\text{cis } \frac{6\pi}{5} = (\text{cis } \frac{2\pi}{5})^3 = w^3 \quad \text{or}$$

$$\text{cis } \frac{8\pi}{5} = (\text{cis } \frac{2\pi}{5})^4 = w^4 \quad \{\text{letting } k = 0, 1, 2, 3, 4\}$$



Hence the five roots can be expressed as  $1, w, w^2, w^3, w^4$  where  $w = \text{cis } \frac{2\pi}{5}$

**b**  $z^5 - 1 = (z - 1)(z^4 + z^3 + z^2 + z + 1) \dots (1)$

Also, since  $1, w, w^2, w^3$  and  $w^4$  are the solutions to  $z^5 = 1$ ,

$$z^5 - 1 = (z - 1)(z - w)(z - w^2)(z - w^3)(z - w^4) \dots (2)$$

Equating (1) and (2),

$$(z - 1)(z^4 + z^3 + z^2 + z + 1) = (z - 1)(z - w)(z - w^2)(z - w^3)(z - w^4)$$

$$\therefore z^4 + z^3 + z^2 + z + 1 = (z - w)(z - w^2)(z - w^3)(z - w^4)$$

**c**  $(2 - w)(2 - w^2)(2 - w^3)(2 - w^4) = 2^4 + 2^3 + 2^2 + 2 + 1 \quad \{\text{letting } z = 2 \text{ in b}\}$   
 $= 16 + 8 + 4 + 2 + 1$   
 $= 31$

**13 a**  $\cos 3\theta + i \sin 3\theta = \text{cis } 3\theta = (\text{cis } \theta)^3$

**b**  $\frac{1}{\cos 2\theta + i \sin 2\theta} = \frac{1}{\text{cis } 2\theta}$   
 $= (\text{cis } 2\theta)^{-1}$   
 $= [(\text{cis } \theta)^2]^{-1}$   
 $= (\text{cis } \theta)^{-2}$

**c**  $\cos \theta - i \sin \theta$   
 $= \cos(-\theta) + i \sin(-\theta)$   
 $= \text{cis}(-\theta)$   
 $= (\text{cis } \theta)^{-1}$

**14** The fifth roots of  $2 + 2i$  are the solutions of  $z^5 = 2 + 2i$

$$\therefore z^5 = 2\sqrt{2} \text{cis} \left( \frac{\pi}{4} + k2\pi \right) \quad \text{where } k \in \mathbb{Z} \quad \{\text{polar form}\}$$

$$\therefore z = \left[ 2^{\frac{3}{2}} \text{cis} \left( \frac{\pi}{4} + k2\pi \right) \right]^{\frac{1}{5}}$$

$$\therefore z = 2^{0.3} \text{cis} \left( \frac{\pi}{20} + \frac{k2\pi}{5} \right) \quad \{\text{De Moivre's theorem}\}$$

$$\therefore z = 2^{0.3} \text{cis} \left( \frac{\pi}{20} + \frac{k8\pi}{20} \right)$$

$$\therefore z = 2^{0.3} \text{cis} \left( -\frac{3\pi}{4} \right), 2^{0.3} \text{cis} \left( -\frac{7\pi}{20} \right), 2^{0.3} \text{cis} \frac{\pi}{20}, 2^{0.3} \text{cis} \frac{9\pi}{20}, 2^{0.3} \text{cis} \frac{17\pi}{20}$$

{letting  $k = -2, -1, 0, 1, 2$ }

$$\begin{aligned}
 \text{15 Let } z = x + iy \quad \therefore z + \frac{1}{z} &= (x + iy) + \left( \frac{1}{x + iy} \right) \times \left( \frac{x - iy}{x - iy} \right) \\
 &= (x + iy) + \frac{(x - iy)}{x^2 + y^2} \\
 &= \frac{(x^2 + y^2)(x + iy) + (x - iy)}{x^2 + y^2} \\
 &= \frac{x^3 + ix^2y + xy^2 + y^3i + x - iy}{x^2 + y^2} \\
 &= \frac{x(x^2 + y^2 + 1) + i(x^2 + y^2 - 1)y}{x^2 + y^2}
 \end{aligned}$$

$$\begin{aligned}
 \text{which is real if } \frac{(x^2 + y^2 - 1)y}{x^2 + y^2} = 0 \quad \therefore x^2 + y^2 - 1 = 0 \quad \text{or } y = 0 \\
 \therefore x^2 + y^2 = 1 \quad \text{or } y = 0 \\
 \therefore |z|^2 = 1 \quad \text{or } y = 0 \\
 \therefore |z| = 1 \quad \text{or } z \text{ is real}
 \end{aligned}$$

$$\begin{aligned}
 \text{16 Since } \left| \frac{z+1}{z-1} \right| = \frac{|z+1|}{|z-1|} = 1, \text{ then } |z+1| &= |z-1| \\
 \text{Letting } z = x + iy, \quad \therefore |(x+1) + iy| &= |(x-1) + iy| \\
 \therefore \sqrt{(x+1)^2 + y^2} &= \sqrt{(x-1)^2 + y^2} \\
 \therefore (x+1)^2 + \cancel{y^2} &= (x-1)^2 + \cancel{y^2} \\
 \therefore x^2 + 2x + 1 &= x^2 - 2x + 1 \\
 \therefore 4x &= 0 \\
 \therefore x &= 0
 \end{aligned}$$

Since  $z \neq 0$ ,  $z$  is purely imaginary.

$$\begin{aligned}
 \text{17 a Let } z = r \operatorname{cis} \theta \\
 \therefore z^n = r^n \operatorname{cis} n\theta \quad \text{for all } n \in \mathbb{Q} \quad \{\text{De Moivre}\} \\
 \text{and so } \arg(z^n) = n\theta \quad \text{for all } n \in \mathbb{Q} \\
 \therefore \arg(z^n) = n \arg z \quad \text{for all } n \in \mathbb{Q}
 \end{aligned}$$

$$\text{b Let } z = a + bi \text{ and } w = c + di \text{ where } c \neq 0 \text{ or } d \neq 0$$

$$\begin{aligned}
 \left( \frac{z}{w} \right)^* &= \left( \frac{a + bi}{c + di} \right)^* & \text{and also } \frac{z^*}{w^*} &= \frac{a - bi}{c - di} \\
 &= \left( \frac{(a + bi)(c - di)}{(c + di)(c - di)} \right)^* & &= \frac{(a - bi)(c + di)}{(c - di)(c + di)} \\
 &= \left( \frac{(ac + bd) + i(bc - ad)}{c^2 + d^2} \right)^* & &= \frac{(ac + bd) - i(bc - ad)}{c^2 + d^2} \\
 &= \frac{(ac + bd) - i(bc - ad)}{c^2 + d^2} & &= \left( \frac{z}{w} \right)^*
 \end{aligned}$$

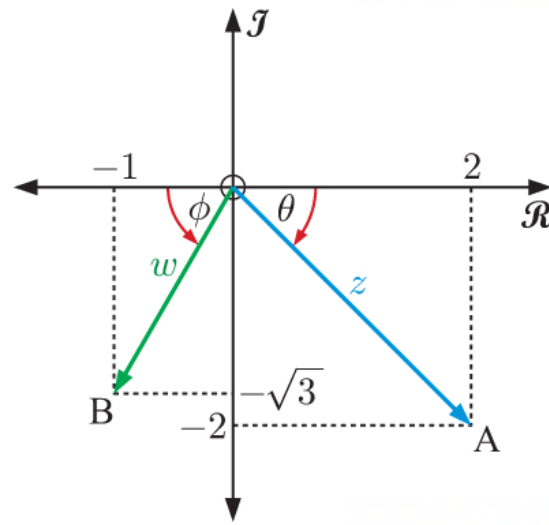
**18 a**  $\tan \theta = \frac{2}{2} = 1$

$$\therefore \theta = \frac{\pi}{4}$$

$$\tan \phi = \frac{\sqrt{3}}{1} = \sqrt{3}$$

$$\therefore \phi = \frac{\pi}{3}$$

$$\begin{aligned} \therefore \widehat{AOB} &= \pi - \phi - \theta \\ &= \pi - \frac{\pi}{3} - \frac{\pi}{4} \\ &= \frac{5\pi}{12} \end{aligned}$$



**b**  $\arg z = -\theta = -\frac{\pi}{4}$

$$\arg w = -\pi + \phi = -\frac{2\pi}{3}$$

$$\begin{aligned} \therefore \arg(zw) &= \arg z + \arg w \\ &= -\frac{\pi}{4} - \frac{2\pi}{3} \\ &= -\frac{11\pi}{12} \end{aligned}$$

**19 a** If  $z = \operatorname{cis} \theta$   
 $= \cos \theta + i \sin \theta$

$$\begin{aligned} \therefore |z| &= \sqrt{\cos^2 \theta + \sin^2 \theta} \\ &= \sqrt{1} \\ &= 1 \end{aligned}$$

**b** If  $z = \operatorname{cis} \theta$   
 then  $z^* = \operatorname{cis}(-\theta)$   
 $= (\operatorname{cis} \theta)^{-1}$  {De Moivre's theorem}  
 $= z^{-1}$   
 $= \frac{1}{z}$

**c**  $z = \operatorname{cis} \theta$   
 $\therefore z^4 = (\operatorname{cis} \theta)^4$   
 $\therefore z^4 = \operatorname{cis} 4\theta$  {De Moivre's theorem}  
 $\therefore z^4 = \cos 4\theta + i \sin 4\theta$  .... (1)

Also,  $z^4 = (\cos \theta + i \sin \theta)^4$   
 $\therefore z^4 = \cos^4 \theta + 4 \cos^3 \theta i \sin \theta + 6 \cos^2 \theta i^2 \sin^2 \theta + 4 \cos \theta i^3 \sin^3 \theta + i^4 \sin^4 \theta$   
 $\therefore z^4 = \cos^4 \theta + 4i \cos^3 \theta \sin \theta - 6 \cos^2 \theta \sin^2 \theta - 4i \cos \theta \sin^3 \theta + \sin^4 \theta$  .... (2)

Equating real parts in (1) and (2) gives

$$\begin{aligned} \cos 4\theta &= \cos^4 \theta - 6 \cos^2 \theta \sin^2 \theta + \sin^4 \theta \\ \therefore \sin^4 \theta &= \cos 4\theta - \cos^4 \theta + 6 \cos^2 \theta \sin^2 \theta \\ &= \cos 4\theta - (1 - \sin^2 \theta)^2 + 6(1 - \sin^2 \theta) \sin^2 \theta \\ &= \cos 4\theta - (1 - 2 \sin^2 \theta + \sin^4 \theta) + 6 \sin^2 \theta - 6 \sin^4 \theta \\ &= \cos 4\theta - 1 + 2 \sin^2 \theta - \sin^4 \theta + 6 \sin^2 \theta - 6 \sin^4 \theta \\ \therefore 8 \sin^4 \theta &= \cos 4\theta - 1 + 8 \sin^2 \theta \\ &= \cos 4\theta - 1 + 8 \left( \frac{1}{2} - \frac{1}{2} \cos 2\theta \right) \\ &= \cos 4\theta - 1 + 4 - 4 \cos 2\theta \\ &= \cos 4\theta - 4 \cos 2\theta + 3 \\ \therefore \sin^4 \theta &= \frac{1}{8} (\cos 4\theta - 4 \cos 2\theta + 3) \end{aligned}$$



- 20 a** If  $w$  is the root of  $z^5 = 1$  with smallest positive argument, then  $w = \text{cis } \frac{2\pi}{5}$  and  $w^4 = \text{cis } \frac{8\pi}{5}$ .

$$\begin{aligned}\text{These have sum} &= \cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5} + \cos \frac{8\pi}{5} + i \sin \frac{8\pi}{5} \\ &= \cos \frac{2\pi}{5} + \cancel{i \sin \frac{2\pi}{5}} + \cos \frac{2\pi}{5} - \cancel{i \sin \frac{2\pi}{5}} \\ &= 2 \cos \frac{2\pi}{5}\end{aligned}$$

$$\text{and product} = \text{cis } \frac{2\pi}{5} \times \text{cis } \frac{8\pi}{5} = \text{cis } \frac{10\pi}{5} = \text{cis } 2\pi = 1$$

$\therefore$  a real quadratic with roots  $w, w^4$  is  $a(z^2 - 2 \cos \frac{2\pi}{5} z + 1) = 0, a \neq 0$

- b** Let  $\alpha = w + w^4$  and  $\beta = w^2 + w^3$

Now we know that  $1 + w + w^2 + w^3 + w^4 = 0 \dots (*)$

$$\therefore 1 + (w + w^4) + (w^2 + w^3) = 0$$

$$\therefore 1 + \alpha + \beta = 0$$

$$\therefore \alpha + \beta = -1$$

$$\begin{aligned}\text{Also } \alpha\beta &= (w + w^4)(w^2 + w^3) \\ &= w^3 + w^4 + w^6 + w^7 \\ &= w^3 + w^4 + w + w^2 \quad \{\text{as } w^5 = 1\} \\ &= w + w^2 + w^3 + w^4 \\ &= -1 \quad \{\text{from } (*)\}\end{aligned}$$

$\therefore$  the quadratic equation is  $a(z^2 + z - 1) = 0, a \neq 0$ .

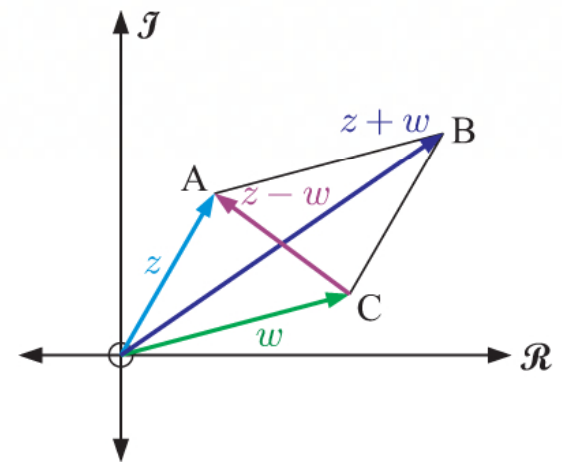
- 21** Consider the diagram which shows vectors  $w, z, z + w$ , and  $z - w$ .

Clearly OABC is a parallelogram with  $\overrightarrow{OB} = z + w$  and  $\overrightarrow{CA} = z - w$ .

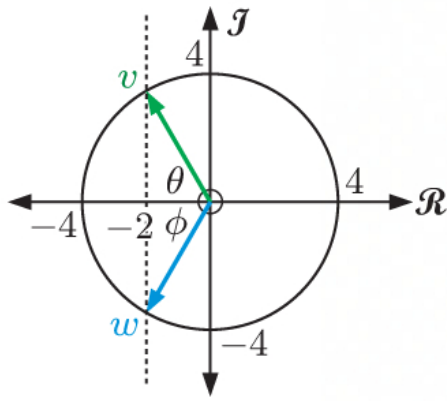
If  $|z + w| = |z - w|$ , the diagonals are equal in length and OABC is actually a rectangle.

So,  $\widehat{COA}$  is a right angle.

$\therefore \arg z$  and  $\arg w$  differ by  $\frac{\pi}{2}$ .



- 22 a**
- $$\begin{aligned}|z| &= |z + 4| \\ \therefore |z|^2 &= |z + 4|^2 \\ \text{Let } z &= x + yi \\ \therefore |x + yi|^2 &= |(x + 4) + yi|^2 \\ \therefore x^2 + y^2 &= (x + 4)^2 + y^2 \\ \therefore \cancel{x^2} + \cancel{y^2} &= \cancel{x^2} + 8x + 16 + \cancel{y^2} \\ \therefore -8x &= 16 \\ \therefore x &= -2 \\ \therefore \text{the real part of } z &\text{ is } -2.\end{aligned}$$

**b i****ii** In the diagram in **b i**,  $\cos \theta = \frac{2}{4} = \frac{1}{2}$ 

$$\therefore \theta = \frac{\pi}{3}$$

$$\therefore \arg v = \pi - \theta = \frac{2\pi}{3}$$

$$\text{iii} \quad \cos \phi = \frac{2}{4} = \frac{1}{2}$$

$$\therefore \phi = \frac{\pi}{3}$$

$$\begin{aligned} \therefore \arg w &= -\pi + \phi \\ &= -\frac{2\pi}{3} \end{aligned}$$

$$\text{iv} \quad v = 4 \operatorname{cis} \frac{2\pi}{3}, \quad w = 4 \operatorname{cis} \left(-\frac{2\pi}{3}\right)$$

$$\therefore \frac{v^m w}{i} = \frac{\left(4 \operatorname{cis} \frac{2\pi}{3}\right)^m 4 \operatorname{cis} \left(-\frac{2\pi}{3}\right)}{\operatorname{cis} \frac{\pi}{2}}$$

$$= \frac{4^m \operatorname{cis} \frac{2m\pi}{3} 4 \operatorname{cis} \left(-\frac{2\pi}{3}\right)}{\operatorname{cis} \frac{\pi}{2}} \quad \{\text{De Moivre}\}$$

$$= 4^{m+1} \operatorname{cis} \left(\frac{2m\pi}{3} - \frac{2\pi}{3} - \frac{\pi}{2}\right)$$

$$\begin{aligned} \therefore \arg \left(\frac{v^m w}{i}\right) &= \frac{2m\pi}{3} - \frac{2\pi}{3} - \frac{\pi}{2} \\ &= \frac{4m\pi}{6} - \frac{4\pi}{6} - \frac{3\pi}{6} \\ &= \frac{\pi(4m-7)}{6} \end{aligned}$$

$$\text{v} \quad \frac{v^m w}{i} \text{ is real when } \arg \left(\frac{v^m w}{i}\right) = 0 + k\pi, \quad k \in \mathbb{Z}$$

$$\therefore \frac{\cancel{\pi}(4m-7)}{6} = k\cancel{\pi}$$

$$\therefore 4m - 7 = 6k$$

$$\therefore m = \frac{7+6k}{4}$$

One such value of  $m$  is  $m = \frac{7}{4}$  {when  $k = 0$ }

# Chapter 15

## LIMITS

### EXERCISE 15A

1 a As  $x \rightarrow 3$ ,  $x + 4 \rightarrow 7$   
 $\therefore \lim_{x \rightarrow 3} (x + 4) = 7$

c As  $x \rightarrow 4$ ,  $3x - 1 \rightarrow 11$   
 $\therefore \lim_{x \rightarrow 4} (3x - 1) = 11$

d As  $x \rightarrow 2$ ,  $5x^2 - 3x + 2 \rightarrow 5(4) - 3(2) + 2 = 16$   
 $\therefore \lim_{x \rightarrow 2} (5x^2 - 3x + 2) = 16$

e As  $h \rightarrow 0$ ,  $h^2 \rightarrow 0$  and  $1 - h \rightarrow 1$   
 $\therefore \lim_{h \rightarrow 0} h^2(1 - h) = 0 \times 1 = 0$

b As  $x \rightarrow -1$ ,  $5 - 2x \rightarrow 7$   
 $\therefore \lim_{x \rightarrow -1} (5 - 2x) = 7$

f As  $x \rightarrow 0$ ,  $x^2 + 5 \rightarrow 5$   
 $\therefore \lim_{x \rightarrow 0} (x^2 + 5) = 5$

2 a  $\lim_{x \rightarrow 0} 5 = 5$

b  $\lim_{h \rightarrow 2} 7 = 7$

c  $\lim_{x \rightarrow 0} c = c$  (where  $c$  is a constant)

3 a  $\frac{x^2 - 3x}{x}$  can be made as close as we like to  $-2$  by making  $x$  sufficiently close to 1.

$\therefore \lim_{x \rightarrow 1} \frac{x^2 - 3x}{x} = -2$

c  $\frac{x - 1}{x + 1}$  can be made as close as we like to  $-1$  by making  $x$  sufficiently close to 0.

$\therefore \lim_{x \rightarrow 0} \frac{x - 1}{x + 1} = -1$

b  $\frac{h^2 + 5h}{h}$  can be made as close as we like to 7 by making  $h$  sufficiently close to 2.

$\therefore \lim_{h \rightarrow 2} \frac{h^2 + 5h}{h} = 7$

4  $\lim_{x \rightarrow 0} \frac{x}{x} = \lim_{x \rightarrow 0} 1$  {since  $x \neq 0$ }  
 $= 1$

5 a

$x$	$\frac{x^2 - 4}{x - 2}$
1.9	3.9
1.99	3.99
1.999	3.999
1.9999	3.9999
1.99999	3.99999

$x$	$\frac{x^2 - 4}{x - 2}$
2.1	4.1
2.01	4.01
2.001	4.001
2.0001	4.0001
2.00001	4.00001

b  $\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2} = 4$

c  $\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2}$   
 $= \lim_{x \rightarrow 2} \frac{(x + 2)(x - 2)}{x - 2}$   
 $= \lim_{x \rightarrow 2} (x + 2)$  {since  $x \neq 2$ }  
 $= 4$

$$\begin{aligned}
 \text{6 a} \quad & \lim_{x \rightarrow 0} \frac{x^2 - 3x}{x} \\
 &= \lim_{x \rightarrow 0} \frac{\cancel{x}(x - 3)}{\cancel{x}} \\
 &= \lim_{x \rightarrow 0} (x - 3) \quad \{\text{since } x \neq 0\} \\
 &= -3
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & \lim_{x \rightarrow 0} \frac{2x^2 - x}{x} \\
 &= \lim_{x \rightarrow 0} \frac{\cancel{x}(2x - 1)}{\cancel{x}} \\
 &= \lim_{x \rightarrow 0} (2x - 1) \quad \{\text{since } x \neq 0\} \\
 &= -1
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & \lim_{h \rightarrow 0} \frac{3h^2 - 4h}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{h}(3h - 4)}{\cancel{h}} \\
 &= \lim_{h \rightarrow 0} (3h - 4) \quad \{\text{since } h \neq 0\} \\
 &= -4
 \end{aligned}$$

$$\begin{aligned}
 \text{g} \quad & \lim_{x \rightarrow 1} \frac{x^2 - x}{x - 1} \\
 &= \lim_{x \rightarrow 1} \frac{x(\cancel{x - 1})}{\cancel{x - 1}} \\
 &= \lim_{x \rightarrow 1} x \quad \{\text{since } x \neq 1\} \\
 &= 1
 \end{aligned}$$

$$\begin{aligned}
 \text{i} \quad & \lim_{x \rightarrow 3} \frac{x^2 - x - 6}{x - 3} \\
 &= \lim_{x \rightarrow 3} \frac{(x + 2)(\cancel{x - 3})}{\cancel{x - 3}} \\
 &= \lim_{x \rightarrow 3} (x + 2) \quad \{\text{since } x \neq 3\} \\
 &= 5
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & \lim_{x \rightarrow 0} \frac{x^2 + 5x}{x} \\
 &= \lim_{x \rightarrow 0} \frac{\cancel{x}(x + 5)}{\cancel{x}} \\
 &= \lim_{x \rightarrow 0} (x + 5) \quad \{\text{since } x \neq 0\} \\
 &= 5
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad & \lim_{h \rightarrow 0} \frac{2h^2 + 6h}{h} \\
 &= \lim_{h \rightarrow 0} \frac{2\cancel{h}(h + 3)}{\cancel{h}} \\
 &= \lim_{h \rightarrow 0} 2(h + 3) \quad \{\text{since } h \neq 0\} \\
 &= 6
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & \lim_{h \rightarrow 0} \frac{h^3 - 8h}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{h}(h^2 - 8)}{\cancel{h}} \\
 &= \lim_{h \rightarrow 0} (h^2 - 8) \quad \{\text{since } h \neq 0\} \\
 &= -8
 \end{aligned}$$

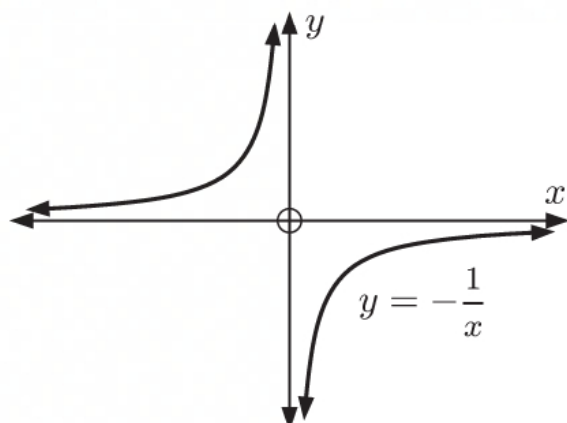
$$\begin{aligned}
 \text{h} \quad & \lim_{x \rightarrow 2} \frac{x^2 - 2x}{x - 2} \\
 &= \lim_{x \rightarrow 2} \frac{x(\cancel{x - 2})}{\cancel{x - 2}} \\
 &= \lim_{x \rightarrow 2} x \quad \{\text{since } x \neq 2\} \\
 &= 2
 \end{aligned}$$

$$\begin{aligned}
 \text{7} \quad \lim_{x \rightarrow a} f(x) = l & \Leftrightarrow \lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} l \\
 & \Leftrightarrow \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} l = 0 \\
 & \Leftrightarrow \lim_{x \rightarrow a} (f(x) - l) = 0
 \end{aligned}$$



## EXERCISE 15B

1 a



b i As  $x \rightarrow 1^+$ ,  $-\frac{1}{x} \rightarrow -1$

As  $x \rightarrow 1^-$ ,  $-\frac{1}{x} \rightarrow -1$

$\therefore \lim_{x \rightarrow 1} \left(-\frac{1}{x}\right) = -1$

iii As  $x \rightarrow 0^-$ ,  $-\frac{1}{x} \rightarrow \infty$

As  $x \rightarrow 0^+$ ,  $-\frac{1}{x} \rightarrow -\infty$

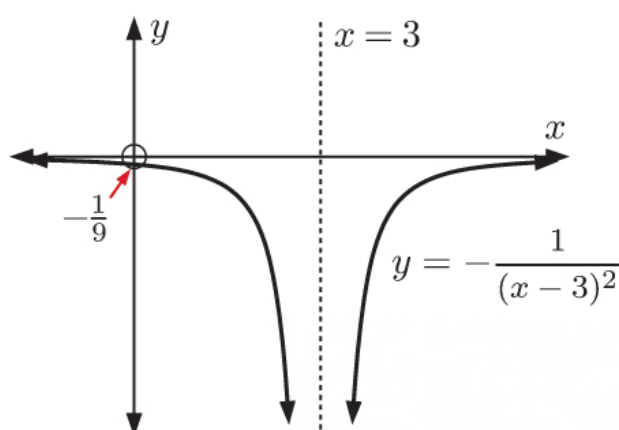
$\therefore \lim_{x \rightarrow 0} \left(-\frac{1}{x}\right)$  does not exist.

ii As  $x \rightarrow -1^+$ ,  $-\frac{1}{x} \rightarrow 1$

As  $x \rightarrow -1^-$ ,  $-\frac{1}{x} \rightarrow 1$

$\therefore \lim_{x \rightarrow -1} \left(-\frac{1}{x}\right) = 1$

2 a



b i As  $x \rightarrow 0^+$ ,  $-\frac{1}{(x-3)^2} \rightarrow -\frac{1}{9}$

As  $x \rightarrow 0^-$ ,  $-\frac{1}{(x-3)^2} \rightarrow -\frac{1}{9}$

$\therefore \lim_{x \rightarrow 0} \left(-\frac{1}{(x-3)^2}\right) = -\frac{1}{9}$

iii As  $x \rightarrow 5^+$ ,  $-\frac{1}{(x-3)^2} \rightarrow -\frac{1}{4}$

As  $x \rightarrow 5^-$ ,  $-\frac{1}{(x-3)^2} \rightarrow -\frac{1}{4}$

$\therefore \lim_{x \rightarrow 5} \left(-\frac{1}{(x-3)^2}\right) = -\frac{1}{4}$

ii As  $x \rightarrow 3^+$ ,  $-\frac{1}{(x-3)^2} \rightarrow -\infty$

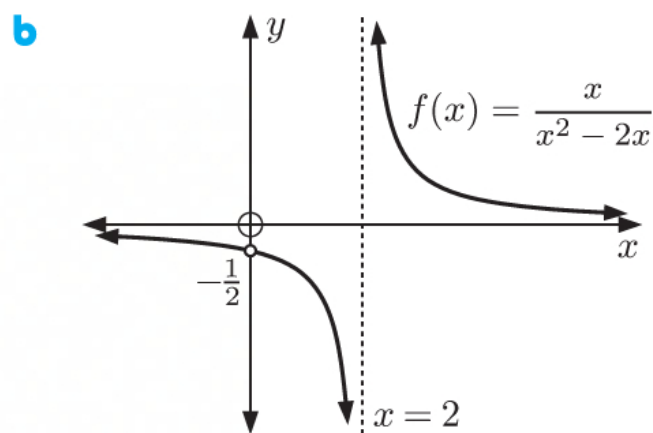
As  $x \rightarrow 3^-$ ,  $-\frac{1}{(x-3)^2} \rightarrow -\infty$

$\therefore \lim_{x \rightarrow 3} \left(-\frac{1}{(x-3)^2}\right)$  does not exist.

**3 a** As  $x \rightarrow -1$ ,  $3x + 2 \rightarrow -1$   
 $\therefore \lim_{x \rightarrow -1} (3x + 2) = -1$

**b** As  $x \rightarrow -2$ ,  $\frac{x^2 + x - 2}{x - 2} \rightarrow \frac{(-2)^2 + (-2) - 2}{(-2) - 2} = 0$   
 $\therefore \lim_{x \rightarrow -2} \frac{x^2 + x - 2}{x - 2} = 0$

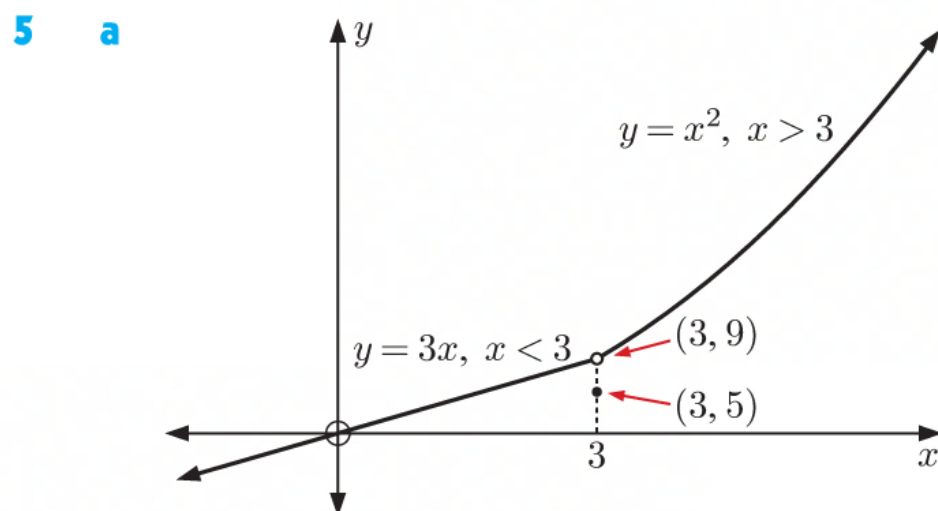
**4 a**  $f(x) = \frac{x}{x^2 - 2x}$  is undefined when  $x^2 - 2x = 0$   
 $\therefore x(x - 2) = 0$   
 $\therefore x = 0$  or  $2$



**c i** As  $x \rightarrow 2^+$ ,  $f(x) \rightarrow \infty$   
 As  $x \rightarrow 2^-$ ,  $f(x) \rightarrow -\infty$   
 $\therefore \lim_{x \rightarrow 2} f(x)$  does not exist.

**ii** As  $x \rightarrow 1$ ,  $f(x) \rightarrow \frac{1}{1^2 - 2(1)} = -1$   
 $\therefore \lim_{x \rightarrow 1} f(x) = -1$

**iii**  $\lim_{x \rightarrow 0} \frac{x}{x^2 - 2x} = \lim_{x \rightarrow 0} \frac{\cancel{x}}{\cancel{x}(x - 2)}$   
 $= \lim_{x \rightarrow 0} \frac{1}{x - 2} \quad \{\text{since } x \neq 0\}$   
 $= \frac{1}{-2}$   
 $\therefore \lim_{x \rightarrow 0} f(x) = -\frac{1}{2}$



**b**  $\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^-} 3x = 9$        $\lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3^+} x^2 = 9$

**c** Yes,  $\lim_{x \rightarrow 3} f(x) = 9$  since  $\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^+} f(x) = 9$ .

**6** No,  $\lim_{x \rightarrow 0} \sqrt{x}$  does not exist since  $\lim_{x \rightarrow 0^-} \sqrt{x}$  does not exist.

$$\begin{aligned} 7 \quad \mathbf{a} \quad \lim_{x \rightarrow 0^-} f(x) &= \lim_{x \rightarrow 0^-} (x-1)^3 \\ &= (-1)^3 \\ &= -1 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \lim_{x \rightarrow 0^+} f(x) &= \lim_{x \rightarrow 0^+} 1 \\ &= 1 \end{aligned}$$

$$\mathbf{c} \quad \lim_{x \rightarrow 0} f(x) \text{ does not exist since } \lim_{x \rightarrow 0^+} f(x) \neq \lim_{x \rightarrow 0^-} f(x).$$

$$\begin{aligned} 8 \quad \mathbf{a} \quad \lim_{x \rightarrow 0^-} f(x) &= \lim_{x \rightarrow 0^-} \sin x \\ &= 0 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \lim_{x \rightarrow 0^+} f(x) &= \lim_{x \rightarrow 0^+} \frac{x^2 - 3x + 5}{5} \\ &= \frac{5}{5} \\ &= 1 \end{aligned}$$

$$\mathbf{c} \quad \lim_{x \rightarrow 0} f(x) \text{ does not exist since } \lim_{x \rightarrow 0^+} f(x) \neq \lim_{x \rightarrow 0^-} f(x).$$

$$\begin{aligned} 9 \quad \mathbf{a} \quad \lim_{x \rightarrow 0} \sin x &= \sin 0 \\ &= 0 \end{aligned}$$

$$\begin{aligned} \lim_{x \rightarrow 0} e^x &= e^0 \\ &= 1 \end{aligned}$$

$$\begin{aligned} \therefore \lim_{x \rightarrow 0} \frac{\sin x}{e^x} &= \frac{0}{1} \\ &= 0 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \lim_{x \rightarrow \pi} \sin x &= \sin \pi \\ &= 0 \end{aligned}$$

$$\begin{aligned} \lim_{x \rightarrow \pi} (1 - \cos x) &= 1 - \cos \pi \\ &= 2 \end{aligned}$$

$$\begin{aligned} \therefore \lim_{x \rightarrow \pi} \frac{\sin x}{1 - \cos x} &= \frac{0}{2} \\ &= 0 \end{aligned}$$

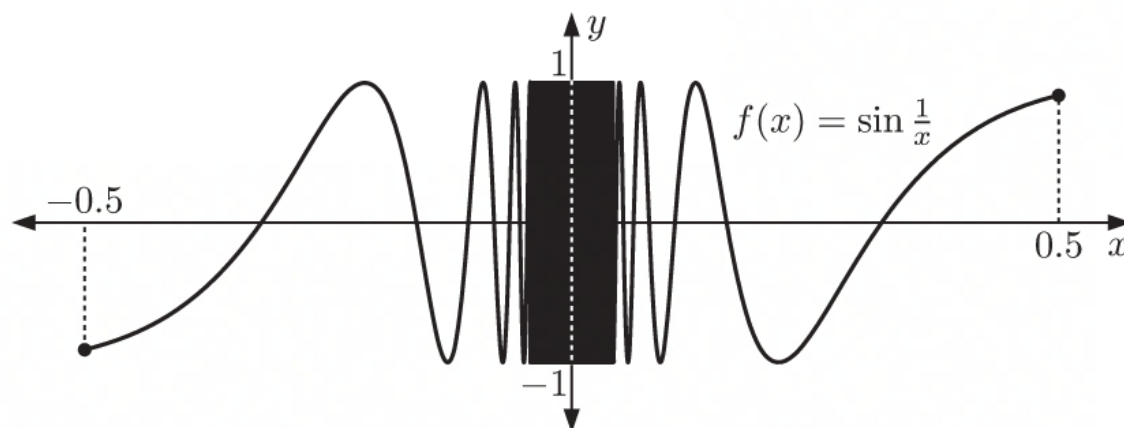
$$\mathbf{c} \quad \lim_{\theta \rightarrow 0} \cos \theta = 1 \quad \text{but} \quad \lim_{\theta \rightarrow 0} \theta = 0$$

$$\therefore \lim_{\theta \rightarrow 0} \frac{\cos \theta}{\theta} \text{ does not exist.}$$

$$\mathbf{d} \quad \lim_{x \rightarrow 2} \ln x = \ln 2 \quad \text{but} \quad \lim_{x \rightarrow 2} \sqrt{2-x} = 0$$

$$\therefore \lim_{x \rightarrow 2} \frac{\ln x}{\sqrt{2-x}} \text{ does not exist.}$$

10 **a**



**b**  $f(x)$  does not approach any value as  $x \rightarrow 0$  from above or below. It oscillates over values between  $-1$  and  $1$ .

**c**  $\lim_{x \rightarrow 0} f(x)$  does not exist since neither  $\lim_{x \rightarrow 0^-} f(x)$  nor  $\lim_{x \rightarrow 0^+} f(x)$  exist.

- 11 a**
- i**  $f(\frac{1}{2}) = \frac{1}{2}$  since  $\frac{1}{2} \in \mathbb{Q}$
  - ii**  $f(\sqrt{2}) = 0$  since  $\sqrt{2} \notin \mathbb{Q}$
  - iii**  $f(0) = 0$  since  $0 \in \mathbb{Q}$
- b**
- i**  $\lim_{x \rightarrow \frac{1}{2}} f(x)$  does not exist. As  $x \rightarrow \frac{1}{2}$  from either side, the function oscillates between  $x$  and 0, never approaching any value.
  - ii**  $\lim_{x \rightarrow \sqrt{2}} f(x)$  does not exist. As  $x \rightarrow \sqrt{2}$  from either side, the function oscillates between  $x$  and 0, never approaching any value.
  - iii**  $\lim_{x \rightarrow 0} f(x) = 0$ . As  $x \rightarrow 0$  from either side, the function oscillates between  $x$  and 0, but eventually approaches 0.

**12 a**

$$f(x) = \sum_{n=1}^{\infty} \frac{x^2}{(1+x^2)^n} = \frac{x^2}{1+x^2} + \frac{x^2}{(1+x^2)^2} + \frac{x^2}{(1+x^2)^3} + \dots$$

$$\therefore f(0) = \frac{0^2}{1+0^2} + \frac{0^2}{(1+0^2)^2} + \frac{0^2}{(1+0^2)^3} + \dots$$

$$= 0$$

- b** If  $x \neq 0$ ,  $f(x)$  is an infinite geometric sequence with  $u_1 = \frac{x^2}{1+x^2}$ ,  $r = \frac{1}{1+x^2}$ .

When  $x \neq 0$ ,  $|r| < 1$ , so  $f(x) = \frac{u_1}{1-r}$

$$= \frac{\frac{x^2}{1+x^2}}{1 - \frac{1}{1+x^2}}$$

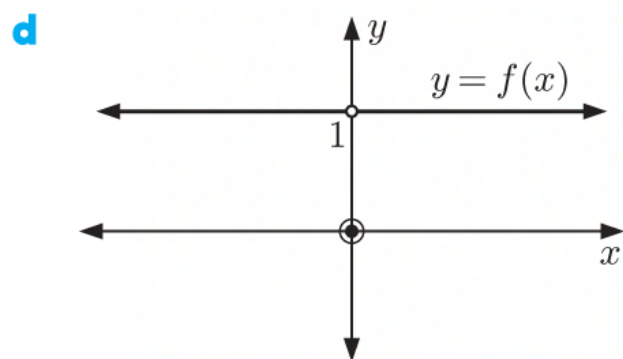
$$= \frac{x^2}{1+x^2-1}$$

$$= \frac{x^2}{x^2}$$

$$= 1 \quad \{\text{since } x \neq 0\}$$

**c**  $\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^-} f(x) = 1$

$$\therefore \lim_{x \rightarrow 0} f(x) = 1$$





**EXERCISE 15C**

1 As  $x \rightarrow \infty$ ,  $\frac{1}{x^2} \rightarrow 0$ .

$$\therefore \lim_{x \rightarrow \infty} \frac{1}{x^2} = 0$$

2 a 
$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{3x-2}{x+1} &= \lim_{x \rightarrow \infty} \frac{3 - \frac{2}{x}}{1 + \frac{1}{x}} \\ &= \frac{3}{1} \\ &= 3 \end{aligned}$$

d 
$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{x^2+3}{x^2-1} &= \lim_{x \rightarrow \infty} \frac{1 + \frac{3}{x^2}}{1 - \frac{1}{x^2}} \\ &= \frac{1}{1} \\ &= 1 \end{aligned}$$

g 
$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{x+5}{x^2-x+4} &= \lim_{x \rightarrow \infty} \frac{\frac{1}{x} + \frac{5}{x^2}}{1 - \frac{1}{x} + \frac{4}{x^2}} \\ &= \frac{0}{1} \\ &= 0 \end{aligned}$$

i 
$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{2x^2-x+1}{5x+3} &= \lim_{x \rightarrow \infty} \frac{2 - \frac{1}{x} + \frac{1}{x^2}}{\frac{5}{x} + \frac{3}{x^2}} \end{aligned}$$

As  $x \rightarrow \infty$ , the numerator  $\rightarrow 2$  but the denominator  $\rightarrow 0$ .

Hence as  $x \rightarrow \infty$ ,  $\frac{2x^2-x+1}{5x+3} \rightarrow \infty$ .

$\therefore \lim_{x \rightarrow \infty} \frac{2x^2-x+1}{5x+3}$  does not exist.

b 
$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{1-2x}{3x+2} &= \lim_{x \rightarrow \infty} \frac{\frac{1}{x} - 2}{3 + \frac{2}{x}} \\ &= -\frac{2}{3} \end{aligned}$$

e 
$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{x^2-2x+4}{4x^2-x-1} &= \lim_{x \rightarrow \infty} \frac{1 - \frac{2}{x} + \frac{4}{x^2}}{4 - \frac{1}{x} - \frac{1}{x^2}} \\ &= \frac{1}{4} \end{aligned}$$

h 
$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{x^2+3x-4}{x-1} &= \lim_{x \rightarrow \infty} \frac{1 + \frac{3}{x} - \frac{4}{x^2}}{\frac{1}{x} - \frac{1}{x^2}} \end{aligned}$$

As  $x \rightarrow \infty$ , the numerator  $\rightarrow 1$  but the denominator  $\rightarrow 0$ .

Hence as  $x \rightarrow \infty$ ,  $\frac{x^2+3x-4}{x-1} \rightarrow \infty$ .

$\therefore \lim_{x \rightarrow \infty} \frac{x^2+3x-4}{x-1}$  does not exist.

c 
$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{x}{1-x} &= \lim_{x \rightarrow \infty} \frac{1}{\frac{1}{x} - 1} \\ &= \frac{1}{-1} \\ &= -1 \end{aligned}$$

f 
$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{x+4}{2x^2+x-1} &= \lim_{x \rightarrow \infty} \frac{\frac{1}{x} + \frac{4}{x^2}}{2 + \frac{1}{x} - \frac{1}{x^2}} \\ &= \frac{0}{2} \\ &= 0 \end{aligned}$$

**3 a**  $f(x) = \frac{1}{x}$

**i** As  $x \rightarrow 0^-$ ,  $f(x) \rightarrow -\infty$

As  $x \rightarrow 0^+$ ,  $f(x) \rightarrow \infty$

As  $x \rightarrow -\infty$ ,  $f(x) \rightarrow 0^-$

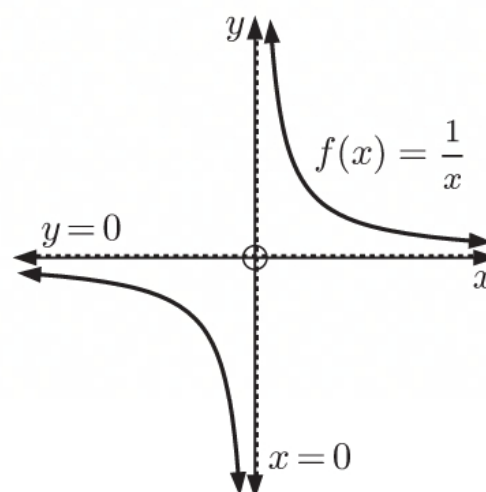
As  $x \rightarrow \infty$ ,  $f(x) \rightarrow 0^+$

The vertical asymptote is  $x = 0$ .

The horizontal asymptote is  $y = 0$ .

**ii**  $\lim_{x \rightarrow -\infty} f(x) = 0$ ,

$\lim_{x \rightarrow \infty} f(x) = 0$



**b**  $f(x) = \frac{3x-2}{x+3}$

**i** As  $x \rightarrow -3^-$ ,  $f(x) \rightarrow \infty$

As  $x \rightarrow -3^+$ ,  $f(x) \rightarrow -\infty$

As  $x \rightarrow -\infty$ ,  $f(x) \rightarrow 3^+$

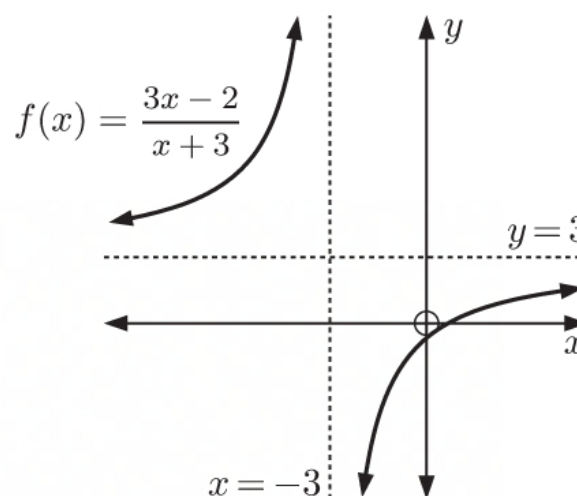
As  $x \rightarrow \infty$ ,  $f(x) \rightarrow 3^-$

The vertical asymptote is  $x = -3$ .

The horizontal asymptote is  $y = 3$ .

**ii**  $\lim_{x \rightarrow -\infty} f(x) = 3$ ,

$\lim_{x \rightarrow \infty} f(x) = 3$



**c**  $f(x) = \frac{1-2x}{3x+2}$

**i** As  $x \rightarrow -\frac{2}{3}^-$ ,  $f(x) \rightarrow -\infty$

As  $x \rightarrow -\frac{2}{3}^+$ ,  $f(x) \rightarrow \infty$

As  $x \rightarrow -\infty$ ,  $f(x) \rightarrow -\frac{2}{3}^-$

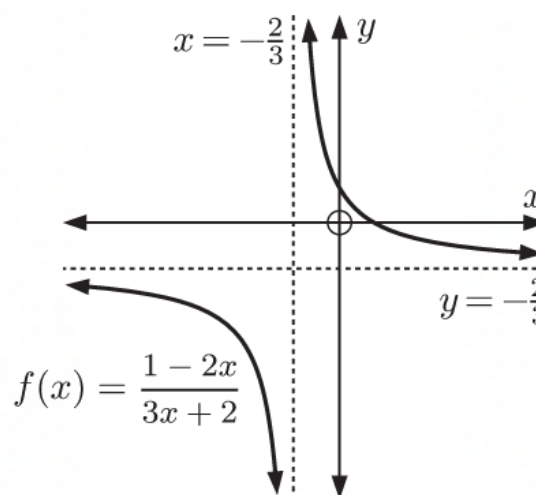
As  $x \rightarrow \infty$ ,  $f(x) \rightarrow -\frac{2}{3}^+$

The vertical asymptote is  $x = -\frac{2}{3}$ .

The horizontal asymptote is  $y = -\frac{2}{3}$ .

**ii**  $\lim_{x \rightarrow -\infty} f(x) = -\frac{2}{3}$ ,

$\lim_{x \rightarrow \infty} f(x) = -\frac{2}{3}$



**d**  $f(x) = \frac{x}{1-x}$

**i** As  $x \rightarrow 1^-$ ,  $f(x) \rightarrow \infty$

As  $x \rightarrow 1^+$ ,  $f(x) \rightarrow -\infty$

As  $x \rightarrow -\infty$ ,  $f(x) \rightarrow -1^+$

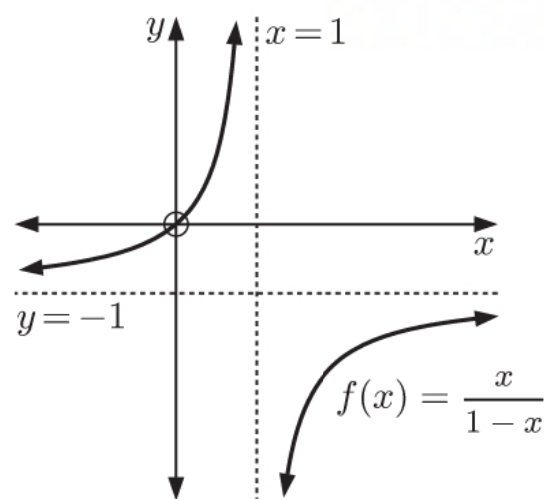
As  $x \rightarrow \infty$ ,  $f(x) \rightarrow -1^-$

The vertical asymptote is  $x = 1$ .

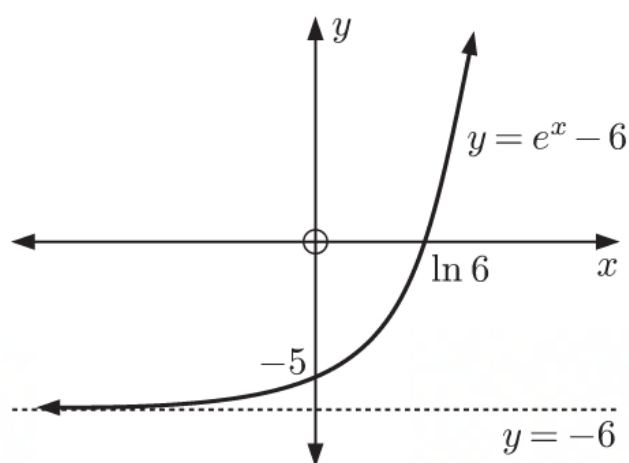
The horizontal asymptote is  $y = -1$ .

**ii**  $\lim_{x \rightarrow -\infty} f(x) = -1$ ,

$\lim_{x \rightarrow \infty} f(x) = -1$



**4 a**



**b i** As  $x \rightarrow -\infty$ ,  $e^x - 6 \rightarrow -6^+$   
 $\therefore \lim_{x \rightarrow -\infty} (e^x - 6) = -6$

$\therefore$  the function has horizontal asymptote  $y = -6$ .

**ii** As  $x \rightarrow \infty$ ,  $e^x - 6 \rightarrow \infty$

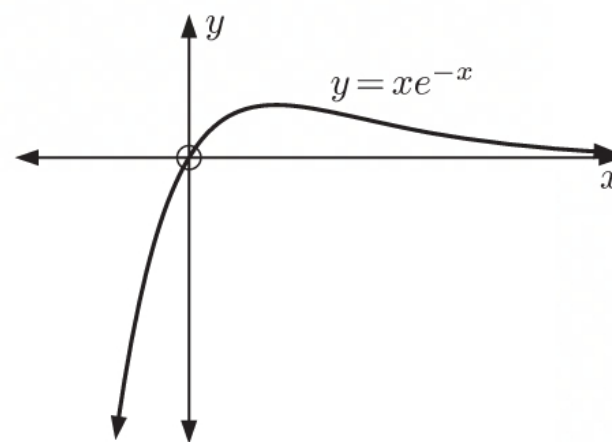
$\therefore \lim_{x \rightarrow \infty} (e^x - 6)$  does not exist.

**5 a**

$x$	$xe^{-x}$
10	$\approx 0.000\,454$
50	$\approx 9.64 \times 10^{-21}$
100	$\approx 3.72 \times 10^{-42}$
200	$\approx 2.77 \times 10^{-85}$

**b** We predict that  $\lim_{x \rightarrow \infty} xe^{-x} = 0$ .

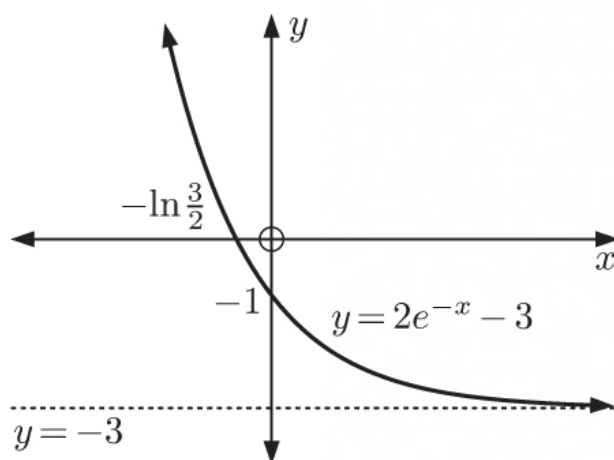
**c**



As  $x \rightarrow \infty$ ,  $y \rightarrow 0^+$

The graph supports our prediction in **b**.

**6 a** We sketch the graph of  $y = 2e^{-x} - 3$ :



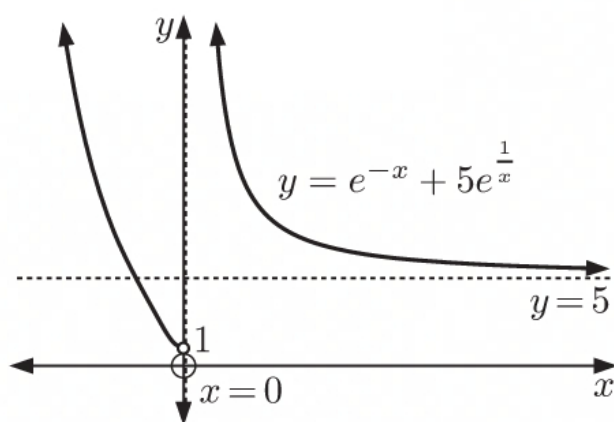
As  $x \rightarrow -\infty$ ,  $2e^{-x} - 3 \rightarrow \infty$

$\therefore \lim_{x \rightarrow -\infty} (2e^{-x} - 3)$  does not exist.

As  $x \rightarrow \infty$ ,  $2e^{-x} - 3 \rightarrow -3^+$

$\therefore \lim_{x \rightarrow \infty} (2e^{-x} - 3) = -3$ .

**b** We sketch the graph of  $y = e^{-x} + 5e^{\frac{1}{x}}$ :



As  $x \rightarrow -\infty$ ,  $e^{-x} + 5e^{\frac{1}{x}} \rightarrow \infty$

$\therefore \lim_{x \rightarrow -\infty} (e^{-x} + 5e^{\frac{1}{x}})$  does not exist.

As  $x \rightarrow \infty$ ,  $e^{-x} + 5e^{\frac{1}{x}} \rightarrow 5^+$

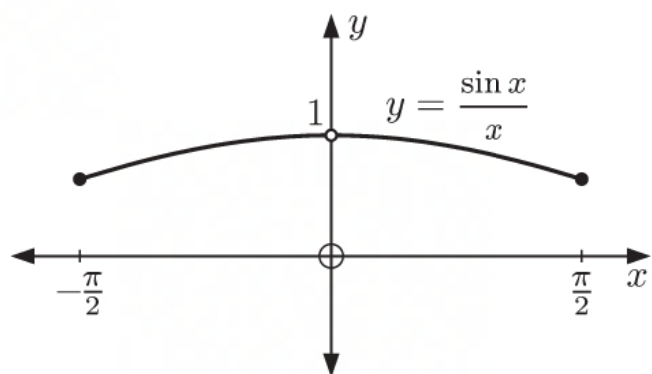
$\therefore \lim_{x \rightarrow \infty} (e^{-x} + 5e^{\frac{1}{x}}) = 5$

$$\begin{aligned}
 7 \quad \lim_{x \rightarrow \infty} \sqrt{x^2 + x} - x &= \lim_{x \rightarrow \infty} (\sqrt{x^2 + x} - x) \times \frac{\sqrt{x^2 + x} + x}{\sqrt{x^2 + x} + x} \\
 &= \lim_{x \rightarrow \infty} \frac{\cancel{x^2} + x - \cancel{x^2}}{\sqrt{x^2 + x} + x} \\
 &= \lim_{x \rightarrow \infty} \frac{x}{x\sqrt{1 + \frac{1}{x}} + x} \\
 &= \lim_{x \rightarrow \infty} \frac{1}{\sqrt{1 + \frac{1}{x}} + 1} \\
 &= \frac{1}{1 + 1} \\
 &= \frac{1}{2}
 \end{aligned}$$

## INVESTIGATION

## EXAMINING $\frac{\sin \theta}{\theta}$ NEAR $\theta = 0$

**1 a**



**b** As  $x \rightarrow 0^-$ ,  $y \rightarrow 1$

As  $x \rightarrow 0^+$ ,  $y \rightarrow 1$

It appears that  $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$ .

**2 a**

$$\begin{aligned}
 f(\theta) &= \frac{\sin \theta}{\theta} \\
 \therefore f(-\theta) &= \frac{\sin(-\theta)}{(-\theta)} \\
 &= \frac{-\sin \theta}{-\theta} \\
 &= \frac{\sin \theta}{\theta} \\
 &= f(\theta) \\
 \therefore f(\theta) &= \frac{\sin \theta}{\theta} \text{ is an even function.}
 \end{aligned}$$

This means that the function is symmetrical about the  $y$ -axis.



$$\begin{aligned}
 \text{b i } \lim_{\theta \rightarrow 0^-} f(\theta) &= \lim_{-\theta \rightarrow 0^-} f(-\theta) \\
 &= \lim_{\theta \rightarrow 0^+} f(\theta) \quad \{f(\theta) \text{ is even, } \theta \rightarrow 0^+ \text{ as } -\theta \rightarrow 0^-\} \\
 &= A
 \end{aligned}$$

$$\begin{aligned}
 \text{ii } \lim_{\theta \rightarrow 0^+} f(\theta) &= \lim_{\theta \rightarrow 0^-} f(\theta) = A \\
 \therefore \lim_{\theta \rightarrow 0} f(\theta) &= A
 \end{aligned}$$

c i If there are  $n$  triangles, each triangle has  $\theta = \frac{2\pi}{n}$ .

The area of one triangle is

$$\frac{1}{2} \times r \times r \times \sin \frac{2\pi}{n} = \frac{1}{2} r^2 \sin \frac{2\pi}{n}.$$

$$\therefore \text{the sum of all } n \text{ triangles is } \frac{n}{2} r^2 \sin \frac{2\pi}{n}$$

As  $n \rightarrow \infty$ , the triangles exactly fit the circle, which has area  $\pi r^2$ .

$$\therefore \lim_{n \rightarrow \infty} \frac{n}{2} r^2 \sin \frac{2\pi}{n} = \pi r^2$$

$$\text{ii (1) } \lim_{n \rightarrow \infty} \frac{n}{2} r^2 \sin \frac{2\pi}{n} = \pi r^2$$

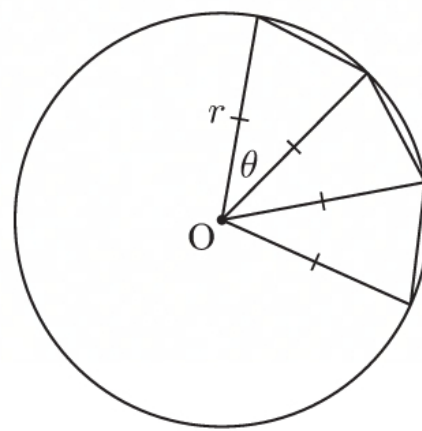
$$\therefore \lim_{n \rightarrow \infty} \frac{\frac{n}{2} \cancel{r^2} \sin \frac{2\pi}{n}}{\pi \cancel{r^2}} = 1$$

$$\therefore \lim_{n \rightarrow \infty} \frac{n \sin \frac{2\pi}{n}}{2\pi} = 1$$

$$\therefore \lim_{n \rightarrow \infty} \frac{\sin \frac{2\pi}{n}}{\frac{2\pi}{n}} = 1$$

(2) Let  $\theta = \frac{2\pi}{n}$ . As  $n \rightarrow \infty$ ,  $\theta \rightarrow 0$

$$\therefore \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1 \quad \{\text{using (1)}\}$$



## EXERCISE 15D

$$\begin{aligned}
 \text{1 a } \lim_{\theta \rightarrow 0} \frac{\sin 2\theta}{\theta} &= \lim_{\theta \rightarrow 0} \frac{\sin 2\theta}{2\theta} \times 2 \\
 &= 2 \times \lim_{2\theta \rightarrow 0} \frac{\sin 2\theta}{2\theta} \quad \{2\theta \rightarrow 0 \text{ as } \theta \rightarrow 0\} \\
 &= 2 \times 1 \\
 &= 2
 \end{aligned}$$

$$\begin{aligned}
 \text{b } \lim_{\theta \rightarrow 0} \frac{\theta}{\sin \theta} &= \lim_{\theta \rightarrow 0} \frac{1}{\frac{\sin \theta}{\theta}} \\
 &= \frac{1}{\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta}} \\
 &= \frac{1}{1} \\
 &= 1
 \end{aligned}$$

$$\begin{aligned}
 \text{c } \lim_{\theta \rightarrow 0} \frac{\tan \theta}{\theta} &= \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} \lim_{\theta \rightarrow 0} \frac{1}{\cos \theta} \\
 &= 1 \times \frac{1}{1} \\
 &= 1
 \end{aligned}$$

$$\begin{aligned}
 \text{d } \lim_{\theta \rightarrow 0} \frac{\sin^2 \theta}{\theta} &= \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} \lim_{\theta \rightarrow 0} \sin \theta \\
 &= 1 \times 0 \\
 &= 0
 \end{aligned}$$

$$\begin{aligned}
 \text{e } \lim_{\theta \rightarrow 0} \frac{\sin \theta \sin 4\theta}{\theta^2} &= \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} \lim_{\theta \rightarrow 0} \frac{\sin 4\theta}{\theta} \\
 &= 1 \times \lim_{\theta \rightarrow 0} \frac{\sin 4\theta}{4\theta} \times 4 \\
 &= 4 \times \lim_{4\theta \rightarrow 0} \frac{\sin 4\theta}{4\theta} \quad \{4\theta \rightarrow 0 \text{ as } \theta \rightarrow 0\} \\
 &= 4 \times 1 \\
 &= 4
 \end{aligned}$$

$$\begin{aligned}
 \text{f } \lim_{h \rightarrow 0} \frac{\sin \frac{h}{2} \cos h}{h} &= \lim_{h \rightarrow 0} \cos h \lim_{h \rightarrow 0} \frac{\sin \frac{h}{2}}{h} \\
 &= 1 \times \lim_{h \rightarrow 0} \frac{\sin \frac{h}{2}}{\frac{h}{2}} \times \frac{1}{2} \\
 &= \frac{1}{2} \times \lim_{\frac{h}{2} \rightarrow 0} \frac{\sin \frac{h}{2}}{\frac{h}{2}} \quad \left\{ \frac{h}{2} \rightarrow 0 \text{ as } h \rightarrow 0 \right\} \\
 &= \frac{1}{2} \times 1 \\
 &= \frac{1}{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{g } \lim_{n \rightarrow \infty} n \sin \frac{3\pi}{n} &= \lim_{\theta \rightarrow 0} \frac{3\pi}{\theta} \sin \theta \quad \left\{ \theta = \frac{3\pi}{n} \rightarrow 0 \text{ as } n \rightarrow \infty \right\} \\
 &= 3\pi \times \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} \\
 &= 3\pi \times 1 \\
 &= 3\pi
 \end{aligned}$$

$$\begin{aligned}
 \text{h } \lim_{x \rightarrow 0} \frac{\sin 7x}{4x} &= \lim_{x \rightarrow 0} \frac{\sin 7x}{7x} \times \frac{7}{4} \\
 &= \frac{7}{4} \times \lim_{7x \rightarrow 0} \frac{\sin 7x}{7x} \quad \{7x \rightarrow 0 \text{ as } x \rightarrow 0\} \\
 &= \frac{7}{4} \times 1 \\
 &= \frac{7}{4}
 \end{aligned}$$

$$\begin{aligned}
 \text{i} \quad \lim_{x \rightarrow 0} \frac{x^2 + x}{\sin 2x} &= \lim_{x \rightarrow 0} \left( \frac{x^2}{\sin 2x} + \frac{x}{\sin 2x} \right) \\
 &= \lim_{x \rightarrow 0} \left( \frac{x}{2} \times \frac{2x}{\sin 2x} + \frac{1}{2} \times \frac{2x}{\sin 2x} \right) \\
 &= 0 \times 1 + \frac{1}{2} \times 1 \quad \{\text{using b}\} \\
 &= \frac{1}{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{2 a} \quad \lim_{x \rightarrow 0} \frac{x + \sin x}{x - \sin x} &= \lim_{x \rightarrow 0} \frac{x + \sin x}{x - \sin x} \times \frac{\frac{1}{x}}{\frac{1}{x}} \\
 &= \lim_{x \rightarrow 0} \frac{1 + \frac{\sin x}{x}}{1 - \frac{\sin x}{x}}
 \end{aligned}$$

As  $x \rightarrow 0$ , numerator  $\rightarrow 2$ , but the denominator  $\rightarrow 0$ .

Hence as  $x \rightarrow \infty$ ,  $\frac{x + \sin x}{x - \sin x} \rightarrow \infty$ .

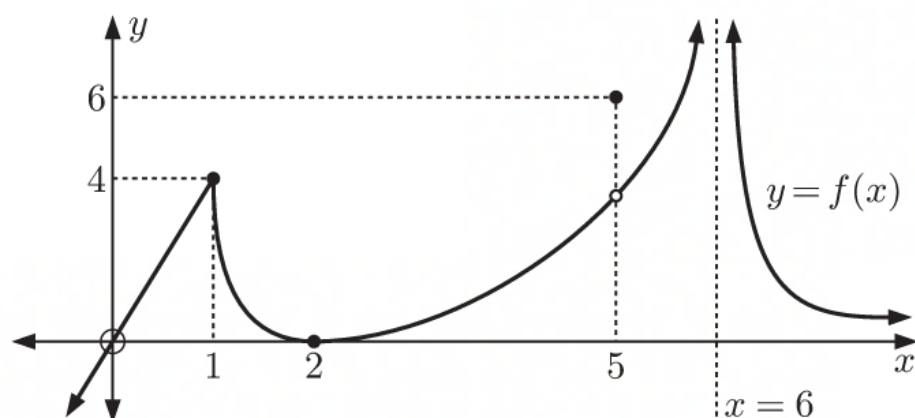
$\therefore \lim_{x \rightarrow 0} \frac{x + \sin x}{x - \sin x}$  does not exist.

$$\begin{aligned}
 \text{b} \quad \lim_{h \rightarrow 0} \frac{\cos h - 1}{h} &= \lim_{h \rightarrow 0} \frac{\cos h - 1}{h} \times \frac{\cos h + 1}{\cos h + 1} \\
 &= \lim_{h \rightarrow 0} \frac{\cos^2 h - 1}{h(\cos h + 1)} \\
 &= \lim_{h \rightarrow 0} \frac{-\sin^2 h}{h(\cos h + 1)} \\
 &= \lim_{h \rightarrow 0} -h \left( \frac{\sin h}{h} \right)^2 \frac{1}{\cos h + 1} \\
 &= 0 \times 1^2 \times \frac{1}{2} \\
 &= 0
 \end{aligned}$$

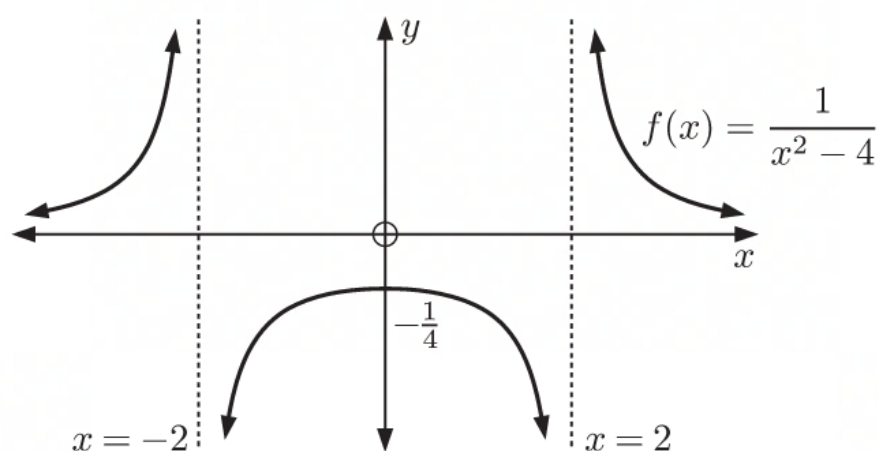
$$\begin{aligned}
 \text{c} \quad \lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} &= \lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} \times \frac{1 + \cos x}{1 + \cos x} \\
 &= \lim_{x \rightarrow 0} \frac{1 - \cos^2 x}{x^2(1 + \cos x)} \\
 &= \lim_{x \rightarrow 0} \frac{\sin^2 x}{x^2(1 + \cos x)} \\
 &= \lim_{x \rightarrow 0} \left( \frac{\sin x}{x} \right)^2 \lim_{x \rightarrow 0} \frac{1}{1 + \cos x} \\
 &= (1)^2 \times \frac{1}{2} \\
 &= \frac{1}{2}
 \end{aligned}$$

## EXERCISE 15E

1



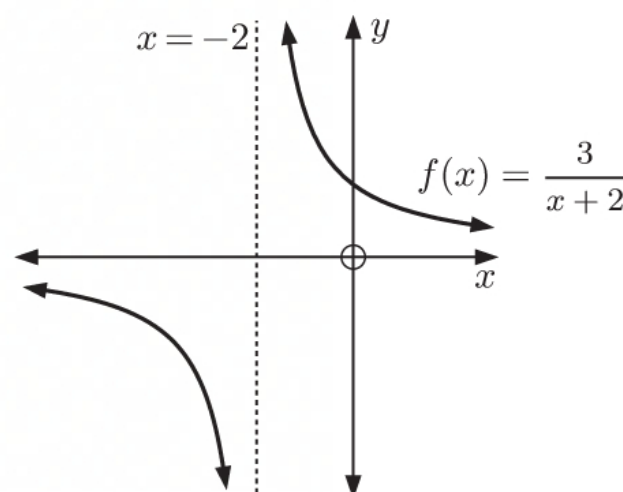
- a**  $f$  is not defined at  $x = 6$ , and  $\lim_{x \rightarrow 6} f(x)$  does not exist.  
 $\therefore f$  has an essential discontinuity at  $x = 6$ .
- b**  $f(5) = 6$ , and  $\lim_{x \rightarrow 5} f(x)$  exists, but  $\lim_{x \rightarrow 5} f(x) \neq f(5)$ .  
 $\therefore f$  has a removable discontinuity at  $x = 5$ .
- c**  $f$  is continuous for all  $x \neq 5$  or  $6$ .

2 **a**

- b**  $f(x)$  is continuous for all  $x \in \mathbb{R}$ ,  $x \neq \pm 2$ .

- 3 a**  $\lim_{x \rightarrow a} \sqrt{x}$  exists when  $a > 0$ .  $\{x \geq 0, \lim_{x \rightarrow 0^-} \sqrt{x} \text{ does not exist}\}$
- b**  $f(x) = \sqrt{x}$  is continuous for  $x \geq 0$ .  $\{\lim_{x \rightarrow a} \sqrt{x} = \sqrt{a} \text{ for all } a > 0, \lim_{x \rightarrow 0^+} \sqrt{x} = 0\}$

- 4 a**  $f$  is not defined at  $x = -2$ ,  
 and  $\lim_{x \rightarrow -2} f(x)$  does not exist.  
 $f$  has an essential discontinuity at  $x = -2$ .  
 $f$  is continuous for all  $x \in \mathbb{R}$ ,  $x \neq -2$ .





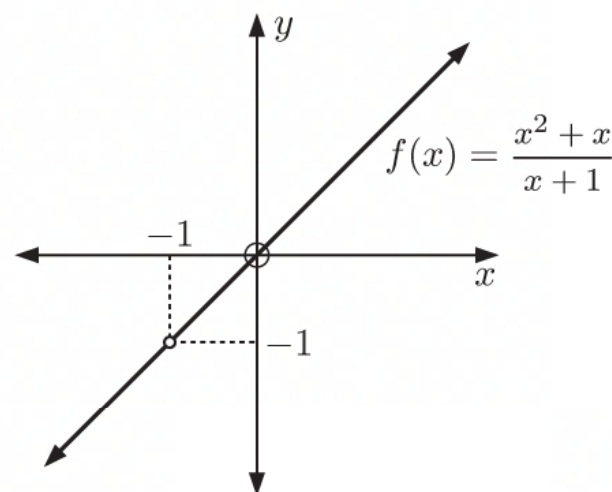
**b**  $f$  is not defined at  $x = -1$ ,

$$\begin{aligned} \text{but } \lim_{x \rightarrow -1} \frac{x^2 + x}{x + 1} &= \lim_{x \rightarrow -1} \frac{x(x + 1)}{x + 1} \\ &= \lim_{x \rightarrow -1} x \quad \{x + 1 \neq 0\} \\ &= -1 \end{aligned}$$

$\therefore f$  has a removable discontinuity at  $x = -1$ .

$f$  is continuous for all  $x \in \mathbb{R}$ ,  $x \neq -1$ .

The discontinuity can be removed by defining the function  $g(x) = x$ .



$$\begin{aligned} \text{c } f(x) &= \frac{x - 4}{x^2 - 2x - 8} \\ &= \frac{x - 4}{(x - 4)(x + 2)} \end{aligned}$$

$f$  is not defined at  $x = 4$ ,

$$\begin{aligned} \text{but } \lim_{x \rightarrow 4} f(x) &= \lim_{x \rightarrow 4} \frac{x - 4}{(x - 4)(x + 2)} \\ &= \lim_{x \rightarrow 4} \frac{1}{x + 2} \quad \{x - 4 \neq 0\} \\ &= \frac{1}{6} \end{aligned}$$

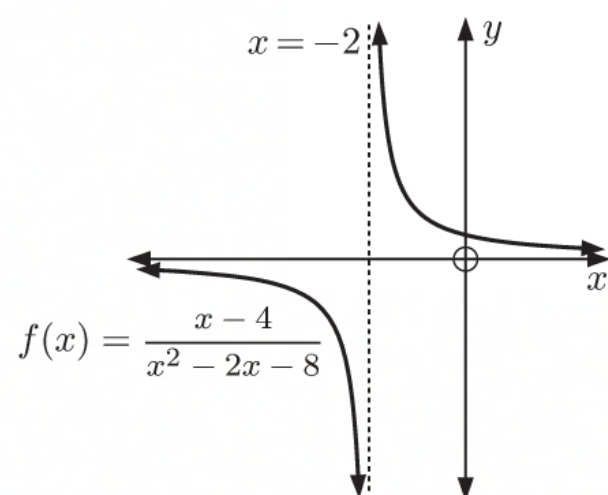
$\therefore f$  has a removable discontinuity at  $x = 4$ .

$f$  is also not defined at  $x = -2$ , and  $\lim_{x \rightarrow -2} f(x)$  does not exist.

$\therefore f$  has an essential discontinuity at  $x = -2$ .

$f$  is continuous for all  $x \in \mathbb{R}$ ,  $x \neq -2$  or  $4$ .

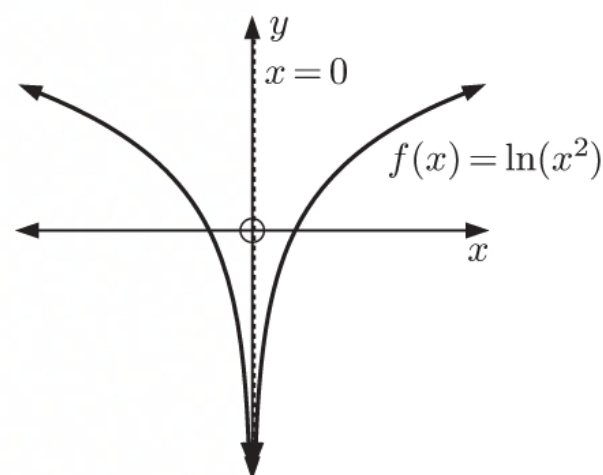
The removable discontinuity can be removed by defining the function  $g(x) = \frac{1}{x + 2}$ .



**d**  $f$  is not defined at  $x = 0$ ,  
and  $\lim_{x \rightarrow 0} f(x)$  does not exist.

$\therefore f$  has an essential discontinuity at  $x = 0$ .

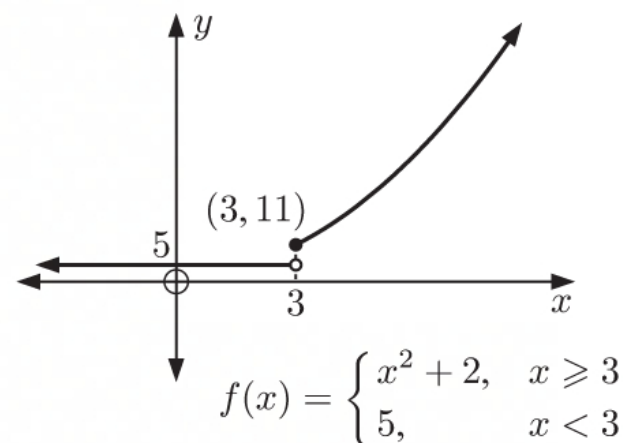
$f$  is continuous for all  $x \in \mathbb{R}$ ,  $x \neq 0$ .



**e**  $f$  is defined for all  $x$ , but there is a “jump” at  $x = 3$ .

$\therefore \lim_{x \rightarrow 3} f(x)$  does not exist, and  $f$  has an essential discontinuity at  $x = 3$ .

$f$  is continuous for all  $x \in \mathbb{R}$ ,  $x \neq 3$ .



$$\begin{aligned} f(x) &= \tan x \cos x \\ &= \frac{\sin x \cos x}{\cos x} \end{aligned}$$

$f$  is not defined at  $x = \frac{\pi}{2} + k\pi$ ,  $k \in \mathbb{Z}$ .

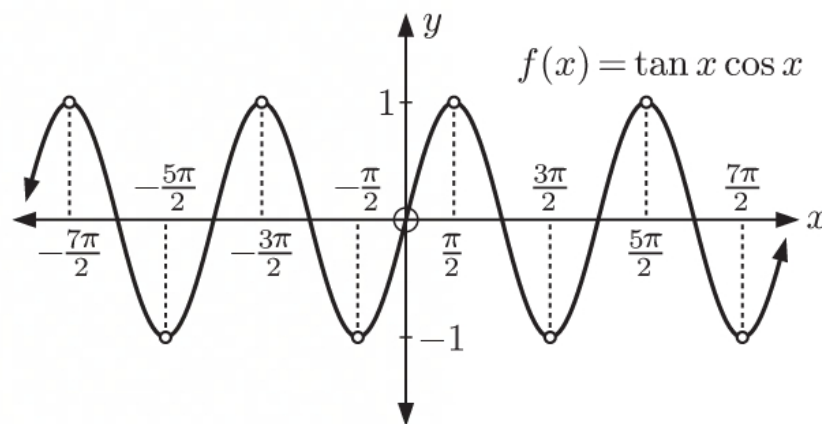
For even  $k$ , the limit as  $x$  approaches  $\frac{\pi}{2} + k\pi$  from each side is 1.

For odd  $k$ , the limit as  $x$  approaches  $\frac{\pi}{2} + k\pi$  from each side is  $-1$ .

$\therefore f$  has removable discontinuities at  $x = \frac{\pi}{2} + k\pi$ ,  $k \in \mathbb{Z}$ .

$f$  is continuous for all  $x \in \mathbb{R}$ ,  $x \neq \frac{\pi}{2} + k\pi$ ,  $k \in \mathbb{Z}$ .

The discontinuities could be removed by defining the function  $g(x) = \sin x$ .



$$\begin{aligned} 5 \quad a \quad \lim_{x \rightarrow 1} \frac{x^3 - 1}{x - 1} &= \lim_{x \rightarrow 1} \frac{(x-1)(x^2 + x + 1)}{(x-1)} \quad \{x \neq 1\} \\ &= 3 \end{aligned}$$

$\therefore f$  is continuous on  $\mathbb{R}$  if  $k = 3$ .

$$b \quad \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

$\therefore f$  is continuous on  $\mathbb{R}$  if  $k = 1$ .

$$c \quad f(2) = 2^2 = 4$$

$\therefore f$  is continuous on  $\mathbb{R}$  if  $k + 1 = 4$   
 $\therefore k = 3$

$$d \quad f(x) = \frac{1}{x} \text{ has an essential discontinuity at } x = 0.$$

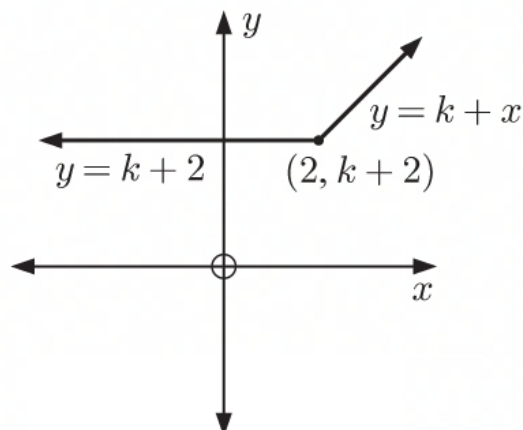
$\therefore$  there is no  $k \in \mathbb{R}$  such that  $f$  is continuous on  $\mathbb{R}$ .

$$e \quad y = kx \text{ always passes through } (0, 0).$$

$\therefore f$  is continuous on  $\mathbb{R}$  for all  $k \in \mathbb{R}$ .

$$f \quad \text{If } k + 2 \geq 0: \\ k \geq -2$$

$$\text{and } f(x) = \begin{cases} k + x, & x \geq 2 \\ k + 2, & x < 2 \end{cases}$$



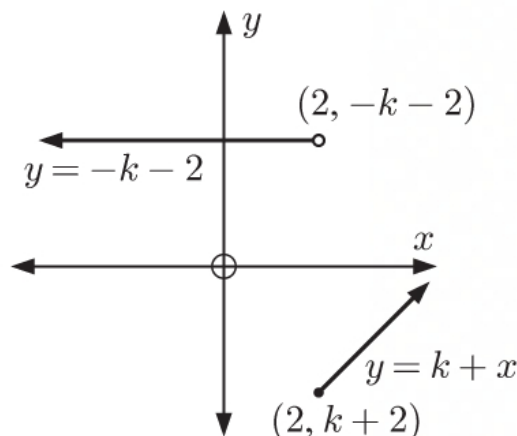
$\therefore f(x)$  is continuous for all  $x \in \mathbb{R}$ .

$\therefore f$  is continuous on  $\mathbb{R}$  if  $k \geq -2$ .

$$\text{If } k + 2 < 0:$$

$$k < -2$$

$$\text{and } f(x) = \begin{cases} k + x, & x \geq 2 \\ -k - 2, & x < 2 \end{cases}$$



$\therefore f(x)$  is discontinuous at  $x = 2$ .

**6**  $g(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}.$

**a**  $x_n = a + \frac{\sqrt{2}}{n}, \quad a \in \mathbb{Q}, \quad n \in \mathbb{Z}^+$

Since  $a \in \mathbb{Q}$  and  $\frac{\sqrt{2}}{n} \notin \mathbb{Q}$ ,  $x_n \notin \mathbb{Q}$

$$\therefore g(x_n) = 0 \quad \text{for all } n \in \mathbb{Z}^+ \quad \{\text{Dirichlet function}\}$$

$$\therefore \lim_{n \rightarrow \infty} g(x_n) = 0$$

But  $g(a) = 1$  since  $a \in \mathbb{Q}$

$$\therefore \lim_{n \rightarrow \infty} g(x_n) \neq g(a)$$

$$\text{and } \lim_{n \rightarrow \infty} x_n = a$$

$\therefore g(x)$  is discontinuous at  $x = a$  (from the alternative definition of continuity given in the question).

**b** If  $x_n$  is the decimal expansion of  $a$  to  $n$  decimal places then  $x_n \in \mathbb{Q}$ .

$$\therefore g(x_n) = 1 \quad \text{for all } n \in \mathbb{Z}^+ \quad \{\text{Dirichlet function}\}$$

$$\therefore \lim_{n \rightarrow \infty} g(x_n) = 1$$

But  $g(a) = 0$  since  $a \notin \mathbb{Q}$

$$\therefore \lim_{n \rightarrow \infty} g(x_n) \neq g(a)$$

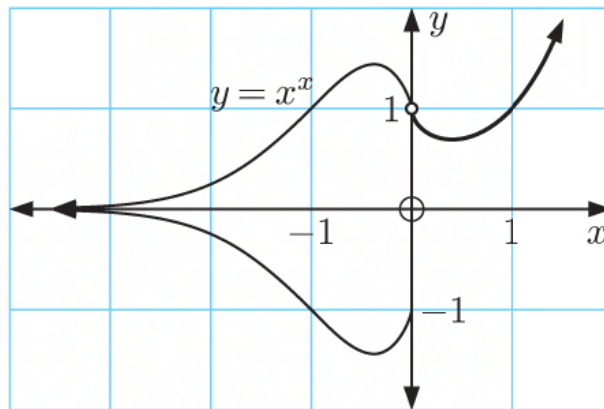
$$\text{and } \lim_{n \rightarrow \infty} x_n = a$$

$\therefore g(x)$  is discontinuous at  $x = a$ .

**c** From **a** and **b**, for all rational and irrational  $a$ ,  $g(x)$  is discontinuous at  $x = a$ .

$\therefore$  the Dirichlet function is continuous nowhere.

**7 a**



**b** The function  $y = x^x$  is defined for all  $x > 0$ , and  $\lim_{x \rightarrow a} x^x = a^a$  for all  $a > 0$ .

$\therefore y = x^x$  is continuous for all  $x > 0$ .

$0^0$  is undefined, so  $y = x^x$  is discontinuous at  $x = 0$ .

If  $x < 0$  and  $x \in \mathbb{Q}$ , then in lowest terms  $x = -\frac{m}{n}$ , where  $m, n \in \mathbb{Z}^+$ ,  $n$  odd.

$$\begin{aligned}
 \therefore x^x &= \left(-\frac{m}{n}\right)^{\left(-\frac{m}{n}\right)} \\
 &= \left(-\frac{n}{m}\right)^{\left(\frac{m}{n}\right)} \\
 &= \left(\sqrt[n]{-\frac{n}{m}}\right)^m \\
 &= \left(-\sqrt[n]{\frac{n}{m}}\right)^m \quad \{n \text{ odd}\} \\
 \therefore x^x &= \left(\sqrt[n]{\frac{n}{m}}\right)^m = (-x)^x \text{ if } m \text{ is even} \\
 \text{and } x^x &= -\left(\sqrt[n]{\frac{n}{m}}\right)^m = -(-x)^x \text{ if } m \text{ is odd.}
 \end{aligned}$$

So for  $a < 0$ , as  $x \rightarrow a$ ,  $y = x^x$  “jumps” between  $(-x)^x$  and  $-(-x)^x$ .

$\therefore \lim_{x \rightarrow a} x^x$  does not exist for  $a < 0$ .

$\therefore y = x^x$  is continuous for  $x > 0$ , and discontinuous for  $x \leq 0$ .

## REVIEW SET 15A

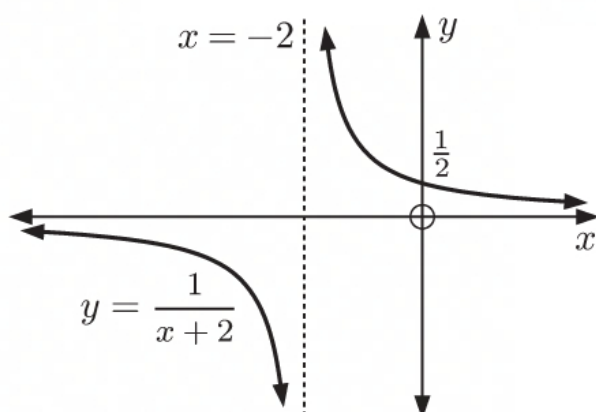
- 1 a** We can make  $6x - 7$  as close as we like to  $-1$  by making  $x$  sufficiently close to 1.

$$\therefore \lim_{x \rightarrow 1} (6x - 7) = -1$$

$$\begin{aligned}
 \text{b } \lim_{h \rightarrow 0} \frac{2h^2 - h}{h} &= \lim_{h \rightarrow 0} \frac{\cancel{h}(2h - 1)}{\cancel{h}} \\
 &= \lim_{h \rightarrow 0} (2h - 1) \quad \{\text{as } h \neq 0\} \\
 &= -1
 \end{aligned}$$

$$\begin{aligned}
 \text{c } \lim_{x \rightarrow 4} \frac{x^2 - 16}{x - 4} &= \lim_{x \rightarrow 4} \frac{(x + 4)\cancel{(x - 4)}}{\cancel{(x - 4)}} \\
 &= \lim_{x \rightarrow 4} (x + 4) \quad \{\text{as } x \neq 4\} \\
 &= 8
 \end{aligned}$$

- 2 a**





**b i** As  $x \rightarrow 0^+$ ,  $\frac{1}{x+2} \rightarrow \frac{1}{2}$

As  $x \rightarrow 0^-$ ,  $\frac{1}{x+2} \rightarrow \frac{1}{2}$

$\therefore \lim_{x \rightarrow 0} \frac{1}{x+2} = \frac{1}{2}$

**ii** As  $x \rightarrow 2^+$ ,  $\frac{1}{x+2} \rightarrow \frac{1}{4}$

As  $x \rightarrow 2^-$ ,  $\frac{1}{x+2} \rightarrow \frac{1}{4}$

$\therefore \lim_{x \rightarrow 2} \frac{1}{x+2} = \frac{1}{4}$

**iii** As  $x \rightarrow -2^+$ ,  $\frac{1}{x+2} \rightarrow \infty$

As  $x \rightarrow -2^-$ ,  $\frac{1}{x+2} \rightarrow -\infty$

$\therefore \lim_{x \rightarrow -2} \frac{1}{x+2}$  does not exist.

**3 a**

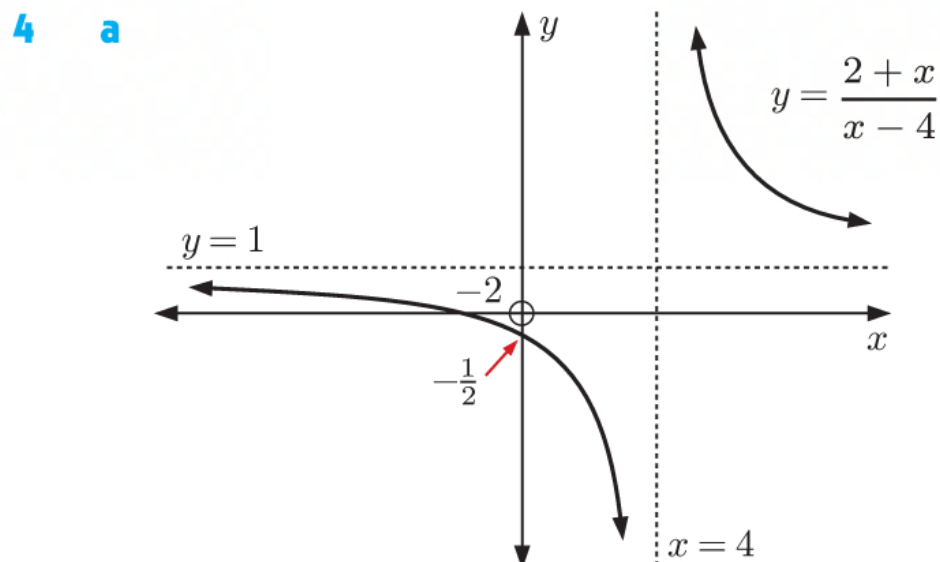
$$\begin{aligned} & \lim_{\theta \rightarrow 0} \frac{\sin 4\theta}{\theta} \\ &= \lim_{\theta \rightarrow 0} \frac{\sin 4\theta}{4\theta} \times 4 \\ &= 4 \lim_{4\theta \rightarrow 0} \frac{\sin 4\theta}{4\theta} \quad \{4\theta \rightarrow 0 \text{ as } \theta \rightarrow 0\} \\ &= 4 \times 1 \\ &= 4 \end{aligned}$$

**b**

$$\begin{aligned} & \lim_{\theta \rightarrow 0} \frac{2\theta}{\sin 3\theta} \\ &= \lim_{\theta \rightarrow 0} \frac{3\theta}{\sin 3\theta} \times \frac{2}{3} \\ &= \frac{2}{3} \lim_{\theta \rightarrow 0} \frac{1}{\frac{\sin 3\theta}{3\theta}} \\ &= \frac{2}{3} \frac{1}{\lim_{3\theta \rightarrow 0} \frac{\sin 3\theta}{3\theta}} \quad \{3\theta \rightarrow 0 \text{ as } \theta \rightarrow 0\} \\ &= \frac{2}{3} \times \frac{1}{1} \\ &= \frac{2}{3} \end{aligned}$$

**c**

$$\begin{aligned} & \lim_{n \rightarrow \infty} n \sin \frac{\pi}{n} \\ &= \lim_{\theta \rightarrow 0} \frac{\pi}{\theta} \sin \theta \quad \{\theta = \frac{\pi}{n} \rightarrow 0 \text{ as } n \rightarrow \infty\} \\ &= \pi \times \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} \\ &= \pi \times 1 \\ &= \pi \end{aligned}$$



**b** As  $x \rightarrow 4^-$ ,  $y \rightarrow -\infty$

As  $x \rightarrow 4^+$ ,  $y \rightarrow \infty$

As  $x \rightarrow -\infty$ ,  $y \rightarrow 1^-$

As  $x \rightarrow \infty$ ,  $y \rightarrow 1^+$

The vertical asymptote is  $x = 4$ .

The horizontal asymptote is  $y = 1$ .

**c**  $\lim_{x \rightarrow -\infty} \frac{2+x}{x-4} = 1$ ,  $\lim_{x \rightarrow \infty} \frac{2+x}{x-4} = 1$

$$5 \quad \sin \frac{\theta}{2} = \frac{b}{2r}$$

$$\therefore \sin \frac{\theta}{2} = \frac{b}{2r}$$

$$\therefore \sin \frac{2\pi}{n} = \frac{b}{2r}$$

$$\therefore \sin \frac{\pi}{n} = \frac{b}{2r}$$

$$\therefore b = 2r \sin \frac{\pi}{n}$$

$$\text{Now, } C = \lim_{n \rightarrow \infty} n \times b$$

$$= \lim_{n \rightarrow \infty} n \times 2r \sin \frac{\pi}{n}$$

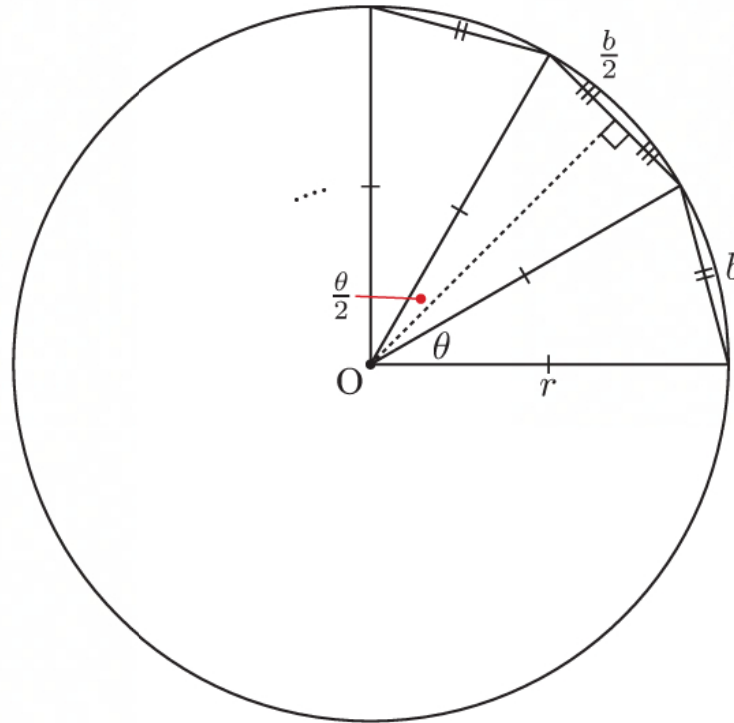
$$= 2r \lim_{n \rightarrow \infty} n \sin \frac{\pi}{n}$$

$$= 2r \lim_{\theta \rightarrow 0} \frac{\pi}{\theta} \sin \theta \quad \left\{ \theta = \frac{\pi}{n} \rightarrow 0 \text{ as } n \rightarrow \infty \right\}$$

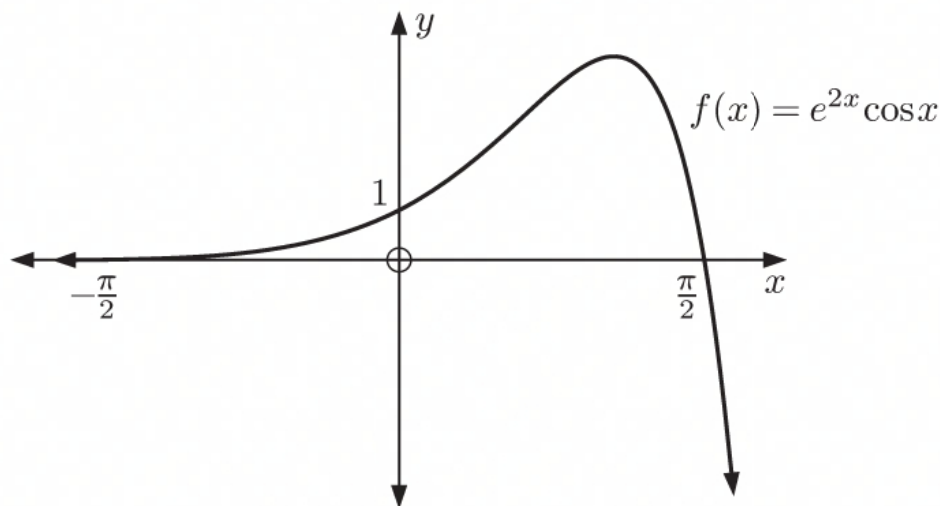
$$= 2\pi r \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta}$$

$$= 2\pi r \times 1$$

$$= 2\pi r$$



6 a



b i As  $x \rightarrow -\infty$ ,  $e^{2x} \cos x \rightarrow 0^+$

$$\therefore \lim_{x \rightarrow -\infty} f(x) = 0$$

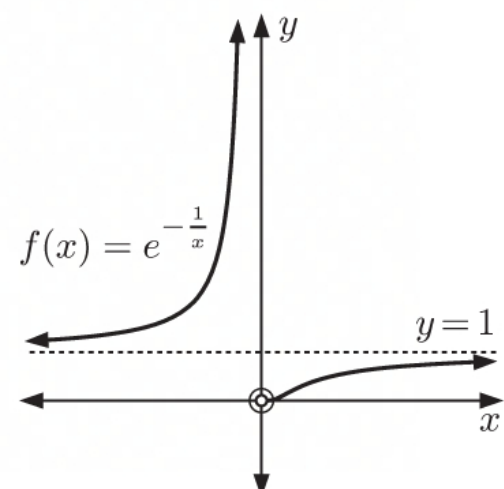
ii As  $x \rightarrow \infty$ ,  $e^{2x} \cos x \rightarrow -\infty$

$$\therefore \lim_{x \rightarrow \infty} f(x) \text{ does not exist}$$

7 a  $f$  is not defined at  $x = 0$ , and  $\lim_{x \rightarrow 0} f(x)$  does not exist.

$\therefore f$  has an essential discontinuity at  $x = 0$ .

$f$  is continuous for all  $x \in \mathbb{R}$ ,  $x \neq 0$ .



$$\begin{aligned} \text{b } g(x) &= \frac{2x-1}{8x^2-10x+3} \\ &= \frac{2x-1}{(2x-1)(4x-3)} \end{aligned}$$

$g$  is not defined at  $x = \frac{1}{2}$ , but

$$\begin{aligned} \lim_{x \rightarrow \frac{1}{2}} g(x) &= \lim_{x \rightarrow \frac{1}{2}} \frac{\cancel{2x-1}}{(\cancel{2x-1})(4x-3)} \\ &= \lim_{x \rightarrow \frac{1}{2}} \frac{1}{4x-3} \quad \{\text{as } x \neq \tfrac{1}{2}\} \\ &= -1 \end{aligned}$$

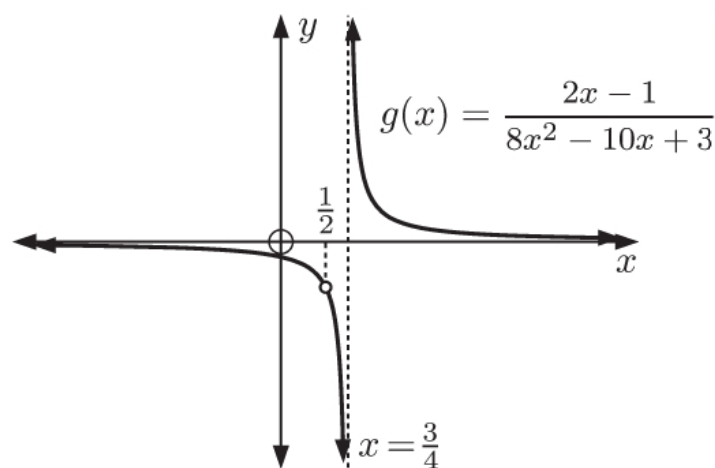
$\therefore g$  has a removable discontinuity at  $x = \frac{1}{2}$ .

$g$  is also not defined at  $x = \frac{3}{4}$ , and  $\lim_{x \rightarrow \frac{3}{4}} g(x)$  does not exist.

$\therefore g$  has an essential discontinuity at  $x = \frac{3}{4}$ .

$g$  is continuous for all  $x \in \mathbb{R}$ ,  $x \neq \frac{1}{2}$  or  $\frac{3}{4}$ .

The removable discontinuity can be removed by defining the function  $h(x) = \frac{1}{4x-3}$ .



## REVIEW SET 15B

$$\begin{aligned} \text{1 a } \lim_{h \rightarrow 0} \frac{h^3 - 3h}{h} &= \lim_{h \rightarrow 0} \frac{\cancel{h}(h^2 - 3)}{\cancel{h}} \\ &= \lim_{h \rightarrow 0} (h^2 - 3) \quad \{\text{as } h \neq 0\} \\ &= -3 \end{aligned}$$

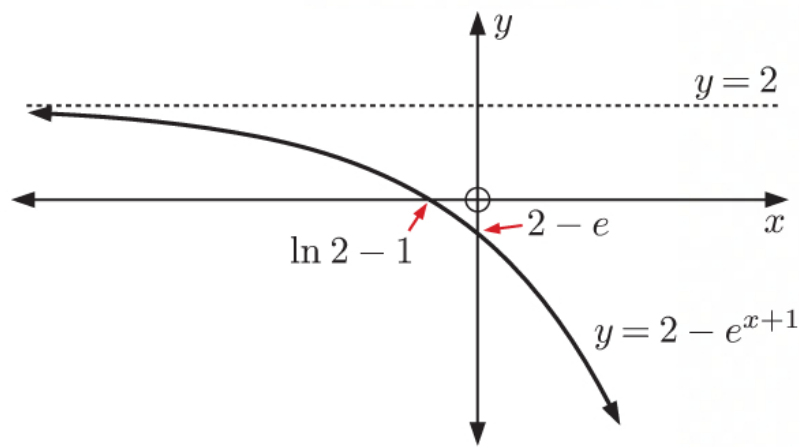
$$\begin{aligned} \text{b } \lim_{x \rightarrow 1} \frac{3x^2 - 3x}{x - 1} &= \lim_{x \rightarrow 1} \frac{3x(\cancel{x-1})}{(\cancel{x-1})} \\ &= \lim_{x \rightarrow 1} 3x \quad \{\text{as } x \neq 1\} \\ &= 3 \end{aligned}$$

$$\begin{aligned} \text{c } \lim_{x \rightarrow 2} \frac{x^2 - 3x + 2}{2 - x} &= \lim_{x \rightarrow 2} \frac{(x-1)\cancel{(x-2)}}{-(\cancel{x-2})} \\ &= \lim_{x \rightarrow 2} -(x-1) \quad \{\text{as } x \neq 2\} \\ &= -1 \end{aligned}$$

$$\begin{aligned} \text{2 a } f(x) &= \frac{x+3}{x^2-x-12} \text{ is undefined when } x^2-x-12=0 \\ &\therefore (x+3)(x-4)=0 \\ &\therefore x = -3 \text{ or } 4 \end{aligned}$$

$$\begin{aligned} \text{b i } \lim_{x \rightarrow -3} f(x) &= \lim_{x \rightarrow -3} \frac{\cancel{x+3}}{(\cancel{x+3})(x-4)} \\ &= \lim_{x \rightarrow -3} \frac{1}{x-4} \quad \{\text{since } x \neq -3\} \\ &= -\frac{1}{7} \end{aligned}$$

$$\begin{aligned} \text{ii As } x \rightarrow 4^+, \quad \frac{1}{x-4} &\rightarrow \infty \\ \text{As } x \rightarrow 4^-, \quad \frac{1}{x-4} &\rightarrow -\infty \\ \therefore \lim_{x \rightarrow 4} f(x) &\text{ does not exist.} \end{aligned}$$

**3 a**

**b**  $\lim_{x \rightarrow -\infty} (2 - e^{x+1}) = 2$ ,  $\lim_{x \rightarrow \infty} (2 - e^{x+1})$  does not exist.

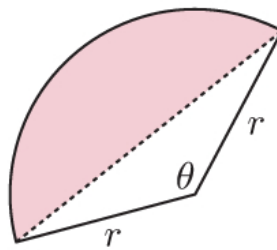
**c** The asymptote is  $y = 2$ .

**4 a** Area of shaded segment

$$= (\text{area of sector}) - (\text{area of triangle})$$

$$= \frac{1}{2}r^2\theta - \frac{1}{2}r^2 \sin \theta$$

$$= \frac{1}{2}r^2(\theta - \sin \theta)$$



**b** As  $\theta \rightarrow 0^+$ , area of shaded segment  $\rightarrow 0$

$$\therefore \frac{1}{2}r^2(\theta - \sin \theta) \rightarrow 0$$

$$\therefore \theta - \sin \theta \rightarrow 0$$

$$\therefore \frac{\theta - \sin \theta}{\theta} \rightarrow 0$$

$$\therefore 1 - \frac{\sin \theta}{\theta} \rightarrow 0$$

$$\therefore \frac{\sin \theta}{\theta} \rightarrow 1$$

$$\therefore \lim_{\theta \rightarrow 0^+} \frac{\sin \theta}{\theta} = 1$$

**5**

$$\lim_{\theta \rightarrow 0} \frac{\sin \frac{\theta}{k}}{\frac{\theta}{k}} = k + 2$$

$$\therefore \lim_{\theta \rightarrow 0} \frac{1}{k} \frac{\sin \frac{\theta}{k}}{\frac{\theta}{k}} = k + 2$$

$$\therefore \frac{1}{k} \lim_{\frac{\theta}{k} \rightarrow 0} \frac{\sin \frac{\theta}{k}}{\frac{\theta}{k}} = k + 2 \quad \left\{ \frac{\theta}{k} \rightarrow 0 \text{ as } \theta \rightarrow 0 \right\}$$

$$\therefore \frac{1}{k} = k + 2$$

$$\therefore 1 = k^2 + 2k$$

$$\therefore k^2 + 2k - 1 = 0$$

$$\therefore k = \frac{-2 \pm \sqrt{2^2 - 4(1)(-1)}}{2(1)}$$

$$\therefore k = \frac{-2 \pm 2\sqrt{2}}{2}$$

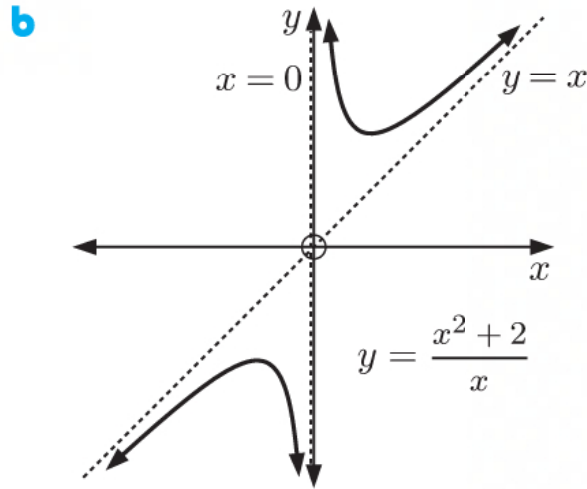
$$\therefore k = -1 \pm \sqrt{2}$$



**6 a**  $y = \frac{x^2 + 2}{x}$   
 $= x + \frac{2}{x}$

$y$  is undefined when  $x = 0$ , and as  $x \rightarrow \pm\infty$ ,  $x + \frac{2}{x} \rightarrow x$ .

$\therefore$  the vertical asymptote is  $x = 0$ , and the oblique asymptote is  $y = x$ .

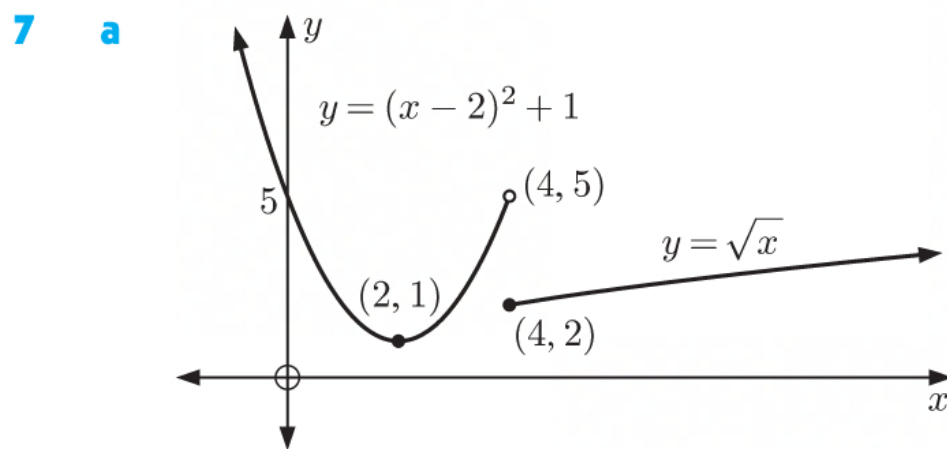


**c i**  $\lim_{x \rightarrow \infty} \frac{x^2 + 2}{x} = \lim_{x \rightarrow \infty} \left( x + \frac{2}{x} \right)$  which does not exist.

**ii**  $\lim_{x \rightarrow \infty} \left( \frac{x^2 + 2}{x} - x \right) = \lim_{x \rightarrow \infty} \frac{x^2 + 2 - x^2}{x}$   
 $= \lim_{x \rightarrow \infty} \frac{2}{x}$   
 $= 0$

**d** As  $x \rightarrow \infty$ ,  $\frac{x^2 + 2}{x} - x \rightarrow 0 \therefore \frac{x^2 + 2}{x} \rightarrow x$ .

This is shown by the oblique asymptote  $y = x$ .



**b**  $\lim_{x \rightarrow 4^-} f(x) = \lim_{x \rightarrow 4^-} [(x - 2)^2 + 1]$   
 $= (4 - 2)^2 + 1$   
 $= 2^2 + 1$   
 $= 5$

$\lim_{x \rightarrow 4^+} f(x) = \lim_{x \rightarrow 4^+} \sqrt{x}$   
 $= \sqrt{4}$   
 $= 2$

**c**  $f$  has an essential discontinuity at  $x = 4$ .

$\lim_{x \rightarrow 4^-} f(x)$  and  $\lim_{x \rightarrow 4^+} f(x)$  both exist, but  $\lim_{x \rightarrow 4^-} f(x) \neq \lim_{x \rightarrow 4^+} f(x)$ .

$$\begin{aligned} \textbf{8 a } g(x) &= \frac{\sin x}{\sin 2x} \\ &= \frac{\sin x}{2 \sin x \cos x} \end{aligned}$$

The removable discontinuities of  $g$  occur when  $\sin x = 0$ , as they can be removed by defining the function  $h(x) = \frac{1}{2 \cos x}$ .

$\therefore$  the removable discontinuities of  $g$  are  $x = n\pi$ ,  $n \in \mathbb{Z}$ .

$$\textbf{b } h(x) = \frac{1}{2 \cos x}$$

$h$  is not defined when  $\cos x = 0$ .

$\therefore$  the essential discontinuities of  $g$  are  $x = (2n + 1)\frac{\pi}{2}$ ,  $n \in \mathbb{Z}$ .

# Chapter 16

## INTRODUCTION TO DIFFERENTIAL CALCULUS

### EXERCISE 16A.1

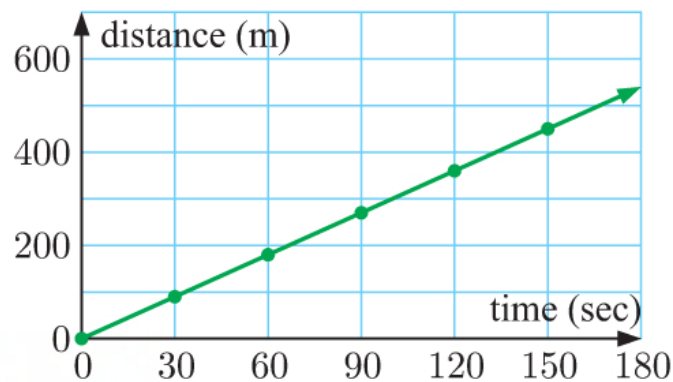
1 a

Time (seconds)	0	30	60	90	120	150
Distance (metres)	0	90	180	270	360	450

$\xrightarrow{+90}$   $\xrightarrow{+90}$   $\xrightarrow{+90}$   $\xrightarrow{+90}$   $\xrightarrow{+90}$

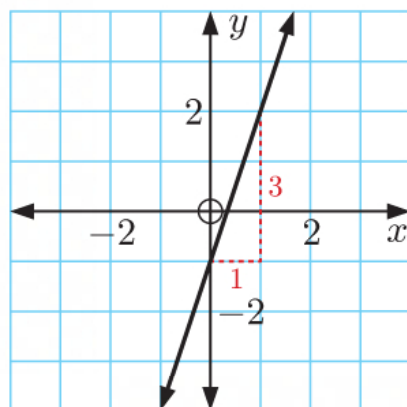
The distance travelled increases by the same amount each time interval.  
 $\therefore$  the jogger is travelling at a constant speed.

b



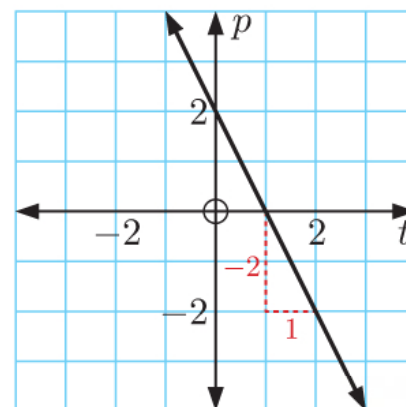
$$\begin{aligned} \text{c speed} &= \frac{(90 - 0) \text{ m}}{(30 - 0) \text{ s}} \\ &= 3 \text{ m per s} \end{aligned}$$

2 a



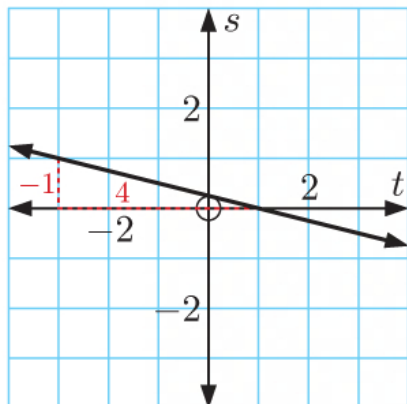
$$\begin{aligned} \text{rate of change} &= \text{gradient of line} \\ &= \frac{3}{1} \\ &= 3 \end{aligned}$$

b



$$\begin{aligned} \text{rate of change} &= \text{gradient of line} \\ &= \frac{-2}{1} \\ &= -2 \end{aligned}$$

c



$$\begin{aligned} \text{rate of change} &= \text{gradient of line} \\ &= \frac{-1}{4} \\ &= -\frac{1}{4} \end{aligned}$$

3 For the function  $f(x) = \frac{5}{2}x - 3$ , the gradient is  $\frac{5}{2}$ , so the rate of change is  $\frac{5}{2}$ .

**EXERCISE 16A.2**

- 1 a** The graph of distance against time is not a straight line.

$\therefore$  Aileen did not travel at a constant speed.

- b i** average speed from  $t = 0$  to  $t = 5$  h

$$= \frac{(300 - 0) \text{ km}}{(5 - 0) \text{ h}}$$

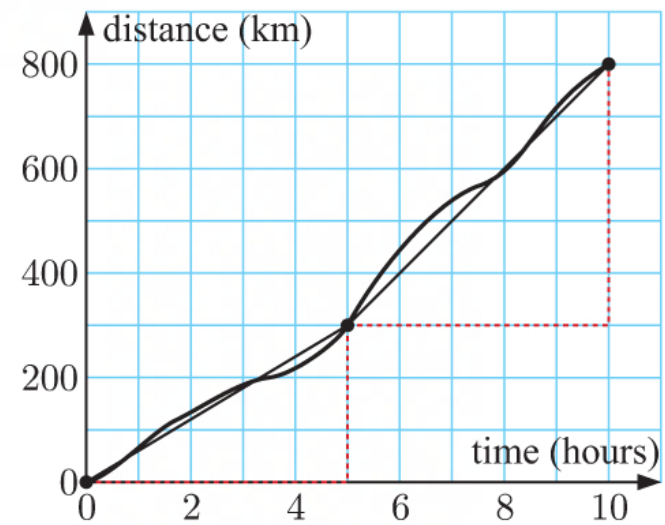
$$= 60 \text{ km per hour}$$

- ii** average speed from  $t = 5$  h to  $t = 10$  h

$$= \frac{(800 - 300) \text{ km}}{(10 - 5) \text{ h}}$$

$$= \frac{500}{5} \text{ km per hour}$$

$$= 100 \text{ km per hour}$$



- 2 a** average rate of change from  $t = 1$  h to  $t = 2.5$  h

$$= \frac{(250 - 100) \text{ m}}{(2.5 - 1) \text{ h}}$$

$$= \frac{150}{1.5} \text{ m per hour}$$

$$= 100 \text{ m per hour}$$

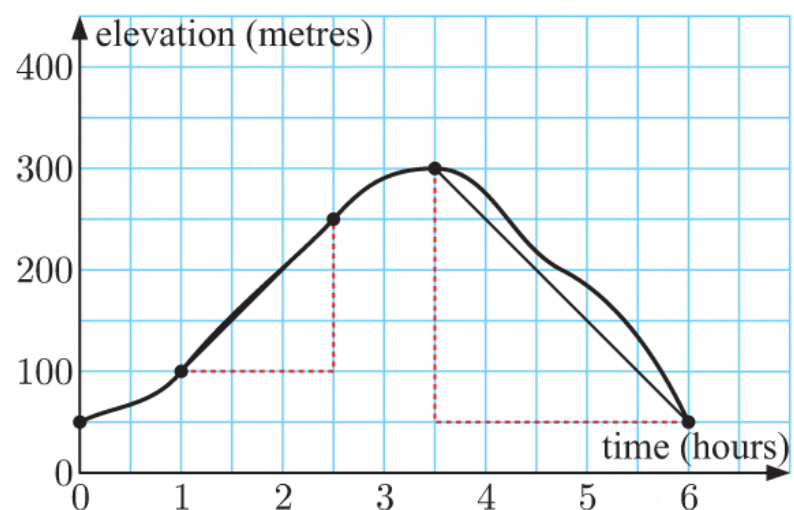
- b** average rate of change from  $t = 3.5$  h to  $t = 6$  h

$$= \frac{(50 - 300) \text{ m}}{(6 - 3.5) \text{ h}}$$

$$= \frac{-250}{2.5} \text{ m per hour}$$

$$= -100 \text{ m per hour}$$

$$= 100 \text{ m per hour (downwards)}$$

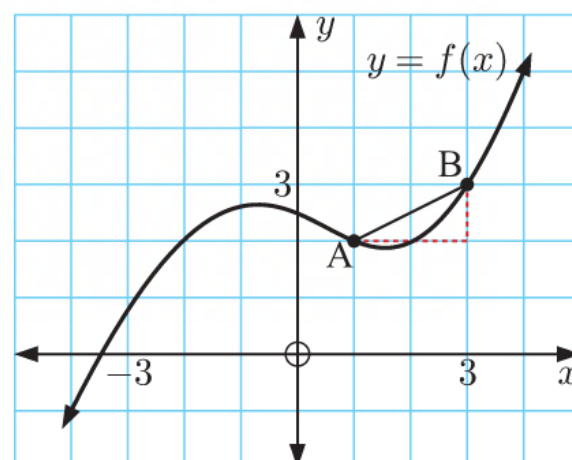


- 3 a** average rate of change in  $f(x)$  from A to B

$$= \frac{f(b) - f(a)}{b - a}$$

$$= \frac{3 - 2}{3 - 1}$$

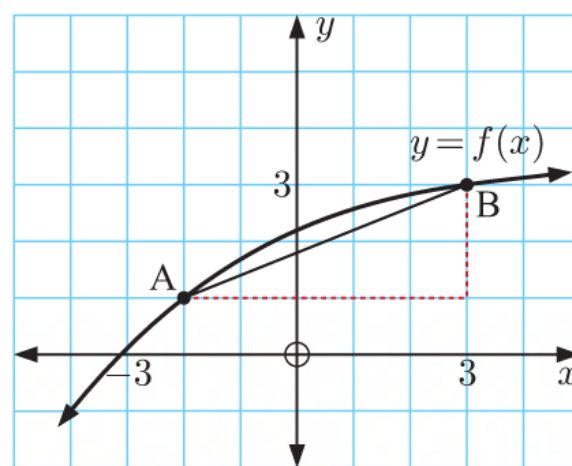
$$= \frac{1}{2}$$





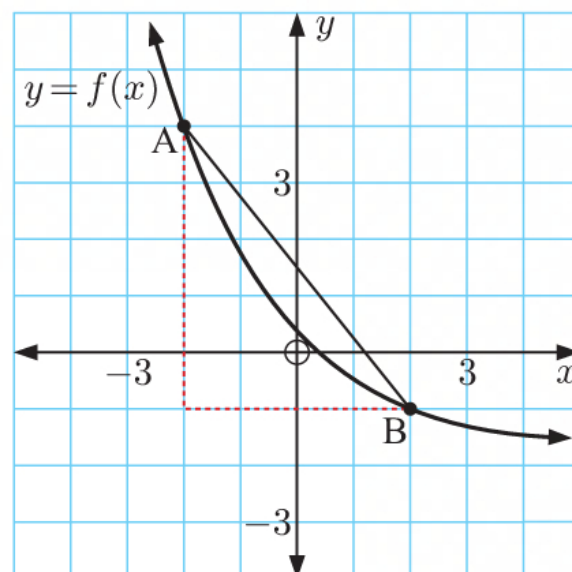
**b** average rate of change in  $f(x)$  from A to B

$$\begin{aligned}
 &= \frac{f(b) - f(a)}{b - a} \\
 &= \frac{3 - 1}{3 - (-2)} \\
 &= \frac{2}{5}
 \end{aligned}$$



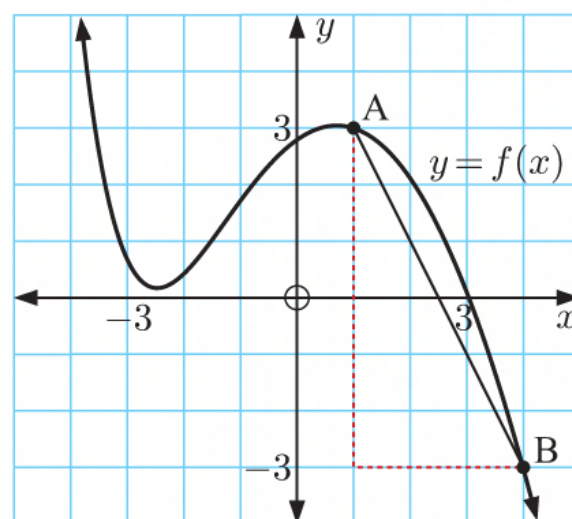
**c** average rate of change in  $f(x)$  from A to B

$$\begin{aligned}
 &= \frac{f(b) - f(a)}{b - a} \\
 &= \frac{-1 - 4}{2 - (-2)} \\
 &= -\frac{5}{4}
 \end{aligned}$$



**d** average rate of change in  $f(x)$  from A to B

$$\begin{aligned}
 &= \frac{f(b) - f(a)}{b - a} \\
 &= \frac{-3 - 3}{4 - 1} \\
 &= \frac{-6}{3} \\
 &= -2
 \end{aligned}$$

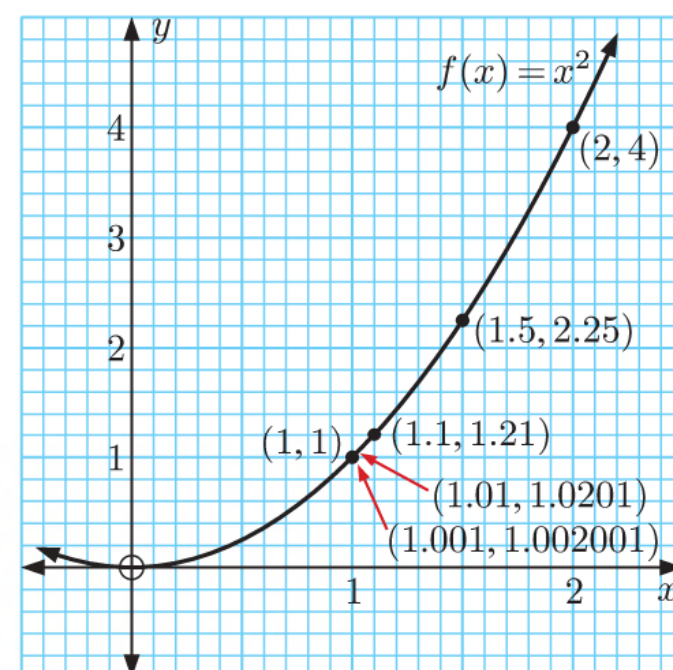


**4 a i** average rate of change in  $f(x)$  from  $x = 1$  to  $x = 2$

$$\begin{aligned}
 &= \frac{f(2) - f(1)}{2 - 1} \\
 &= \frac{4 - 1}{1} \\
 &= 3
 \end{aligned}$$

**ii** average rate of change in  $f(x)$  from  $x = 1$  to  $x = 1.5$

$$\begin{aligned}
 &= \frac{f(1.5) - f(1)}{1.5 - 1} \\
 &= \frac{2.25 - 1}{0.5} \\
 &= 2.5
 \end{aligned}$$



iii average rate of change in  $f(x)$  from  $x = 1$  to  $x = 1.1$

$$\begin{aligned} &= \frac{f(1.1) - f(1)}{1.1 - 1} \\ &= \frac{1.21 - 1}{0.1} \\ &= 2.1 \end{aligned}$$

iv average rate of change in  $f(x)$  from  $x = 1$  to  $x = 1.01$

$$\begin{aligned} &= \frac{f(1.01) - f(1)}{1.01 - 1} \\ &= \frac{1.0201 - 1}{0.01} \\ &= 2.01 \end{aligned}$$

v average rate of change in  $f(x)$  from  $x = 1$  to  $x = 1.001$

$$\begin{aligned} &= \frac{f(1.001) - f(1)}{1.001 - 1} \\ &= \frac{1.002001 - 1}{0.001} \\ &= 2.001 \end{aligned}$$

b The average rate of change approaches 2.

## INVESTIGATION 1

## INSTANTANEOUS SPEED

2

$t$	gradient of [FM]
4	30
3	25
2.5	22.5
2.1	20.5
2.01	20.05

3 As  $M$  approaches  $F$ , the gradient of [FM] approaches 20. However, when  $M$  reaches  $F$ , the gradient is undefined since we cannot divide by zero.

4 As  $t$  approaches 2 from the right, the gradient of [FM] approaches 20.

We suspect that the instantaneous speed of the ball bearing when  $t = 2$  seconds is  $20 \text{ m s}^{-1}$ .

5

$t$	gradient of [FM]
0	10
1.5	17.5
1.9	19.5
1.99	19.95

6 As  $t$  approaches 2 from the left, the gradient of [FM] approaches 20.

The instantaneous speed of the ball bearing when  $t = 2$  seconds appears to be  $20 \text{ m s}^{-1}$ , which agrees with our result in 4.

## EXERCISE 16B

- 1 a The tangent at A has gradient

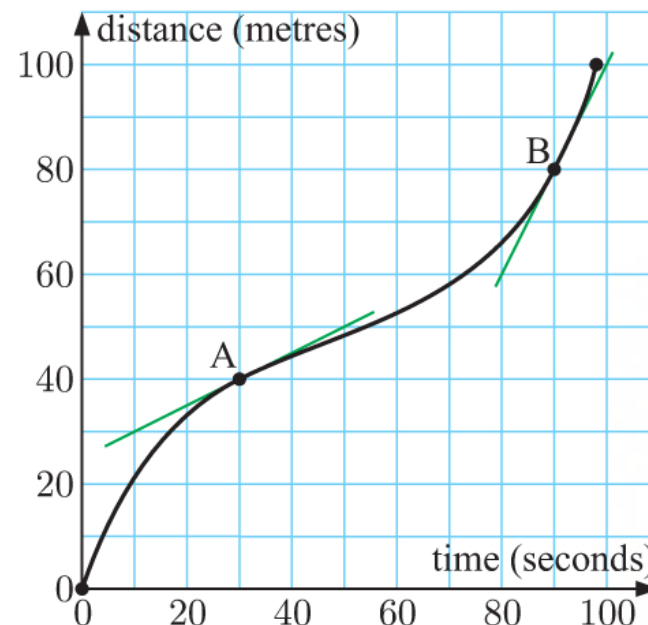
$$\frac{50 - 40}{50 - 30} = \frac{10}{20} = \frac{1}{2}.$$

$\therefore$  the swimmer's instantaneous speed after 30 seconds is  $0.5 \text{ m s}^{-1}$ .

- b The tangent at B has gradient

$$\frac{80 - 60}{90 - 80} = \frac{20}{10} = 2.$$

$\therefore$  the swimmer's instantaneous speed after 90 seconds is  $2 \text{ m s}^{-1}$ .



- 2 a The tangent at  $x = -1$  has gradient

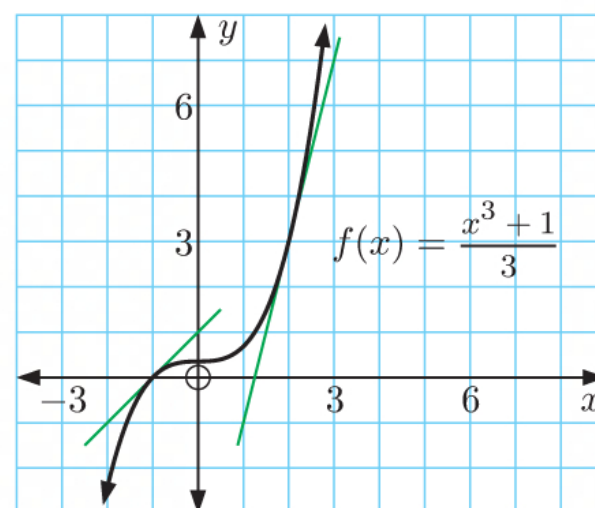
$$\frac{1 - (-1)}{0 - (-2)} = \frac{2}{2} = 1.$$

$\therefore$  the instantaneous rate of change in  $f(x)$  at  $x = -1$  is 1.

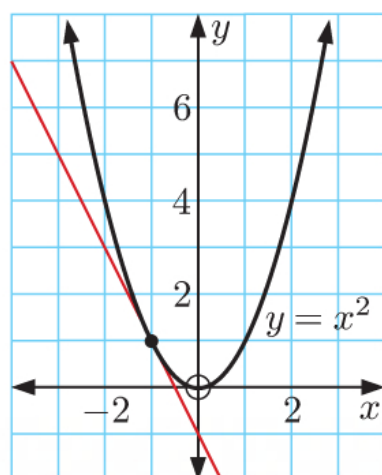
- b The tangent at  $x = 2$  has gradient

$$\frac{3 - (-1)}{2 - 1} = \frac{4}{1} = 4.$$

$\therefore$  the instantaneous rate of change in  $f(x)$  at  $x = 2$  is 4.



- 3 a, b



- c The tangent at  $x = -1$  has gradient  $\frac{1 - (-1)}{-1 - 0} = -2$ .

$\therefore$  the instantaneous rate of change in  $y = x^2$  when  $x = -1$  is  $-2$ .

## EXERCISE 16C

- 1 a M has  $x$ -coordinate  $3 + h$  and lies on the graph of  $f(x) = x^2$ .

$\therefore$  its  $y$ -coordinate is  $(3 + h)^2$ .

$$\begin{aligned}
 \text{b The gradient of [FM]} &= \frac{y_M - y_F}{x_M - x_F} \\
 &= \frac{(3+h)^2 - 9}{(3+h) - 3} \\
 &= \frac{\cancel{9} + 6h + h^2 - \cancel{9}}{h} \\
 &= \frac{6h + h^2}{h} \\
 &= \frac{\cancel{h}(6+h)}{\cancel{h}} \\
 &= 6 + h \quad \text{provided } h \neq 0
 \end{aligned}$$

- c i** M has  $x$ -coordinate  $3 + h$ .  
 $\therefore$  at the point  $(4, 16)$ ,  $3 + h = 4$   
 $\therefore h = 1$   
 The gradient of [FM] is  $6 + h$ .  
{from **b**}  
 $\therefore$  the gradient of [FM] at  $(4, 16)$  is  
 $6 + 1 = 7$ .

- iii** M has  $x$ -coordinate  $3 + h$ .  
 $\therefore$  at the point  $(3.1, 9.61)$ ,  
 $3 + h = 3.1$   
 $\therefore h = 0.1$   
 The gradient of [FM] is  $6 + h$ .  
{from **b**}  
 $\therefore$  the gradient of [FM] at  $(3.1, 9.61)$   
 is  $6 + 0.1 = 6.1$ .

- ii** M has  $x$ -coordinate  $3 + h$ .  
 $\therefore$  at the point  $(3.5, 12.25)$ ,  
 $3 + h = 3.5$   
 $\therefore h = 0.5$   
 The gradient of [FM] is  $6 + h$ .  
{from **b**}  
 $\therefore$  the gradient of [FM] at  
 $(3.5, 12.25)$  is  $6 + 0.5 = 6.5$ .

- iv** M has  $x$ -coordinate  $3 + h$ .  
 $\therefore$  at the point  $(3.01, 9.0601)$ ,  
 $3 + h = 3.01$   
 $\therefore h = 0.01$   
 The gradient of [FM] is  $6 + h$ .  
{from **b**}  
 $\therefore$  the gradient of [FM] at  
 $(3.01, 9.0601)$  is  
 $6 + 0.01 = 6.01$ .

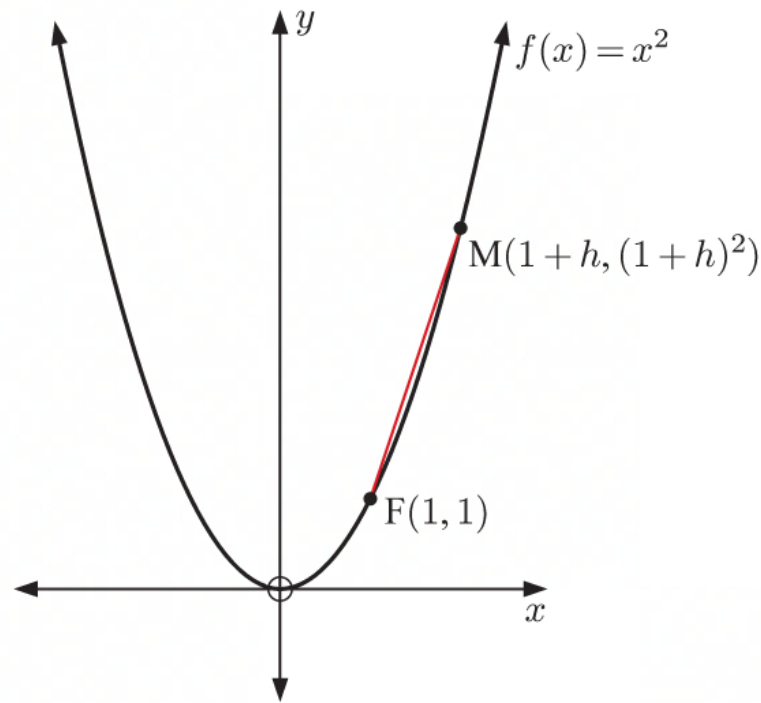
- d** Using limit theory, the gradient of the tangent to  $f(x) = x^2$  at the point  $(3, 9)$  is

$$\begin{aligned}
 \lim_{h \rightarrow 0} \frac{f(3+h) - f(3)}{h} &= \lim_{h \rightarrow 0} \frac{(3+h)^2 - 3^2}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{9} + 6h + h^2 - \cancel{9}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{6h + h^2}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{h}(6+h)}{\cancel{h}} \\
 &= \lim_{h \rightarrow 0} (6 + h) \quad \{\text{as } h \neq 0\} \\
 &= 6
 \end{aligned}$$



**2 a i** At  $x = 1$ ,  $f(1) = 1^2 = 1$ .

Let F be the point  $(1, 1)$  and M have  $x$ -coordinate  $1 + h$ , so M is  $(1 + h, (1 + h)^2)$ .

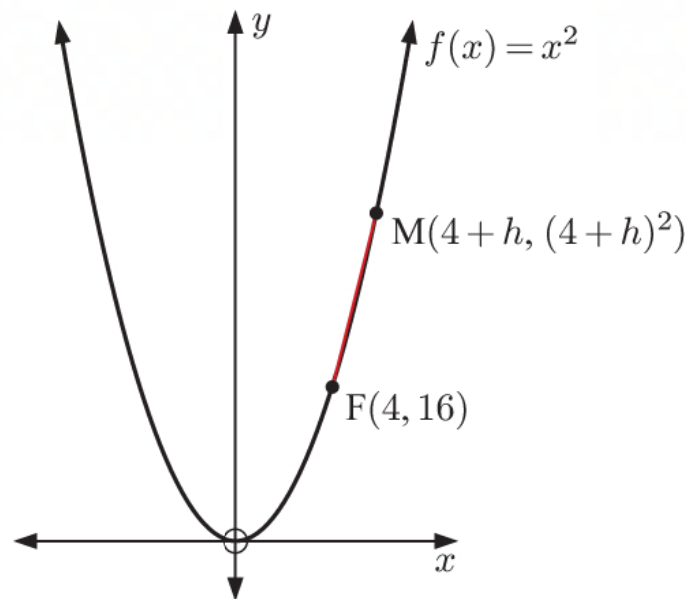


The gradient of the tangent at F

$$\begin{aligned}
 &= \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(1+h)^2 - 1^2}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{1} + 2h + h^2 - \cancel{1}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{2h + h^2}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{h}(2+h)}{\cancel{h}} \\
 &= \lim_{h \rightarrow 0} (2+h) \quad \{\text{as } h \neq 0\} \\
 &= 2
 \end{aligned}$$

**ii** At  $x = 4$ ,  $f(4) = 4^2 = 16$ .

Let F be the point  $(4, 16)$  and M have  $x$ -coordinate  $4 + h$ , so M is  $(4 + h, (4 + h)^2)$ .



The gradient of the tangent at F

$$\begin{aligned}
 &= \lim_{h \rightarrow 0} \frac{f(4+h) - f(4)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(4+h)^2 - 4^2}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{16} + 8h + h^2 - \cancel{16}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{8h + h^2}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{h}(8+h)}{\cancel{h}} \\
 &= \lim_{h \rightarrow 0} (8+h) \quad \{\text{as } h \neq 0\} \\
 &= 8
 \end{aligned}$$

- b** The gradient of the tangent to  $f(x) = x^2$  at the point where  $x = 2$

$$\begin{aligned}
 &= \lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(2+h)^2 - 2^2}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{4} + 4h + h^2 - \cancel{4}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{4h + h^2}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{h}(4+h)}{\cancel{h}} \\
 &= \lim_{h \rightarrow 0} (4+h) \quad \{\text{as } h \neq 0\} \\
 &= 4
 \end{aligned}$$

- The gradient of the tangent to  $f(x) = x^2$  at the point where  $x = 3$

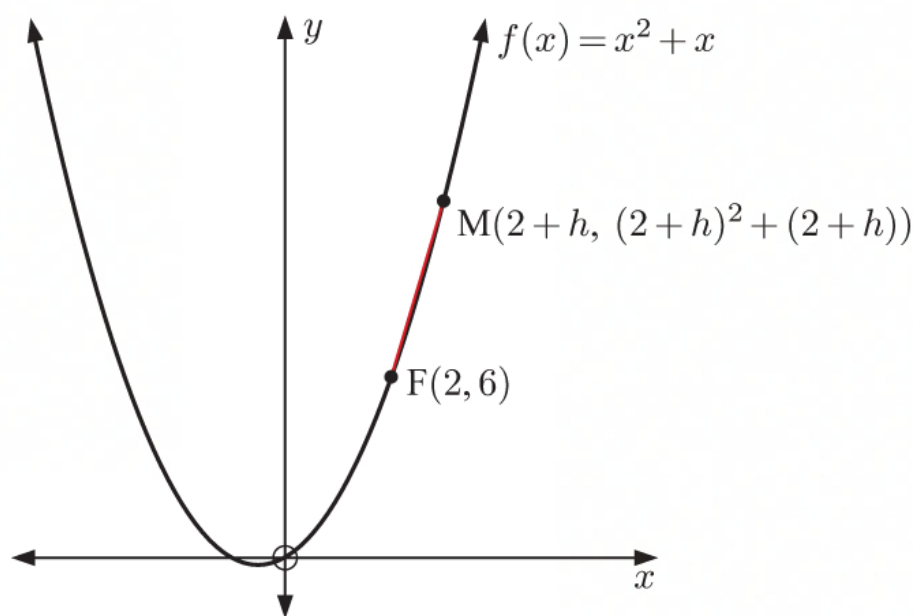
$$\begin{aligned}
 &= \lim_{h \rightarrow 0} \frac{f(3+h) - f(3)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(3+h)^2 - 3^2}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{9} + 6h + h^2 - \cancel{9}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{6h + h^2}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{h}(6+h)}{\cancel{h}} \\
 &= \lim_{h \rightarrow 0} (6+h) \quad \{\text{as } h \neq 0\} \\
 &= 6
 \end{aligned}$$

Using the results from **a**, the table is:

$x$ -coordinate	Gradient of tangent to $f(x) = x^2$
1	2
2	4
3	6
4	8

- c** The gradient of the tangent is equal to twice the  $x$ -coordinate in each case in **b**. So, we predict the gradient of the tangent to  $f(x) = x^2$  at the point where  $x = a$  will be  $2a$ .

- 3 a** Let F be the point  $(2, 6)$  and M have  $x$ -coordinate  $2 + h$ , so M is  $(2 + h, (2 + h)^2 + (2 + h))$ .

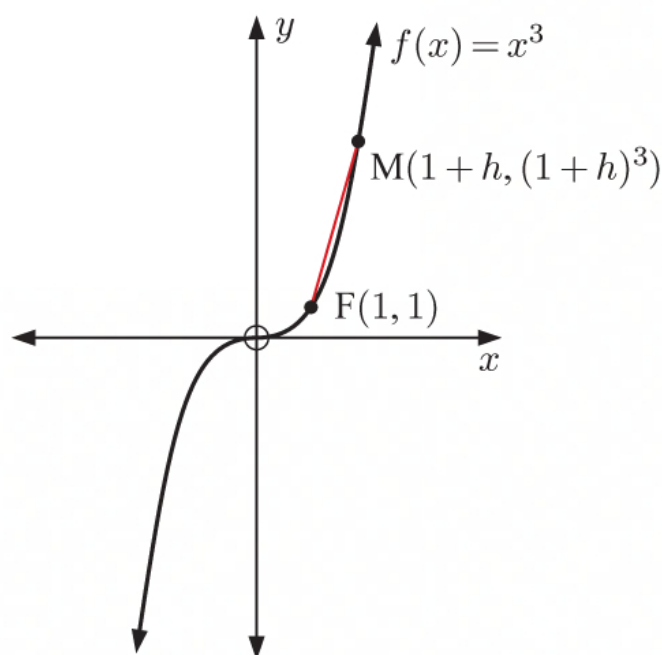


The gradient of the tangent at F

$$\begin{aligned}
 &= \lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(2+h)^2 + (2+h) - (2^2 + 2)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{4 + 4h + h^2 + 2 + h - 6}{h} \\
 &= \lim_{h \rightarrow 0} \frac{5h + h^2}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{h}(5+h)}{\cancel{h}} \\
 &= \lim_{h \rightarrow 0} (5+h) \quad \{\text{as } h \neq 0\} \\
 &= 5
 \end{aligned}$$

- b** At  $x = 1$ ,  $f(1) = 1^3 = 1$ .

Let F be the point  $(1, 1)$  and M have  $x$ -coordinate  $1 + h$ , so M is  $(1 + h, (1 + h)^3)$ .

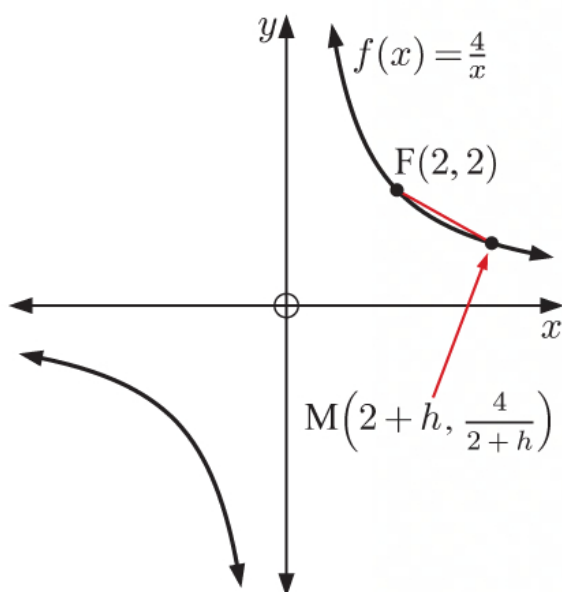


The gradient of the tangent at F

$$\begin{aligned}
 &= \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(1+h)^3 - 1^3}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{1} + 3h + 3h^2 + h^3 - \cancel{1}}{h} \quad \{\text{binomial expansion}\} \\
 &= \lim_{h \rightarrow 0} \frac{3h + 3h^2 + h^3}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{h}(3 + 3h + h^2)}{\cancel{h}} \\
 &= \lim_{h \rightarrow 0} (3 + 3h + h^2) \quad \{\text{as } h \neq 0\} \\
 &= 3
 \end{aligned}$$

- c** At  $x = 2$ ,  $f(2) = \frac{4}{2} = 2$ .

Let F be the point  $(2, 2)$  and M have  $x$ -coordinate  $2 + h$ , so M is  $(2 + h, \frac{4}{2+h})$ .

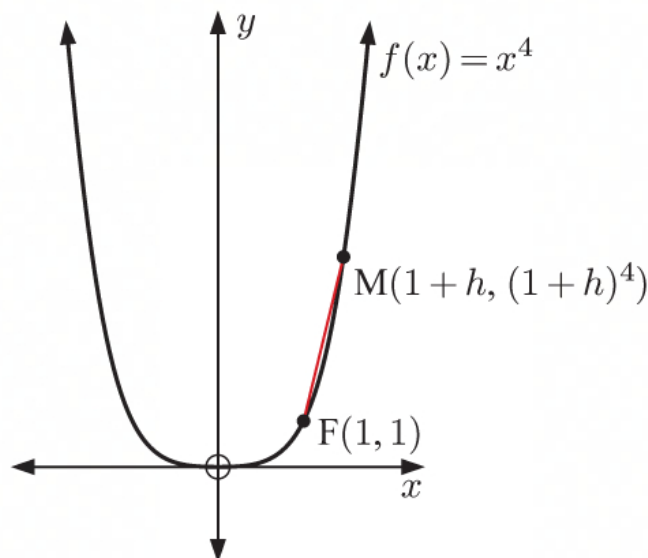


The gradient of the tangent at F

$$\begin{aligned}
 &= \lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\frac{4}{2+h} - \frac{4}{2}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\frac{4}{2+h} - 2\left(\frac{2+h}{2+h}\right)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\frac{4 - 2(2+h)}{2+h}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{4 - 4 - 2h}{h(2+h)} \\
 &= \lim_{h \rightarrow 0} \frac{-2\cancel{h}}{\cancel{h}(2+h)} \\
 &= \lim_{h \rightarrow 0} \frac{-2}{2+h} \quad \{\text{as } h \neq 0\} \\
 &= \frac{-2}{2} \\
 &= -1
 \end{aligned}$$

- d** At  $x = 1$ ,  $f(1) = 1^4 = 1$ .

Let F be the point  $(1, 1)$  and M have  $x$ -coordinate  $1 + h$ , so M is  $(1 + h, (1 + h)^4)$ .

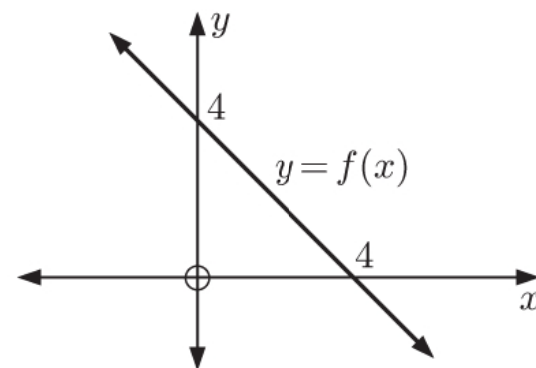


The gradient of the tangent at F

$$\begin{aligned}
 &= \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(1+h)^4 - 1^4}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{x} + 4h + 6h^2 + 4h^3 + h^4 - \cancel{x}}{h} \\
 &\quad \text{\{binomial expansion\}} \\
 &= \lim_{h \rightarrow 0} \frac{4h + 6h^2 + 4h^3 + h^4}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{h}(4 + 6h + 4h^2 + h^3)}{\cancel{h}} \\
 &= \lim_{h \rightarrow 0} (4 + 6h + 4h^2 + h^3) \quad \{\text{as } h \neq 0\} \\
 &= 4
 \end{aligned}$$

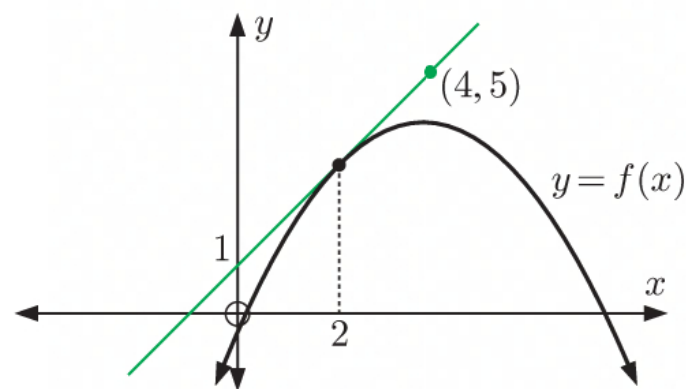
## EXERCISE 16D

- 1 a**  $f(0) = 4$
- b**  $f'(0)$  is the gradient of the tangent to  $f(x)$  at the point where  $x = 0$ .  
 Since  $f(x)$  is a straight line, this is the same as the gradient of  $f(x)$  itself.  
 $f(x)$  goes through  $(0, 4)$  and  $(4, 0)$ , so it has  
 gradient  $= \frac{0 - 4}{4 - 0} = -1$   
 $\therefore f'(0) = -1$



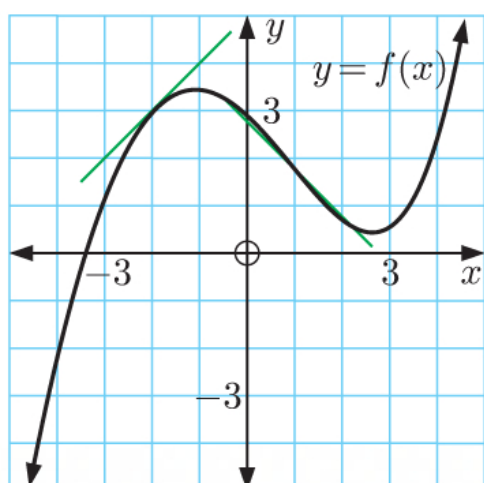
- 2** The graph shows the tangent to the curve  $y = f(x)$  at the point where  $x = 2$ .  
 The tangent passes through  $(0, 1)$  and  $(4, 5)$ .  
 $\therefore f'(2) = \text{gradient of the tangent}$

$$\begin{aligned}
 &= \frac{5 - 1}{4 - 0} \\
 &= 1
 \end{aligned}$$





3



- a  $f(3)$  is above the  $x$ -axis, so  $f(3)$  is positive.
- b  $f'(1)$  is the gradient of the tangent to  $f(x)$  at the point where  $x = 1$ . Since the curve is decreasing at  $x = 1$ , then  $f'(1)$  is negative.
- c  $f(-4)$  is below the  $x$ -axis, so  $f(-4)$  is negative.
- d  $f'(-2)$  is the gradient of the tangent to  $f(x)$  at the point where  $x = -2$ . Since the curve is increasing at  $x = -2$ , then  $f'(-2)$  is positive.

4

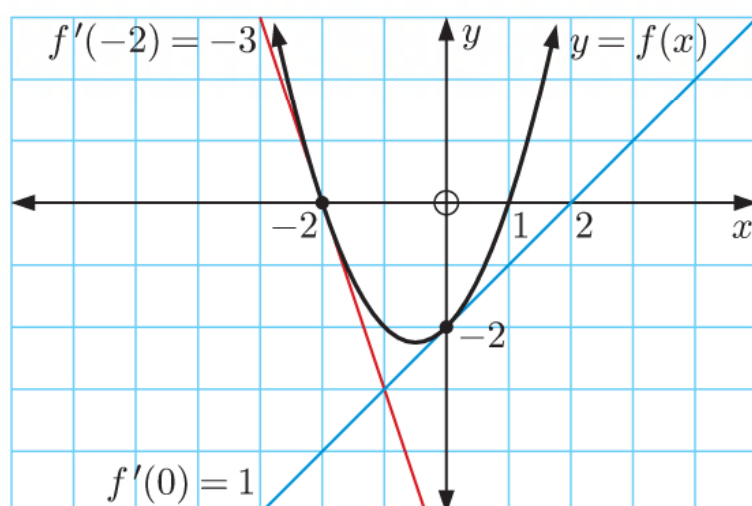
a i  $f'(x) = 2x + 1$   
 $f'(-2) = 2(-2) + 1$   
 $= -3$

The gradient of the tangent to  $y = f(x)$  at the point where  $x = -2$  is  $-3$ .

ii  $f'(0) = 2(0) + 1$   
 $= 1$

The gradient of the tangent to  $y = f(x)$  at the point where  $x = 0$  is 1.

b

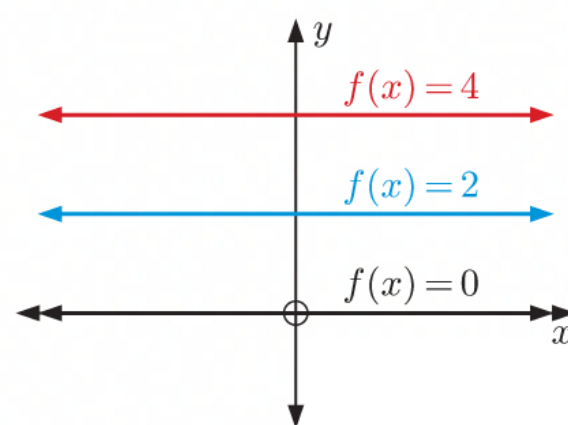


## INVESTIGATION 2

## GRADIENT FUNCTIONS

1

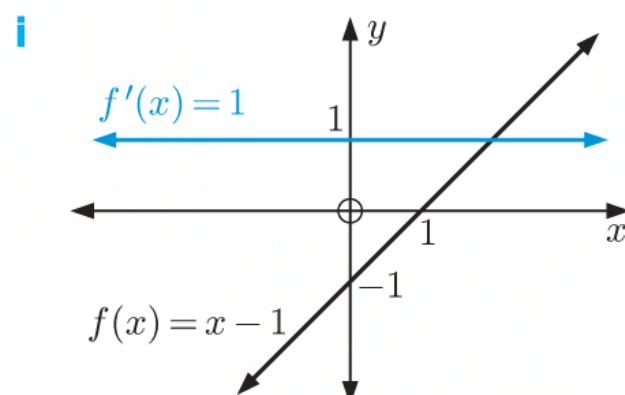
- a  $f(x) = 0$ ,  $f(x) = 2$ , and  $f(x) = 4$  are all horizontal lines and hence all have gradient 0.
- b Yes, the gradient is constant for all values of  $x$ .



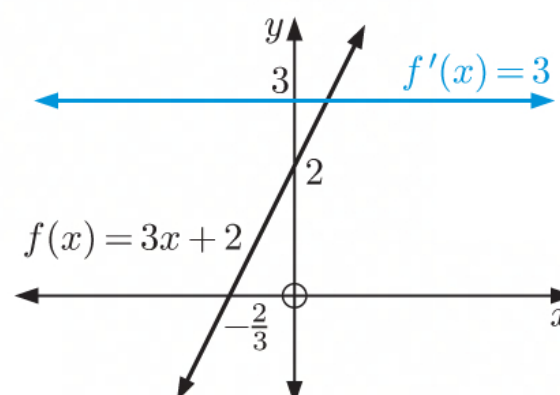
2

- a The gradient of  $f(x) = mx + c$  is  $m$ .
- b The gradient  $m$  is constant for all values of  $x$ .

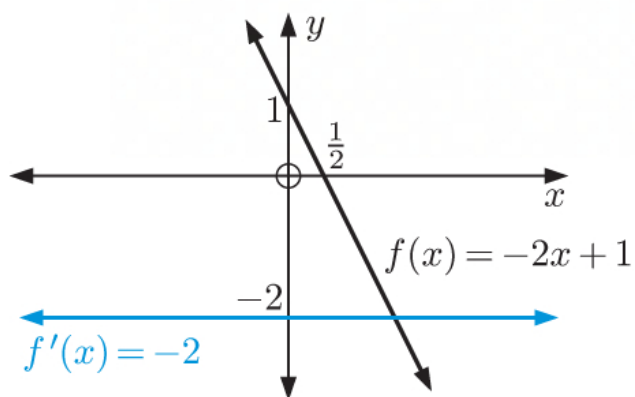
c



ii



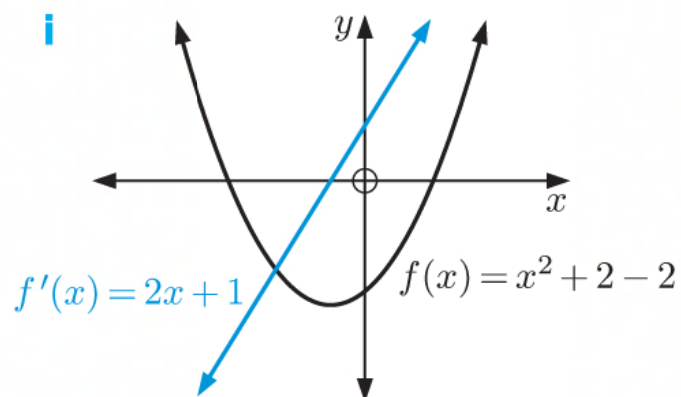
iii



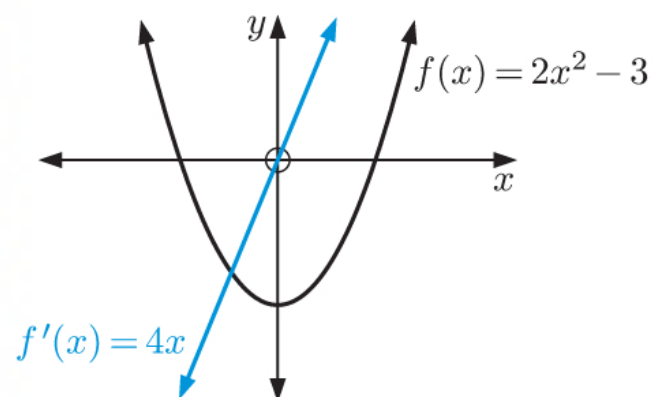
$f'(x)$  is constant for all  $x$ .

3 a  $f'(x)$  is a linear function.

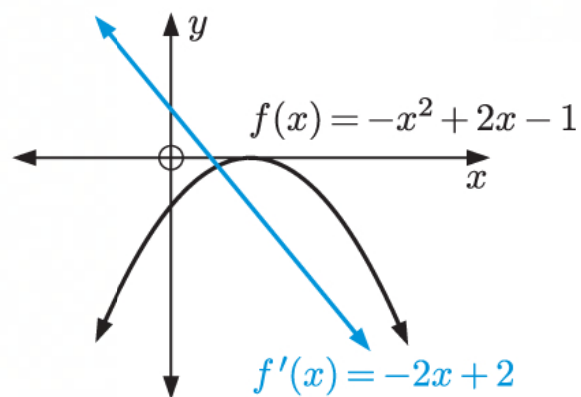
b i



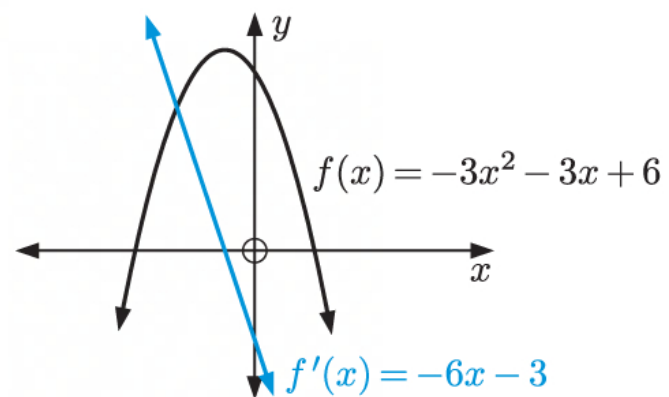
ii



iii

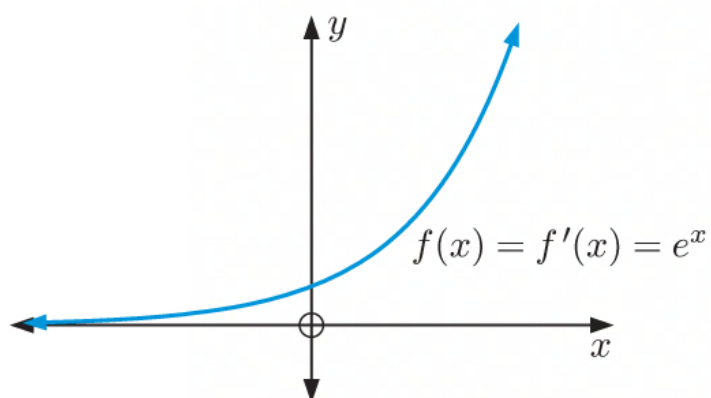


iv



c The gradient functions  $f'(x)$  in b are all linear functions.

4 a



b The gradient function is  $f'(x) = f(x) = e^x$ .

**EXERCISE 16E**

**1 a**  $f(x) = x$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(x+h) - x}{h} \\ &= \lim_{h \rightarrow 0} \frac{h}{h} \\ &= \lim_{h \rightarrow 0} 1 \quad \{\text{as } h \neq 0\} \\ &= 1 \end{aligned}$$

**b**  $f(x) = 1$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{1 - 1}{h} \\ &= \lim_{h \rightarrow 0} \frac{0}{h} \\ &= \lim_{h \rightarrow 0} 0 \quad \{\text{as } h \neq 0\} \\ &= 0 \end{aligned}$$

**c**  $f(x) = x^3$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(x+h)^3 - x^3}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 - \cancel{x^3}}{h} \\ &= \lim_{h \rightarrow 0} \frac{3x^2h + 3xh^2 + h^3}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cancel{h}(3x^2 + 3xh + h^2)}{\cancel{h}} \\ &= \lim_{h \rightarrow 0} (3x^2 + 3xh + h^2) \quad \{\text{as } h \neq 0\} \\ &= 3x^2 \end{aligned}$$

**2 a**  $f(x) = 2x + 5$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(2(x+h) + 5) - (2x + 5)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cancel{2x} + 2h + \cancel{5} - \cancel{2x} - \cancel{5}}{h} \\ &= \lim_{h \rightarrow 0} \frac{2\cancel{h}}{\cancel{h}} \\ &= \lim_{h \rightarrow 0} 2 \quad \{\text{as } h \neq 0\} \\ &= 2 \end{aligned}$$

**b**  $f(x) = x^2 - 3x$

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{[(x+h)^2 - 3(x+h)] - [x^2 - 3x]}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{3x} - 3h - \cancel{x^2} + \cancel{3x}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{2xh + h^2 - 3h}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{h}(2x + h - 3)}{\cancel{h}} \\
 &= \lim_{h \rightarrow 0} (2x + h - 3) \quad \{\text{as } h \neq 0\} \\
 &= 2x - 3
 \end{aligned}$$

**c**  $f(x) = -x^2 + 5x - 3$

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{[-(x+h)^2 + 5(x+h) - 3] - [-x^2 + 5x - 3]}{h} \\
 &= \lim_{h \rightarrow 0} \frac{-\cancel{x^2} - 2xh - h^2 + \cancel{5x} + 5h - \cancel{3} + \cancel{x^2} - \cancel{5x} + \cancel{3}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{-2xh - h^2 + 5h}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{h}(-2x - h + 5)}{\cancel{h}} \\
 &= \lim_{h \rightarrow 0} (-2x + 5 - h) \quad \{\text{as } h \neq 0\} \\
 &= -2x + 5
 \end{aligned}$$

**3 a**  $y = f(x) = 4 - x$

$$\begin{aligned}
 \frac{dy}{dx} &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{[4 - (x+h)] - [4 - x]}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{4} - \cancel{x} - h - \cancel{4} + \cancel{x}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{-h}{h} \\
 &= \lim_{h \rightarrow 0} -1 \quad \{\text{as } h \neq 0\} \\
 &= -1
 \end{aligned}$$



**b**  $y = f(x) = 2x^2 + x - 1$

$$\begin{aligned}
 \frac{dy}{dx} &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{[2(x+h)^2 + (x+h) - 1] - [2x^2 + x - 1]}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{2x^2} + 4xh + 2h^2 + \cancel{x} + h - \cancel{1} - \cancel{2x^2} - \cancel{x} + \cancel{1}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{4xh + 2h^2 + h}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{h}(4x + 2h + 1)}{\cancel{h}} \\
 &= \lim_{h \rightarrow 0} (4x + 1 + 2h) \quad \{\text{as } h \neq 0\} \\
 &= 4x + 1
 \end{aligned}$$

**c**  $y = f(x) = x^3 - 2x^2 + 3$

$$\begin{aligned}
 \frac{dy}{dx} &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{[(x+h)^3 - 2(x+h)^2 + 3] - [x^3 - 2x^2 + 3]}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 - \cancel{2x^2} - 4xh - 2h^2 + \cancel{3} - \cancel{x^3} + \cancel{2x^2} - \cancel{3}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{3x^2h + 3xh^2 + h^3 - 4xh - 2h^2}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{h}(3x^2 + 3xh + h^2 - 4x - 2h)}{\cancel{h}} \\
 &= \lim_{h \rightarrow 0} (3x^2 + 3xh + h^2 - 4x - 2h) \quad \{\text{as } h \neq 0\} \\
 &= 3x^2 - 4x
 \end{aligned}$$

**4 a**  $f(x) = x^3$

$$\begin{aligned}
 f'(a) &= \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} \\
 \therefore f'(2) &= \lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h} \quad \text{where } f(2) = 2^3 = 8 \\
 &= \lim_{h \rightarrow 0} \frac{(2+h)^3 - 8}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{8} + 12h + 6h^2 + h^3 - \cancel{8}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{12h + 6h^2 + h^3}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{h}(12 + 6h + h^2)}{\cancel{h}} \\
 &= \lim_{h \rightarrow 0} (12 + 6h + h^2) \quad \{\text{as } h \neq 0\} \\
 &= 12
 \end{aligned}$$

**b**  $f(x) = x^4$

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

$$\therefore f'(3) = \lim_{h \rightarrow 0} \frac{f(3+h) - f(3)}{h} \quad \text{where } f(3) = 3^4 = 81$$

$$= \lim_{h \rightarrow 0} \frac{(3+h)^4 - 81}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\cancel{81} + 108h + 54h^2 + 12h^3 + h^4 - \cancel{81}}{h}$$

$$= \lim_{h \rightarrow 0} \frac{108h + 54h^2 + 12h^3 + h^4}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\cancel{h}(108 + 54h + 12h^2 + h^3)}{\cancel{h}}$$

$$= \lim_{h \rightarrow 0} (108 + 54h + 12h^2 + h^3) \quad \{\text{as } h \neq 0\}$$

$$= 108$$

**5 a**  $f(x) = \frac{1}{x}$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\left(\frac{x}{x}\right) \frac{1}{x+h} - \frac{1}{x} \left(\frac{x+h}{x+h}\right)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\frac{x-(x+h)}{x(x+h)}}{h}$$

$$= \lim_{h \rightarrow 0} \frac{-\cancel{h}}{\cancel{h}x(x+h)}$$

$$= \lim_{h \rightarrow 0} \frac{-1}{x(x+h)} \quad \{\text{as } h \neq 0\}$$

$$= -\frac{1}{x^2}$$

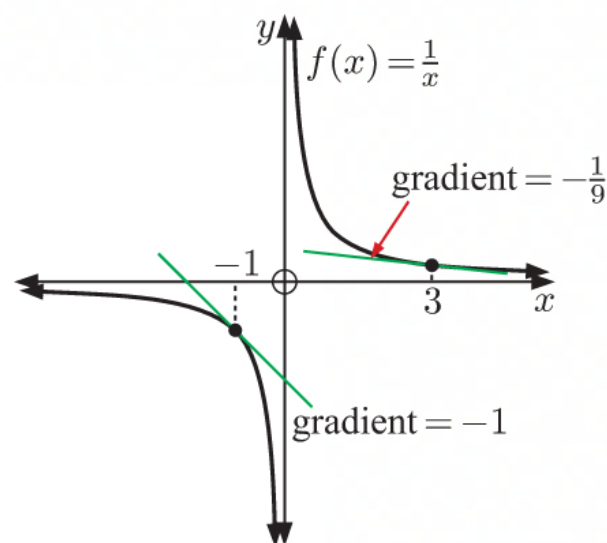
**b**  $f'(-1) = -\frac{1}{(-1)^2}$   
 $= -1$

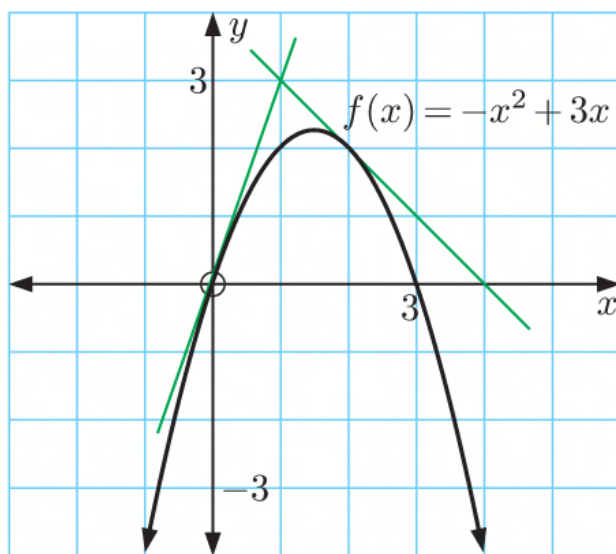
The tangent to  $f(x) = \frac{1}{x}$  at the point where  $x = -1$  has gradient  $-1$ .

$$f'(3) = -\frac{1}{3^2}$$

$$= -\frac{1}{9}$$

The tangent to  $f(x) = \frac{1}{x}$  at the point where  $x = 3$  has gradient  $-\frac{1}{9}$ .



**6 a**

- i** The tangent to  $f(x) = -x^2 + 3x$  at the point where  $x = 0$  has gradient  $\approx \frac{3-0}{1-0} \approx 3$ .
- ii** The tangent to  $f(x) = -x^2 + 3x$  at the point where  $x = 2$  has gradient  $\approx \frac{0-3}{4-1} \approx -1$ .

$$\begin{aligned}
 \text{b } f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{-(x+h)^2 + 3(x+h) - (-x^2 + 3x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{-\cancel{x^2} - 2xh - h^2 + \cancel{3x} + 3h + \cancel{x^2} - \cancel{3x}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{-2xh - h^2 + 3h}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{h}(-2x - h + 3)}{\cancel{h}} \\
 &= \lim_{h \rightarrow 0} (-2x - h + 3) \quad \{\text{as } h \neq 0\} \\
 &= -2x + 3
 \end{aligned}$$

$$\begin{aligned}
 \text{c } f'(0) &= -2(0) + 3 & f'(2) &= -2(2) + 3 \\
 &= 3 & &= -1
 \end{aligned}$$

Both values are the same as the estimates in **a**.

**7 a**  $y = f(x) = x^3 - 3x$ 

$$\begin{aligned}
 \frac{dy}{dx} &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{[(x+h)^3 - 3(x+h)] - [x^3 - 3x]}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 - \cancel{3x} - 3h - \cancel{x^3} + \cancel{3x}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{3x^2h + 3xh^2 + h^3 - 3h}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{h}(3x^2 + 3xh + h^2 - 3)}{\cancel{h}} \\
 &= \lim_{h \rightarrow 0} (3x^2 + 3xh + h^2 - 3) \quad \{\text{as } h \neq 0\} \\
 &= 3x^2 - 3
 \end{aligned}$$

**b** The tangent has zero gradient when  $\frac{dy}{dx} = 0$

$$\begin{aligned}\therefore 3x^2 - 3 &= 0 \\ \therefore 3x^2 &= 3 \\ \therefore x^2 &= 1 \\ \therefore x &= \pm 1\end{aligned}$$

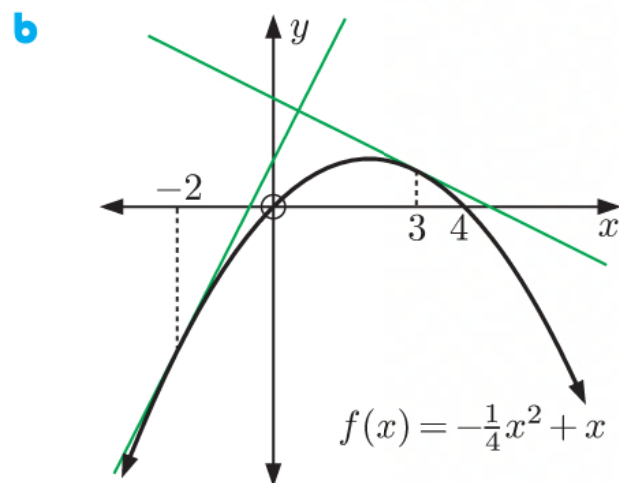
When  $x = -1$ ,  $y = (-1)^3 - 3(-1) = 2$

When  $x = 1$ ,  $y = (1)^3 - 3(1) = -2$

So, the points on the graph at which the tangent has zero gradient are  $(-1, 2)$  and  $(1, -2)$ .

**8 a**  $f(x) = -\frac{1}{4}x^2 + x$

$$\begin{aligned}f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\left[-\frac{1}{4}(x+h)^2 + (x+h)\right] - \left[-\frac{1}{4}x^2 + x\right]}{h} \\ &= \lim_{h \rightarrow 0} \frac{-\cancel{\frac{1}{4}x^2} - \frac{1}{2}xh - \frac{1}{4}h^2 + \cancel{x} + h + \cancel{\frac{1}{4}x^2} - \cancel{x}}{h} \\ &= \lim_{h \rightarrow 0} \frac{-\frac{1}{2}xh - \frac{1}{4}h^2 + h}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cancel{h}\left(-\frac{1}{2}x - \frac{1}{4}h + 1\right)}{\cancel{h}} \\ &= \lim_{h \rightarrow 0} \left(-\frac{1}{2}x - \frac{1}{4}h + 1\right) \quad \{\text{as } h \neq 0\} \\ &= -\frac{1}{2}x + 1\end{aligned}$$



The illustrated tangents are perpendicular if the product of their gradients is  $-1$ .

One tangent passes through the point where  $x = -2$  and the other tangent passes through the point where  $x = 3$ .

The tangent at  $x = -2$  has gradient

$$\begin{aligned}f'(-2) &= -\frac{1}{2}(-2) + 1 \\ &= 2\end{aligned}$$

and the tangent at  $x = 3$  has gradient

$$\begin{aligned}f'(3) &= -\frac{1}{2}(3) + 1 \\ &= -\frac{1}{2}\end{aligned}$$

Since  $2 \times \left(-\frac{1}{2}\right) = -1$ , the two tangents are perpendicular.



**9 a**  $f(x) = mx + c, \quad m \neq 0$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{[m(x+h) + c] - [mx + c]}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cancel{mx} + mh + \cancel{c} - \cancel{mx} - \cancel{c}}{h} \\ &= \lim_{h \rightarrow 0} \frac{m\cancel{h}}{\cancel{h}} \\ &= m \quad \{\text{as } h \neq 0\} \end{aligned}$$

$\therefore f'(x) = m, \quad \text{a constant function}$

**b**  $f(x) = ax^2 + bx + c, \quad a \neq 0$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{[a(x+h)^2 + b(x+h) + c] - [ax^2 + bx + c]}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cancel{ax^2} + 2axh + ah^2 + \cancel{bx} + bh + \cancel{c} - \cancel{ax^2} - \cancel{bx} - \cancel{c}}{h} \\ &= \lim_{h \rightarrow 0} \frac{2axh + ah^2 + bh}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cancel{h}(2ax + ah + b)}{\cancel{h}} \\ &= \lim_{h \rightarrow 0} (2ax + ah + b) \quad \{\text{as } h \neq 0\} \\ &= 2ax + b \end{aligned}$$

$\therefore f'(x) = 2ax + b, \quad \text{a linear function}$

**c**  $f(x) = ax^3 + bx^2 + cx + d, \quad a \neq 0$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{[a(x+h)^3 + b(x+h)^2 + c(x+h) + d] - [ax^3 + bx^2 + cx + d]}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cancel{ax^3} + 3ax^2h + 3axh^2 + ah^3 + \cancel{bx^2} + 2bxh + bh^2 + \cancel{cx} + ch + \cancel{d} - \cancel{ax^3} - \cancel{bx^2} - \cancel{cx} - \cancel{d}}{h} \\ &= \lim_{h \rightarrow 0} \frac{3ax^2h + 3axh^2 + ah^3 + 2bxh + bh^2 + ch}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cancel{h}(3ax^2 + 3axh + ah^2 + 2bx + bh + c)}{\cancel{h}} \\ &= \lim_{h \rightarrow 0} (3ax^2 + 3axh + ah^2 + 2bx + bh + c) \quad \{\text{as } h \neq 0\} \\ &= 3ax^2 + 2bx + c \end{aligned}$$

$\therefore f'(x) = 3ax^2 + 2bx + c, \quad \text{a quadratic function}$

**10**  $y = f(x) = \sqrt{x}$

$$\begin{aligned}\frac{dy}{dx} &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \left( \frac{\sqrt{x+h} - \sqrt{x}}{h} \times \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \right) \\ &= \lim_{h \rightarrow 0} \frac{\cancel{x} + h - \cancel{x}}{h(\sqrt{x+h} + \sqrt{x})} \\ &= \lim_{h \rightarrow 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{x+h} + \sqrt{x})} \\ &= \lim_{h \rightarrow 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} \quad \{\text{as } h \neq 0\} \\ &= \frac{1}{\sqrt{x} + \sqrt{x}} \\ &= \frac{1}{2\sqrt{x}} \quad \text{which is defined for } x > 0\end{aligned}$$

$\therefore y = \sqrt{x}$  is differentiable for all  $x > 0$ .

**11 a**

$f(x)$	$x^4$	$x^3$	$x^2$	$x^1$	$x^{\frac{1}{2}}$	$x^0$	$x^{-1}$
$f'(x)$	$4x^3$	$3x^2$	$2x$	$1$	$\frac{1}{2}x^{-\frac{1}{2}}$	$0$	$-x^{-2}$

**b** If  $f(x) = x^n$ ,  
then  $f'(x) = nx^{n-1}$ .

**12**  $f(x) = \cos x$

$$\begin{aligned}f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cos(x+h) - \cos x}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h} \\ &= \lim_{h \rightarrow 0} \frac{(\cos h - 1) \cos x - \sin x \sin h}{h} \\ &= \lim_{h \rightarrow 0} \frac{(\cos h - 1) \cos x}{h} - \lim_{h \rightarrow 0} \frac{\sin x \sin h}{h} \\ &= \cos x \left( \lim_{h \rightarrow 0} \frac{\cos h - 1}{h} \right) - \sin x \left( \lim_{h \rightarrow 0} \frac{\sin h}{h} \right) \quad \{\text{since } x \text{ is independent of } h\} \\ &= \cos x \times 0 - \sin x \times 1 \quad \{\text{as } \lim_{h \rightarrow 0} \frac{\cos h - 1}{h} = 0\} \\ &= -\sin x\end{aligned}$$

**13 a**  $\cos(A+B) - \cos(A-B) = \cos A \cos B - \sin A \sin B - (\cos A \cos B + \sin A \sin B)$   
 $= \cancel{\cos A \cos B} - \sin A \sin B - \cancel{\cos A \cos B} - \sin A \sin B$   
 $= -2 \sin A \sin B$

**b**  $S = A + B$  and  $D = A - B$

$$\begin{aligned}\frac{S+D}{2} &= \frac{A + \cancel{B} + A - \cancel{B}}{2} & \frac{S-D}{2} &= \frac{\cancel{A} + B - (\cancel{A} - B)}{2} \\ &= \frac{2A}{2} & &= \frac{2B}{2} \\ &= A & &= B\end{aligned}$$

Now,  $\cos(A + B) - \cos(A - B) = -2 \sin A \sin B$  {from **a**}

$$\therefore \cos S - \cos D = -2 \sin \left( \frac{S+D}{2} \right) \cos \left( \frac{S-D}{2} \right)$$

{substituting corresponding terms}

**c**  $f(x) = \cos x$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cos(x+h) - \cos x}{h} \end{aligned}$$

If  $S = x + h$  and  $D = x$ , then  $\frac{S+D}{2} = \frac{2x+h}{2}$  and  $\frac{S-D}{2} = \frac{h}{2}$ .

$$\begin{aligned} \therefore f'(x) &= \lim_{h \rightarrow 0} \frac{-2 \sin \left( \frac{2x+h}{2} \right) \sin \frac{h}{2}}{h} && \text{{using **b**}} \\ &= \lim_{h \rightarrow 0} \frac{-\sin \left( \frac{2x+h}{2} \right) \sin \frac{h}{2}}{\frac{h}{2}} \\ &= \lim_{h \rightarrow 0} \left[ -\sin \left( \frac{2x+h}{2} \right) \right] \times \lim_{h \rightarrow 0} \frac{\sin \frac{h}{2}}{\frac{h}{2}} && \text{{by the limit laws since both limits exist}} \\ &= \lim_{h \rightarrow 0} \left[ -\sin \left( \frac{2x+h}{2} \right) \right] \times \lim_{\frac{h}{2} \rightarrow 0} \frac{\sin \frac{h}{2}}{\frac{h}{2}} \\ &= -\sin x \times 1 \\ &= -\sin x \end{aligned}$$

## EXERCISE 16F

**1**  $f(x) = \begin{cases} x + 2, & x \geq 0 \\ x^2 + 3x, & x < 0 \end{cases}$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} (x^2 + 3x) = 0$$

and  $\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} (x + 2) = 2$

$$\therefore \lim_{x \rightarrow 0^-} f(x) \neq \lim_{x \rightarrow 0^+} f(x)$$

$\therefore f$  is not continuous at  $x = 0$ .

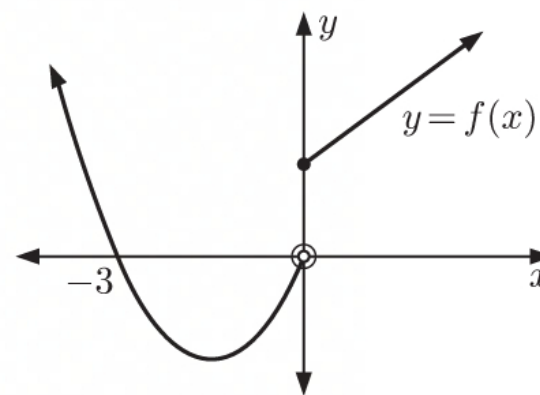
$\therefore f$  is not differentiable at  $x = 0$ .

**2**  $f(x) = \frac{x^2 + x}{x}$

$f(0)$  is undefined.

$\therefore f$  is not continuous at  $x = 0$ .

$\therefore f$  is not differentiable at  $x = 0$ .



$$3 \quad f(x) = |x - 5| = \begin{cases} x - 5, & x \geq 5 \\ 5 - x, & x < 5 \end{cases}$$

$$\lim_{x \rightarrow 5^-} f(x) = \lim_{x \rightarrow 5^-} (5 - x) = 0$$

$$\text{and } \lim_{x \rightarrow 5^+} f(x) = \lim_{x \rightarrow 5^+} (x - 5) = 0$$

$$\therefore \lim_{x \rightarrow 5} f(x) = 0 = f(5)$$

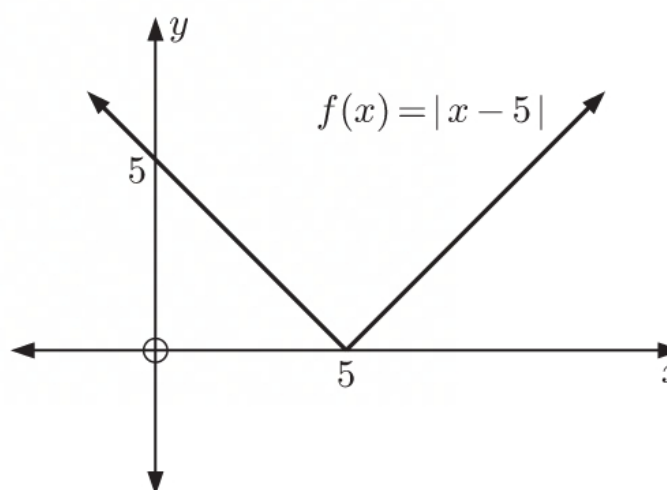
$\therefore f$  is continuous at  $x = 5$ .

$$\text{Now, } f'(x) = \begin{cases} 1, & x > 5 \\ -1, & x < 5 \end{cases} \quad \text{since the derivative of } f \text{ exists on the intervals } x < 5 \text{ and } x > 5.$$

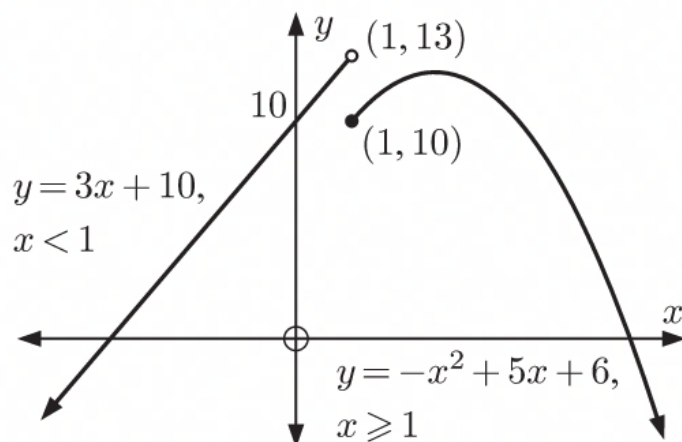
$$\therefore f'_-(5) = -1 \quad \text{and} \quad f'_+(5) = 1$$

$$\therefore f'_-(5) \neq f'_+(5)$$

$\therefore f$  is not differentiable at  $x = 5$ .



4 a



$$b \quad f(x) = \begin{cases} -x^2 + 5x + 6, & x \geq 1 \\ 3x + 10, & x < 1 \end{cases}$$

$$\therefore f'(x) = \begin{cases} -2x + 5, & x > 1 \\ 3, & x < 1 \end{cases} \quad \text{since the derivative of } f \text{ exists on the intervals } x < 1 \text{ and } x > 1.$$

$$i \quad f'_-(1) = 3$$

$$ii \quad f'_+(1) = -2(1) + 5 = 3$$

c No, although  $f'_-(1) = f'_+(1)$ ,  $f(x)$  is not continuous at  $x = 1$ , and hence is not differentiable at  $x = 1$ .

$$5 \quad f(x) = \begin{cases} x^2, & x \leq 1 \\ cx + d, & x > 1 \end{cases}$$

$$\therefore f'(x) = \begin{cases} 2x, & x < 1 \\ c, & x > 1 \end{cases} \quad \text{since the derivative of } f \text{ exists on the intervals } x < 1 \text{ and } x > 1.$$

If  $f(x)$  is differentiable at  $x = 1$ ,  $f'_-(1) = f'_+(1)$

$$\therefore 2(1) = c$$

$$\therefore c = 2$$



$f(x)$  must also be continuous at  $x = 1$ .

$$\therefore \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} f(x)$$

$$\therefore 1^2 = 2(1) + d$$

$$\therefore 1 = 2 + d$$

$$\therefore d = -1$$

$$6 \quad f(x) = \begin{cases} ax^2, & x \geq 2 \\ x + b, & x < 2 \end{cases}$$

$$\therefore f'(x) = \begin{cases} 2ax, & x > 2 \\ 1, & x < 2 \end{cases} \quad \text{since the derivative of } f \text{ exists on the intervals } x < 2 \text{ and } x > 2.$$

If  $f(x)$  is differentiable at  $x = 2$ ,  $f'_-(2) = f'_+(2)$

$$\therefore 1 = 2a(2)$$

$$\therefore 1 = 4a$$

$$\therefore a = \frac{1}{4}$$

$f(x)$  must also be continuous at  $x = 2$ .

$$\therefore \lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^+} f(x)$$

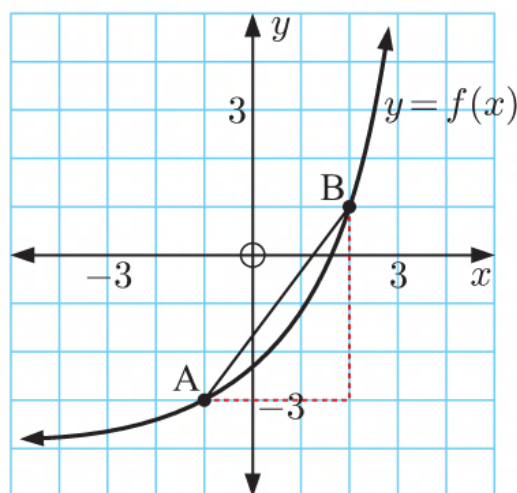
$$\therefore 2 + b = \frac{1}{4}(2)^2$$

$$\therefore 2 + b = 1$$

$$\therefore b = -1$$

## REVIEW SET 16A

1



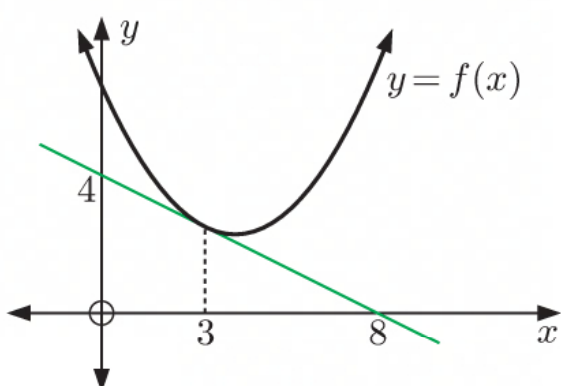
average rate of change in  $f(x)$  from A to B

$$= \frac{f(b) - f(a)}{b - a}$$

$$= \frac{1 - (-3)}{2 - (-1)}$$

$$= \frac{4}{3}$$

2



The tangent to  $y = f(x)$  at the point where  $x = 3$  has

$$\text{gradient } \frac{4 - 0}{0 - 8} = -\frac{1}{2}.$$

$$\therefore f'(3) = -\frac{1}{2}$$

$$\begin{aligned}
 \mathbf{3} \quad \mathbf{a} \quad & f(x) = 2x^2 \\
 & \frac{f(x+h) - f(x)}{h} = \frac{2(x+h)^2 - 2x^2}{h} \\
 & = \frac{2(x^2 + 2xh + h^2) - 2x^2}{h} \\
 & = \frac{\cancel{2x^2} + 4xh + 2h^2 - \cancel{2x^2}}{h} \\
 & = \frac{\cancel{h}(4x + 2h)}{\cancel{h}} \\
 & = 4x + 2h \quad \text{provided } h \neq 0
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad \text{If } x = 3 \text{ then } & \frac{f(3+h) - f(3)}{h} = 4(3) + 2h \quad \{\text{using } \mathbf{a}\} \\
 & = 12 + 2h
 \end{aligned}$$

When  $h = 0.1$ ,

$$\begin{aligned}
 \frac{f(3+h) - f(3)}{h} &= 12 + 2(0.1) \\
 &= 12 + 0.2 \\
 &= 12.2
 \end{aligned}$$

When  $h = 0.001$ ,

$$\begin{aligned}
 \frac{f(3+h) - f(3)}{h} &= 12 + 2(0.001) \\
 &= 12 + 0.002 \\
 &= 12.002
 \end{aligned}$$

When  $h = 0.01$ ,

$$\begin{aligned}
 \frac{f(3+h) - f(3)}{h} &= 12 + 2(0.01) \\
 &= 12 + 0.02 \\
 &= 12.02
 \end{aligned}$$

When  $h = 0.0001$ ,

$$\begin{aligned}
 \frac{f(3+h) - f(3)}{h} &= 12 + 2(0.0001) \\
 &= 12 + 0.0002 \\
 &= 12.0002
 \end{aligned}$$

$h$	$\frac{f(3+h) - f(3)}{h}$
0.1	12.2
0.01	12.02
0.001	12.002
0.0001	12.0002

$$\begin{aligned}
 \mathbf{c} \quad \lim_{h \rightarrow 0} \frac{f(3+h) - f(3)}{h} &= \lim_{h \rightarrow 0} (12 + 2h) \\
 &= 12
 \end{aligned}$$

The gradient of the tangent to  $y = 2x^2$  at the point  $(3, 18)$  is 12.

**4 a**  $f(x) = x^2 + 2x$

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{[(x+h)^2 + 2(x+h)] - [x^2 + 2x]}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{x^2} + 2xh + h^2 + \cancel{2x} + 2h - \cancel{x^2} - \cancel{2x}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{2xh + h^2 + 2h}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{h}(2x + h + 2)}{\cancel{h}} \\
 &= \lim_{h \rightarrow 0} (2x + h + 2) \quad \{\text{as } h \neq 0\} \\
 &= 2x + 2
 \end{aligned}$$

**b**  $y = f(x) = 4 - 3x^2$

$$\begin{aligned}
 \frac{dy}{dx} &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{[4 - 3(x+h)^2] - [4 - 3x^2]}{h} \\
 &= \lim_{h \rightarrow 0} \frac{4 - 3(x^2 + 2xh + h^2) - 4 + 3x^2}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{4} - \cancel{3x^2} - 6xh - 3h^2 - \cancel{4} + \cancel{3x^2}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{h}(-6x - 3h)}{\cancel{h}} \\
 &= \lim_{h \rightarrow 0} (-6x - 3h) \quad \{\text{as } h \neq 0\} \\
 &= -6x
 \end{aligned}$$

**5 a**  $y = f(x) = 2x^2 - 1$

$$\begin{aligned}
 \frac{dy}{dx} &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{[2(x+h)^2 - 1] - [2x^2 - 1]}{h} \\
 &= \lim_{h \rightarrow 0} \frac{2(x^2 + 2xh + h^2) - 1 - 2x^2 + 1}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{2x^2} + 4xh + 2h^2 - \cancel{1} - \cancel{2x^2} + \cancel{1}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{4xh + 2h^2}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{h}(4x + 2h)}{\cancel{h}} \\
 &= \lim_{h \rightarrow 0} (4x + 2h) \quad \{\text{as } h \neq 0\} \\
 &= 4x
 \end{aligned}$$

**b** The gradient of the tangent to  $y = 2x^2 - 1$  at the point where  $x = 4$  is  $4 \times 4 = 16$ .

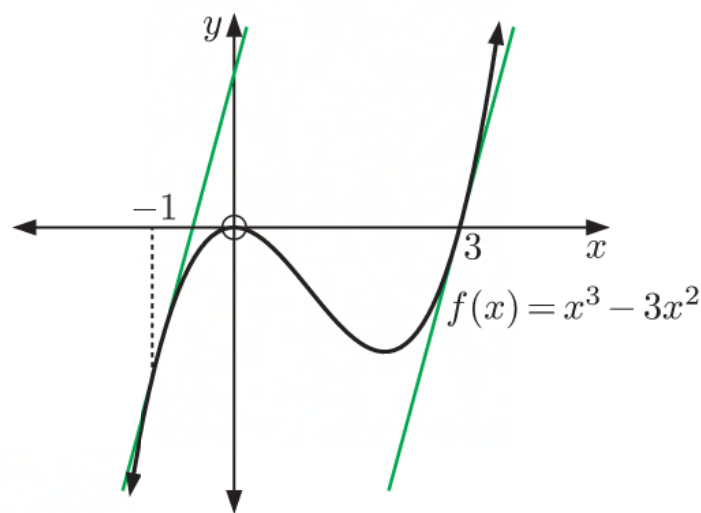
**c** The gradient of the tangent is equal to  $-12$  when  $4x = -12$

$$\therefore x = -3$$

**6 a**  $f(x) = x^3 - 3x^2$

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{[(x+h)^3 - 3(x+h)^2] - [x^3 - 3x^2]}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 - \cancel{3x^2} - 6xh - 3h^2 - \cancel{x^3} + \cancel{3x^2}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{3x^2h + 3xh^2 + h^3 - 6xh - 3h^2}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{h}(3x^2 + 3xh + h^2 - 6x - 3h)}{\cancel{h}} \\
 &= \lim_{h \rightarrow 0} (3x^2 + 3xh + h^2 - 6x - 3h) \quad \{\text{as } h \neq 0\} \\
 &= 3x^2 - 6x
 \end{aligned}$$

**b**



The illustrated tangents pass through the point where  $x = -1$  and the point where  $x = 3$ .

The gradient of the tangent at  $x = -1$  is

$$\begin{aligned}
 f'(-1) &= 3(-1)^2 - 6(-1) \\
 &= 9
 \end{aligned}$$

and the gradient of the tangent at  $x = 3$  is

$$\begin{aligned}
 f'(3) &= 3(3)^2 - 6(3) \\
 &= 9
 \end{aligned}$$

Since  $f'(-1) = f'(3)$ , the gradients of the tangents are equal.

This means the tangents are parallel.

**7 a**  $y = f(x) = 9x - x^3$

$$\begin{aligned}
 \frac{dy}{dx} &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{[9(x+h) - (x+h)^3] - [9x - x^3]}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{9x} + 9h - \cancel{x^3} - 3x^2h - 3xh^2 - h^3 - \cancel{9x} + \cancel{x^3}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{9h - 3x^2h - 3xh^2 - h^3}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{h}(9 - 3x^2 - 3xh - h^2)}{\cancel{h}} \\
 &= \lim_{h \rightarrow 0} (9 - 3x^2 - 3xh - h^2) \quad \{\text{as } h \neq 0\} \\
 &= 9 - 3x^2
 \end{aligned}$$



**b** The tangent has zero gradient when  $\frac{dy}{dx} = 0$

$$\therefore 9 - 3x^2 = 0$$

$$\therefore 3x^2 = 9$$

$$\therefore x^2 = 3$$

$$\therefore x = \pm\sqrt{3}$$

$$\begin{aligned}\text{When } x = \sqrt{3}, \quad y &= 9\sqrt{3} - (\sqrt{3})^3 \\ &= 9\sqrt{3} - 3\sqrt{3} \\ &= 6\sqrt{3}\end{aligned}$$

$$\begin{aligned}\text{When } x = -\sqrt{3}, \quad y &= -9\sqrt{3} - (-\sqrt{3})^3 \\ &= -9\sqrt{3} + 3\sqrt{3} \\ &= -6\sqrt{3}\end{aligned}$$

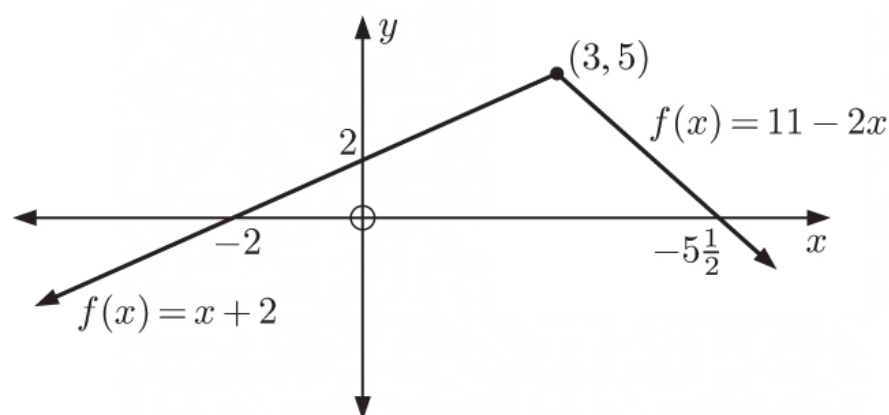
So, the points on the graph at which the tangent has zero gradient are  $(\sqrt{3}, 6\sqrt{3})$  and  $(-\sqrt{3}, -6\sqrt{3})$ .

$$\begin{aligned}\mathbf{8} \quad f(2) &= \frac{2}{2^2 - 2(2)} \\ &= \frac{2}{4 - 4} \\ &= \frac{2}{0} \text{ which is undefined}\end{aligned}$$

$\therefore f(x)$  is not continuous at  $x = 2$ .

$\therefore f(x)$  is not differentiable at  $x = 2$ .

**9 a**



$$\mathbf{b} \quad f(x) = \begin{cases} x + 2, & x \geq 3 \\ 11 - 2x, & x < 3 \end{cases}$$

$$\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^-} (11 - 2x) = 11 - 2(3) = 5$$

$$\text{and } \lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3^+} (x + 2) = 3 + 2 = 5$$

$$\therefore \lim_{x \rightarrow 3} f(x) = 5 = f(3)$$

$\therefore f$  is continuous at  $x = 3$ .

$$\text{Now, } f'(x) = \begin{cases} 1, & x > 3 \\ -2, & x < 3 \end{cases} \quad \begin{array}{l} \text{since the derivative of } f \text{ exists on the intervals } x < 3 \\ \text{and } x > 3. \end{array}$$

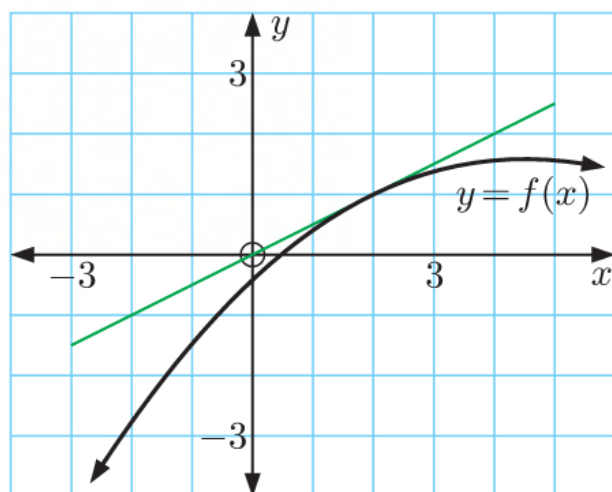
$$\therefore f'_-(3) = -2 \quad \text{and} \quad f'_+(3) = 1$$

$$\therefore f'_-(3) \neq f'_+(3)$$

$\therefore f$  is not differentiable at  $x = 3$ .

## REVIEW SET 16B

1



The tangent at  $x = 2$  has gradient  $\frac{1-0}{2-0} = \frac{1}{2}$ .

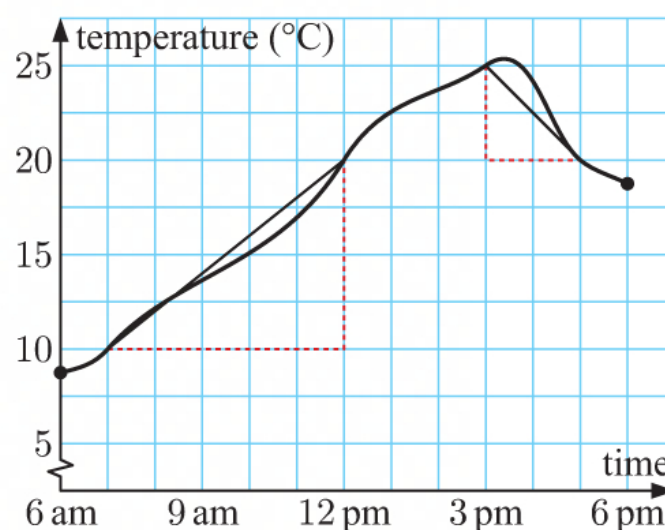
$\therefore$  the instantaneous rate of change in  $f(x)$  at  $x = 2$  is  $\frac{1}{2}$ .

- 2 a average rate of change in temperature from 7 am to noon

$$\begin{aligned} &= \frac{(20 - 10)^\circ\text{C}}{(6 - 1) \text{ hour}} \\ &= \frac{10}{5}^\circ\text{C per hour} \\ &= 2^\circ\text{C per hour} \end{aligned}$$

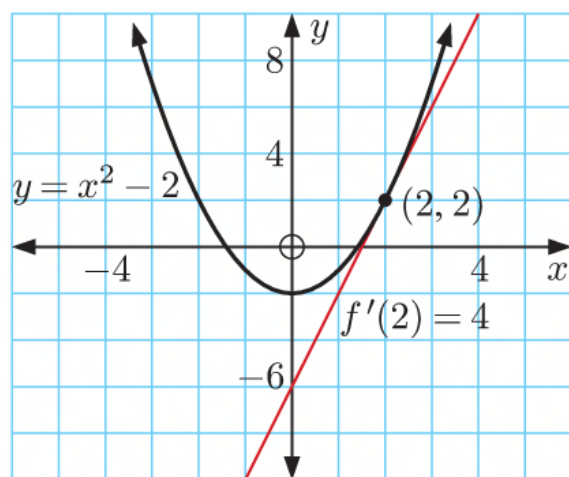
- b average rate of change in temperature from 3 pm to 5 pm

$$\begin{aligned} &= \frac{(20 - 25)^\circ\text{C}}{(11 - 9) \text{ hour}} \\ &= \frac{-5}{2}^\circ\text{C per hour} \\ &= -2.5^\circ\text{C per hour or } -2\frac{1}{2}^\circ\text{C per hour} \end{aligned}$$



3 a, b

$x$	-3	-2	-1	0	1	2	3
$f(x) = x^2 - 2$	7	2	-1	-2	-1	2	7



- c The tangent to  $f(x) = x^2 - 2$  at the point where  $x = 2$  has gradient  $\frac{6 - (-2)}{3 - 1} = \frac{8}{2} = 4$ .
- $\therefore$  the instantaneous rate of change in  $f(x) = x^2 - 2$  when  $x = 2$  is 4.

$$\begin{aligned}
 \text{d} \quad f'(a) &= \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} \\
 \therefore f'(2) &= \lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{[(2+h)^2 - 2] - [2^2 - 2]}{h} \\
 &= \lim_{h \rightarrow 0} \frac{4 + 4h + h^2 - 2 - 2}{h} \\
 &= \lim_{h \rightarrow 0} \frac{4h + h^2}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{h}(4+h)}{\cancel{h}} \\
 &= \lim_{h \rightarrow 0} (4+h) \quad \{\text{as } h \neq 0\} \\
 &= 4 \quad \checkmark
 \end{aligned}$$

$$4 \quad \text{a} \quad f(x) = x^4 - 2x$$

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{[(x+h)^4 - 2(x+h)] - [x^4 - 2x]}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{x^4} + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 - \cancel{2x} - 2h - \cancel{x^4} + \cancel{2x}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{4x^3h + 6x^2h^2 + 4xh^3 + h^4 - 2h}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{h}(4x^3 + 6x^2h + 4xh^2 + h^3 - 2)}{\cancel{h}} \\
 &= \lim_{h \rightarrow 0} (4x^3 + 6x^2h + 4xh^2 + h^3 - 2) \quad \{\text{as } h \neq 0\} \\
 &= 4x^3 - 2
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad f'(-2) &= 4(-2)^3 - 2 \\
 &= -34
 \end{aligned}$$

The gradient of the tangent to  $y = f(x)$  at the point where  $x = -2$  is  $-34$ .

$$5 \quad \text{a} \quad y = f(x) = x^2 + 5x - 2$$

$$\begin{aligned}
 \frac{dy}{dx} &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{[(x+h)^2 + 5(x+h) - 2] - [x^2 + 5x - 2]}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{x^2} + 2xh + h^2 + \cancel{5x} + 5h - \cancel{2} - \cancel{x^2} - \cancel{5x} + \cancel{2}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{2xh + h^2 + 5h}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{h}(2x + h + 5)}{\cancel{h}} \\
 &= \lim_{h \rightarrow 0} (2x + h + 5) \quad \{\text{as } h \neq 0\} \\
 &= 2x + 5
 \end{aligned}$$

**b** The tangent to  $f(x) = x^2 + 5x - 2$  has gradient  $-3$  when  $f'(x) = \frac{dy}{dx} = -3$

$$\begin{aligned}\therefore 2x + 5 &= -3 \\ \therefore 2x &= -8 \\ \therefore x &= -4\end{aligned}$$

Now,  $f(-4) = (-4)^2 + 5(-4) - 2$   
 $= -6$

So, the tangent has gradient  $-3$  at the point  $(-4, -6)$ .

**6**  $f(t) = 452 - 4.8t^2$  metres,  $0 \leq t \leq 3$  seconds

**a i**  $f(1) = 452 - 4.8(1)^2$   
 $= 447.2$

The jumper is 447.2 m above ground level after 1 second.

**ii**  $f(2) = 452 - 4.8(2)^2$   
 $= 432.8$

The jumper is 432.8 m above ground level after 2 seconds.

**b**  $f'(t) = \lim_{h \rightarrow 0} \frac{f(t+h) - f(t)}{h}$

$$\begin{aligned}&= \lim_{h \rightarrow 0} \frac{[452 - 4.8(t+h)^2] - [452 - 4.8t^2]}{h} \\&= \lim_{h \rightarrow 0} \frac{\cancel{452} - \cancel{4.8t^2} - 9.6th - 4.8h^2 - \cancel{452} + \cancel{4.8t^2}}{h} \\&= \lim_{h \rightarrow 0} \frac{-9.6th - 4.8h^2}{h} \\&= \lim_{h \rightarrow 0} \frac{\cancel{h}(-9.6t - 4.8h)}{\cancel{h}} \\&= \lim_{h \rightarrow 0} (-9.6t - 4.8h) \quad \{\text{as } h \neq 0\} \\&= -9.6t\end{aligned}$$

**c** The speed of the jumper is equal to the rate of change in the jumper's altitude which is given by  $f'(t)$ .

**i**  $f'(1) = -9.6(1)$   
 $= -9.6$

The jumper's speed was  $9.6 \text{ m s}^{-1}$  after 1 second.

(The negative sign indicates the jumper is moving downwards.)

**ii**  $f'(2) = -9.6(2)$   
 $= -19.2$

The jumper's speed was  $19.2 \text{ m s}^{-1}$  after 2 seconds.

(The negative sign indicates the jumper is moving downwards.)



**7 a**  $f(x) = 2x^2 + 2x - 12$

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{[2(x+h)^2 + 2(x+h) - 12] - [2x^2 + 2x - 12]}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{2x^2} + 4xh + 2h^2 + \cancel{2x} + 2h - \cancel{12} - \cancel{2x^2} - \cancel{2x} + \cancel{12}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{4xh + 2h^2 + 2h}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{h}(4x + 2h + 2)}{\cancel{h}} \\
 &= \lim_{h \rightarrow 0} (4x + 2h + 2) \quad \{\text{as } h \neq 0\} \\
 &= 4x + 2
 \end{aligned}$$

**b** The tangent has gradient  $-2$  when  $f'(x) = -2$

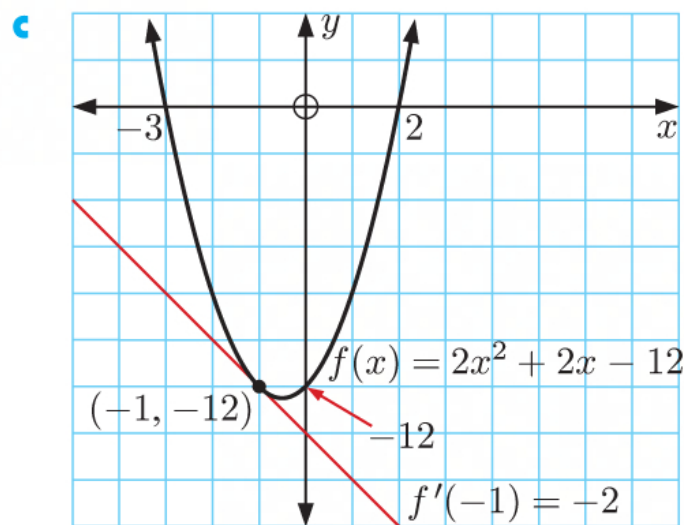
$$\therefore 4x + 2 = -2$$

$$\therefore 4x = -4$$

$$\therefore x = -1$$

$$\begin{aligned}
 \text{Now, } f(-1) &= 2(-1)^2 + 2(-1) - 12 \\
 &= 2 - 2 - 12 \\
 &= -12
 \end{aligned}$$

So, the tangent has gradient  $-2$  at  $(-1, -12)$ .



**8 a**  $\sin(A+B) - \sin(A-B) = \sin A \cos B + \cos A \sin B - (\sin A \cos B - \cos A \sin B)$   
 $= \cancel{\sin A \cos B} + \cos A \sin B - \cancel{\sin A \cos B} + \cos A \sin B$   
 $= 2 \cos A \sin B$

**b**  $S = A + B$  and  $D = A - B$

$$\begin{aligned}
 \frac{S+D}{2} &= \frac{A + \cancel{B} + A - \cancel{B}}{2} & \frac{S-D}{2} &= \frac{\cancel{A} + B - (\cancel{A} - B)}{2} \\
 &= \frac{2A}{2} & &= \frac{2B}{2} \\
 &= A & &= B
 \end{aligned}$$

Now,  $\sin(A+B) - \sin(A-B) = 2 \cos A \sin B$  {from **a**}

$$\therefore \sin S - \sin D = 2 \cos \left( \frac{S+D}{2} \right) \sin \left( \frac{S-D}{2} \right)$$

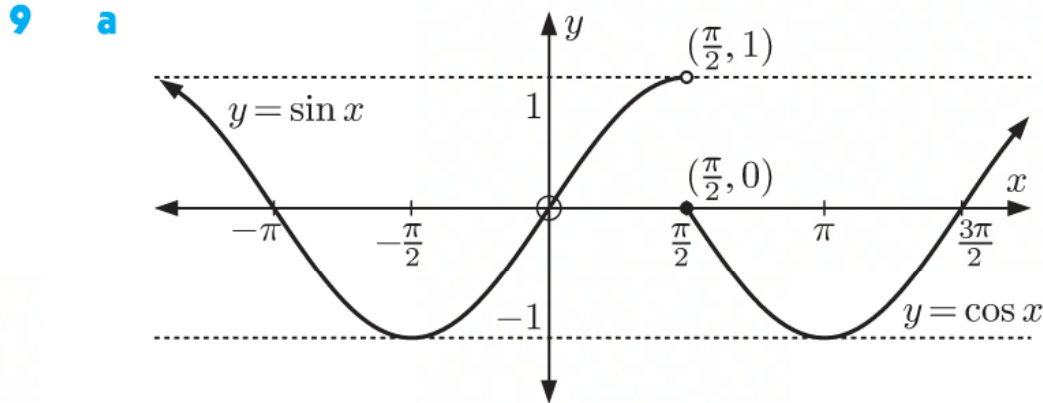
{substituting corresponding terms}

**c**  $f(x) = \sin x$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin x}{h} \end{aligned}$$

If  $S = x + h$  and  $D = x$ , then  $\frac{S+D}{2} = \frac{2x+h}{2}$  and  $\frac{S-D}{2} = \frac{h}{2}$ .

$$\begin{aligned} \therefore f'(x) &= \lim_{h \rightarrow 0} \frac{2 \cos\left(\frac{2x+h}{2}\right) \sin \frac{h}{2}}{h} && \{\text{using b}\} \\ &= \lim_{h \rightarrow 0} \frac{\cos\left(\frac{2x+h}{2}\right) \sin \frac{h}{2}}{\frac{h}{2}} \\ &= \lim_{h \rightarrow 0} \cos\left(\frac{2x+h}{2}\right) \times \lim_{h \rightarrow 0} \frac{\sin \frac{h}{2}}{\frac{h}{2}} && \{\text{by the limit laws since both limits exist}\} \\ &= \lim_{h \rightarrow 0} \cos\left(\frac{2x+h}{2}\right) \times \lim_{\frac{h}{2} \rightarrow 0} \frac{\sin \frac{h}{2}}{\frac{h}{2}} \\ &= \cos x \times 1 \\ &= \cos x \end{aligned}$$



**b** There is a “jump” at  $x = \frac{\pi}{2}$ .

$\therefore f(x)$  is not continuous at  $x = \frac{\pi}{2}$ .

$\therefore f(x)$  is not differentiable at  $x = \frac{\pi}{2}$ .

# Chapter 17

## RULES OF DIFFERENTIATION

### INVESTIGATION 1

### SIMPLE RULES OF DIFFERENTIATION

1  $f(x) = x^n$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(x+h)^n - x^n}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cancel{\binom{n}{0} x^n} + \binom{n}{1} x^{n-1} h + \binom{n}{2} x^{n-2} h^2 + \dots + \binom{n}{n} h^n - \cancel{x^n}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cancel{h} \left[ \binom{n}{1} x^{n-1} + \binom{n}{2} x^{n-2} h + \dots + \binom{n}{n} h^{n-1} \right]}{\cancel{h}} \\ &= \lim_{h \rightarrow 0} \left[ \binom{n}{1} x^{n-1} + \binom{n}{2} x^{n-2} h + \dots + \binom{n}{n} h^{n-1} \right] \quad \{\text{as } h \neq 0\} \\ &= \binom{n}{1} x^{n-1} \\ &= nx^{n-1} \end{aligned}$$

2 a i  $f(x) = 4x^2$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{4(x+h)^2 - 4x^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{4(x^2 + 2xh + h^2) - 4x^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cancel{4x^2} + 8xh + 4h^2 - \cancel{4x^2}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cancel{h}(8x + 4h)}{\cancel{h}} \\ &= \lim_{h \rightarrow 0} (8x + 4h) \quad \{\text{as } h \neq 0\} \\ &= 8x \end{aligned}$$

ii  $f(x) = 2x^3$

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{2(x+h)^3 - 2x^3}{h} \\
 &= \lim_{h \rightarrow 0} \frac{2(x^3 + 3x^2h + 3xh^2 + h^3) - 2x^3}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{2x^3} + 6x^2h + 6xh^2 + 2h^3 - \cancel{2x^3}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{h}(6x^2 + 6xh + 2h^2)}{\cancel{h}} \\
 &= \lim_{h \rightarrow 0} (6x^2 + 6xh + 2h^2) \quad \{\text{as } h \neq 0\} \\
 &= 6x^2
 \end{aligned}$$

iii  $f(x) = 7x^4$

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{7(x+h)^4 - 7x^4}{h} \\
 &= \lim_{h \rightarrow 0} \frac{7(x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4) - 7x^4}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{7x^4} + 28x^3h + 42x^2h^2 + 28xh^3 + 7h^4 - \cancel{7x^4}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{h}(28x^3 + 42x^2h + 28xh^2 + 7h^3)}{\cancel{h}} \\
 &= \lim_{h \rightarrow 0} (28x^3 + 42x^2h + 28xh^2 + 7h^3) \quad \{\text{as } h \neq 0\} \\
 &= 28x^3
 \end{aligned}$$

b If  $f(x) = cx^n$ , then  $f'(x) = cnx^{n-1}$ .

3 a i  $f(x) = x^2 + 3x$

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(x+h)^2 + 3(x+h) - (x^2 + 3x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{x^2} + 2xh + h^2 + \cancel{3x} + 3h - \cancel{x^2} - \cancel{3x}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{h}(2x + h + 3)}{\cancel{h}} \\
 &= \lim_{h \rightarrow 0} (2x + h + 3) \quad \{\text{as } h \neq 0\} \\
 &= 2x + 3
 \end{aligned}$$



ii  $f(x) = x^3 - 2x^2$

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(x+h)^3 - 2(x+h)^2 - (x^3 - 2x^2)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - 2(x^2 + 2xh + h^2) - x^3 + 2x^2}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 - \cancel{2x^2} - 4xh - 2h^2 - \cancel{x^3} + \cancel{2x^2}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{h}(3x^2 + 3xh + h^2 - 4x - 2h)}{\cancel{h}} \quad \{\text{as } h \neq 0\} \\
 &= \lim_{h \rightarrow 0} (3x^2 + 3xh + h^2 - 4x - 2h) \\
 &= 3x^2 - 4x
 \end{aligned}$$

b If  $f(x) = u(x) + v(x)$ , then  $f'(x) = u'(x) + v'(x)$ .

## EXERCISE 17A

1 a  $f(x) = x^3$   
 $\therefore f'(x) = 3x^2$

d  $f(x) = 6x$   
 $\therefore f'(x) = 6(1)$   
 $= 6$

g  $f(x) = 3x^5$   
 $\therefore f'(x) = 3(5x^4)$   
 $= 15x^4$

j  $f(x) = x^2 + 3$   
 $\therefore f'(x) = 2x$

m  $f(x) = 2x^2 + x - 1$   
 $\therefore f'(x) = 2(2x) + 1$   
 $= 4x + 1$

o  $f(x) = 4 - 2x^2$   
 $\therefore f'(x) = -2(2x)$   
 $= -4x$

q  $f(x) = x^3 - 4x^2 + 6x$   
 $\therefore f'(x) = 3x^2 - 4(2x) + 6(1)$   
 $= 3x^2 - 8x + 6$

s  $f(x) = 7 - x - 4x^3$   
 $\therefore f'(x) = -1 - 4(3x^2)$   
 $= -1 - 12x^2$

b  $f(x) = x^8$   
 $\therefore f'(x) = 8x^7$

e  $f(x) = 2x^3$   
 $\therefore f'(x) = 2(3x^2)$   
 $= 6x^2$

h  $f(x) = 5x^6$   
 $\therefore f'(x) = 5(6x^5)$   
 $= 30x^5$

k  $f(x) = x^2 + x$   
 $\therefore f'(x) = 2x + 1$

n  $f(x) = 3x^2 - 7x + 8$   
 $\therefore f'(x) = 3(2x) - 7(1)$   
 $= 6x - 7$

p  $f(x) = \frac{1}{2}x^4 - 6x^2$   
 $\therefore f'(x) = \frac{1}{2}(4x^3) - 6(2x)$   
 $= 2x^3 - 12x$

r  $f(x) = 2x^3 + x - 1$   
 $\therefore f'(x) = 2(3x^2) + (1)$   
 $= 6x^2 + 1$

t  $f(x) = \frac{1}{5}x^3 - \frac{7}{2}x^2 - 2$   
 $\therefore f'(x) = \frac{1}{5}(3x^2) - \frac{7}{2}(2x)$   
 $= \frac{3}{5}x^2 - 7x$

c  $f(x) = x^{11}$   
 $\therefore f'(x) = 11x^{10}$

f  $f(x) = 7x^2$   
 $\therefore f'(x) = 7(2x)$   
 $= 14x$

i  $f(x) = 5x - 2$   
 $\therefore f'(x) = 5(1)$   
 $= 5$

l  $f(x) = x^2 + 3x - 5$   
 $\therefore f'(x) = 2x + 3(1)$   
 $= 2x + 3$

$$\mathbf{2} \quad \mathbf{a} \quad \text{Let } f(x) = \frac{1}{x^2}$$

$$= x^{-2}$$

$$\therefore f'(x) = -2x^{-3}$$

$$= -\frac{2}{x^3}$$

$$\mathbf{b} \quad \text{Let } f(x) = \frac{1}{x^5}$$

$$= x^{-5}$$

$$\therefore f'(x) = -5x^{-6}$$

$$= -\frac{5}{x^6}$$

$$\mathbf{c} \quad \text{Let } f(x) = \frac{3}{x}$$

$$= 3x^{-1}$$

$$\therefore f'(x) = 3(-1x^{-2})$$

$$= -3x^{-2}$$

$$= -\frac{3}{x^2}$$

$$\mathbf{d} \quad \text{Let } f(x) = \frac{4}{x^3}$$

$$= 4x^{-3}$$

$$\therefore f'(x) = 4(-3x^{-4})$$

$$= -12x^{-4}$$

$$= -\frac{12}{x^4}$$

$$\mathbf{e} \quad \text{Let } f(x) = -\frac{7}{x^4}$$

$$= -7x^{-4}$$

$$\therefore f'(x) = -7(-4x^{-5})$$

$$= 28x^{-5}$$

$$= \frac{28}{x^5}$$

$$\mathbf{f} \quad \text{Let } f(x) = 2x + \frac{3}{x^2}$$

$$= 2x + 3x^{-2}$$

$$\therefore f'(x) = 2(1) + 3(-2x^{-3})$$

$$= 2 - 6x^{-3}$$

$$= 2 - \frac{6}{x^3}$$

$$\mathbf{g} \quad \text{Let } f(x) = x^2 - \frac{6}{x}$$

$$= x^2 - 6x^{-1}$$

$$\therefore f'(x) = 2x - 6(-1x^{-2})$$

$$= 2x + 6x^{-2}$$

$$= 2x + \frac{6}{x^2}$$

$$\mathbf{h} \quad \text{Let } f(x) = 9 - \frac{2}{x^3}$$

$$= 9 - 2x^{-3}$$

$$\therefore f'(x) = -2(-3x^{-4})$$

$$= 6x^{-4}$$

$$= \frac{6}{x^4}$$

$$\mathbf{i} \quad \text{Let } f(x) = \frac{2}{x^2} + \frac{9}{x^4}$$

$$= 2x^{-2} + 9x^{-4}$$

$$\therefore f'(x) = 2(-2x^{-3}) + 9(-4x^{-5})$$

$$= -4x^{-3} - 36x^{-5}$$

$$= -\frac{4}{x^3} - \frac{36}{x^5}$$

$$\mathbf{j} \quad \text{Let } f(x) = 3x - \frac{1}{x} + \frac{2}{x^2}$$

$$= 3x - x^{-1} + 2x^{-2}$$

$$\therefore f'(x) = 3(1) - (-1x^{-2}) + 2(-2x^{-3})$$

$$= 3 + x^{-2} - 4x^{-3}$$

$$= 3 + \frac{1}{x^2} - \frac{4}{x^3}$$

$$\mathbf{k} \quad \text{Let } f(x) = 5 - \frac{8}{x^2} + \frac{4}{x^3}$$

$$= 5 - 8x^{-2} + 4x^{-3}$$

$$\therefore f'(x) = -8(-2x^{-3}) + 4(-3x^{-4})$$

$$= 16x^{-3} - 12x^{-4}$$

$$= \frac{16}{x^3} - \frac{12}{x^4}$$

$$\mathbf{l} \quad \text{Let } f(x) = \frac{1}{5x^2}$$

$$= \frac{1}{5}x^{-2}$$

$$\therefore f'(x) = \frac{1}{5}(-2x^{-3})$$

$$= -\frac{2}{5}x^{-3}$$

$$= -\frac{2}{5x^3}$$

$$\mathbf{m} \quad \text{Let } f(x) = 4x - \frac{1}{4x}$$

$$= 4x - \frac{1}{4}x^{-1}$$

$$\therefore f'(x) = 4(1) - \frac{1}{4}(-1x^{-2})$$

$$= 4 + \frac{1}{4}x^{-2}$$

$$= 4 + \frac{1}{4x^2}$$

$$\begin{aligned}
 \text{n Let } f(x) &= \frac{x^2 - 3}{x} \\
 &= \frac{x^2}{x} - \frac{3}{x} \\
 &= x - 3x^{-1} \\
 \therefore f'(x) &= 1 - 3(-1x^{-2}) \\
 &= 1 + 3x^{-2} \\
 &= 1 + \frac{3}{x^2}
 \end{aligned}$$

$$\begin{aligned}
 \text{p Let } f(x) &= \frac{2x - 5}{x^2} \\
 &= \frac{2x}{x^2} - \frac{5}{x^2} \\
 &= \frac{2}{x} - \frac{5}{x^2} \\
 &= 2x^{-1} - 5x^{-2} \\
 \therefore f'(x) &= 2(-1x^{-2}) - 5(-2x^{-3}) \\
 &= -2x^{-2} + 10x^{-3} \\
 &= -\frac{2}{x^2} + \frac{10}{x^3}
 \end{aligned}$$

$$\begin{aligned}
 \text{3 a } f(x) &= \sqrt{x} \\
 &= x^{\frac{1}{2}} \\
 \therefore f'(x) &= \frac{1}{2}x^{-\frac{1}{2}} \\
 &= \frac{1}{2\sqrt{x}}
 \end{aligned}$$

$$\begin{aligned}
 \text{c } f(x) &= \frac{1}{\sqrt{x}} \\
 &= x^{-\frac{1}{2}} \\
 \therefore f'(x) &= -\frac{1}{2}x^{-\frac{3}{2}} \\
 &= -\frac{1}{2x\sqrt{x}}
 \end{aligned}$$

$$\begin{aligned}
 \text{e } f(x) &= \frac{1}{x^2} + 6\sqrt{x} \\
 &= x^{-2} + 6x^{\frac{1}{2}} \\
 \therefore f'(x) &= -2x^{-3} + 6(\frac{1}{2}x^{-\frac{1}{2}}) \\
 &= -2x^{-3} + 3x^{-\frac{1}{2}} \\
 &= -\frac{2}{x^3} + \frac{3}{\sqrt{x}}
 \end{aligned}$$

$$\begin{aligned}
 \text{• Let } f(x) &= \frac{x^3 + 4}{x} \\
 &= \frac{x^3}{x} + \frac{4}{x} \\
 &= x^2 + 4x^{-1} \\
 \therefore f'(x) &= 2x + 4(-1x^{-2}) \\
 &= 2x - 4x^{-2} \\
 &= 2x - \frac{4}{x^2}
 \end{aligned}$$

$$\begin{aligned}
 \text{b } f(x) &= \sqrt[3]{x} \\
 &= x^{\frac{1}{3}} \\
 \therefore f'(x) &= \frac{1}{3}x^{-\frac{2}{3}} \\
 &= \frac{1}{3\sqrt[3]{x^2}}
 \end{aligned}$$

$$\begin{aligned}
 \text{d } f(x) &= x^3 - \frac{1}{2}\sqrt{x} \\
 &= x^3 - \frac{1}{2}x^{\frac{1}{2}} \\
 \therefore f'(x) &= 3x^2 - \frac{1}{2}(\frac{1}{2}x^{-\frac{1}{2}}) \\
 &= 3x^2 - \frac{1}{4}x^{-\frac{1}{2}} \\
 &= 3x^2 - \frac{1}{4\sqrt{x}}
 \end{aligned}$$

$$\begin{aligned}
 \text{f } f(x) &= 2x - \sqrt{x} \\
 &= 2x - x^{\frac{1}{2}} \\
 \therefore f'(x) &= 2(1) - \frac{1}{2}x^{-\frac{1}{2}} \\
 &= 2 - \frac{1}{2\sqrt{x}}
 \end{aligned}$$

$$\begin{aligned}\mathbf{g} \quad f(x) &= x\sqrt{x} \\ &= x^{\frac{3}{2}}\end{aligned}$$

$$\begin{aligned}\therefore f'(x) &= \frac{3}{2}x^{\frac{1}{2}} \\ &= \frac{3}{2}\sqrt{x}\end{aligned}$$

$$\begin{aligned}\mathbf{i} \quad f(x) &= 2x^2 - \frac{3}{\sqrt{x}} \\ &= 2x^2 - 3x^{-\frac{1}{2}}\end{aligned}$$

$$\begin{aligned}\therefore f'(x) &= 2(2x) - 3(-\frac{1}{2}x^{-\frac{3}{2}}) \\ &= 4x + \frac{3}{2}x^{-\frac{3}{2}} \\ &= 4x + \frac{3}{2x\sqrt{x}}\end{aligned}$$

$$\begin{aligned}\mathbf{k} \quad f(x) &= \frac{x+5}{\sqrt{x}} \\ &= \frac{x}{\sqrt{x}} + \frac{5}{\sqrt{x}} \\ &= \frac{x}{x^{\frac{1}{2}}} + \frac{5}{x^{\frac{1}{2}}} \\ &= x^{\frac{1}{2}} + 5x^{-\frac{1}{2}}\end{aligned}$$

$$\begin{aligned}\therefore f'(x) &= \frac{1}{2}x^{-\frac{1}{2}} + 5(-\frac{1}{2}x^{-\frac{3}{2}}) \\ &= \frac{1}{2\sqrt{x}} - \frac{5}{2x\sqrt{x}}\end{aligned}$$

$$\begin{aligned}\mathbf{m} \quad f(x) &= 3x^2 - x\sqrt{x} \\ &= 3x^2 - x^{\frac{3}{2}}\end{aligned}$$

$$\begin{aligned}\therefore f'(x) &= 3(2x) - \frac{3}{2}x^{\frac{1}{2}} \\ &= 6x - \frac{3\sqrt{x}}{2}\end{aligned}$$

$$\begin{aligned}\mathbf{h} \quad f(x) &= \frac{1}{x\sqrt{x}} \\ &= x^{-\frac{3}{2}}\end{aligned}$$

$$\begin{aligned}\therefore f'(x) &= -\frac{3}{2}x^{-\frac{5}{2}} \\ &= \frac{-3}{2x^2\sqrt{x}}\end{aligned}$$

$$\begin{aligned}\mathbf{j} \quad f(x) &= \frac{\sqrt{x}-4}{x} \\ &= \frac{\sqrt{x}}{x} - \frac{4}{x} \\ &= \frac{x^{\frac{1}{2}}}{x} - \frac{4}{x} \\ &= x^{-\frac{1}{2}} - 4x^{-1}\end{aligned}$$

$$\begin{aligned}\therefore f'(x) &= -\frac{1}{2}x^{-\frac{3}{2}} - 4(-x^{-2}) \\ &= -\frac{1}{2x\sqrt{x}} + \frac{4}{x^2}\end{aligned}$$

$$\begin{aligned}\mathbf{l} \quad f(x) &= \frac{7-x^2}{\sqrt{x}} \\ &= \frac{7}{\sqrt{x}} - \frac{x^2}{\sqrt{x}} \\ &= 7x^{-\frac{1}{2}} - \frac{x^2}{x^{\frac{1}{2}}} \\ &= 7x^{-\frac{1}{2}} - x^{\frac{3}{2}}\end{aligned}$$

$$\begin{aligned}\therefore f'(x) &= 7(-\frac{1}{2}x^{-\frac{3}{2}}) - \frac{3}{2}x^{\frac{1}{2}} \\ &= -\frac{7}{2x\sqrt{x}} - \frac{3\sqrt{x}}{2}\end{aligned}$$

$$\begin{aligned}\mathbf{n} \quad f(x) &= \frac{4}{x^2\sqrt{x}} \\ &= \frac{4}{x^{\frac{5}{2}}}\end{aligned}$$

$$\begin{aligned}&= 4x^{-\frac{5}{2}} \\ \therefore f'(x) &= 4(-\frac{5}{2}x^{-\frac{7}{2}}) \\ &= -\frac{10}{x^3\sqrt{x}}\end{aligned}$$



$$\begin{aligned}
 \bullet \quad f(x) &= 2x - \frac{3}{x\sqrt{x}} \\
 &= 2x - \frac{3}{x^{\frac{3}{2}}} \\
 &= 2x - 3x^{-\frac{3}{2}} \\
 \therefore f'(x) &= 2(1) - 3(-\frac{3}{2}x^{-\frac{5}{2}}) \\
 &= 2 + \frac{9}{2}x^{-\frac{5}{2}} \\
 &= 2 + \frac{9}{2x^2\sqrt{x}}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{p} \quad f(x) &= \frac{x^2 - x + 2}{\sqrt[3]{x}} \\
 &= \frac{x^2}{\sqrt[3]{x}} - \frac{x}{\sqrt[3]{x}} + \frac{2}{\sqrt[3]{x}} \\
 &= \frac{x^2}{x^{\frac{1}{3}}} - \frac{x}{x^{\frac{1}{3}}} + \frac{2}{x^{\frac{1}{3}}} \\
 &= x^{\frac{5}{3}} - x^{\frac{2}{3}} + 2x^{-\frac{1}{3}} \\
 \therefore f'(x) &= \frac{5}{3}x^{\frac{2}{3}} - \frac{2}{3}x^{-\frac{1}{3}} + 2(-\frac{1}{3}x^{-\frac{4}{3}}) \\
 &= \frac{5}{3}x^{\frac{2}{3}} - \frac{2}{3}x^{-\frac{1}{3}} - \frac{2}{3}x^{-\frac{4}{3}} \\
 &= \frac{5\sqrt[3]{x^2}}{3} - \frac{2}{3\sqrt[3]{x}} - \frac{2}{3x\sqrt[3]{x}}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{4} \quad \mathbf{a} \quad y &= \pi x^2 \\
 \therefore \frac{dy}{dx} &= \pi(2x) \\
 &= 2\pi x
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad y &= 3x^2 - \frac{8}{x^2} \\
 &= 3x^2 - 8x^{-2} \\
 \therefore \frac{dy}{dx} &= 3(2x) - 8(-2x^{-3}) \\
 &= 6x + 16x^{-3} \\
 &= 6x + \frac{16}{x^3}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad y &= 6\sqrt{x} + \frac{5}{x} \\
 &= 6x^{\frac{1}{2}} + 5x^{-1} \\
 \therefore \frac{dy}{dx} &= 6(\frac{1}{2}x^{-\frac{1}{2}}) + 5(-1x^{-2}) \\
 &= 3x^{-\frac{1}{2}} - 5x^{-2} \\
 &= \frac{3}{\sqrt{x}} - \frac{5}{x^2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad y &= 4\pi x^3 \\
 \therefore \frac{dy}{dx} &= 4\pi(3x^2) \\
 &= 12\pi x^2
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{e} \quad y &= 2.5x^3 - 1.4x^2 - 1.3 \\
 \therefore \frac{dy}{dx} &= 2.5(3x^2) - 1.4(2x) \\
 &= 7.5x^2 - 2.8x
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{f} \quad y &= 10(x + 1) \\
 &= 10x + 10 \\
 \therefore \frac{dy}{dx} &= 10(1) \\
 &= 10
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{g} \quad y &= (x + 1)(x - 2) \\
 &= x^2 - x - 2 \\
 \therefore \frac{dy}{dx} &= 2x - 1
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{h} \quad y &= (2x + 1)(3x - 2) \\
 &= 6x^2 - x - 2 \\
 \therefore \frac{dy}{dx} &= 12x - 1
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{i} \quad y &= (5 - x)^2 \\
 &= 25 - 10x + x^2 \\
 \therefore \frac{dy}{dx} &= -10(1) + 2x \\
 &= 2x - 10
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{j} \quad y &= (2x - 1)^2 \\
 &= 4x^2 - 4x + 1 \\
 \therefore \frac{dy}{dx} &= 4(2x) - 4(1) \\
 &= 8x - 4
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{k} \quad y &= x(x+1)(2x-5) \\
 &= x(2x^2 - 3x - 5) \\
 &= 2x^3 - 3x^2 - 5x \\
 \therefore \frac{dy}{dx} &= 2(3x^2) - 3(2x) - 5(1) \\
 &= 6x^2 - 6x - 5
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{l} \quad y &= \frac{(x-3)^2}{\sqrt{x}} \\
 &= \frac{x^2 - 6x + 9}{\sqrt{x}} \\
 &= \frac{x^2}{\sqrt{x}} - \frac{6x}{\sqrt{x}} + \frac{9}{\sqrt{x}} \\
 &= \frac{x^2}{x^{\frac{1}{2}}} - \frac{6x}{x^{\frac{1}{2}}} + \frac{9}{x^{\frac{1}{2}}} \\
 &= x^{\frac{3}{2}} - 6x^{\frac{1}{2}} + 9x^{-\frac{1}{2}} \\
 \therefore \frac{dy}{dx} &= \frac{3}{2}x^{\frac{1}{2}} - 6\left(\frac{1}{2}x^{-\frac{1}{2}}\right) + 9\left(-\frac{1}{2}x^{-\frac{3}{2}}\right) \\
 &= \frac{3}{2}x^{\frac{1}{2}} - 3x^{-\frac{1}{2}} - \frac{9}{2}x^{-\frac{3}{2}} \\
 &= \frac{3}{2}\sqrt{x} - \frac{3}{\sqrt{x}} - \frac{9}{2x\sqrt{x}}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{5} \quad \mathbf{a} \quad f(x) &= (1-x)^3 \\
 &= 1 - 3x + 3x^2 - x^3 \\
 \therefore f'(x) &= -3 + 6x - 3x^2
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad f(x) &= \left(3x - \frac{1}{\sqrt{x}}\right)^3 \\
 &= (3x)^3 + 3(3x)^2 \left(-\frac{1}{\sqrt{x}}\right) + 3(3x) \left(-\frac{1}{\sqrt{x}}\right)^2 + \left(-\frac{1}{\sqrt{x}}\right)^3 \\
 &= 27x^3 - \frac{27x^2}{\sqrt{x}} + 9 - \frac{1}{x\sqrt{x}} \\
 &= 27x^3 - 27x^{\frac{3}{2}} + 9 - x^{-\frac{3}{2}} \\
 \therefore f'(x) &= 27(3x^2) - 27\left(\frac{3}{2}x^{\frac{1}{2}}\right) - \left(-\frac{3}{2}x^{-\frac{5}{2}}\right) \\
 &= 81x^2 - \frac{81}{2}x^{\frac{1}{2}} + \frac{3}{2}x^{-\frac{5}{2}} \\
 &= 81x^2 - \frac{81\sqrt{x}}{2} + \frac{3}{2x^2\sqrt{x}}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{6} \quad \mathbf{a} \quad y &= \frac{1}{2}t^4 - \frac{1}{3}t \\
 \therefore \frac{dy}{dt} &= \frac{1}{2}(4t^3) - \frac{1}{3}(1) \\
 &= 2t^3 - \frac{1}{3}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad y &= 7 - \frac{6}{\sqrt{t}} \\
 &= 7 - 6t^{-\frac{1}{2}} \\
 \therefore \frac{dy}{dt} &= -6\left(-\frac{1}{2}t^{-\frac{3}{2}}\right) \\
 &= 3t^{-\frac{3}{2}} \\
 &= \frac{3}{t\sqrt{t}}
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad T &= \sqrt[3]{t} - \frac{2}{t^2} \\
 &= t^{\frac{1}{3}} - 2t^{-2} \\
 \therefore \frac{dT}{dt} &= \frac{1}{3}t^{-\frac{2}{3}} - 2(-2t^{-3}) \\
 &= \frac{1}{3}t^{-\frac{2}{3}} + 4t^{-3} \\
 &= \frac{1}{3\sqrt[3]{t^2}} + \frac{4}{t^3}
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad P &= \frac{5}{u} - 10u\sqrt{u} \\
 &= 5u^{-1} - 10u^{\frac{3}{2}} \\
 \therefore \frac{dP}{du} &= 5(-u^{-2}) - 10\left(\frac{3}{2}u^{\frac{1}{2}}\right) \\
 &= -5u^{-2} - 15u^{\frac{1}{2}} \\
 &= -\frac{5}{u^2} - 15\sqrt{u}
 \end{aligned}$$

$$\begin{aligned}
 7 \quad \text{a} \quad y &= x^2 \\
 \therefore \frac{dy}{dx} &= 2x \\
 \text{When } x &= 2, \quad \frac{dy}{dx} = 2(2) = 4 \\
 \text{So, the tangent has gradient } &4.
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad y &= x^3 - 5x + 2 \\
 \therefore \frac{dy}{dx} &= 3x^2 - 5(1) \\
 &= 3x^2 - 5 \\
 \text{At the point } (3, 14), \\
 \frac{dy}{dx} &= 3(3)^2 - 5 = 22 \\
 \text{So, the tangent has gradient } &22.
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad y &= \frac{8}{x^2} \\
 &= 8x^{-2} \\
 \therefore \frac{dy}{dx} &= 8(-2x^{-3}) \\
 &= -16x^{-3} \\
 &= -\frac{16}{x^3} \\
 \text{At the point } (9, \frac{8}{81}), \\
 \frac{dy}{dx} &= -\frac{16}{9^3} = -\frac{16}{729} \\
 \text{So, the tangent has gradient } &-\frac{16}{729}.
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad y &= 2x^2 - 3x + 7 \\
 \therefore \frac{dy}{dx} &= 2(2x) - 3(1) \\
 &= 4x - 3 \\
 \text{When } x &= -1, \\
 \frac{dy}{dx} &= 4(-1) - 3 \\
 &= -7 \\
 \text{So, the tangent has gradient } &-7.
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad y &= 3\sqrt{x} \\
 &= 3x^{\frac{1}{2}} \\
 \therefore \frac{dy}{dx} &= 3\left(\frac{1}{2}x^{-\frac{1}{2}}\right) \\
 &= \frac{3}{2\sqrt{x}} \\
 \text{At the point } (1, 3), \\
 \frac{dy}{dx} &= \frac{3}{2\sqrt{1}} = \frac{3}{2} \\
 \text{So, the tangent has gradient } &\frac{3}{2}.
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad y &= 2x - \frac{5}{x} \\
 &= 2x - 5x^{-1} \\
 \therefore \frac{dy}{dx} &= 2(1) - 5(-1x^{-2}) \\
 &= 2 + \frac{5}{x^2} \\
 \text{At the point } (2, \frac{3}{2}), \\
 \frac{dy}{dx} &= 2 + \frac{5}{2^2} \\
 &= 2 + \frac{5}{4} \\
 &= \frac{13}{4} \\
 \text{So, the tangent has gradient } &\frac{13}{4}.
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{g} \quad y &= \frac{x^2 - 4}{x^2} \\
 &= \frac{x^2}{x^2} - \frac{4}{x^2} \\
 &= 1 - 4x^{-2} \\
 \therefore \frac{dy}{dx} &= -4(-2x^{-3}) \\
 &= \frac{8}{x^3}
 \end{aligned}$$

At the point  $(4, \frac{3}{4})$ ,

$$\frac{dy}{dx} = \frac{8}{4^3} = \frac{8}{64} = \frac{1}{8}.$$

So, the tangent has gradient  $\frac{1}{8}$ .

$$\begin{aligned}
 \mathbf{h} \quad y &= \frac{x^3 - 4x - 8}{x^2} \\
 &= \frac{x^3}{x^2} - \frac{4x}{x^2} - \frac{8}{x^2} \\
 &= x - 4x^{-1} - 8x^{-2} \\
 \therefore \frac{dy}{dx} &= 1 - 4(-1x^{-2}) - 8(-2x^{-3}) \\
 &= 1 + \frac{4}{x^2} + \frac{16}{x^3}
 \end{aligned}$$

When  $x = -1$ ,

$$\begin{aligned}
 \frac{dy}{dx} &= 1 + \frac{4}{(-1)^2} + \frac{16}{(-1)^3} \\
 &= -11
 \end{aligned}$$

So, the tangent has gradient  $-11$ .

$$\mathbf{8} \quad \mathbf{a} \quad f(x) = x^2 + (b+1)x + 2c, \quad f(2) = 4, \quad \text{and} \quad f'(-1) = 2$$

$$\therefore f'(x) = 2x + (b+1)$$

$$\text{But } f'(-1) = 2, \text{ so } 2(-1) + b + 1 = 2$$

$$\therefore -1 + b = 2$$

$$\therefore b = 3$$

$$\begin{aligned}
 \text{So, } f(x) &= x^2 + (3+1)x + 2c \\
 &= x^2 + 4x + 2c
 \end{aligned}$$

$$\text{But } f(2) = 4, \text{ so } 2^2 + 4(2) + 2c = 4$$

$$\therefore 2c = -8$$

$$\therefore c = -4$$

$$\begin{aligned}
 \mathbf{b} \quad f(x) &= bx + \frac{c}{x}, \quad f(3) = 5, \quad \text{and} \quad f'(1) = 5 \\
 &= bx + cx^{-1} \\
 \therefore f'(x) &= b + c(-x^{-2}) \\
 &= b - \frac{c}{x^2}
 \end{aligned}$$

$$\text{But } f'(1) = 5, \text{ so } b - \frac{c}{(1)^2} = 5$$

$$\therefore b - c = 5$$

$$\therefore b = c + 5 \quad \dots (*)$$

$$\text{and } f(3) = 5, \text{ so } b(3) + \frac{c}{3} = 5$$

$$\therefore 3b + \frac{c}{3} = 5$$

$$\therefore 3(c+5) + \frac{c}{3} = 5 \quad \{\text{using } (*)\}$$

$$\therefore 3c + 15 + \frac{c}{3} = 5$$

$$\therefore \frac{10}{3}c = -10$$

$$\therefore c = -3$$

$$\text{and so } b = -3 + 5 = 2$$



$$9 \quad f(x) = \begin{cases} 4x^2 - 3, & x \geq 2 \\ x^3 + 2x + 1, & x < 2 \end{cases}$$

$$\therefore f'(x) = \begin{cases} 8x, & x > 2 \\ 3x^2 + 2, & x < 2 \end{cases} \quad \begin{array}{l} \text{since the derivative of } f \text{ exists on the intervals } x < 2 \\ \text{and } x > 2. \end{array}$$

$$\therefore f'_-(2) = 3(2)^2 + 2 = 14 \quad \text{and} \quad f'_+(2) = 8(2) = 16$$

$$\therefore f'_-(2) \neq f'_+(2)$$

$\therefore f$  is not differentiable at  $x = 2$ .

$$10 \quad \begin{aligned} y &= 4x - \frac{3}{x} \\ &= 4x - 3x^{-1} \\ \therefore \frac{dy}{dx} &= 4 + 3x^{-2} \\ &= 4 + \frac{3}{x^2} \end{aligned}$$

$\frac{dy}{dx}$  is the gradient function of  $y = 4x - \frac{3}{x}$  from which the gradient of the tangent at any point can be found. It is also the instantaneous rate of change of  $y$  with respect to  $x$ .

$$11 \quad f(x) = \sqrt{x} - \frac{4}{\sqrt{x}}$$

**a** The domain of  $f(x)$  is  $\{x \mid x > 0\}$ .

$$\begin{aligned} \mathbf{b} \quad f(x) &= \sqrt{x} - \frac{4}{\sqrt{x}} \\ &= x^{\frac{1}{2}} - 4x^{-\frac{1}{2}} \\ \therefore f'(x) &= \frac{1}{2}x^{-\frac{1}{2}} + 2x^{-\frac{3}{2}} \\ &= \frac{1}{2\sqrt{x}} + \frac{2}{x\sqrt{x}} \end{aligned}$$

**c** The domain of  $f'(x)$  is  $\{x \mid x > 0\}$ .

$$\begin{aligned} \mathbf{d} \quad f'(1) &= \frac{1}{2\sqrt{1}} + \frac{2}{1\sqrt{1}} \\ &= \frac{1}{2} + 2 \\ &= \frac{5}{2} \\ &= 2.5 \end{aligned}$$

The gradient of the tangent to the curve  $f(x) = \sqrt{x} - \frac{4}{\sqrt{x}}$  at  $x = 1$  is 2.5.

$$12 \quad \mathbf{a} \quad S = 2t^2 + 4t \text{ m}$$

$$\therefore \frac{dS}{dt} = 4t + 4 \text{ m s}^{-1}$$

$\frac{dS}{dt}$  is the instantaneous rate of change in position at time  $t$ . It is the velocity function of the car.

**b** When  $t = 3$ ,  $\frac{dS}{dt} = 4(3) + 4$   
 $= 16$

The instantaneous rate of change in position at time  $t = 3$  seconds is  $16 \text{ m s}^{-1}$ , or the velocity of the car at  $t = 3$  seconds is  $16 \text{ m s}^{-1}$ .

**13**  $C = 1785 + 3x + 0.002x^2$  pounds  
 $\therefore \frac{dC}{dx} = 3 + 0.002(2x)$   
 $= 3 + 0.004x$  pounds per toaster

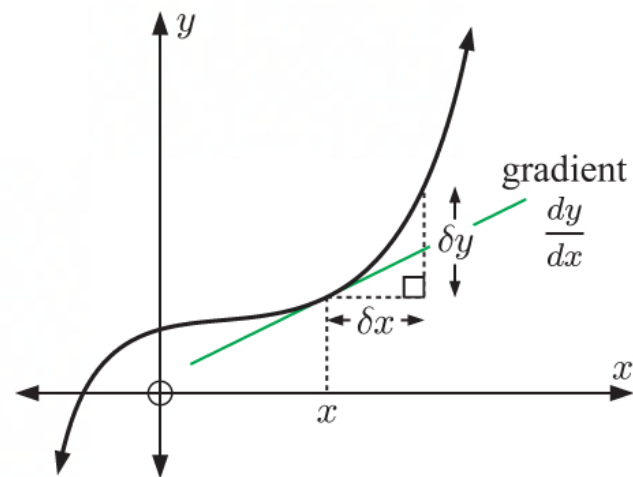
When  $x = 1000$ ,  $\frac{dC}{dx} = 3 + 0.004(1000)$   
 $= 7$

When 1000 toasters are being produced each week, the cost of production is increasing by £7 per toaster.

**14 a** We are estimating  $\delta y$  by multiplying the small increment  $\delta x$  by the gradient of the tangent at  $x$ .

The gradient of the tangent is  $\frac{dy}{dx} \approx \frac{\delta y}{\delta x}$   
 $\therefore \delta y \approx \frac{dy}{dx} \times \delta x$

**b** When  $\delta x$  is small, the value of the graph at  $x + \delta x$  is close to the tangent drawn at  $x$ . Our estimate for  $\delta y$  will hence be more accurate.



**c i** Suppose  $y = x^2$ , and therefore  $\frac{dy}{dx} = 2x$ .

To estimate the value of  $5.01^2$ , we let  $x = 5$  and  $\delta x = 0.01$ .

Now  $\delta y \approx \frac{dy}{dx} \times \delta x$   
 $\approx 2x \times \delta x$   
 $\approx 2 \times 5 \times 0.01$   
 $\approx 0.1$

Since  $5^2 = 25$ , we estimate that  $5.01^2 \approx 25 + 0.1 \approx 25.1$ .

Using technology,  $5.01^2 = 25.1001$ .

**ii** Suppose  $y = x^6$ , and therefore  $\frac{dy}{dx} = 6x^5$ .

To estimate the value of  $2.01^6$ , we let  $x = 2$  and  $\delta x = 0.01$ .

Now  $\delta y \approx \frac{dy}{dx} \times \delta x$   
 $\approx 6x^5 \times \delta x$   
 $\approx 6 \times 2^5 \times 0.01$   
 $\approx 1.92$

Since  $2^6 = 64$ , we estimate that  $2.01^6 \approx 64 + 1.92 \approx 65.92$ .

Using technology,  $2.01^6 \approx 65.9442$ .

- iii Suppose  $y = x^3$ , and therefore  $\frac{dy}{dx} = 3x^2$ .

To estimate the value of  $2.98^3$ , we let  $x = 3$  and  $\delta x = -0.02$ .

$$\begin{aligned}\text{Now } \delta y &\approx \frac{dy}{dx} \times \delta x \\ &\approx 3x^2 \times \delta x \\ &\approx 3 \times 3^2 \times (-0.02) \\ &\approx -0.54\end{aligned}$$

Since  $3^3 = 27$ , we estimate that  $2.98^3 \approx 27 - 0.54 \approx 26.46$ .

Using technology,  $2.98^3 \approx 26.4636$ .

- iv Suppose  $y = x^4$ , and therefore  $\frac{dy}{dx} = 4x^3$ .

To estimate the value of  $1.95^4$ , we let  $x = 2$  and  $\delta x = -0.05$ .

$$\begin{aligned}\text{Now } \delta y &\approx \frac{dy}{dx} \times \delta x \\ &\approx 4x^3 \times \delta x \\ &\approx 4 \times 2^3 \times (-0.05) \\ &\approx -1.6\end{aligned}$$

Since  $2^4 = 16$ , we estimate that  $1.95^4 \approx 16 - 1.6 \approx 14.4$ .

Using technology,  $1.95^4 \approx 14.4590$ .

## EXERCISE 17B.1

1 a  $g(x) = x^2$  and  $f(x) = 2x + 7$

$$\begin{aligned}\therefore g(f(x)) &= g(2x + 7) \\ &= (2x + 7)^2\end{aligned}$$

c  $g(x) = \sqrt{x}$  and  $f(x) = 3 - 4x$

$$\begin{aligned}\therefore g(f(x)) &= g(3 - 4x) \\ &= \sqrt{3 - 4x}\end{aligned}$$

e  $g(x) = \frac{2}{x}$  and  $f(x) = x^2 + 3$

$$\begin{aligned}\therefore g(f(x)) &= g(x^2 + 3) \\ &= \frac{2}{x^2 + 3}\end{aligned}$$

b  $g(x) = 2x + 7$  and  $f(x) = x^2$

$$\begin{aligned}\therefore g(f(x)) &= g(x^2) \\ &= 2(x^2) + 7 \\ &= 2x^2 + 7\end{aligned}$$

d  $g(x) = 3 - 4x$  and  $f(x) = \sqrt{x}$

$$\begin{aligned}\therefore g(f(x)) &= g(\sqrt{x}) \\ &= 3 - 4\sqrt{x}\end{aligned}$$

f  $g(x) = x^2 + 3$  and  $f(x) = \frac{2}{x}$

$$\begin{aligned}\therefore g(f(x)) &= g\left(\frac{2}{x}\right) \\ &= \left(\frac{2}{x}\right)^2 + 3 \\ &= \frac{4}{x^2} + 3\end{aligned}$$

2 **Note:** There may be other answers.

a  $g(f(x)) = (3x + 10)^3$

If we let  $f(x) = 3x + 10$  then

$$g(f(x)) = (f(x))^3$$

$$\therefore g(x) = x^3 \text{ and } f(x) = 3x + 10$$

b  $g(f(x)) = (7 - 2x)^5$

If we let  $f(x) = 7 - 2x$  then

$$g(f(x)) = (f(x))^5$$

$$\therefore g(x) = x^5 \text{ and } f(x) = 7 - 2x$$



$$\text{c } g(f(x)) = \frac{1}{2x+4}$$

If we let  $f(x) = 2x + 4$  then

$$g(f(x)) = \frac{1}{f(x)}$$

$$\therefore g(x) = \frac{1}{x} \text{ and } f(x) = 2x + 4$$

$$\text{e } g(f(x)) = \frac{1}{(5x-1)^4}$$

If we let  $f(x) = 5x - 1$  then

$$g(f(x)) = \frac{1}{(f(x))^4}$$

$$\therefore g(x) = \frac{1}{x^4} \text{ and } f(x) = 5x - 1$$

$$\text{d } g(f(x)) = \sqrt{x^2 - 3x}$$

If we let  $f(x) = x^2 - 3x$  then

$$g(f(x)) = \sqrt{f(x)}$$

$$\therefore g(x) = \sqrt{x} \text{ and } f(x) = x^2 - 3x$$

$$\text{f } g(f(x)) = \frac{10}{(3x-x^2)^3}$$

If we let  $f(x) = 3x - x^2$  then

$$g(f(x)) = \frac{10}{(f(x))^3}$$

$$\therefore g(x) = \frac{10}{x^3} \text{ and } f(x) = 3x - x^2$$

## INVESTIGATION 2

## DIFFERENTIATING COMPOSITE FUNCTIONS

$$\begin{aligned} \text{1 } y &= (2x+1)^2 \\ &= 4x^2 + 4x + 1 \end{aligned}$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= 4(2x) + 4(1) \\ &= 8x + 4 \\ &= 2 \times 2(2x+1)^1 \end{aligned}$$

$$\begin{aligned} \text{2 } y &= (3x+1)^2 \\ &= 9x^2 + 6x + 1 \end{aligned}$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= 9(2x) + 6(1) \\ &= 18x + 6 \\ &= 3 \times 2(3x+1)^1 \end{aligned}$$

$$\begin{aligned} \text{3 } y &= (ax+1)^2 \\ &= a^2x^2 + 2ax + 1 \end{aligned}$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= a^2(2x) + 2a(1) \\ &= 2a^2x + 2a \\ &= a \times 2(ax+1)^1 \end{aligned}$$

$$\begin{aligned} \text{4 a } y &= u^2 \\ \therefore \frac{dy}{du} &= 2u \end{aligned}$$

$$\text{b } u = ax + 1, \quad y = (ax + 1)^2$$

$$\text{i } \frac{du}{dx} = a$$

$$\begin{aligned} \text{ii } \frac{dy}{du} &= 2u \\ &= 2(ax + 1) \\ &= 2ax + 2 \end{aligned}$$

$$\begin{aligned} \text{iii } \frac{dy}{du} \times \frac{du}{dx} &= (2ax + 2) \times a \\ &= a \times 2(ax + 1)^1 \end{aligned}$$

**iv** Our answer to **iii** is the same as the result in **3**.

$$\text{c } \text{If } y = u^2 \text{ where } u \text{ is a function of } x, \text{ we suspect that } \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}.$$

$$\begin{aligned} \text{5 } y &= (x^2 + 3x)^2 \\ &= (x^2)^2 + 2(x^2)(3x) + (3x)^2 \\ &= x^4 + 6x^3 + 9x^2 \end{aligned}$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= 4x^3 + 6(3x^2) + 9(2x) \\ &= 4x^3 + 18x^2 + 18x \\ &= 2(2x^3 + 9x^2 + 9x) \\ &= 2(x^2 + 3x)(2x + 3) \quad \dots (*) \end{aligned}$$

$$\text{Now, consider } y = u^2 \text{ where } u = x^2 + 3x. \quad \therefore \frac{du}{dx} = 2x + 3$$



Comparing with (\*),  $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

Our answer agrees with the rule we suggested in 4 c.

$$\begin{aligned} \text{6 a } y &= (2x + 1)^3 \\ &= (2x)^3 + 3(2x)^2(1) + 3(2x)(1)^2 + 1^3 \\ &= 8x^3 + 12x^2 + 6x + 1 \\ \therefore \frac{dy}{dx} &= 24x^2 + 24x + 6 \end{aligned}$$

$$\text{b } u = 2x + 1, \quad y = u^3$$

$$\text{i } \frac{du}{dx} = 2$$

$$\text{ii } \frac{dy}{du} = 3u^2 \\ = 3(2x + 1)^2$$

$$\begin{aligned} \text{iii } \frac{dy}{du} \times \frac{du}{dx} &= 3(2x + 1)^2 \times 2 \\ &= 3(4x^2 + 4x + 1) \times 2 \\ &= (12x^2 + 12x + 3) \times 2 \\ &= 24x^2 + 24x + 6 \end{aligned}$$

iv Our answer to iii is the same as the result in a.

7 If  $y$  is a function of  $u$ , and  $u$  is a function of  $x$ , then  $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$ .

## EXERCISE 17B.2

$$\text{1 a } \frac{1}{(2x - 1)^2} = u^{-2} \text{ where } u = 2x - 1$$

$$\text{c } \frac{2}{\sqrt{2 - x^2}} = 2u^{-\frac{1}{2}} \text{ where } u = 2 - x^2$$

$$\text{e } \frac{4}{(3 - x)^3} = 4u^{-3} \text{ where } u = 3 - x$$

$$\text{b } \sqrt{x^2 - 3x} = u^{\frac{1}{2}} \text{ where } u = x^2 - 3x$$

$$\text{d } \sqrt[3]{x^3 - x^2} = u^{\frac{1}{3}} \text{ where } u = x^3 - x^2$$

$$\text{f } \frac{10}{x^2 - 3} = 10u^{-1} \text{ where } u = x^2 - 3$$

$$\begin{aligned} \text{2 a } y &= (2x + 3)^2 \\ \therefore y &= u^2 \text{ where } u = 2x + 3 \end{aligned}$$

$$\begin{aligned} \text{Now } \frac{dy}{dx} &= \frac{dy}{du} \frac{du}{dx} \quad \{\text{chain rule}\} \\ &= 2u(2) \\ &= 4u \\ &= 4(2x + 3) \\ &= 8x + 12 \end{aligned}$$

$$\begin{aligned} \text{b } y &= (2x + 3)^2 \\ &= 4x^2 + 12x + 9 \\ \therefore \frac{dy}{dx} &= 4(2x) + 12(1) \\ &= 8x + 12 \end{aligned}$$

$$\begin{aligned} \text{3 a } y &= (4x - 5)^2 \\ \therefore y &= u^2 \text{ where } u = 4x - 5 \end{aligned}$$

$$\begin{aligned} \text{Now } \frac{dy}{dx} &= \frac{dy}{du} \frac{du}{dx} \quad \{\text{chain rule}\} \\ &= 2u(4) \\ &= 8u \\ &= 8(4x - 5) \end{aligned}$$

$$\begin{aligned} \text{b } y &= \frac{1}{5 - 2x} \\ \therefore y &= u^{-1} \text{ where } u = 5 - 2x \end{aligned}$$

$$\begin{aligned} \text{Now } \frac{dy}{dx} &= \frac{dy}{du} \frac{du}{dx} \quad \{\text{chain rule}\} \\ &= -u^{-2}(-2) \\ &= 2u^{-2} \\ &= 2(5 - 2x)^{-2} \end{aligned}$$

$$\mathbf{c} \quad y = \sqrt{3x - x^2}$$

$$\therefore y = u^{\frac{1}{2}} \quad \text{where} \quad u = 3x - x^2$$

$$\begin{aligned} \text{Now} \quad \frac{dy}{dx} &= \frac{dy}{du} \frac{du}{dx} \quad \{\text{chain rule}\} \\ &= \frac{1}{2} u^{-\frac{1}{2}} (3 - 2x) \\ &= \frac{1}{2} (3x - x^2)^{-\frac{1}{2}} (3 - 2x) \end{aligned}$$

$$\mathbf{e} \quad y = 6(5 - x)^3$$

$$\therefore y = 6u^3 \quad \text{where} \quad u = 5 - x$$

$$\begin{aligned} \text{Now} \quad \frac{dy}{dx} &= \frac{dy}{du} \frac{du}{dx} \quad \{\text{chain rule}\} \\ &= 6(3u^2)(-1) \\ &= -18u^2 \\ &= -18(5 - x)^2 \end{aligned}$$

$$\mathbf{g} \quad y = \frac{6}{(5x - 4)^2}$$

$$\therefore y = 6u^{-2} \quad \text{where} \quad u = 5x - 4$$

$$\begin{aligned} \text{Now} \quad \frac{dy}{dx} &= \frac{dy}{du} \frac{du}{dx} \quad \{\text{chain rule}\} \\ &= 6(-2u^{-3})(5) \\ &= -60u^{-3} \\ &= -60(5x - 4)^{-3} \end{aligned}$$

$$\mathbf{i} \quad y = 2\left(x^2 - \frac{2}{x}\right)^3$$

$$\therefore y = 2u^3 \quad \text{where} \quad u = x^2 - 2x^{-1}$$

$$\begin{aligned} \text{Now} \quad \frac{dy}{dx} &= \frac{dy}{du} \frac{du}{dx} \quad \{\text{chain rule}\} \\ &= 2(3u^2)(2x - 2(-1x^{-2})) \\ &= 6u^2(2x + 2x^{-2}) \\ &= 6(x^2 - 2x^{-1})^2(2x + 2x^{-2}) \\ &= 6\left(x^2 - \frac{2}{x}\right)^2\left(2x + \frac{2}{x^2}\right) \end{aligned}$$

$$\mathbf{d} \quad y = (1 - 3x)^4$$

$$\therefore y = u^4 \quad \text{where} \quad u = 1 - 3x$$

$$\begin{aligned} \text{Now} \quad \frac{dy}{dx} &= \frac{dy}{du} \frac{du}{dx} \quad \{\text{chain rule}\} \\ &= 4u^3(-3) \\ &= -12u^3 \\ &= -12(1 - 3x)^3 \end{aligned}$$

$$\mathbf{f} \quad y = \sqrt[3]{2x^3 - x^2}$$

$$\therefore y = u^{\frac{1}{3}} \quad \text{where} \quad u = 2x^3 - x^2$$

$$\begin{aligned} \text{Now} \quad \frac{dy}{dx} &= \frac{dy}{du} \frac{du}{dx} \quad \{\text{chain rule}\} \\ &= \frac{1}{3} u^{-\frac{2}{3}} (2(3x^2) - 2x) \\ &= \frac{1}{3} (2x^3 - x^2)^{-\frac{2}{3}} (6x^2 - 2x) \end{aligned}$$

$$\mathbf{h} \quad y = (x^2 - 5x + 8)^5$$

$$\therefore y = u^5 \quad \text{where} \quad u = x^2 - 5x + 8$$

$$\begin{aligned} \text{Now} \quad \frac{dy}{dx} &= \frac{dy}{du} \frac{du}{dx} \quad \{\text{chain rule}\} \\ &= 5u^4(2x - 5) \\ &= 5(x^2 - 5x + 8)^4(2x - 5) \end{aligned}$$

**4 a**  $y = \sqrt{1 - x^2}$   
 $\therefore y = u^{\frac{1}{2}}$  where  $u = 1 - x^2$

Now  $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$  {chain rule}  
 $= \frac{1}{2} u^{-\frac{1}{2}} (-2x)$   
 $= \frac{1}{2} (1 - x^2)^{-\frac{1}{2}} (-2x)$   
 $= -x(1 - x^2)^{-\frac{1}{2}}$

At  $x = \frac{1}{2}$ ,  $\frac{dy}{dx} = -\frac{1}{2} (1 - (\frac{1}{2})^2)^{-\frac{1}{2}}$   
 $= -\frac{1}{2} (\frac{3}{4})^{-\frac{1}{2}}$   
 $= -\frac{1}{\sqrt{3}}$

$\therefore$  the gradient of the tangent to  
 $y = \sqrt{1 - x^2}$  at  $x = \frac{1}{2}$  is  $-\frac{1}{\sqrt{3}}$ .

**c**  $y = \frac{1}{(2x - 1)^4}$   
 $\therefore y = u^{-4}$  where  $u = 2x - 1$

Now  $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$  {chain rule}  
 $= -4u^{-5} (2)$   
 $= -8u^{-5}$   
 $= -8(2x - 1)^{-5}$

At  $x = 1$ ,  $\frac{dy}{dx} = -8(2(1) - 1)^{-5}$   
 $= -8$

$\therefore$  the gradient of the tangent to  
 $y = \frac{1}{(2x - 1)^4}$  at  $x = 1$  is  $-8$ .

**b**  $y = (3x + 2)^6$   
 $\therefore y = u^6$  where  $u = 3x + 2$

Now  $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$  {chain rule}  
 $= 6u^5 (3)$   
 $= 18u^5$   
 $= 18(3x + 2)^5$

At  $x = -1$ ,  $\frac{dy}{dx} = 18(3(-1) + 2)^5$   
 $= 18(-1)^5$   
 $= -18$

$\therefore$  the gradient of the tangent to  
 $y = (3x + 2)^6$  at  $x = -1$  is  $-18$ .

**d**  $y = 6 \times \sqrt[3]{1 - 2x}$   
 $\therefore y = 6u^{\frac{1}{3}}$  where  $u = 1 - 2x$

Now  $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$  {chain rule}  
 $= 6(\frac{1}{3} u^{-\frac{2}{3}}) (-2)$   
 $= -4u^{-\frac{2}{3}}$   
 $= -4(1 - 2x)^{-\frac{2}{3}}$

At  $x = 0$ ,  $\frac{dy}{dx} = -4(1 - 2(0))^{-\frac{2}{3}}$   
 $= -4$

$\therefore$  the gradient of the tangent to  
 $y = 6 \times \sqrt[3]{1 - 2x}$  at  $x = 0$  is  $-4$ .

$$\text{e} \quad y = \frac{4}{x + 2\sqrt{x}}$$

$$\therefore y = 4u^{-1} \quad \text{where} \quad u = x + 2x^{\frac{1}{2}}$$

$$\begin{aligned} \text{Now} \quad \frac{dy}{dx} &= \frac{dy}{du} \frac{du}{dx} \quad \{\text{chain rule}\} \\ &= 4(-1u^{-2})(1 + 2(\frac{1}{2}x^{-\frac{1}{2}})) \\ &= -4u^{-2}(1 + x^{-\frac{1}{2}}) \\ &= -4\left(x + 2x^{\frac{1}{2}}\right)^{-2} (1 + x^{-\frac{1}{2}}) \end{aligned}$$

At  $x = 4$ ,

$$\begin{aligned} \frac{dy}{dx} &= -4\left(4 + 2(4)^{\frac{1}{2}}\right)^{-2} (1 + 4^{-\frac{1}{2}}) \\ &= -4(8)^{-2} \left(\frac{3}{2}\right) \\ &= -\frac{3}{32} \end{aligned}$$

$\therefore$  the gradient of the tangent to

$$y = \frac{4}{x + 2\sqrt{x}} \quad \text{at } x = 4 \text{ is } -\frac{3}{32}.$$

$$\text{f} \quad y = \left(x + \frac{1}{x}\right)^3$$

$$\therefore y = u^3 \quad \text{where} \quad u = x + x^{-1}$$

$$\begin{aligned} \text{Now} \quad \frac{dy}{dx} &= \frac{dy}{du} \frac{du}{dx} \quad \{\text{chain rule}\} \\ &= 3u^2(1 - x^{-2}) \\ &= 3(x + x^{-1})^2(1 - x^{-2}) \end{aligned}$$

$$\begin{aligned} \text{At } x = 1, \quad \frac{dy}{dx} &= 3(1 + 1^{-1})^2(1 - 1^{-2}) \\ &= 3(4)(0) \\ &= 0 \end{aligned}$$

$\therefore$  the gradient of the tangent to

$$y = \left(x + \frac{1}{x}\right)^3 \quad \text{at } x = 1 \text{ is } 0.$$

$$\text{5} \quad y = f(x) = (2x - b)^a$$

$$\therefore y = u^a \quad \text{where} \quad u = 2x - b$$

$$\begin{aligned} \text{Now} \quad \frac{dy}{dx} &= \frac{dy}{du} \frac{du}{dx} \quad \{\text{chain rule}\} \\ &= au^{a-1}(2) \\ &= 2au^{a-1} \\ &= 2a(2x - b)^{a-1} \end{aligned}$$

$$\text{But } f'(x) = 24x^2 - 24x + 6 \quad \{\text{given}\}$$

$$\therefore 2a(2x - b)^{a-1} = 24x^2 - 24x + 6$$

$$\begin{aligned} \therefore a(2x - b)^{a-1} &= 12x^2 - 12x + 3 \\ &= 3(4x^2 - 4x + 1) \\ &= 3(2x - 1)^2 \end{aligned}$$

Solving by inspection:

$$\begin{aligned} \therefore a = 3 \quad 2x - b &= 2x - 1 \quad a - 1 = 2 \\ \therefore b &= 1 \quad \therefore a = 3 \end{aligned}$$

So,  $a = 3$  and  $b = 1$ .



**6**

$$y = \frac{a}{\sqrt{1+bx}}$$

$$\therefore y = au^{-\frac{1}{2}} \quad \text{where } u = 1+bx$$

$$\text{Now } \frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} \quad \{\text{chain rule}\}$$

$$= a\left(-\frac{1}{2}u^{-\frac{3}{2}}\right)(b)$$

$$= -\frac{1}{2}abu^{-\frac{3}{2}}$$

$$= -\frac{1}{2}ab(1+bx)^{-\frac{3}{2}}$$

$$\text{When } x=3, y=1, \text{ and } \frac{dy}{dx} = -\frac{1}{8}$$

$$\therefore 1 = \frac{a}{\sqrt{1+b(3)}} \quad \text{and} \quad -\frac{1}{8} = -\frac{1}{2}ab(1+b(3))^{-\frac{3}{2}}$$

$$\therefore a = \sqrt{1+3b} \quad \dots (*) \quad \therefore \frac{1}{4} = ab(1+3b)^{-\frac{3}{2}}$$

$$= \frac{\cancel{\sqrt{1+3b}}(b)}{\cancel{\sqrt{1+3b}}(1+3b)} \quad \{\text{using } (*)\}$$

$$\therefore \frac{1}{4} = \frac{b}{1+3b}$$

$$\therefore 1+3b = 4b$$

$$\therefore b = 1$$

$$\therefore a = \sqrt{1+3(1)} \quad \{\text{substituting } b=1 \text{ into } (*)\}$$

$$= 2$$

So,  $a=2$  and  $b=1$ .

**7**

$$y = f(x) = 3\left(ax - \frac{b}{x}\right)^3$$

$$\therefore y = 3u^3 \quad \text{where } u = ax - bx^{-1}$$

$$\text{Now } \frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} \quad \{\text{chain rule}\}$$

$$= 3(3u^2)(a - b(-1x^{-2}))$$

$$= 9u^2(a + bx^{-2})$$

$$= 9(ax - bx^{-1})^2(a + bx^{-2})$$

$$f\left(\frac{3}{2}\right) = 3 \quad \{\text{given}\}$$

$$\therefore 3\left(a\left(\frac{3}{2}\right) - \frac{b}{\frac{3}{2}}\right)^3 = 3$$

$$\therefore \left(\frac{3}{2}a - \frac{2}{3}b\right)^3 = 1$$

$$\therefore \frac{3}{2}a - \frac{2}{3}b = 1$$

$$\therefore \frac{3}{2}a = \frac{2}{3}b + 1$$

$$\therefore a = \frac{2}{3}\left(\frac{2}{3}b + 1\right)$$

$$\therefore a = \frac{4}{9}b + \frac{2}{3} \quad \dots (*)$$

Also,  $f'(\frac{3}{2}) = 30$

$$\therefore \frac{dy}{dx} = 30 \quad \text{when } x = \frac{3}{2}$$

$$\therefore 9\left(a\left(\frac{3}{2}\right) - b\left(\frac{3}{2}\right)^{-1}\right)^2 \left(a + b\left(\frac{3}{2}\right)^{-2}\right) = 30$$

$$\therefore \left(\frac{3}{2}a - \frac{2}{3}b\right)^2 \left(a + \frac{4}{9}b\right) = \frac{10}{3}$$

$$\therefore \left(\frac{3}{2}\left(\frac{4}{9}b + \frac{2}{3}\right) - \frac{2}{3}b\right)^2 \left(\left(\frac{4}{9}b + \frac{2}{3}\right) + \frac{4}{9}b\right) = \frac{10}{3} \quad \{\text{using } (*)\}$$

$$\therefore \left(\frac{2}{3}b + 1 - \frac{2}{3}b\right)^2 \left(\frac{8}{9}b + \frac{2}{3}\right) = \frac{10}{3}$$

$$\therefore \frac{8}{9}b + \frac{2}{3} = \frac{10}{3}$$

$$\therefore \frac{8}{9}b = \frac{8}{3}$$

$$\therefore b = 3$$

$$\therefore a = \frac{4}{9}(3) + \frac{2}{3} \quad \{\text{substituting } b = 3 \text{ into } (*)\}$$

$$= 2$$

So,  $a = 2$  and  $b = 3$ .

**8**  $y = x^3$  and  $x = y^{\frac{1}{3}}$

**a**  $\frac{dy}{dx} = 3x^2$  and  $\frac{dx}{dy} = \frac{1}{3}y^{-\frac{2}{3}}$

$$\begin{aligned} \frac{dy}{dx} \times \frac{dx}{dy} &= 3x^2 \left(\frac{1}{3}y^{-\frac{2}{3}}\right) \\ &= 3x^2 \left(\frac{1}{3}(x^3)^{-\frac{2}{3}}\right) \\ &= 3x^2 \left(\frac{1}{3}x^{-2}\right) \\ &= 3x^2 \times \frac{1}{3x^2} \\ &= 1 \quad \text{as required} \end{aligned}$$

**b** We know that

$$\frac{dy}{du} \frac{du}{dx} = \frac{dy}{dx} \quad \{\text{chain rule}\}$$

Letting  $x = y$ ,  $\frac{dy}{du} \frac{du}{dy} = \frac{dy}{dy}$

$$\therefore \frac{dy}{du} \frac{du}{dy} = 1$$

Letting  $u = x$ ,  $\frac{dy}{dx} \frac{dx}{dy} = 1$

## INVESTIGATION 3

## THE PRODUCT RULE

**1**  $u(x) = x$ ,  $v(x) = x$ ,  $f(x) = u(x)v(x) = x^2$

**a**  $f'(x) = 2x$

**b**  $u'(x) = 1$ ,  $v'(x) = 1$

**c**  $f'(x) \neq u'(x)v'(x)$

**2**  $u(x) = x$ ,  $v(x) = \sqrt{x} = x^{\frac{1}{2}}$ ,  $f(x) = x\sqrt{x} = x^{\frac{3}{2}}$

**a**  $f'(x) = \frac{3}{2}x^{\frac{1}{2}}$

$$= \frac{3}{2\sqrt{x}}$$

**b**  $u'(x) = 1$ ,  $v'(x) = \frac{1}{2}x^{-\frac{1}{2}}$

$$= \frac{1}{2\sqrt{x}}$$

**c**  $f'(x) \neq u'(x)v'(x)$

3	$f(x)$	$f'(x)$	$u(x)$	$v(x)$	$u'(x)$	$v'(x)$	$u'(x)v(x) + u(x)v'(x)$
	$x^2$	$2x$	$x$	$x$	1	1	$2x$
	$x^{\frac{3}{2}}$	$\frac{3}{2}\sqrt{x}$	$x$	$\sqrt{x}$	1	$\frac{1}{2\sqrt{x}}$	$\frac{3}{2}\sqrt{x}$
	$x(x+1)$	$2x+1$	$x$	$x+1$	1	1	$2x+1$
	$(x-1)(2-x^2)$	$-3x^2+2x+2$	$x-1$	$2-x^2$	1	$-2x$	$-3x^2+2x+2$

4 If  $f(x) = u(x)v(x)$  then  $f'(x) = u'(x)v(x) + u(x)v'(x)$ .

## EXERCISE 17C

1 a  $f(x) = x(x-1)$  is the product of  $u(x) = x$  and  $v(x) = x-1$   
 $\therefore u'(x) = 1$  and  $v'(x) = 1$

$$\begin{aligned}
 \text{Now } f'(x) &= u'(x)v(x) + u(x)v'(x) && \{\text{product rule}\} \\
 &= 1(x-1) + x(1) \\
 &= x-1+x \\
 &= 2x-1
 \end{aligned}$$

b  $f(x) = 2x(x+1)$  is the product of  $u(x) = 2x$  and  $v(x) = x+1$   
 $\therefore u'(x) = 2$  and  $v'(x) = 1$

$$\begin{aligned}
 \text{Now } f'(x) &= u'(x)v(x) + u(x)v'(x) && \{\text{product rule}\} \\
 &= 2(x+1) + 2x(1) \\
 &= 2x+2+2x \\
 &= 4x+2
 \end{aligned}$$

c  $f(x) = x^2\sqrt{x+1}$  is the product of  $u(x) = x^2$  and  $v(x) = (x+1)^{\frac{1}{2}}$   
 $\therefore u'(x) = 2x$  and  $v'(x) = \frac{1}{2}(x+1)^{-\frac{1}{2}}(1)$  {chain rule}  
 $= \frac{1}{2}(x+1)^{-\frac{1}{2}}$

$$\begin{aligned}
 \text{Now } f'(x) &= u'(x)v(x) + u(x)v'(x) && \{\text{product rule}\} \\
 &= 2x(x+1)^{\frac{1}{2}} + x^2\left(\frac{1}{2}(x+1)^{-\frac{1}{2}}\right) \\
 &= 2x(x+1)^{\frac{1}{2}} + \frac{1}{2}x^2(x+1)^{-\frac{1}{2}}
 \end{aligned}$$

d  $f(x) = (x+3)(x-1)$  is the product of  $u(x) = x+3$  and  $v(x) = x-1$   
 $\therefore u'(x) = 1$  and  $v'(x) = 1$

$$\begin{aligned}
 \text{Now } f'(x) &= u'(x)v(x) + u(x)v'(x) && \{\text{product rule}\} \\
 &= 1(x-1) + (x+3)(1) \\
 &= x-1+x+3 \\
 &= 2x+2
 \end{aligned}$$

**e**  $f(x) = x\sqrt{x^2 - 1}$  is the product of  $u(x) = x$  and  $v(x) = (x^2 - 1)^{\frac{1}{2}}$

$$\therefore u'(x) = 1 \quad \text{and} \quad v'(x) = \frac{1}{2}(x^2 - 1)^{-\frac{1}{2}}(2x) \quad \{\text{chain rule}\}$$

$$= x(x^2 - 1)^{-\frac{1}{2}}$$

Now  $f'(x) = u'(x)v(x) + u(x)v'(x) \quad \{\text{product rule}\}$

$$= 1(x^2 - 1)^{\frac{1}{2}} + x(x^2 - 1)^{-\frac{1}{2}}$$

$$= (x^2 - 1)^{\frac{1}{2}} + x^2(x^2 - 1)^{-\frac{1}{2}}$$

**f**  $f(x) = x(x + 1)^2$  is the product of  $u(x) = x$  and  $v(x) = (x + 1)^2$

$$\therefore u'(x) = 1 \quad \text{and} \quad v'(x) = 2(x + 1)(1) \quad \{\text{chain rule}\}$$

$$= 2x + 2$$

Now  $f'(x) = u'(x)v(x) + u(x)v'(x) \quad \{\text{product rule}\}$

$$= 1(x + 1)^2 + x(2x + 2)$$

$$= (x + 1)^2 + 2x(x + 1)$$

**2 a**  $y = x^2(2x - 1)$  is the product of  $u = x^2$  and  $v = 2x - 1$

$$\therefore u' = 2x \quad \text{and} \quad v' = 2$$

Now  $\frac{dy}{dx} = u'v + uv' \quad \{\text{product rule}\}$

$$= 2x(2x - 1) + x^2(2)$$

$$= 2x(2x - 1) + 2x^2$$

**b**  $y = 4x(2x + 1)^3$  is the product of  $u = 4x$  and  $v = (2x + 1)^3$

$$\therefore u' = 4 \quad \text{and} \quad v' = 3(2x + 1)^2(2) \quad \{\text{chain rule}\}$$

$$= 6(2x + 1)^2$$

Now  $\frac{dy}{dx} = u'v + uv' \quad \{\text{product rule}\}$

$$= 4(2x + 1)^3 + 4x(6(2x + 1)^2)$$

$$= 4(2x + 1)^3 + 24x(2x + 1)^2$$

**c**  $y = x^2\sqrt{3 - x}$  is the product of  $u = x^2$  and  $v = (3 - x)^{\frac{1}{2}}$

$$\therefore u' = 2x \quad \text{and} \quad v' = \frac{1}{2}(3 - x)^{-\frac{1}{2}}(-1) \quad \{\text{chain rule}\}$$

$$= -\frac{1}{2}(3 - x)^{-\frac{1}{2}}$$

Now  $\frac{dy}{dx} = u'v + uv' \quad \{\text{product rule}\}$

$$= 2x(3 - x)^{\frac{1}{2}} + x^2(-\frac{1}{2}(3 - x)^{-\frac{1}{2}})$$

$$= 2x(3 - x)^{\frac{1}{2}} - \frac{1}{2}x^2(3 - x)^{-\frac{1}{2}}$$

**d**  $y = \sqrt{x}(x - 3)^2$  is the product of  $u = x^{\frac{1}{2}}$  and  $v = (x - 3)^2$

$$\therefore u' = \frac{1}{2}x^{-\frac{1}{2}} \quad \text{and} \quad v' = 2(x - 3)(1) \quad \{\text{chain rule}\}$$

$$= 2(x - 3)$$

Now  $\frac{dy}{dx} = u'v + uv' \quad \{\text{product rule}\}$

$$= \frac{1}{2}x^{-\frac{1}{2}}(x - 3)^2 + x^{\frac{1}{2}}(2(x - 3))$$

$$= \frac{1}{2}x^{-\frac{1}{2}}(x - 3)^2 + 2\sqrt{x}(x - 3)$$



**e**  $y = 5x^2(3x^2 - 1)^2$  is the product of  $u = 5x^2$  and  $v = (3x^2 - 1)^2$   
 $\therefore u' = 5(2x)$  and  $v' = 2(3x^2 - 1)(3(2x))$  {chain rule}  
 $= 10x$   $= 12x(3x^2 - 1)$

Now  $\frac{dy}{dx} = u'v + uv'$  {product rule}  
 $= 10x(3x^2 - 1)^2 + 5x^2(12x(3x^2 - 1))$   
 $= 10x(3x^2 - 1)^2 + 60x^3(3x^2 - 1)$

**f**  $y = \sqrt{x}(x - x^2)^3$  is the product of  $u = x^{\frac{1}{2}}$  and  $v = (x - x^2)^3$   
 $\therefore u' = \frac{1}{2}x^{-\frac{1}{2}}$  and  $v' = 3(x - x^2)^2(1 - 2x)$  {chain rule}

Now  $\frac{dy}{dx} = u'v + uv'$  {product rule}  
 $= \frac{1}{2}x^{-\frac{1}{2}}(x - x^2)^3 + x^{\frac{1}{2}}(3(x - x^2)^2(1 - 2x))$   
 $= \frac{1}{2}x^{-\frac{1}{2}}(x - x^2)^3 + 3\sqrt{x}(x - x^2)^2(1 - 2x)$

**3 a**  $y = x^4(1 - 2x)^2$  is the product of  $u = x^4$  and  $v = (1 - 2x)^2$   
 $\therefore u' = 4x^3$  and  $v' = 2(1 - 2x)(-2)$  {chain rule}  
 $= -4(1 - 2x)$

Now  $\frac{dy}{dx} = u'v + uv'$  {product rule}  
 $= 4x^3(1 - 2x)^2 + x^4(-4(1 - 2x))$   
 $= 4x^3(1 - 2x)^2 - 4x^4(1 - 2x)$

At  $x = -1$ ,  $\frac{dy}{dx} = 4(-1)^3(1 - 2(-1))^2 - 4(-1)^4(1 - 2(-1))$   
 $= -4(9) - 4(3)$   
 $= -48$

$\therefore$  the gradient of the tangent to  $y = x^4(1 - 2x)^2$  at  $x = -1$  is  $-48$ .

**b**  $y = \sqrt{x}(x^2 - x + 1)^2$  is the product of  
 $u = x^{\frac{1}{2}}$  and  $v = (x^2 - x + 1)^2$   
 $\therefore u' = \frac{1}{2}x^{-\frac{1}{2}}$  and  $v' = 2(x^2 - x + 1)(2x - 1)$  {chain rule}

Now  $\frac{dy}{dx} = u'v + uv'$  {product rule}  
 $= \frac{1}{2}x^{-\frac{1}{2}}(x^2 - x + 1)^2 + x^{\frac{1}{2}}(2(x^2 - x + 1)(2x - 1))$

At  $x = 4$ ,  $\frac{dy}{dx} = \frac{1}{2}(4)^{-\frac{1}{2}}(4^2 - 4 + 1)^2 + 4^{\frac{1}{2}}(2(4^2 - 4 + 1)(2(4) - 1))$   
 $= \frac{1}{4}(169) + 2(2(13)(7))$   
 $= \frac{169}{4} + 364$   
 $= 406\frac{1}{4}$

$\therefore$  the gradient of the tangent to  $y = \sqrt{x}(x^2 - x + 1)^2$  at  $x = 4$  is  $406\frac{1}{4}$ .

**c**  $y = x\sqrt{1-2x}$  is the product of  $u = x$  and  $v = (1-2x)^{\frac{1}{2}}$

$$\begin{aligned}\therefore u' &= 1 \quad \text{and} \quad v' = \frac{1}{2}(1-2x)^{-\frac{1}{2}}(-2) && \{\text{chain rule}\} \\ &= -(1-2x)^{-\frac{1}{2}}\end{aligned}$$

$$\begin{aligned}\text{Now } \frac{dy}{dx} &= u'v + uv' && \{\text{product rule}\} \\ &= 1(1-2x)^{\frac{1}{2}} + x(-(1-2x)^{-\frac{1}{2}}) \\ &= (1-2x)^{\frac{1}{2}} - x(1-2x)^{-\frac{1}{2}}\end{aligned}$$

$$\begin{aligned}\text{At } x = -4, \quad \frac{dy}{dx} &= (1-2(-4))^{\frac{1}{2}} - (-4)(1-2(-4))^{-\frac{1}{2}} \\ &= 3 + 4\left(\frac{1}{3}\right) \\ &= \frac{13}{3}\end{aligned}$$

$\therefore$  the gradient of the tangent to  $y = x\sqrt{1-2x}$  at  $x = -4$  is  $\frac{13}{3}$ .

**d**  $y = x^3\sqrt{5-x^2}$  is the product of  $u = x^3$  and  $v = (5-x^2)^{\frac{1}{2}}$

$$\begin{aligned}\therefore u' &= 3x^2 \quad \text{and} \quad v' = \frac{1}{2}(5-x^2)^{-\frac{1}{2}}(-2x) && \{\text{chain rule}\} \\ &= -x(5-x^2)^{-\frac{1}{2}}\end{aligned}$$

$$\begin{aligned}\text{Now } \frac{dy}{dx} &= u'v + uv' && \{\text{product rule}\} \\ &= 3x^2(5-x^2)^{\frac{1}{2}} + x^3(-x(5-x^2)^{-\frac{1}{2}}) \\ &= 3x^2(5-x^2)^{\frac{1}{2}} - x^4(5-x^2)^{-\frac{1}{2}}\end{aligned}$$

$$\begin{aligned}\text{At } x = 1, \quad \frac{dy}{dx} &= 3(1)^2(5-1^2)^{\frac{1}{2}} - 1^4(5-1^2)^{-\frac{1}{2}} \\ &= 3(2) - 1\left(\frac{1}{2}\right) \\ &= \frac{11}{2}\end{aligned}$$

$\therefore$  the gradient of the tangent to  $y = x^3\sqrt{5-x^2}$  at  $x = 1$  is  $\frac{11}{2}$ .

**4 a**  $y = \sqrt{x}(3-x)^2$  is the product of  $u = x^{\frac{1}{2}}$  and  $v = (3-x)^2$

$$\begin{aligned}\therefore u' &= \frac{1}{2}x^{-\frac{1}{2}} \quad \text{and} \quad v' = 2(3-x)(-1) && \{\text{chain rule}\} \\ &= -2(3-x)\end{aligned}$$

$$\begin{aligned}\text{Now } \frac{dy}{dx} &= u'v + uv' && \{\text{product rule}\} \\ &= \frac{1}{2}x^{-\frac{1}{2}}(3-x)^2 + x^{\frac{1}{2}}(-2(3-x)) \\ &= \frac{1}{2}x^{-\frac{1}{2}}(3-x)^2 - 2x^{\frac{1}{2}}(3-x) \\ &= (3-x) \left[ \frac{1}{2\sqrt{x}}(3-x) - 2\sqrt{x} \right] \\ &= (3-x) \left[ \frac{3-x}{2\sqrt{x}} - 2\sqrt{x} \times \frac{2\sqrt{x}}{2\sqrt{x}} \right] \\ &= (3-x) \left( \frac{3-x-4x}{2\sqrt{x}} \right) \\ &= \frac{(3-x)(3-5x)}{2\sqrt{x}} \quad \text{as required}\end{aligned}$$

**b** The tangent is horizontal when its gradient is zero.

$$\therefore \frac{dy}{dx} = \frac{(3-x)(3-5x)}{2\sqrt{x}} = 0$$

$$\therefore (3-x)(3-5x) = 0$$

$$\therefore x = 3 \text{ or } \frac{3}{5}$$

**c**  $\frac{dy}{dx}$  is defined if its denominator is greater than zero.

$$\therefore 2\sqrt{x} > 0$$

$$\therefore x > 0$$

$\therefore$  the domain of  $\frac{dy}{dx}$  is  $\{x \mid x > 0\}$  and the domain of the original function is  $\{x \mid x \geq 0\}$ .

$\frac{dy}{dx}$  is undefined when  $x = 0$ .

**5**  $y = -2x^2(x+4)$  is the product of  $u = -2x^2$  and  $v = x+4$   
 $\therefore u' = -2(2x)$  and  $v' = 1$   
 $= -4x$

$$\begin{aligned} \text{Now } \frac{dy}{dx} &= u'v + uv' \quad \{\text{product rule}\} \\ &= -4x(x+4) + (-2x^2)(1) \\ &= -4x^2 - 16x - 2x^2 \\ &= -6x^2 - 16x \end{aligned}$$

$$\begin{aligned} \frac{dy}{dx} = 10 \quad \text{when} \quad -6x^2 - 16x &= 10 \\ \therefore 6x^2 + 16x + 10 &= 0 \\ \therefore 3x^2 + 8x + 5 &= 0 \\ \therefore (3x+5)(x+1) &= 0 \\ \therefore x = -1 \text{ or } -\frac{5}{3} \end{aligned}$$

**6**  $y = (x+3)(x-2)^2$  is the product of  $u = x+3$  and  $v = (x-2)^2$   
 $\therefore u' = 1$  and  $v' = 2(x-2)$   
 $= 2x-4$

$$\begin{aligned} \text{Now } \frac{dy}{dx} &= u'v + uv' \quad \{\text{product rule}\} \\ &= 1(x-2)^2 + (x+3)(2x-4) \\ &= x^2 - 4x + 4 + 2x^2 - 4x + 6x - 12 \\ &= 3x^2 - 2x - 8 \end{aligned}$$

$$\begin{aligned} \frac{dy}{dx} = -7 \quad \text{when} \quad 3x^2 - 2x - 8 &= -7 \\ \therefore 3x^2 - 2x - 1 &= 0 \\ \therefore (3x+1)(x-1) &= 0 \\ \therefore x = 1 \text{ or } x = -\frac{1}{3} \end{aligned}$$

**7**  $f(x) = ax\sqrt{1-x}$  is the product of  $u(x) = ax$  and  $v(x) = (1-x)^{\frac{1}{2}}$

$$\therefore u'(x) = a \quad \text{and} \quad v'(x) = \frac{1}{2}(1-x)^{-\frac{1}{2}}(-1) \quad \{\text{chain rule}\}$$

$$= -\frac{1}{2}(1-x)^{-\frac{1}{2}}$$

Now  $f'(x) = u'(x)v(x) + u(x)v'(x) \quad \{\text{product rule}\}$

$$= a(1-x)^{\frac{1}{2}} + ax\left(-\frac{1}{2}(1-x)^{-\frac{1}{2}}\right)$$

$$= a(1-x)^{\frac{1}{2}} - \frac{1}{2}ax(1-x)^{-\frac{1}{2}}$$

**a** The tangent to  $f(x) = ax\sqrt{1-x}$  has gradient 0 when  $f'(x) = 0$

$$\therefore a(1-x)^{\frac{1}{2}} - \frac{1}{2}ax(1-x)^{-\frac{1}{2}} = 0$$

$$\therefore a\sqrt{1-x} - \frac{ax}{2\sqrt{1-x}} = 0$$

$$\therefore a\sqrt{1-x} = \frac{ax}{2\sqrt{1-x}}$$

$$\therefore 2a(1-x) = ax$$

$$\therefore 2 - 2x = x$$

$$\therefore 3x = 2$$

$$\therefore x = \frac{2}{3}$$

**b** The tangent to  $f(x) = ax\sqrt{1-x}$  has gradient  $a$  when  $f'(x) = a$

$$\therefore a(1-x)^{\frac{1}{2}} - \frac{1}{2}ax(1-x)^{-\frac{1}{2}} = a$$

$$\therefore a\sqrt{1-x} - \frac{ax}{2\sqrt{1-x}} = a$$

$$\therefore \sqrt{1-x} - \frac{x}{2\sqrt{1-x}} = 1$$

$$\therefore \frac{2(1-x) - x}{2\sqrt{1-x}} = 1$$

$$\therefore 2 - 3x = 2\sqrt{1-x}$$

$$\therefore (2 - 3x)^2 = 4(1-x)$$

{squaring both sides assuming both sides are  $\geq 0$ }

$$\therefore 4 - 12x + 9x^2 = 4 - 4x$$

$$\therefore 9x^2 - 8x = 0$$

$$\therefore x(9x - 8) = 0$$

$$\therefore x = 0 \text{ or } \frac{8}{9}$$

*Check:* If  $x = 0$ ,  $f'(0) = a(1-0)^{\frac{1}{2}} - \frac{1}{2}a(0)(1-0)^{-\frac{1}{2}} = a \quad \checkmark$

If  $x = \frac{8}{9}$ ,  $f'\left(\frac{8}{9}\right) = a\left(1 - \frac{8}{9}\right)^{\frac{1}{2}} - \frac{1}{2}a\left(\frac{8}{9}\right)\left(1 - \frac{8}{9}\right)^{-\frac{1}{2}} = \frac{1}{3}a - \frac{4}{3}a = -a \quad \times$

So,  $x = 0$ .



$$\begin{aligned}
 \text{8 } f(x) = x^2 \sqrt{x^2 + a} \text{ is the product of } u(x) = x^2 \text{ and } v(x) = (x^2 + a)^{\frac{1}{2}} \\
 \therefore u'(x) = 2x \text{ and } v'(x) = \frac{1}{2}(x^2 + a)^{-\frac{1}{2}}(2x) \quad \{\text{chain rule}\} \\
 = x(x^2 + a)^{-\frac{1}{2}}
 \end{aligned}$$

$$\text{Now } f'(x) = u'(x)v(x) + u(x)v'(x) \quad \{\text{product rule}\}$$

$$\begin{aligned}
 &= 2x(x^2 + a)^{\frac{1}{2}} + x^2(x(x^2 + a)^{-\frac{1}{2}}) \\
 &= 2x\sqrt{x^2 + a} + \frac{x^3}{\sqrt{x^2 + a}}
 \end{aligned}$$

$$\text{and } f'(-2) = -\frac{34}{3}$$

$$\therefore 2(-2)\sqrt{(-2)^2 + a} + \frac{(-2)^3}{\sqrt{(-2)^2 + a}} = -\frac{34}{3}$$

$$\therefore -4\sqrt{a+4} - \frac{8}{\sqrt{a+4}} = -\frac{34}{3}$$

$$\therefore -4(a+4) - 8 = -\frac{34}{3}\sqrt{a+4}$$

$$\therefore -4(a+4) + \frac{34}{3}\sqrt{a+4} - 8 = 0$$

$$\therefore -12(a+4) + 34\sqrt{a+4} - 24 = 0$$

$$\therefore -12X^2 + 34X - 24 = 0 \quad \{\text{letting } X = \sqrt{a+4}\}$$

$$\therefore X = \frac{-34 \pm \sqrt{(34)^2 - 4(-12)(-24)}}{2(-12)}$$

$$= \frac{-34 \pm 2}{-24}$$

$$= \frac{-32}{-24} \text{ or } \frac{-36}{-24}$$

$$= \frac{4}{3} \text{ or } \frac{3}{2}$$

$$\therefore \sqrt{a+4} = \frac{4}{3} \quad \text{or} \quad \sqrt{a+4} = \frac{3}{2}$$

$$\therefore a+4 = \frac{16}{9} \quad \text{or} \quad a+4 = \frac{9}{4}$$

$$\therefore a = -\frac{20}{9} \quad \text{or} \quad a = -\frac{7}{4}$$

## EXERCISE 17D

$$\text{1 a } y = \frac{1+3x}{2-x} \text{ is a quotient with}$$

$$u = 1+3x \quad \text{and} \quad v = 2-x$$

$$\therefore u' = 3 \quad \text{and} \quad v' = -1$$

$$\text{Now } \frac{dy}{dx} = \frac{u'v - uv'}{v^2} \quad \{\text{quotient rule}\}$$

$$= \frac{3(2-x) - (1+3x)(-1)}{(2-x)^2}$$

$$= \frac{6-3x+1+3x}{(2-x)^2}$$

$$= \frac{7}{(2-x)^2}$$

$$\text{b } y = \frac{x^2}{2x+1} \text{ is a quotient with}$$

$$u = x^2 \quad \text{and} \quad v = 2x+1$$

$$\therefore u' = 2x \quad \text{and} \quad v' = 2$$

$$\text{Now } \frac{dy}{dx} = \frac{u'v - uv'}{v^2} \quad \{\text{quotient rule}\}$$

$$= \frac{2x(2x+1) - x^2(2)}{(2x+1)^2}$$

$$= \frac{4x^2 + 2x - 2x^2}{(2x+1)^2}$$

$$= \frac{2x^2 + 2x}{(2x+1)^2}$$

**c**  $y = \frac{x}{x^2 - 3}$  is a quotient with

$$u = x \quad \text{and} \quad v = x^2 - 3$$

$$\therefore u' = 1 \quad \text{and} \quad v' = 2x$$

$$\begin{aligned} \text{Now } \frac{dy}{dx} &= \frac{u'v - uv'}{v^2} \quad \{\text{quotient rule}\} \\ &= \frac{1(x^2 - 3) - x(2x)}{(x^2 - 3)^2} \\ &= \frac{x^2 - 3 - 2x^2}{(x^2 - 3)^2} \\ &= \frac{-x^2 - 3}{(x^2 - 3)^2} \end{aligned}$$

**e**  $y = \frac{x^2 - 3}{3x - x^2}$  is a quotient with

$$u = x^2 - 3 \quad \text{and} \quad v = 3x - x^2$$

$$\therefore u' = 2x \quad \text{and} \quad v' = 3 - 2x$$

$$\begin{aligned} \text{Now } \frac{dy}{dx} &= \frac{u'v - uv'}{v^2} \quad \{\text{quotient rule}\} \\ &= \frac{2x(3x - x^2) - (x^2 - 3)(3 - 2x)}{(3x - x^2)^2} \\ &= \frac{6x^2 - 2x^3 - (3x^2 - 2x^3 - 9 + 6x)}{(3x - x^2)^2} \\ &= \frac{3x^2 - 6x + 9}{(3x - x^2)^2} \end{aligned}$$

**f**  $y = \frac{x}{\sqrt{1-3x}}$  is a quotient with  $u = x$  and  $v = (1-3x)^{\frac{1}{2}}$

$$\therefore u' = 1 \quad \text{and} \quad v' = \frac{1}{2}(1-3x)^{-\frac{1}{2}}(-3) \quad \{\text{chain rule}\}$$

$$= -\frac{3}{2}(1-3x)^{-\frac{1}{2}}$$

$$\begin{aligned} \text{Now } \frac{dy}{dx} &= \frac{u'v - uv'}{v^2} \quad \{\text{quotient rule}\} \\ &= \frac{1(1-3x)^{\frac{1}{2}} - x(-\frac{3}{2}(1-3x)^{-\frac{1}{2}})}{(\sqrt{1-3x})^2} \\ &= \frac{(1-3x)^{\frac{1}{2}} + \frac{3}{2}x(1-3x)^{-\frac{1}{2}}}{1-3x} \\ &= \frac{\sqrt{1-3x} + \frac{3x}{2\sqrt{1-3x}}}{1-3x} \\ &= \frac{\sqrt{1-3x} \times \frac{2\sqrt{1-3x}}{2\sqrt{1-3x}} + \frac{3x}{2\sqrt{1-3x}}}{1-3x} \\ &= \frac{2(1-3x) + 3x}{2\sqrt{1-3x}(1-3x)} \\ &= \frac{2-3x}{2(1-3x)^{\frac{3}{2}}} \end{aligned}$$

**d**  $y = \frac{\sqrt{x}}{1-2x}$  is a quotient with

$$u = x^{\frac{1}{2}} \quad \text{and} \quad v = 1 - 2x$$

$$\therefore u' = \frac{1}{2}x^{-\frac{1}{2}} \quad \text{and} \quad v' = -2$$

$$\begin{aligned} \text{Now } \frac{dy}{dx} &= \frac{u'v - uv'}{v^2} \quad \{\text{quotient rule}\} \\ &= \frac{\frac{1}{2}x^{-\frac{1}{2}}(1-2x) - \sqrt{x}(-2)}{(1-2x)^2} \\ &= \frac{\frac{1-2x}{2\sqrt{x}} + 2\sqrt{x} \times \frac{2\sqrt{x}}{2\sqrt{x}}}{(1-2x)^2} \\ &= \frac{1-2x+4x}{2\sqrt{x}(1-2x)^2} \\ &= \frac{2x+1}{2\sqrt{x}(1-2x)^2} \end{aligned}$$

**2 a**  $\frac{x+1}{3-x}$  is a quotient with  $u = x+1$  and  $v = 3-x$   
 $\therefore u' = 1$  and  $v' = -1$

$$\begin{aligned}\text{Now } \frac{d}{dx} \left( \frac{x+1}{3-x} \right) &= \frac{u'v - uv'}{v^2} \quad \{\text{quotient rule}\} \\ &= \frac{1(3-x) - (x+1)(-1)}{(3-x)^2} \\ &= \frac{3 - \cancel{x} + \cancel{x} + 1}{(3-x)^2} \\ &= \frac{4}{(3-x)^2}\end{aligned}$$

**b**  $\frac{3x}{x^2-1}$  is a quotient with  $u = 3x$  and  $v = x^2-1$   
 $\therefore u' = 3$  and  $v' = 2x$

$$\begin{aligned}\text{Now } \frac{d}{dx} \left( \frac{3x}{x^2-1} \right) &= \frac{u'v - uv'}{v^2} \quad \{\text{quotient rule}\} \\ &= \frac{3(x^2-1) - (3x)(2x)}{(x^2-1)^2} \\ &= \frac{3x^2 - 3 - 6x^2}{(x^2-1)^2} \\ &= \frac{-3x^2 - 3}{(x^2-1)^2}\end{aligned}$$

**c**  $\frac{x^3}{2x-1}$  is a quotient with  $u = x^3$  and  $v = 2x-1$   
 $\therefore u' = 3x^2$  and  $v' = 2$

$$\begin{aligned}\text{Now } \frac{d}{dx} \left( \frac{x^3}{2x-1} \right) &= \frac{u'v - uv'}{v^2} \\ &= \frac{3x^2(2x-1) - x^3(2)}{(2x-1)^2} \\ &= \frac{6x^3 - 3x^2 - 2x^3}{(2x-1)^2} \\ &= \frac{4x^3 - 3x^2}{(2x-1)^2}\end{aligned}$$

**d**  $\frac{4x}{\sqrt{x-5}}$  is a quotient with  $u = 4x$  and  $v = (x-5)^{\frac{1}{2}}$

$$\therefore u' = 4 \quad \text{and} \quad v' = \frac{1}{2}(x-5)^{-\frac{1}{2}}(1) \quad \{\text{chain rule}\}$$

$$= \frac{1}{2}(x-5)^{-\frac{1}{2}}$$

Now  $\frac{d}{dx} \left( \frac{4x}{\sqrt{x-5}} \right) = \frac{u'v - uv'}{v^2}$

$$= \frac{4(x-5)^{\frac{1}{2}} - 4x(\frac{1}{2}(x-5)^{-\frac{1}{2}})}{\left( (x-5)^{\frac{1}{2}} \right)^2}$$

$$= \frac{4(x-5)^{\frac{1}{2}} - 2x(x-5)^{-\frac{1}{2}}}{x-5} \times \frac{(x-5)^{\frac{1}{2}}}{(x-5)^{\frac{1}{2}}}$$

$$= \frac{4(x-5) - 2x}{(x-5)^{\frac{3}{2}}}$$

$$= \frac{4x - 20 - 2x}{(x-5)^{\frac{3}{2}}}$$

$$= \frac{2x - 20}{(x-5)^{\frac{3}{2}}}$$

**e**  $\frac{\sqrt{x}}{3-x^2}$  is a quotient with  $u = x^{\frac{1}{2}}$  and  $v = 3-x^2$

$$\therefore u' = \frac{1}{2}x^{-\frac{1}{2}} \quad \text{and} \quad v' = -2x$$

Now  $\frac{d}{dx} \left( \frac{\sqrt{x}}{3-x^2} \right) = \frac{u'v - uv'}{v^2}$

$$= \frac{\frac{1}{2}x^{-\frac{1}{2}}(3-x^2) - x^{\frac{1}{2}}(-2x)}{(3-x^2)^2}$$

$$= \frac{\frac{3}{2}x^{-\frac{1}{2}} - \frac{1}{2}x^{\frac{3}{2}} + 2x^{\frac{3}{2}}}{(3-x^2)^2}$$

$$= \frac{\frac{3}{2\sqrt{x}} + \frac{3x\sqrt{x}}{2}}{(3-x^2)^2} \times \frac{2\sqrt{x}}{2\sqrt{x}}$$

$$= \frac{3x^2 + 3}{2\sqrt{x}(3-x^2)^2}$$



**f**  $-\frac{x^2}{\sqrt{x^2+3}}$  is a quotient with  $u = -x^2$  and  $v = (x^2+3)^{\frac{1}{2}}$

$$\therefore u' = -2x \quad \text{and} \quad v' = \frac{1}{2}(x^2+3)^{-\frac{1}{2}} \times (2x) \quad \{\text{chain rule}\}$$

$$= x(x^2+3)^{-\frac{1}{2}}$$

Now  $\frac{d}{dx} \left( -\frac{x^2}{\sqrt{x^2+3}} \right) = \frac{u'v - uv'}{v^2}$

$$= \frac{(-2x)(x^2+3)^{\frac{1}{2}} - (-x^2)(x(x^2+3)^{-\frac{1}{2}})}{\left( (x^2+3)^{\frac{1}{2}} \right)^2}$$

$$= \frac{(-2x)\sqrt{x^2+3} + \frac{x^3}{\sqrt{x^2+3}}}{x^2+3} \times \frac{\sqrt{x^2+3}}{\sqrt{x^2+3}}$$

$$= \frac{(-2x)(x^2+3) + x^3}{(x^2+3)^{\frac{3}{2}}}$$

$$= \frac{-2x^3 - 6x + x^3}{(x^2+3)^{\frac{3}{2}}}$$

$$= \frac{-x^3 - 6x}{(x^2+3)^{\frac{3}{2}}}$$

**3 a**  $y = \frac{x}{1-2x}$  is a quotient with

$$u = x \quad \text{and} \quad v = 1-2x$$

$$\therefore u' = 1 \quad \text{and} \quad v' = -2$$

Now  $\frac{dy}{dx} = \frac{u'v - uv'}{v^2} \quad \{\text{quotient rule}\}$

$$= \frac{1(1-2x) - x(-2)}{(1-2x)^2}$$

$$= \frac{1}{(1-2x)^2}$$

At  $x = 1$ ,  $\frac{dy}{dx} = \frac{1}{(1-2(1))^2}$

$$= \frac{1}{(-1)^2}$$

$$= 1$$

$\therefore$  the gradient of the tangent = 1

**b**  $y = \frac{x^3}{x^2+1}$  is a quotient with

$$u = x^3 \quad \text{and} \quad v = x^2+1$$

$$\therefore u' = 3x^2 \quad \text{and} \quad v' = 2x$$

Now  $\frac{dy}{dx} = \frac{u'v - uv'}{v^2} \quad \{\text{quotient rule}\}$

$$= \frac{3x^2(x^2+1) - x^3(2x)}{(x^2+1)^2}$$

$$= \frac{3x^4 + 3x^2 - 2x^4}{(x^2+1)^2}$$

$$= \frac{x^4 + 3x^2}{(x^2+1)^2}$$

At  $x = -1$ ,  $\frac{dy}{dx} = \frac{(-1)^4 + 3(-1)^2}{((-1)^2+1)^2}$

$$= \frac{4}{4}$$

$$= 1$$

$\therefore$  the gradient of the tangent = 1

$$\begin{aligned}
 \text{c } y &= \frac{\sqrt{x}}{2x+1} \text{ is a quotient with} \\
 u &= x^{\frac{1}{2}} \quad \text{and} \quad v = 2x+1 \\
 \therefore u' &= \frac{1}{2}x^{-\frac{1}{2}} \quad \text{and} \quad v' = 2 \\
 \text{Now } \frac{dy}{dx} &= \frac{u'v - uv'}{v^2} \quad \{\text{quotient rule}\} \\
 &= \frac{\frac{1}{2\sqrt{x}}(2x+1) - \sqrt{x}(2)}{(2x+1)^2}
 \end{aligned}$$

$$\begin{aligned}
 \text{At } x = 4, \quad \frac{dy}{dx} &= \frac{\frac{1}{2\sqrt{4}}(2(4)+1) - 2\sqrt{4}}{(2(4)+1)^2} \\
 &= \frac{\frac{9}{4} - 4}{81} \\
 &= \frac{-\frac{7}{4}}{81} \\
 &= -\frac{7}{324}
 \end{aligned}$$

$$\therefore \text{ the gradient of the tangent} = -\frac{7}{324}$$

$$\begin{aligned}
 \text{d } y &= \frac{x^2}{\sqrt{x^2+5}} \text{ is a quotient with} \\
 u &= x^2 \quad \text{and} \quad v = (x^2+5)^{\frac{1}{2}} \\
 \therefore u' &= 2x \quad \text{and} \quad v' = \frac{1}{2}(x^2+5)^{-\frac{1}{2}}(2x) \\
 &= x(x^2+5)^{-\frac{1}{2}} \\
 \text{Now } \frac{dy}{dx} &= \frac{u'v - uv'}{v^2} \quad \{\text{quotient rule}\} \\
 &= \frac{2x\sqrt{x^2+5} - x^2\left(\frac{x}{\sqrt{x^2+5}}\right)}{(\sqrt{x^2+5})^2} \\
 &= \frac{2x\sqrt{x^2+5} - \frac{x^3}{\sqrt{x^2+5}}}{x^2+5}
 \end{aligned}$$

$$\begin{aligned}
 \text{At } x = -2, \\
 \frac{dy}{dx} &= \frac{2(-2)\sqrt{(-2)^2+5} - \frac{(-2)^3}{\sqrt{(-2)^2+5}}}{(-2)^2+5} \\
 &= \frac{-4(3) - \frac{-8}{3}}{9} \\
 &= \frac{-\frac{28}{3}}{9} \\
 &= -\frac{28}{27}
 \end{aligned}$$

$$\therefore \text{ the gradient of the tangent} = -\frac{28}{27}$$

$$\begin{aligned}
 \text{4 } f(x) &= \frac{x}{\sqrt{x-1}} \text{ is a quotient with} \\
 u(x) &= x \quad \text{and} \quad v(x) = (x-1)^{\frac{1}{2}} \\
 \therefore u'(x) &= 1 \quad \text{and} \quad v'(x) = \frac{1}{2}(x-1)^{-\frac{1}{2}}(1) \quad \{\text{chain rule}\} \\
 &= \frac{1}{2\sqrt{x-1}}
 \end{aligned}$$

$$\begin{aligned}
 \text{Now } f'(x) &= \frac{u'(x)v(x) + u(x)v'(x)}{[v(x)]^2} \quad \{\text{quotient rule}\} \\
 &= \frac{1\sqrt{x-1} - x\left(\frac{1}{2\sqrt{x-1}}\right)}{(\sqrt{x-1})^2} \\
 &= \frac{\sqrt{x-1} - \frac{x}{2\sqrt{x-1}}}{x-1} \\
 &= \frac{\sqrt{x-1} \times \frac{2\sqrt{x-1}}{2\sqrt{x-1}} - \frac{x}{2\sqrt{x-1}}}{x-1} \\
 &= \frac{2(x-1) - x}{2\sqrt{x-1}(x-1)} \\
 &= \frac{x-2}{2(x-1)^{\frac{3}{2}}}
 \end{aligned}$$

Check:  $f(x) = \frac{x}{\sqrt{x-1}}$

$$= \frac{x-1}{\sqrt{x-1}} + \frac{1}{\sqrt{x-1}}$$

$$= (x-1)^{\frac{1}{2}} + (x-1)^{-\frac{1}{2}}$$

$$\therefore f'(x) = \frac{1}{2}(x-1)^{-\frac{1}{2}}(1) + \left(-\frac{1}{2}(x-1)^{-\frac{3}{2}}(1)\right) \quad \{\text{chain rule}\}$$

$$= \frac{1}{2\sqrt{x-1}} - \frac{1}{2(x-1)^{\frac{3}{2}}}$$

$$= \frac{1}{2\sqrt{x-1}} \times \frac{(x-1)}{(x-1)} - \frac{1}{2(x-1)^{\frac{3}{2}}}$$

$$= \frac{x-1-1}{2(x-1)^{\frac{3}{2}}}$$

$$= \frac{x-2}{2(x-1)^{\frac{3}{2}}} \quad \checkmark$$

- 5 a**  $y = \frac{2x+3}{x+1}$  is a quotient with  $u = 2x+3$  and  $v = x+1$   
 $\therefore u' = 2$  and  $v' = 1$

Now  $\frac{dy}{dx} = \frac{u'v - uv'}{v^2} \quad \{\text{quotient rule}\}$

$$= \frac{2(x+1) - (2x+3)(1)}{(x+1)^2}$$

$$= \frac{2x+2-2x-3}{(x+1)^2}$$

$$= -\frac{1}{(x+1)^2}$$

- b** The illustrated tangents to the graph of  $y = \frac{2x+3}{x+1}$  meet the graph at the points where  $x = -2$  and  $x = 0$ .

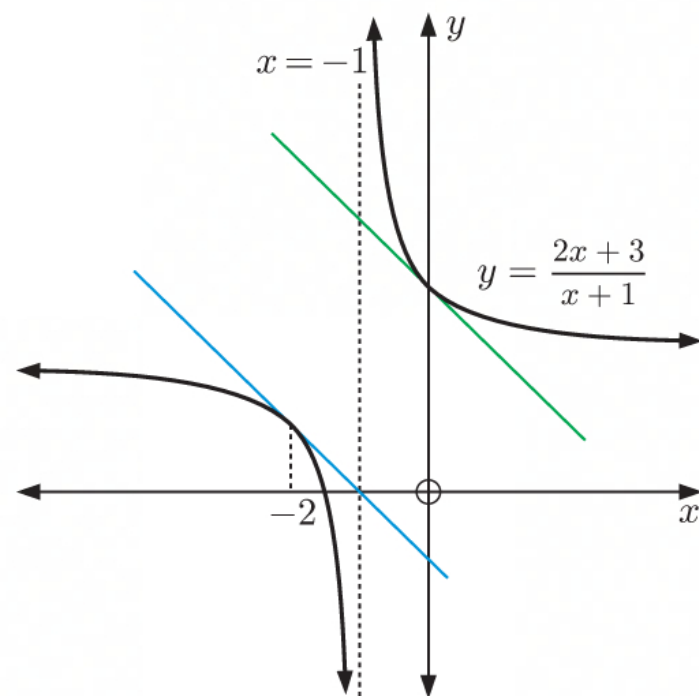
Now  $\frac{dy}{dx}$  is the gradient of the tangent to the graph at any point.

At  $x = -2$ ,  $\frac{dy}{dx} = -\frac{1}{(-2+1)^2} = -1$

At  $x = 0$ ,  $\frac{dy}{dx} = -\frac{1}{(0+1)^2} = -1$

So, the gradients of the two tangents to the graph of  $y = \frac{2x+3}{x+1}$  are equal.

$\therefore$  the tangents are parallel.



**6 a**  $y = \frac{2\sqrt{x}}{1-x}$  is a quotient with  $u = 2x^{\frac{1}{2}}$  and  $v = 1 - x$   
 $\therefore u' = x^{-\frac{1}{2}}$  and  $v' = -1$

Now  $\frac{dy}{dx} = \frac{u'v - uv'}{v^2}$  {quotient rule}  
 $= \frac{\frac{1}{\sqrt{x}}(1-x) - 2\sqrt{x}(-1)}{(1-x)^2} \times \left( \frac{\sqrt{x}}{\sqrt{x}} \right)$   
 $= \frac{(1-x) + 2x}{\sqrt{x}(1-x)^2}$   
 $= \frac{x+1}{\sqrt{x}(1-x)^2}$  as required

**b i**  $\frac{dy}{dx} = 0$  when  $x+1=0 \therefore x=-1$ .

However  $\frac{dy}{dx}$  is not defined for  $x \leq 0$  because of the  $\sqrt{x}$  term. Hence  $\frac{dy}{dx}$  never equals 0.

**ii**  $\frac{dy}{dx}$  is undefined when  $x < 0$  and when  $\sqrt{x}(1-x)^2 = 0$   
 $\therefore$  when  $x \leq 0$  and when  $x = 1$

**7 a**  $y = \frac{x^2+6}{2x+1}$  is a quotient with  $u = x^2+6$  and  $v = 2x+1$   
 $\therefore u' = 2x$  and  $v' = 2$

Now  $\frac{dy}{dx} = \frac{u'v - uv'}{v^2}$  {quotient rule}  
 $= \frac{2x(2x+1) - (x^2+6)(2)}{(2x+1)^2}$   
 $= \frac{4x^2 + 2x - 2x^2 - 12}{(2x+1)^2}$   
 $= \frac{2x^2 + 2x - 12}{(2x+1)^2}$  as required

**b i**  $\frac{dy}{dx} = 0$  when  $2x^2 + 2x - 12 = 0$   
 $\therefore x^2 + x - 6 = 0$   
 $\therefore (x+3)(x-2) = 0$   
 $\therefore x = -3$  or  $x = 2$

**ii**  $\frac{dy}{dx}$  is undefined when  $(2x+1)^2 = 0$   
 $\therefore 2x+1 = 0$   
 $\therefore x = -\frac{1}{2}$



**8 a**  $y = \frac{x^2 - 3x + 1}{x + 2}$  is a quotient with  $u = x^2 - 3x + 1$  and  $v = x + 2$   
 $\therefore u' = 2x - 3$  and  $v' = 1$

Now  $\frac{dy}{dx} = \frac{u'v - uv'}{v^2}$  {quotient rule}  
 $= \frac{(2x - 3)(x + 2) - (x^2 - 3x + 1)(1)}{(x + 2)^2}$   
 $= \frac{2x^2 + 4x - 3x - 6 - x^2 + 3x - 1}{(x + 2)^2}$   
 $= \frac{x^2 + 4x - 7}{(x + 2)^2}$  as required

**b i**  $\frac{dy}{dx} = 0$  when  $x^2 + 4x - 7 = 0$   
 $\therefore x = \frac{-4 \pm \sqrt{4^2 - 4(1)(-7)}}{2(1)}$   
 $= \frac{-4 \pm \sqrt{44}}{2}$   
 $= -2 \pm \sqrt{11}$

**ii**  $\frac{dy}{dx}$  is undefined when  $(x + 2)^2 = 0$   
 $\therefore x = -2$

**INVESTIGATION 4****THE DERIVATIVE OF  $b^x$** 

**1**  $y = 2^x$

$x$	$y$	$\frac{dy}{dx}$	$\frac{dy}{dx} \div y$
0	1	0.6931	0.6931
0.5	1.4142	0.9803	0.6931
1	2	1.3863	0.6931
1.5	2.8284	1.9605	0.6931
2	4	2.7726	0.6931

**2 a**  $y = 3^x$

$x$	$y$	$\frac{dy}{dx}$	$\frac{dy}{dx} \div y$
0	1	1.0986	1.0986
0.5	1.7321	1.9029	1.0986
1	3	3.2958	1.0986
1.5	5.1962	5.7086	1.0986
2	9	9.8875	1.0986

**b**  $y = 5^x$

$x$	$y$	$\frac{dy}{dx}$	$\frac{dy}{dx} \div y$
0	1	1.6094	1.6094
0.5	2.2361	3.5988	1.6094
1	5	8.0472	1.6094
1.5	11.1803	17.9941	1.6094
2	25	40.2359	1.6094

**c**  $y = (0.5)^x$

$x$	$y$	$\frac{dy}{dx}$	$\frac{dy}{dx} \div y$
0	1	-0.6931	-0.6931
0.5	0.7071	-0.4901	-0.6931
1	0.5	-0.3466	-0.6931
1.5	0.3536	-0.2451	-0.6931
2	0.25	-0.1733	-0.6931

**3** From **2 a**, **b**, and **c**, we can see that  $\frac{dy}{dx} \div y$  is always equal to the value of  $\frac{dy}{dx}$  at  $x = 0$ .

So, if  $f(x) = b^x$ , then  $\frac{f'(x)}{b^x} = f'(0)$   
 $\therefore f'(x) = f'(0) \times b^x$

## EXERCISE 17E.1

**1 a** If  $f(x) = e^{4x}$   
 then  $f'(x) = e^{4x}(4)$   
 $= 4e^{4x}$

**c** If  $f(x) = e^{-2x}$   
 then  $f'(x) = e^{-2x}(-2)$   
 $= -2e^{-2x}$

**e** If  $f(x) = 2e^{-\frac{x}{2}}$   
 then  $f'(x) = 2e^{-\frac{x}{2}}(-\frac{1}{2})$   
 $= -e^{-\frac{x}{2}}$

**g** If  $f(x) = 4e^{\frac{x}{2}} - 3e^{-x}$   
 then  $f'(x) = 4e^{\frac{x}{2}}(\frac{1}{2}) - 3e^{-x}(-1)$   
 {addition rule}  
 $= 2e^{\frac{x}{2}} + 3e^{-x}$

**i** If  $f(x) = e^{-x^2}$   
 then  $f'(x) = e^{-x^2}(-2x)$   
 $= -2xe^{-x^2}$

**b** If  $f(x) = e^x + 3$   
 then  $f'(x) = e^x + 0$  {addition rule}  
 $= e^x$

**d** If  $f(x) = e^{\frac{x}{2}}$   
 then  $f'(x) = e^{\frac{x}{2}}(\frac{1}{2})$   
 $= \frac{1}{2}e^{\frac{x}{2}}$

**f** If  $f(x) = 1 - 2e^{-x}$   
 then  $f'(x) = 0 - 2e^{-x}(-1)$   
 {addition rule}  
 $= 2e^{-x}$

**h** If  $f(x) = \frac{e^x + e^{-x}}{2}$   
 $= \frac{1}{2}e^x + \frac{1}{2}e^{-x}$   
 then  $f'(x) = \frac{1}{2}e^x + \frac{1}{2}e^{-x}(-1)$   
 $= \frac{e^x - e^{-x}}{2}$

**j** If  $f(x) = e^{\frac{1}{x}}$   
 then  $f'(x) = e^{\frac{1}{x}}(-\frac{1}{x^2})$   
 $= -\frac{e^{\frac{1}{x}}}{x^2}$

$$\begin{aligned}
 \mathbf{k} \quad & \text{If } f(x) = 10(1 + e^{2x}) \\
 & = 10 + 10e^{2x} \\
 \text{then } & f'(x) = 0 + 10e^{2x}(2) \\
 & \quad \{\text{addition rule}\} \\
 & = 20e^{2x}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{m} \quad & \text{If } f(x) = e^{2x+1} \\
 \text{then } & f'(x) = e^{2x+1}(2) \\
 & = 2e^{2x+1}
 \end{aligned}$$

$$\begin{aligned}
 \circ \quad & \text{If } f(x) = e^{1-2x^2} \\
 \text{then } & f'(x) = e^{1-2x^2}(-4x) \\
 & = -4xe^{1-2x^2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{2} \quad \mathbf{a} \quad & \text{If } f(x) = xe^x \\
 \text{then } & f'(x) = 1e^x + xe^x \\
 & \quad \{\text{product rule}\} \\
 & = e^x + xe^x
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & \text{If } f(x) = \frac{e^x}{x} \\
 \text{then } & f'(x) = \frac{e^x x - e^x(1)}{x^2} \\
 & \quad \{\text{quotient rule}\} \\
 & = \frac{xe^x - e^x}{x^2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{e} \quad & \text{If } f(x) = x^2 e^{3x} \\
 \text{then } & f'(x) = 2xe^{3x} + x^2 e^{3x}(3) \\
 & \quad \{\text{product rule}\} \\
 & = 2xe^{3x} + 3x^2 e^{3x}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{g} \quad & \text{If } f(x) = 20xe^{-0.5x} \\
 \text{then } & f'(x) = 20e^{-0.5x} + 20xe^{-0.5x}(-0.5) \quad \{\text{product rule}\} \\
 & = 20e^{-0.5x} - 10xe^{-0.5x}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{l} \quad & \text{If } f(x) = 20(1 - e^{-2x}) \\
 & = 20 - 20e^{-2x} \\
 \text{then } & f'(x) = 0 - 20e^{-2x}(-2) \\
 & \quad \{\text{addition rule}\} \\
 & = 40e^{-2x}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{n} \quad & \text{If } f(x) = e^{\frac{x}{4}} \\
 \text{then } & f'(x) = e^{\frac{x}{4}}\left(\frac{1}{4}\right) \\
 & = \frac{1}{4}e^{\frac{x}{4}}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{p} \quad & \text{If } f(x) = e^{-0.02x} \\
 \text{then } & f'(x) = e^{-0.02x} \times (-0.02) \\
 & = -0.02e^{-0.02x}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \text{If } f(x) = x^3 e^{-x} \\
 \text{then } & f'(x) = 3x^2 e^{-x} + x^3(e^{-x})(-1) \\
 & \quad \{\text{product rule}\} \\
 & = 3x^2 e^{-x} - x^3 e^{-x}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad & \text{If } f(x) = \frac{x}{e^x} \\
 \text{then } & f'(x) = \frac{1e^x - xe^x}{(e^x)^2} \\
 & \quad \{\text{quotient rule}\} \\
 & = \frac{e^x(1-x)}{(e^x)^2} \\
 & = \frac{1-x}{e^x}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{f} \quad & \text{If } f(x) = \frac{e^x}{\sqrt{x}} \\
 \text{then } & f'(x) = \frac{e^x \sqrt{x} - \frac{e^x}{2\sqrt{x}}}{(\sqrt{x})^2} \\
 & \quad \{\text{quotient rule}\} \\
 & = \frac{e^x \sqrt{x} \times \left(\frac{\sqrt{x}}{\sqrt{x}}\right) - \frac{e^x}{2\sqrt{x}}}{x} \\
 & = \frac{xe^x - \frac{1}{2}e^x}{x\sqrt{x}}
 \end{aligned}$$

**h** If  $f(x) = \frac{e^x + 2}{e^{-x} + 1}$   
 then  $f'(x) = \frac{e^x(e^{-x} + 1) - (e^x + 2)(e^{-x})(-1)}{(e^{-x} + 1)^2}$  {quotient rule}  

$$= \frac{1 + e^x + 1 + 2e^{-x}}{(e^{-x} + 1)^2}$$
  

$$= \frac{e^x + 2 + 2e^{-x}}{(e^{-x} + 1)^2}$$

**3 a**  $y = (2 + e^x)^4$   
 $= u^4$  where  $u = 2 + e^x$   
 $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$  {chain rule}  
 $= 4u^3 \frac{du}{dx}$   
 $= 4(2 + e^x)^3 \times e^x$   
 $= 4e^x(2 + e^x)^3$

**c**  $y = (e^x + e^{-x})^{\frac{3}{2}}$   
 $= u^{\frac{3}{2}}$  where  $u = e^x + e^{-x}$   
 $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$  {chain rule}  
 $= \frac{3}{2}u^{\frac{1}{2}} \frac{du}{dx}$   
 $= \frac{3}{2}(e^x + e^{-x})^{\frac{1}{2}} \times (e^x + e^{-x}(-1))$   
 $= \frac{3}{2}(e^x + e^{-x})^{\frac{1}{2}}(e^x - e^{-x})$

**4 a**  $y = (e^x + 2)^4$   
 $= u^4$  where  $u = e^x + 2$   
 $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$  {chain rule}  
 $= 4u^3 \frac{du}{dx}$   
 $= 4(e^x + 2)^3 \times e^x$   
 $= 4e^x(e^x + 2)^3$   
 At  $x = 0$ ,  $\frac{dy}{dx} = 4e^0(e^0 + 2)^3$   
 $= 108$   
 $\therefore$  gradient of tangent = 108

**b**  $y = \sqrt{e^x - 1}$   
 $= u^{\frac{1}{2}}$  where  $u = e^x - 1$   
 $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$  {chain rule}  
 $= \frac{1}{2}u^{-\frac{1}{2}} \frac{du}{dx}$   
 $= \frac{1}{2}(e^x - 1)^{-\frac{1}{2}} \times e^x$   
 $= \frac{e^x}{2\sqrt{e^x - 1}}$

**d**  $y = \frac{1}{\sqrt{e^{2x} + 2}}$   
 $= u^{-\frac{1}{2}}$  where  $u = e^{2x} + 2$   
 $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$  {chain rule}  
 $= -\frac{1}{2}u^{-\frac{3}{2}} \frac{du}{dx}$   
 $= -\frac{1}{2}(e^{2x} + 2)^{-\frac{3}{2}} \times e^{2x}(2)$   
 $= -e^{2x}(e^{2x} + 2)^{-\frac{3}{2}}$

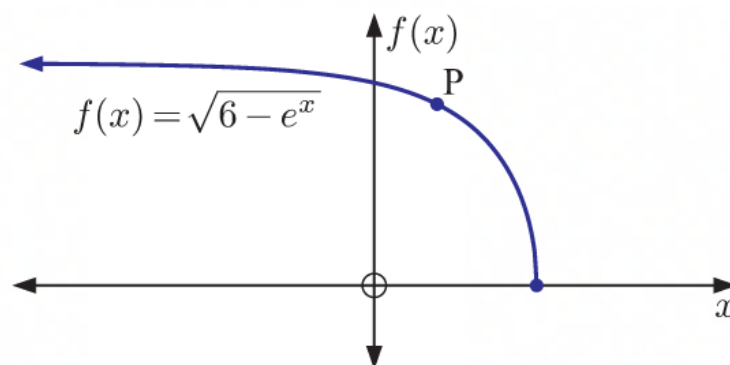
**b**  $y = \frac{1}{2 - e^{-x}}$   
 $= u^{-1}$  where  $u = 2 - e^{-x}$   
 $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$  {chain rule}  
 $= -u^{-2} \frac{du}{dx}$   
 $= -\frac{1}{(2 - e^{-x})^2} \times (-e^{-x}(-1))$   
 $= -\frac{e^{-x}}{(2 - e^{-x})^2}$   
 At  $x = 0$ ,  $\frac{dy}{dx} = -\frac{e^0}{(2 - e^0)^2} = -1$   
 $\therefore$  gradient of tangent = -1



$$\begin{aligned}
 \text{c} \quad y &= \sqrt{e^{2x} + 10} \\
 &= u^{\frac{1}{2}} \quad \text{where } u = e^{2x} + 10 \\
 \frac{dy}{dx} &= \frac{dy}{du} \frac{du}{dx} \quad \{\text{chain rule}\} \\
 &= \frac{1}{2} u^{-\frac{1}{2}} \frac{du}{dx} \\
 &= \frac{1}{2\sqrt{e^{2x} + 10}} \times e^{2x}(2) \\
 &= \frac{e^{2x}}{\sqrt{e^{2x} + 10}} \\
 \text{At } x = \ln 3, \quad \frac{dy}{dx} &= \frac{e^{2 \ln 3}}{\sqrt{e^{2 \ln 3} + 10}} \\
 &= \frac{e^{\ln 3^2}}{\sqrt{e^{\ln 3^2} + 10}} \\
 &= \frac{3^2}{\sqrt{3^2 + 10}} = \frac{9}{\sqrt{19}} \\
 \therefore \text{gradient of tangent} &= \frac{9}{\sqrt{19}}
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad y &= \frac{2-x}{e^{3x}} \\
 \therefore \frac{dy}{dx} &= \frac{(-1)e^{3x} - (2-x)e^{3x}(3)}{(e^{3x})^2} \quad \{\text{quotient rule}\} \\
 &= \frac{-e^{3x} - 3(2-x)e^{3x}}{e^{6x}} \\
 \text{At } x = 1, \quad \frac{dy}{dx} &= \frac{-e^{3(1)} - 3(2-1)e^{3(1)}}{e^{6(1)}} \\
 &= \frac{-e^3 - 3e^3}{e^6} \\
 &= \frac{-4e^3}{e^6} \\
 &= -\frac{4}{e^3} \\
 \therefore \text{gradient of tangent} &= -\frac{4}{e^3}
 \end{aligned}$$

$$\begin{aligned}
 5 \quad \text{a} \quad f(x) &= \sqrt{6 - e^x} \text{ is defined when} \\
 6 - e^x &\geq 0 \\
 \therefore e^x &\leq 6 \\
 \therefore x &\leq \ln 6 \\
 \text{So, the domain is } &\{x \mid x \leq \ln 6\}.
 \end{aligned}$$



$$\begin{aligned}
 \text{b} \quad \text{i} \quad \text{P has } y\text{-coordinate } 2. \\
 \therefore 2 &= \sqrt{6 - e^x} \\
 \therefore 6 - e^x &= 4 \\
 \therefore e^x &= 2 \\
 \therefore x &= \ln 2 \\
 \text{So, P is at } &(\ln 2, 2).
 \end{aligned}$$

$$\begin{aligned}
 \text{ii} \quad f(x) &= \sqrt{6 - e^x} = (6 - e^x)^{\frac{1}{2}} \\
 \therefore f'(x) &= \frac{1}{2}(6 - e^x)^{-\frac{1}{2}} \times (-e^x) \quad \{\text{chain rule}\} \\
 &= -\frac{1}{2}e^x(6 - e^x)^{-\frac{1}{2}} \\
 \text{Now } f'(\ln 2) &= -\frac{1}{2}e^{\ln 2}(6 - e^{\ln 2})^{-\frac{1}{2}} \\
 &= -\frac{1}{2} \times 2 \times \frac{1}{\sqrt{6-2}} \\
 &= -1 \times \frac{1}{\sqrt{4}} \\
 &= -1 \times \frac{1}{2} \\
 &= -\frac{1}{2} \\
 \therefore \text{the gradient of the tangent at P} &= -\frac{1}{2}.
 \end{aligned}$$

$$\begin{aligned}
 6 \quad f(x) &= e^{kx} + x \quad \therefore f'(x) = ke^{kx} + 1 \\
 \text{Now } f'(0) &= -8, \text{ so } ke^0 + 1 = -8 \\
 \therefore k &= -9
 \end{aligned}$$

$$\begin{aligned}
 7 \quad a \quad y &= 2^x \\
 &= (e^{\ln 2})^x \\
 &= e^{x \ln 2} \\
 \therefore \frac{dy}{dx} &= e^{x \ln 2} \times \ln 2 \\
 &= e^{\ln 2^x} \times \ln 2 \\
 &= 2^x \ln 2
 \end{aligned}$$

$$\begin{aligned}
 b \quad y &= a^x \\
 &= (e^{\ln a})^x \\
 &= e^{x \ln a} \\
 \therefore \frac{dy}{dx} &= e^{x \ln a} \times \ln a \\
 &= e^{\ln a^x} \times \ln a \\
 &= a^x \times \ln a
 \end{aligned}$$

$$\begin{aligned}
 8 \quad f(x) &= x^2 e^{-x} \text{ is the product of } u(x) = x^2 \text{ and } v(x) = e^{-x} \\
 \therefore u'(x) &= 2x \text{ and } v'(x) = -e^{-x}
 \end{aligned}$$

$$\begin{aligned}
 \therefore f'(x) &= u'(x)v(x) + u(x)v'(x) \quad \{\text{product rule}\} \\
 &= 2x(e^{-x}) + x^2(-e^{-x}) \\
 &= 2xe^{-x} - x^2e^{-x}
 \end{aligned}$$

The tangent to  $f(x) = x^2 e^{-x}$  is horizontal at point P.

$\therefore$  the gradient of the tangent at point P is zero.

$\therefore f'(x) = 0$  at point P.

$$\begin{aligned}
 \text{Now } f'(x) &= 0 \text{ when } 2xe^{-x} - x^2e^{-x} = 0 \\
 &\therefore xe^{-x}(2 - x) = 0 \\
 &\therefore x = 0 \text{ or } 2 - x = 0 \quad \{\text{as } e^{-x} > 0 \text{ for all } x\} \\
 &\therefore x = 0 \text{ or } x = 2
 \end{aligned}$$

$$\begin{aligned}
 f(0) &= 0^2 e^0 \text{ and } f(2) = 2^2 e^{-2} \\
 &= 0 \qquad \qquad \qquad = 4e^{-2} \\
 &\qquad \qquad \qquad = \frac{4}{e^2}
 \end{aligned}$$

So, the possible coordinates of P are  $(0, 0)$  and  $(2, \frac{4}{e^2})$ .

$$9 \quad S(x) = \frac{1}{2}(e^x - e^{-x}), \quad C(x) = \frac{1}{2}(e^x + e^{-x})$$

$$\begin{aligned}
 a \quad [C(x)]^2 - [S(x)]^2 &= \left[\frac{1}{2}(e^x + e^{-x})\right]^2 - \left[\frac{1}{2}(e^x - e^{-x})\right]^2 \\
 &= \frac{1}{4}(e^x + e^{-x})^2 - \frac{1}{4}(e^x - e^{-x})^2 \\
 &= \frac{1}{4}(e^{2x} + 2e^0 + e^{-2x}) - \frac{1}{4}(e^{2x} - 2e^0 + e^{-2x}) \\
 &= \frac{1}{4}(e^{2x} + 2 + e^{-2x}) - \frac{1}{4}(e^{2x} - 2 + e^{-2x}) \\
 &= \frac{1}{4}e^{2x} + \frac{1}{2} + \frac{1}{4}e^{-2x} - \frac{1}{4}e^{2x} + \frac{1}{2} - \frac{1}{4}e^{-2x} \\
 &= 1
 \end{aligned}$$

$$\begin{aligned}
 b \quad \frac{d}{dx} [S(x)] &= \frac{d}{dx} \left[\frac{1}{2}(e^x - e^{-x})\right] \\
 &= \frac{d}{dx} \left[\frac{1}{2}e^x - \frac{1}{2}e^{-x}\right] \\
 &= \frac{1}{2}e^x + \frac{1}{2}e^{-x} \\
 &= \frac{1}{2}(e^x + e^{-x}) \\
 &= C(x)
 \end{aligned}$$

$$\begin{aligned}
 c \quad \frac{d}{dx} [C(x)] &= \frac{d}{dx} \left[\frac{1}{2}(e^x + e^{-x})\right] \\
 &= \frac{d}{dx} \left[\frac{1}{2}e^x + \frac{1}{2}e^{-x}\right] \\
 &= \frac{1}{2}e^x - \frac{1}{2}e^{-x} \\
 &= \frac{1}{2}(e^x - e^{-x}) \\
 &= S(x)
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad \frac{d}{dx} [T(x)] &= \frac{d}{dx} \left[ \frac{S(x)}{C(x)} \right] \\
 &= \frac{S'(x) C(x) - S(x) C'(x)}{[C(x)]^2} && \{\text{quotient rule}\} \\
 &= \frac{C(x) C(x) - S(x) S(x)}{[C(x)]^2} && \{\text{using b and c}\} \\
 &= \frac{[C(x)]^2 - [S(x)]^2}{[C(x)]^2} \\
 &= \frac{1}{[C(x)]^2} && \{\text{using a}\}
 \end{aligned}$$

## EXERCISE 17E.2

$$\begin{aligned}
 \text{1 a} \quad y &= 2^x \\
 \therefore \frac{dy}{dx} &= 2^x \ln 2
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad y &= 5^x \\
 \therefore \frac{dy}{dx} &= 5^x \ln 5
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad y &= 3^x + 2 \times 7^x + e^x \\
 \therefore \frac{dy}{dx} &= 3^x \ln 3 + 2 \times 7^x \ln 7 + e^x
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad y &= x 2^x \\
 \therefore \frac{dy}{dx} &= 2^x + x 2^x \ln 2 && \{\text{product rule}\}
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad y &= \frac{2^x}{x} \\
 \therefore \frac{dy}{dx} &= \frac{2^x \ln 2 \times x - 2^x \times 1}{x^2} && \{\text{quotient rule}\} \\
 &= \frac{2^x \ln 2}{x} - \frac{2^x}{x^2}
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad y &= \frac{x}{3^x} \\
 \therefore \frac{dy}{dx} &= \frac{1 \times 3^x - x \times 3^x \ln 3}{(3^x)^2} && \{\text{quotient rule}\} \\
 &= \frac{3^x(1 - x \ln 3)}{(3^x)^2} \\
 &= \frac{1 - x \ln 3}{3^x}
 \end{aligned}$$

$$\begin{aligned}
 \text{2 a} \quad y &= 4^{-x} \\
 \therefore \frac{dy}{dx} &= 4^{-x} \ln 4 \times (-1) \\
 &= -\frac{\ln 4}{4^x}
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad y &= 2^{x+3} + 3^{2x} \\
 \therefore \frac{dy}{dx} &= 2^{x+3} \ln 2 \times 1 + 3^{2x} \ln 3 \times 2 \\
 &= 2^{x+3} \ln 2 + 2 \times (3^2)^x \ln 3 \\
 &= 2^{x+3} \ln 2 + 2 \times 9^x \ln 3
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad y &= 5^{x^3+x} \\
 \therefore \frac{dy}{dx} &= 5^{x^3+x} \ln 5 \times (3x^2 + 1)
 \end{aligned}$$

**d**  $y = x^3 6^{-x}$

$$\begin{aligned}\therefore \frac{dy}{dx} &= 3x^2 \times 6^{-x} + x^3 \times 6^{-x} \ln 6 \times (-1) \quad \{\text{product rule}\} \\ &= 6^{-x}(3x^2 - x^3 \ln 6) \\ &= \frac{3x^2 - x^3 \ln 6}{6^x}\end{aligned}$$

**e**  $y = \frac{3x}{2^{5x-1}}$

$$\begin{aligned}\therefore \frac{dy}{dx} &= \frac{3 \times 2^{5x-1} - 3x \times 2^{5x-1} \ln 2 \times 5}{(2^{5x-1})^2} \quad \{\text{quotient rule}\} \\ &= \frac{3 \times \cancel{2^{5x-1}}(1 - 5x \ln 2)}{(\cancel{2^{5x-1}})^2} \\ &= \frac{3(1 - 5x \ln 2)}{2^{5x-1}}\end{aligned}$$

**f**  $y = \frac{e^{2x} - 3\sqrt{x}}{5x^2}$

$$\begin{aligned}\therefore \frac{dy}{dx} &= \frac{[2e^{2x} - 3x^{\frac{1}{2}} \ln 3 \times (\frac{1}{2}x^{-\frac{1}{2}})] \times \cancel{5x^2} - (e^{2x} - 3x^{\frac{1}{2}}) \times \cancel{5x^2} \ln 5 \times (2x)}{(\cancel{5x^2})^2} \quad \{\text{quotient rule}\} \\ &= \frac{2e^{2x} - 3x^{\frac{1}{2}} \ln 3 \times \frac{1}{2}x^{-\frac{1}{2}} - 2x \ln 5(e^{2x} - 3x^{\frac{1}{2}})}{5x^2} \times \frac{\cancel{2x^{\frac{1}{2}}}}{\cancel{2x^{\frac{1}{2}}}} \\ &= \frac{4x^{\frac{1}{2}}e^{2x} - 3x^{\frac{1}{2}} \ln 3 - 4x^{\frac{3}{2}} \ln 5(e^{2x} - 3x^{\frac{1}{2}})}{2x^{\frac{1}{2}} 5x^2} \\ &= \frac{4e^{2x}x^{\frac{1}{2}}(1 - x \ln 5) - 3x^{\frac{1}{2}}(\ln 3 - 4x^{\frac{3}{2}} \ln 5)}{2x^{\frac{1}{2}} 5x^2} \\ &= \frac{4e^{2x}\sqrt{x}(1 - x \ln 5) - 3\sqrt{x}(\ln 3 - 4x\sqrt{x} \ln 5)}{2\sqrt{x} 5x^2}\end{aligned}$$

**3 a**  $f(x) = 4^x$

$$\therefore f'(x) = 4^x \ln 4$$

$$\begin{aligned}\therefore f'(2) &= 4^2 \ln 4 = 16 \ln 4 \\ &= 32 \ln 2\end{aligned}$$

$\therefore$  the gradient of the tangent to  $f(x) = 4^x$  at  $x = 2$  is  $32 \ln 2$ .

**b**  $f(x) = 3^{-x}$

$$\begin{aligned}\therefore f'(x) &= 3^{-x} \ln 3 \times (-1) \\ &= -3^{-x} \ln 3\end{aligned}$$

$$\begin{aligned}\therefore f'(-1) &= -3^{-(-1)} \ln 3 \\ &= -3 \ln 3\end{aligned}$$

$\therefore$  the gradient of the tangent to  $f(x) = 3^{-x}$  at  $x = -1$  is  $-3 \ln 3$ .



$$\text{c} \quad f(x) = 3 \times 2^{3x}$$

$$\therefore f'(x) = 3 \times 2^{3x} \ln 2 \times 3$$

$$\text{When } y = 10, \quad 3 \times 2^{3x} = 10$$

$$\therefore 2^{3x} = \frac{10}{3}$$

$$\therefore 3x = \log_2\left(\frac{10}{3}\right)$$

$$\therefore x = \frac{1}{3} \log_2\left(\frac{10}{3}\right)$$

$$\begin{aligned} f'\left(\frac{1}{3} \log_2\left(\frac{10}{3}\right)\right) &= 3 \times 2^{\log_2\left(\frac{10}{3}\right)} \times \ln 2 \times 3 \\ &= 3 \times \frac{10}{3} \times \ln 2 \times 3 \\ &= 30 \ln 2 \end{aligned}$$

$\therefore$  the gradient of the tangent to  $f(x) = 3 \times 2^{3x}$  at  $y = 10$  is  $30 \ln 2$ .

$$\text{d} \quad f(x) = x^2 5^{1-x}$$

$$\begin{aligned} \therefore f'(x) &= 2x \times 5^{1-x} + x^2 \times 5^{1-x} \ln 5 \times (-1) \quad \{\text{product rule}\} \\ &= 2x \times 5^{1-x} - x^2 \times 5^{1-x} \ln 5 \end{aligned}$$

$$\begin{aligned} \therefore f'(3) &= 2(3) \times 5^{1-3} - 3^2 \times 5^{1-3} \ln 5 \\ &= 6 \times 5^{-2} - 9 \times 5^{-2} \ln 5 \\ &= \frac{6}{25} - \frac{9 \ln 5}{25} \\ &= \frac{3}{25}(2 - 3 \ln 5) \end{aligned}$$

$\therefore$  the gradient of the tangent to  $f(x) = x^2 5^{1-x}$  at  $x = 3$  is  $\frac{3}{25}(2 - 3 \ln 5)$ .

$$\text{4} \quad \begin{aligned} f(x) &= 2^x + 4^x \\ &= 2^x + 2^{2x} \end{aligned}$$

$$\begin{aligned} \therefore f'(x) &= 2^x \ln 2 + 2^{2x} \ln 2 \times 2 \\ &= \ln 2(2 \times 2^{2x} + 2^x) \end{aligned}$$

The tangent at point P has gradient  $\ln 2$ .

$$\therefore \text{the } x\text{-coordinate of P satisfies } 2 \times 2^{2x} + 2^x = 1$$

$$\therefore 2 \times 2^{2x} + 2^x - 1 = 0$$

$$\therefore 2X^2 + X - 1 = 0 \quad \{\text{letting } X = 2^x\}$$

$$\therefore (2X - 1)(X + 1) = 0$$

$$\therefore 2X - 1 = 0 \quad \{X > 0\}$$

$$\therefore 2 \times 2^x - 1 = 0$$

$$\therefore 2^x = \frac{1}{2}$$

$$\therefore x = -1$$

$$\text{and } f(-1) = 2^{-1} + 4^{-1}$$

$$= \frac{1}{2} + \frac{1}{4}$$

$$= \frac{3}{4}$$

$\therefore$  the coordinates of P are  $(-1, \frac{3}{4})$ .

## INVESTIGATION 6

THE DERIVATIVE OF  $\ln x$ 

**1** The gradient function has a vertical asymptote  $x = 0$ , and as  $x$  increases, it approaches its horizontal asymptote  $y = 0$ .

**2** We predict that for  $y = \ln x$ ,  $\frac{dy}{dx} = \frac{1}{x}$ .

**3** It appears that  $f'(x) = \frac{1}{x}$ , which agrees with our prediction in **2**.

$x$	$f'(x)$
0.25	4
0.5	2
1	1
2	0.5
3	0.3333
4	0.25
5	0.2

## EXERCISE 17F.1

**1 a**  $y = \ln(7x)$  *or*  $y = \ln(7x)$   
 $= \ln 7 + \ln x$   
 $\therefore \frac{dy}{dx} = 0 + \frac{1}{x} = \frac{1}{x}$   $\therefore \frac{dy}{dx} = \frac{7}{7x} = \frac{1}{x}$

**b**  $y = \ln(2x + 1)$   
 $\therefore \frac{dy}{dx} = \frac{2}{2x + 1}$

**c**  $y = \ln(x - x^2)$   
 $\therefore \frac{dy}{dx} = \frac{1 - 2x}{x - x^2}$

**d**  $y = 3 - 2 \ln x$   
 $\therefore \frac{dy}{dx} = 0 - 2\left(\frac{1}{x}\right) = -\frac{2}{x}$

**e**  $y = x^2 \ln x$   
 $\therefore \frac{dy}{dx} = 2x \ln x + x^2 \left(\frac{1}{x}\right)$  {product rule}  
 $= 2x \ln x + x$

**f**  $y = \frac{\ln x}{2x}$   
 $\therefore \frac{dy}{dx} = \frac{\left(\frac{1}{x}\right)2x - \ln x \times 2}{(2x)^2}$  {quotient rule}  
 $= \frac{2 - 2 \ln x}{4x^2}$   
 $= \frac{1 - \ln x}{2x^2}$

**g**  $y = e^x \ln x$   
 $\therefore \frac{dy}{dx} = e^x \ln x + e^x \left(\frac{1}{x}\right)$  {product rule}  
 $= e^x \ln x + \frac{e^x}{x}$

**h**  $y = (\ln x)^2$   
 $\therefore \frac{dy}{dx} = 2(\ln x)^1 \left(\frac{1}{x}\right)$  {chain rule}  
 $= \frac{2 \ln x}{x}$

$$\begin{aligned}
 \text{i} \quad y &= \sqrt{\ln x} = (\ln x)^{\frac{1}{2}} \\
 \therefore \frac{dy}{dx} &= \frac{1}{2}(\ln x)^{-\frac{1}{2}} \left( \frac{1}{x} \right) \quad \{\text{chain rule}\} \\
 &= \frac{1}{2x\sqrt{\ln x}}
 \end{aligned}$$

$$\begin{aligned}
 \text{j} \quad y &= e^{-x} \ln x \\
 \therefore \frac{dy}{dx} &= e^{-x}(-1) \ln x + e^{-x} \left( \frac{1}{x} \right) \quad \{\text{product rule}\} \\
 &= \frac{e^{-x}}{x} - e^{-x} \ln x
 \end{aligned}$$

$$\begin{aligned}
 \text{k} \quad y &= \sqrt{x} \ln(2x) = x^{\frac{1}{2}} \ln(2x) \\
 \therefore \frac{dy}{dx} &= \frac{1}{2}x^{-\frac{1}{2}} \ln(2x) + x^{\frac{1}{2}} \left( \frac{2}{2x} \right) \quad \{\text{product rule}\} \\
 &= \frac{1}{2\sqrt{x}} \ln(2x) + \sqrt{x} \left( \frac{1}{x} \right) \\
 &= \frac{\ln(2x)}{2\sqrt{x}} + \frac{1}{\sqrt{x}}
 \end{aligned}$$

$$\begin{aligned}
 \text{l} \quad y &= \frac{2\sqrt{x}}{\ln x} = \frac{2x^{\frac{1}{2}}}{\ln x} \\
 \therefore \frac{dy}{dx} &= \frac{x^{-\frac{1}{2}} \ln x - 2x^{\frac{1}{2}} \left( \frac{1}{x} \right)}{(\ln x)^2} \quad \{\text{quotient rule}\} \\
 &= \frac{\frac{1}{\sqrt{x}} \ln x - 2\sqrt{x} \left( \frac{1}{x} \right)}{(\ln x)^2} \\
 &= \frac{\frac{1}{\sqrt{x}} \ln x - \frac{2}{\sqrt{x}}}{(\ln x)^2} \\
 &= \frac{\ln x - 2}{\sqrt{x}(\ln x)^2}
 \end{aligned}$$

$$\begin{aligned}
 \text{m} \quad y &= 3 - 4 \ln(1 - x) \\
 \therefore \frac{dy}{dx} &= -4 \left( \frac{-1}{1 - x} \right) \\
 &= \frac{4}{1 - x}
 \end{aligned}$$

$$\begin{aligned}
 \text{n} \quad y &= x \ln(x^2 + 1) \\
 \therefore \frac{dy}{dx} &= 1(\ln(x^2 + 1)) + x \left( \frac{2x}{x^2 + 1} \right) \quad \{\text{product rule}\} \\
 &= \ln(x^2 + 1) + \frac{2x^2}{x^2 + 1}
 \end{aligned}$$

$$\begin{aligned}
 \circ \quad y &= \frac{\ln x}{x^2} \\
 \therefore \frac{dy}{dx} &= \frac{\left( \frac{1}{x} \right) x^2 - \ln x(2x)}{(x^2)^2} \quad \{\text{quotient rule}\} \\
 &= \frac{x - 2x \ln x}{x^4} \\
 &= \frac{1 - 2 \ln x}{x^3}
 \end{aligned}$$

$$\begin{aligned}
 \text{2 a} \quad f(x) &= \ln(kx), \quad k \neq 0 \\
 \therefore f'(x) &= \frac{k}{kx} \\
 &= \frac{1}{x}
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad f(x) &= \ln(kx) \\
 &= \ln x + \ln k \\
 \therefore f'(x) &= \frac{1}{x} + 0 \quad \{\ln k \text{ is constant}\} \\
 &= \frac{1}{x}
 \end{aligned}$$

$$\begin{aligned}
 \text{3 a} \quad y &= x \ln 5 \\
 \therefore \frac{dy}{dx} &= \ln 5
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad y &= \ln(x^3) = 3 \ln x \\
 &\quad \{\ln(a^n) = n \ln a\} \\
 \therefore \frac{dy}{dx} &= 3 \left( \frac{1}{x} \right) = \frac{3}{x}
 \end{aligned}$$

$$\text{c} \quad y = \ln(x^4 + x)$$

$$\therefore \frac{dy}{dx} = \frac{4x^3 + 1}{x^4 + x}$$

$$\text{e} \quad y = [\ln(2x + 1)]^3$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= 3[\ln(2x + 1)]^2 \times \frac{2}{2x + 1} \\ &\quad \{\text{chain rule}\} \\ &= \frac{6}{2x + 1} [\ln(2x + 1)]^2 \end{aligned}$$

$$\begin{aligned} \text{g} \quad y &= \ln\left(\frac{1}{x}\right) = \ln(x^{-1}) \\ &= -\ln x \quad \{\ln(a^n) = n \ln a\} \\ \therefore \frac{dy}{dx} &= -\frac{1}{x} \end{aligned}$$

$$\begin{aligned} \text{i} \quad y &= \frac{1}{\ln x} = (\ln x)^{-1} \\ \therefore \frac{dy}{dx} &= -1(\ln x)^{-2} \times \frac{1}{x} \quad \{\text{chain rule}\} \\ &= \frac{-1}{x(\ln x)^2} \end{aligned}$$

$$\begin{aligned} \text{4 a} \quad y &= \ln \sqrt{1 - 2x} \\ &= \ln\left((1 - 2x)^{\frac{1}{2}}\right) \\ &= \frac{1}{2} \ln(1 - 2x) \quad \{\ln(a^n) = n \ln a\} \\ \therefore \frac{dy}{dx} &= \frac{1}{2} \times \frac{-2}{1 - 2x} \\ &= \frac{-1}{1 - 2x} \\ &= \frac{1}{2x - 1} \end{aligned}$$

$$\begin{aligned} \text{c} \quad y &= \ln(e^x \sqrt{x}) \\ &= \ln(e^x) + \ln(x^{\frac{1}{2}}) \quad \{\ln(ab) = \ln a + \ln b\} \\ &= \ln(e^x) + \frac{1}{2} \ln x \quad \{\ln(a^n) = n \ln a\} \\ &= x + \frac{1}{2} \ln x \quad \{\ln e^a = a\} \\ \therefore \frac{dy}{dx} &= 1 + \frac{1}{2} \left(\frac{1}{x}\right) \\ &= 1 + \frac{1}{2x} \end{aligned}$$

$$\begin{aligned} \text{d} \quad y &= \ln(10 - 5x) \\ \therefore \frac{dy}{dx} &= \frac{-5}{10 - 5x} = \frac{1}{x - 2} \end{aligned}$$

$$\begin{aligned} \text{f} \quad y &= \frac{\ln(4x)}{x} \\ \therefore \frac{dy}{dx} &= \frac{\left(\frac{4}{4x}\right)x - \ln(4x) \times 1}{x^2} \\ &\quad \{\text{quotient rule}\} \\ &= \frac{1 - \ln(4x)}{x^2} \end{aligned}$$

$$\begin{aligned} \text{h} \quad y &= \ln(\ln x) \\ \therefore \frac{dy}{dx} &= \frac{\frac{1}{x}}{\ln x} \\ &= \frac{1}{x \ln x} \end{aligned}$$

$$\begin{aligned} \text{b} \quad y &= \ln\left(\frac{1}{2x + 3}\right) \\ &= \ln((2x + 3)^{-1}) \\ &= -\ln(2x + 3) \quad \{\ln(a^n) = n \ln a\} \\ \therefore \frac{dy}{dx} &= -\frac{2}{2x + 3} \end{aligned}$$



$$\begin{aligned}
 \text{d} \quad y &= \ln(x\sqrt{2-x}) \\
 &= \ln x + \ln\left((2-x)^{\frac{1}{2}}\right) \\
 &= \ln x + \frac{1}{2} \ln(2-x) \\
 &\quad \{\ln(a^n) = n \ln a\} \\
 \therefore \frac{dy}{dx} &= \frac{1}{x} + \frac{1}{2} \left(\frac{-1}{2-x}\right) \\
 &= \frac{1}{x} - \frac{1}{2(2-x)}
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad y &= \ln\left(\frac{x^2}{3-x}\right) \\
 &= \ln(x^2) - \ln(3-x) \quad \{\ln\left(\frac{a}{b}\right) = \ln a - \ln b\} \\
 &= 2 \ln x - \ln(3-x) \quad \{\ln(a^n) = n \ln a\} \\
 \therefore \frac{dy}{dx} &= 2\left(\frac{1}{x}\right) - \frac{-1}{3-x} \\
 &= \frac{2}{x} + \frac{1}{3-x}
 \end{aligned}$$

$$\begin{aligned}
 \text{5 a} \quad f(x) &= \ln((3x-4)^3) \\
 &= 3 \ln(3x-4) \\
 &\quad \{\ln(a^n) = n \ln a\} \\
 \therefore f'(x) &= 3 \times \frac{3}{3x-4} = \frac{9}{3x-4}
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad f(x) &= \ln(x(x^2+1)) \\
 &= \ln x + \ln(x^2+1) \\
 &\quad \{\ln(ab) = \ln a + \ln b\} \\
 \therefore f'(x) &= \frac{1}{x} + \frac{2x}{x^2+1}
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad f(x) &= \ln\left(\frac{x^2+2x}{x-5}\right) \\
 &= \ln(x^2+2x) - \ln(x-5) \quad \{\ln\left(\frac{a}{b}\right) = \ln a - \ln b\} \\
 \therefore f'(x) &= \frac{2x+2}{x^2+2x} - \frac{1}{x-5}
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad f(x) &= \ln\left(\frac{x^3}{(x+4)(x-1)}\right) \\
 &= \ln(x^3) - \ln[(x+4)(x-1)] \quad \{\ln\left(\frac{a}{b}\right) = \ln a - \ln b\} \\
 &= \ln(x^3) - [\ln(x+4) + \ln(x-1)] \quad \{\ln(ab) = \ln a + \ln b\} \\
 &= 3 \ln x - \ln(x+4) - \ln(x-1) \quad \{\ln(a^n) = n \ln a\} \\
 \therefore f'(x) &= 3\left(\frac{1}{x}\right) - \frac{1}{x+4} - \frac{1}{x-1} \\
 &= \frac{3}{x} - \frac{1}{x+4} - \frac{1}{x-1}
 \end{aligned}$$

$$\begin{aligned}
 6 \quad y &= x \ln x \\
 \therefore \frac{dy}{dx} &= 1 \ln x + x \times \frac{1}{x} \quad \{\text{product rule}\} \\
 &= \ln x + 1
 \end{aligned}$$

$$\begin{aligned}
 \text{At } x = e, \quad \frac{dy}{dx} &= \ln e + 1 \\
 &= 1 + 1 \\
 &= 2
 \end{aligned}$$

$$\therefore \text{gradient of tangent} = 2$$

$$7 \quad f(x) = a \ln(bx^2)$$

$$\text{Now } f(e) = 3, \quad \therefore 3 = a \ln(be^2)$$

$$\begin{aligned}
 \therefore a &= \frac{3}{\ln(be^2)} \\
 &= \frac{3}{\ln b + \ln(e^2)} \\
 &= \frac{3}{\ln b + 2} \quad \dots (1)
 \end{aligned}$$

$$\begin{aligned}
 \text{Now } f'(x) &= a \times \frac{2bx}{bx^2} \\
 &= \frac{2a}{x} \quad \text{and} \quad f'(1) = 6
 \end{aligned}$$

$$\therefore 6 = 2a$$

$$\therefore a = 3 \quad \dots (2)$$

$$\therefore 3 = \frac{3}{\ln b + 2} \quad \{\text{equating (1) and (2)}\}$$

$$\therefore \ln b + 2 = 1$$

$$\therefore \ln b = -1$$

$$\therefore b = e^{-1}$$

$$\text{So, } a = 3, \quad b = \frac{1}{e}.$$

$$8 \quad f(x) = ax \ln(bx)$$

$$\text{Now } f(1) = 12, \quad \therefore 12 = a \ln b \quad \dots (*)$$

$$\text{Now } f'(x) = a \ln(bx) + ax \times \frac{b}{bx} \quad \{\text{product rule}\}$$

$$= a \ln(bx) + a \quad \text{and} \quad f'(1) = 16$$

$$\therefore 16 = a \ln b + a$$

$$\therefore 16 = 12 + a \quad \{\text{using (*)}\}$$

$$\therefore a = 4$$

$$\text{Substituting } a = 4 \text{ into (*) gives } 12 = 4 \ln b$$

$$\therefore \ln b = 3$$

$$\therefore b = e^3$$

$$\text{So, } a = 4, \quad b = e^3.$$

$$9 \quad y = \ln(15 - x^2)$$

$$\therefore \frac{dy}{dx} = \frac{-2x}{15 - x^2}$$

$$\begin{aligned} \text{Now } \frac{dy}{dx} = 1 \text{ when } \frac{-2x}{15 - x^2} = 1 \\ \therefore -2x = 15 - x^2 \\ \therefore x^2 - 2x - 15 = 0 \\ \therefore (x + 3)(x - 5) = 0 \\ \therefore x = -3 \text{ or } 5 \end{aligned}$$

But  $y = \ln(15 - x^2)$  is undefined for  $x = 5$ .

$$\begin{aligned} \text{When } x = -3, \quad y &= \ln(15 - (-3)^2) \\ &= \ln(15 - 9) \\ &= \ln 6 \end{aligned}$$

$\therefore$  the tangent at  $(-3, \ln 6)$  has gradient 1.

$$10 \quad f(x) = \begin{cases} \ln(2x + a), & x \geq 0 \\ bx, & x < 0 \end{cases}$$

$$\therefore f'(x) = \begin{cases} \frac{2}{2x + a}, & x > 0 \\ b, & x < 0 \end{cases} \quad \begin{array}{l} \text{since the derivative of } f \text{ exists on the intervals } x < 0 \\ \text{and } x > 0. \end{array}$$

$f$  is continuous at  $x = 0$ .

$$\begin{aligned} \therefore \lim_{x \rightarrow 0^+} f(x) &= \lim_{x \rightarrow 0^-} f(x) \\ \therefore \ln a &= 0 \\ \therefore a &= 1 \end{aligned}$$

$f$  is differentiable at  $x = 0$ .

$$\begin{aligned} \therefore f'_+(0) &= f'_-(0) \\ \therefore \frac{2}{1} &= b \\ \therefore b &= 2 \end{aligned}$$

## EXERCISE 17F.2

$$1 \quad \mathbf{a} \quad f(x) = \log_2 x \\ \therefore f'(x) = \frac{1}{x \ln 2}$$

$$\mathbf{b} \quad f(x) = \log_{10} x \\ \therefore f'(x) = \frac{1}{x \ln 10}$$

$$\begin{aligned} \mathbf{c} \quad f(x) &= \log_3 x - \frac{1}{2} \log_5 x \\ \therefore f'(x) &= \frac{1}{x \ln 3} - \frac{1}{2} \times \frac{1}{x \ln 5} \\ &= \frac{1}{x \ln 3} - \frac{1}{2x \ln 5} \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad f(x) &= x \log_3 x \\ \therefore f'(x) &= \log_3 x + x \times \frac{1}{x \ln 3} \quad \{\text{product rule}\} \\ &= \log_3 x + \frac{1}{\ln 3} \end{aligned}$$

$$\begin{aligned} \text{e} \quad f(x) &= \frac{\log_5 x}{x^2 - 3x} \\ \therefore f'(x) &= \frac{\frac{x^2 - 3x}{x \ln 5} - (2x - 3) \log_5 x}{(x^2 - 3x)^2} \quad \{\text{quotient rule}\} \end{aligned}$$

$$\begin{aligned} \text{f} \quad f(x) &= (\log_2 x)^4 \\ \therefore f'(x) &= 4(\log_2 x)^3 \times \frac{1}{x \ln 2} \\ &= \frac{4(\log_2 x)^3}{x \ln 2} \end{aligned}$$

$$\begin{aligned} \text{2 a} \quad y &= \log_2(3x - 1) \\ \therefore \frac{dy}{dx} &= \frac{3}{(3x - 1) \ln 2} \end{aligned}$$

$$\begin{aligned} \text{b} \quad y &= \log_5(4x^2 + 3x) \\ \therefore \frac{dy}{dx} &= \frac{8x + 3}{(4x^2 + 3x) \ln 5} \end{aligned}$$

$$\begin{aligned} \text{c} \quad y &= 3^x \log_3(2x) \\ \therefore \frac{dy}{dx} &= 3^x \ln 3 \times \log_3(2x) + 3^x \times \frac{2}{2x \ln 3} \quad \{\text{product rule}\} \\ &= 3^x \left[ \ln 3 \times \log_3(2x) + \frac{1}{x \ln 3} \right] \end{aligned}$$

$$\begin{aligned} \text{d} \quad y &= \log_5 \left( \frac{2}{\sqrt{x}} \right) \\ &= \log_5(2x^{-\frac{1}{2}}) \\ \therefore \frac{dy}{dx} &= \frac{-x^{-\frac{3}{2}}}{2x^{-\frac{1}{2}} \times \ln 5} \times \frac{x^{\frac{3}{2}}}{x^{\frac{3}{2}}} \\ &= -\frac{1}{2x \ln 5} \end{aligned}$$

$$\text{e} \quad \text{Consider } y = \log_3 x,$$

$$\therefore \frac{dy}{dx} = \frac{1}{x \ln 3}$$

$$\text{So, if } y = \log_2(\log_3 x),$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{\frac{1}{x \ln 3}}{\log_3 x \times \ln 2} \\ &= \frac{1}{x \ln 3 \times \log_3 x \times \ln 2} \\ &= \frac{1}{x \ln 3 \times \frac{\ln x}{\ln 3} \times \ln 2} \\ &= \frac{1}{x \ln x \times \ln 2} \end{aligned}$$

$$\begin{aligned} \text{f} \quad y &= \frac{\log(4 - x)}{x^2 - 1} \\ \therefore \frac{dy}{dx} &= \frac{\frac{-1}{(4-x) \ln 10} \times (x^2 - 1) - \log(4 - x) \times 2x}{(x^2 - 1)^2} \quad \{\text{quotient rule}\} \\ &= \frac{\frac{1-x^2}{(4-x) \ln 10} - 2x \log(4 - x)}{(x^2 - 1)^2} \end{aligned}$$



**3 a**  $y = \log_2(x - 3)$

$$\therefore \frac{dy}{dx} = \frac{1}{(x - 3) \ln 2}$$

$$\begin{aligned} \text{When } x = 5, \quad \frac{dy}{dx} &= \frac{1}{(5 - 3) \ln 2} \\ &= \frac{1}{2 \ln 2} \end{aligned}$$

$\therefore$  the gradient of the tangent to  $y = \log_2(x - 3)$  at  $x = 5$  is  $\frac{1}{2 \ln 2}$ .

**b**  $y = x^2 \log_3(2x)$

$$\begin{aligned} \therefore \frac{dy}{dx} &= 2x \log_3(2x) + x^2 \times \frac{2}{2x \ln 3} \quad \{\text{product rule}\} \\ &= 2x \log_3(2x) + \frac{x}{\ln 3} \end{aligned}$$

$$\begin{aligned} \text{When } x = \frac{9}{2}, \quad \frac{dy}{dx} &= 2\left(\frac{9}{2}\right) \log_3\left(2\left(\frac{9}{2}\right)\right) + \frac{\frac{9}{2}}{\ln 3} \\ &= 9 \log_3 9 + \frac{9}{2 \ln 3} \\ &= 9 \log_3(3^2) + \frac{9}{2 \ln 3} \\ &= 9 \times 2 + \frac{9}{2 \ln 3} \\ &= 18 + \frac{9}{2 \ln 3} \end{aligned}$$

**c**  $y = \frac{\log_2(\sqrt{x})}{e^x}$

$$= \frac{\log_2(x^{\frac{1}{2}})}{e^x}$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= \frac{\frac{\frac{1}{2}x^{-\frac{1}{2}}}{x^{\frac{1}{2}} \ln 2} \times \cancel{e^x} - \log_2(x^{\frac{1}{2}}) \times \cancel{e^x}}{(e^x)^2} \quad \{\text{quotient rule}\} \\ &= \frac{\frac{1}{2x \ln 2} - \log_2(\sqrt{x})}{e^x} \end{aligned}$$

$$\begin{aligned} \text{When } x = 2, \quad \frac{dy}{dx} &= \frac{\frac{1}{4 \ln 2} - \log_2(2^{\frac{1}{2}})}{e^2} \\ &= \frac{\frac{1}{4 \ln 2} - \frac{1}{2}}{e^2} \\ &= \frac{\frac{1 - 2 \ln 2}{4 \ln 2}}{e^2} \\ &= \frac{1 - 2 \ln 2}{4e^2 \ln 2} \end{aligned}$$

$\therefore$  the gradient of the tangent to  $y = \frac{\log_2(\sqrt{x})}{e^x}$  at  $x = 2$  is  $\frac{1 - 2 \ln 2}{4e^2 \ln 2}$ .

$$\begin{aligned} \mathbf{d} \quad y &= \frac{1}{2} \log_5(5^x + 4) \\ \therefore \frac{dy}{dx} &= \frac{1}{2} \times \frac{5^x \ln 5}{(5^x + 4) \ln 5} \\ &= \frac{5^x}{2(5^x + 4)} \end{aligned}$$

$$\begin{aligned} \text{When } y = 1, \quad 1 &= \frac{1}{2} \log_5(5^x + 4) \\ \therefore 2 &= \log_5(5^x + 4) \\ \therefore 5^x + 4 &= 25 \\ \therefore 5^x &= 21 \\ \therefore \frac{dy}{dx} &= \frac{21}{2(21 + 4)} \\ &= \frac{21}{50} \end{aligned}$$

$\therefore$  the gradient of the tangent to  $y = \frac{1}{2} \log_5(5^x + 4)$  at  $y = 1$  is  $\frac{21}{50}$ .

$$\begin{aligned} \mathbf{4} \quad \mathbf{a} \quad y &= \log_3 \left( \frac{1}{x^2} \right) \\ &= \log_3 1 - \log_3(x^2) \\ &= -\log_3(x^2) \\ \therefore \frac{dy}{dx} &= -\frac{2x}{x^2 \ln 3} \\ &= -\frac{2}{x \ln 3} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad y &= \log_2[(x+5)(x-1)] \\ &= \log_2(x+5) + \log_2(x-1) \\ \therefore \frac{dy}{dx} &= \frac{1}{(x+5) \ln 2} + \frac{1}{(x-1) \ln 2} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad y &= \log(xe^x) \\ &= \log x + \log(e^x) \\ \therefore \frac{dy}{dx} &= \frac{1}{x \ln 10} + \frac{e^x}{e^x \ln 10} \\ &= \frac{1}{x \ln 10} + \frac{1}{\ln 10} \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad y &= \log_5 \left[ \frac{1}{(2x-1)(x+2)} \right] \\ &= \log_5 1 - \log_5(2x-1) - \log_5(x+2) \\ &= -\log_5(2x-1) - \log_5(x+2) \\ \therefore \frac{dy}{dx} &= -\frac{2}{(2x-1) \ln 5} - \frac{1}{(x+2) \ln 5} \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad y &= \log_2 \left( \frac{x^2 - x}{x + 3} \right) \\ &= \log_2(x^2 - x) - \log_2(x + 3) \\ \therefore \frac{dy}{dx} &= \frac{2x - 1}{(x^2 - x) \ln 2} - \frac{1}{(x + 3) \ln 2} \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad y &= \log_4[(\sqrt{x} - 1)^3] \\ &= 3 \log_4(x^{\frac{1}{2}} - 1) \\ \therefore \frac{dy}{dx} &= 3 \times \frac{\frac{1}{2} x^{-\frac{1}{2}}}{(x^{\frac{1}{2}} - 1) \ln 4} \\ &= \frac{3x^{-\frac{1}{2}}}{2(x^{\frac{1}{2}} - 1) \times 2 \ln 2} \times \frac{x^{\frac{1}{2}}}{x^{\frac{1}{2}}} \\ &= \frac{3}{(4x - 4\sqrt{x}) \ln 2} \end{aligned}$$

$$\begin{aligned}
 5 \quad & f(x) = \log_3(ax + b) \\
 & f(2) = -1 \\
 \therefore & \log_3(2a + b) = -1 \\
 \therefore & 2a + b = 3^{-1} \\
 \therefore & b = \frac{1}{3} - 2a \quad \dots (*)
 \end{aligned}$$

$$\begin{aligned}
 \text{Now } f'(x) &= \frac{a}{(ax + b) \ln 3} \\
 f'(3) &= \frac{3}{4 \ln 3} \\
 \therefore \frac{a}{(3a + b) \ln 3} &= \frac{3}{4 \ln 3} \\
 \therefore \frac{a}{3a + b} &= \frac{3}{4} \\
 \therefore 4a &= 3(3a + b) \\
 \therefore 4a &= 3\left(3a + \frac{1}{3} - 2a\right) \quad \{\text{using } (*)\} \\
 \therefore 4a &= 3\left(a + \frac{1}{3}\right) \\
 \therefore 4a &= 3a + 1 \\
 \therefore a &= 1
 \end{aligned}$$

$$\begin{aligned}
 \text{Substituting into } (*): \quad b &= \frac{1}{3} - 2(1) \\
 \therefore b &= -\frac{5}{3}
 \end{aligned}$$

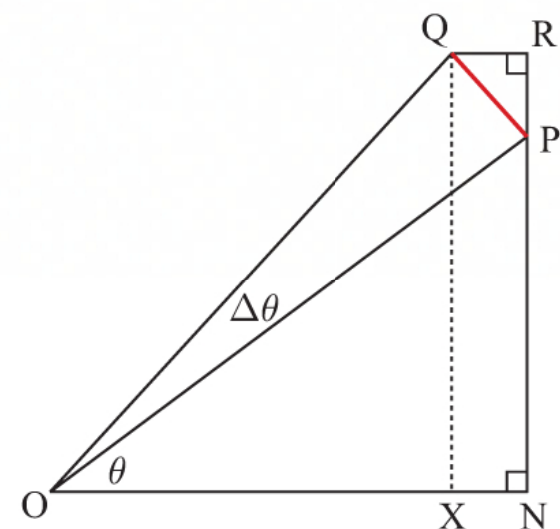
## INVESTIGATION 7

## DERIVATIVES OF $\sin x$ AND $\cos x$

1 We predict that if  $y = \sin x$ , then  $\frac{dy}{dx} = \cos x$ .

2 We predict that if  $y = \cos x$ , then  $\frac{dy}{dx} = -\sin x$ .

$$\begin{aligned}
 3 \quad a \quad \sin(\theta + \Delta\theta) &= \frac{QX}{OQ} = NR \quad \{OQ = 1, QX = NR\} \\
 \sin \theta &= \frac{NP}{OP} = NP \quad \{OP = 1\} \\
 \therefore \sin(\theta + \Delta\theta) - \sin \theta &= NR - NP \\
 &= PR
 \end{aligned}$$



b i For very small  $\Delta\theta$ , the line segment  $[PQ]$  is a good approximation of the arc  $PQ$ .  
 $\therefore$  as  $Q$  approaches  $P$ , the arc  $PQ$  resembles line segment  $[PQ]$ .

$$\begin{aligned}
 \text{ii} \quad \text{arc } PQ &= 1 \times \Delta\theta \quad \{l = r\theta\} \\
 &= \Delta\theta
 \end{aligned}$$

$\therefore$  as  $Q$  approaches  $P$ ,  $PQ \approx \Delta\theta$ .

$$\text{iii } \widehat{QPO} = \frac{\pi}{2} - \frac{\Delta\theta}{2} \quad \{\text{isosceles triangle}\}$$

$$\text{As } \Delta\theta \rightarrow 0, \widehat{QPO} \rightarrow \frac{\pi}{2}$$

$\therefore$  as Q approaches P,  $\widehat{QPO}$  approaches a right angle.

$$\text{iv } \text{As Q approaches P, } \widehat{QPO} \text{ approaches a right angle} \quad \{\text{from iii}\}$$

$$\begin{aligned} \therefore \widehat{QPR} &\approx \pi - \frac{\pi}{2} - \widehat{OPN} \quad \{\text{angles on a line}\} \\ &\approx \pi - \frac{\pi}{2} - (\pi - \frac{\pi}{2} - \theta) \\ &\approx \cancel{\pi} - \cancel{\frac{\pi}{2}} - \cancel{\pi} + \cancel{\frac{\pi}{2}} + \theta \\ &\approx \theta \end{aligned}$$

$$\begin{aligned} \text{c } \text{For small } \Delta\theta, \cos \theta &\approx \frac{PR}{PQ} \\ &\approx \frac{\sin(\theta + \Delta\theta) - \sin \theta}{\Delta\theta} \quad \{\text{using a, b ii}\} \end{aligned}$$

$$\begin{aligned} \therefore \lim_{\Delta\theta \rightarrow 0} \cos \theta &= \lim_{\Delta\theta \rightarrow 0} \frac{\sin(\theta + \Delta\theta) - \sin \theta}{\Delta\theta} \\ \therefore \cos \theta &= \frac{d}{d\theta} (\sin \theta) \end{aligned}$$

## EXERCISE 17G.1

$$1 \quad \text{a } y = \sin 2x$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= (\cos 2x) \times 2 \\ &= 2 \cos 2x \end{aligned}$$

$$\text{c } y = \cos 3x - \sin x$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= (-\sin 3x) \times 3 - \cos x \\ &= -3 \sin 3x - \cos x \end{aligned}$$

$$\text{e } y = \cos(3 - 2x)$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= (-\sin(3 - 2x)) \times (-2) \\ &= 2 \sin(3 - 2x) \end{aligned}$$

$$\text{g } y = \sin \frac{x}{2} - 3 \cos x$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= \left(\cos \frac{x}{2}\right) \left(\frac{1}{2}\right) - 3(-\sin x) \\ &= \frac{1}{2} \cos \frac{x}{2} + 3 \sin x \end{aligned}$$

$$\text{i } y = \frac{1}{2} \cos 6x - 5 \sin 4x$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= \frac{1}{2}(-\sin 6x) \times 6 - 5(\cos 4x) \times 4 \\ &= -3 \sin 6x - 20 \cos 4x \end{aligned}$$

$$2 \quad \text{a } y = x^2 + \cos x$$

$$\therefore \frac{dy}{dx} = 2x - \sin x$$

$$\text{b } y = \sin x + \cos x$$

$$\therefore \frac{dy}{dx} = \cos x - \sin x$$

$$\text{d } y = \sin(x + 1)$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= (\cos(x + 1)) \times 1 \\ &= \cos(x + 1) \end{aligned}$$

$$\text{f } y = \tan 5x$$

$$\therefore \frac{dy}{dx} = 5 \sec^2 5x$$

$$\text{h } y = 4 \sin x - \cos 2x$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= 4 \cos x + (\sin 2x) \times 2 \\ &= 4 \cos x + 2 \sin 2x \end{aligned}$$

$$\text{b } y = \tan x - 3 \sin x$$

$$\therefore \frac{dy}{dx} = \sec^2 x - 3 \cos x$$



$$\begin{aligned}
 \text{c} \quad y &= e^x \cos x \\
 \therefore \frac{dy}{dx} &= e^x \cos x + e^x(-\sin x) \\
 &\quad \{\text{product rule}\} \\
 &= e^x \cos x - e^x \sin x
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad y &= \ln(\sin x) \\
 \therefore \frac{dy}{dx} &= \frac{\cos x}{\sin x}
 \end{aligned}$$

$$\begin{aligned}
 \text{g} \quad y &= 3 \tan \pi x \\
 \therefore \frac{dy}{dx} &= 3\pi \sec^2 \pi x \quad \{\text{chain rule}\}
 \end{aligned}$$

$$\begin{aligned}
 \text{i} \quad y &= 3 \tan 2x \\
 \therefore \frac{dy}{dx} &= 3 \sec^2 2x \times 2 \quad \{\text{chain rule}\} \\
 &= 6 \sec^2 2x
 \end{aligned}$$

$$\begin{aligned}
 \text{k} \quad y &= \frac{\sin x}{x} \\
 \therefore \frac{dy}{dx} &= \frac{(\cos x)(x) - \sin x \times 1}{x^2} \\
 &\quad \{\text{quotient rule}\} \\
 &= \frac{x \cos x - \sin x}{x^2}
 \end{aligned}$$

$$\begin{aligned}
 \text{m} \quad y &= 3^x \sin 2x \\
 \therefore \frac{dy}{dx} &= 3^x \ln 3 \times \sin 2x + 3^x \times \cos 2x \times 2 \\
 &\quad \{\text{product rule}\} \\
 &= 3^x (\ln 3 \sin 2x + 2 \cos 2x)
 \end{aligned}$$

$$\begin{aligned}
 \text{o} \quad y &= \frac{\cos x + \sin 2x}{x^3} \\
 \therefore \frac{dy}{dx} &= \frac{(-\sin x + 2 \cos 2x) \times x^3 - (\cos x + \sin 2x) \times 3x^2}{x^6} \quad \{\text{quotient rule}\} \\
 &= \frac{2x \cos 2x - x \sin x - 3 \cos x - 3 \sin 2x}{x^4}
 \end{aligned}$$

$$\begin{aligned}
 \text{3 a} \quad y &= \sin(x^2) \\
 \therefore \frac{dy}{dx} &= (\cos(x^2)) \times 2x \quad \{\text{chain rule}\} \\
 &= 2x \cos(x^2)
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad y &= e^{-x} \sin x \\
 \therefore \frac{dy}{dx} &= -e^{-x} \sin x + e^{-x} \cos x \\
 &\quad \{\text{product rule}\}
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad y &= e^{2x} \tan x \\
 \therefore \frac{dy}{dx} &= 2e^{2x} \tan x + e^{2x} \sec^2 x \\
 &\quad \{\text{product rule}\}
 \end{aligned}$$

$$\begin{aligned}
 \text{h} \quad y &= \cos \frac{x}{2} \\
 \therefore \frac{dy}{dx} &= \left(-\sin \frac{x}{2}\right) \times \left(\frac{1}{2}\right) \quad \{\text{chain rule}\} \\
 &= -\frac{1}{2} \sin \frac{x}{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{j} \quad y &= x \cos x \\
 \therefore \frac{dy}{dx} &= 1 \times \cos x + x(-\sin x) \\
 &\quad \{\text{product rule}\} \\
 &= \cos x - x \sin x
 \end{aligned}$$

$$\begin{aligned}
 \text{l} \quad y &= x \tan x \\
 \therefore \frac{dy}{dx} &= \tan x + x \sec^2 x \quad \{\text{product rule}\}
 \end{aligned}$$

$$\begin{aligned}
 \text{n} \quad y &= \log(\cos 2x) \\
 \therefore \frac{dy}{dx} &= \frac{-2 \sin 2x}{\cos 2x \times \ln 10} \\
 &= -\frac{2 \tan 2x}{\ln 10}
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad y &= \cos(\sqrt{x}) = \cos(x^{\frac{1}{2}}) \\
 \therefore \frac{dy}{dx} &= -\sin(x^{\frac{1}{2}}) \times \frac{1}{2} x^{-\frac{1}{2}} \quad \{\text{chain rule}\} \\
 &= -\frac{1}{2\sqrt{x}} \sin(\sqrt{x})
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad y &= \sqrt{\cos x} = (\cos x)^{\frac{1}{2}} \\
 \therefore \frac{dy}{dx} &= \frac{1}{2}(\cos x)^{-\frac{1}{2}} \times (-\sin x) \\
 &\quad \{\text{chain rule}\} \\
 &= -\frac{\sin x}{2\sqrt{\cos x}}
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad y &= \cos^3 x = (\cos x)^3 \\
 \therefore \frac{dy}{dx} &= 3(\cos x)^2 \times (-\sin x) \\
 &\quad \{\text{chain rule}\} \\
 &= -3 \sin x \cos^2 x
 \end{aligned}$$

$$\begin{aligned}
 \text{g} \quad y &= \cos(\cos x) \\
 \therefore \frac{dy}{dx} &= -\sin(\cos x) \times (-\sin x) \\
 &\quad \{\text{chain rule}\} \\
 &= \sin x \sin(\cos x)
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad y &= \sin^2 x = (\sin x)^2 \\
 \therefore \frac{dy}{dx} &= 2 \sin x \cos x \quad \{\text{chain rule}\}
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad y &= \cos x \sin 2x \\
 \therefore \frac{dy}{dx} &= (-\sin x) \sin 2x + \cos x (2 \cos 2x) \\
 &\quad \{\text{product rule}\} \\
 &= -\sin x \sin 2x + 2 \cos x \cos 2x
 \end{aligned}$$

$$\begin{aligned}
 \text{h} \quad y &= \cos^3 4x = (\cos 4x)^3 \\
 \therefore \frac{dy}{dx} &= 3(\cos 4x)^2 \times (-4 \sin 4x) \\
 &\quad \{\text{chain rule}\} \\
 &= -12 \sin 4x \cos^2 4x
 \end{aligned}$$

$$\begin{aligned}
 \text{4 a} \quad \frac{d}{dx} (\operatorname{cosec} x) &= \frac{d}{dx} \left( \frac{1}{\sin x} \right) \\
 &= \frac{0 \times \sin x - 1 \times \cos x}{\sin^2 x} \\
 &\quad \{\text{quotient rule}\} \\
 &= -\frac{\cos x}{\sin^2 x} \\
 &= -\frac{1}{\sin x} \times \frac{\cos x}{\sin x} \\
 &= -\operatorname{cosec} x \cot x
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad \frac{d}{dx} (\operatorname{cosec} x) &= \frac{d}{dx} (\sin x)^{-1} \\
 &= -(\sin x)^{-2} \times \cos x \\
 &\quad \{\text{chain rule}\} \\
 &= -\frac{\cos x}{\sin^2 x} \\
 &= -\frac{1}{\sin x} \times \frac{\cos x}{\sin x} \\
 &= -\operatorname{cosec} x \cot x
 \end{aligned}$$

$$\begin{aligned}
 \text{5 a} \quad \frac{d}{dx} (\sec x) &= \frac{d}{dx} \left( \frac{1}{\cos x} \right) \\
 &= \frac{d}{dx} (\cos x)^{-1} \\
 &= -(\cos x)^{-2} \times (-\sin x) \\
 &\quad \{\text{chain rule}\} \\
 &= \frac{\sin x}{\cos^2 x} \\
 &= \frac{1}{\cos x} \times \frac{\sin x}{\cos x} \\
 &= \sec x \tan x
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad \frac{d}{dx} (\cot x) &= \frac{d}{dx} \left( \frac{1}{\tan x} \right) \\
 &= \frac{d}{dx} (\tan x)^{-1} \\
 &= -(\tan x)^{-2} \times \sec^2 x \\
 &\quad \{\text{chain rule}\} \\
 &= -\frac{\cancel{\cos^2 x}}{\sin^2 x} \times \frac{1}{\cancel{\cos^2 x}} \\
 &= -\frac{1}{\sin^2 x} \\
 &= -\operatorname{cosec}^2 x
 \end{aligned}$$

$$\begin{aligned}
 \text{6 a} \quad f(x) &= \sin^3 x \\
 &= (\sin x)^3 \\
 \therefore f'(x) &= 3(\sin x)^2(\cos x) \quad \{\text{chain rule}\} \\
 &= 3\sin^2 x \cos x \\
 \therefore f'\left(\frac{2\pi}{3}\right) &= 3\sin^2\left(\frac{2\pi}{3}\right)\cos\frac{2\pi}{3} \\
 &= 3\left(\frac{\sqrt{3}}{2}\right)^2\left(-\frac{1}{2}\right) \\
 &= -\frac{9}{8} \\
 \therefore \text{gradient of tangent} &= -\frac{9}{8}
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad f(x) &= \cos x \sin x \\
 \therefore f'(x) &= -\sin x \sin x + \cos x \cos x \quad \{\text{product rule}\} \\
 &= \cos^2 x - \sin^2 x \\
 \therefore f'\left(\frac{\pi}{4}\right) &= \cos^2\left(\frac{\pi}{4}\right) - \sin^2\left(\frac{\pi}{4}\right) \\
 &= 0 \\
 \therefore \text{gradient of tangent} &= 0
 \end{aligned}$$

$$\begin{aligned}
 \text{7 a} \quad f(x) &= 2\cos^2 x + 2\sin^2 x + 1 \\
 &= 2(\cos x)^2 + 2(\sin x)^2 + 1 \\
 \therefore f'(x) &= 2(2\cos x)(-\sin x) + 2(2\sin x)(\cos x) \\
 &= -4\cos x \sin x + 4\sin x \cos x \\
 &= 0
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad f(x) &= 2\cos^2 x + 2\sin^2 x + 1 \\
 &= 2(\cos^2 x + \sin^2 x) + 1 \\
 &= 2(1) + 1 \quad \{\cos^2 x + \sin^2 x = 1\} \\
 &= 3 \quad \text{which is a constant and the derivative of a constant is zero.}
 \end{aligned}$$

8 a Tangent **B** appears to have the steeper gradient.

$$\begin{aligned}
 \text{b} \quad y &= \cos x + 2\sin 2x \\
 \therefore \frac{dy}{dx} &= -\sin x + 2\cos 2x \times 2 \\
 &= 4\cos 2x - \sin x
 \end{aligned}$$

Tangent **A** meets the graph at  $x = \frac{\pi}{6}$ .

$$\begin{aligned}
 \text{When } x = \frac{\pi}{6}, \quad \frac{dy}{dx} &= 4\cos\left(2 \times \frac{\pi}{6}\right) - \sin \frac{\pi}{6} \\
 &= 4\cos \frac{\pi}{3} - \sin \frac{\pi}{6} \\
 &= 2 - \frac{1}{2} \\
 &= \frac{3}{2}
 \end{aligned}$$

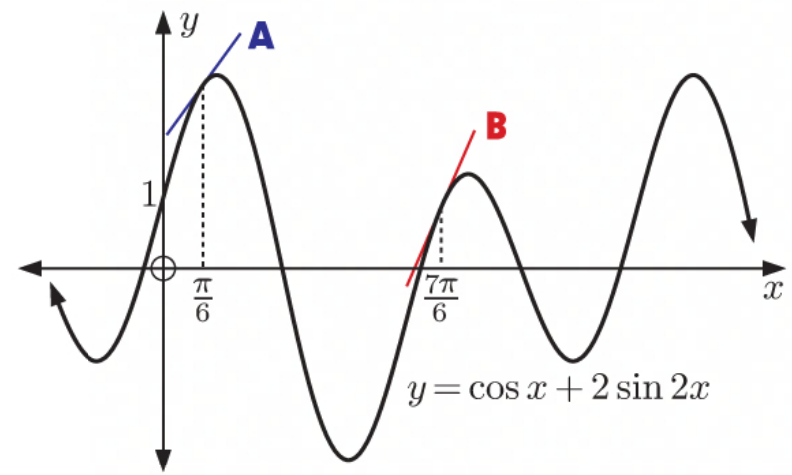
$\therefore$  gradient of tangent **A** is  $\frac{3}{2}$ .

Tangent **B** meets the graph at  $x = \frac{7\pi}{6}$ .

$$\begin{aligned}
 \text{When } x = \frac{7\pi}{6}, \quad \frac{dy}{dx} &= 4\cos\left(2 \times \frac{7\pi}{6}\right) - \sin \frac{7\pi}{6} \\
 &= 4\cos \frac{7\pi}{3} - \sin \frac{7\pi}{6} \\
 &= 2 - \left(-\frac{1}{2}\right) \\
 &= \frac{5}{2}
 \end{aligned}$$

$\therefore$  gradient of tangent **B** is  $\frac{5}{2}$ .

Since  $\frac{5}{2} > \frac{3}{2}$ , tangent **B** has the steeper gradient. ✓



$$9 \quad f(x) = \begin{cases} ax^n, & x < \frac{\pi}{4} \\ \tan^2 x, & \frac{\pi}{4} \leq x \leq \frac{\pi}{2} \end{cases}$$

$$\therefore f'(x) = \begin{cases} anx^{n-1}, & x < \frac{\pi}{4} \\ 2 \tan x \sec^2 x, & \frac{\pi}{4} < x < \frac{\pi}{2} \end{cases} \quad \text{since the derivative of } f \text{ exists on the intervals } x < \frac{\pi}{4} \text{ and } \frac{\pi}{4} < x < \frac{\pi}{2}.$$

$f$  is continuous at  $x = \frac{\pi}{4}$ .

$$\therefore \lim_{x \rightarrow \frac{\pi}{4}^-} f(x) = \lim_{x \rightarrow \frac{\pi}{4}^+} f(x)$$

$$\therefore a \left(\frac{\pi}{4}\right)^n = 1 \quad \dots (*)$$

$f$  is differentiable at  $x = \frac{\pi}{4}$ .

$$\therefore f'_- \left(\frac{\pi}{4}\right) = f'_+ \left(\frac{\pi}{4}\right)$$

$$\therefore an \left(\frac{\pi}{4}\right)^{n-1} = 2 \tan \frac{\pi}{4} \sec^2 \frac{\pi}{4}$$

$$\therefore an \left(\frac{\pi}{4}\right)^{n-1} = 2 \times 1 \times 2$$

$$\therefore an \left(\frac{\pi}{4}\right)^{n-1} \times \frac{\pi}{4} = 4 \times \frac{\pi}{4}$$

$$\therefore an \left(\frac{\pi}{4}\right)^n = \pi$$

$$\therefore n = \pi \quad \{\text{using } (*)\}$$

Substituting into (\*),  $a \left(\frac{\pi}{4}\right)^\pi = 1$

$$\therefore a = \left(\frac{\pi}{4}\right)^{-\pi}$$

$$\therefore a = \left(\frac{4}{\pi}\right)^\pi$$

$$10 \quad a \quad f(x) = \begin{cases} \sin x, & x \geq 0 \\ x^2 + 5x, & x < 0 \end{cases}$$

$$\lim_{x \rightarrow 0^-} f(x) = 0^2 + 5(0) = 0$$

$$\text{and } \lim_{x \rightarrow 0^+} f(x) = \sin 0 = 0$$

$$\therefore \lim_{x \rightarrow 0} f(x) = 0 = f(0)$$

$\therefore f$  is continuous at  $x = 0$ .

$$\text{Now, } f'(x) = \begin{cases} \cos x, & x > 0 \\ 2x + 5, & x < 0 \end{cases} \quad \text{since the derivative of } f \text{ exists on the intervals } x < 0 \text{ and } x > 0.$$

$$\therefore f'_-(0) = 2(0) + 5 = 5 \quad \text{and} \quad f'_+(0) = \cos 0 = 1$$

$$\therefore f'_-(0) \neq f'_+(0)$$

$\therefore f$  is not differentiable at  $x = 0$ .

$$b \quad f'(x) = \begin{cases} \cos x, & x > 0 \\ 2x + 5, & x < 0 \end{cases}$$



**11 a**  $\frac{d}{d\theta} \left( \frac{\cos \theta + i \sin \theta}{e^{i\theta}} \right) = \frac{(-\sin \theta + i \cos \theta) \times \cancel{e^{i\theta}} - (\cos \theta + i \sin \theta) \times i \cancel{e^{i\theta}}}{(e^{i\theta})^2} \quad \{\text{quotient rule}\}$

$$= \frac{-\sin \theta + \cancel{i \cos \theta} - i \cancel{\cos \theta} - i^2 \sin \theta}{e^{i\theta}}$$

$$= \frac{-\sin \theta + \sin \theta}{e^{i\theta}}$$

$$= 0$$

$\therefore \frac{\cos \theta + i \sin \theta}{e^{i\theta}}$  is a constant.

**b**  $\frac{\cos 0 + i \sin 0}{e^0} = \frac{1 + 0}{1} = 1$

**c**  $\frac{\cos \theta + i \sin \theta}{e^{i\theta}} = 1$   
 $\therefore e^{i\theta} = \cos \theta + i \sin \theta$

## EXERCISE 17G.2

**1 a** Suppose  $y = \operatorname{cosec}[f(x)]$   
 $\therefore y = \operatorname{cosec} u$  where  $u = f(x)$   
 But  $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} \quad \{\text{chain rule}\}$   
 $\therefore \frac{dy}{dx} = -\operatorname{cosec} u \cot u \times f'(x)$   
 $= -\operatorname{cosec}[f(x)] \cot[f(x)] f'(x)$

**b** Suppose  $y = \cot[f(x)]$   
 $\therefore y = \cot u$  where  $u = f(x)$   
 But  $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} \quad \{\text{chain rule}\}$   
 $\therefore \frac{dy}{dx} = -\operatorname{cosec}^2 u \times f'(x)$   
 $= -\operatorname{cosec}^2[f(x)] f'(x)$

**2 a**  $\frac{d}{dx} (\operatorname{cosec} 4x) = -\operatorname{cosec} 4x \cot 4x \frac{d}{dx} (4x)$   
 $= -4 \operatorname{cosec} 4x \cot 4x$

**b**  $\frac{d}{dx} (\sec 2x) = \sec 2x \tan 2x \frac{d}{dx} (2x)$   
 $= 2 \sec 2x \tan 2x$

**c**  $\frac{d}{dx} \left( \cot \frac{x}{3} \right) = -\operatorname{cosec}^2 \left( \frac{x}{3} \right) \times \frac{d}{dx} \left( \frac{x}{3} \right)$   
 $= -\frac{1}{3} \operatorname{cosec}^2 \left( \frac{x}{3} \right)$

**d**  $\frac{d}{dx} (2 \operatorname{cosec}^2 2x) = 2 \times \frac{d}{dx} (\operatorname{cosec} 2x)^2$   
 $= 2 \times 2(\operatorname{cosec} 2x) \times (-\operatorname{cosec} 2x \cot 2x) \times 2 \quad \{\text{chain rule}\}$   
 $= -8 \operatorname{cosec}^2 2x \cot 2x$

**e**  $\frac{d}{dx} \left( 8 \cot^3 \left( \frac{x}{2} \right) \right) = 8 \times \frac{d}{dx} \left( \cot \frac{x}{2} \right)^3$   
 $= 8 \times 3 \times \left( \cot \frac{x}{2} \right)^2 \times \left( -\operatorname{cosec}^2 \left( \frac{x}{2} \right) \right) \times \frac{1}{2} \quad \{\text{chain rule}\}$   
 $= -12 \cot^2 \left( \frac{x}{2} \right) \operatorname{cosec}^2 \left( \frac{x}{2} \right)$

$$\begin{aligned}
 \text{f} \quad \frac{d}{dx} \left( \sqrt{\sec \frac{3x}{2}} \right) &= \frac{d}{dx} \left( \sec \frac{3x}{2} \right)^{\frac{1}{2}} \\
 &= \frac{1}{2} \left( \sec \frac{3x}{2} \right)^{-\frac{1}{2}} \times \sec \frac{3x}{2} \tan \frac{3x}{2} \times \frac{3}{2} \quad \{\text{chain rule}\} \\
 &= \frac{3}{4} \left( \sec \frac{3x}{2} \right)^{\frac{1}{2}} \tan \frac{3x}{2} \\
 &= \frac{3}{4} \sqrt{\sec \frac{3x}{2}} \tan \frac{3x}{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{3 a} \quad y &= x \sec x \\
 \therefore \frac{dy}{dx} &= 1 \sec x + x(\sec x \tan x) \quad \{\text{product rule}\} \\
 &= \sec x + x \sec x \tan x \\
 &= \sec x(1 + x \tan x)
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad y &= e^x \cot x \\
 \therefore \frac{dy}{dx} &= e^x \cot x + e^x(-\operatorname{cosec}^2 x) \quad \{\text{product rule}\} \\
 &= e^x(\cot x - \operatorname{cosec}^2 x)
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad y &= 4 \sec 2x \\
 \therefore \frac{dy}{dx} &= 4(\sec 2x \tan 2x \times 2) \quad \{\text{chain rule}\} \\
 &= 8 \sec 2x \tan 2x
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad y &= e^{-x} \cot \frac{x}{2} \\
 \therefore \frac{dy}{dx} &= (-e^{-x}) \cot \frac{x}{2} + e^{-x} \left( -\operatorname{cosec}^2 \left( \frac{x}{2} \right) \times \frac{1}{2} \right) \quad \{\text{product rule, chain rule}\} \\
 &= -e^{-x} \left( \cot \frac{x}{2} + \frac{1}{2} \operatorname{cosec}^2 \left( \frac{x}{2} \right) \right)
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad y &= x^2 \operatorname{cosec} x \\
 \therefore \frac{dy}{dx} &= 2x \operatorname{cosec} x + x^2(-\operatorname{cosec} x \cot x) \quad \{\text{product rule}\} \\
 &= x \operatorname{cosec} x(2 - x \cot x)
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad y &= x \sqrt{\operatorname{cosec} x} \\
 &= x(\operatorname{cosec} x)^{\frac{1}{2}} \\
 \therefore \frac{dy}{dx} &= 1(\operatorname{cosec} x)^{\frac{1}{2}} + x \times \frac{1}{2}(\operatorname{cosec} x)^{-\frac{1}{2}} \times (-\operatorname{cosec} x \cot x) \quad \{\text{product rule, chain rule}\} \\
 &= (\operatorname{cosec} x)^{\frac{1}{2}} - \frac{x}{2}(\operatorname{cosec} x)^{\frac{1}{2}} \cot x \\
 &= \sqrt{\operatorname{cosec} x} \left[ 1 - \frac{x}{2} \cot x \right]
 \end{aligned}$$

$$\begin{aligned}
 \text{g} \quad y &= \ln(\sec x) \\
 \therefore \frac{dy}{dx} &= \frac{\cancel{\sec x} \tan x}{\cancel{\sec x}} \quad \{\text{chain rule}\} \\
 &= \tan x
 \end{aligned}$$

**h**  $y = x \operatorname{cosec}(x^2)$

$$\begin{aligned}\therefore \frac{dy}{dx} &= 1 \operatorname{cosec}(x^2) + x(-\operatorname{cosec}(x^2) \cot(x^2) \times 2x) \quad \{\text{product rule, chain rule}\} \\ &= \operatorname{cosec}(x^2) - 2x^2 \operatorname{cosec}(x^2) \cot(x^2) \\ &= \operatorname{cosec}(x^2)[1 - 2x^2 \cot(x^2)]\end{aligned}$$

**i**  $y = \frac{\cot x}{\sqrt{x}}$

$$\begin{aligned}&= \frac{\cot x}{x^{\frac{1}{2}}} \\ \therefore \frac{dy}{dx} &= \frac{(-\operatorname{cosec}^2 x) \times x^{\frac{1}{2}} - \cot x \times \frac{1}{2}x^{-\frac{1}{2}}}{x} \quad \{\text{quotient rule}\} \\ &= \frac{-x^{\frac{1}{2}} \operatorname{cosec}^2 x - \frac{1}{2}x^{-\frac{1}{2}} \cot x}{x} \\ &= \frac{-\sqrt{x} \times \frac{1}{\sin^2 x} - \frac{1}{2\sqrt{x}} \times \frac{\cos x}{\sin x}}{x} \times \frac{2\sqrt{x} \sin^2 x}{2\sqrt{x} \sin^2 x} \\ &= -\frac{\cos x \sin x + 2x}{2x\sqrt{x} \sin^2 x}\end{aligned}$$

## EXERCISE 17H

**1** If  $y = \arccos x$  then  $x = \cos y$

$$\begin{aligned}\therefore \frac{dx}{dy} &= -\sin y \\ &= -\sqrt{1 - \cos^2 y} \\ \therefore \frac{dx}{dy} &= -\sqrt{1 - x^2}, \quad -1 < x < 1\end{aligned}$$

From the chain rule,  $\frac{dy}{dx} \frac{dx}{dy} = \frac{dy}{dy} = 1$ , so  $\frac{dy}{dx}$  and  $\frac{dx}{dy}$  are reciprocals.

$$\begin{aligned}\therefore \frac{dy}{dx} &= \frac{1}{-\sqrt{1 - x^2}}, \quad -1 < x < 1 \\ &= \frac{-1}{\sqrt{1 - x^2}}, \quad -1 < x < 1\end{aligned}$$

**2** If  $y = \arctan x$  then  $x = \tan y$

$$\begin{aligned}\therefore \frac{dx}{dy} &= \sec^2 y \\ &= 1 + \tan^2 y \\ \therefore \frac{dx}{dy} &= 1 + x^2\end{aligned}$$

From the chain rule,  $\frac{dy}{dx} \frac{dx}{dy} = \frac{dy}{dy} = 1$ , so  $\frac{dy}{dx}$  and  $\frac{dx}{dy}$  are reciprocals.

$$\therefore \frac{dy}{dx} = \frac{1}{1 + x^2}, \quad x \in \mathbb{R}$$

**3 a** Suppose  $y = \arcsin[f(x)]$

$$\therefore y = \arcsin u \quad \text{where } u = f(x)$$

But  $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$  {chain rule}

$$\begin{aligned} \therefore \frac{dy}{dx} &= \frac{1}{\sqrt{1-u^2}} \times f'(x) \\ &= \frac{f'(x)}{\sqrt{1-[f(x)]^2}} \end{aligned}$$

**b** Suppose  $y = \arctan[f(x)]$

$$\therefore y = \arctan u \quad \text{where } u = f(x)$$

But  $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$  {chain rule}

$$\begin{aligned} \therefore \frac{dy}{dx} &= \frac{1}{1+u^2} \times f'(x) \\ &= \frac{f'(x)}{1+[f(x)]^2} \end{aligned}$$

**4 a**  $y = \arctan 2x$

$$\begin{aligned} \therefore \frac{dy}{dx} &= \frac{2}{1+(2x)^2} \\ &= \frac{2}{1+4x^2} \end{aligned}$$

**c**  $y = \arcsin \frac{x}{4}$

$$\begin{aligned} \therefore \frac{dy}{dx} &= \frac{\frac{1}{4}}{\sqrt{1-(\frac{x}{4})^2}} \\ &= \frac{\frac{1}{4}}{\sqrt{1-\frac{x^2}{16}}} \times \frac{\sqrt{16}}{\sqrt{16}} \\ &= \frac{1}{\sqrt{16-x^2}} \end{aligned}$$

**e**  $y = \arctan(x^2)$

$$\begin{aligned} \therefore \frac{dy}{dx} &= \frac{2x}{1+(x^2)^2} \\ &= \frac{2x}{1+x^4} \end{aligned}$$

**b**  $y = \arccos 3x$

$$\begin{aligned} \therefore \frac{dy}{dx} &= \frac{-3}{\sqrt{1-(3x)^2}} \\ &= \frac{-3}{\sqrt{1-9x^2}} \end{aligned}$$

**d**  $y = \arccos \frac{x}{5}$

$$\begin{aligned} \therefore \frac{dy}{dx} &= \frac{-\frac{1}{5}}{\sqrt{1-(\frac{x}{5})^2}} \\ &= \frac{-\frac{1}{5}}{\sqrt{1-\frac{x^2}{25}}} \times \frac{\sqrt{25}}{\sqrt{25}} \\ &= \frac{-1}{\sqrt{25-x^2}} \end{aligned}$$

**f**  $y = \arccos(\sin x)$

$$\begin{aligned} \therefore \frac{dy}{dx} &= \frac{-\cos x}{\sqrt{1-\sin^2 x}} \\ &= \frac{-\cos x}{\sqrt{\cos^2 x}} \\ &= \frac{-\cos x}{\cos x} \\ &= -1 \end{aligned}$$

**5 a**  $y = x \arcsin x$

$$\begin{aligned} \therefore \frac{dy}{dx} &= 1 \arcsin x + x \left( \frac{1}{\sqrt{1-x^2}} \right) \quad \{\text{product rule}\} \\ &= \arcsin x + \frac{x}{\sqrt{1-x^2}} \end{aligned}$$

**b**  $y = e^x \arccos x$

$$\begin{aligned} \therefore \frac{dy}{dx} &= e^x \arccos x + e^x \left( \frac{-1}{\sqrt{1-x^2}} \right) \quad \{\text{product rule}\} \\ &= e^x \arccos x - \frac{e^x}{\sqrt{1-x^2}} \end{aligned}$$



**c**  $y = e^{-x} \arctan x$

$$\begin{aligned}\therefore \frac{dy}{dx} &= (-e^{-x}) \arctan x + e^{-x} \left( \frac{1}{1+x^2} \right) \quad \{\text{product rule}\} \\ &= -e^{-x} \arctan x + \frac{e^{-x}}{1+x^2}\end{aligned}$$

**d**  $y = x^3 \arcsin x$

$$\begin{aligned}\therefore \frac{dy}{dx} &= 3x^2 \arcsin x + x^3 \left( \frac{1}{\sqrt{1-x^2}} \right) \quad \{\text{product rule}\} \\ &= 3x^2 \arcsin x + \frac{x^3}{\sqrt{1-x^2}}\end{aligned}$$

**e**  $y = (x^2 + 1) \arctan 3x$

$$\begin{aligned}\therefore \frac{dy}{dx} &= 2x \arctan 3x + (x^2 + 1) \frac{3}{1+(3x)^2} \quad \{\text{product rule}\} \\ &= 2x \arctan 3x + \frac{3(x^2 + 1)}{1+9x^2}\end{aligned}$$

**f**  $y = 3^x \arccos 2x$

$$\begin{aligned}\therefore \frac{dy}{dx} &= (3^x \ln 3) \arccos 2x + 3^x \left( \frac{-2}{\sqrt{1-(2x)^2}} \right) \quad \{\text{product rule}\} \\ &= 3^x \ln 3 \arccos 2x + 3^x \left( -\frac{2}{\sqrt{1-4x^2}} \right) \\ &= 3^x \left( \ln 3 \arccos 2x - \frac{2}{\sqrt{1-4x^2}} \right)\end{aligned}$$

**6**  $f(x) = \arccos x$  has domain  $\{x \mid -1 \leq x \leq 1\}$

$\therefore \arccos(1+x-x^2)$  is defined when:

$$\begin{array}{ll} 1+x-x^2 \geq -1 & \text{and} \quad 1+x-x^2 \leq 1 \\ \therefore -x^2+x+2 \geq 0 & \therefore x-x^2 \leq 0 \\ \therefore x^2-x-2 \leq 0 & \therefore x(1-x) \leq 0 \\ \therefore (x+1)(x-2) \leq 0 & \end{array}$$

$$\begin{array}{c} + \quad | \quad - \quad | \quad + \\ \leftarrow \quad -1 \quad 2 \quad x \end{array}$$

$$\begin{array}{c} - \quad | \quad + \quad | \quad - \\ \leftarrow \quad 0 \quad 1 \quad x \end{array}$$

$$\therefore x \leq 0 \text{ or } x \geq 1$$

$$\therefore -1 \leq x \leq 2$$

$\therefore f(x) = \arccos(1+x-x^2)$  has domain  $\{x \mid -1 \leq x \leq 0 \text{ or } 1 \leq x \leq 2\}$

$$\begin{aligned}f'(x) &= \frac{-(1-2x)}{\sqrt{1-(1+x-x^2)^2}} \\ &= \frac{2x-1}{\sqrt{1-(1+x-x^2)^2}}\end{aligned}$$

**7 a** P has  $y$ -coordinate  $\frac{\pi}{6}$  and lies on the graph of  $y = \arctan \frac{x}{3}$

$$\therefore \frac{\pi}{6} = \arctan \frac{x}{3}$$

$$\therefore \tan \frac{\pi}{6} = \frac{x}{3}$$

$$\therefore \frac{x}{3} = \frac{1}{\sqrt{3}}$$

$$\therefore x = \sqrt{3}$$

$\therefore$  P has coordinates  $(\sqrt{3}, \frac{\pi}{6})$ .

**b**  $f(x) = \arctan \frac{x}{3}$

$$\begin{aligned} \therefore f'(x) &= \frac{\frac{1}{3}}{1 + (\frac{x}{3})^2} \\ &= \frac{\frac{1}{3}}{1 + \frac{x^2}{9}} \times \frac{9}{9} \end{aligned}$$

$$\therefore f'(x) = \frac{3}{9 + x^2}$$

$$\begin{aligned} \therefore f'(\sqrt{3}) &= \frac{3}{9 + (\sqrt{3})^2} \\ &= \frac{3}{12} \\ &= \frac{1}{4} \end{aligned}$$

$\therefore$  the gradient of the tangent to  $y = f(x)$  at P is  $\frac{1}{4}$ .

**8**  $f(x) = \arcsin x + \arccos x$

$$\therefore f'(x) = \frac{1}{\sqrt{1-x^2}} + \frac{-1}{\sqrt{1-x^2}} = 0$$

To show why this is the case, let  $\arcsin x = \theta$

$$\therefore x = \sin \theta$$

$$= \cos\left(\frac{\pi}{2} - \theta\right)$$

$$\therefore \arccos x = \frac{\pi}{2} - \theta$$

$$\therefore \arccos x = \frac{\pi}{2} - \arcsin x$$

$$\therefore \arcsin x + \arccos x = \frac{\pi}{2}, \text{ which is a constant}$$

**9 a**  $y = \arcsin \frac{x}{a}, \quad a \in \mathbb{R}, \quad a > 0$

$$\begin{aligned} \therefore \frac{dy}{dx} &= \frac{\frac{1}{a}}{\sqrt{1 - (\frac{x}{a})^2}}, \quad -1 < \frac{x}{a} < 1 \\ &= \frac{\frac{1}{a}}{\sqrt{1 - \frac{x^2}{a^2}}} \times \frac{\sqrt{a^2}}{\sqrt{a^2}}, \quad -1 < \frac{x}{a} < 1 \\ &= \frac{1}{\sqrt{a^2 - x^2}}, \quad -a < x < a \end{aligned}$$

**b**  $y = \arctan \frac{x}{a}, \quad a \in \mathbb{R}, \quad a > 0$

$$\begin{aligned} \therefore \frac{dy}{dx} &= \frac{\frac{1}{a}}{1 + (\frac{x}{a})^2} \\ &= \frac{\frac{1}{a}}{1 + \frac{x^2}{a^2}} \times \frac{a^2}{a^2} \\ &= \frac{a}{a^2 + x^2} \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad y &= \arccos \frac{x}{a}, \quad a \in \mathbb{R}, \quad a > 0 \\
 \therefore \frac{dy}{dx} &= \frac{-\frac{1}{a}}{\sqrt{1 - (\frac{x}{a})^2}}, \quad -1 < \frac{x}{a} < 1 \\
 &= \frac{-\frac{1}{a}}{\sqrt{1 - \frac{x^2}{a^2}}} \times \frac{\sqrt{a^2}}{\sqrt{a^2}}, \quad -1 < \frac{x}{a} < 1 \\
 &= -\frac{1}{\sqrt{a^2 - x^2}}, \quad -a < x < a
 \end{aligned}$$

## EXERCISE 17I

$$\begin{aligned}
 \text{1 a} \quad f(x) &= 3x^2 - 6x + 2 \\
 \therefore f'(x) &= 6x - 6 \\
 \therefore f''(x) &= 6
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad f(x) &= \frac{2}{\sqrt{x}} - 1 = 2x^{-\frac{1}{2}} - 1 \\
 \therefore f'(x) &= -x^{-\frac{3}{2}} \\
 f''(x) &= \frac{3}{2}x^{-\frac{5}{2}} \\
 &= \frac{3}{2x^{\frac{5}{2}}} = \frac{3}{2x^2\sqrt{x}}
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad f(x) &= 2x^3 - 3x^2 - x + 5 \\
 \therefore f'(x) &= 6x^2 - 6x - 1 \\
 \therefore f''(x) &= 12x - 6
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad f(x) &= \frac{2-3x}{x^2} = 2x^{-2} - 3x^{-1} \\
 \therefore f'(x) &= -4x^{-3} + 3x^{-2} \\
 \therefore f''(x) &= 12x^{-4} - 6x^{-3} \\
 &= \frac{12-6x}{x^4}
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad f(x) &= (1-2x)^3 \\
 \therefore f'(x) &= 3(1-2x)^2(-2) \quad \{\text{chain rule}\} \\
 &= -6(1-2x)^2 \\
 \therefore f''(x) &= -12(1-2x)(-2) \quad \{\text{chain rule}\} \\
 &= 24(1-2x) \\
 &= 24 - 48x
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad f(x) &= \frac{x+2}{2x-1} \\
 \therefore f'(x) &= \frac{1(2x-1) - (x+2)(2)}{(2x-1)^2} \quad \{\text{quotient rule}\} \\
 &= \frac{2x-1-2x-4}{(2x-1)^2} \\
 &= \frac{-5}{(2x-1)^2} \\
 &= -5(2x-1)^{-2} \\
 \therefore f''(x) &= 10(2x-1)^{-3}(2) \quad \{\text{chain rule}\} \\
 &= \frac{20}{(2x-1)^3}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{2} \quad \mathbf{a} \quad y &= x - x^3 \\
 \therefore \frac{dy}{dx} &= 1 - 3x^2 \\
 \therefore \frac{d^2y}{dx^2} &= -6x
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad y &= x^2 - \frac{5}{x^2} \\
 &= x^2 - 5x^{-2} \\
 \therefore \frac{dy}{dx} &= 2x + 10x^{-3} \\
 \therefore \frac{d^2y}{dx^2} &= 2 - 30x^{-4} \\
 &= 2 - \frac{30}{x^4}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad y &= 2 - \frac{3}{\sqrt{x}} \\
 &= 2 - 3x^{-\frac{1}{2}} \\
 \therefore \frac{dy}{dx} &= \frac{3}{2}x^{-\frac{3}{2}} \\
 \therefore \frac{d^2y}{dx^2} &= -\frac{9}{4}x^{-\frac{5}{2}} = -\frac{9}{4x^2\sqrt{x}}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad y &= \frac{4-x}{x} \\
 &= 4x^{-1} - 1 \\
 \therefore \frac{dy}{dx} &= -4x^{-2} \\
 \therefore \frac{d^2y}{dx^2} &= 8x^{-3} \\
 &= \frac{8}{x^3}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{e} \quad y &= (x^2 - 3x)^3 \\
 \therefore \frac{dy}{dx} &= 3(x^2 - 3x)^2(2x - 3) && \{\text{chain rule}\} \\
 \therefore \frac{d^2y}{dx^2} &= 6(x^2 - 3x)(2x - 3)^2 + 3(x^2 - 3x)^2 \times 2 && \{\text{product rule}\} \\
 &= 6(x^2 - 3x)[(2x - 3)^2 + (x^2 - 3x)] \\
 &= 6(x^2 - 3x)(4x^2 - 12x + 9 + x^2 - 3x) \\
 &= 6(x^2 - 3x)(5x^2 - 15x + 9)
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{f} \quad y &= x^2 - x + \frac{1}{1-x} \\
 &= x^2 - x + (1-x)^{-1} \\
 \therefore \frac{dy}{dx} &= 2x - 1 + (-1)(1-x)^{-2}(-1) \\
 &\hspace{15em} \{\text{chain rule}\} \\
 &= 2x - 1 + (1-x)^{-2} \\
 \therefore \frac{d^2y}{dx^2} &= 2 - 2(1-x)^{-3}(-1) \\
 &= 2 + \frac{2}{(1-x)^3}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{g} \quad y &= e^{3x} + 2x \\
 \therefore \frac{dy}{dx} &= 3e^{3x} + 2 \\
 \therefore \frac{d^2y}{dx^2} &= 3(3e^{3x}) \\
 &= 9e^{3x}
 \end{aligned}$$



**h**  $y = \frac{1 - e^{-x}}{x}$

$$\begin{aligned} \therefore \frac{dy}{dx} &= \frac{(e^{-x})(x) - (1 - e^{-x})(1)}{x^2} && \{\text{quotient rule}\} \\ &= \frac{xe^{-x} - 1 + e^{-x}}{x^2} \\ &= \frac{(x+1)e^{-x} - 1}{x^2} \end{aligned}$$

$$\begin{aligned} \therefore \frac{d^2y}{dx^2} &= \frac{[e^{-x} - (x+1)e^{-x}](x^2) - [(x+1)e^{-x} - 1](2x)}{x^4} && \{\text{quotient rule}\} \\ &= \frac{\cancel{x^2e^{-x}} - x^3e^{-x} - \cancel{2x^2e^{-x}} - 2x^2e^{-x} - 2xe^{-x} + 2x}{x^4} \\ &= \frac{x[-x^2e^{-x} - 2xe^{-x} + 2 - 2e^{-x}]}{x^4} \\ &= \frac{-x^2e^{-x} - 2xe^{-x} + 2 - 2e^{-x}}{x^3} \end{aligned}$$

**i**  $y = \frac{3-x}{xe^x}$

$$\begin{aligned} \therefore \frac{dy}{dx} &= \frac{(-1)(xe^x) - (3-x)e^x(x+1)}{x^2e^{2x}} && \{\text{quotient rule}\} \\ &= \frac{-xe^x - e^x(3x+3-x^2-x)}{x^2e^{2x}} \\ &= \frac{-xe^x - 2xe^x - 3e^x + x^2e^x}{x^2e^{2x}} \\ &= \frac{e^x(x^2 - 3x - 3)}{x^2e^{2x}} \\ &= \frac{x^2 - 3x - 3}{x^2e^x} \end{aligned}$$

$$\begin{aligned} \therefore \frac{d^2y}{dx^2} &= \frac{(2x-3)(x^2e^x) - (x^2-3x-3)(2xe^x + x^2e^x)}{x^4e^{2x}} && \{\text{quotient rule}\} \\ &= \frac{2x^3e^x - 3x^2e^x - (2x^3e^x + x^4e^x - 6x^2e^x - 3x^3e^x - 6xe^x - 3x^2e^x)}{x^4e^{2x}} \\ &= \frac{\cancel{2x^3e^x} - \cancel{3x^2e^x} - \cancel{2x^3e^x} - x^4e^x + 6x^2e^x + 3x^3e^x + 6xe^x + \cancel{3x^2e^x}}{x^4e^{2x}} \\ &= \frac{-xe^x(x^3 - 3x^2 - 6x - 6)}{x^4e^{2x}} \\ &= -\frac{x^3 - 3x^2 - 6x - 6}{x^3e^x} \end{aligned}$$

**3 a**  $f(x) = x^3 - 2x + 5$

$$\begin{aligned} \therefore f(2) &= (2)^3 - 2(2) + 5 \\ &= 9 \end{aligned}$$

**b**  $f(x) = x^3 - 2x + 5$

$$\begin{aligned} \therefore f'(x) &= 3x^2 - 2 \\ \therefore f'(2) &= 3(2)^2 - 2 \\ &= 10 \end{aligned}$$

$$\begin{aligned} \text{c} \quad f'(x) &= 3x^2 - 2 \quad \{\text{from b}\} \\ \therefore f''(x) &= 6x \\ \therefore f''(2) &= 6(2) \\ &= 12 \end{aligned}$$

$$\begin{aligned} \text{d} \quad f''(x) &= 6x \quad \{\text{from c}\} \\ \therefore f^{(3)}(x) &= 6 \\ \therefore f^{(3)}(2) &= 6 \end{aligned}$$

4  $P_n$  is: If  $y = Ae^{bx}$  where  $A$  and  $b$  are constants, then  $\frac{d^n y}{dx^n} = b^n y$ , for  $n \in \mathbb{Z}^+$ .

**Proof:** (By the principle of mathematical induction)

$$(1) \text{ If } n = 1, \text{ LHS} = \frac{dy}{dx} = Ae^{bx} \times b = bAe^{bx} = by \text{ and RHS} = by \quad \therefore P_1 \text{ is true.}$$

$$(2) \text{ If } P_k \text{ is true, then } \frac{d^k y}{dx^k} = b^k y.$$

$$\begin{aligned} \text{Now, } \frac{d^{k+1} y}{dx^{k+1}} &= \frac{d}{dx} \left( \frac{d^k y}{dx^k} \right) \\ &= \frac{d}{dx} (b^k y) \quad \{\text{using } P_k\} \\ &= b^k \times \frac{d}{dx} (y) \quad \{\text{since } b^k \text{ is a constant}\} \\ &= b^k \times bAe^{bx} \\ &= b^{k+1} Ae^{bx} \\ &= b^{k+1} y \end{aligned}$$

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

$$\begin{aligned} \text{5 a} \quad f(x) &= 2x^3 - 6x^2 + 5x + 1 \\ \therefore f'(x) &= 6x^2 - 12x + 5 \\ \therefore f''(x) &= 12x - 12 \\ f''(x) &= 0 \text{ when } 12x - 12 = 0 \\ &\therefore 12x = 12 \\ &\therefore x = 1 \end{aligned}$$

$$\begin{aligned} \text{b} \quad f(x) &= \frac{x}{x^2 + 2} \\ \therefore f'(x) &= \frac{1(x^2 + 2) - x(2x)}{(x^2 + 2)^2} \quad \{\text{quotient rule}\} \\ &= \frac{2 - x^2}{(x^2 + 2)^2} \end{aligned}$$

This is another quotient, this time with  $u = 2 - x^2$  and  $v = (x^2 + 2)^2$   
 $\therefore u' = -2x$  and  $v' = 2(x^2 + 2)(2x)$   
 $= 4x(x^2 + 2)$

$$\begin{aligned}
 \therefore f''(x) &= \frac{-2x(x^2+2)^2 - 4x(x^2+2)(2-x^2)}{(x^2+2)^4} && \{\text{quotient rule}\} \\
 &= \frac{-2x(x^2+2)[x^2+2+2(2-x^2)]}{(x^2+2)^4} \\
 &= \frac{-2x(-x^2+6)}{(x^2+2)^3} \\
 &= \frac{2x(x^2-6)}{(x^2+2)^3}
 \end{aligned}$$

$$\text{So, } f''(x) = 0 \quad \text{when} \quad 2x(x^2-6) = 0$$

$$\therefore x = 0 \quad \text{or} \quad x^2 - 6 = 0$$

$$\therefore x = 0 \quad \text{or} \quad x = \pm\sqrt{6}$$

$$\mathbf{6} \quad f(x) = 2x^3 - x$$

$$\therefore f'(x) = 6x^2 - 1$$

$$\therefore f''(x) = 12x$$

$$f(-1) = 2(-1)^3 - (-1) = -1 \quad \therefore -$$

$$f'(-1) = 6(-1)^2 - 1 = 5 \quad \therefore +$$

$$f''(-1) = 12(-1) = -12 \quad \therefore -$$

$$f(0) = 2(0)^3 - 0 = 0 \quad \therefore 0$$

$$f'(0) = 6(0)^2 - 1 = -1 \quad \therefore -$$

$$f''(0) = 12(0) = 0 \quad \therefore 0$$

$$f(1) = 2(1)^3 - 1 = 1 \quad \therefore +$$

$$f'(1) = 6(1)^2 - 1 = 5 \quad \therefore +$$

$$f''(1) = 12(1) = 12 \quad \therefore +$$

We can fill in the table as follows:

$x$	-1	0	1
$f(x)$	-	0	+
$f'(x)$	+	-	+
$f''(x)$	-	0	+

$$\mathbf{7} \quad f(x) = \frac{2}{3} \sin 3x$$

$$\begin{aligned}
 \therefore f'(x) &= \frac{2}{3} \times (\cos 3x) \times 3 \\
 &= 2 \cos 3x
 \end{aligned}$$

$$\begin{aligned}
 \therefore f''(x) &= 2 \times (-\sin 3x) \times 3 \\
 &= -6 \sin 3x
 \end{aligned}$$

$$\begin{aligned}
 \therefore f^{(3)}(x) &= -6 \times (\cos 3x) \times 3 \\
 &= -18 \cos 3x
 \end{aligned}$$

$$\begin{aligned}
 \therefore f^{(3)}\left(\frac{2\pi}{9}\right) &= -18 \cos\left(3 \times \frac{2\pi}{9}\right) \\
 &= -18 \cos \frac{2\pi}{3} \\
 &= 9
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{8} \quad \mathbf{a} \quad f(x) &= 2 \sin^3 x - 3 \sin x \\
 &= 2(\sin x)^3 - 3 \sin x
 \end{aligned}$$

$$\begin{aligned}
 \therefore f'(x) &= 2 \times 3(\sin x)^2 \times (\cos x) - 3 \cos x && \{\text{chain rule}\} \\
 &= -3 \cos x(1 - 2 \sin^2 x) \\
 &= -3 \cos x \cos 2x
 \end{aligned}$$

$$\begin{aligned} \text{b} \quad f''(x) &= -3(-\sin x \times \cos 2x + \cos x \times (-\sin 2x) \times 2) \quad \{\text{product rule}\} \\ &= 3 \sin x \cos 2x + 6 \cos x \sin 2x \end{aligned}$$

$$\begin{aligned} \text{9 a} \quad y &= -\ln x \\ \therefore \frac{dy}{dx} &= -1 \times \frac{1}{x} \\ &= -x^{-1} \\ \therefore \frac{d^2y}{dx^2} &= -(-x^{-2}) \\ &= x^{-2} = \frac{1}{x^2} \end{aligned}$$

$$\begin{aligned} \text{b} \quad y &= x \ln x \\ \therefore \frac{dy}{dx} &= 1 \times \ln x + x \times \frac{1}{x} \quad \{\text{product rule}\} \\ &= \ln x + 1 \\ \therefore \frac{d^2y}{dx^2} &= \frac{1}{x} \end{aligned}$$

$$\begin{aligned} \text{c} \quad y &= (\ln x)^2 \\ \therefore \frac{dy}{dx} &= 2(\ln x) \left( \frac{1}{x} \right) \\ &= \frac{2 \ln x}{x} \\ \therefore \frac{d^2y}{dx^2} &= \frac{\frac{2}{x} \times x - 2 \ln x \times 1}{x^2} \quad \{\text{quotient rule}\} \\ &= \frac{2 - 2 \ln x}{x^2} \\ &= \frac{2}{x^2} (1 - \ln x) \end{aligned}$$

$$\begin{aligned} \text{10 a} \quad f(x) &= x^2 - \frac{1}{x} \\ \therefore f(1) &= (1)^2 - \frac{1}{1} \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

$$\begin{aligned} \text{b} \quad f(x) &= x^2 - \frac{1}{x} \\ &= x^2 - x^{-1} \\ \therefore f'(x) &= 2x - (-x^{-2}) \\ &= 2x + x^{-2} \\ &= 2x + \frac{1}{x^2} \\ \therefore f'(1) &= 2(1) + \frac{1}{1^2} \\ &= 2 + 1 \\ &= 3 \end{aligned}$$

$$\begin{aligned} \text{c} \quad f'(x) &= 2x + x^{-2} \quad \{\text{from b}\} \\ \therefore f''(x) &= 2 - 2x^{-3} \\ &= 2 - \frac{2}{x^3} \\ \therefore f''(1) &= 2 - \frac{2}{1^3} \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

$$\begin{aligned} \text{d} \quad f''(x) &= 2 - 2x^{-3} \quad \{\text{from c}\} \\ \therefore f^{(3)}(x) &= 6x^{-4} \\ &= \frac{6}{x^4} \\ \therefore f^{(3)}(1) &= \frac{6}{1^4} \\ &= 6 \end{aligned}$$



**11**  $y = 2e^{3x} + 5e^{4x}$

$$\therefore \frac{dy}{dx} = 2e^{3x}(3) + 5e^{4x}(4) \quad \text{and} \quad \frac{d^2y}{dx^2} = 6e^{3x}(3) + 20e^{4x}(4)$$

$$= 6e^{3x} + 20e^{4x} \qquad \qquad \qquad = 18e^{3x} + 80e^{4x}$$

Now  $\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 12y = (18e^{3x} + 80e^{4x}) - 7(6e^{3x} + 20e^{4x}) + 12(2e^{3x} + 5e^{4x})$

$$= 18e^{3x} + 80e^{4x} - 42e^{3x} - 140e^{4x} + 24e^{3x} + 60e^{4x}$$

$$= e^{3x}(18 - 42 + 24) + e^{4x}(80 - 140 + 60)$$

$$= e^{3x}(0) + e^{4x}(0)$$

$$= 0$$

$$\therefore \frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 12y = 0 \quad \text{as required}$$

**12** If  $y = \sin(2x + 3)$ , then  $\frac{dy}{dx} = (\cos(2x + 3))(2) \quad \text{and} \quad \frac{d^2y}{dx^2} = (-2\sin(2x + 3))(2)$

$$= 2\cos(2x + 3) \qquad \qquad \qquad = -4\sin(2x + 3)$$

$$\therefore \frac{d^2y}{dx^2} + 4y = -4\sin(2x + 3) + 4\sin(2x + 3) = 0 \quad \text{as required}$$

**13**  $y = \sin x$

$$\therefore \frac{dy}{dx} = \cos x$$

$$\therefore \frac{d^2y}{dx^2} = -\sin x$$

$$\therefore \frac{d^3y}{dx^3} = -\cos x$$

$$\therefore \frac{d^4y}{dx^4} = \sin x = y$$

**14**  $y = 2\sin x + 3\cos x$

$$\therefore y' = 2\cos x - 3\sin x$$

$$\therefore y'' = -2\sin x - 3\cos x$$

Now  $y'' + y = (-2\sin x - 3\cos x) + (2\sin x + 3\cos x)$

$$= 0 \quad \text{as required}$$

**15**  $P_n$  is: If  $f(x) = e^{ax}(x + 1)$ ,  $a \in \mathbb{R}$ , then  $f^{(n)}(x) = a^{n-1}e^{ax}(a[x + 1] + n)$ ,  $n \in \mathbb{Z}^+$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ , LHS  $= f'(x) = ae^{ax}(x + 1) + e^{ax}$  {product rule}

$$= e^{ax}(a[x + 1] + 1)$$

and RHS  $= e^{ax}(a[x + 1] + 1)$

$$\therefore P_1 \text{ is true.}$$

(2) If  $P_k$  is true, then  $f^{(k)}(x) = a^{k-1}e^{ax}(a[x+1] + k)$ .

$$\begin{aligned}
 \text{Now } f^{(k+1)}(x) &= \frac{d}{dx} (f^{(k)}(x)) \\
 &= \frac{d}{dx} (a^{k-1}e^{ax}(a[x+1] + k)) && \{\text{using } P_k\} \\
 &= a^{k-1} \times ae^{ax}(a[x+1] + k) + a^{k-1}e^{ax} \times a && \{\text{product rule}\} \\
 &= a^k e^{ax}(a[x+1] + k) + a^k e^{ax} \\
 &= a^k e^{ax}(a[x+1] + k + 1)
 \end{aligned}$$

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,  
 $P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

<p><b>16 a i</b> <math>f(x) = e^{-x}(x+2)</math></p> <p><math>\therefore f'(x) = (-e^{-x})(x+2) + e^{-x}(1)</math></p> <p style="text-align: center;">{product rule}</p> <p><math>= -xe^{-x} - 2e^{-x} + e^{-x}</math></p> <p><math>= -xe^{-x} - e^{-x}</math></p> <p><math>= -e^{-x}(x+1)</math></p>	<p><b>ii</b> <math>f'(x) = -e^{-x}(x+1)</math> {from i}</p> <p><math>\therefore f''(x) = e^{-x}(x+1) - e^{-x}(1)</math></p> <p style="text-align: center;">{product rule}</p> <p><math>= xe^{-x} + \cancel{e^{-x}} - \cancel{e^{-x}}</math></p> <p><math>= e^{-x}(x)</math></p>
<p><b>iii</b> <math>f''(x) = e^{-x}(x)</math> {from ii}</p> <p><math>\therefore f^{(3)}(x) = (-e^{-x})(x) + e^{-x}(1)</math></p> <p style="text-align: center;">{product rule}</p> <p><math>= -xe^{-x} + e^{-x}</math></p> <p><math>= -e^{-x}(x-1)</math></p>	<p><b>iv</b> <math>f^{(3)}(x) = -e^{-x}(x-1)</math> {from iii}</p> <p><math>\therefore f^{(4)}(x) = e^{-x}(x-1) - e^{-x}(1)</math></p> <p style="text-align: center;">{product rule}</p> <p><math>= xe^{-x} - e^{-x} - e^{-x}</math></p> <p><math>= xe^{-x} - 2e^{-x}</math></p> <p><math>= e^{-x}(x-2)</math></p>

**b** We conjecture that  $f^{(n)}(x) = (-1)^n e^{-x}(x-n+2)$ ,  $n \in \mathbb{Z}^+$ .

**c**  $P_n$  is: If  $f(x) = e^{-x}(x+2)$ , then  $f^{(n)}(x) = (-1)^n e^{-x}(x-n+2)$ ,  $n \in \mathbb{Z}^+$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ , LHS =  $f'(x) = -e^{-x}(x+1)$  {from a i}

$$\begin{aligned}
 \text{RHS} &= f(x) = (-1)^1 e^{-x}(x-1+2) \\
 &= -e^{-x}(x+1)
 \end{aligned}$$

$\therefore P_1$  is true.

(2) If  $P_k$  is true, then  $f^{(k)}(x) = (-1)^k e^{-x}(x-k+2)$ .

$$\begin{aligned}
 \text{Now } f^{(k+1)}(x) &= \frac{d}{dx} (f^{(k)}(x)) \\
 &= \frac{d}{dx} ((-1)^k e^{-x}(x-k+2)) && \{\text{using } P_k\} \\
 &= (-1)^k (-e^{-x})(x-k+2) + (-1)^k e^{-x}(1) \\
 &= (-1)^{k+1} e^{-x}(x-k+2) - (-1)^{k+1} e^{-x} \\
 &= (-1)^{k+1} e^{-x}(x-k+1) \\
 &= (-1)^{k+1} e^{-x}(x-(k+1)+2)
 \end{aligned}$$

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,  
 $P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

$$\begin{aligned}
 \mathbf{17} \quad \mathbf{a} \quad \cos\left(x - \frac{\pi}{2}\right) &= \cos x \cos \frac{\pi}{2} + \sin x \sin \frac{\pi}{2} \\
 &= \cos x \times 0 + \sin x \times 1 \\
 &= \sin x
 \end{aligned}$$

$\mathbf{b}$   $P_n$  is: If  $f(x) = \cos ax$ ,  $a \in \mathbb{R}$ , then  $f^{(n)}(x) = (-a)^n \cos(ax - \frac{n\pi}{2})$  for all  $n \in \mathbb{Z}^+$ .

**Proof:** (By the principle of mathematical induction)

$$\begin{aligned}
 (1) \quad \text{If } n = 1, \quad \text{LHS} &= f'(x) = -a \sin ax \\
 \text{RHS} &= f'(x) = (-a)^1 \cos(ax - \frac{\pi}{2}) \\
 &= -a \sin ax \quad \{\text{using } \mathbf{a}\}
 \end{aligned}$$

$\therefore P_1$  is true.

$$(2) \quad \text{If } P_k \text{ is true, then } f^{(k)}(x) = (-a)^k \cos(ax - \frac{k\pi}{2}).$$

$$\begin{aligned}
 \text{Now } f^{(k+1)}(x) &= \frac{d}{dx} ((-a)^k \cos(ax - \frac{k\pi}{2})) \quad \{\text{using } P_k\} \\
 &= (-a)^k \times (-\sin(ax - \frac{k\pi}{2})) \times a \\
 &= (-a)^{k+1} \sin(ax - \frac{k\pi}{2}) \\
 &= (-a)^{k+1} \cos(ax - \frac{k\pi}{2} - \frac{\pi}{2}) \quad \{\text{using } \mathbf{a}\} \\
 &= (-a)^{k+1} \cos(ax - \frac{(k+1)\pi}{2})
 \end{aligned}$$

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

$\mathbf{18}$   $P_n$  is: If  $f(x)$  is an even differentiable function for  $x \in \mathbb{R}$ , then

$$f^{(n)}(x) \text{ is } \begin{cases} \text{even if } n \text{ is even} \\ \text{odd if } n \text{ is odd} \end{cases}, \text{ for all } n \in \mathbb{Z}^+.$$

**Proof:** (By the principle of mathematical induction)

$$(1) \quad \text{Since } f(x) \text{ is even, } f(x) = f(-x)$$

$$\begin{aligned}
 \therefore f'(x) &= \frac{d}{dx} f(-x) \\
 &= f'(-x) \times (-1) \quad \{\text{chain rule}\} \\
 &= -f'(-x) \\
 \therefore f'(-x) &= -f'(x) \quad \text{so } f'(x) \text{ is odd} \quad \therefore P_1 \text{ is true.}
 \end{aligned}$$

$$(2) \quad \text{If } P_k \text{ is true, } f^{(k)}(x) \text{ is } \begin{cases} \text{even if } k \text{ is even} \\ \text{odd if } k \text{ is odd} \end{cases}.$$

*Case 1:* If  $k+1$  is even, then  $k$  is odd and  $f^{(k)}(x)$  is odd. {using  $P_k$ }

$$\text{So, } f^{(k)}(-x) = -f^{(k)}(x)$$

$$\therefore f^{(k)}(x) = -f^{(k)}(-x)$$

$$\begin{aligned}
 \therefore f^{(k+1)}(x) &= -\frac{d}{dx} f^{(k)}(-x) \\
 &= -f^{(k+1)}(-x) \times (-1) \quad \{\text{chain rule}\} \\
 &= f^{(k+1)}(-x) \quad \text{so } f^{(k+1)}(x) \text{ is even.}
 \end{aligned}$$



Case 2: If  $k + 1$  is odd, then  $k$  is even and  $f^{(k)}(x)$  is even. {using  $P_k$ }

$$\text{So, } f^{(k)}(x) = f^{(k)}(-x)$$

$$\begin{aligned}\therefore f^{(k+1)}(x) &= \frac{d}{dx} f^{(k)}(-x) \\ &= f^{(k+1)}(-x) \times (-1) \quad \{\text{chain rule}\} \\ &= -f^{(k+1)}(-x)\end{aligned}$$

$$\therefore f^{(k+1)}(-x) = -f^{(k+1)}(x) \quad \text{so } f^{(k+1)}(x) \text{ is odd.}$$

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

## EXERCISE 17J

$$1 \quad \mathbf{a} \quad \frac{d}{dy}(2y) = 2 \frac{dy}{dx} \qquad \mathbf{b} \quad \frac{d}{dx}(-3y) = -3 \frac{dy}{dx} \qquad \mathbf{c} \quad \frac{d}{dx}(2y^3) = 6y^2 \frac{dy}{dx}$$

$$\begin{aligned}\mathbf{d} \quad \frac{d}{dx}\left(-\frac{4}{y}\right) &= \frac{d}{dx}(-4y^{-1}) & \mathbf{e} \quad \frac{d}{dx}(y^4) &= 4y^3 \frac{dy}{dx} & \mathbf{f} \quad \frac{d}{dx}(\sqrt{y}) &= \frac{d}{dx}(y^{\frac{1}{2}}) \\ &= 4y^{-2} \frac{dy}{dx} & & & &= \frac{1}{2}y^{-\frac{1}{2}} \frac{dy}{dx} \\ &= \frac{4}{y^2} \frac{dy}{dx} & & & &= \frac{1}{2\sqrt{y}} \frac{dy}{dx}\end{aligned}$$

$$\begin{aligned}\mathbf{g} \quad \frac{d}{dx}\left(\frac{1}{y^2}\right) &= \frac{d}{dx}(y^{-2}) & \mathbf{h} \quad \frac{d}{dx}(xy) &= 1 \times y + x \times \frac{dy}{dx} \\ &= -2y^{-3} \frac{dy}{dx} & & \{\text{product rule}\} \\ &= -\frac{2}{y^3} \frac{dy}{dx} & & = y + x \frac{dy}{dx}\end{aligned}$$

$$\begin{aligned}\mathbf{i} \quad \frac{d}{dx}(x^2y) &= 2x \times y + x^2 \times \frac{dy}{dx} & \mathbf{j} \quad \frac{d}{dx}(xy^3) &= 1 \times y^3 + x \times 3y^2 \frac{dy}{dx} \\ & \{\text{product rule}\} & & \{\text{product rule}\} \\ &= 2xy + x^2 \frac{dy}{dx} & & = y^3 + 3xy^2 \frac{dy}{dx}\end{aligned}$$

$$\begin{aligned}2 \quad \mathbf{a} \quad & x^2 + y^2 = 25 \\ \therefore \frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) &= \frac{d}{dx}(25) \\ \therefore 2x + 2y \frac{dy}{dx} &= 0 \\ \therefore 2y \frac{dy}{dx} &= -2x \\ \therefore \frac{dy}{dx} &= -\frac{2x}{2y} = -\frac{x}{y}\end{aligned}$$

$$\begin{aligned}\mathbf{b} \quad & x^2 + 3y^2 = 9 \\ \therefore \frac{d}{dx}(x^2) + \frac{d}{dx}(3y^2) &= \frac{d}{dx}(9) \\ \therefore 2x + 6y \frac{dy}{dx} &= 0 \\ \therefore 6y \frac{dy}{dx} &= -2x \\ \therefore \frac{dy}{dx} &= -\frac{2x}{6y} = -\frac{x}{3y}\end{aligned}$$



$$\begin{aligned}
 \text{c} \quad & y^2 - x^2 = 8 \\
 \therefore \frac{d}{dx}(y^2) - \frac{d}{dx}(x^2) &= \frac{d}{dx}(8) \\
 \therefore 2y \frac{dy}{dx} - 2x &= 0 \\
 \therefore 2y \frac{dy}{dx} &= 2x \\
 \therefore \frac{dy}{dx} &= \frac{2x}{2y} = \frac{x}{y}
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad & x^2 - y^3 = 10 \\
 \therefore \frac{d}{dx}(x^2) - \frac{d}{dx}(y^3) &= \frac{d}{dx}(10) \\
 \therefore 2x - 3y^2 \frac{dy}{dx} &= 0 \\
 \therefore 3y^2 \frac{dy}{dx} &= 2x \\
 \therefore \frac{dy}{dx} &= \frac{2x}{3y^2}
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & x^2 + xy = 4 \\
 \therefore 2x + \left(y + x \frac{dy}{dx}\right) &= 0 \\
 \therefore x \frac{dy}{dx} &= -2x - y \\
 \therefore \frac{dy}{dx} &= \frac{-2x - y}{x}
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & x^3 - 2xy = 5 \\
 \therefore 3x^2 - \left(2y + 2x \frac{dy}{dx}\right) &= 0 \\
 \therefore 3x^2 - 2y &= 2x \frac{dy}{dx} \\
 \therefore \frac{dy}{dx} &= \frac{3x^2 - 2y}{2x}
 \end{aligned}$$

$$\begin{aligned}
 \text{g} \quad & xy + \frac{2}{x} = 12 \\
 \therefore \left(y + x \frac{dy}{dx}\right) - \frac{2}{x^2} &= 0 \\
 \therefore x \frac{dy}{dx} &= \frac{2}{x^2} - y \\
 \therefore \frac{dy}{dx} &= \frac{2}{x^3} - \frac{y}{x}
 \end{aligned}$$

$$\begin{aligned}
 \text{h} \quad & x + y^2 + 2xy = 7 \\
 \therefore 1 + 2y \frac{dy}{dx} + \left(2y + 2x \frac{dy}{dx}\right) &= 0 \\
 \therefore (2x + 2y) \frac{dy}{dx} &= -1 - 2y \\
 \therefore \frac{dy}{dx} &= \frac{-1 - 2y}{2x + 2y}
 \end{aligned}$$

$$\begin{aligned}
 \text{i} \quad & x^3 - 2y^2 + xy^2 = x \\
 \therefore 3x^2 - 4y \frac{dy}{dx} + \left(y^2 + 2xy \frac{dy}{dx}\right) &= 1 \\
 \therefore (2xy - 4y) \frac{dy}{dx} &= 1 - 3x^2 - y^2 \\
 \therefore \frac{dy}{dx} &= \frac{1 - 3x^2 - y^2}{2xy - 4y}
 \end{aligned}$$

$$\begin{aligned}
 \text{j} \quad & \frac{x}{y} + y = 0 \\
 \therefore \left(\frac{y - x \frac{dy}{dx}}{y^2}\right) + \frac{dy}{dx} &= 0 \\
 & \text{\{quotient rule\}} \\
 \therefore y - x \frac{dy}{dx} + y^2 \frac{dy}{dx} &= 0 \\
 \therefore (y^2 - x) \frac{dy}{dx} &= -y \\
 \therefore \frac{dy}{dx} &= \frac{y}{x - y^2}
 \end{aligned}$$

$$\begin{aligned}
 \text{k} \quad & x + \cos y = 1 \\
 \therefore 1 - \sin y \frac{dy}{dx} &= 0 \\
 \therefore \sin y \frac{dy}{dx} &= 1 \\
 \therefore \frac{dy}{dx} &= \frac{1}{\sin y}
 \end{aligned}$$

$$\begin{aligned}
 \text{l} \quad & \sin y + xe^y = 2y \\
 \therefore \cos y \frac{dy}{dx} + \left(e^y + xe^y \frac{dy}{dx}\right) &= 2 \frac{dy}{dx} \\
 \therefore (\cos y + xe^y - 2) \frac{dy}{dx} &= -e^y \\
 \therefore \frac{dy}{dx} &= \frac{-e^y}{\cos y + xe^y - 2}
 \end{aligned}$$

**m**

$$2^x + 3^y = 10$$

$$\therefore 2^x \ln 2 + 3^y \ln 3 \times \frac{dy}{dx} = 0$$

$$\therefore 3^y \ln 3 \times \frac{dy}{dx} = -2^x \ln 2$$

$$\therefore \frac{dy}{dx} = -\frac{2^x \ln 2}{3^y \ln 3}$$

**n**

$$\ln(xy) = e^y$$

$$\therefore \ln x + \ln y = e^y$$

$$\therefore \frac{1}{x} + \frac{\frac{dy}{dx}}{y} = e^y \frac{dy}{dx}$$

$$\therefore e^y \frac{dy}{dx} - \frac{1}{y} \frac{dy}{dx} = \frac{1}{x}$$

$$\therefore ye^y \frac{dy}{dx} - \frac{dy}{dx} = \frac{y}{x}$$

$$\therefore (ye^y - 1) \frac{dy}{dx} = \frac{y}{x}$$

$$\therefore \frac{dy}{dx} = \frac{y}{x(ye^y - 1)}$$

**o**

$$y + x \sec y = \arccos x$$

$$\therefore \frac{dy}{dx} + \left( \sec y + x \sec y \tan y \frac{dy}{dx} \right) = \frac{-1}{\sqrt{1-x^2}}$$

$$\therefore (1 + x \sec y \tan y) \frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}} - \sec y$$

$$\therefore \frac{dy}{dx} = -\frac{\frac{1}{\sqrt{1-x^2}} + \sec y}{1 + x \sec y \tan y} \times \frac{\sqrt{1-x^2}}{\sqrt{1-x^2}}$$

$$= -\frac{1 + \sec y \sqrt{1-x^2}}{(1 + x \sec y \tan y) \sqrt{1-x^2}}$$

**3**

$$y = \ln x$$

$$\therefore e^y = x$$

$$\therefore e^y \frac{dy}{dx} = 1$$

$$\therefore \frac{dy}{dx} = \frac{1}{e^y}$$

$$= \frac{1}{x} \quad \{\text{since } e^y = x\}$$

**4 a**

$$x^3 e^{3y} + 4x^2 y^3 = 27e^{-2x}$$

$$\therefore 3x^2 e^{3y} + x^3 \left( 3e^{3y} \frac{dy}{dx} \right) + 8xy^3 + 4x^2 \left( 3y^2 \frac{dy}{dx} \right) = -54e^{-2x}$$

$$\therefore 3x^2 e^{3y} + 3x^3 e^{3y} \frac{dy}{dx} + 8xy^3 + 12x^2 y^2 \frac{dy}{dx} = -54e^{-2x}$$

$$\therefore 3x^2 (xe^{3y} + 4y^2) \frac{dy}{dx} = -54e^{-2x} - 3x^2 e^{3y} - 8xy^3$$

$$\therefore \frac{dy}{dx} = \frac{-(54e^{-2x} + 3x^2 e^{3y} + 8xy^3)}{3x^2 (xe^{3y} + 4y^2)}$$

**b**

$$e^{2a} \ln(b^2) - a^3 b + \ln(ab) = 21$$

$$\therefore 2e^{2a} \ln b - a^3 b + \ln a + \ln b = 21$$

$$\therefore 4e^{2a} \frac{da}{db} \times \ln b + 2e^{2a} \times \frac{1}{b} - \left( 3a^2 b \frac{da}{db} + a^3 \right) + \frac{\frac{da}{db}}{a} + \frac{1}{b} = 0$$

$$\therefore 4abe^{2a} \ln b \frac{da}{db} + 2ae^{2a} - 3a^3 b^2 \frac{da}{db} - a^4 b + b \frac{da}{db} + a = 0$$

$$\therefore \frac{da}{db} (4abe^{2a} \ln b - 3a^3 b^2 + b) = a^4 b - 2ae^{2a} - a$$

$$\therefore \frac{da}{db} = \frac{a^4 b - 2ae^{2a} - a}{4abe^{2a} \ln b - 3a^3 b^2 + b}$$

**5**

$$y = x^x$$

$$\therefore \ln y = \ln(x^x)$$

$$\therefore \ln y = x \ln x$$

$$\therefore \frac{\frac{dy}{dx}}{y} = \ln x + 1$$

$$\therefore \frac{dy}{dx} = y(\ln x + 1)$$

$$\therefore \frac{dy}{dx} = x^x (\ln x + 1)$$

**6 a**

$$x^2 + y^2 = 25$$

$$\therefore 2x + 2y \frac{dy}{dx} = 0 \quad \text{or} \quad \frac{dy}{dx} = -\frac{x}{y} \quad \dots (*)$$

$$\therefore 2 + 2 \frac{dy}{dx} \times \frac{dy}{dx} + 2y \frac{d^2 y}{dx^2} = 0$$

$$\therefore 1 + \left( \frac{dy}{dx} \right)^2 + y \frac{d^2 y}{dx^2} = 0$$

$$\therefore 1 + \left( -\frac{x}{y} \right)^2 + y \frac{d^2 y}{dx^2} = 0 \quad \{\text{using } (*)\}$$

$$\therefore 1 + \frac{x^2}{y^2} + y \frac{d^2 y}{dx^2} = 0$$

$$\therefore y^2 + x^2 + y^3 \frac{d^2 y}{dx^2} = 0$$

$$\therefore y^3 \frac{d^2 y}{dx^2} = -y^2 - x^2$$

$$\therefore \frac{d^2 y}{dx^2} = \frac{-y^2 - x^2}{y^3}$$

**b**

$$x^2 - y^2 = 10$$

$$\therefore 2x - 2y \frac{dy}{dx} = 0 \quad \text{or} \quad \frac{dy}{dx} = \frac{x}{y} \quad \dots (*)$$

$$\therefore 2 - \left( 2 \frac{dy}{dx} \times \frac{dy}{dx} + 2y \frac{d^2y}{dx^2} \right) = 0$$

$$\therefore 2 - 2 \left( \frac{dy}{dx} \right)^2 - 2y \frac{d^2y}{dx^2} = 0$$

$$\therefore 1 - \left( \frac{x}{y} \right)^2 - y \frac{d^2y}{dx^2} = 0 \quad \{\text{using } (*)\}$$

$$\therefore 1 - \frac{x^2}{y^2} - y \frac{d^2y}{dx^2} = 0$$

$$\therefore y^2 - x^2 - y^3 \frac{d^2y}{dx^2} = 0$$

$$\therefore y^3 \frac{d^2y}{dx^2} = y^2 - x^2$$

$$\therefore \frac{d^2y}{dx^2} = \frac{y^2 - x^2}{y^3}$$

**c**

$$x^3 + 2xy = 4$$

$$\therefore 3x^2 + 2y + 2x \frac{dy}{dx} = 0 \quad \text{or} \quad 2x \frac{dy}{dx} = -3x^2 - 2y$$

$$\frac{dy}{dx} = \frac{-3x^2 - 2y}{2x} \quad \dots (*)$$

$$\therefore 6x + 2 \frac{dy}{dx} + 2 \frac{dy}{dx} + 2x \frac{d^2y}{dx^2} = 0$$

$$\therefore 6x + 4 \frac{dy}{dx} + 2x \frac{d^2y}{dx^2} = 0$$

$$\therefore 6x + 4 \left( \frac{-3x^2 - 2y}{2x} \right) + 2x \frac{d^2y}{dx^2} = 0 \quad \{\text{using } (*)\}$$

$$\therefore 6x - \frac{6x^2 + 4y}{x} + 2x \frac{d^2y}{dx^2} = 0$$

$$\therefore \cancel{6x^2} - \cancel{6x^2} - 4y + 2x^2 \frac{d^2y}{dx^2} = 0$$

$$\therefore 2x^2 \frac{d^2y}{dx^2} = 4y$$

$$\therefore \frac{d^2y}{dx^2} = \frac{2y}{x^2}$$

**7 a**

$$3V^2 + 2q = 2Vq$$

$$\therefore 6V \frac{dV}{dq} + 2 = 2 \frac{dV}{dq} q + 2V$$

$$\therefore (6V - 2q) \frac{dV}{dq} = 2V - 2$$

$$\therefore \frac{dV}{dq} = \frac{2V - 2}{6V - 2q}$$

$$\therefore \frac{dV}{dq} = \frac{V - 1}{3V - q}$$



**b**

$$3V^2 + 2q = 2Vq$$

$$\therefore 6V + 2 \frac{dq}{dV} = 2q + 2V \frac{dq}{dV} \quad \text{or} \quad (2 - 2V) \frac{dq}{dV} = 2q - 6V$$

$$\begin{aligned} \therefore \frac{dq}{dV} &= \frac{2q - 6V}{2 - 2V} \\ &= \frac{q - 3V}{1 - V} \quad \dots (*) \end{aligned}$$

$$\therefore 6 + 2 \frac{d^2q}{dV^2} = 2 \frac{dq}{dV} + 2 \frac{dq}{dV} + 2V \frac{d^2q}{dV^2} \quad \{\text{product rule}\}$$

$$\therefore (2 - 2V) \frac{d^2q}{dV^2} = 4 \frac{dq}{dV} - 6$$

$$\therefore (2 - 2V) \frac{d^2q}{dV^2} = 4 \left( \frac{q - 3V}{1 - V} \right) - 6 \quad \{\text{using } (*)\}$$

$$\therefore 2(1 - V) \frac{d^2q}{dV^2} = \frac{4q - 12V}{1 - V} - \frac{6(1 - V)}{1 - V}$$

$$\therefore 2(1 - V) \frac{d^2q}{dV^2} = \frac{4q - 6V - 6}{1 - V}$$

$$\therefore \frac{d^2q}{dV^2} = \frac{2q - 3V - 3}{(1 - V)^2}$$

**8 a**

$$xy^2 - 3x - y = 0$$

$$\therefore \left( y^2 + 2xy \frac{dy}{dx} \right) - 3 - \frac{dy}{dx} = 0 \quad \{\text{product rule}\}$$

$$\therefore (2xy - 1) \frac{dy}{dx} = 3 - y^2$$

$$\therefore \frac{dy}{dx} = \frac{3 - y^2}{2xy - 1}$$

**b i** At the point (0, 0),

$$\frac{dy}{dx} = \frac{3 - (0)^2}{2(0)(0) - 1} = \frac{3}{-1} = -3$$

$\therefore$  the gradient of the tangent to the relation at (0, 0) is  $-3$ .

**ii** At the point (2, 2),

$$\frac{dy}{dx} = \frac{3 - (2)^2}{2(2)(2) - 1} = \frac{-1}{7}$$

$\therefore$  the gradient of the tangent to the relation at (2, 2) is  $-\frac{1}{7}$ .

**9 a**

$$x + y^2 = 5$$

$$\therefore 1 + 2y \frac{dy}{dx} = 0$$

$$\therefore 2y \frac{dy}{dx} = -1$$

$$\therefore \frac{dy}{dx} = \frac{-1}{2y}$$

At the point (1, 2),  $\frac{dy}{dx} = \frac{-1}{2(2)} = -\frac{1}{4}$

$\therefore$  the gradient of the tangent to  $x + y^2 = 5$  at (1, 2) is  $-\frac{1}{4}$ .

**b**

$$x^3 - y^2 = -1$$

$$\therefore 3x^2 - 2y \frac{dy}{dx} = 0$$

$$\therefore 2y \frac{dy}{dx} = 3x^2$$

$$\therefore \frac{dy}{dx} = \frac{3x^2}{2y}$$

At the point (2, -3),

$$\frac{dy}{dx} = \frac{3(2)^2}{2(-3)} = \frac{12}{-6} = -2$$

$\therefore$  the gradient of the tangent to  $x^3 - y^2 = -1$  at (2, -3) is  $-2$ .

$$\text{c} \quad x + y^3 = 4y$$

$$\therefore 1 + 3y^2 \frac{dy}{dx} = 4 \frac{dy}{dx}$$

$$\therefore (4 - 3y^2) \frac{dy}{dx} = 1$$

$$\therefore \frac{dy}{dx} = \frac{1}{4 - 3y^2}$$

$$\text{When } y = 1, \quad \frac{dy}{dx} = \frac{1}{4 - 3(1)^2} = 1$$

$\therefore$  the gradient of the tangent to  $x + y^3 = 4y$  at  $y = 1$  is 1.

$$\text{d} \quad x + y = 8xy$$

$$\therefore 1 + \frac{dy}{dx} = 8y + 8x \frac{dy}{dx} \quad \{\text{product rule}\}$$

$$\therefore (1 - 8x) \frac{dy}{dx} = 8y - 1$$

$$\therefore \frac{dy}{dx} = \frac{8y - 1}{1 - 8x}$$

$$\text{When } x = \frac{1}{2}, \quad \frac{1}{2} + y = 8\left(\frac{1}{2}\right)y \quad \therefore \text{ at the point } \left(\frac{1}{2}, \frac{1}{6}\right), \quad \frac{dy}{dx} = \frac{8\left(\frac{1}{6}\right) - 1}{1 - 8\left(\frac{1}{2}\right)}$$

$$\therefore \frac{1}{2} + y = 4y$$

$$\therefore -3y = -\frac{1}{2}$$

$$\therefore y = \frac{1}{6}$$

$$= \frac{\frac{4}{3} - 1}{1 - 4}$$

$$= \frac{\frac{1}{3}}{-3}$$

$$= -\frac{1}{9}$$

$\therefore$  the gradient of the tangent to  $x + y = 8xy$  at  $x = \frac{1}{2}$  is  $-\frac{1}{9}$ .

$$\text{e} \quad \frac{x}{y^2} - x = 2$$

$$\therefore \frac{y^2 - 2xy \frac{dy}{dx}}{(y^2)^2} - 1 = 0 \quad \{\text{quotient rule}\}$$

$$\therefore y^2 - 2xy \frac{dy}{dx} = y^4$$

$$\therefore -2xy \frac{dy}{dx} = y^4 - y^2$$

$$\therefore \frac{dy}{dx} = \frac{y^4 - y^2}{-2xy} = \frac{y - y^3}{2x}$$

$$\text{When } y = 2, \quad \frac{x}{2^2} - x = 2 \quad \text{At the point } \left(-\frac{8}{3}, 2\right), \quad \frac{dy}{dx} = \frac{2 - 2^3}{2\left(-\frac{8}{3}\right)}$$

$$\therefore x - 4x = 8$$

$$\therefore -3x = 8$$

$$\therefore x = -\frac{8}{3}$$

$$= \frac{-6}{-\frac{16}{3}}$$

$$= \frac{18}{16}$$

$$= \frac{9}{8}$$

$\therefore$  the gradient of the tangent to  $\frac{x}{y^2} - x = 2$  at  $y = 2$  is  $\frac{9}{8}$ .

**f**

$$x^3 - xy^3 = y - 1$$

$$\therefore 3x^2 - \left(y^3 + 3xy^2 \frac{dy}{dx}\right) = \frac{dy}{dx} \quad \{\text{product rule}\}$$

$$\therefore (1 + 3xy^2) \frac{dy}{dx} = 3x^2 - y^3$$

$$\therefore \frac{dy}{dx} = \frac{3x^2 - y^3}{1 + 3xy^2}$$

When  $x = 1$ ,  $1^3 - (1)y^3 = y - 1$

$$\therefore y^3 + y - 2 = 0$$

We observe that  $y = 1$  is a solution.

$$\therefore (y - 1)(y^2 + y + 2) = 0$$

But  $y^2 + y + 2 > 0$  for all  $y$

$$\therefore y = 1$$

At the point  $(1, 1)$ ,  $\frac{dy}{dx} = \frac{3(1)^2 - 1^3}{1 + 3(1)(1)^2} = \frac{2}{4} = \frac{1}{2}$

$\therefore$  the gradient of the tangent to  $x^3 - xy^3 = y - 1$  at  $x = 1$  is  $\frac{1}{2}$ .

**10 a**

$$\frac{2x^2}{y} - \frac{y}{x} = 1$$

$$\therefore \left( \frac{4xy - 2x^2 \frac{dy}{dx}}{y^2} \right) - \left( \frac{x \frac{dy}{dx} - y}{x^2} \right) = 0 \quad \{\text{quotient rule}\}$$

$$\therefore x^2 \left( 4xy - 2x^2 \frac{dy}{dx} \right) - y^2 \left( x \frac{dy}{dx} - y \right) = 0$$

$$\therefore 4x^3y - 2x^4 \frac{dy}{dx} - xy^2 \frac{dy}{dx} + y^3 = 0$$

$$\therefore (2x^4 + xy^2) \frac{dy}{dx} = 4x^3y + y^3$$

$$\therefore \frac{dy}{dx} = \frac{4x^3y + y^3}{2x^4 + xy^2}$$

**b i** When  $x = 1$ ,  $\frac{2}{y} - y = 1$

$$\therefore 2 - y^2 = y$$

$$\therefore y^2 + y - 2 = 0$$

$$\therefore (y + 2)(y - 1) = 0$$

$$\therefore y = 1 \text{ or } -2$$

$\therefore$  P has coordinates  $(1, 1)$ , and Q has coordinates  $(1, -2)$ .

ii At the point  $P(1, 1)$ ,

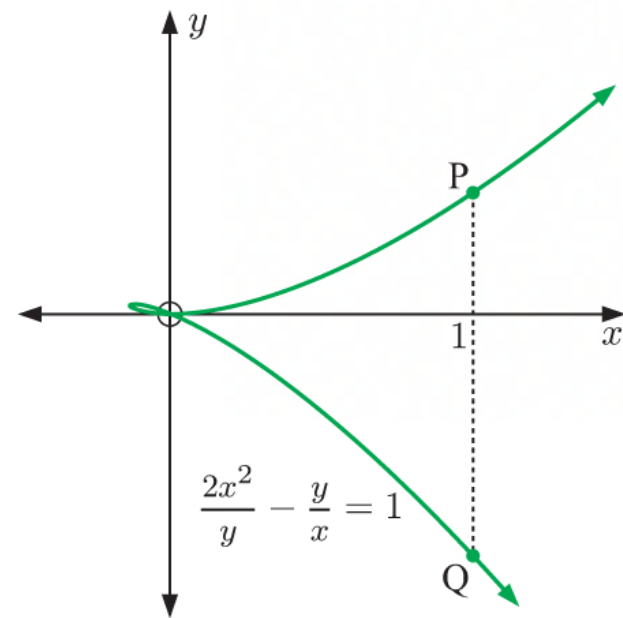
$$\frac{dy}{dx} = \frac{4(1)^3(1) + 1^3}{2(1)^4 + (1)(1)^2} = \frac{5}{3}$$

$\therefore$  the gradient of the tangent to the graph at  $P$  is  $\frac{5}{3}$ .

At the point  $Q(1, -2)$ ,

$$\frac{dy}{dx} = \frac{4(1)^3(-2) + (-2)^3}{2(1)^4 + (1)(-2)^2} = \frac{-16}{6} = -\frac{8}{3}$$

$\therefore$  the gradient of the tangent to the graph at  $Q$  is  $-\frac{8}{3}$ .



11 a

$$x^2 - 3xy + y^2 = 7$$

$$\therefore 2x - 3y - 3x \frac{dy}{dx} + 2y \frac{dy}{dx} = 0 \quad \{\text{product rule}\}$$

$$\therefore (2y - 3x) \frac{dy}{dx} = 3y - 2x$$

$$\therefore \frac{dy}{dx} = \frac{3y - 2x}{2y - 3x}$$

b The gradient of the tangent to the curve is  $\frac{2}{3}$  at the points  $(x, y)$  where

$$\frac{dy}{dx} = \frac{3y - 2x}{2y - 3x} = \frac{2}{3}$$

$$\therefore 3(3y - 2x) = 2(2y - 3x)$$

$$\therefore 9y - 6x = 4y - 6x$$

$$\therefore 9y = 4y$$

$$\therefore 5y = 0$$

$$\therefore y = 0$$

$$\text{When } y = 0, \quad x^2 - 3x(0) + 0^2 = 7$$

$$\therefore x^2 = 7$$

$$\therefore x = \pm\sqrt{7}$$

$\therefore$  the gradient of the tangent to the curve is  $\frac{2}{3}$  at  $(\sqrt{7}, 0)$  and  $(-\sqrt{7}, 0)$ .

12 a  $(x^2 + y^2)^2 = x^2 - y^2$

The graph cuts the  $x$ -axis when  $y = 0$ .

$$\therefore x^4 = x^2$$

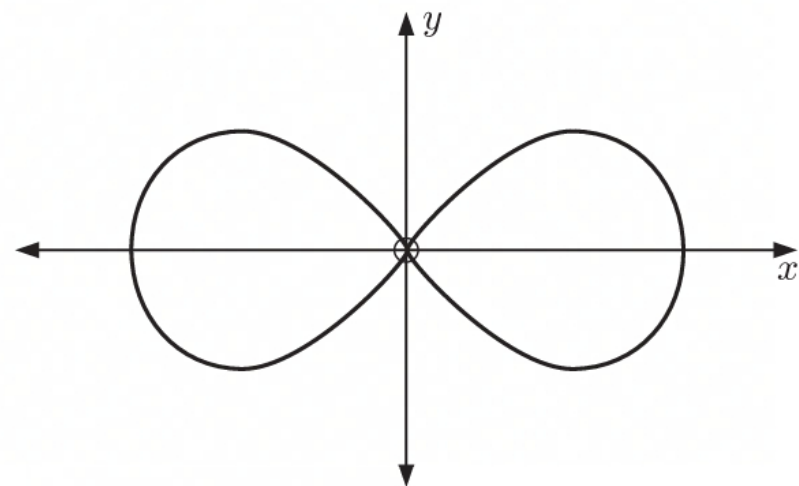
$$\therefore x^4 - x^2 = 0$$

$$\therefore x^2(x^2 - 1) = 0$$

$$\therefore x^2(x + 1)(x - 1) = 0$$

$$\therefore x = 0, \pm 1$$

$\therefore$  the  $x$ -intercepts are  $-1, 0$ , and  $1$ .





The graph cuts the  $y$ -axis when  $x = 0$ .

$$\therefore y^4 = -y^2$$

$$\therefore y^4 + y^2 = 0$$

$$\therefore y^2(y^2 + 1) = 0$$

$$\therefore y^2 = 0 \quad \{y^2 + 1 > 0\}$$

$\therefore$  the  $y$ -intercept is 0.

**b**

$$(x^2 + y^2)^2 = x^2 - y^2$$

$$\therefore 2(x^2 + y^2) \left[ 2x + 2y \frac{dy}{dx} \right] = 2x - 2y \frac{dy}{dx} \quad \{\text{chain rule}\}$$

$$\therefore 4x(x^2 + y^2) + 4y(x^2 + y^2) \frac{dy}{dx} = 2x - 2y \frac{dy}{dx}$$

$$\therefore [4y(x^2 + y^2) + 2y] \frac{dy}{dx} = 2x - 4x(x^2 + y^2)$$

$$\therefore \frac{dy}{dx} = \frac{x - 2x(x^2 + y^2)}{2y(x^2 + y^2) + y}$$

which is 0 when  $x[1 - 2(x^2 + y^2)] = 0$

But  $x$  is clearly not 0, since if  $x = 0$ ,  $y = 0$  and  $\frac{dy}{dx}$  is undefined.

$$\therefore x^2 + y^2 = \frac{1}{2} \quad \dots (1)$$

Substituting into the original equation,

$$\left(\frac{1}{2}\right)^2 = x^2 - y^2$$

$$\therefore x^2 - y^2 = \frac{1}{4} \quad \dots (2)$$

Adding (1) and (2) gives  $2x^2 = \frac{3}{4}$

$$\therefore x^2 = \frac{3}{8}$$

$$\therefore x = \pm \frac{\sqrt{3}}{2\sqrt{2}} = \pm \frac{\sqrt{6}}{4}$$

Subtracting (2) from (1) gives  $2y^2 = \frac{1}{4}$

$$\therefore y^2 = \frac{1}{8}$$

$$\therefore y = \pm \frac{1}{2\sqrt{2}} = \pm \frac{\sqrt{2}}{4}$$

$\therefore$  the four points on the graph at which the tangent is horizontal are  $\left(\frac{\sqrt{6}}{4}, \frac{\sqrt{2}}{4}\right)$ ,  $\left(\frac{\sqrt{6}}{4}, -\frac{\sqrt{2}}{4}\right)$ ,  $\left(-\frac{\sqrt{6}}{4}, \frac{\sqrt{2}}{4}\right)$ , and  $\left(-\frac{\sqrt{6}}{4}, -\frac{\sqrt{2}}{4}\right)$ .

**13**  $\sin x + \cos y = 0.5 \quad \dots (1)$

When  $x = 0$ ,  $\sin 0 + \cos y = 0.5$

$$\therefore \cos y = 0.5$$

$$\therefore y = \pm \frac{\pi}{3} \quad \left\{ -\frac{2\pi}{3} \leq y \leq \frac{2\pi}{3} \right\}$$

$\therefore$  P is at  $(0, \frac{\pi}{3})$ .

Now, differentiating (1) with respect to  $x$  gives

$$\cos x - \sin y \times \frac{dy}{dx} = 0 \quad \{\text{chain rule}\}$$

$$\therefore -\sin y \frac{dy}{dx} = -\cos x$$

$$\therefore \frac{dy}{dx} = \frac{\cos x}{\sin y}$$

$\therefore$  the gradient of the tangent through P is  $\frac{\cos 0}{\sin \frac{\pi}{3}} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}}$

Since the tangents through P and Q are perpendicular, the gradient of the tangent through Q is  $-\frac{\sqrt{3}}{2}$ .

$\therefore$  at  $Q(x, y)$ ,  $\frac{dy}{dx} = \frac{\cos x}{\sin y} = -\frac{\sqrt{3}}{2}$

$$\therefore -2 \cos x = \sqrt{3} \sin y \quad \dots (2)$$

$$= \sqrt{3} \sqrt{1 - \cos^2 y} \quad \{0 < y < \frac{2\pi}{3} \text{ from the graph}\}$$

$$\therefore 4 \cos^2 x = 3(1 - \cos^2 y)$$

$$\therefore 4(1 - \sin^2 x) = 3(1 - (\frac{1}{2} - \sin x)^2) \quad \{\text{using (1)}\}$$

$$\begin{aligned} \therefore 4 - 4 \sin^2 x &= 3 - 3(\frac{1}{4} - \sin x + \sin^2 x) \\ &= \frac{9}{4} + 3 \sin x - 3 \sin^2 x \end{aligned}$$

$$\therefore \sin^2 x + 3 \sin x - \frac{7}{4} = 0$$

$$\therefore 4 \sin^2 x + 12 \sin x - 7 = 0$$

$$\therefore (2 \sin x + 7)(2 \sin x - 1) = 0$$

$$\therefore \sin x = \frac{1}{2} \quad \{-1 \leq \sin x \leq 1\}$$

$$\therefore x = \frac{\pi}{6} \text{ or } \frac{5\pi}{6} \quad \{0 < x < \frac{7\pi}{6} \text{ from the graph}\}$$

If  $x = \frac{\pi}{6}$ , then  $\sin y = -\frac{2}{\sqrt{3}} \left( \frac{\sqrt{3}}{2} \right) = -1 \quad \{\text{using (2)}\}$

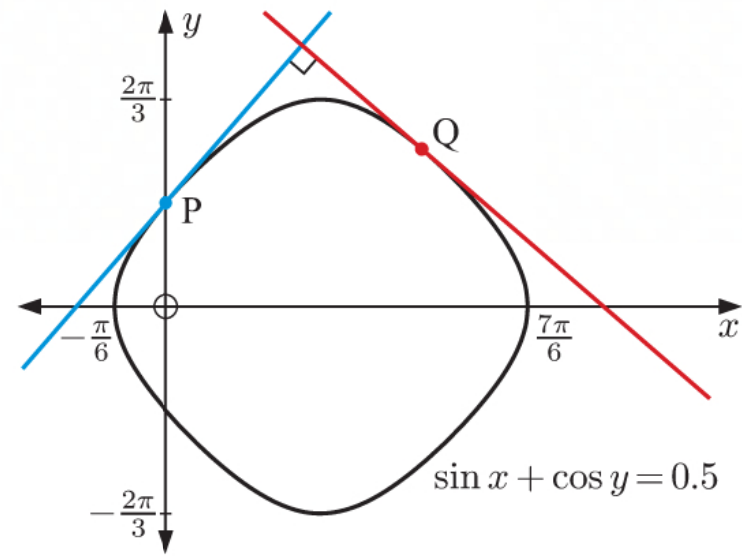
$$\therefore y = -\frac{\pi}{2} \quad \left\{ -\frac{2\pi}{3} \leq y \leq \frac{2\pi}{3} \right\}$$

But we know that  $y > 0$  from the graph, so  $x = \frac{5\pi}{6}$ .

$$\therefore \sin y = -\frac{2}{\sqrt{3}} \left( -\frac{\sqrt{3}}{2} \right) = 1 \quad \{\text{using (2)}\}$$

$$\therefore y = \frac{\pi}{2} \quad \left\{ -\frac{2\pi}{3} \leq y \leq \frac{2\pi}{3} \right\}$$

So, Q is at  $(\frac{5\pi}{6}, \frac{\pi}{2})$ .



$$14 \quad x^2 + 3y^2 = 48$$

$$\therefore 2x + 6y \frac{dy}{dx} = 0$$

$$\therefore 6y \frac{dy}{dx} = -2x$$

$$\therefore \frac{dy}{dx} = -\frac{x}{3y}$$

$\therefore$  the gradient of the tangent to a point  $(a, b)$  on the curve is  $-\frac{a}{3b}$ .

$\therefore$  the normal to a point  $(a, b)$  on the curve has gradient  $\frac{3b}{a}$ .

$\therefore$  the equation of the normal to  $(a, b)$  is  $3bx - ay = 3b(a) - a(b)$

$$\therefore 3bx - ay = 3ab - ab$$

$$\therefore 3bx - ay = 2ab$$

The normal passes through  $(0, 2)$ , so  $3b(0) - a(2) = 2ab$

$$\therefore -2a = 2ab$$

$$\therefore 2a + 2ab = 0$$

$$\therefore 2a(1 + b) = 0$$

$$\therefore a = 0 \text{ or } b = -1$$

When  $a = 0$ ,  $3b^2 = 48$

$$\therefore b^2 = 16$$

$$\therefore b = \pm 4$$

When  $b = -1$ ,  $a^2 + 3(-1)^2 = 48$

$$\therefore a^2 + 3 = 48$$

$$\therefore a^2 = 45$$

$$\therefore a = \pm 3\sqrt{5}$$

$\therefore$  the points on the ellipse such that the normal at those points passes through  $(0, 2)$  are  $(0, 4)$ ,  $(0, -4)$ ,  $(-3\sqrt{5}, -1)$ , and  $(3\sqrt{5}, -1)$ .

## REVIEW SET 17A

$$1 \quad \mathbf{a} \quad f(x) = 5x^3$$

$$\therefore f'(x) = 15x^2$$

$$\mathbf{c} \quad f(x) = 7x^2 - \frac{3}{x}$$

$$= 7x^2 - 3x^{-1}$$

$$\therefore f'(x) = 7(2x) - 3(-x^{-2})$$

$$= 14x + 3x^{-2}$$

$$= 14x + \frac{3}{x^2}$$

$$\mathbf{e} \quad f(x) = 2x\sqrt{x} = 2x^{\frac{3}{2}}$$

$$\therefore f'(x) = 2\left(\frac{3}{2}x^{\frac{1}{2}}\right)$$

$$= 3x^{\frac{1}{2}}$$

$$= 3\sqrt{x}$$

$$\mathbf{b} \quad f(x) = x^6 - 5x$$

$$\therefore f'(x) = 6x^5 - 5$$

$$\mathbf{d} \quad f(x) = 3x - \frac{4}{x^2}$$

$$= 3x - 4x^{-2}$$

$$\therefore f'(x) = 3 - 4(-2x^{-3})$$

$$= 3 + 8x^{-3}$$

$$= 3 + \frac{8}{x^3}$$

$$\mathbf{f} \quad f(x) = 4\sqrt{x} - \frac{1}{\sqrt{x}} = 4x^{\frac{1}{2}} - x^{-\frac{1}{2}}$$

$$\therefore f'(x) = 4\left(\frac{1}{2}x^{-\frac{1}{2}}\right) - \left(-\frac{1}{2}x^{-\frac{3}{2}}\right)$$

$$= 2x^{-\frac{1}{2}} + \frac{1}{2}x^{-\frac{3}{2}}$$

$$= \frac{2}{\sqrt{x}} + \frac{1}{2x\sqrt{x}}$$

$$2 \quad a \quad y = 3x^2 - x^4$$

$$\therefore \frac{dy}{dx} = 3(2x) - 4x^3 \\ = 6x - 4x^3$$

$$b \quad y = \frac{x^3 - x}{x^2}$$

$$= x - x^{-1} \\ \therefore \frac{dy}{dx} = 1 - (-x^{-2}) \\ = 1 + x^{-2} \\ = 1 + \frac{1}{x^2}$$

$$c \quad y = x^2\sqrt{x-2} \text{ is the product of}$$

$$u = x^2 \quad \text{and} \quad v = (x-2)^{\frac{1}{2}}$$

$$\therefore u' = 2x \quad \text{and} \quad v' = \frac{1}{2}(x-2)^{-\frac{1}{2}}(1) \quad \{\text{chain rule}\} \\ = \frac{1}{2}(x-2)^{-\frac{1}{2}}$$

$$\text{Now } \frac{dy}{dx} = u'v + uv' \quad \{\text{product rule}\} \\ = 2x(x-2)^{\frac{1}{2}} + x^2\left(\frac{1}{2}(x-2)^{-\frac{1}{2}}\right) \\ = 2x(x-2)^{\frac{1}{2}} + \frac{1}{2}x^2(x-2)^{-\frac{1}{2}} \\ = 2x\sqrt{x-2} + \frac{x^2}{2\sqrt{x-2}}$$

$$3 \quad a \quad f(x) = \frac{x}{\sqrt{x^2+1}} \text{ is a quotient with } u = x \quad \text{and} \quad v = (x^2+1)^{\frac{1}{2}} \\ \therefore u' = 1 \quad \text{and} \quad v' = \frac{1}{2}(x^2+1)^{-\frac{1}{2}} \times 2x \\ = x(x^2+1)^{-\frac{1}{2}}$$

$$\text{Now } f'(x) = \frac{u'v - uv'}{v^2} \quad \{\text{quotient rule}\} \\ = \frac{1 \times (x^2+1)^{\frac{1}{2}} - x \times x(x^2+1)^{-\frac{1}{2}}}{\left((x^2+1)^{\frac{1}{2}}\right)^2} \\ = \frac{\sqrt{x^2+1} - \frac{x^2}{\sqrt{x^2+1}}}{x^2+1} \\ = \frac{\sqrt{x^2+1} \times \frac{\sqrt{x^2+1}}{\sqrt{x^2+1}} - \frac{x^2}{\sqrt{x^2+1}}}{x^2+1} \\ = \frac{(x^2+1) - x^2}{(x^2+1)\sqrt{x^2+1}} \\ = \frac{1}{(x^2+1)\sqrt{x^2+1}} \\ = (x^2+1)^{-\frac{3}{2}}$$



**b** The tangent to  $f(x)$  has gradient 1 when  $f'(x) = 1$

$$\therefore (x^2 + 1)^{-\frac{3}{2}} = 1$$

$$\therefore x^2 + 1 = 1$$

$$\therefore x^2 = 0$$

$$\therefore x = 0$$

$$\text{and } f(0) = \frac{0}{\sqrt{0^2 + 1}} = 0$$

$\therefore$  the tangent to  $f(x) = \frac{x}{\sqrt{x^2 + 1}}$  has gradient 1 at the point  $(0, 0)$ .

**4 a**  $y = e^{x^3+2}$   
 $= e^u$  where  $u = x^3 + 2$

Now  $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$  {chain rule}

$$= e^u \frac{du}{dx}$$

$$= e^{x^3+2} \times 3x^2$$

$$= 3x^2 e^{x^3+2}$$

**b**  $y = \ln\left(\frac{x+3}{x^2}\right)$   
 $= \ln(x+3) - \ln(x^2)$   
 $\left\{ \ln\left(\frac{a}{b}\right) = \ln a - \ln b \right\}$

$$\therefore \frac{dy}{dx} = \frac{1}{x+3} - \frac{2x}{x^2}$$

$$= \frac{1}{x+3} - \frac{2}{x}$$

**c**  $y = x^3 e^{2x}$   
 $\therefore \frac{dy}{dx} = 3x^2 e^{2x} + x^3 e^{2x}(2)$  {product rule}  
 $= 3x^2 e^{2x} + 2x^3 e^{2x}$

**5 a**  $f(x) = 5^{\sin x}$   
 $\therefore f'(x) = 5^{\sin x} \ln 5 \times \cos x$

**b**  $f(x) = \log_2(x^2 + 4)$   
 $\therefore f'(x) = \frac{2x}{(x^2 + 4) \ln 2}$

**c**  $f(x) = \log_3(4^x)$   
 $\therefore f'(x) = \frac{\cancel{4^x} \ln 4}{\cancel{4^x} \ln 3}$   
 $= \frac{\ln 4}{\ln 3}$

**6 a**  $f(x) = -x^2 + 4x - 2$   
 $\therefore f'(x) = -2x + 4$   
 At the point  $(-3, -23)$ ,  
 $f'(-3) = -2(-3) + 4$   
 $= 10$   
 So, the gradient of the tangent is 10.

**b**  $y = (2 - 3x)^5$   
 $\therefore \frac{dy}{dx} = 5(2 - 3x)^4 \times (-3)$  {chain rule}  
 $= -15(2 - 3x)^4$   
 When  $x = 1$ ,  $\frac{dy}{dx} = -15(2 - 3(1))^4$   
 $= -15(-1)^4$   
 $= -15$

So, the gradient of the tangent is  $-15$ .

$$\begin{array}{ll}
 \text{7 a} & y = 5x - 3x^{-1} \\
 & \therefore \frac{dy}{dx} = 5 + 3x^{-2} \\
 \text{b} & y = (3x^2 + \sqrt{x})^4 = \left(3x^2 + x^{\frac{1}{2}}\right)^4 \\
 & \therefore \frac{dy}{dx} = 4 \left(3x^2 + x^{\frac{1}{2}}\right)^3 \left(6x + \frac{1}{2}x^{-\frac{1}{2}}\right) \quad \{\text{chain rule}\}
 \end{array}$$

$$\begin{array}{l}
 \text{c } y = (x^2 + 1)(1 - x^2)^3 \text{ is a product with } u = x^2 + 1 \text{ and } v = (1 - x^2)^3 \\
 \therefore u' = 2x \quad \text{and} \quad v' = 3(1 - x^2)^2(-2x) \\
 \qquad \qquad \qquad = -6x(1 - x^2)^2 \\
 \therefore \frac{dy}{dx} = 2x(1 - x^2)^3 + (x^2 + 1)[-6x(1 - x^2)^2] \quad \{\text{product rule}\} \\
 \qquad \qquad \qquad = 2x(1 - x^2)^3 - 6x(x^2 + 1)(1 - x^2)^2
 \end{array}$$

$$\begin{array}{l}
 \text{8} \quad y = 2x^3 + 3x^2 - 10x + 3 \\
 \therefore \frac{dy}{dx} = 6x^2 + 6x - 10
 \end{array}$$

$$\begin{array}{l}
 \text{The gradient of the tangent is 2 when } 6x^2 + 6x - 10 = 2 \\
 \therefore 6x^2 + 6x - 12 = 0 \\
 \therefore x^2 + x - 2 = 0 \\
 \therefore (x + 2)(x - 1) = 0 \\
 \therefore x = -2 \text{ or } 1
 \end{array}$$

$$\begin{array}{l}
 \text{When } x = -2, \quad y = 2(-2)^3 + 3(-2)^2 - 10(-2) + 3 \\
 \qquad \qquad \qquad = 19
 \end{array}$$

$$\begin{array}{l}
 \text{When } x = 1, \quad y = 2(1)^3 + 3(1)^2 - 10(1) + 3 \\
 \qquad \qquad \qquad = -2
 \end{array}$$

So, the gradient of the tangent to  $y = 2x^3 + 3x^2 - 10x + 3$  is 2 at the points  $(-2, 19)$  and  $(1, -2)$ .

$$\begin{array}{ll}
 \text{9 a} & \frac{d}{dx} (\sin 5x \ln x) \\
 & = (\cos 5x)(5) \ln x + \sin 5x \left(\frac{1}{x}\right) \\
 & \qquad \qquad \qquad \{\text{product rule}\} \\
 & = (5 \cos 5x) \ln x + \frac{\sin 5x}{x} \\
 \text{b} & \frac{d}{dx} (\sin x \cos 2x) \\
 & = \cos x \cos 2x + \sin x(-\sin 2x)(2) \\
 & \qquad \qquad \qquad \{\text{product rule}\} \\
 & = \cos x \cos 2x - 2 \sin x \sin 2x
 \end{array}$$

$$\begin{array}{l}
 \text{c} \quad \frac{d}{dx} (e^{-2x} \tan x) \\
 = (-2e^{-2x})(\tan x) + (e^{-2x})(\sec^2 x) \quad \{\text{product rule}\} \\
 = -2e^{-2x} \tan x + e^{-2x} \sec^2 x
 \end{array}$$

$$\begin{array}{l}
 \text{10} \quad y = \sin^2 x \\
 \qquad \qquad = (\sin x)^2 \\
 \therefore \frac{dy}{dx} = 2 \sin x \cos x \quad \{\text{chain rule}\}
 \end{array}$$

$$\begin{array}{l}
 \text{When } x = \frac{\pi}{3}, \quad \frac{dy}{dx} = 2 \sin \frac{\pi}{3} \cos \frac{\pi}{3} \\
 \qquad \qquad \qquad = \frac{\sqrt{3}}{2}
 \end{array}$$

$$\therefore \text{gradient of tangent} = \frac{\sqrt{3}}{2}$$

$$\begin{aligned}
 11 \quad y &= 3e^x - e^{-x} \\
 \therefore \frac{dy}{dx} &= 3e^x + e^{-x} \\
 \therefore \frac{d^2y}{dx^2} &= 3e^x - e^{-x} = y \quad \text{as required}
 \end{aligned}$$

$$\begin{aligned}
 12 \quad a \quad f(x) &= \frac{x^2 - 4x - 1}{e^x} \\
 \therefore f'(x) &= \frac{(2x - 4)e^x - (x^2 - 4x - 1)e^x}{(e^x)^2} \quad \{\text{quotient rule}\} \\
 &= \frac{e^x(2x - 4 - x^2 + 4x + 1)}{(e^x)^2} \\
 &= \frac{-x^2 + 6x - 3}{e^x}
 \end{aligned}$$

$$\begin{aligned}
 b \quad f'(1) &= \frac{-1^2 + 6(1) - 3}{e^1} \\
 &= \frac{2}{e} \\
 \therefore \text{gradient of tangent} &= \frac{2}{e}
 \end{aligned}$$

c The tangent to  $y = f(x)$  is horizontal when  $f'(x) = 0$ .

$$\therefore -x^2 + 6x - 3 = 0$$

$$\therefore x^2 - 6x + 3 = 0$$

$$\begin{aligned}
 \therefore x &= \frac{6 \pm \sqrt{(-6)^2 - 4(1)(3)}}{2(1)} \\
 &= \frac{6 \pm \sqrt{24}}{2} \\
 &= 3 \pm \sqrt{6}
 \end{aligned}$$

$$\begin{aligned}
 13 \quad a \quad f(x) &= (x^2 + 3)^4 \\
 \therefore f'(x) &= 4(x^2 + 3)^3(2x) \quad \{\text{chain rule}\} \\
 &= 8x(x^2 + 3)^3
 \end{aligned}$$

$$\begin{aligned}
 b \quad g(x) &= \frac{\sqrt{x+5}}{x^2} \text{ is a quotient with } u = (x+5)^{\frac{1}{2}} \quad \text{and} \quad v = x^2 \\
 \therefore u' &= \frac{1}{2}(x+5)^{-\frac{1}{2}} \quad \text{and} \quad v' = 2x
 \end{aligned}$$

$$\begin{aligned}
 \therefore g'(x) &= \frac{\frac{1}{2}(x+5)^{-\frac{1}{2}}(x^2) - (x+5)^{\frac{1}{2}}(2x)}{(x^2)^2} \quad \{\text{quotient rule}\} \\
 &= \frac{\frac{1}{2}x(x+5)^{-\frac{1}{2}} - 2(x+5)^{\frac{1}{2}}}{x^3}
 \end{aligned}$$

$$\begin{aligned}
 c \quad h(x) &= \frac{e^{4x}}{1-2x} \text{ is a quotient with } u = e^{4x} \quad \text{and} \quad v = 1-2x \\
 \therefore u' &= 4e^{4x} \quad \text{and} \quad v' = -2
 \end{aligned}$$

$$\begin{aligned}
 \therefore h'(x) &= \frac{4e^{4x}(1-2x) - e^{4x}(-2)}{(1-2x)^2} \quad \{\text{quotient rule}\} \\
 &= \frac{4e^{4x} - 8xe^{4x} + 2e^{4x}}{(1-2x)^2} \\
 &= \frac{6e^{4x} - 8xe^{4x}}{(1-2x)^2}
 \end{aligned}$$

**14 a**  $y = \sin x \cos x$

$$\therefore \frac{dy}{dx} = \cos x \cos x + \sin x(-\sin x)$$

{product rule}

$$= \cos^2 x - \sin^2 x$$

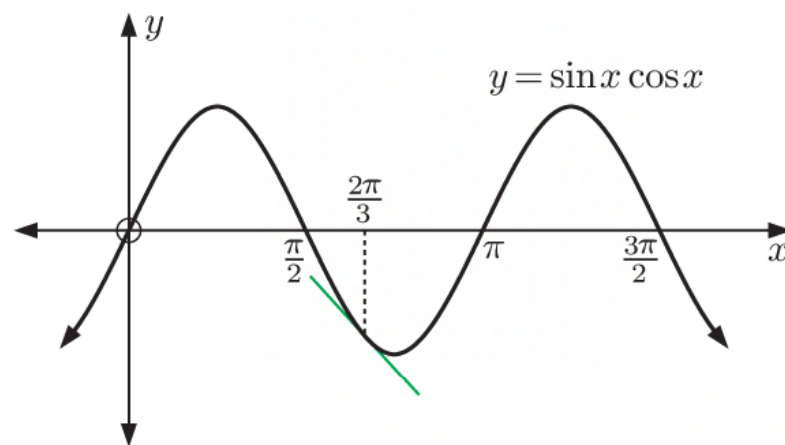
**b**  $y = \frac{1}{2} \sin 2x$

$$\therefore \frac{dy}{dx} = \left(\frac{1}{2} \cos 2x\right)(2)$$

$$= \cos 2x$$

$$= \cos^2 x - \sin^2 x \quad \{\text{double angle formula}\}$$

which is the same derivative as in **a** ✓



- c** The tangent meets the graph of  $y = \sin x \cos x$  at the point where  $x = \frac{2\pi}{3}$ .

$$\begin{aligned} \text{When } x = \frac{2\pi}{3}, \quad \frac{dy}{dx} &= \cos^2\left(\frac{2\pi}{3}\right) - \sin^2\left(\frac{2\pi}{3}\right) \\ &= \frac{1}{4} - \frac{3}{4} \\ &= -\frac{1}{2} \end{aligned}$$

$$\therefore \text{gradient of tangent} = -\frac{1}{2}$$

**15 a**  $f(x) = 2 \sin x + \cos 2x$

$$\begin{aligned} \therefore f\left(\frac{\pi}{2}\right) &= 2 \sin \frac{\pi}{2} + \cos\left(2 \times \frac{\pi}{2}\right) \\ &= 2(1) + (-1) \\ &= 1 \end{aligned}$$

**b**  $f'(x) = 2 \cos x - (\sin 2x)(2)$

$$= 2 \cos x - 2 \sin 2x$$

$$\begin{aligned} \therefore f'\left(\frac{\pi}{2}\right) &= 2 \cos \frac{\pi}{2} - 2 \sin\left(2 \times \frac{\pi}{2}\right) \\ &= 2(0) - 2(0) \\ &= 0 \end{aligned}$$

**c**  $f'(x) = 2 \cos x - 2 \sin 2x$  {from **b**}

$$\begin{aligned} \therefore f''(x) &= -2 \sin x - (2 \cos 2x)(2) \\ &= -2 \sin x - 4 \cos 2x \end{aligned}$$

$$\begin{aligned} \therefore f''\left(\frac{\pi}{2}\right) &= -2 \sin \frac{\pi}{2} - 4 \cos\left(2 \times \frac{\pi}{2}\right) \\ &= -2(1) - 4(-1) \\ &= 2 \end{aligned}$$



$$\begin{aligned}
 \mathbf{16} \quad y &= 3 \tan \frac{x}{2} \\
 \therefore \frac{dy}{dx} &= 3 \sec^2 \frac{x}{2} \times \frac{1}{2} \quad \{\text{chain rule}\} \\
 &= \frac{3}{2} \sec^2 \frac{x}{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{When } x = \frac{\pi}{3}, \quad \frac{dy}{dx} &= \frac{3}{2} \sec^2 \left( \frac{\frac{\pi}{3}}{2} \right) \\
 &= \frac{3}{2} \times \frac{1}{\cos^2 \left( \frac{\pi}{6} \right)} \\
 &= \frac{3}{2} \times \frac{1}{\left( \frac{\sqrt{3}}{2} \right)^2} \\
 &= \frac{3}{2} \times \frac{1}{\frac{3}{4}} \\
 &= \frac{3}{2} \times \frac{4}{3} \\
 &= 2
 \end{aligned}$$

$\therefore$  the gradient of the tangent to  $y = 3 \tan \frac{x}{2}$  at  $x = \frac{\pi}{3}$  is 2.

$$\begin{aligned}
 \mathbf{17} \quad \text{Suppose } y &= \sec[f(x)] \\
 \therefore y &= \sec u \quad \text{where } u = f(x)
 \end{aligned}$$

$$\begin{aligned}
 \text{But } \frac{dy}{dx} &= \frac{dy}{du} \frac{du}{dx} \quad \{\text{chain rule}\} \\
 \therefore \frac{dy}{dx} &= \sec u \tan u \times f'(x) \\
 &= \sec[f(x)] \tan[f(x)] f'(x)
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{18} \quad \mathbf{a} \quad y &= \operatorname{cosec} 4x \\
 \therefore \frac{dy}{dx} &= -\operatorname{cosec} 4x \cot 4x \times 4 \quad \{\text{chain rule}\} \\
 &= -4 \operatorname{cosec} 4x \cot 4x
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad y &= \frac{x}{\sqrt{\sec x}} \\
 &= x \times (\sec x)^{-\frac{1}{2}} \\
 \therefore \frac{dy}{dx} &= 1 \times (\sec x)^{-\frac{1}{2}} + x \times \left( -\frac{1}{2} (\sec x)^{-\frac{3}{2}} \right) \times (\sec x \tan x) \quad \{\text{product rule, chain rule}\} \\
 &= (\sec x)^{-\frac{1}{2}} - \frac{x}{2} \times (\sec x)^{-\frac{1}{2}} \times \tan x \\
 &= \left( \frac{1}{\sqrt{\cos x}} - \frac{x \sin x}{2\sqrt{\cos x} \cos x} \right) \times \frac{\cos x}{\cos x} \\
 &= \sqrt{\cos x} - \frac{x \sin x}{2\sqrt{\cos x}}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad y &= e^x \cot 2x \\
 \therefore \frac{dy}{dx} &= e^x \cot 2x + e^x (-\operatorname{cosec}^2 2x) \times 2 \quad \{\text{product rule, chain rule}\} \\
 &= e^x (\cot 2x - 2 \operatorname{cosec}^2 2x)
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad y &= \arccos \frac{x}{2} \\
 \therefore \frac{dy}{dx} &= \frac{-1 \times \frac{1}{2}}{\sqrt{1 - (\frac{x}{2})^2}} \quad \{\text{chain rule}\} \\
 &= \frac{-\frac{1}{2}}{\sqrt{1 - \frac{x^2}{4}}} \times \frac{\sqrt{4}}{\sqrt{4}} \\
 &= \frac{-1}{\sqrt{4 - x^2}}
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad y &= x^5 \arcsin 3x \\
 \therefore \frac{dy}{dx} &= (5x^4) \times \arcsin 3x + x^5 \times \left( \frac{1 \times 3}{\sqrt{1 - (3x)^2}} \right) \quad \{\text{product rule, chain rule}\} \\
 &= 5x^4 \arcsin 3x + \frac{3x^5}{\sqrt{1 - 9x^2}}
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad y &= 2^x \arctan(x - \frac{\pi}{3}) \\
 \therefore \frac{dy}{dx} &= (2^x \ln 2) \times \arctan(x - \frac{\pi}{3}) + 2^x \times \left( \frac{1}{1 + (x - \frac{\pi}{3})^2} \right) \quad \{\text{product rule}\} \\
 &= 2^x \ln 2 \times \arctan(x - \frac{\pi}{3}) + \frac{2^x}{1 + (x - \frac{\pi}{3})^2}
 \end{aligned}$$

$$\begin{aligned}
 19 \quad \text{a} \quad y &= \frac{1}{8}x^4 + \frac{1}{6}x^3 - \frac{1}{4}x^2 \\
 \therefore \frac{dy}{dx} &= \frac{1}{2}x^3 + \frac{1}{2}x^2 - \frac{1}{2}x \\
 \therefore \frac{d^2y}{dx^2} &= \frac{3}{2}x^2 + x - \frac{1}{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad y &= xe^{-x} \\
 \therefore \frac{dy}{dx} &= (1)e^{-x} + x(e^{-x})(-1) \quad \{\text{product rule}\} \\
 &= e^{-x} - xe^{-x} \\
 \therefore \frac{d^2y}{dx^2} &= e^{-x}(-1) - [(1)e^{-x} + xe^{-x}(-1)] \\
 &= -e^{-x} - e^{-x} + xe^{-x} \\
 &= -2e^{-x} + xe^{-x}
 \end{aligned}$$

$$20 \quad f(x) = \sqrt{x} \cos 4x = x^{\frac{1}{2}} \cos 4x$$

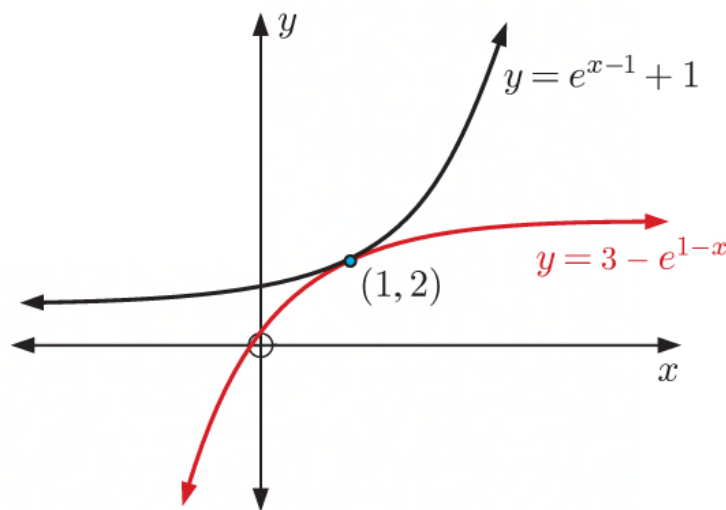
$$\begin{aligned}
 \text{a} \quad f'(x) &= \frac{1}{2}x^{-\frac{1}{2}} \cos 4x + x^{\frac{1}{2}} \times (-\sin 4x) \times 4 \quad \{\text{product rule}\} \\
 &= \frac{1}{2}x^{-\frac{1}{2}} \cos 4x - 4x^{\frac{1}{2}} \sin 4x \\
 &= \frac{1}{2\sqrt{x}} \cos 4x - 4\sqrt{x} \sin 4x
 \end{aligned}$$

$$\begin{aligned}
 \therefore f''(x) &= -\frac{1}{4}x^{-\frac{3}{2}} \cos 4x + \frac{1}{2}x^{-\frac{1}{2}} \times (-\sin 4x) \times 4 - [2x^{-\frac{1}{2}} \sin 4x + 4x^{\frac{1}{2}} \times (\cos 4x) \times 4] \\
 &\quad \{\text{product rule}\} \\
 &= -\frac{1}{4}x^{-\frac{3}{2}} \cos 4x - 2x^{-\frac{1}{2}} \sin 4x - 2x^{-\frac{1}{2}} \sin 4x - 16x^{\frac{1}{2}} \cos 4x \\
 &= -\frac{1}{4}x^{-\frac{3}{2}} \cos 4x - 4x^{-\frac{1}{2}} \sin 4x - 16x^{\frac{1}{2}} \cos 4x \\
 &= -\frac{1}{4x\sqrt{x}} \cos 4x - \frac{4}{\sqrt{x}} \sin 4x - 16\sqrt{x} \cos 4x
 \end{aligned}$$

$$\begin{aligned}
 \text{b i } f'\left(\frac{\pi}{16}\right) &= \frac{1}{2\sqrt{\frac{\pi}{16}}} \cos\left(4 \times \frac{\pi}{16}\right) - 4\sqrt{\frac{\pi}{16}} \sin\left(4 \times \frac{\pi}{16}\right) \\
 &= \frac{1}{2 \times \frac{\sqrt{\pi}}{4}} \cos \frac{\pi}{4} - 4 \times \frac{\sqrt{\pi}}{4} \sin \frac{\pi}{4} \\
 &= \frac{1}{\frac{\sqrt{\pi}}{2}} \times \frac{1}{\sqrt{2}} - \sqrt{\pi} \times \frac{1}{\sqrt{2}} \\
 &= \frac{2}{\sqrt{\pi}} \times \frac{1}{\sqrt{2}} - \sqrt{\pi} \times \frac{1}{\sqrt{2}} \\
 &= \frac{1}{\sqrt{2}} \left( \frac{2}{\sqrt{\pi}} - \sqrt{\pi} \right)
 \end{aligned}$$

$$\begin{aligned}
 \text{ii } f''\left(\frac{\pi}{8}\right) &= -\frac{1}{4 \times \frac{\pi}{8} \times \sqrt{\frac{\pi}{8}}} \cos\left(4 \times \frac{\pi}{8}\right) - \frac{4}{\sqrt{\frac{\pi}{8}}} \sin\left(4 \times \frac{\pi}{8}\right) - 16\sqrt{\frac{\pi}{8}} \cos\left(4 \times \frac{\pi}{8}\right) \\
 &= -\frac{1}{\frac{\pi}{2} \times \frac{\sqrt{\pi}}{\sqrt{8}}} \cos \frac{\pi}{2} - \frac{4}{\frac{\sqrt{\pi}}{\sqrt{8}}} \sin \frac{\pi}{2} - 16\frac{\sqrt{\pi}}{\sqrt{8}} \cos \frac{\pi}{2} \\
 &= -\frac{1}{\frac{\pi}{2} \times \frac{\sqrt{\pi}}{\sqrt{8}}} (0) - 4 \times \frac{\sqrt{8}}{\sqrt{\pi}} (1) - 16\frac{\sqrt{\pi}}{\sqrt{8}} (0) \\
 &= -4 \times \frac{2\sqrt{2}}{\sqrt{\pi}} \\
 &= -\frac{8\sqrt{2}}{\sqrt{\pi}}
 \end{aligned}$$

21 a



$$\begin{aligned}
 \text{b The graphs intersect when } e^{x-1} + 1 &= 3 - e^{1-x} \\
 \therefore e^{x-1} - 2 + e^{1-x} &= 0 \\
 \therefore \frac{1}{e} \times e^x - 2 + e^1 \times e^{-x} &= 0 \\
 \therefore e^x - 2e + e^2 \times e^{-x} &= 0 \\
 \therefore e^{2x} - 2e \times e^x + e^2 &= 0 \\
 \therefore (e^x - e)^2 &= 0 \\
 \therefore e^x &= e \\
 \therefore x &= 1
 \end{aligned}$$

$$\begin{aligned}
 \text{Substituting } x = 1 \text{ into } y = e^{x-1} + 1 \text{ gives } y &= e^0 + 1 \\
 &= 1 + 1 \\
 &= 2
 \end{aligned}$$

So, the curves intersect at (1, 2).

• If  $y = e^{x-1} + 1$ ,

$$\frac{dy}{dx} = e^{x-1} \times 1 \quad \{\text{chain rule}\}$$

$$= e^{x-1}$$

When  $x = 1$ ,  $\frac{dy}{dx} = e^0$

$$= 1$$

If  $y = 3 - e^{1-x}$ ,

$$\frac{dy}{dx} = -e^{1-x} \times (-1) \quad \{\text{chain rule}\}$$

$$= e^{1-x}$$

When  $x = 1$ ,  $\frac{dy}{dx} = e^0$

$$= 1$$

The gradient of both curves at  $(1, 2)$  is  $e^0 = 1$ .

$\therefore$  the tangents to each of the curves at this point are the same line.

22

$$x^2 + 2xy + y^2 = 4$$

$$\therefore 2x + 2y + 2x \frac{dy}{dx} + 2y \frac{dy}{dx} = 0 \quad \{\text{product rule}\}$$

$$\therefore (2x + 2y) \frac{dy}{dx} = -2x - 2y$$

$$\therefore (2x + 2y) \frac{dy}{dx} = -(2x + 2y)$$

$$\therefore \frac{dy}{dx} = -1$$

$$\therefore \frac{d^2y}{dx^2} = 0$$

23

$$y = (1 - \frac{1}{3}x)^3$$

$$= 1 - 3(1)^2(-\frac{1}{3}x) + 3(1)(-\frac{1}{3}x)^2 + (-\frac{1}{3}x)^3$$

$$= 1 - x + \frac{1}{3}x^2 - \frac{1}{27}x^3$$

$$\therefore \frac{dy}{dx} = -1 + \frac{2}{3}x - \frac{1}{9}x^2$$

$$\therefore \frac{d^2y}{dx^2} = \frac{2}{3} - \frac{2}{9}x$$

$$\therefore \frac{d^3y}{dx^3} = -\frac{2}{9} \quad \checkmark$$

24  $P_n$  is: If  $y = \frac{1}{2x+1} = (2x+1)^{-1}$ , then  $\frac{d^n y}{dx^n} = \frac{(-2)^n n!}{(2x+1)^{n+1}}$  for all  $n \in \mathbb{Z}^+$ .

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ , LHS =  $\frac{dy}{dx}$  and RHS =  $\frac{(-2)^1 1!}{(2x+1)^{1+1}}$

$$= -(2x+1)^{-2}(2)$$

$$= \frac{-2}{(2x+1)^2}$$

$$= \frac{(-2)^1 1!}{(2x+1)^{1+1}}$$

$\therefore P_1$  is true.



(2) If  $P_k$  is true, then  $\frac{d^k y}{dx^k} = \frac{(-2)^k k!}{(2x+1)^{k+1}}$

$$\begin{aligned}\text{Now } \frac{d^{k+1} y}{dx^{k+1}} &= \frac{d}{dx} \left( \frac{d^k y}{dx^k} \right) \\ &= \frac{d}{dx} \left( \frac{(-2)^k k!}{(2x+1)^{k+1}} \right) \quad \{\text{using } P_k\} \\ &= \frac{d}{dx} [(-2)^k k! (2x+1)^{-(k+1)}] \\ &= (-2)^k k! [-(k+1)] (2x+1)^{-(k+1)-1} (2) \\ &= -2(-2)^k (k+1)! (2x+1)^{-(k+2)} \\ &= \frac{(-2)^{k+1} (k+1)!}{(2x+1)^{(k+1)+1}}\end{aligned}$$

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,  
 $P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

**25 a**  $f(x) = x 3^x$

**i**  $f'(x) = 1 \times 3^x + x \times (3^x \ln 3) \quad \{\text{product rule}\}$   
 $= 3^x (1 + x \ln 3)$

**ii**  $f''(x) = (3^x \ln 3)(1 + x \ln 3) + 3^x \times \ln 3 \quad \{\text{product rule}\}$   
 $= 3^x \ln 3 (2 + x \ln 3)$

**iii**  $f^{(3)}(x) = (3^x \ln 3) \times \ln 3 (2 + x \ln 3) + 3^x \ln 3 \times \ln 3 \quad \{\text{product rule}\}$   
 $= 3^x (\ln 3)^2 (3 + x \ln 3)$

**iv**  $f^{(4)}(x) = (3^x \ln 3) \times (\ln 3)^2 (3 + x \ln 3) + 3^x (\ln 3)^2 \times \ln 3$   
 $= 3^x (\ln 3)^3 (4 + x \ln 3)$

**b** We conjecture that  $f^{(n)}(x) = 3^x (\ln 3)^{n-1} (n + x \ln 3)$ ,  $n \in \mathbb{Z}^+$ .

**c**  $P_n$  is: If  $f(x) = x 3^x$ , then  $f^{(n)}(x) = 3^x (\ln 3)^{n-1} (n + x \ln 3)$  for all  $n \in \mathbb{Z}^+$

**Proof:** (By the principle of mathematical induction)

(1) If  $n = 1$ , LHS =  $f'(x)$  and RHS =  $3^x (\ln 3)^{1-1} (1 + x \ln 3)$   
 $= 3^x (1 + x \ln 3) \quad \{\text{from a i}\}$   
 $= 3^x (\ln 3)^{1-1} (1 + x \ln 3)$

$\therefore P_1$  is true.

(2) If  $P_k$  is true, then  $f^{(k)}(x) = 3^x (\ln 3)^{k-1} (k + x \ln 3)$

$$\begin{aligned}\text{Now } f^{(k+1)}(x) &= \frac{d}{dx} [f^{(k)}(x)] \\ &= \frac{d}{dx} [3^x (\ln 3)^{k-1} (k + x \ln 3)] \quad \{\text{using } P_k\} \\ &= 3^x \ln 3 \times (\ln 3)^{k-1} (k + x \ln 3) + 3^x (\ln 3)^{k-1} (\ln 3) \quad \{\text{product rule}\} \\ &= 3^x (\ln 3)^k (k + x \ln 3) + 3^x (\ln 3)^k \\ &= 3^x (\ln 3)^{(k+1)-1} (k + 1 + x \ln 3)\end{aligned}$$

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,  
 $P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

**26**  $\arcsin(xy) + y = \frac{2\pi}{3}$

When  $y = \frac{\pi}{2}$ ,  $\arcsin \frac{\pi x}{2} + \frac{\pi}{2} = \frac{2\pi}{3}$

$$\therefore \arcsin \frac{\pi x}{2} = \frac{\pi}{6}$$

$$\therefore \frac{\pi x}{2} = \sin \frac{\pi}{6}$$

$$\therefore \frac{\pi x}{2} = \frac{1}{2}$$

$$\therefore x = \frac{1}{\pi}$$

$\therefore$  the point is  $(\frac{1}{\pi}, \frac{\pi}{2})$ .

Differentiating the original equation with respect to  $x$  gives

$$\frac{1}{\sqrt{1-(xy)^2}} \times \left(y + x \frac{dy}{dx}\right) + \frac{dy}{dx} = 0 \quad \{\text{chain rule, product rule}\}$$

$$\therefore \frac{y}{\sqrt{1-x^2y^2}} + \frac{x}{\sqrt{1-x^2y^2}} \frac{dy}{dx} + \frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} \left( \frac{x}{\sqrt{1-x^2y^2}} + 1 \right) = -\frac{y}{\sqrt{1-x^2y^2}}$$

$$\therefore \frac{dy}{dx} = -\frac{y}{\sqrt{1-x^2y^2}} \times \frac{1}{\frac{x}{\sqrt{1-x^2y^2}} + 1}$$

$$= -\frac{y}{x + \sqrt{1-x^2y^2}}$$

Substituting  $x = \frac{1}{\pi}$ ,  $y = \frac{\pi}{2}$ ,  $\frac{dy}{dx} = -\frac{\frac{\pi}{2}}{\frac{1}{\pi} + \sqrt{1 - (\frac{1}{\pi})^2(\frac{\pi}{2})^2}}$

$$= -\frac{\pi}{\frac{2}{\pi} + 2\sqrt{1 - \frac{1}{4}}}$$

$$= -\frac{\pi}{\frac{2}{\pi} + \sqrt{3}} \times \frac{\pi}{\pi}$$

$$= -\frac{\pi^2}{2 + \sqrt{3}\pi}$$

$\therefore$  the gradient of the tangent to the curve  $\arcsin(xy) + y = \frac{2\pi}{3}$  at the point where  $y = \frac{\pi}{2}$  is  $-\frac{\pi^2}{2 + \sqrt{3}\pi}$ .

## REVIEW SET 17B

**1 a**  $f(x) = 3x^2 - 7x + 4$   
 $\therefore f'(x) = 3(2x) - 7(1)$   
 $= 6x - 7$

**b**  $f(x) = (x + 5)^2$   
 $= x^2 + 10x + 25$   
 $\therefore f'(x) = 2x + 10(1)$   
 $= 2x + 10$

$$\begin{aligned}
 \text{c} \quad f(x) &= 2\sqrt{x} - \frac{3}{x} \\
 &= 2x^{\frac{1}{2}} - 3x^{-1} \\
 \therefore f'(x) &= 2\left(\frac{1}{2}x^{-\frac{1}{2}}\right) - 3(-x^{-2}) \\
 &= x^{-\frac{1}{2}} + 3x^{-2} \\
 &= \frac{1}{\sqrt{x}} + \frac{3}{x^2}
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad f(x) &= 6x^2\sqrt{x} \\
 &= 6x^{\frac{5}{2}} \\
 \therefore f'(x) &= 6\left(\frac{5}{2}x^{\frac{3}{2}}\right) \\
 &= 15x^{\frac{3}{2}} \\
 &= 15x\sqrt{x}
 \end{aligned}$$

$$\begin{aligned}
 \text{2 a} \quad y &= 2x^3 - 6x^2 + 7x - 4 \\
 \therefore \frac{dy}{dx} &= 2(3x^2) - 6(2x) + 7(1) \\
 &= 6x^2 - 12x + 7
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad y &= \frac{3}{x} - \frac{5}{x^3} \\
 &= 3x^{-1} - 5x^{-3} \\
 \therefore \frac{dy}{dx} &= 3(-x^{-2}) - 5(-3x^{-4}) \\
 &= -3x^{-2} + 15x^{-4} \\
 &= -\frac{3}{x^2} + \frac{15}{x^4}
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad y &= \frac{15}{\sqrt[3]{x}} \\
 &= 15x^{-\frac{1}{3}} \\
 \therefore \frac{dy}{dx} &= 15\left(-\frac{1}{3}x^{-\frac{4}{3}}\right) \\
 &= -5x^{-\frac{4}{3}} \\
 &= -\frac{5}{x^{\frac{4}{3}}}
 \end{aligned}$$

$$\begin{aligned}
 \text{3 a} \quad f(x) &= 7 + x - 3x^2 \\
 \therefore f(3) &= 7 + 3 - 3(3)^2 \\
 &= 7 + 3 - 27 \\
 &= -17
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad f(x) &= 7 + x - 3x^2 \\
 \therefore f'(x) &= 1 - 6x \\
 \therefore f'(3) &= 1 - 6(3) \\
 &= 1 - 18 \\
 &= -17
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad f'(x) &= 1 - 6x \quad \{\text{from b}\} \\
 \therefore f''(x) &= -6 \\
 \therefore f''(3) &= -6
 \end{aligned}$$

$$\begin{aligned}
 \text{4 a} \quad y &= x^3\sqrt{1-x^2} \text{ is the product of} \\
 u &= x^3 \quad \text{and} \quad v = (1-x^2)^{\frac{1}{2}} \\
 \therefore u' &= 3x^2 \quad \text{and} \quad v' = \frac{1}{2}(1-x^2)^{-\frac{1}{2}} \times (-2x) \quad \{\text{chain rule}\} \\
 &= -x(1-x^2)^{-\frac{1}{2}}
 \end{aligned}$$

$$\begin{aligned}
 \text{Now } \frac{dy}{dx} &= u'v + uv' \quad \{\text{product rule}\} \\
 &= 3x^2(1-x^2)^{\frac{1}{2}} + x^3 \times \left[-x(1-x^2)^{-\frac{1}{2}}\right] \\
 &= 3x^2(1-x^2)^{\frac{1}{2}} - x^4(1-x^2)^{-\frac{1}{2}}
 \end{aligned}$$

**b**  $y = \frac{x^2 - 3x}{\sqrt{x+1}}$  is a quotient with  $u = x^2 - 3x$  and  $v = (x+1)^{\frac{1}{2}}$   
 $\therefore u' = 2x - 3$  and  $v' = \frac{1}{2}(x+1)^{-\frac{1}{2}}$

Now  $\frac{dy}{dx} = \frac{u'v - uv'}{v^2}$  {quotient rule}  

$$= \frac{(2x-3)(x+1)^{\frac{1}{2}} - (x^2-3x) \times \frac{1}{2}(x+1)^{-\frac{1}{2}}}{(\sqrt{x+1})^2}$$

$$= \frac{(2x-3)(x+1)^{\frac{1}{2}} - \frac{1}{2}(x^2-3x)(x+1)^{-\frac{1}{2}}}{x+1}$$

**5 a**  $y = xe^x$  is the product of  $u = x$  and  $v = e^x$   
 $\therefore u' = 1$  and  $v' = e^x$

Now  $\frac{dy}{dx} = u'v + uv'$  {product rule}  

$$= 1 \times e^x + x \times e^x$$

$$= e^x + xe^x$$

**b**  $\frac{dy}{dx} = e^x + xe^x = (1+x)e^x$  {from **a**}

$\frac{dy}{dx} = 2e$  when  $(1+x)e^x = 2e$

Solving by inspection, we find  $x = 1$ .

When  $x = 1$ ,  $y = 1 \times e^1 = e$ .

$\therefore$  the gradient of  $y = xe^x$  is  $2e$  at the point  $(1, e)$ .

**6 a**  $f(x) = \ln(e^x + 3)$  **b**  $f(x) = \ln \left[ \frac{(x+2)^3}{x} \right]$   
 $\therefore f'(x) = \frac{e^x}{e^x + 3}$ 

$$= \ln(x+2)^3 - \ln x$$
 {  $\ln \left( \frac{a}{b} \right) = \ln a - \ln b$  }  

$$= 3 \ln(x+2) - \ln x$$
 {  $\ln a^n = n \ln a$  }  
 $\therefore f'(x) = \frac{3}{x+2} - \frac{1}{x}$

**c**  $f(x) = x^{x^2}$   
 $\therefore \ln[f(x)] = \ln(x^{x^2})$   
 $\therefore \ln[f(x)] = x^2 \ln x$   
 $\therefore \frac{f'(x)}{f(x)} = 2x \ln x + x^2 \left( \frac{1}{x} \right)$  {product rule}  
 $\therefore f'(x) = f(x)[2x \ln x + x]$   

$$= x^{x^2}(2x \ln x + x)$$

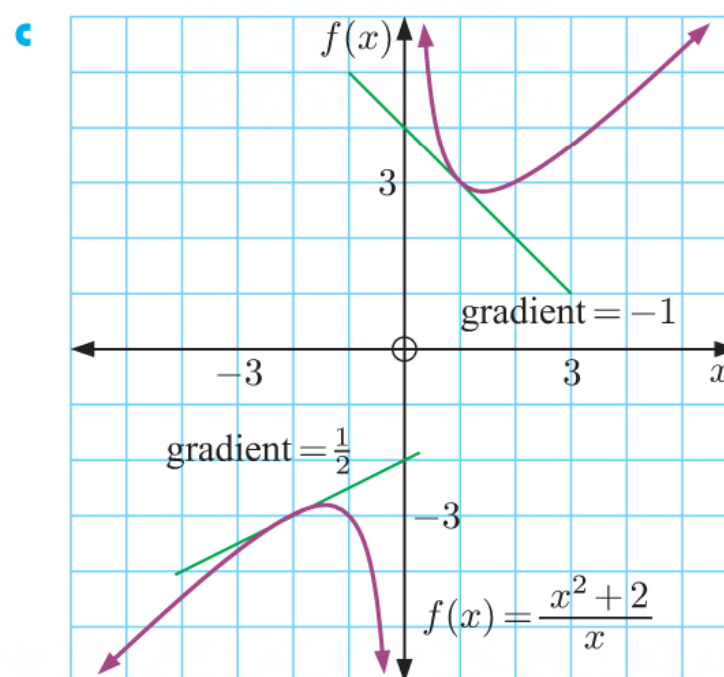
$$= x^{x^2+1}(2 \ln x + 1)$$



$$\begin{aligned}
 7 \quad a \quad f(x) &= \frac{x^2 + 2}{x} \\
 &= x + 2x^{-1} \\
 \therefore f'(x) &= 1 - 2x^{-2} \\
 &= 1 - \frac{2}{x^2}
 \end{aligned}$$

$$\begin{aligned}
 b \quad i \quad f'(1) &= 1 - \frac{2}{1^2} \\
 &= -1 \\
 \therefore \text{gradient of tangent} &= -1
 \end{aligned}$$

$$\begin{aligned}
 ii \quad f'(-2) &= 1 - \frac{2}{(-2)^2} \\
 &= 1 - \frac{2}{4} \\
 &= \frac{1}{2} \\
 \therefore \text{gradient of tangent} &= \frac{1}{2}
 \end{aligned}$$



$$\begin{aligned}
 8 \quad y &= \left(x - \frac{1}{x}\right)^4 \\
 &= (x - x^{-1})^4 \\
 \therefore \frac{dy}{dx} &= 4(x - x^{-1})^3(1 + x^{-2}) \quad \{\text{chain rule}\} \\
 &= 4\left(x - \frac{1}{x}\right)^3\left(1 + \frac{1}{x^2}\right)
 \end{aligned}$$

$$\begin{aligned}
 \text{When } x = 1, \quad \frac{dy}{dx} &= 4\left(1 - \frac{1}{1}\right)^3\left(1 + \frac{1}{1^2}\right) \\
 &= 4 \times 0 \times 2 \\
 &= 0
 \end{aligned}$$

$$\begin{aligned}
 9 \quad a \quad y &= \ln(x^3 - 3x) \\
 \therefore \frac{dy}{dx} &= \frac{3x^2 - 3}{x^3 - 3x}
 \end{aligned}$$

$$\begin{aligned}
 b \quad y &= \frac{e^x}{x^2} \\
 \therefore \frac{dy}{dx} &= \frac{e^x(x^2) - e^x(2x)}{(x^2)^2} \quad \{\text{quotient rule}\} \\
 &= \frac{e^x(x - 2)}{x^3}
 \end{aligned}$$

$$\begin{aligned}
 c \quad e^{x+y} &= \ln(y^2 + 1) \\
 \therefore e^x e^y &= \ln(y^2 + 1) \\
 \therefore e^x e^y + e^x e^y \times \frac{dy}{dx} &= \frac{2y \frac{dy}{dx}}{y^2 + 1} \quad \{\text{product rule}\} \\
 \therefore (y^2 + 1)e^{x+y} + (y^2 + 1)e^{x+y} \frac{dy}{dx} &= 2y \frac{dy}{dx} \\
 \therefore \frac{dy}{dx} (2y - e^{x+y}(y^2 + 1)) &= e^{x+y}(y^2 + 1) \\
 \therefore \frac{dy}{dx} &= \frac{e^{x+y}(y^2 + 1)}{2y - e^{x+y}(y^2 + 1)}
 \end{aligned}$$

$$\begin{aligned} \mathbf{10} \quad \mathbf{a} \quad f(x) &= 2^{x^2-5x} \\ \therefore f'(x) &= 2^{x^2-5x} \ln 2 \times (2x-5) \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad f(x) &= \log(\tan x) \\ \therefore f'(x) &= \frac{\sec^2 x}{\tan x \ln 10} \\ &= \frac{1}{\cos^2 x} \times \frac{\cos x}{\sin x} \times \frac{1}{\ln 10} \\ &= \frac{1}{\cos x \sin x} \times \frac{1}{\ln 10} \\ &= \frac{\operatorname{cosec} x \sec x}{\ln 10} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad f(x) &= \log_3 \left( \frac{2}{(x+5)(x-4)} \right) \\ &= \log_3 2 - \log_3(x+5) - \log_3(x-4) \\ \therefore f'(x) &= -\frac{1}{(x+5) \ln 3} - \frac{1}{(x-4) \ln 3} \\ &= -\frac{1}{\ln 3} \left( \frac{1}{x+5} + \frac{1}{x-4} \right) \end{aligned}$$

$$\begin{aligned} \mathbf{11} \quad \mathbf{a} \quad f(x) &= 2x^4 - 4x^3 - 9x^2 + 4x + 7 \\ \therefore f'(x) &= 8x^3 - 12x^2 - 18x + 4 \\ \therefore f''(x) &= 24x^2 - 24x - 18 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad f''(x) &= 0 \quad \text{when} \\ 24x^2 - 24x - 18 &= 0 \\ \therefore 4x^2 - 4x - 3 &= 0 \\ \therefore (2x+1)(2x-3) &= 0 \\ \therefore x &= -\frac{1}{2} \quad \text{or} \quad x = \frac{3}{2} \end{aligned}$$

$$\begin{aligned} \mathbf{12} \quad \mathbf{a} \quad y &= 10x - \sin 10x \\ \therefore \frac{dy}{dx} &= 10 - 10 \cos 10x \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad y &= \ln \left( \frac{1}{\cos x} \right) \\ &= \ln[(\cos x)^{-1}] \\ \therefore \frac{dy}{dx} &= \frac{-(\cos x)^{-2}(-\sin x)}{(\cos x)^{-1}} \quad \{\text{chain rule}\} \\ &= \frac{\sin x \cos x}{\cos^2 x} \\ &= \frac{\sin x}{\cos x} \\ &= \tan x \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad y &= \sin 5x \ln(2x) \\ \therefore \frac{dy}{dx} &= (5 \cos 5x) \ln(2x) + \sin 5x \times \frac{2}{2x} \\ &\quad \{\text{product rule}\} \\ &= (5 \cos 5x) \ln(2x) + \frac{\sin 5x}{x} \end{aligned}$$

$$\begin{aligned}
 \mathbf{13} \quad \mathbf{a} \quad y &= \frac{x^3}{x+1} \\
 \therefore \frac{dy}{dx} &= \frac{3x^2(x+1) - x^3(1)}{(x+1)^2} \quad \{\text{quotient rule}\} \\
 &= \frac{3x^3 + 3x^2 - x^3}{(x+1)^2} \\
 &= \frac{2x^3 + 3x^2}{(x+1)^2}
 \end{aligned}$$

$$\begin{aligned}
 \text{At } x = 2, \quad \frac{dy}{dx} &= \frac{2(2)^3 + 3(2)^2}{(2+1)^2} \\
 &= \frac{16 + 12}{9} \\
 &= \frac{28}{9}
 \end{aligned}$$

$\therefore$  the gradient of the tangent to  $y = \frac{x^3}{x+1}$  at  $x = 2$  is  $\frac{28}{9}$ .

$$\begin{aligned}
 \mathbf{b} \quad y &= \tan 2x \\
 \therefore \frac{dy}{dx} &= \sec^2 2x \times 2 \quad \{\text{chain rule}\} \\
 &= 2 \sec^2 2x
 \end{aligned}$$

$$\begin{aligned}
 \text{At } x = \frac{\pi}{6}, \quad \frac{dy}{dx} &= 2 \sec^2(2 \times \frac{\pi}{6}) \\
 &= 2 \times \frac{1}{\cos^2(\frac{\pi}{3})} \\
 &= 2 \times \frac{1}{(\frac{1}{2})^2} \\
 &= 2 \times 4 \\
 &= 8
 \end{aligned}$$

$\therefore$  the gradient of the tangent to  $y = \tan 2x$  at  $x = \frac{\pi}{6}$  is 8.

$$\mathbf{14} \quad f(x) = a \ln(bx)$$

$$\text{Now } f(e) = 12$$

$$\begin{aligned}
 \therefore 12 &= a \ln(be) \\
 &= a(\ln b + \ln e) \quad \{\ln(ab) = \ln a + \ln b\} \\
 &= a(\ln b + 1)
 \end{aligned}$$

$$\therefore a = \frac{12}{\ln b + 1} \quad \dots (*)$$

$$\begin{aligned}
 f(x) &= a \ln(bx) \\
 &= a(\ln b + \ln x) \quad \{\ln(ab) = \ln a + \ln b\}
 \end{aligned}$$

$$\begin{aligned}
 \therefore f'(x) &= a \left( 0 + \frac{1}{x} \right) \\
 &= \frac{a}{x}
 \end{aligned}$$

Now  $f'(2) = 2$

$$\therefore \frac{a}{2} = 2$$

$$\therefore a = 4$$

Substituting  $a = 4$  into (\*) gives:

$$4 = \frac{12}{\ln b + 1}$$

$$\therefore 4 \ln b + 4 = 12$$

$$\therefore 4 \ln b = 8$$

$$\therefore \ln b = 2$$

$$\therefore b = e^2$$

So,  $a = 4$  and  $b = e^2$ .

**15 a**

$$y = \frac{\cos x}{\sin x + 2}$$

$$\therefore \frac{dy}{dx} = \frac{(-\sin x)(\sin x + 2) - \cos x(\cos x)}{(\sin x + 2)^2}$$

{quotient rule}

The tangent meets the graph at  $x = 0$ .

At  $x = 0$ ,

$$\begin{aligned} \frac{dy}{dx} &= \frac{(-\sin 0)(\sin 0 + 2) - \cos 0(\cos 0)}{(\sin 0 + 2)^2} \\ &= \frac{0 - 1}{4} \\ &= -\frac{1}{4} \end{aligned}$$

$$\therefore \text{gradient of tangent} = -\frac{1}{4}$$

**b**  $\frac{dy}{dx} = \frac{(-\sin x)(\sin x + 2) - \cos x(\cos x)}{(\sin x + 2)^2}$  {from a}

$$= \frac{-\sin^2 x - 2\sin x - \cos^2 x}{(\sin x + 2)^2}$$

$$= \frac{-(\sin^2 x + \cos^2 x) - 2\sin x}{(\sin x + 2)^2}$$

$$= -\frac{2\sin x + 1}{(\sin x + 2)^2}$$

A tangent of gradient  $-\frac{1}{2}$  occurs when  $\frac{dy}{dx} = -\frac{1}{2}$ .

$$\therefore -\frac{2\sin x + 1}{(\sin x + 2)^2} = -\frac{1}{2}$$

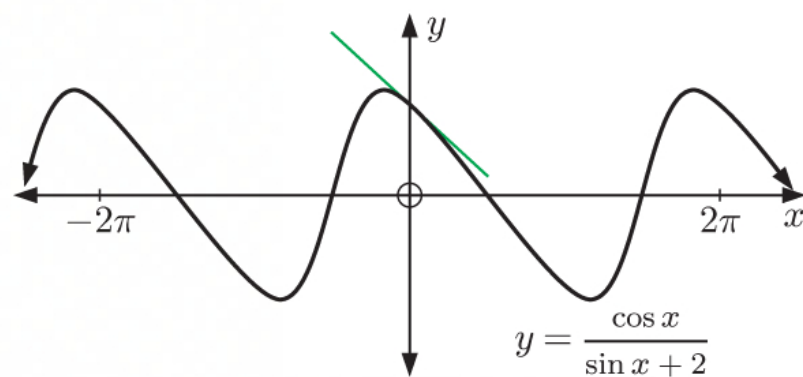
$$\therefore \frac{2\sin x + 1}{(\sin x + 2)^2} = \frac{1}{2}$$

$$\therefore 2(2\sin x + 1) = (\sin x + 2)^2$$

$$\therefore \cancel{4\sin x} + 2 = \sin^2 x + \cancel{4\sin x} + 4$$

$$\therefore \sin^2 x = -2 \quad \text{which has no real solutions}$$

$\therefore$  it is impossible to draw a tangent to the graph with gradient  $-\frac{1}{2}$ .





$$16 \quad a \quad y = \frac{e^x}{\sqrt{x}} = \frac{e^x}{x^{\frac{1}{2}}}$$

$$\therefore \frac{dy}{dx} = \frac{e^x x^{\frac{1}{2}} - e^x (\frac{1}{2} x^{-\frac{1}{2}})}{\left(x^{\frac{1}{2}}\right)^2} \quad \{\text{quotient rule}\}$$

$$= \frac{e^x \sqrt{x} \times \frac{2\sqrt{x}}{2\sqrt{x}} - \frac{e^x}{2\sqrt{x}}}{x}$$

$$= \frac{2xe^x - e^x}{2x\sqrt{x}}$$

$$= \frac{e^x(2x - 1)}{2x\sqrt{x}} \quad \text{as required}$$

$$b \quad i \quad \frac{dy}{dx} = 0 \quad \text{when} \quad e^x(2x - 1) = 0$$

$$\therefore e^x = 0 \quad \text{or} \quad 2x - 1 = 0$$

$$\therefore x = \frac{1}{2} \quad \{\text{as } e^x > 0 \text{ for all } x\}$$

$$ii \quad \frac{dy}{dx} \text{ is undefined when } 2x\sqrt{x} = 0 \quad \text{or } \sqrt{x} \text{ is undefined}$$

$$\therefore x \leq 0$$

$$17 \quad \text{Suppose } y = \arccos[f(x)]$$

$$\therefore y = \arccos u \quad \text{where } u = f(x)$$

$$\text{But } \frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} \quad \{\text{chain rule}\}$$

$$\therefore \frac{dy}{dx} = \frac{-1}{\sqrt{1-u^2}} \times f'(x)$$

$$= -\frac{f'(x)}{\sqrt{1-[f(x)]^2}}$$

$$18 \quad a \quad y = \cot \frac{x}{4}$$

$$\therefore \frac{dy}{dx} = -\operatorname{cosec}^2\left(\frac{x}{4}\right) \times \frac{1}{4} \quad \{\text{chain rule}\}$$

$$= -\frac{1}{4} \operatorname{cosec}^2\left(\frac{x}{4}\right)$$

$$b \quad y = x^2 \sec 3x$$

$$\therefore \frac{dy}{dx} = 2x \times \sec 3x + x^2 \times (\sec 3x \tan 3x) \times 3 \quad \{\text{product rule, chain rule}\}$$

$$= 2x \sec 3x + 3x^2 \sec 3x \tan 3x$$

$$= x \sec 3x(3x \tan 3x + 2)$$

$$c \quad y = \frac{\operatorname{cosec}(e^x)}{e^x}$$

$$\therefore \frac{dy}{dx} = \frac{(-\operatorname{cosec}(e^x) \cot(e^x) \times e^x) \times \cancel{e^x} - \operatorname{cosec}(e^x) \times \cancel{e^x}}{(e^x)^2} \quad \{\text{quotient rule, chain rule}\}$$

$$= \frac{-\operatorname{cosec}(e^x) \cot(e^x) \times e^x - \operatorname{cosec}(e^x)}{e^x}$$

$$= \frac{-\operatorname{cosec}(e^x)[e^x \cot(e^x) + 1]}{e^x}$$

$$= -e^{-x} \operatorname{cosec}(e^x)[e^x \cot(e^x) + 1]$$

**d**  $y = \arcsin 5x$

$$\begin{aligned}\therefore \frac{dy}{dx} &= \frac{5}{\sqrt{1-(5x)^2}} \quad \{\text{chain rule}\} \\ &= \frac{5}{\sqrt{1-25x^2}}\end{aligned}$$

**e**  $y = e^{2x} \arctan 2x$

$$\begin{aligned}\therefore \frac{dy}{dx} &= 2e^{2x} \times \arctan 2x + e^{2x} \times \left( \frac{2}{1+(2x)^2} \right) \quad \{\text{product rule, chain rule}\} \\ &= 2e^{2x} \arctan 2x + \frac{2e^{2x}}{1+4x^2} \\ &= 2e^{2x} \left( \arctan 2x + \frac{1}{1+4x^2} \right)\end{aligned}$$

**f**  $y = \arccos(\tan x)$

$$\begin{aligned}\therefore \frac{dy}{dx} &= \frac{-\sec^2 x}{\sqrt{1-(\tan x)^2}} \quad \{\text{chain rule}\} \\ &= -\frac{\sec^2 x}{\sqrt{1-\tan^2 x}}\end{aligned}$$

**19 a**

$$y = \frac{3x^2 - 2}{1 - 2x}$$

$$\begin{aligned}\therefore \frac{dy}{dx} &= \frac{6x(1-2x) - (3x^2 - 2)(-2)}{(1-2x)^2} \quad \{\text{quotient rule}\} \\ &= \frac{6x - 12x^2 + 6x^2 - 4}{(1-2x)^2} \\ &= \frac{-6x^2 + 6x - 4}{(1-2x)^2} \\ \therefore \frac{d^2y}{dx^2} &= \frac{(-12x + 6)(1-2x)^2 - (-6x^2 + 6x - 4) \times 2(1-2x) \times (-2)}{(1-2x)^4}\end{aligned}$$

{\text{quotient rule, chain rule}}

$$\begin{aligned}&= \frac{(1-2x)[(-12x+6)(1-2x) + 4(-6x^2+6x-4)]}{(1-2x)^4} \\ &= \frac{-12x + 24x^2 + 6 - 12x - 24x^2 + 24x - 16}{(1-2x)^3} \\ &= \frac{-10}{(1-2x)^3} \\ &= -\frac{10}{(1-2x)^3}\end{aligned}$$

**b**  $y = x^3 - x + \frac{1}{\sqrt{x}} = x^3 - x + x^{-\frac{1}{2}}$

$$\therefore \frac{dy}{dx} = 3x^2 - 1 - \frac{1}{2}x^{-\frac{3}{2}}$$

$$\therefore \frac{d^2y}{dx^2} = 6x + \frac{3}{4}x^{-\frac{5}{2}}$$

$$\begin{aligned}
& \text{c} \quad 3^x + 2^y = 20 \\
& \quad \therefore 2^y = 20 - 3^x \quad \dots (*) \\
& \therefore 2^y \ln 2 \frac{dy}{dx} = -3^x \ln 3 \quad \{\text{chain rule}\} \\
& \quad \therefore \frac{dy}{dx} = \frac{-3^x \ln 3}{2^y \ln 2} \\
& \quad \quad = -\frac{3^x \ln 3}{(20 - 3^x) \ln 2} \quad \{\text{using } (*)\} \\
& \therefore \frac{d^2y}{dx^2} = -\frac{\ln 3}{\ln 2} \left( \frac{3^x \ln 3 \times (20 - 3^x) - 3^x(-3^x \ln 3)}{(20 - 3^x)^2} \right) \quad \{\text{quotient rule}\} \\
& \quad \quad = -\frac{\ln 3}{\ln 2} \left( \frac{3^x \ln 3 \times 20 - \cancel{(3^x)^2 \ln 3} + \cancel{(3^x)^2 \ln 3}}{(20 - 3^x)^2} \right) \\
& \quad \quad = -\frac{3^x (\ln 3)^2 \times 20}{(20 - 3^x)^2 \times \ln 2}
\end{aligned}$$

$$\begin{aligned}
& \text{20} \quad y = 3 \sin 2x + 2 \cos 2x \\
& \therefore \frac{dy}{dx} = 3 \times (\cos 2x) \times 2 + 2 \times (-\sin 2x) \times 2 \\
& \quad \quad = 6 \cos 2x - 4 \sin 2x \\
& \therefore \frac{d^2y}{dx^2} = 6 \times (-\sin 2x) \times 2 - 4 \times (\cos 2x) \times 2 \\
& \quad \quad = -12 \sin 2x - 8 \cos 2x \\
& \therefore 4y + \frac{d^2y}{dx^2} = 4(3 \sin 2x + 2 \cos 2x) + (-12 \sin 2x - 8 \cos 2x) \\
& \quad \quad = 12 \sin 2x + 8 \cos 2x - 12 \sin 2x - 8 \cos 2x \\
& \quad \quad = 0 \quad \text{as required}
\end{aligned}$$

$$\begin{aligned}
& \text{21} \quad \text{a} \quad f(x) = -\frac{1}{2} \quad \text{when} \quad \frac{6x}{3+x^2} = -\frac{1}{2} \\
& \quad \quad \therefore 12x = -(3+x^2) \\
& \quad \quad \therefore 12x = -3 - x^2 \\
& \quad \quad \therefore x^2 + 12x + 3 = 0 \\
& \quad \quad \therefore x = \frac{-12 \pm \sqrt{(12)^2 - 4(1)(3)}}{2(1)} \\
& \quad \quad \quad = \frac{-12 \pm \sqrt{144 - 12}}{2} \\
& \quad \quad \quad = \frac{-12 \pm \sqrt{132}}{2} \\
& \quad \quad \quad = \frac{-12 \pm 2\sqrt{33}}{2} \\
& \quad \quad \quad = -6 \pm \sqrt{33}
\end{aligned}$$

$$\begin{aligned}
 \text{b} \quad f(x) &= \frac{6x}{3+x^2} \\
 \therefore f'(x) &= \frac{6(3+x^2) - 6x(2x)}{(3+x^2)^2} \quad \{\text{quotient rule}\} \\
 &= \frac{18 + 6x^2 - 12x^2}{(3+x^2)^2} \\
 &= \frac{18 - 6x^2}{(3+x^2)^2}
 \end{aligned}$$

$$\begin{aligned}
 f'(x) = 0 \quad \text{when} \quad \frac{18 - 6x^2}{(3+x^2)^2} &= 0 \\
 \therefore 18 - 6x^2 &= 0 \\
 \therefore 6x^2 &= 18 \\
 \therefore x^2 &= 3 \\
 \therefore x &= \pm\sqrt{3}
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad f'(x) &= \frac{18 - 6x^2}{(3+x^2)^2} \quad \{\text{from b}\} \\
 \therefore f''(x) &= \frac{(-12x)(3+x^2)^2 - (18 - 6x^2) \times 2(3+x^2) \times (2x)}{(3+x^2)^4} \quad \{\text{quotient rule}\} \\
 &= \frac{-12x(9 + 6x^2 + x^4) - 4x(18 - 6x^2)(3+x^2)}{(3+x^2)^4} \\
 &= \frac{-108x - 72x^3 - 12x^5 - 4x(54 + 18x^2 - 18x^2 - 6x^4)}{(3+x^2)^4} \\
 &= \frac{-12x^5 - 72x^3 - 108x - 4x(-6x^4 + 54)}{(3+x^2)^4} \\
 &= \frac{-12x^5 - 72x^3 - 108x + 24x^5 - 216x}{(3+x^2)^4} \\
 &= \frac{12x^5 - 72x^3 - 324x}{(3+x^2)^4}
 \end{aligned}$$

$$\begin{aligned}
 f''(x) = 0 \quad \text{when} \quad \frac{12x^5 - 72x^3 - 324x}{(3+x^2)^4} &= 0 \\
 \therefore 12x^5 - 72x^3 - 324x &= 0 \\
 \therefore x^5 - 6x^3 - 27x &= 0 \\
 \therefore x(x^4 - 6x^2 - 27) &= 0 \\
 \therefore x(x^2 - 9)(x^2 + 3) &= 0 \\
 \therefore x = 0 \quad \text{or} \quad x^2 - 9 &= 0 \\
 \therefore x^2 &= 9 \\
 \therefore x &= \pm 3 \\
 \therefore x &= -3, 0, \text{ or } 3
 \end{aligned}$$

$$\begin{aligned}
 \text{22 a} \quad f(x) &= -10 \sin 2x \cos 2x, \quad 0 \leq x \leq \pi \\
 \therefore f(x) &= -5 \sin 4x \quad \{2 \sin A \cos A = \sin 2A\}
 \end{aligned}$$



$$\begin{aligned}
 \text{b} \quad f'(x) &= -5 \cos 4x \times 4 \quad \{\text{chain rule}\} \\
 \therefore f'(x) &= -20 \cos 4x \\
 f'(x) &= 0 \quad \text{when} \quad -20 \cos 4x = 0 \\
 &\therefore \cos 4x = 0 \\
 &\therefore 4x = \frac{\pi}{2} + n\pi, \quad n \in \mathbb{Z} \\
 &\therefore x = \frac{\pi}{8} + \frac{n\pi}{4} \\
 \text{So, on the domain } 0 \leq x \leq \pi, \quad x &= \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}
 \end{aligned}$$

$$\begin{aligned}
 \text{23 a} \quad e^x y - xy^2 &= 1 \\
 \therefore e^x y + e^x \times \frac{dy}{dx} - y^2 - 2xy \times \frac{dy}{dx} &= 0 \quad \{\text{product rule, chain rule}\} \\
 \therefore \frac{dy}{dx} (e^x - 2xy) &= y^2 - e^x y \\
 \therefore \frac{dy}{dx} &= \frac{y^2 - e^x y}{e^x - 2xy}
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad \text{When } x = 0, \quad e^0 y - (0)y^2 &= 1 \\
 \therefore y &= 1
 \end{aligned}$$

So, the point of contact is  $(0, 1)$ .

$$\begin{aligned}
 \text{When } x = 0, \quad y = 1, \quad \frac{dy}{dx} &= \frac{1^2 - e^0(1)}{e^0 - 2(0)(1)} \\
 &= \frac{1 - 1}{1 - 0} \\
 &= 0
 \end{aligned}$$

$\therefore$  the gradient of the tangent to the curve at  $x = 0$  is 0.

$$\text{24 } P_n \text{ is: If } y = x^n, \text{ then } \frac{dy}{dx} = nx^{n-1} \text{ for all } n \in \mathbb{Z}^+.$$

**Proof:** (By the principle of mathematical induction)

$$\begin{aligned}
 (1) \quad \text{If } n = 1, \text{ then } \text{LHS} &= \frac{d}{dx}(x) = 1 = 1x^{1-1} \quad \text{and} \quad \text{RHS} = 1x^{1-1} \\
 \therefore P_1 &\text{ is true.}
 \end{aligned}$$

$$(2) \quad \text{If } P_k \text{ is true, then } \frac{d}{dx}(x^k) = kx^{k-1}$$

$$\text{If } y = x^{k+1} = x^k x,$$

$$\begin{aligned}
 \text{then } \frac{dy}{dx} &= \frac{d}{dx}(x^k)x + x^k \frac{d}{dx}(x) \quad \{\text{product rule}\} \\
 &= kx^{k-1}x + x^k \times 1 \quad \{\text{using } P_k\} \\
 &= kx^k + x^k \\
 &= (k+1)x^k
 \end{aligned}$$

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,  
 $P_n$  is true for all  $n \in \mathbb{Z}^+$ .  $\{\text{principle of mathematical induction}\}$

**25**  $y = e^{2t}(A \cos t + B \sin t), \quad A, B \in \mathbb{R}$

$$\therefore \frac{dy}{dt} = 2e^{2t}(A \cos t + B \sin t) + e^{2t}(-A \sin t + B \cos t) \quad \{\text{product rule}\}$$

$$\therefore \frac{dy}{dt} = 2y + e^{2t}(-A \sin t + B \cos t) \quad \dots (*)$$

$$\therefore \frac{d^2y}{dt^2} = 2 \frac{dy}{dt} + 2e^{2t}(-A \sin t + B \cos t) + e^{2t}(-A \cos t - B \sin t) \quad \{\text{product rule}\}$$

$$= 2 \frac{dy}{dt} + 2 \left( \frac{dy}{dt} - 2y \right) - e^{2t}(A \cos t + B \sin t) \quad \{\text{using } (*)\}$$

$$= 2 \frac{dy}{dt} + 2 \frac{dy}{dt} - 4y - y$$

$$\therefore \frac{d^2y}{dt^2} - 4 \frac{dy}{dt} + 5y = 0 \quad \text{as required}$$

**26 a**  $y = uv$

$$\therefore \frac{dy}{dx} = \left( \frac{du}{dx} \right) v + u \left( \frac{dv}{dx} \right) \quad \{\text{product rule}\}$$

$$\therefore \frac{d^2y}{dx^2} = \left[ \left( \frac{d^2u}{dx^2} \right) v + \frac{du}{dx} \frac{dv}{dx} \right] + \left[ \frac{du}{dx} \frac{dv}{dx} + u \left( \frac{d^2v}{dx^2} \right) \right] \quad \{\text{product rule}\}$$

$$= \left( \frac{d^2u}{dx^2} \right) v + 2 \frac{du}{dx} \frac{dv}{dx} + u \left( \frac{d^2v}{dx^2} \right)$$

**b**  $y = uvw = u(vw)$

$$\therefore \frac{dy}{dx} = \frac{du}{dx} (vw) + u \left[ \frac{d}{dx} (vw) \right] \quad \{\text{product rule}\}$$

$$= \frac{du}{dx} vw + u \left( \frac{dv}{dx} w + v \frac{dw}{dx} \right) \quad \{\text{product rule}\}$$

$$= \frac{du}{dx} vw + u \frac{dv}{dx} w + uv \frac{dw}{dx}$$

**27 a**  $f(x) = xe^{ax}$

$$\therefore f'(x) = e^{ax} + x(ae^{ax}) \quad \{\text{product rule}\}$$

$$= e^{ax}(ax + 1)$$

$$\therefore f''(x) = ae^{ax}(ax + 1) + e^{ax}(a) \quad \{\text{product rule}\}$$

$$= ae^{ax}(ax + 1 + 1)$$

$$= ae^{ax}(ax + 2)$$

$$\therefore f^{(3)}(x) = a^2e^{ax}(ax + 2) + ae^{ax}(a) \quad \{\text{product rule}\}$$

$$= a^2e^{ax}(ax + 2 + 1)$$

$$= a^2e^{ax}(ax + 3)$$

$$\therefore f^{(4)}(x) = a^3e^{ax}(ax + 3) + a^2e^{ax}(a) \quad \{\text{product rule}\}$$

$$= a^3e^{ax}(ax + 3 + 1)$$

$$= a^3e^{ax}(ax + 4)$$

**b** We conjecture that  $f^{(n)}(x) = a^{n-1}e^{ax}(ax + n), \quad n \in \mathbb{Z}^+.$

•  $P_n$  is: If  $f(x) = xe^{ax}$ , then  $f^{(n)}(x) = a^{n-1}e^{ax}(ax + n)$  for all  $n \in \mathbb{Z}^+$ .

**Proof:** (By the principle of mathematical induction)

$$\begin{aligned} (1) \quad \text{For } n = 1, \quad \text{LHS} &= f'(x) & \text{and} \quad \text{RHS} &= a^{1-1}e^{ax}(ax + 1) \\ &= e^{ax}(ax + 1) & \{\text{using } \bullet\} \\ &= a^{1-1}e^{ax}(ax + 1) \end{aligned}$$

$\therefore P_1$  is true.

$$(2) \quad \text{If } P_k \text{ is true, then } f^{(k)}(x) = a^{k-1}e^{ax}(ax + k)$$

$$\begin{aligned} \text{Now } f^{(k+1)}(x) &= \frac{d}{dx} [a^{k-1}e^{ax}(ax + k)] & \{\text{using } P_k\} \\ &= a^{k-1}(a)e^{ax}(ax + k) + a^{k-1}e^{ax}(a) & \{\text{product rule}\} \\ &= a^k e^{ax}(ax + k) + a^k e^{ax} \\ &= a^{(k+1)-1}e^{ax}(ax + [k + 1]) \end{aligned}$$

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

# Chapter 18

## PROPERTIES OF CURVES

### EXERCISE 18A

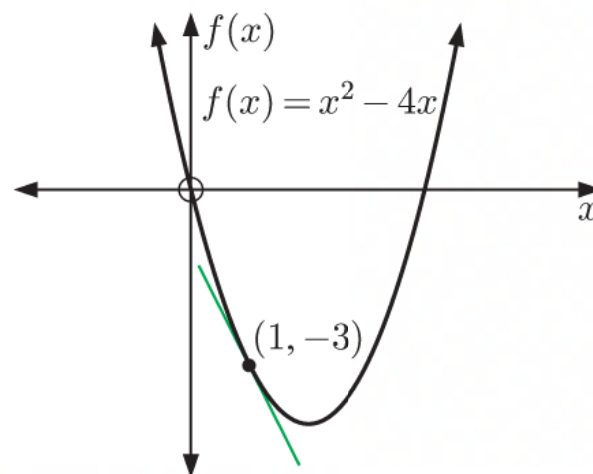
1 a  $f(x) = x^2 - 4x$

$$\therefore f'(x) = 2x - 4$$

b The point of contact is  $(1, -3)$ .

$$\begin{aligned} f'(1) &= 2(1) - 4 \\ &= -2 \end{aligned}$$

So, the tangent has equation  $y = -2(x - 1) - 3$   
 $\therefore y = -2x - 1$



2 a  $y = x - 2x^2 + 3$

When  $x = 2$ ,

$$y = 2 - 2(2)^2 + 3 = -3$$

So, the point of contact is  $(2, -3)$ .

Now  $\frac{dy}{dx} = 1 - 4x$ , so at  $x = 2$ ,

$$\frac{dy}{dx} = 1 - 4(2) = -7$$

So, the tangent has gradient  $-7$ .

The tangent has equation

$$y = f'(a)(x - a) + f(a)$$

$$\therefore y = -7(x - 2) + (-3)$$

$$\therefore y = -7x + 14 - 3$$

$$\therefore y = -7x + 11$$

c  $y = x^3 - 5x$

When  $x = 1$ ,

$$y = 1^3 - 5(1) = -4$$

So, the point of contact is  $(1, -4)$ .

Now  $\frac{dy}{dx} = 3x^2 - 5$ , so at  $x = 1$ ,

$$\frac{dy}{dx} = 3(1)^2 - 5 = -2$$

The tangent has equation

$$y = f'(a)(x - a) + f(a)$$

$$\therefore y = -2(x - 1) + (-4)$$

$$\therefore y = -2x - 2$$

b  $y = \sqrt{x} + 1$

$$= x^{\frac{1}{2}} + 1$$

When  $x = 4$ ,

$$y = \sqrt{4} + 1 = 3$$

So, the point of contact is  $(4, 3)$ .

Now  $\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$ , so at  $x = 4$ ,

$$\frac{dy}{dx} = \frac{1}{2\sqrt{4}} = \frac{1}{4}$$

So, the tangent has gradient  $\frac{1}{4}$ .

The tangent has equation

$$y = f'(a)(x - a) + f(a)$$

$$\therefore y = \frac{1}{4}(x - 4) + 3$$

$$\therefore y = \frac{1}{4}x + 2$$

d  $y = \frac{4}{\sqrt{x}} = 4x^{-\frac{1}{2}}$

Now  $\frac{dy}{dx} = -2x^{-\frac{3}{2}}$ , so at  $x = 1$ ,

$$\frac{dy}{dx} = -2(1^{-\frac{3}{2}}) = -2$$

The tangent has equation

$$y = f'(a)(x - a) + f(a)$$

$$\therefore y = -2(x - 1) + 4$$

$$\therefore y = -2x + 6$$



$$\begin{aligned} \text{e} \quad y &= \frac{3}{x} - \frac{1}{x^2} \\ &= 3x^{-1} - x^{-2} \end{aligned}$$

$$\begin{aligned} \text{Now } \frac{dy}{dx} &= -3x^{-2} + 2x^{-3} \\ &= -\frac{3}{x^2} + \frac{2}{x^3}, \text{ so at } (-1, -4), \\ \frac{dy}{dx} &= -\frac{3}{(-1)^2} + \frac{2}{(-1)^3} \\ &= -3 - 2 \\ &= -5 \end{aligned}$$

The tangent has equation

$$\begin{aligned} y &= f'(a)(x - a) + f(a) \\ \therefore y &= -5(x + 1) + (-4) \\ \therefore y &= -5x - 9 \end{aligned}$$

$$\begin{aligned} \text{g} \quad y &= \frac{1}{(x^2 + 1)^2} \\ &= (x^2 + 1)^{-2} \\ \text{Now } \frac{dy}{dx} &= -2(x^2 + 1)^{-3}(2x) \\ &\quad \{\text{chain rule}\} \\ &= -\frac{4x}{(x^2 + 1)^3}, \text{ so at } (1, \frac{1}{4}), \\ \frac{dy}{dx} &= -\frac{4(1)}{(1^2 + 1)^3} \\ &= -\frac{4}{2^3} \\ &= -\frac{1}{2} \end{aligned}$$

The tangent has equation

$$\begin{aligned} y &= f'(a)(x - a) + f(a) \\ \therefore y &= -\frac{1}{2}(x - 1) + \frac{1}{4} \\ \therefore y &= -\frac{1}{2}x + \frac{3}{4} \end{aligned}$$

$$\begin{aligned} \text{f} \quad y &= 3x^2 - \frac{1}{x} \\ &= 3x^2 - x^{-1} \\ \text{When } x &= -1, \\ y &= 3(-1)^2 - \frac{1}{(-1)} = 4 \\ \text{So, the point of contact is } &(-1, 4). \\ \text{Now } \frac{dy}{dx} &= 6x + x^{-2} \\ &= 6x + \frac{1}{x^2}, \text{ so at } x = -1, \\ \frac{dy}{dx} &= 6(-1) + \frac{1}{(-1)^2} = -5 \end{aligned}$$

The tangent has equation

$$\begin{aligned} y &= f'(a)(x - a) + f(a) \\ \therefore y &= -5(x + 1) + 4 \\ \therefore y &= -5x - 1 \end{aligned}$$

$$\begin{aligned} \text{h} \quad y &= \frac{1}{\sqrt{3 - 2x}} \\ &= (3 - 2x)^{-\frac{1}{2}} \\ \text{When } x &= -3, \\ y &= \frac{1}{\sqrt{3 - 2(-3)}} = \frac{1}{3} \\ \text{So, the point of contact is } &(-3, \frac{1}{3}). \\ \text{Now } \frac{dy}{dx} &= -\frac{1}{2}(3 - 2x)^{-\frac{3}{2}}(-2) \\ &\quad \{\text{chain rule}\} \\ &= \frac{1}{(\sqrt{3 - 2x})^3}, \text{ so at } (-3, \frac{1}{3}), \\ \frac{dy}{dx} &= \frac{1}{(\sqrt{3 - 2(-3)})^3} \\ &= \frac{1}{27} \end{aligned}$$

The tangent has equation

$$\begin{aligned} y &= f'(a)(x - a) + f(a) \\ \therefore y &= \frac{1}{27}(x - (-3)) + \frac{1}{3} \\ \therefore y &= \frac{1}{27}x + \frac{4}{9} \end{aligned}$$

**3 a**  $y = 2x^3 + 3x^2 - 12x + 1$

$$\therefore \frac{dy}{dx} = 6x^2 + 6x - 12$$

Horizontal tangents have gradient 0,

$$\text{so } 6x^2 + 6x - 12 = 0$$

$$\therefore 6(x^2 + x - 2) = 0$$

$$\therefore 6(x + 2)(x - 1) = 0$$

$$\therefore x = -2 \text{ or } 1$$

When  $x = -2$ ,

$$\begin{aligned} y &= 2(-2)^3 + 3(-2)^2 - 12(-2) + 1 \\ &= 21 \end{aligned}$$

When  $x = 1$ ,

$$\begin{aligned} y &= 2(1)^3 + 3(1)^2 - 12(1) + 1 \\ &= -6 \end{aligned}$$

$\therefore$  the points of contact are  $(-2, 21)$  and  $(1, -6)$ .

$\therefore$  the tangents are  $y = 21$  and  $y = -6$ .

**b**  $y = -x^3 + 3x^2 + 9x - 4$

$$\therefore \frac{dy}{dx} = -3x^2 + 6x + 9$$

Horizontal tangents have gradient 0,

$$\text{so } -3x^2 + 6x + 9 = 0$$

$$\therefore -3(x^2 - 2x - 3) = 0$$

$$\therefore -3(x + 1)(x - 3) = 0$$

$$\therefore x = -1 \text{ or } 3$$

When  $x = -1$ ,

$$\begin{aligned} y &= -(-1)^3 + 3(-1)^2 + 9(-1) - 4 \\ &= -9 \end{aligned}$$

When  $x = 3$ ,

$$\begin{aligned} y &= -3^3 + 3(3)^2 + 9(3) - 4 \\ &= 23 \end{aligned}$$

$\therefore$  the points of contact are  $(-1, -9)$  and  $(3, 23)$ .

$\therefore$  the tangents are  $y = -9$  and  $y = 23$ .

**c**  $y = \sqrt{x} + \frac{1}{\sqrt{x}}$

$$= x^{\frac{1}{2}} + x^{-\frac{1}{2}}$$

$$\therefore \frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}x^{-\frac{3}{2}}$$

$$= \frac{1}{2\sqrt{x}} - \frac{1}{2x\sqrt{x}}$$

Horizontal tangents have gradient 0, so  $\frac{1}{2\sqrt{x}} - \frac{1}{2x\sqrt{x}} = 0$

$$\therefore \frac{x - 1}{2x\sqrt{x}} = 0$$

$$\therefore x - 1 = 0$$

$$\therefore x = 1$$

$$\begin{aligned} \text{When } x = 1, \quad y &= \sqrt{1} + \frac{1}{\sqrt{1}} \\ &= 2 \end{aligned}$$

$\therefore$  the point of contact is  $(1, 2)$ .

$\therefore$  the tangent is  $y = 2$ .

**4**  $y = 2x^3 + kx^2 - 3$

**a**  $\frac{dy}{dx} = 6x^2 + 2kx$

When  $x = 2$ ,  $\frac{dy}{dx} = 4$

$$\therefore 6(2)^2 + 2k(2) = 4$$

$$\therefore 24 + 4k = 4$$

$$\therefore 4k = -20$$

$$\therefore k = -5$$

**b** Since  $k = -5$ ,

$$y = 2x^3 - 5x^2 - 3$$

When  $x = 2$ ,

$$\begin{aligned} y &= 2(2)^3 - 5(2)^2 - 3 \\ &= -7 \end{aligned}$$

So, the point of contact is  $(2, -7)$ .

The tangent has equation

$$y = 4(x - 2) + (-7)$$

$$\therefore y = 4x - 15$$

**5**  $y = 1 - 3x + 12x^2 - 8x^3$

$$\therefore \frac{dy}{dx} = -3 + 24x - 24x^2$$

When  $x = 1$ ,  $\frac{dy}{dx} = -3 + 24(1) - 24(1)^2$   
 $= -3$

So, the tangent at  $(1, 2)$  has gradient  $-3$ .

The tangents to the curve have gradient  $-3$  when  $-3 + 24x - 24x^2 = -3$

$$\therefore 24x^2 - 24x = 0$$

$$\therefore 24x(x - 1) = 0$$

$$\therefore x = 0 \text{ or } 1$$

So the other  $x$ -value for which the tangent to the curve has gradient  $-3$  is  $x = 0$ ,  
 and when  $x = 0$ ,  $y = 1 - 3(0) + 12(0)^2 - 8(0)^3 = 1$ .

$\therefore$  the tangent to the curve at  $(0, 1)$  is parallel to the tangent at  $(1, 2)$ .

This tangent has equation  $y = -3(x - 0) + 1$

$$\therefore y = -3x + 1$$

**6** The tangent to the curve  $y = x^2 + ax + b$  at the point where  $x = 1$  is  $2x + y = 6$  or  $y = -2x + 6$ .

$\therefore$  the tangent has gradient  $-2$ , and the point of contact is  $(1, -2(1) + 6)$  which is  $(1, 4)$ .

Now,  $y = x^2 + ax + b$

$$\therefore \frac{dy}{dx} = 2x + a$$

When  $x = 1$ ,  $\frac{dy}{dx} = -2$

$$\therefore 2(1) + a = -2$$

$$\therefore 2 + a = -2$$

$$\therefore a = -4 \quad \dots (*)$$

and  $y = 4$

$$\therefore 1^2 + a(1) + b = 4$$

$$\therefore 1 + (-4) + b = 4 \quad \{\text{using } (*)\}$$

$$\therefore b = 7$$

So,  $a = -4$  and  $b = 7$ .

- 7** The tangent to the curve  $y = a\sqrt{x} + bx$  at the point where  $x = 4$  is  $y = x + 2$ .

$\therefore$  the tangent has gradient 1, and the point of contact is  $(4, 4 + 2)$  which is  $(4, 6)$ .

Now,  $y = a\sqrt{x} + bx$

$$= ax^{\frac{1}{2}} + bx$$

$$\therefore \frac{dy}{dx} = \frac{1}{2}ax^{-\frac{1}{2}} + b$$

$$= \frac{a}{2\sqrt{x}} + b$$

When  $x = 4$ ,  $\frac{dy}{dx} = 1$

$$\therefore \frac{a}{2\sqrt{4}} + b = 1$$

$$\therefore \frac{a}{4} + b = 1$$

$$\therefore a + 4b = 4$$

$$\therefore a = 4 - 4b \quad \dots (*)$$

and  $y = 6$

$$\therefore a\sqrt{4} + b(4) = 6$$

$$\therefore 2(4 - 4b) + 4b = 6 \quad \{\text{using } (*)\}$$

$$\therefore 4 - 4b + 2b = 3$$

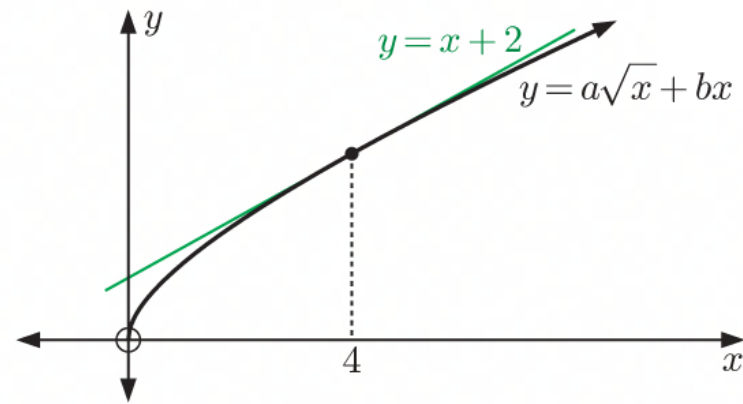
$$\therefore -2b = -1$$

$$\therefore b = \frac{1}{2}$$

Substituting  $b = \frac{1}{2}$  into  $(*)$  gives

$$\begin{aligned} a &= 4 - 4\left(\frac{1}{2}\right) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

So,  $a = 2$  and  $b = \frac{1}{2}$ .



**8**  $y = 2x^2 - 1$

$$\therefore \frac{dy}{dx} = 4x$$

$$\therefore \text{at the point where } x = a, \frac{dy}{dx} = 4a$$

$\therefore$  the gradient of the tangent at the point where  $x = a$  is  $4a$ .

Also, at  $x = a$ ,  $y = 2a^2 - 1$ .

$$\therefore \text{the tangent has equation } y = 4a(x - a) + (2a^2 - 1)$$

$$\therefore y = 4ax - 4a^2 + 2a^2 - 1$$

$$\therefore 4ax - y = 2a^2 + 1$$



$$\begin{aligned}
 9 \quad f(x) &= x^2 + \frac{4}{x^2} \\
 &= x^2 + 4x^{-2} \\
 \therefore f'(x) &= 2x - 8x^{-3} \\
 &= 2x - \frac{8}{x^3}
 \end{aligned}$$

Horizontal tangents have gradient 0, so  $2x - \frac{8}{x^3} = 0$

$$\begin{aligned}
 \therefore \frac{2x^4 - 8}{x^3} &= 0 \\
 \therefore 2x^4 - 8 &= 0 \\
 \therefore 2x^4 &= 8 \\
 \therefore x^4 &= 4 \\
 \therefore x &= \pm\sqrt{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{When } x = \sqrt{2}, \quad f(\sqrt{2}) &= (\sqrt{2})^2 + \frac{4}{(\sqrt{2})^2} \\
 &= 2 + \frac{4}{2} \\
 &= 4
 \end{aligned}$$

$\therefore$  the horizontal tangent at  $(\sqrt{2}, 4)$  is  $y = 4$ .

$$\begin{aligned}
 \text{When } x = -\sqrt{2}, \quad f(-\sqrt{2}) &= (-\sqrt{2})^2 + \frac{4}{(-\sqrt{2})^2} \\
 &= 2 + \frac{4}{2} \\
 &= 4
 \end{aligned}$$

$\therefore$  the horizontal tangent at  $(-\sqrt{2}, 4)$  is  $y = 4$ .

So, there is a unique horizontal tangent  $y = 4$ , which touches the curve at  $(-\sqrt{2}, 4)$  and  $(\sqrt{2}, 4)$ .

- 10** The tangent to the curve  $y = a\sqrt{1-bx}$  at the point where  $x = -1$  is  $3x + y = 5$  or  $y = -3x + 5$ .  
 $\therefore$  the tangent has gradient  $-3$ , and the point of contact is  $(-1, -3(-1) + 5)$  which is  $(-1, 8)$ .

$$\begin{aligned}
 \text{Now } y &= a\sqrt{1-bx} \\
 &= a(1-bx)^{\frac{1}{2}} \\
 \therefore \frac{dy}{dx} &= \frac{1}{2}a(1-bx)^{-\frac{1}{2}}(-b) \\
 &= -\frac{ab}{2\sqrt{1-bx}}
 \end{aligned}$$

When  $x = -1$ ,  $\frac{dy}{dx} = -3$

and

$$y = 8$$

$$\therefore -\frac{ab}{2\sqrt{1-b(-1)}} = -3$$

$$\therefore ab = 6\sqrt{1+b}$$

$$\therefore a = \frac{6\sqrt{1+b}}{b} \dots (*)$$

$$\therefore a\sqrt{1-b(-1)} = 8$$

$$\therefore \left(\frac{6\sqrt{1+b}}{b}\right)\sqrt{1+b} = 8 \quad \{\text{using } (*)\}$$

$$\therefore \frac{6(1+b)}{b} = 8$$

$$\therefore 6 + 6b = 8b$$

$$\therefore -2b = -6$$

$$\therefore b = 3$$

Substituting  $b = 3$  into  $(*)$  gives  $a = \frac{6\sqrt{1+3}}{3}$   
 $= 2\sqrt{4}$   
 $= 4$

So,  $a = 4$  and  $b = 3$ .

**11 a**  $f(x) = (2x - 1)^4$   
 $\therefore f'(x) = 4(2x - 1)^3(2) \quad \{\text{chain rule}\}$   
 $= 8(2x - 1)^3$

The tangent to  $f(x) = (2x - 1)^4$  at  $x = k$  has gradient 8.

$$\therefore f'(k) = 8$$

$$\therefore 8(2k - 1)^3 = 8$$

$$\therefore (2k - 1)^3 = 1$$

$$\therefore 2k - 1 = 1$$

$$\therefore 2k = 2$$

$$\therefore k = 1$$

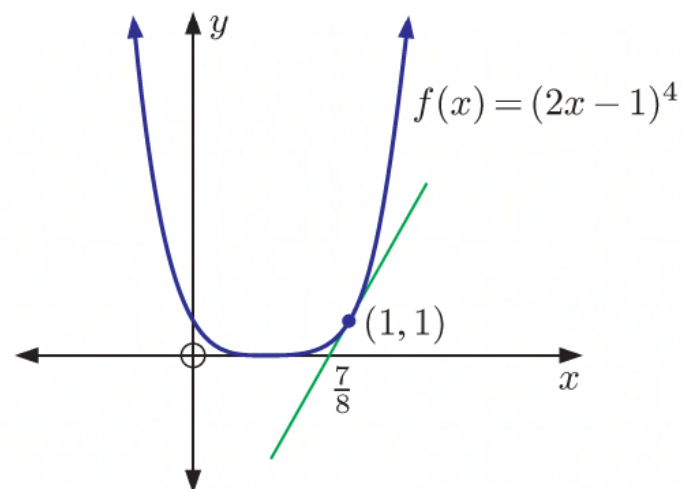
**b**  $f(1) = (2(1) - 1)^4 = 1$   
 $\therefore$  the point of contact is  $(1, 1)$ .

$$f'(1) = 8 \quad \{\text{using a}\}$$

So, the tangent has equation  $y = 8(x - 1) + 1$   
 $\therefore y = 8x - 7$

**c** When  $y = 0$ ,  $0 = 8x - 7$   
 $\therefore 8x = 7$   
 $\therefore x = \frac{7}{8}$

$\therefore$  the  $x$ -intercept of this tangent is  $\frac{7}{8}$ .



**12 a**  $x^3 + y^2 = 100$

$$\therefore 3x^2 + 2y \frac{dy}{dx} = 0$$

$$\therefore 2y \frac{dy}{dx} = -3x^2$$

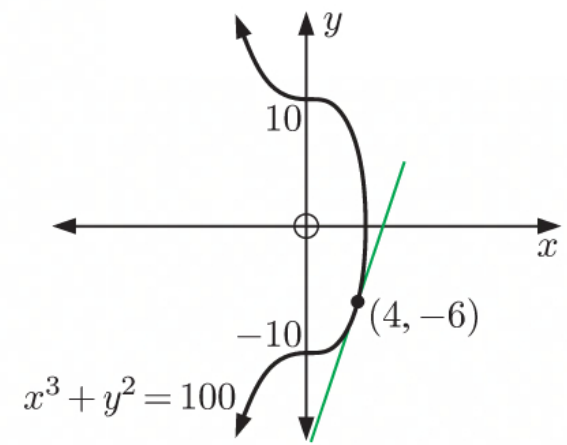
$$\therefore \frac{dy}{dx} = -\frac{3x^2}{2y}$$

**b** At the point  $(4, -6)$ ,  $\frac{dy}{dx} = -\frac{3(4)^2}{2(-6)} = -\frac{48}{-12} = 4$

So, the illustrated tangent has equation

$$y = 4(x - 4) - 6$$

$$\therefore y = 4x - 22$$



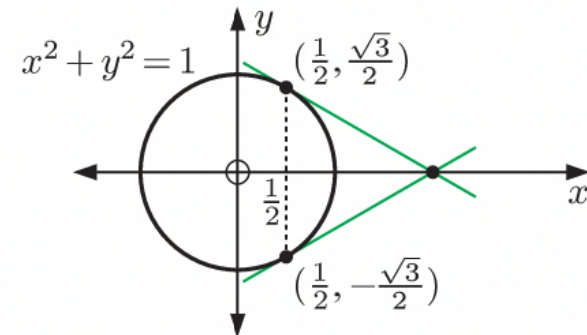
**13 a**  $x^2 + y^2 = 1$

When  $x = \frac{1}{2}$ ,  $(\frac{1}{2})^2 + y^2 = 1$

$$\therefore \frac{1}{4} + y^2 = 1$$

$$\therefore y^2 = \frac{3}{4}$$

$$\therefore y = \pm \frac{\sqrt{3}}{2}$$



So, we need to find the tangents at  $(\frac{1}{2}, \frac{\sqrt{3}}{2})$  and  $(\frac{1}{2}, -\frac{\sqrt{3}}{2})$ .

Now  $2x + 2y \frac{dy}{dx} = 0$

$$\therefore 2y \frac{dy}{dx} = -2x$$

$$\therefore \frac{dy}{dx} = -\frac{x}{y}$$

The tangent at  $(\frac{1}{2}, \frac{\sqrt{3}}{2})$  has gradient  $-\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = -\frac{1}{\sqrt{3}}$ .

$$\begin{aligned} \therefore \text{the tangent has equation } y &= -\frac{1}{\sqrt{3}}(x - \frac{1}{2}) + \frac{\sqrt{3}}{2} \\ &= -\frac{1}{\sqrt{3}}x + \frac{1}{2\sqrt{3}} + \frac{\sqrt{3}}{2} \times \left(\frac{\sqrt{3}}{\sqrt{3}}\right) \\ &= -\frac{1}{\sqrt{3}}x + \frac{2}{\sqrt{3}} \end{aligned}$$

The tangent at  $(\frac{1}{2}, -\frac{\sqrt{3}}{2})$  has gradient  $-\frac{\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}}$ .

$$\begin{aligned} \therefore \text{the tangent has equation } y &= \frac{1}{\sqrt{3}}(x - \frac{1}{2}) - \frac{\sqrt{3}}{2} \\ &= \frac{1}{\sqrt{3}}x - \frac{1}{2\sqrt{3}} - \frac{\sqrt{3}}{2} \times \left(\frac{\sqrt{3}}{\sqrt{3}}\right) \\ &= \frac{1}{\sqrt{3}}x - \frac{2}{\sqrt{3}} \end{aligned}$$

**b** The tangents intersect when  $-\frac{1}{\sqrt{3}}x + \frac{2}{\sqrt{3}} = \frac{1}{\sqrt{3}}x - \frac{2}{\sqrt{3}}$

$$\therefore \frac{2}{\sqrt{3}}x = \frac{4}{\sqrt{3}}$$

$$\therefore x = 2$$

$$\begin{aligned} \text{When } x = 2, \quad y &= -\frac{1}{\sqrt{3}}(2) + \frac{2}{\sqrt{3}} \\ &= 0 \end{aligned}$$

So, the tangents intersect at  $(2, 0)$ .

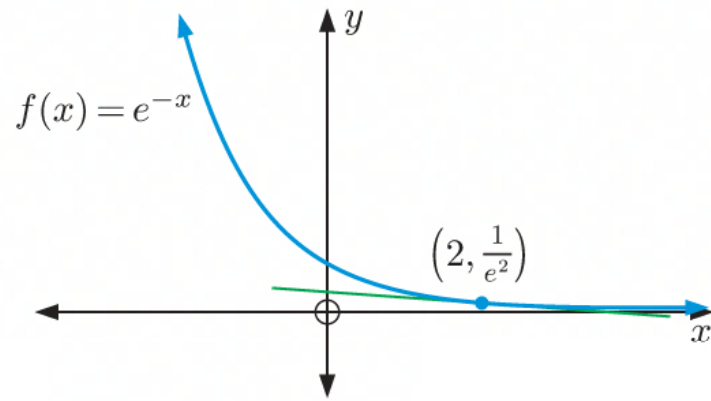
**14 a**  $f(x) = e^{-x}$   
 $\therefore f(2) = e^{-2}$   
 $= \frac{1}{e^2}$

$\therefore$  the point of contact is  $\left(2, \frac{1}{e^2}\right)$ .

Now  $f(x) = e^{-x}$  has derivative  $f'(x) = -e^{-x}$   
 $= -\frac{1}{e^x}$

$\therefore$  the tangent at  $\left(2, \frac{1}{e^2}\right)$  has gradient  $-\frac{1}{e^2}$ .

$\therefore$  the tangent has equation  $y = -\frac{1}{e^2}(x - 2) + \frac{1}{e^2}$   
 $= -\frac{x}{e^2} + \frac{2}{e^2} + \frac{1}{e^2}$   
 $= -\frac{x}{e^2} + \frac{3}{e^2}$   
 $\therefore y = -e^{-2}x + 3e^{-2}$



**b**  $y = \ln(2 - x)$

When  $x = -1$ ,  $y = \ln(2 - (-1))$   
 $= \ln 3$

$\therefore$  the point of contact is  $(-1, \ln 3)$ .

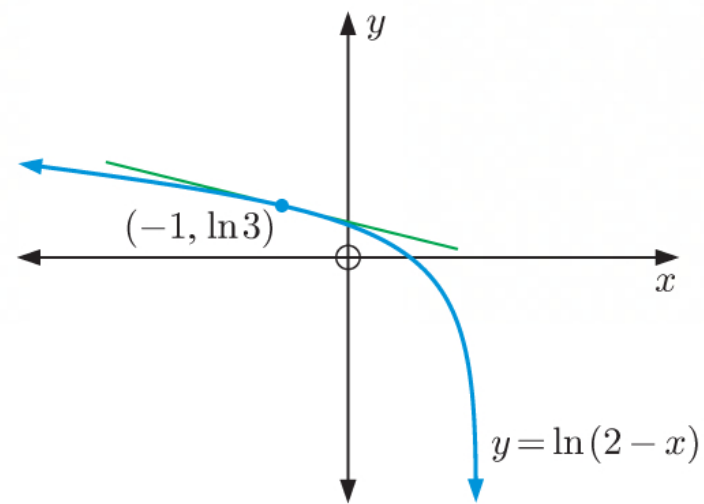
Now  $y = \ln(2 - x)$  has derivative

$$\frac{dy}{dx} = \frac{-1}{2 - x} = \frac{1}{x - 2}$$

$\therefore$  the tangent at  $(-1, \ln 3)$  has gradient

$$\frac{1}{-1 - 2} = -\frac{1}{3}$$

$\therefore$  the tangent has equation  $y = -\frac{1}{3}(x + 1) + \ln 3$   
 which is  $y = -\frac{1}{3}x - \frac{1}{3} + \ln 3$



**c**  $y = (x + 2)e^x$

When  $x = 1$ ,  $y = (1 + 2)e^1$   
 $= 3e$

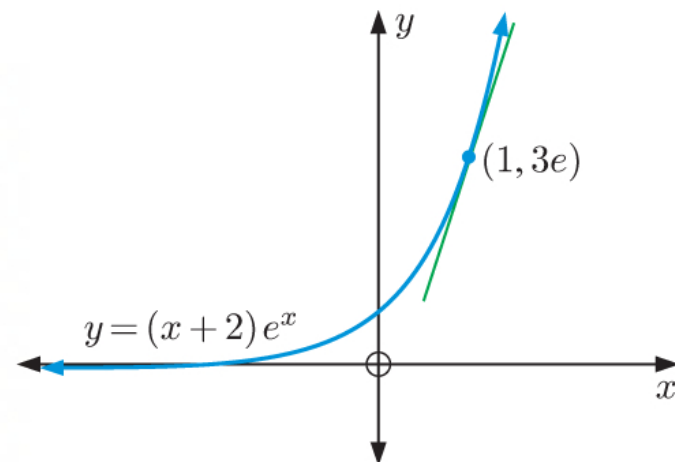
$\therefore$  the point of contact is  $(1, 3e)$ .

Now  $y = (x + 2)e^x$  has derivative

$$\frac{dy}{dx} = (1)e^x + (x + 2)e^x \quad \{\text{product rule}\}$$

$\therefore$  the tangent at  $(1, 3e)$  has gradient  
 $e^1 + (1 + 2)e^1 = 4e$

$\therefore$  the tangent has equation  $y = 4e(x - 1) + 3e$   
 which is  $y = 4ex - e$





**d**  $y = \ln \sqrt{x}$

When  $y = -1$ ,  $\ln \sqrt{x} = -1$

$$\therefore \sqrt{x} = e^{-1}$$

$$\therefore x = e^{-2} = \frac{1}{e^2}$$

$\therefore$  the point of contact is  $\left(\frac{1}{e^2}, -1\right)$ .

Now  $y = \ln \sqrt{x} = \ln(x^{\frac{1}{2}})$  has derivative

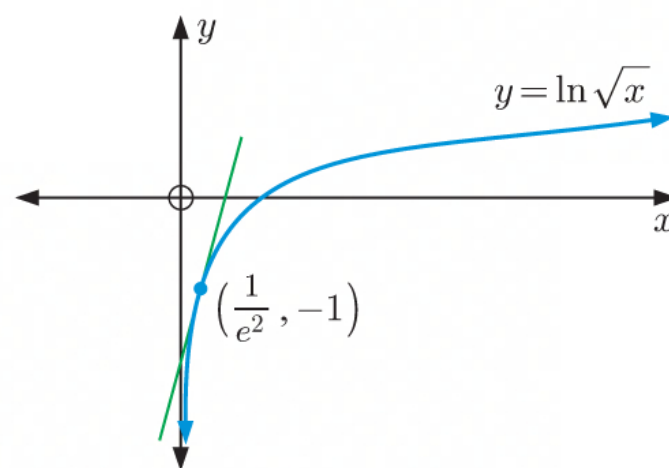
$$\frac{dy}{dx} = \frac{\frac{1}{2}x^{-\frac{1}{2}}}{x^{\frac{1}{2}}} = \frac{1}{2x}$$

$\therefore$  the tangent at  $\left(\frac{1}{e^2}, -1\right)$  has gradient  $\frac{1}{\frac{2}{e^2}} = \frac{e^2}{2}$

$\therefore$  the tangent has equation  $y = \frac{e^2}{2}\left(x - \frac{1}{e^2}\right) - 1$

$$= \frac{e^2}{2}x - \frac{1}{2} - 1$$

$$\therefore y = \frac{e^2}{2}x - \frac{3}{2}$$



**e**  $y = e^{3x-5}$

When  $y = e$ ,  $e^{3x-5} = e$

$$\therefore 3x - 5 = 1$$

$$\therefore 3x = 6$$

$$\therefore x = 2$$

$\therefore$  the point of contact is  $(2, e)$ .

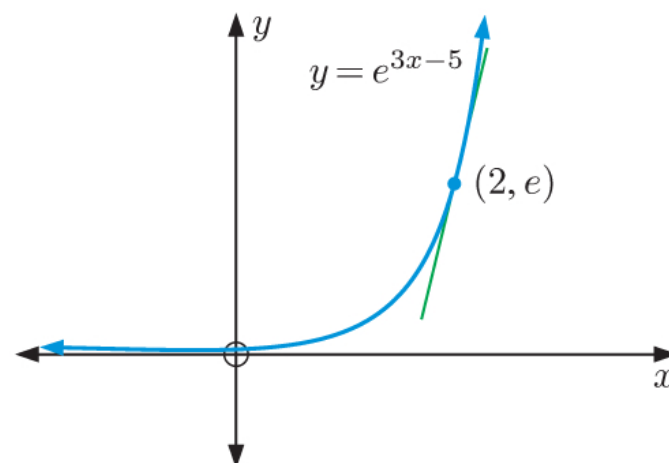
Now  $y = e^{3x-5}$  has derivative  $\frac{dy}{dx} = 3e^{3x-5}$

$\therefore$  the tangent at  $(2, e)$  has gradient

$$3e^{3(2)-5} = 3e$$

$\therefore$  the tangent has equation  $y = 3e(x - 2) + e$

which is  $y = 3ex - 5e$



**15 a**  $f(x) = \ln(x(x-2))$  is defined when  $x(x-2) > 0$

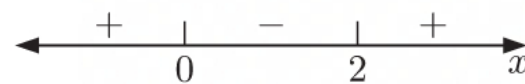
$$\therefore x < 0 \text{ or } x > 2$$

$\therefore$  the domain of  $f(x)$  is  $\{x \mid x < 0 \text{ or } x > 2\}$

**b**  $f(x) = \ln(x(x-2))$

$$= \ln x + \ln(x-2) \quad \{\ln(ab) = \ln a + \ln b\}$$

$$\therefore f'(x) = \frac{1}{x} + \frac{1}{x-2}$$



$$\begin{aligned} \bullet \quad f(3) &= \ln(3(3-2)) \\ &= \ln 3 \end{aligned}$$

$\therefore$  the point of contact is  $(3, \ln 3)$ .

$$\begin{aligned} \text{Now } f'(3) &= \frac{1}{3} + \frac{1}{3-2} \\ &= \frac{1}{3} + 1 \\ &= \frac{4}{3} \end{aligned}$$

$\therefore$  the tangent at  $(3, \ln 3)$  has gradient  $\frac{4}{3}$ .

$$\begin{aligned} \therefore \text{ the tangent has equation } y &= \frac{4}{3}(x-3) + \ln 3 \\ \text{which is } y &= \frac{4}{3}x - 4 + \ln 3 \end{aligned}$$

**16 a**  $f(x) = x \ln x$

$$\begin{aligned} \therefore f'(x) &= (1) \ln x + x \left( \frac{1}{x} \right) \quad \{\text{product rule}\} \\ &= \ln x + 1 \end{aligned}$$

**i**  $f(1) = (1) \ln 1 = 0$

$\therefore$  the point of contact is  $(1, 0)$ .

$$f'(1) = \ln 1 + 1 = 1$$

So, the tangent has equation

$$y = 1(x-1) + 0$$

$$\therefore y = x - 1$$

When  $x = 0$ ,  $y = -1$

$\therefore$  the  $y$ -intercept of the tangent is  $-1$ .

**ii**  $f(2) = 2 \ln 2$

$\therefore$  the point of contact is  $(2, 2 \ln 2)$ .

$$f'(2) = \ln 2 + 1$$

So, the tangent has equation

$$\begin{aligned} y &= (\ln 2 + 1)(x-2) + 2 \ln 2 \\ &= x \ln 2 - \cancel{2 \ln 2} + x - 2 + \cancel{2 \ln 2} \end{aligned}$$

$$\therefore y = x(\ln 2 + 1) - 2$$

When  $x = 0$ ,  $y = -2$

$\therefore$  the  $y$ -intercept of the tangent is  $-2$ .

**iii**  $f(3) = 3 \ln 3$

$\therefore$  the point of contact is  $(3, 3 \ln 3)$ .

$$f'(3) = \ln 3 + 1$$

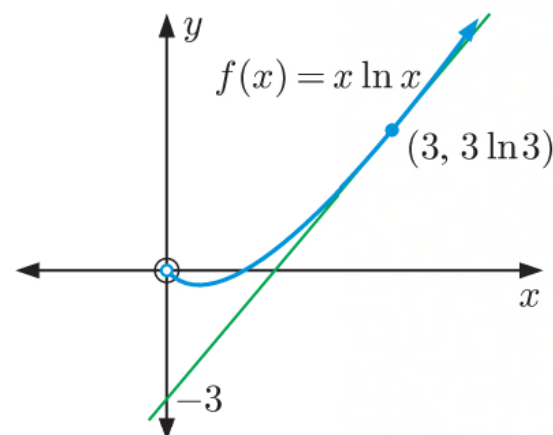
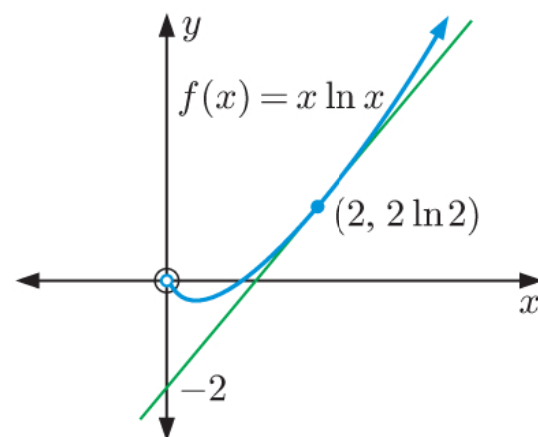
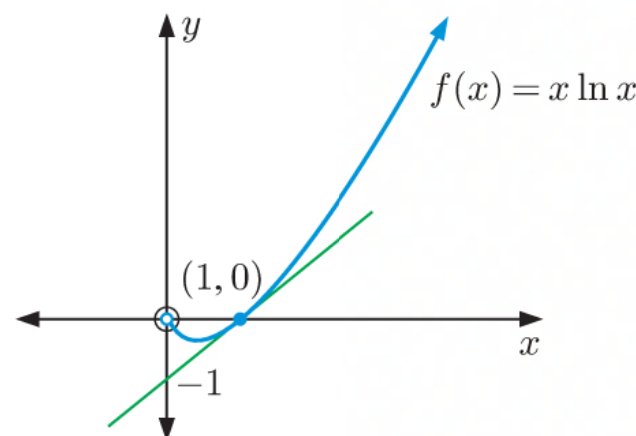
So, the tangent has equation

$$\begin{aligned} y &= (\ln 3 + 1)(x-3) + 3 \ln 3 \\ &= x \ln 3 - \cancel{3 \ln 3} + x - 3 + \cancel{3 \ln 3} \end{aligned}$$

$$\therefore y = x(\ln 3 + 1) - 3$$

When  $x = 0$ ,  $y = -3$

$\therefore$  the  $y$ -intercept of the tangent is  $-3$ .



- b** We conjecture that the  $y$ -intercept of the tangent to  $f(x) = x \ln x$  at the point where  $x = a$ ,  $a > 0$ , is  $-a$ .

$$f(x) = x \ln x$$

$$\therefore f(a) = a \ln a$$

$\therefore$  the point of contact is  $(a, a \ln a)$ .

$$f'(x) = \ln x + 1$$

$$\therefore f'(a) = \ln a + 1$$

$$\begin{aligned} \text{So, the tangent has equation } y &= (\ln a + 1)(x - a) + a \ln a \\ &= x \ln a - \cancel{a \ln a} + x - a + \cancel{a \ln a} \\ &= x(\ln a + 1) - a \end{aligned}$$

When  $x = 0$ ,  $y = -a$

$\therefore$  the  $y$ -intercept of the tangent to  $f(x) = x \ln x$  at the point where  $x = a$ ,  $a > 0$ , is  $-a$ .

**17**  $y = x^2 e^x$

When  $x = 1$ ,  $y = (1)^2 e^1 = e$

$\therefore$  the point of contact is  $(1, e)$ .

Now  $y = x^2 e^x$

$$\therefore \frac{dy}{dx} = 2xe^x + x^2 e^x \quad \{\text{product rule}\}$$

When  $x = 1$ ,  $\frac{dy}{dx} = 2e + e = 3e$

$$\begin{aligned} \text{So, the tangent has equation } y &= 3e(x - 1) + e \\ \therefore y &= 3ex - 2e \end{aligned}$$

When  $y = 0$ ,  $0 = 3ex - 2e$

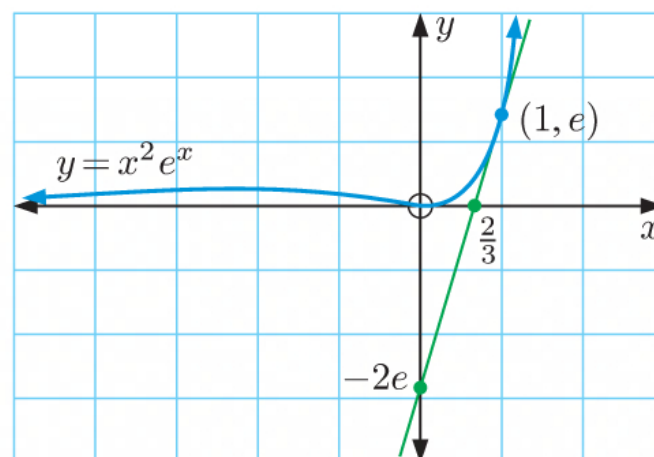
$$\therefore 3ex = 2e$$

$$\therefore x = \frac{2}{3}$$

$\therefore$  the  $x$ -intercept is  $\frac{2}{3}$ .

When  $x = 0$ ,  $y = -2e$

$\therefore$  the  $y$ -intercept is  $-2e$ .



**18**  $y = 3xe^{\frac{x}{2}}$

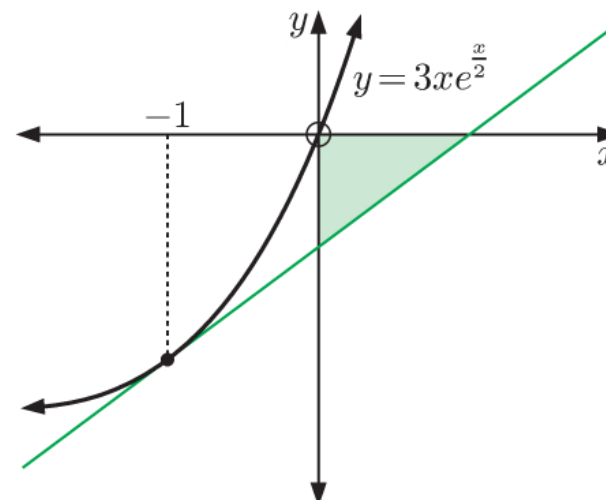
$$\begin{aligned} \text{When } x = -1, \quad y &= 3(-1)e^{\frac{-1}{2}} \\ &= -\frac{3}{\sqrt{e}} \end{aligned}$$

$\therefore$  the point of contact is  $\left(-1, -\frac{3}{\sqrt{e}}\right)$ .

Now  $y = 3xe^{\frac{x}{2}}$

$$\begin{aligned} \therefore \frac{dy}{dx} &= 3e^{\frac{x}{2}} + 3x\left(\frac{1}{2}e^{\frac{x}{2}}\right) \quad \{\text{product rule}\} \\ &= 3e^{\frac{x}{2}} + \frac{3}{2}xe^{\frac{x}{2}} \end{aligned}$$

$$\begin{aligned} \text{When } x = -1, \quad \frac{dy}{dx} &= 3e^{-\frac{1}{2}} - \frac{3}{2}e^{-\frac{1}{2}} \\ &= \frac{3}{\sqrt{e}} - \frac{3}{2\sqrt{e}} \\ &= \frac{3}{2\sqrt{e}} \end{aligned}$$



$$\begin{aligned}
 \text{So, the tangent has equation } y &= \frac{3}{2\sqrt{e}}(x+1) - \frac{3}{\sqrt{e}} \\
 &= \frac{3}{2\sqrt{e}}x + \frac{3}{2\sqrt{e}} - \frac{3}{\sqrt{e}} \\
 &= \frac{3}{2\sqrt{e}}x - \frac{3}{2\sqrt{e}} \\
 &= \frac{3}{2\sqrt{e}}(x-1)
 \end{aligned}$$

$$\text{When } y = 0, \quad 0 = \frac{3}{2\sqrt{e}}(x-1)$$

$$\therefore x = 1$$

$\therefore$  the  $x$ -intercept is 1.

$$\text{When } x = 0, \quad y = \frac{3}{2\sqrt{e}}(0-1)$$

$$= -\frac{3}{2\sqrt{e}}$$

$$\therefore \text{ the } y\text{-intercept is } -\frac{3}{2\sqrt{e}}.$$

$$\therefore \text{ the shaded triangle has area } = \frac{1}{2} \times \text{base} \times \text{height}$$

$$= \frac{1}{2} \times 1 \times \frac{3}{2\sqrt{e}}$$

$$= \frac{3}{4\sqrt{e}} \text{ units}^2$$

**19 a**  $y = \sin x$  has derivative  $\frac{dy}{dx} = \cos x$

$$\therefore \text{ the tangent at } (0, 0) \text{ has gradient } \cos 0 = 1$$

$$\therefore \text{ the tangent has equation } y = 1(x-0) + 0$$

$$\text{which is } y = x$$

**b**  $y = \tan x$  has derivative  $\frac{dy}{dx} = \sec^2 x = \frac{1}{\cos^2 x}$

$$\therefore \text{ the tangent at } (0, 0) \text{ has gradient } \frac{1}{\cos^2 0} = 1$$

$$\therefore \text{ the tangent has equation } y = 1(x-0) + 0$$

$$\text{which is } y = x$$

**c**  $y = \cos x$

$$\text{When } x = \frac{\pi}{6}, \quad y = \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

$$\therefore \text{ the point of contact is } \left(\frac{\pi}{6}, \frac{\sqrt{3}}{2}\right).$$

$$\text{Now } y = \cos x \text{ has derivative } \frac{dy}{dx} = -\sin x$$

$$\therefore \text{ the tangent at } \left(\frac{\pi}{6}, \frac{\sqrt{3}}{2}\right) \text{ has gradient } -\sin \frac{\pi}{6} = -\frac{1}{2}$$

$$\therefore \text{ the tangent has equation } y = -\frac{1}{2}\left(x - \frac{\pi}{6}\right) + \frac{\sqrt{3}}{2}$$

$$\text{which is } y = -\frac{1}{2}x + \frac{\pi}{12} + \frac{\sqrt{3}}{2}$$



**d**  $y = \frac{1}{\sin 2x}$

When  $x = \frac{\pi}{4}$ ,  $y = \frac{1}{\sin \frac{\pi}{2}} = 1$

$\therefore$  the point of contact is  $(\frac{\pi}{4}, 1)$ .

Now  $y = \frac{1}{\sin 2x} = (\sin 2x)^{-1}$  has derivative

$$\begin{aligned} \frac{dy}{dx} &= -(\sin 2x)^{-2}(2 \cos 2x) \quad \{\text{chain rule}\} \\ &= -\frac{2 \cos 2x}{\sin^2 2x} \end{aligned}$$

$\therefore$  the tangent at  $(\frac{\pi}{4}, 1)$  has gradient  $-\frac{2 \cos \frac{\pi}{2}}{\sin^2(\frac{\pi}{2})} = 0$

$\therefore$  the tangent has equation  $y = 0(x - \frac{\pi}{4}) + 1$   
which is  $y = 1$

**e**  $y = \cos 2x + 3 \sin x$

When  $x = \frac{\pi}{2}$ ,  $y = \cos \pi + 3 \sin \frac{\pi}{2}$   
 $= -1 + 3$   
 $= 2$

$\therefore$  the point of contact is  $(\frac{\pi}{2}, 2)$ .

Now  $y = \cos 2x + 3 \sin x$  has derivative  $\frac{dy}{dx} = -2 \sin 2x + 3 \cos x$

$\therefore$  the tangent at  $(\frac{\pi}{2}, 2)$  has gradient  $-2 \sin \pi + 3 \cos \frac{\pi}{2} = 0$

$\therefore$  the tangent has equation  $y = 0(x - \frac{\pi}{2}) + 2$   
which is  $y = 2$

**20**

$y = \frac{\cos x}{1 + \sin x}$

$$\begin{aligned} \therefore \frac{dy}{dx} &= \frac{-\sin x(1 + \sin x) - \cos x(\cos x)}{(1 + \sin x)^2} \quad \{\text{quotient rule}\} \\ &= \frac{-\sin x - \sin^2 x - \cos^2 x}{(1 + \sin x)^2} \\ &= \frac{-(\sin^2 x + \cos^2 x + \sin x)}{(1 + \sin x)^2} \\ &= \frac{-(1 + \sin x)}{(1 + \sin x)^2} \\ &= -\frac{1}{1 + \sin x} \neq 0 \end{aligned}$$

$\therefore$  there are no tangents which have gradient 0.

$\therefore$  there are no horizontal tangents to  $y = \frac{\cos x}{1 + \sin x}$ .

**21**  $y = e^{\cos x}$

When  $x = \frac{\pi}{2}$ ,  $y = e^{\cos \frac{\pi}{2}}$   
 $= e^0$   
 $= 1$

$\therefore$  the point of contact is  $(\frac{\pi}{2}, 1)$ .

Now  $y = e^{\cos x}$  has derivative

$$\frac{dy}{dx} = -\sin x e^{\cos x}$$

$\therefore$  the tangent at  $(\frac{\pi}{2}, 1)$  has gradient

$$-\sin \frac{\pi}{2} e^{\cos \frac{\pi}{2}} = -e^0 = -1$$

$\therefore$  the tangent has equation  $y = -1(x - \frac{\pi}{2}) + 1$   
 which is  $y = -x + \frac{\pi}{2} + 1$

Let the  $y$ -intercept be A and the  $x$ -intercept be B.

When  $x = 0$ ,  $y = \frac{\pi}{2} + 1$

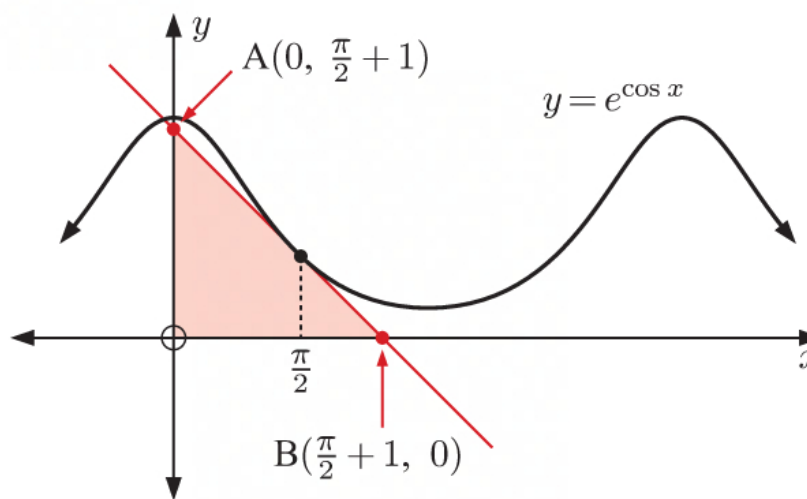
$\therefore$  A is  $(0, \frac{\pi}{2} + 1)$ .

When  $y = 0$ ,  $0 = -x + \frac{\pi}{2} + 1$

$$\therefore x = \frac{\pi}{2} + 1$$

$\therefore$  B is  $(\frac{\pi}{2} + 1, 0)$ .

$$\begin{aligned} \text{Area } \triangle AOB &= \frac{1}{2} \times \text{base} \times \text{height} \\ &= \frac{1}{2} \left( \frac{\pi}{2} + 1 \right)^2 \text{ units}^2 \end{aligned}$$



**22 a**  $y = \sec x$

$$\therefore \frac{dy}{dx} = \sec x \tan x$$

When  $x = \frac{\pi}{4}$ ,  $y = \sec \frac{\pi}{4}$

$$= \frac{1}{\cos \frac{\pi}{4}}$$

$$= \frac{1}{(\frac{1}{\sqrt{2}})} = \sqrt{2}$$

and  $\frac{dy}{dx} = \sec \frac{\pi}{4} \tan \frac{\pi}{4}$

$$= \sqrt{2} \times 1 = \sqrt{2}$$

$\therefore$  the gradient of the tangent at  $(\frac{\pi}{4}, \sqrt{2})$

is  $\sqrt{2}$ , and the equation of the tangent

$$\text{is } y = \sqrt{2}(x - \frac{\pi}{4}) + \sqrt{2}$$

$$= \sqrt{2}x - \sqrt{2}(\frac{\pi}{4} - 1)$$

**b**  $y = \cot \frac{x}{2}$

$$\therefore \frac{dy}{dx} = -\frac{1}{2} \operatorname{cosec}^2\left(\frac{x}{2}\right) \quad \{\text{chain rule}\}$$

When  $x = \frac{\pi}{3}$ ,  $y = \cot \frac{\pi}{6}$

$$= \frac{1}{\tan \frac{\pi}{6}}$$

$$= \frac{1}{(\frac{1}{\sqrt{3}})} = \sqrt{3}$$

and  $\frac{dy}{dx} = -\frac{1}{2 \sin^2(\frac{\pi}{6})}$

$$= -\frac{1}{2(\frac{1}{2})^2} = -2$$

$\therefore$  the gradient of the tangent at  $(\frac{\pi}{3}, \sqrt{3})$

is  $-2$ , and the equation of the tangent

$$\text{is } y = -2(x - \frac{\pi}{3}) + \sqrt{3}$$

$$= -2x + \frac{2\pi}{3} + \sqrt{3}$$

**c**  $y = \arctan x$

$$\therefore \frac{dy}{dx} = \frac{1}{1+x^2}$$

When  $x = 1$ ,  $y = \arctan 1$

$$= \frac{\pi}{4}$$

and  $\frac{dy}{dx} = \frac{1}{1+1^2} = \frac{1}{2}$

$\therefore$  the gradient of the tangent at  $(1, \frac{\pi}{4})$  is  $\frac{1}{2}$ , and the equation of the tangent

is  $y = \frac{1}{2}(x-1) + \frac{\pi}{4}$

$$= \frac{1}{2}x + \frac{\pi}{4} - \frac{1}{2}$$

**d**  $y = x \arccos \frac{x}{2}$

$$\therefore \frac{dy}{dx} = \arccos \frac{x}{2} - \frac{x}{\sqrt{4-x^2}} \quad \{\text{product rule}\}$$

When  $x = -1$ ,  $y = -\arccos(-\frac{1}{2})$

$$= -\frac{2\pi}{3}$$

and  $\frac{dy}{dx} = \arccos(-\frac{1}{2}) - \frac{-1}{\sqrt{4-(-1)^2}}$   
 $= \frac{2\pi}{3} + \frac{1}{\sqrt{3}}$

$\therefore$  the gradient of the tangent at  $(-1, -\frac{2\pi}{3})$  is  $\frac{2\pi}{3} + \frac{1}{\sqrt{3}}$ , and the equation of the tangent is

$$y = (\frac{2\pi}{3} + \frac{1}{\sqrt{3}})(x - (-1)) - \frac{2\pi}{3}$$

$$= (\frac{2\pi}{3} + \frac{1}{\sqrt{3}})x + \frac{2\pi}{3} + \frac{1}{\sqrt{3}} - \frac{2\pi}{3}$$

$$= (\frac{1}{\sqrt{3}} + \frac{2\pi}{3})x + \frac{1}{\sqrt{3}}$$

**23** Consider the tangent to  $y = x^3$  at  $x = 2$ .

When  $x = 2$ ,  $y = 2^3 = 8$  so the point of contact is  $(2, 8)$ .

Now  $\frac{dy}{dx} = 3x^2$  and so at  $x = 2$ ,

$$\frac{dy}{dx} = 3(2)^2 = 12$$

So, the tangent at  $(2, 8)$  has gradient 12 and its equation is

$$12x - y = 12(2) - 8$$

$$\therefore 12x - y = 16$$

$$\therefore y = 12x - 16$$

The tangent meets the curve where

$$12x - 16 = x^3$$

$$\therefore x^3 - 12x + 16 = 0$$

Because the tangent touches the curve at  $x = 2$ , there must be a repeated solution at this point.

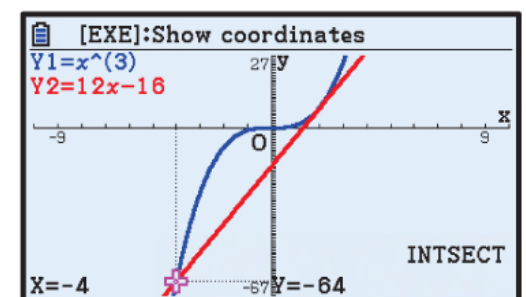
$\therefore (x-2)^2$  must be a factor of this cubic

$$\therefore (x-2)^2(x+4) = 0$$

$\therefore$  the tangent meets the curve again when  $x = -4$ .

When  $x = -4$ ,  $y = (-4)^3 = -64$

$\therefore$  the tangent meets the curve again at  $(-4, -64)$ .



**24** Consider the tangent to  $y = -x^3 + 2x^2 + 1$  at  $x = -1$ .

When  $x = -1$ ,  $y = -(-1)^3 + 2(-1)^2 + 1 = 4$  and so the point of contact is  $(-1, 4)$ .

Now  $\frac{dy}{dx} = -3x^2 + 4x$  and so at  $x = -1$ ,

$$\frac{dy}{dx} = -3(-1)^2 + 4(-1) = -7$$

So, the tangent at  $(-1, 4)$  has gradient  $-7$  and its equation is

$$-7x - y = -7(-1) - 4$$

$$\therefore 7x + y = -3$$

$$\therefore y = -7x - 3$$

The tangent meets the curve where  $-7x - 3 = -x^3 + 2x^2 + 1$

$$\therefore x^3 - 2x^2 - 7x - 4 = 0$$

Because the tangent touches the curve at  $x = -1$ , there must be a repeated solution at this point.

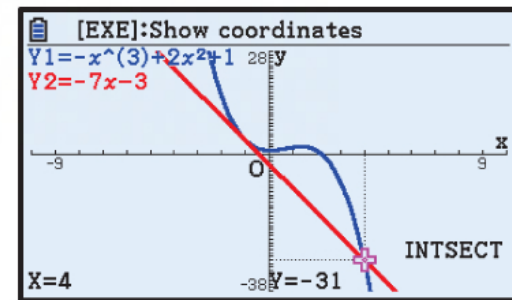
$\therefore (x + 1)^2$  must be a factor of this cubic

$$\therefore (x + 1)^2(x - 4) = 0$$

$\therefore$  the tangent meets the curve again when  $x = 4$ .

$$\begin{aligned} \text{When } x = 4, \quad y &= -(4)^3 + 2(4)^2 + 1 \\ &= -64 + 32 + 1 \\ &= -31 \end{aligned}$$

$\therefore$  the tangent meets the curve again at  $(4, -31)$ .



**25** Consider the tangent to  $y = \frac{1}{x} - \frac{1}{x^2}$  at  $x = 1$ .

When  $x = 1$ ,  $y = \frac{1}{1} - \frac{1}{1^2} = 0$  and so the point of contact is  $(1, 0)$ .

Now  $y = x^{-1} - x^{-2}$

$$\therefore \frac{dy}{dx} = -x^{-2} + 2x^{-3}$$

$$= -\frac{1}{x^2} + \frac{2}{x^3} \quad \text{and so at } x = 1,$$

$$\frac{dy}{dx} = -\frac{1}{1^2} + \frac{2}{1^3} = 1$$

So, the tangent at  $(1, 0)$  has gradient 1 and its equation is

$$y = 1 \times (x - 1) + 0$$

$$\therefore y = x - 1$$

The tangent meets the curve where  $x - 1 = \frac{1}{x} - \frac{1}{x^2}$

$$\therefore x^3 - x^2 = x - 1$$

$$\therefore x^3 - x^2 - x + 1 = 0$$

Because the tangent touches the curve at  $x = 1$ , there must be a repeated solution at this point.

$\therefore (x - 1)^2$  must be a factor of this cubic

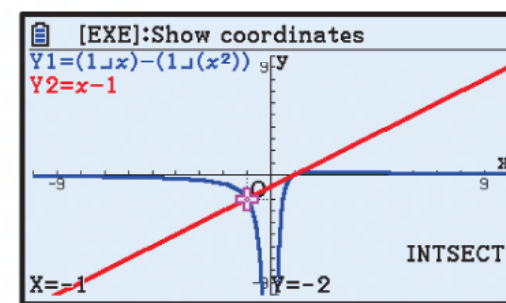
$$\therefore (x - 1)^2(x + 1) = 0$$

$\therefore$  the tangent meets the curve again when  $x = -1$ .



$$\begin{aligned}\text{When } x = -1, \quad y &= \frac{1}{(-1)} - \frac{1}{(-1)^2} \\ &= -1 - 1 \\ &= -2\end{aligned}$$

$\therefore$  the tangent meets the curve again at  $(-1, -2)$ .



**26 a**  $P(x) = x^3 - 3x^2 - x + 3$   
 $\therefore P(1) = 1^3 - 3(1)^2 - 1 + 3$   
 $= 0$

So  $x = 1$  is a zero of  $P(x)$ , which means we can find real coefficients  $a$ ,  $b$ , and  $c$  such that  $P(x) = (x - 1)(ax^2 + bx + c)$ .

$$\begin{aligned}\text{Expanding, we get } x^3 - 3x^2 - x + 3 &= ax^3 + bx^2 + cx \\ &\quad - ax^2 - bx - c \\ &= ax^3 + (b - a)x^2 + (c - b)x - c\end{aligned}$$

$$\text{Equating coefficients of } x^3: \quad a = 1$$

$$\text{Equating coefficients of } x^2: \quad b - a = -3$$

$$\therefore b - 1 = -3$$

$$\therefore b = -2$$

$$\text{Equating constants:} \quad -c = 3$$

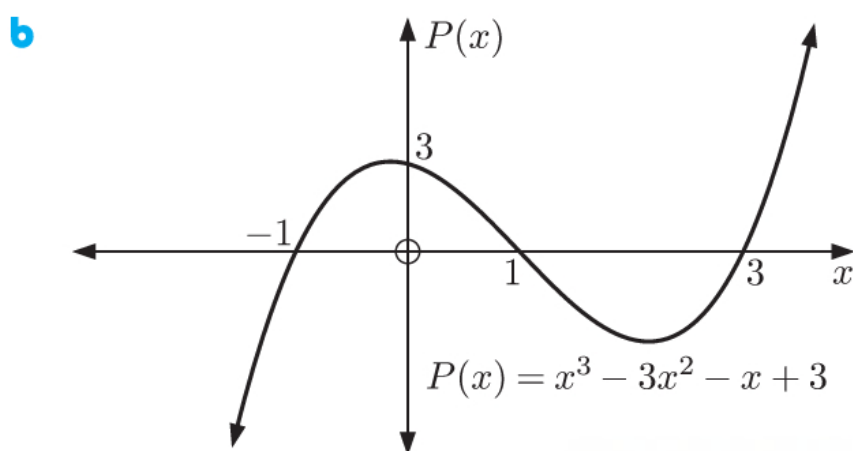
$$\therefore c = -3$$

$$\text{Equating coefficients of } x: \quad c - b = -1$$

$$-3 - (-2) = -1 \quad \checkmark$$

$$\begin{aligned}\text{So, } P(x) &= (x - 1)(x^2 - 2x - 3) \\ &= (x - 1)(x + 1)(x - 3)\end{aligned}$$

$\therefore$  the three real zeros are 1,  $-1$ , and 3.



**c**  $P'(x) = 3x^2 - 6x - 1$

$$\begin{aligned}P(2) &= 2^3 - 3(2)^2 - 2 + 3 & \text{and} & & P'(2) &= 3(2)^2 - 6(2) - 1 \\ &= 8 - 12 - 2 + 3 & & & &= 12 - 12 - 1 \\ &= -3 & & & &= -1\end{aligned}$$

So the gradient of the tangent at  $(2, -3)$  is  $-1$ , and its equation is  $y = -(x - 2) - 3$   
 $= -x + 2 - 3$   
 $= -x - 1$

- d** The curve meets the tangent again when  $x^3 - 3x^2 - x + 3 = -x - 1$   
 $\therefore x^3 - 3x^2 + 4 = 0$

$(x - 2)^2$  must be a factor since we are using the tangent at  $x = 2$ .

$\therefore (x - 2)^2(x + 1) = 0$ , so the tangent crosses the curve again when  $x = -1$ .

$$\begin{aligned} P(-1) &= (-1)^3 - 3(-1)^2 - (-1) + 3 \\ &= -1 - 3 + 1 + 3 \\ &= 0 \end{aligned}$$

So the tangent crosses the curve again at  $(-1, 0)$ .

- e** If a cubic  $Q(x)$  has zeros  $a$ ,  $b$ , and  $c$ , then

$$\begin{aligned} Q(x) &= \alpha(x - a)(x - b)(x - c) \\ \therefore Q(x) &= \alpha(x^2 - bx - ax + ab)(x - c) \\ &= \alpha(x^3 - cx^2 - bx^2 + bcx - ax^2 + acx + abx - abc) \\ &= \alpha(x^3 - (a + b + c)x^2 + (ab + ac + bc)x - abc) \\ &= \alpha x^3 - \alpha(a + b + c)x^2 + \alpha(ab + ac + bc)x - \alpha abc \\ \therefore Q'(x) &= 3\alpha x^2 - 2\alpha(a + b + c)x + \alpha(ab + ac + bc) \end{aligned}$$

$$\begin{aligned} \text{When } x = \frac{a+b}{2}, \quad Q(x) &= \alpha \left( \frac{a+b}{2} - a \right) \left( \frac{a+b}{2} - b \right) \left( \frac{a+b}{2} - c \right) \\ &= \alpha \left( \frac{b-a}{2} \right) \left( \frac{a-b}{2} \right) \left( \frac{a+b}{2} - c \right) \\ \text{and } Q'(x) &= 3\alpha \left( \frac{a+b}{2} \right)^2 - 2\alpha(a + b + c) \left( \frac{a+b}{2} \right) + \alpha(ab + ac + bc) \\ &= 3\alpha \left( \frac{a^2 + 2ab + b^2}{4} \right) - \alpha(a^2 + ab + ab + b^2 + ac + bc) \\ &\quad + \alpha(ab + ac + bc) \\ &= \alpha \left( \frac{3}{4}a^2 + \frac{3}{2}ab + \frac{3}{4}b^2 - a^2 - 2ab - b^2 - ac - bc + ab + ac + bc \right) \\ &= \alpha \left( -\frac{1}{4}a^2 + \frac{1}{2}ab - \frac{1}{4}b^2 \right) \\ &= -\frac{\alpha}{4}(a^2 - 2ab + b^2) \\ &= -\frac{\alpha}{4}(a - b)^2 \end{aligned}$$

$\therefore$  the equation of the tangent to  $Q(x)$  at  $x = \frac{a+b}{2}$  has gradient  $-\frac{\alpha}{4}(a - b)^2$ , and its

$$\begin{aligned} \text{equation is } y &= -\frac{\alpha}{4}(a - b)^2 \left[ x - \left( \frac{a+b}{2} \right) \right] + \alpha \left[ -\left( \frac{a-b}{2} \right) \right] \left[ \left( \frac{a-b}{2} \right) \right] \left[ \left( \frac{a+b}{2} \right) - c \right] \\ &= -\frac{\alpha}{4}(a - b)^2 \left[ x - \left( \frac{a+b}{2} \right) \right] - \frac{\alpha}{4}(a - b)^2 \left[ \left( \frac{a+b}{2} \right) - c \right] \\ &= -\frac{\alpha}{4}(a - b)^2 \left[ x - \left( \frac{a+b}{2} \right) + \left( \frac{a+b}{2} \right) - c \right] \\ &= -\frac{\alpha}{4}(a - b)^2(x - c) \end{aligned}$$

So the tangent also has  $x$ -intercept  $c$ .

$\therefore$  the tangent to the cubic at  $x = \frac{a+b}{2}$  meets the cubic again at  $x = c$ .

- 27 a** Consider the tangent to  $y = x^2 - x + 9$  at  $x = a$ .

When  $x = a$ ,  $y = a^2 - a + 9$ , so the point of contact is  $(a, a^2 - a + 9)$ .

Now  $\frac{dy}{dx} = 2x - 1$  and so at  $x = a$ ,  $\frac{dy}{dx} = 2a - 1$

So, the gradient of the tangent at  $(a, a^2 - a + 9)$  is  $2a - 1$

$$\begin{aligned}\therefore \text{ the equation of the tangent is } y &= (2a - 1)(x - a) + (a^2 - a + 9) \\ &= 2ax - 2a^2 - x + a + a^2 - a + 9 \\ &= 2ax - a^2 - x + 9 \\ \therefore y &= (2a - 1)x - a^2 + 9 \quad \dots (*)\end{aligned}$$

- b** This tangent passes through  $(0, 0)$ , so  $0 = -a^2 + 9$

$$\therefore a^2 = 9$$

$$\therefore a = \pm 3$$

When  $a = 3$ ,  $y = (2(3) - 1)x - 3^2 + 9$  {from (\*)}

$$\therefore y = 5x$$

The tangent is  $y = 5x$  with point of contact  $(3, 15)$ .

When  $a = -3$ ,  $y = (2(-3) - 1)x - (-3)^2 + 9$  {from (\*)}

$$\therefore y = -7x$$

The tangent is  $y = -7x$  with point of contact  $(-3, 21)$ .

- 28 a** Consider the tangent to  $y = x^2 + 4x$  at the point where  $x = a$ .

When  $x = a$ ,  $y = a^2 + 4a$ , so the point of contact is  $(a, a^2 + 4a)$ .

Now  $\frac{dy}{dx} = 2x + 4$  and so at  $x = a$ ,  $\frac{dy}{dx} = 2a + 4$

So, the gradient of the tangent at  $(a, a^2 + 4a)$  is  $2a + 4$

$$\begin{aligned}\therefore \text{ the equation of the tangent is } y &= (2a + 4)(x - a) + (a^2 + 4a) \\ &= 2ax - 2a^2 + 4x - 4a + a^2 + 4a \\ &= 2ax + 4x - a^2 \\ \therefore y &= (2a + 4)x - a^2 \quad \dots (*)\end{aligned}$$

- b** This tangent passes through  $(1, -4)$ , so  $-4 = (2a + 4)(1) - a^2$

$$\therefore -4 = 2a + 4 - a^2$$

$$\therefore a^2 - 2a - 8 = 0$$

$$\therefore (a + 2)(a - 4) = 0$$

$$\therefore a = -2 \text{ or } 4$$

When  $a = -2$ ,  $y = (2(-2) + 4)x - (-2)^2$  {from (\*)}

$$\therefore y = (-4 + 4)x - 4$$

$$\therefore y = -4$$

The tangent is  $y = -4$  with point of contact  $(-2, -4)$ .

When  $a = 4$ ,  $y = (2(4) + 4)x - (4)^2$

$$\therefore y = (8 + 4)x - 16$$

$$\therefore y = 12x - 16$$

When  $x = 4$ ,  $y = 12(4) - 16$

$$= 48 - 16$$

$$= 32$$

The tangent is  $y = 12x - 16$  with point of contact  $(4, 32)$ .



**29** Consider the tangent to  $y = x^2 - 3x + 1$  at the point where  $x = a$ .

When  $x = a$ ,  $y = a^2 - 3a + 1$ , so the point of contact is  $(a, a^2 - 3a + 1)$ .

Now  $\frac{dy}{dx} = 2x - 3$ , and so at  $x = a$ ,  $\frac{dy}{dx} = 2a - 3$

So, the gradient of the tangent at  $(a, a^2 - 3a + 1)$  is  $2a - 3$

$$\begin{aligned}\therefore \text{ the equation of the tangent is } y &= (2a - 3)(x - a) + a^2 - 3a + 1 \\ &= 2ax - 2a^2 - 3x + 3a + a^2 - 3a + 1 \\ &= 2ax - 3x - a^2 + 1 \\ \therefore y &= (2a - 3)x - a^2 + 1 \quad \dots (*)\end{aligned}$$

This tangent passes through  $(1, -10)$ , so  $-10 = (2a - 3)(1) - a^2 + 1$

$$\begin{aligned}\therefore -10 &= 2a - 3 - a^2 + 1 \\ \therefore a^2 - 2a - 8 &= 0 \\ \therefore (a + 2)(a - 4) &= 0 \\ \therefore a &= -2 \text{ or } 4\end{aligned}$$

When  $a = -2$ ,  $y = (2(-2) - 3)x - (-2)^2 + 1$  {from (\*)}

$$\therefore y = (-4 - 3)x - 4 + 1$$

$$\therefore y = -7x - 3$$

The tangent is  $y = -7x - 3$ .

When  $a = 4$ ,  $y = (2(4) - 3)x - (4)^2 + 1$

$$\therefore y = (8 - 3)x - 16 + 1$$

$$\therefore y = 5x - 15$$

The tangent is  $y = 5x - 15$ .

**30 a**  $y = e^x$

When  $x = a$ ,  $y = e^a$

$\therefore$  the point of contact is  $(a, e^a)$ .

Now  $y = e^x$

$$\therefore \frac{dy}{dx} = e^x$$

When  $x = a$ ,  $\frac{dy}{dx} = e^a$

So, the tangent has equation

$$\begin{aligned}y &= e^a(x - a) + e^a \\ &= e^a x - ae^a + e^a\end{aligned}$$

$$\therefore y = e^a x + e^a(1 - a)$$

**b** The tangent passes through the origin when  $0 = e^a(1 - a)$

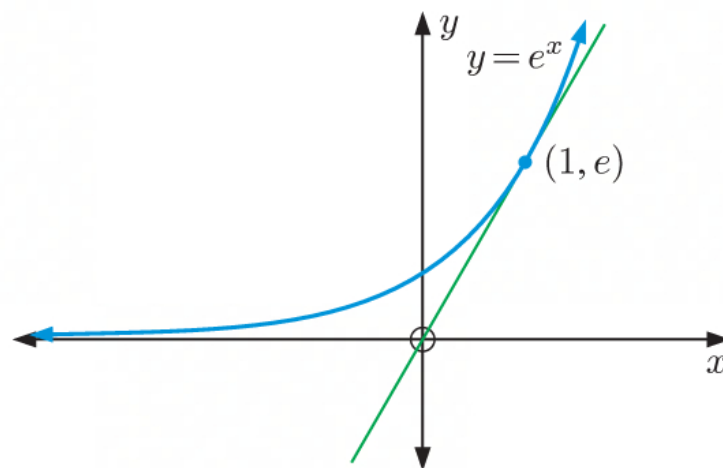
$$\therefore 1 - a = 0 \quad \{\text{as } e^a > 0 \text{ for all } a\}$$

$$\therefore a = 1$$

When  $a = 1$ ,  $y = e^1 x + e^1(1 - 1)$

$$= ex$$

$\therefore$  the tangent to  $y = e^x$  which passes through the origin is  $y = ex$ .





**31 a**  $y = 2x^2$

When  $x = a$ ,  $y = 2a^2$

$\therefore$  the point of contact is  $(a, 2a^2)$ .

Now  $y = 2x^2$

$$\therefore \frac{dy}{dx} = 4x$$

When  $x = a$ ,  $\frac{dy}{dx} = 4a$

So, the tangent has equation

$$\begin{aligned} y &= 4a(x - a) + 2a^2 \\ &= 4ax - 4a^2 + 2a^2 \end{aligned}$$

$$\therefore y = 4ax - 2a^2$$

The tangent passes through  $(1, -6)$  when  $-6 = 4a(1) - 2a^2$

$$\therefore 2a^2 - 4a - 6 = 0$$

$$\therefore a^2 - 2a - 3 = 0$$

$$\therefore (a + 1)(a - 3) = 0$$

$$\therefore a = -1 \text{ or } 3$$

When  $a = -1$ ,  $y = 4(-1)x - 2(-1)^2$   
 $= -4x - 2$

When  $a = 3$ ,  $y = 4(3)x - 2(3)^2$   
 $= 12x - 18$

$\therefore$  the tangents to  $y = 2x^2$  which passes through  $(1, -6)$  are  $y = -4x - 2$  and  $y = 12x - 18$ .

**b**  $y = 12x - 18$  meets  $y = 2x^2$

where  $12x - 18 = 2x^2$

$$\therefore 2x^2 - 12x + 18 = 0$$

$$\therefore x^2 - 6x + 9 = 0$$

$$\therefore (x - 3)^2 = 0$$

$$\therefore x = 3$$

When  $x = 3$ ,  $y = 2(3)^2$   
 $= 18$

So  $y = 12x - 18$  has point of contact  $(3, 18)$ .

$y = -4x - 2$  meets  $y = 2x^2$

where  $-4x - 2 = 2x^2$

$$\therefore 2x^2 + 4x + 2 = 0$$

$$\therefore x^2 + 2x + 1 = 0$$

$$\therefore (x + 1)^2 = 0$$

$$\therefore x = -1$$

When  $x = -1$ ,  $y = 2(-1)^2$   
 $= 2$

So  $y = -4x - 2$  has point of contact  $(-1, 2)$ .

**c** The tangent to  $y = 2x^2$  at a general point  $(a, 2a^2)$  has equation  $y = 4ax - 2a^2$ .

The tangent passes through  $(1, 4)$  when  $4 = 4a(1) - 2a^2$

$$\therefore 4 = 4a - 2a^2$$

$$\therefore 2a^2 - 4a + 4 = 0$$

$$\therefore a^2 - 2a + 2 = 0$$

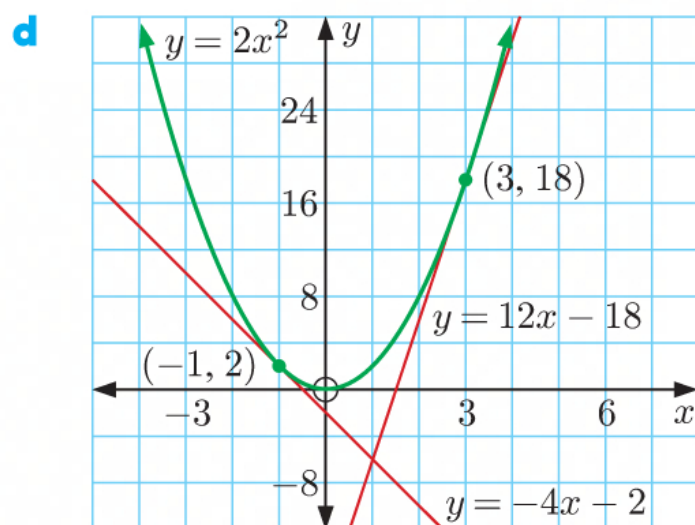
which has  $\Delta = (-2)^2 - 4(1)(2)$

$$= 4 - 8$$

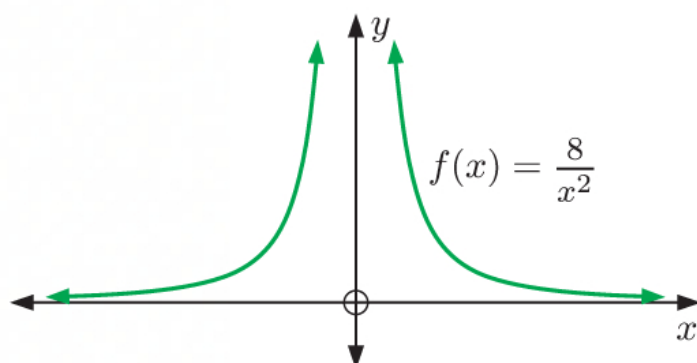
$$= -4$$

$\therefore$  there are no real solutions as  $\Delta < 0$ .

$\therefore$  there are no tangents to the function that pass through the point  $(1, 4)$ .



**32 a**



**b**  $f(x) = \frac{8}{x^2} = 8x^{-2}$

$$\therefore f(a) = \frac{8}{a^2}$$

So, the point of contact is  $\left(a, \frac{8}{a^2}\right)$ .

Now  $f'(x) = -16x^{-3} = -\frac{16}{x^3}$

$$\therefore f'(a) = -\frac{16}{a^3}$$

So, the gradient of the tangent at

$$\left(a, \frac{8}{a^2}\right) \text{ is } -\frac{16}{a^3}$$

$\therefore$  the equation of the tangent is

$$-16x - a^3y = -16a - a^3\left(\frac{8}{a^2}\right)$$

$$\therefore -16x - a^3y = -16a - 8a$$

$$\therefore 16x + a^3y = 24a$$

**c** The tangent cuts the  $x$ -axis when  $y = 0$

$$\therefore 16x = 24a$$

$$\therefore x = \frac{3}{2}a$$

$\therefore$  A is  $\left(\frac{3}{2}a, 0\right)$ .

The tangent cuts the  $y$ -axis when  $x = 0$

$$\therefore a^3y = 24a$$

$$\therefore y = \frac{24}{a^2}$$

$\therefore$  B is  $\left(0, \frac{24}{a^2}\right)$ .

**d** Area of triangle OAB =  $\left|\frac{1}{2} \times \left(\frac{3}{2}a\right) \times \left(\frac{24}{a^2}\right)\right|$

$$= \frac{18}{|a|} \text{ units}^2$$

As  $a \rightarrow \infty$ ,  $\frac{18}{|a|} \rightarrow 0$

$$\therefore \text{area} \rightarrow 0$$

**33** Let  $f(x) = \sqrt{x+a}$  and  $g(x) = \sqrt{2x-x^2}$

$$\begin{aligned} \therefore f'(x) &= \frac{1}{2}(x+a)^{-\frac{1}{2}} & \therefore g'(x) &= \frac{1}{2}(2x-x^2)^{-\frac{1}{2}}(2-2x) \quad \{\text{chain rule}\} \\ &= \frac{1}{2\sqrt{x+a}} & &= \frac{1-x}{\sqrt{2x-x^2}} \end{aligned}$$

The point of intersection occurs when  $f(x) = g(x)$

$$\begin{aligned} \therefore \sqrt{x+a} &= \sqrt{2x-x^2} \\ \therefore x+a &= 2x-x^2 \quad \dots (1) \end{aligned}$$

But  $f'(x) = g'(x)$  at the point of intersection as their gradients are equal

$$\begin{aligned} \therefore \frac{1}{2\sqrt{x+a}} &= \frac{1-x}{\sqrt{2x-x^2}} \\ \therefore \frac{1}{2\sqrt{2x-x^2}} &= \frac{1-x}{\sqrt{2x-x^2}} \quad \{\text{using (1)}\} \\ \therefore \frac{1}{2} &= 1-x \\ \therefore x &= \frac{1}{2} \end{aligned}$$

Substituting  $x = \frac{1}{2}$  into (1), we get  $\frac{1}{2} + a = 2(\frac{1}{2}) - (\frac{1}{2})^2$

$$\begin{aligned} \therefore a &= 1 - \frac{1}{4} - \frac{1}{2} \\ &= \frac{1}{4} \end{aligned}$$

$$\therefore f(\frac{1}{2}) = \sqrt{\frac{1}{2} + \frac{1}{4}} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$$

$$\therefore a = \frac{1}{4}, \text{ and the point of intersection is } (\frac{1}{2}, \frac{\sqrt{3}}{2}).$$

**34**  $y = 3e^{-x}$  and  $y = 2 + e^x$  meet where  $3e^{-x} = 2 + e^x$

$$\begin{aligned} \therefore 3 &= 2e^x + e^{2x} \quad \{\text{multiplying both sides by } e^x\} \\ \therefore e^{2x} + 2e^x - 3 &= 0 \\ \therefore (e^x + 3)(e^x - 1) &= 0 \\ \therefore e^x &= 1 \quad \{\text{as } e^x > 0\} \\ \therefore x &= 0 \end{aligned}$$

Now when  $x = 0$ ,  $y = 3e^0 = 3$ , so the graphs intersect at  $(0, 3)$ .

For  $y = 2 + e^x$ ,  $\frac{dy}{dx} = e^x$

When  $x = 0$ ,  $\frac{dy}{dx} = e^0 = 1$

$\therefore$  the gradient of the tangent at  $(0, 3)$  is 1.

$\therefore$  the equation of the tangent is  $y = x + 3$ .

For  $y = 3e^{-x}$ ,  $\frac{dy}{dx} = -3e^{-x}$

When  $x = 0$ ,  $\frac{dy}{dx} = -3$

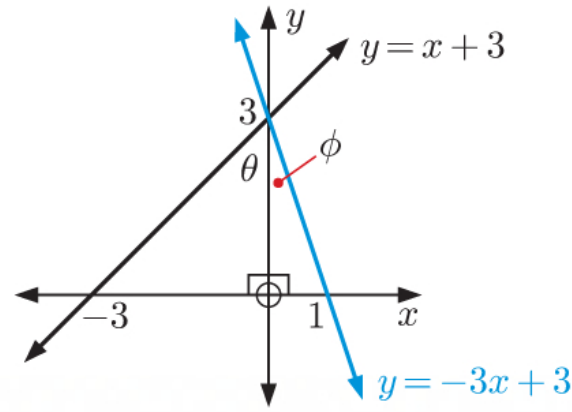
$\therefore$  the gradient of the tangent at  $(0, 3)$  is  $-3$ .

$\therefore$  the equation of the tangent is

$$\begin{aligned} y &= -3(x-0) + 3 \\ &= -3x + 3 \end{aligned}$$

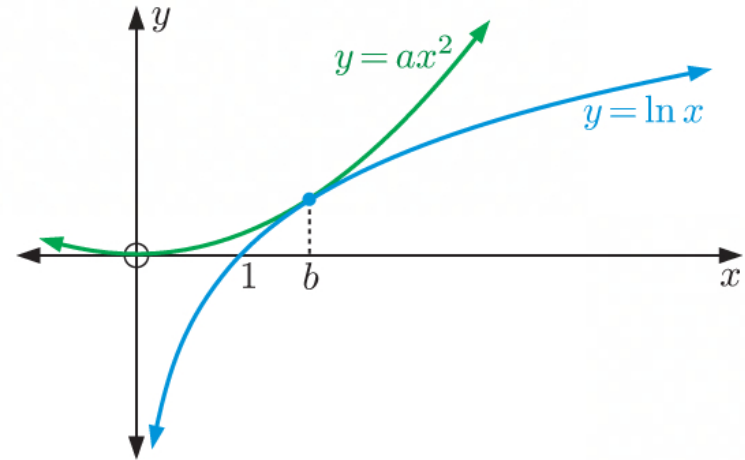
We graph the tangents on the same set of axes.

$$\begin{aligned}\tan \theta &= \frac{3}{3} & \text{and} & & \tan \phi &= \frac{1}{3} \\ &= 1 & & & \therefore \phi &\approx 18.43^\circ \\ \therefore \theta &= 45^\circ\end{aligned}$$



So, the acute angle between the tangents to  $y = 3e^{-x}$  and  $y = 2 + e^x$  at their point of intersection is about  $45^\circ + 18.43^\circ \approx 63.43^\circ$ .

- 35 a**  $y = ax^2$ ,  $a > 0$  touches  $y = \ln x$  when  $ax^2 = \ln x$   
If the curves touch when  $x = b$  then  $ab^2 = \ln b$  .... (1)



$$\begin{aligned}\text{Now for } y = ax^2, \quad \frac{dy}{dx} &= 2ax & \text{and} & & \text{for } y = \ln x, \quad \frac{dy}{dx} &= \frac{1}{x} \\ \therefore \text{ when } x = b, \quad \frac{dy}{dx} &= 2ab & \therefore \text{ when } x = b, \quad \frac{dy}{dx} &= \frac{1}{b}\end{aligned}$$

Since the curves touch each other, they share a common tangent.

$$\therefore 2ab = \frac{1}{b} \quad \dots (2)$$

**b** Now  $ab^2 = \frac{1}{2}$  {from (2)}  
and  $ab^2 = \ln b$  {from (1)}  
 $\therefore \ln b = \frac{1}{2}$

$$\therefore b = e^{\frac{1}{2}} = \sqrt{e}$$

$$\text{When } x = b = \sqrt{e}, \quad y = \ln x = \ln e^{\frac{1}{2}} = \frac{1}{2}$$

$$\therefore \text{ the point of contact is } (\sqrt{e}, \frac{1}{2}).$$

**d** The tangent has gradient  $\frac{1}{b} = \frac{1}{\sqrt{e}}$  and passes through  $(\sqrt{e}, \frac{1}{2})$

$$\therefore \text{ the tangent is } \frac{y - \frac{1}{2}}{x - \sqrt{e}} = \frac{1}{\sqrt{e}} \quad \therefore y - \frac{1}{2} = \frac{1}{\sqrt{e}}(x - \sqrt{e})$$

$$\therefore y - \frac{1}{2} = \frac{1}{\sqrt{e}}x - 1$$

$$\therefore y = e^{-\frac{1}{2}}x - \frac{1}{2}$$

**c**  $a = \frac{1}{2b^2}$  {from (2)}

$$\therefore a = \frac{1}{2(\sqrt{e})^2} = \frac{1}{2e}$$



**36**  $p(x) = ax^2, \quad a \neq 0$

**a**  $p(s) = a(s)^2$   
 $= as^2$

So the point on the curve where  $x = s$  is  $(s, as^2)$ .

Now  $p'(x) = 2ax$

$\therefore p'(s) = 2as$

$\therefore$  the equation of the tangent at  $(s, as^2)$  is  $y = 2as(x - s) + as^2$   
 $= 2asx - 2as^2 + as^2$   
 $= 2asx - as^2$

Similarly, the equation of the tangent when  $x = t$ , at  $(t, at^2)$ , is  $y = 2atx - at^2$ .

**b**  $y = 2asx - as^2$  and  $y = 2atx - at^2$  meet where  $2asx - as^2 = 2atx - at^2$   
 $\therefore 2asx - 2atx = as^2 - at^2$   
 $\therefore 2ax(s - t) = a(s^2 - t^2)$   
 $\therefore 2x(s - t) = (s + t)(s - t) \quad \{a \neq 0\}$   
 $\therefore 2x = s + t$   
 $\therefore x = \frac{s + t}{2}$

**c** If the tangent lines  $y = 2asx - as^2$  and  $y = 2atx - at^2$  are perpendicular, then their gradients are negative reciprocals of each other.

$\therefore 2as = -\frac{1}{2at}$

$\therefore 2ast = -\frac{1}{2a}$

$\therefore ast = -\frac{1}{4a} \quad \dots (*)$

The tangents  $y = 2asx - as^2$  and  $y = 2atx - at^2$  intersect at  $x = \frac{s + t}{2}$ . {from **b**}

When  $x = \frac{s + t}{2}$ ,  $y = 2as\left(\frac{s + t}{2}\right) - as^2$   
 $= as(s + t) - as^2$   
 $= as^2 + ast - as^2$   
 $= ast$   
 $= -\frac{1}{4a} \quad \{\text{from } (*)\}$

$\therefore$  if the tangent lines are perpendicular then they intersect at  $y = -\frac{1}{4a}$ .

## EXERCISE 18B

**1 a**  $y = x^2$

Now  $\frac{dy}{dx} = 2x$ , so at  $(4, 16)$ ,

$$\frac{dy}{dx} = 2(4) = 8 = \frac{8}{1}$$

$\therefore$  the normal at  $(4, 16)$  has gradient  $-\frac{1}{8}$ .

$\therefore$  the equation of the normal is

$$-x - 8y = -(4) - 8(16)$$

$$\therefore x + 8y = 132$$

**c**  $y = \frac{5}{\sqrt{x}} - \sqrt{x}$

$$= 5x^{-\frac{1}{2}} - x^{\frac{1}{2}}$$

$\therefore \frac{dy}{dx} = -\frac{5}{2}x^{-\frac{3}{2}} - \frac{1}{2}x^{-\frac{1}{2}}$ , so at  $(1, 4)$ ,

$$\frac{dy}{dx} = -\frac{5}{2}(1^{-\frac{3}{2}}) - \frac{1}{2}(1^{-\frac{1}{2}})$$

$$= -\frac{5}{2} - \frac{1}{2}$$

$$= -3$$

$\therefore$  the normal at  $(1, 4)$  has gradient  $\frac{1}{3}$ .

$\therefore$  the equation of the normal is

$$x - 3y = 1 - 3(4)$$

$$\therefore x - 3y = -11$$

**b**  $y = x^3 - 5x + 2$

When  $x = -2$ ,

$$y = (-2)^3 - 5(-2) + 2 \\ = 4$$

So, the point of contact is  $(-2, 4)$ .

Now  $\frac{dy}{dx} = 3x^2 - 5$ , so at  $x = -2$ ,

$$\frac{dy}{dx} = 3(-2)^2 - 5 = 7 = \frac{7}{1}$$

$\therefore$  the normal at  $(-2, 4)$  has gradient  $-\frac{1}{7}$ .

$\therefore$  the equation of the normal is

$$-x - 7y = -(-2) - 7(4)$$

$$\therefore x + 7y = 26$$

**d**  $y = 8\sqrt{x} - \frac{1}{x^2}$

When  $x = 1$ ,

$$y = 8\sqrt{1} - \frac{1}{1^2} = 7$$

So, the point of contact is  $(1, 7)$ .

Now  $y = 8\sqrt{x} - \frac{1}{x^2}$

$$= 8x^{\frac{1}{2}} - x^{-2}$$

$\therefore \frac{dy}{dx} = 4x^{-\frac{1}{2}} + 2x^{-3}$ , so at  $x = 1$ ,

$$\frac{dy}{dx} = 4(1^{-\frac{1}{2}}) + 2(1^{-3})$$

$$= 4 + 2$$

$$= 6$$

$\therefore$  the normal at  $(1, 7)$  has gradient  $-\frac{1}{6}$ .

$\therefore$  the equation of the normal is

$$-x - 6y = -(1) - 6(7)$$

$$\therefore x + 6y = 43$$

$$\begin{aligned}
 \text{e} \quad f(x) &= \frac{x}{1-3x} \\
 &= x(1-3x)^{-1} \\
 \therefore f'(x) &= (1-3x)^{-1} - x(1-3x)^{-2}(-3) \quad \{\text{product rule}\} \\
 &= \frac{1}{1-3x} + \frac{3x}{(1-3x)^2} \\
 \therefore f'(-1) &= \frac{1}{1-3(-1)} + \frac{3(-1)}{(1-3(-1))^2} \\
 &= \frac{1}{1+3} - \frac{3}{(1+3)^2} \\
 &= \frac{1}{4} - \frac{3}{16} \\
 &= \frac{1}{16}
 \end{aligned}$$

$\therefore$  the normal at  $(-1, -\frac{1}{4})$  has gradient  $-16$ .

$$\begin{aligned}
 \therefore \text{ the equation of the normal is } 16x + y &= 16(-1) + (-\frac{1}{4}) \\
 &= -16 - \frac{1}{4} \\
 \therefore 16x + y &= -\frac{65}{4} \\
 \therefore 64x + 4y &= -65
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad f(x) &= \frac{x^2}{1-x} \\
 &= x^2(1-x)^{-1} \\
 \therefore f'(x) &= 2x(1-x)^{-1} - x^2(1-x)^{-2}(-1) \quad \{\text{product rule}\} \\
 &= \frac{2x}{1-x} + \frac{x^2}{(1-x)^2} \\
 \therefore f'(2) &= \frac{2(2)}{1-2} + \frac{(2)^2}{(1-2)^2} \\
 &= \frac{4}{-1} + \frac{4}{1} \\
 &= 0
 \end{aligned}$$

$\therefore$  the tangent at  $(2, -4)$  is a horizontal line.

$\therefore$  the normal at  $(2, -4)$  is a vertical line passing through  $(2, -4)$ .

$\therefore$  the equation of the normal is  $x = 2$ .

**g**  $y = \sqrt{x}(1-x)^2$

When  $x = 4$ ,  $y = \sqrt{4}(1-4)^2$   
 $= 2(-3)^2 = 18$

So, the point of contact is  $(4, 18)$ .

Now  $\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}}(1-x)^2 + 2x^{\frac{1}{2}}(1-x)(-1)$  {product rule, chain rule}  
 $= \frac{(1-x)^2}{2\sqrt{x}} - 2\sqrt{x}(1-x)$

so at  $x = 4$ ,  $\frac{dy}{dx} = \frac{(1-4)^2}{2\sqrt{4}} - 2\sqrt{4}(1-4)$   
 $= \frac{9}{4} - 4(-3)$   
 $= \frac{9}{4} + 12$   
 $= \frac{57}{4}$

$\therefore$  the normal at  $(4, 18)$  has gradient  $-\frac{4}{57}$ .

$\therefore$  the equation of the normal is  $y = -\frac{4}{57}(x-4) + 18$   
 $\therefore 57y = -4x + 16 + 1026$   
 $\therefore 4x + 57y = 1042$

**h**  $f(x) = \frac{x^2 - 1}{2x + 3}$

$\therefore f(-1) = \frac{(-1)^2 - 1}{2(-1) + 3} = 0$

So, the point of contact is  $(-1, 0)$ .

Now  $f'(x) = \frac{2x(2x+3) - (x^2-1)(2)}{(2x+3)^2}$  {quotient rule}  
 $= \frac{4x^2 + 6x - 2x^2 + 2}{(2x+3)^2}$   
 $= \frac{2x^2 + 6x + 2}{(2x+3)^2}$

$\therefore f'(-1) = \frac{2(-1)^2 + 6(-1) + 2}{(2(-1) + 3)^2} = \frac{2 - 6 + 2}{1} = -2$

$\therefore$  the normal at  $(-1, 0)$  has gradient  $\frac{1}{2}$ .

$\therefore$  the equation of the normal is  $y = \frac{1}{2}(x - (-1)) + 0$   
 $\therefore y = \frac{1}{2}x + \frac{1}{2}$   
 $\therefore x - 2y = -1$



**2 a**  $f(x) = x^2 - \frac{8}{x}$

$$\begin{aligned}\therefore f(-2) &= (-2)^2 - \frac{8}{(-2)} \\ &= 4 + 4 \\ &= 8\end{aligned}$$

So, the point of contact is  $(-2, 8)$ .

Now  $f(x) = x^2 - 8x^{-1}$

$$\begin{aligned}\therefore f'(x) &= 2x + 8x^{-2} \\ &= 2x + \frac{8}{x^2}\end{aligned}$$

$$\begin{aligned}\therefore f'(-2) &= 2(-2) + \frac{8}{(-2)^2} \\ &= -4 + 2 \\ &= -2\end{aligned}$$

$\therefore$  the tangent at  $(-2, 8)$  has gradient  $-2$ .

$\therefore$  the equation of the tangent is

$$\begin{aligned}y &= -2(x + 2) + 8 \\ \therefore y &= -2x - 4 + 8 \\ \therefore y &= 4 - 2x\end{aligned}$$

**b**  $f(x) = x^2 - \frac{8}{x}$

$$\begin{aligned}\therefore f(3) &= (3)^2 - \frac{8}{3} \\ &= 9 - \frac{8}{3} \\ &= \frac{19}{3}\end{aligned}$$

So, the point of contact is  $(3, \frac{19}{3})$ .

Now  $f'(x) = 2x + \frac{8}{x^2}$  {from **a**}

$$\begin{aligned}\therefore f'(3) &= 2(3) + \frac{8}{(3)^2} \\ &= 6 + \frac{8}{9} \\ &= \frac{62}{9}\end{aligned}$$

$\therefore$  the normal at  $(3, \frac{19}{3})$  has gradient  $-\frac{9}{62}$ .

$\therefore$  the equation of the normal is

$$\begin{aligned}y &= -\frac{9}{62}(x - 3) + \frac{19}{3} \\ \therefore y &= -\frac{9}{62}x + \frac{27}{62} + \frac{19}{3} \\ \therefore y &= -\frac{9}{62}x + \frac{1259}{186}\end{aligned}$$

**3 a**  $f$  is defined when  $2 - x > 0$

$$\therefore x < 2$$

So, the domain of  $f$  is  $\{x \mid x < 2\}$ .

**b**  $f(x) = \frac{x}{\sqrt{2-x}}$

$$= x(2-x)^{-\frac{1}{2}}$$

$$\therefore f'(x) = (2-x)^{-\frac{1}{2}} - \frac{1}{2}x(2-x)^{-\frac{3}{2}}(-1) \quad \{\text{product rule}\}$$

$$\begin{aligned}&= \frac{1}{(2-x)^{\frac{1}{2}}} + \frac{x}{2(2-x)^{\frac{3}{2}}} \\ &= \frac{2(2-x) + x}{2(2-x)^{\frac{3}{2}}} \\ &= \frac{4-x}{2(2-x)^{\frac{3}{2}}}\end{aligned}$$

$$\begin{aligned} \text{c} \quad f(1) &= \frac{1}{\sqrt{2-1}} & \text{and} \quad f'(1) &= \frac{4-1}{2(2-1)^{\frac{3}{2}}} \\ &= \frac{1}{\sqrt{1}} = 1 & &= \frac{3}{2(1)^{\frac{3}{2}}} = \frac{3}{2} \end{aligned}$$

$\therefore$  the gradient of the tangent to  $f$  at  $(1, 1)$  is  $\frac{3}{2}$ , and its equation is

$$y = \frac{3}{2}(x-1) + 1$$

$$\therefore 2y = 3x - 3 + 2$$

$$\therefore 3x - 2y = 1$$

$$\begin{aligned} \text{d} \quad f(x) &= -1 \\ \therefore \frac{x}{\sqrt{2-x}} &= -1 \\ \therefore x &= -\sqrt{2-x} \\ \therefore x^2 &= 2-x \quad \{\text{squaring both sides}\} \\ \therefore x^2 + x - 2 &= 0 \\ \therefore (x+2)(x-1) &= 0 \\ \therefore x &= -2 \text{ or } 1 \end{aligned}$$

$$\text{Check: } f(1) = \frac{1}{\sqrt{2-1}} = 1 \quad \times$$

$$\begin{aligned} f(-2) &= \frac{-2}{\sqrt{2-(-2)}} \\ &= \frac{-2}{\sqrt{4}} \\ &= -1 \quad \checkmark \end{aligned}$$

$$\therefore x = -2$$

$$\begin{aligned} \text{Now } f'(-2) &= \frac{4-(-2)}{2(2-(-2))^{\frac{3}{2}}} \\ &= \frac{6}{2(4)^{\frac{3}{2}}} \\ &= \frac{6}{2(2)^3} \\ &= \frac{3}{8} \end{aligned}$$

$\therefore$  the gradient of the normal to  $f$  at  $(-2, -1)$  is  $-\frac{8}{3}$ , and its equation is

$$y = -\frac{8}{3}(x-(-2)) - 1$$

$$\therefore 3y = -8x - 16 - 3$$

$$\therefore 8x + 3y = -19$$

- 4 a** When  $x = 0$ ,  $y = e^0 = 1$ . So, the point of contact is  $(0, 1)$ .

$$\text{Now as } y = e^{-x}, \quad \frac{dy}{dx} = -e^{-x}$$

$$\therefore \text{ when } x = 0, \quad \frac{dy}{dx} = -e^0 = -1$$

$\therefore$  the normal at  $(0, 1)$  has gradient 1.

$\therefore$  the equation of the normal is

$$y = x + 1.$$

- c** When  $x = 1$ ,  $y = e^{2(1)-1} = e$ . So, the point of contact is  $(1, e)$ .

$$\text{Now as } y = e^{2x-1}, \quad \frac{dy}{dx} = 2e^{2x-1}$$

$$\therefore \text{ when } x = 1, \quad \frac{dy}{dx} = 2e^{2(1)-1} = 2e$$

$\therefore$  the normal at  $(1, e)$  has gradient  $-\frac{1}{2e}$ .

$\therefore$  the equation of the normal is

$$y = -\frac{1}{2e}(x - 1) + e$$

$$\therefore 2ey = -(x - 1) + 2e^2$$

$$\therefore 2ey = -x + 1 + 2e^2$$

$$\therefore x + 2ey = 1 + 2e^2$$

- e** When  $x = 0$ ,  $y = 4^0 = 1$ . So, the point of contact is  $(0, 1)$ .

$$\text{Now as } y = 4^x, \quad \frac{dy}{dx} = 4^x \ln 4$$

$$\therefore \text{ when } x = 0, \quad \frac{dy}{dx} = 4^0 \ln 4 = \ln 4$$

$\therefore$  the normal at  $(0, 1)$  has gradient  $-\frac{1}{\ln 4}$ .

$\therefore$  the equation of the normal is

$$y = -\frac{1}{\ln 4}(x - 0) + 1$$

$$\therefore y = 1 - \frac{1}{\ln 4}x$$

- b** When  $x = e$ ,  $y = \ln e = 1$ . So, the point of contact is  $(e, 1)$ .

$$\text{Now as } y = \ln x, \quad \frac{dy}{dx} = \frac{1}{x}$$

$$\therefore \text{ when } x = e, \quad \frac{dy}{dx} = \frac{1}{e}$$

$\therefore$  the normal at  $(e, 1)$  has gradient  $-e$ .

$\therefore$  the equation of the normal is

$$y = -e(x - e) + 1$$

$$\therefore y = -ex + e^2 + 1$$

$$\therefore ex + y = e^2 + 1$$

- d** When  $x = \frac{\pi}{3}$ ,  $y = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$ . So, the point of contact is  $\left(\frac{\pi}{3}, \frac{\sqrt{3}}{2}\right)$ .

$$\text{Now as } y = \sin x, \quad \frac{dy}{dx} = \cos x$$

$$\therefore \text{ when } x = \frac{\pi}{3}, \quad \frac{dy}{dx} = \cos \frac{\pi}{3} = \frac{1}{2}$$

$\therefore$  the normal at  $\left(\frac{\pi}{3}, \frac{\sqrt{3}}{2}\right)$  has gradient  $-2$ .

$\therefore$  the equation of the normal is

$$y = -2\left(x - \frac{\pi}{3}\right) + \frac{\sqrt{3}}{2}$$

$$\therefore y = -2x + \frac{2\pi}{3} + \frac{\sqrt{3}}{2}$$

$$\therefore 2x + y = \frac{2\pi}{3} + \frac{\sqrt{3}}{2}$$

- f**  $y = \tan x$

$$\therefore \frac{dy}{dx} = \sec^2 x$$

$$\therefore \text{ when } x = 0, \quad \frac{dy}{dx} = \frac{1}{\cos^2 0} = \frac{1}{1^2} = 1$$

$\therefore$  the normal at  $(0, 0)$  has gradient  $-1$ .

$\therefore$  the equation of the normal is

$$y = -1(x - 0) + 0$$

$$\therefore y = -x$$

$$\begin{aligned} \text{g When } x = \frac{\pi}{2}, \quad y &= \cos\left(\pi - \frac{\pi}{3}\right) \\ &= \cos \frac{2\pi}{3} = -\frac{1}{2} \end{aligned}$$

So, the point of contact is  $\left(\frac{\pi}{2}, -\frac{1}{2}\right)$ .

Now as  $y = \cos(2x - \frac{\pi}{3})$ ,

$$\frac{dy}{dx} = -2 \sin(2x - \frac{\pi}{3}) \quad \{\text{chain rule}\}$$

$$\begin{aligned} \therefore \text{ when } x = \frac{\pi}{2}, \quad \frac{dy}{dx} &= -2 \sin\left(\pi - \frac{\pi}{3}\right) \\ &= -2 \sin \frac{2\pi}{3} \\ &= -\sqrt{3} \end{aligned}$$

$\therefore$  the normal at  $\left(\frac{\pi}{2}, -\frac{1}{2}\right)$  has gradient  $\frac{1}{\sqrt{3}}$ .

$\therefore$  the equation of the normal is

$$y = \frac{1}{\sqrt{3}}\left(x - \frac{\pi}{2}\right) - \frac{1}{2}$$

$$\therefore \sqrt{3}y = x - \frac{\pi}{2} - \frac{\sqrt{3}}{2}$$

$$\therefore 2x - 2\sqrt{3}y = \pi + \sqrt{3}$$

$$\begin{aligned} \text{i When } x = 2, \quad y &= \log_3(2^2 + 2) \\ &= \log_3 6 \\ &= \log_3 3 + \log_3 2 \\ &= 1 + \frac{\ln 2}{\ln 3} \end{aligned}$$

So, the point of contact is  $\left(2, 1 + \frac{\ln 2}{\ln 3}\right)$ .

Now as  $y = \log_3(x^2 + 2)$ ,

$$\frac{dy}{dx} = \frac{2x}{(x^2 + 2) \ln 3} \quad \{\text{chain rule}\}$$

$$\begin{aligned} \therefore \text{ when } x = 2, \quad \frac{dy}{dx} &= \frac{2(2)}{(2^2 + 2) \ln 3} \\ &= \frac{2}{3 \ln 3} \end{aligned}$$

$\therefore$  the normal at  $\left(2, 1 + \frac{\ln 2}{\ln 3}\right)$  has gradient  $-\frac{3 \ln 3}{2}$ .

$\therefore$  the equation of the normal is

$$y = -\frac{3 \ln 3}{2}(x - 2) + 1 + \frac{\ln 2}{\ln 3}$$

$$\therefore 2y = (-3 \ln 3)x + 6 \ln 3 + 2 + \frac{2 \ln 2}{\ln 3}$$

$$\therefore (3 \ln 3)x + 2y = 6 \ln 3 + \frac{2 \ln 3}{\ln 3} + \frac{2 \ln 2}{\ln 3}$$

$$\therefore (3 \ln 3)x + 2y = 6 \ln 3 + \frac{2 \ln 6}{\ln 3}$$

$$\begin{aligned} \text{h When } x = \frac{\pi}{4}, \quad y &= \sin \frac{3\pi}{4} - \cos \frac{\pi}{4} \\ &= \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \\ &= 0 \end{aligned}$$

So, the point of contact is  $\left(\frac{\pi}{4}, 0\right)$ .

Now as  $y = \sin 3x - \cos x$ ,

$$\frac{dy}{dx} = 3 \cos 3x + \sin x \quad \{\text{chain rule}\}$$

$$\begin{aligned} \therefore \text{ when } x = \frac{\pi}{4}, \quad \frac{dy}{dx} &= 3 \cos \frac{3\pi}{4} + \sin \frac{\pi}{4} \\ &= 3\left(-\frac{1}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}} \\ &= -\frac{2}{\sqrt{2}} = -\sqrt{2} \end{aligned}$$

$\therefore$  the normal at  $\left(\frac{\pi}{4}, 0\right)$  has gradient  $\frac{1}{\sqrt{2}}$ .

$\therefore$  the equation of the normal is

$$y = \frac{1}{\sqrt{2}}\left(x - \frac{\pi}{4}\right) + 0$$

$$\therefore \sqrt{2}y = x - \frac{\pi}{4}$$

$$\therefore x - \sqrt{2}y = \frac{\pi}{4}$$

$$\begin{aligned} \text{j When } x = \frac{\pi}{6}, \quad y &= \operatorname{cosec} \frac{\pi}{6} \\ &= \frac{1}{\sin \frac{\pi}{6}} \\ &= \frac{1}{\left(\frac{1}{2}\right)} \\ &= 2 \end{aligned}$$

So, the point of contact is  $\left(\frac{\pi}{6}, 2\right)$ .

Now as  $y = \operatorname{cosec} x$ ,

$$\frac{dy}{dx} = -\operatorname{cosec} x \cot x$$

$$\begin{aligned} \therefore \text{ when } x = \frac{\pi}{6}, \quad \frac{dy}{dx} &= -\operatorname{cosec} \frac{\pi}{6} \cot \frac{\pi}{6} \\ &= -\frac{\cos \frac{\pi}{6}}{\sin^2\left(\frac{\pi}{6}\right)} \\ &= \frac{-\left(\frac{\sqrt{3}}{2}\right)}{\left(\frac{1}{2}\right)^2} \\ &= -2\sqrt{3} \end{aligned}$$

$\therefore$  the normal at  $\left(\frac{\pi}{6}, 2\right)$  has gradient  $\frac{1}{2\sqrt{3}}$ .

$\therefore$  the equation of the normal is

$$y = \frac{1}{2\sqrt{3}}\left(x - \frac{\pi}{6}\right) + 2$$

$$\therefore 2\sqrt{3}y = x - \frac{\pi}{6} + 4\sqrt{3}$$

$$\therefore x - 2\sqrt{3}y = \frac{\pi}{6} - 4\sqrt{3}$$



$$\begin{aligned}
 \text{k When } x = \pi, \quad y &= \sqrt{\sec \frac{\pi}{3}} \\
 &= \sqrt{\frac{1}{\cos \frac{\pi}{3}}} \\
 &= \sqrt{\frac{1}{(\frac{1}{2})}} \\
 &= \sqrt{2}
 \end{aligned}$$

So, the point of contact is  $(\pi, \sqrt{2})$ .

Now as  $y = \sqrt{\sec \frac{x}{3}} = (\cos \frac{x}{3})^{-\frac{1}{2}}$ ,

$$\frac{dy}{dx} = -\frac{1}{2}(\cos \frac{x}{3})^{-\frac{3}{2}}(-\frac{1}{3} \sin \frac{x}{3}) \quad \{\text{chain rule}\}$$

$$\begin{aligned}
 &= \frac{\sin \frac{x}{3}}{6 \cos \frac{x}{3} \sqrt{\cos \frac{x}{3}}} \\
 &= \frac{\tan \frac{x}{3}}{6 \sqrt{\cos \frac{x}{3}}}
 \end{aligned}$$

$$\begin{aligned}
 \therefore \text{ when } x = \pi, \quad \frac{dy}{dx} &= \frac{\tan \frac{\pi}{3}}{6 \sqrt{\cos \frac{\pi}{3}}} \\
 &= \frac{\sqrt{3}}{6 \sqrt{\frac{1}{2}}} \\
 &= \frac{\sqrt{6}}{6} \\
 &= \frac{1}{\sqrt{6}}
 \end{aligned}$$

$\therefore$  the normal at  $(\pi, \sqrt{2})$  has gradient  $-\sqrt{6}$ .

$\therefore$  the equation of the normal is

$$y = -\sqrt{6}(x - \pi) + \sqrt{2}$$

$$\therefore y = -\sqrt{6}x + \sqrt{6}\pi + \sqrt{2}$$

$$\therefore \sqrt{6}x + y = \sqrt{6}\pi + \sqrt{2}$$

$$\begin{aligned} 5 \quad y &= a\sqrt{x} + \frac{b}{\sqrt{x}} \\ &= ax^{\frac{1}{2}} + bx^{-\frac{1}{2}} \end{aligned}$$

$$\therefore \frac{dy}{dx} = \frac{a}{2}x^{-\frac{1}{2}} - \frac{b}{2}x^{-\frac{3}{2}}$$

$$\begin{aligned} \text{When } x = 4, \quad \frac{dy}{dx} &= \frac{a}{2}\left(4^{-\frac{1}{2}}\right) - \frac{b}{2}\left(4^{-\frac{3}{2}}\right) \\ &= \frac{a}{2}\left(\frac{1}{2}\right) - \frac{b}{2}\left(\frac{1}{8}\right) \\ &= \frac{a}{4} - \frac{b}{16} \\ &= \frac{4a - b}{16} \end{aligned}$$

The gradient of the normal to the curve at  $x = 4$  will be  $\frac{16}{b - 4a}$ .

However, the equation of the normal is  $4x + y = 22$  or  $y = -4x + 22$  which has gradient  $-4$ .

$$\therefore \frac{16}{b - 4a} = -4$$

$$\therefore b - 4a = -4$$

$$\therefore b = 4a - 4 \quad \dots (*)$$

Also, at  $x = 4$  the normal line intersects the curve.

$$\therefore a\sqrt{4} + \frac{b}{\sqrt{4}} = -4(4) + 22$$

$$\therefore 2a + \frac{b}{2} = 6$$

$$\text{Consequently, } 2a + \frac{4a - 4}{2} = 6 \quad \{\text{using } (*)\}$$

$$\therefore 2a + 2a - 2 = 6$$

$$\therefore 4a = 8$$

$$\therefore a = 2$$

$$\text{and so } b = 4(2) - 4 = 4 \quad \{\text{from } (*)\}$$

6 When  $x = 1$ ,  $y = (1)^3 - 2(1)^2 + 1 = 0$ . So the point of contact is  $(1, 0)$ .

$$\text{Now as } y = x^3 - 2x^2 + 1, \quad \frac{dy}{dx} = 3x^2 - 4x$$

$$\therefore \text{when } x = 1, \quad \frac{dy}{dx} = 3(1)^2 - 4(1) = -1$$

$\therefore$  the normal at  $(1, 0)$  has gradient 1.

$\therefore$  the equation of the normal is  $y = x - 1$ .

This line meets the curve where  $x - 1 = x^3 - 2x^2 + 1$

$$\therefore x^3 - 2x^2 - x + 2 = 0$$

$$\therefore x = 2, 1, \text{ or } -1 \quad \{\text{using technology}\}$$

$$\begin{aligned} \text{When } x = 2, \quad y &= (2)^3 - 2(2)^2 + 1 \\ &= 8 - 8 + 1 \\ &= 1 \end{aligned}$$

$$\begin{aligned} \text{When } x = -1, \quad y &= (-1)^3 - 2(-1)^2 + 1 \\ &= -1 - 2 + 1 \\ &= -2 \end{aligned}$$

$\therefore$  the normal meets the curve again at  $(2, 1)$  and  $(-1, -2)$ .

- 7** Let  $(a, \cos a)$  be a general point on  $f(x) = \cos x$ .

Now  $f'(x) = -\sin x$ , so  $f'(a) = -\sin a$

$\therefore$  the normal at  $(a, \cos a)$  has gradient  $\frac{1}{\sin a}$ .

$\therefore$  the equation of the normal is  $y = \frac{1}{\sin a}(x - a) + \cos a$ .

The normal passes through the origin when

$$0 = \frac{1}{\sin a}(0 - a) + \cos a$$

$$\therefore 0 = -\frac{a}{\sin a} + \cos a$$

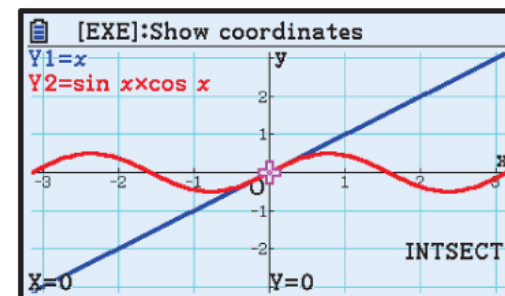
$$\therefore \frac{a}{\sin a} = \cos a$$

$$\therefore a = \sin a \cos a$$

$$\therefore a = 0$$

So, the normal at  $(0, \cos 0)$ , or  $(0, 1)$ , has gradient  $\frac{1}{\sin 0}$  which is undefined. The normal is a vertical line.

$\therefore$  the equation of the normal to  $f(x) = \cos x$  which passes through the origin is the vertical line  $x = 0$ .



- 8** Let  $(a, \sqrt{a})$  be a general point on  $y = \sqrt{x}$ .

$$\begin{aligned} \text{Now } \frac{dy}{dx} &= \frac{1}{2}x^{-\frac{1}{2}} \\ &= \frac{1}{2\sqrt{x}} \end{aligned}$$

$$\text{so at } x = a, \quad \frac{dy}{dx} = \frac{1}{2\sqrt{a}}$$

So, the gradient of the normal at this point is  $-2\sqrt{a}$ .

$\therefore$  the normal has equation  $y = -2\sqrt{a}(x - a) + \sqrt{a}$ .

But this normal passes through  $(4, 0)$ , so  $0 = -2\sqrt{a}(4 - a) + \sqrt{a}$

$$\therefore 2\sqrt{a}(4 - a) - \sqrt{a} = 0$$

$$\therefore \sqrt{a}(8 - 2a - 1) = 0$$

$$\therefore \sqrt{a}(7 - 2a) = 0$$

$$\therefore a = 0 \text{ or } \frac{7}{2}$$

But  $a = 0$  is the end point of the function, so there is no normal here.

$$\text{When } a = \frac{7}{2}, \quad y = -2\sqrt{\frac{7}{2}}(x - \frac{7}{2}) + \sqrt{\frac{7}{2}}$$

$$\therefore y + 2\sqrt{\frac{7}{2}}(x - \frac{7}{2}) = \sqrt{\frac{7}{2}}$$

$$\therefore \sqrt{2}y + 2\sqrt{7}(x - \frac{7}{2}) = \sqrt{7}$$

$$\therefore \sqrt{2}y + 2\sqrt{7}x - 7\sqrt{2} = \sqrt{7}$$

$$\therefore \sqrt{2}y + 2\sqrt{7}x = 8\sqrt{7}$$

$$\therefore 2y + 2\sqrt{14}x = 8\sqrt{14}$$

$$\therefore y + \sqrt{14}x = 4\sqrt{14}$$

$$\therefore y = -\sqrt{14}x + 4\sqrt{14} \text{ is the normal to } y = \sqrt{x} \text{ from } (4, 0), \text{ with}$$

$$\text{contact point } \left(\frac{7}{2}, \sqrt{\frac{7}{2}}\right).$$

**9 a**  $x = (y + 1)e^y$ , so  $\frac{dx}{dy} = (1)e^y + (y + 1)e^y$  {product rule}  
 $= (y + 2)e^y$

**b** When  $y = 2$ ,  $x = (2 + 1)e^2 = 3e^2$

So, the point of contact is  $(3e^2, 2)$ .

Now  $\frac{dx}{dy} = (2 + 2)e^2 = 4e^2$  and  $\frac{dy}{dx} \frac{dx}{dy} = 1$  {chain rule}

$$\therefore \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

$\therefore$  the tangent at  $(3e^2, 2)$  has gradient  $(4e^2)^{-1} = \frac{1}{4}e^{-2}$

$\therefore$  the normal at  $(3e^2, 2)$  has gradient  $-(\frac{1}{4}e^{-2})^{-1} = -4e^2$

$\therefore$  the normal has equation  $y = -4e^2(x - 3e^2) + 2$

$$\therefore y = -4e^2x + 12e^4 + 2$$

$$\therefore 4e^2x + y = 12e^4 + 2$$

**10** The line through  $P(4, \frac{19}{4})$  and  $Q(-2, -\frac{13}{4})$  has gradient  $\frac{-\frac{13}{4} - \frac{19}{4}}{-2 - 4} = \frac{-\frac{32}{4}}{-6} = \frac{4}{3}$ .

So, the line (PQ) has equation  $y = \frac{4}{3}x + c$ .

The line passes through  $(4, \frac{19}{4})$ , so  $\frac{19}{4} = \frac{4}{3}(4) + c$

$$\therefore c = \frac{19}{4} - \frac{16}{3}$$

$$= -\frac{7}{12}$$

$$\therefore y = \frac{4}{3}x - \frac{7}{12}$$

The lines  $y = \frac{b}{(x+1)^2}$  and  $y = \frac{4}{3}x - \frac{7}{12}$  intersect when  $\frac{b}{(x+1)^2} = \frac{4}{3}x - \frac{7}{12}$

$$\therefore b = (x+1)^2 \left( \frac{16x-7}{12} \right) \dots (1)$$

Now  $y = \frac{b}{(x+1)^2} = b(x+1)^{-2}$ , so  $\frac{dy}{dx} = (-2)b(x+1)^{-3}(1)$   
 $= -\frac{2b}{(x+1)^3}$

(PQ) has gradient  $\frac{4}{3}$  and is normal to  $y = \frac{b}{(x+1)^2}$

$$\therefore -\frac{2b}{(x+1)^3} = -\frac{3}{4}$$

$$\therefore b = \frac{3}{8}(x+1)^3 \dots (2)$$

$$\therefore \frac{3}{8}(x+1)^3 = (x+1)^2 \left( \frac{16x-7}{12} \right) \quad \{\text{equating (1) and (2)}\}$$

$$\therefore \frac{3}{8}(x+1) = \left( \frac{16x-7}{12} \right) \quad \{x \neq -1\}$$

$$\therefore \frac{9}{2}(x+1) = 16x-7$$

$$\therefore 9x+9 = 32x-14$$

$$\therefore 23x = 23$$

$$\therefore x = 1$$

$$\therefore b = \frac{3}{8}(1+1)^3 \quad \{\text{substituting into (2)}\}$$

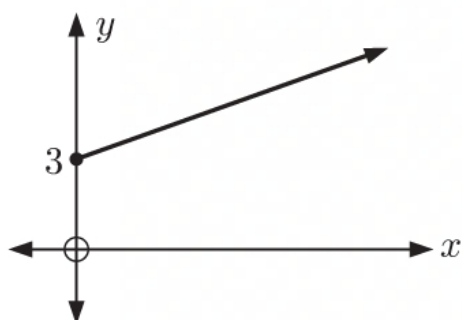
$$\therefore b = \frac{3}{8}(8)$$

$$\therefore b = 3$$



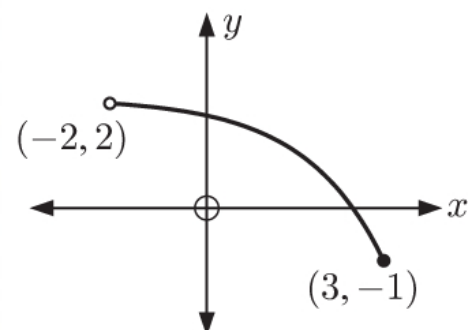
## EXERCISE 18C

1 a



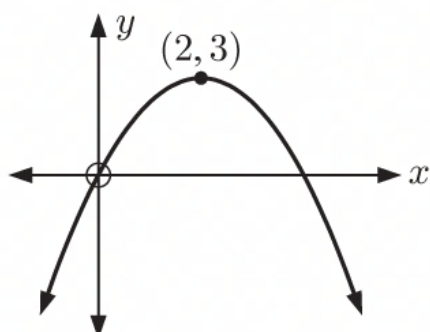
- i The graph is increasing for  $x \geq 0$ .
- ii The graph is never decreasing.

b



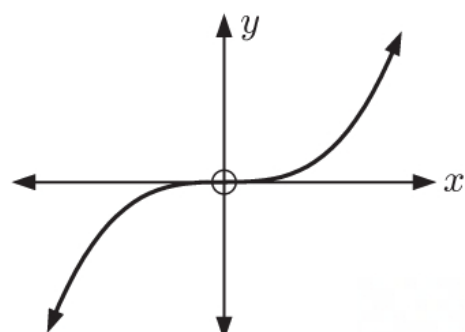
- i The graph is never increasing.
- ii The graph is decreasing for  $-2 < x \leq 3$ .

c



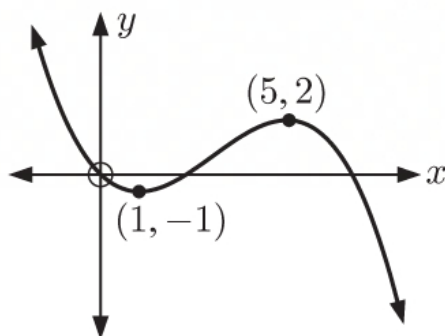
- i The graph is increasing for  $x \leq 2$ .
- ii The graph is decreasing for  $x \geq 2$ .

d



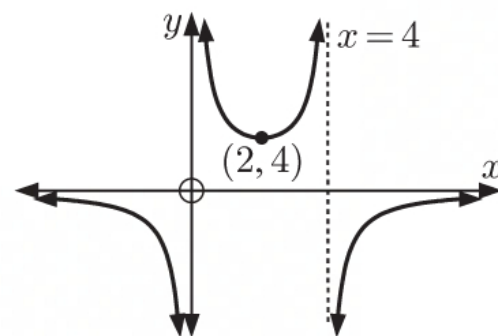
- i The graph is increasing for all real  $x$ .
- ii The graph is never decreasing.

e



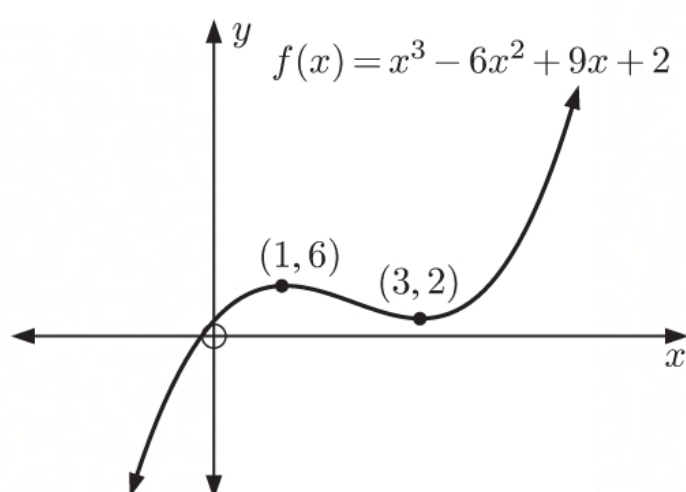
- i The graph is increasing for  $1 \leq x \leq 5$ .
- ii The graph is decreasing for  $x \leq 1$ ,  $x \geq 5$ .

f



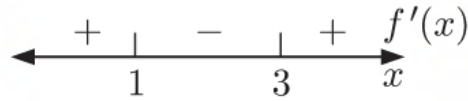
- i The graph is increasing for  $2 \leq x < 4$ ,  $x > 4$ .
- ii The graph is decreasing for  $x < 0$ ,  $0 < x \leq 2$ .

2 a



- i The function is increasing for  $x \leq 1$  and  $x \geq 3$ .
- ii The function is decreasing for  $1 \leq x \leq 3$ .

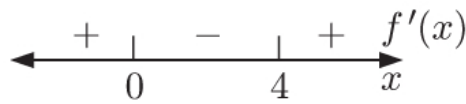
$$\begin{aligned}
 \text{b} \quad f(x) &= x^3 - 6x^2 + 9x + 2 \\
 \therefore f'(x) &= 3x^2 - 12x + 9 \\
 &= 3(x^2 - 4x + 3) \\
 &= 3(x-1)(x-3)
 \end{aligned}$$

which has sign diagram: 

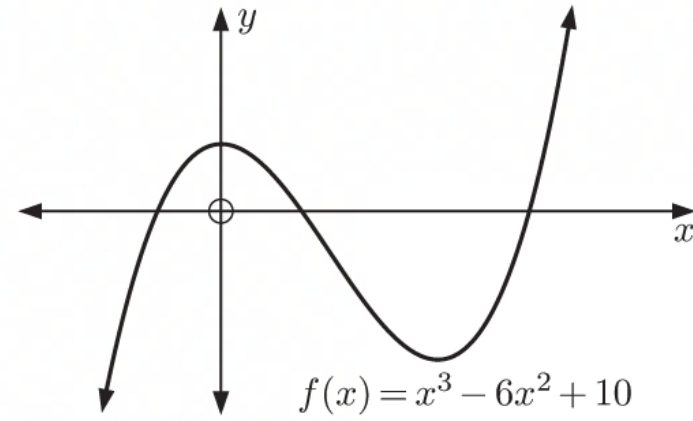
From the sign diagram,  $f(x)$  is increasing for  $x \leq 1$  and  $x \geq 3$ , and decreasing for  $1 \leq x \leq 3$ .

$$\begin{aligned}
 \text{3 a} \quad f(x) &= x^3 - 6x^2 + 10 \\
 \therefore f'(x) &= 3x^2 - 12x \\
 &= 3x(x-4)
 \end{aligned}$$

which has sign diagram:

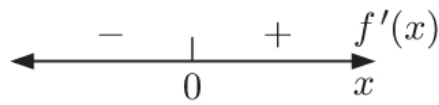


**b**  $f(x)$  is increasing for  $x \leq 0$  and for  $x \geq 4$ .  
 $f(x)$  is decreasing for  $0 \leq x \leq 4$ .

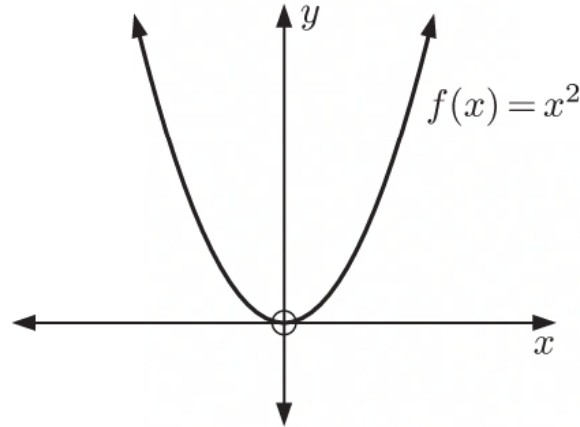


$$\begin{aligned}
 \text{4 a} \quad f(x) &= x^2 \\
 \therefore f'(x) &= 2x
 \end{aligned}$$

which has sign diagram:

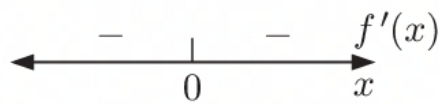


So,  $f(x)$  is increasing for  $x \geq 0$ ,  
 and decreasing for  $x \leq 0$ .

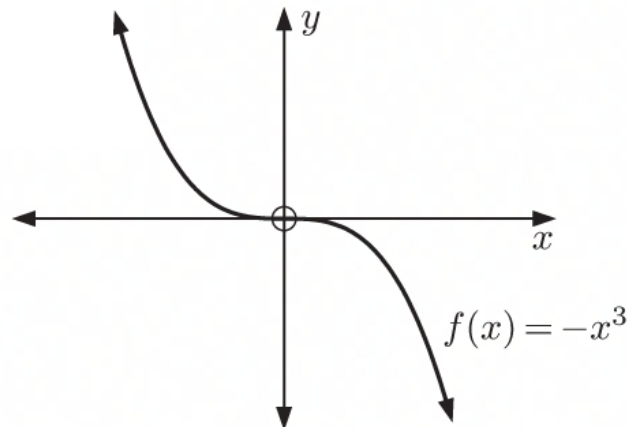


$$\begin{aligned}
 \text{b} \quad f(x) &= -x^3 \\
 \therefore f'(x) &= -3x^2
 \end{aligned}$$

which has sign diagram:

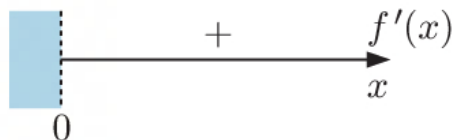


So,  $f(x)$  is decreasing for all  $x \in \mathbb{R}$ .

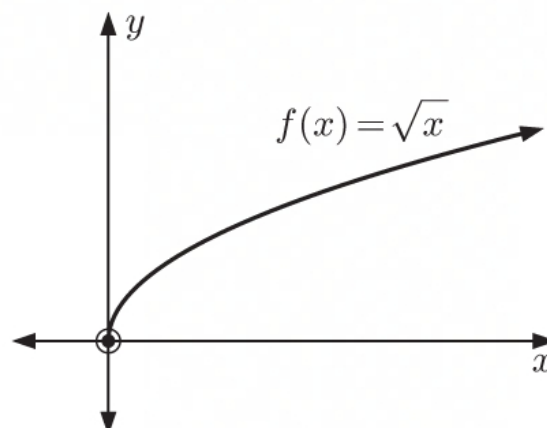


$$\begin{aligned}
 \text{c} \quad f(x) &= \sqrt{x} = x^{\frac{1}{2}} \\
 \therefore f'(x) &= \frac{1}{2}x^{-\frac{1}{2}} \\
 &= \frac{1}{2\sqrt{x}}
 \end{aligned}$$

which has sign diagram:



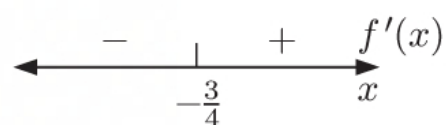
So,  $f(x)$  is increasing for  $x \geq 0$ ,  
 and is never decreasing.



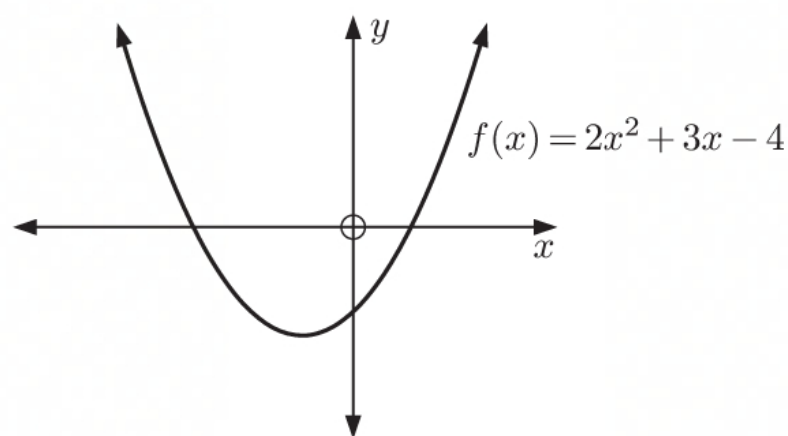
**d**  $f(x) = 2x^2 + 3x - 4$

$\therefore f'(x) = 4x + 3$

which has sign diagram:



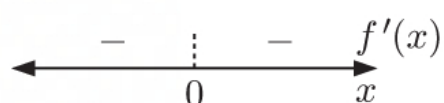
So,  $f(x)$  is increasing for  $x \geq -\frac{3}{4}$ ,  
and decreasing for  $x \leq -\frac{3}{4}$ .



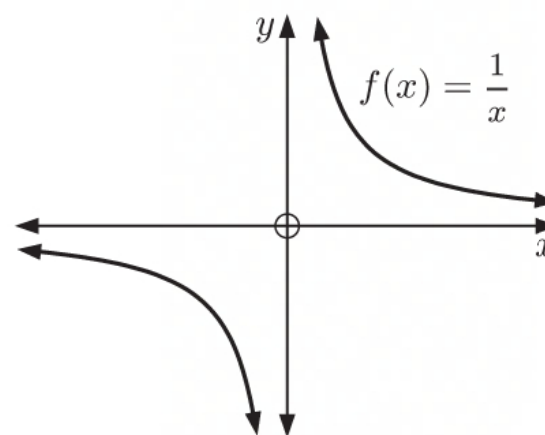
**e**  $f(x) = \frac{1}{x} = x^{-1}$

$\therefore f'(x) = -x^{-2} = -\frac{1}{x^2}$

which has sign diagram:



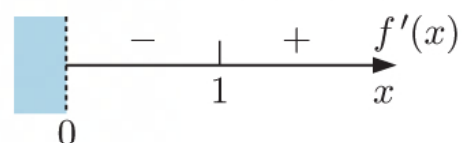
So,  $f(x)$  is decreasing for all  $x \neq 0$ .



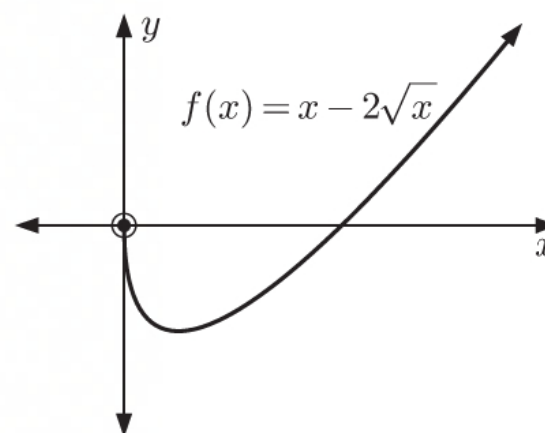
**f**  $f(x) = x - 2\sqrt{x} = x - 2x^{\frac{1}{2}}$

$\therefore f'(x) = 1 - (\frac{1}{2})2x^{-\frac{1}{2}}$   
 $= 1 - \frac{1}{\sqrt{x}}$

which has sign diagram:



So,  $f(x)$  is increasing for  $x \geq 1$ , and  
decreasing for  $0 \leq x \leq 1$ .



**g**  $f(x) = \frac{2}{\sqrt{x}} = 2x^{-\frac{1}{2}}$

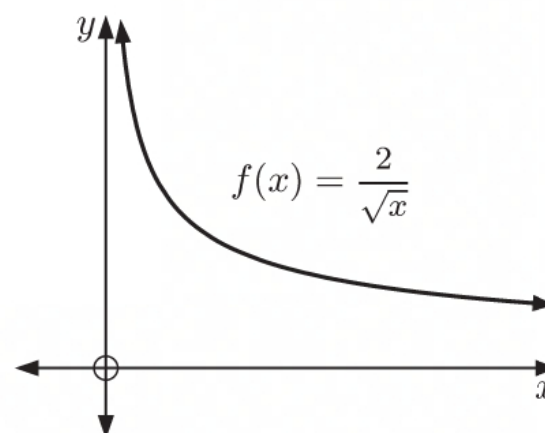
$\therefore f'(x) = -x^{-\frac{3}{2}} = -\frac{1}{x\sqrt{x}}$

which has sign diagram:



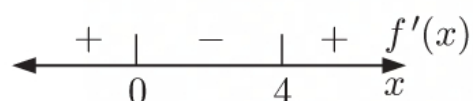
So,  $f(x)$  is only defined for  $x > 0$ .

$f(x)$  is never increasing, but is decreasing  
for  $x > 0$ .

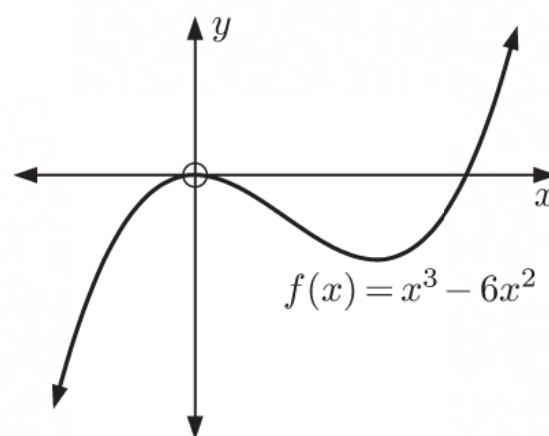


**h**  $f(x) = x^3 - 6x^2$   
 $\therefore f'(x) = 3x^2 - 12x$   
 $= 3x(x - 4)$

which has sign diagram:

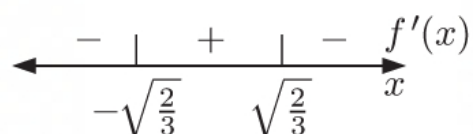


So,  $f(x)$  is increasing for  $x \leq 0$  and  $x \geq 4$ , and decreasing for  $0 \leq x \leq 4$ .



**i**  $f(x) = -2x^3 + 4x$   
 $\therefore f'(x) = -6x^2 + 4$   
 $= -2(3x^2 - 2)$   
 $= -2(\sqrt{3}x + \sqrt{2})(\sqrt{3}x - \sqrt{2})$

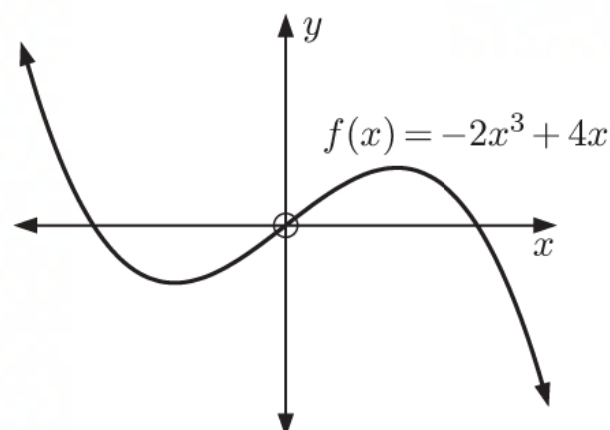
which has sign diagram:



So,  $f(x)$  is increasing for

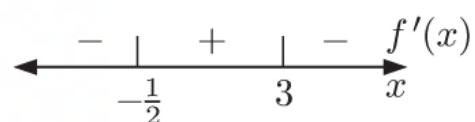
$$-\sqrt{\frac{2}{3}} \leq x \leq \sqrt{\frac{2}{3}}, \text{ and decreasing for}$$

$$x \leq -\sqrt{\frac{2}{3}} \text{ and } x \geq \sqrt{\frac{2}{3}}.$$

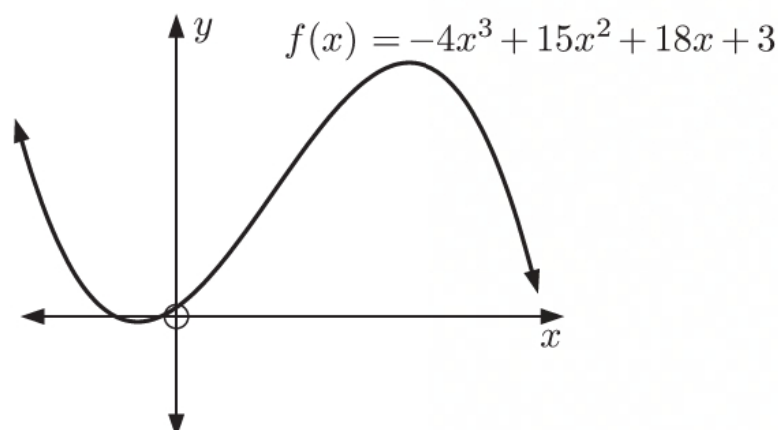


**j**  $f(x) = -4x^3 + 15x^2 + 18x + 3$   
 $\therefore f'(x) = -12x^2 + 30x + 18$   
 $= -6(2x^2 - 5x - 3)$   
 $= -6(2x + 1)(x - 3)$

which has sign diagram:

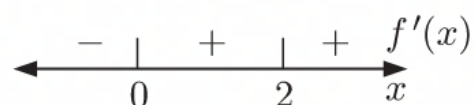


So,  $f(x)$  is increasing for  $-\frac{1}{2} \leq x \leq 3$ , and decreasing for  $x \leq -\frac{1}{2}$  or  $x \geq 3$ .

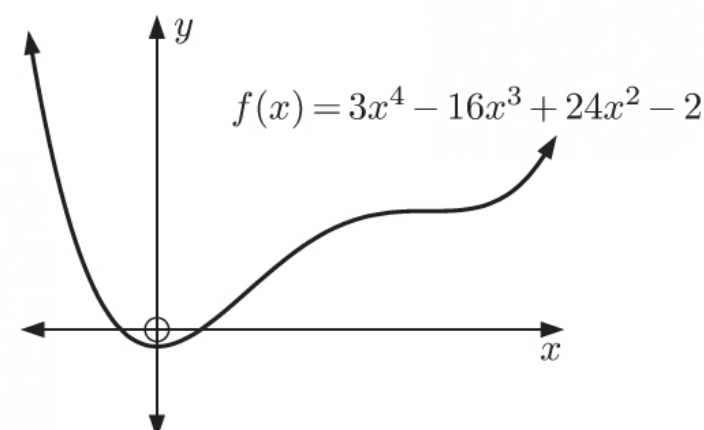


**k**  $f(x) = 3x^4 - 16x^3 + 24x^2 - 2$   
 $\therefore f'(x) = 12x^3 - 48x^2 + 48x$   
 $= 12x(x^2 - 4x + 4)$   
 $= 12x(x - 2)^2$

which has sign diagram:



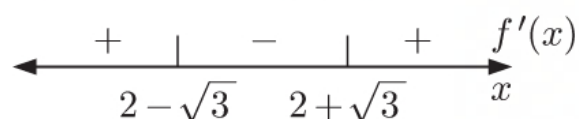
So,  $f(x)$  is increasing for  $x \geq 0$ , and decreasing for  $x \leq 0$ .



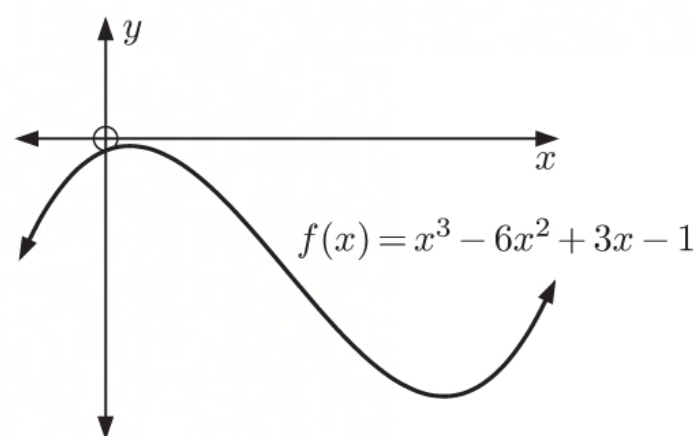


$$\begin{aligned}
 f(x) &= x^3 - 6x^2 + 3x - 1 \\
 \therefore f'(x) &= 3x^2 - 12x + 3 \\
 &= 3(x^2 - 4x + 1) \\
 f'(x) = 0 \text{ when } x &= \frac{4 \pm \sqrt{16 - 4}}{2} \\
 &= 2 \pm \sqrt{3}
 \end{aligned}$$

Sign diagram of  $f'(x)$ :



So,  $f(x)$  is increasing for  $x \leq 2 - \sqrt{3}$  and  $x \geq 2 + \sqrt{3}$ ,  
and decreasing for  $2 - \sqrt{3} \leq x \leq 2 + \sqrt{3}$ .



**5 a**  $f(x) = x^3 - 3x^2 + 5x + 2$   
 $\therefore f'(x) = 3x^2 - 6x + 5$

**b**  $\Delta = b^2 - 4ac$   
 $= (-6)^2 - 4(3)(5)$   
 $= 36 - 60$   
 $= -24$  which is  $< 0$

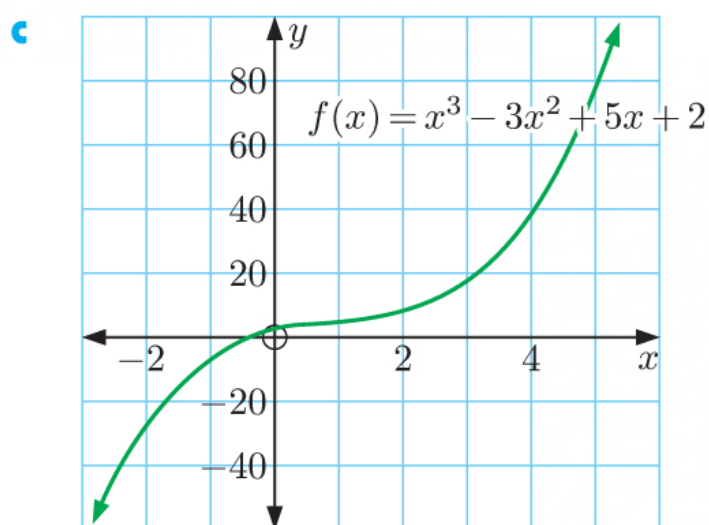
$\therefore f'(x)$  has no real roots.

Also,  $a > 0$  which means  $f'(x)$  is concave up .

$\therefore f'(x)$  lies entirely above the  $x$ -axis.

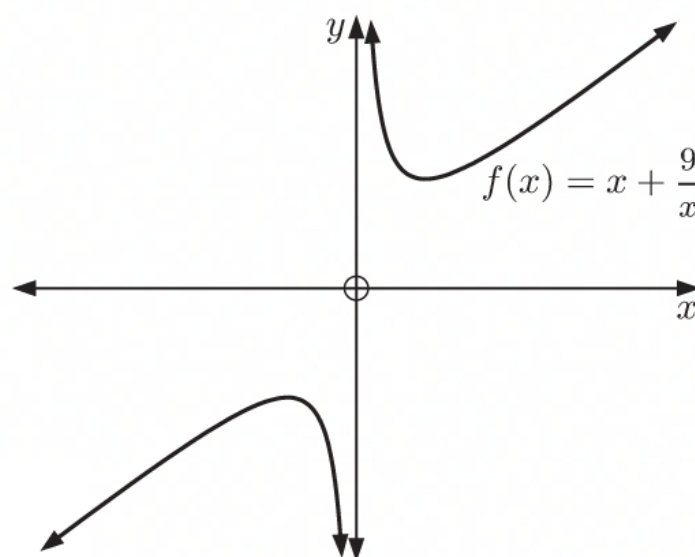
$\therefore f'(x) > 0$  for all  $x$ .

$\therefore f(x)$  is increasing for all  $x$ .

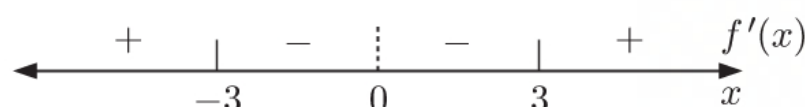


As we can see from the graph of  $y = f(x)$ ,  
 $f(x)$  is increasing for all  $x$ .

$$\begin{aligned}
 6 \quad a \quad f(x) &= x + \frac{9}{x} \\
 &= x + 9x^{-1} \\
 \therefore f'(x) &= 1 - 9x^{-2} \\
 &= 1 - \frac{9}{x^2} \\
 &= \frac{x^2 - 9}{x^2} \\
 &= \frac{(x+3)(x-3)}{x^2}
 \end{aligned}$$

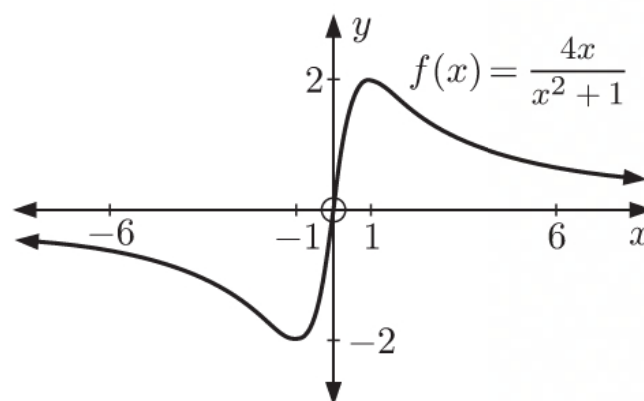


which has sign diagram:



- b**  $y = f(x)$  is increasing for  $x \leq -3$  and  $x \geq 3$ , and decreasing for  $-3 \leq x < 0$  and  $0 < x \leq 3$ .

$$\begin{aligned}
 7 \quad a \quad f(x) &= \frac{4x}{x^2 + 1} \\
 \therefore f'(x) &= \frac{4(x^2 + 1) - 4x(2x)}{(x^2 + 1)^2} \quad \{\text{quotient rule}\} \\
 &= \frac{4x^2 + 4 - 8x^2}{(x^2 + 1)^2} \\
 &= \frac{-4x^2 + 4}{(x^2 + 1)^2} \\
 &= \frac{-4(x^2 - 1)}{(x^2 + 1)^2} \\
 &= \frac{-4(x+1)(x-1)}{(x^2 + 1)^2}
 \end{aligned}$$

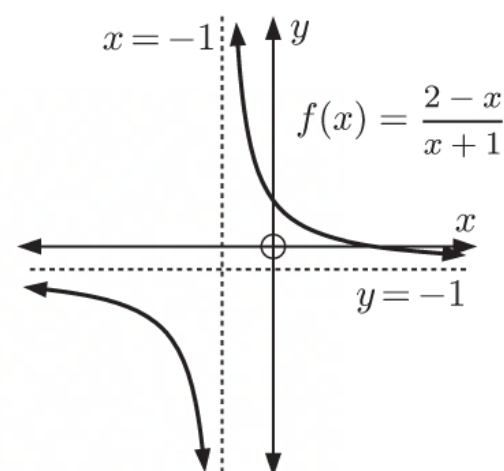
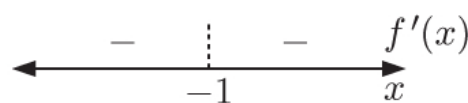


which has sign diagram:

- b**  $y = f(x)$  is increasing for  $-1 \leq x \leq 1$ , and decreasing for  $x \leq -1$  and for  $x \geq 1$ .

$$\begin{aligned}
 8 \quad a \quad f(x) &= \frac{2-x}{x+1} \\
 \therefore f'(x) &= \frac{(-1)(x+1) - (2-x)(1)}{(x+1)^2} \quad \{\text{quotient rule}\} \\
 &= \frac{-x-1-2+x}{(x+1)^2} \\
 &= \frac{-3}{(x+1)^2}
 \end{aligned}$$

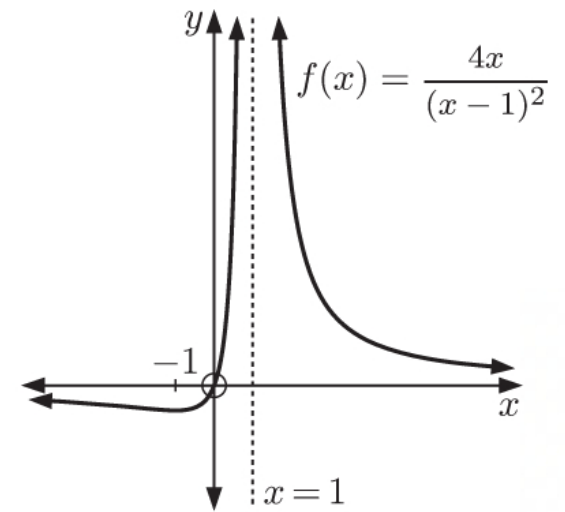
which has sign diagram:



$\therefore f(x)$  is never increasing, and is decreasing for all  $x \neq -1$ .

**b**  $f(x) = \frac{4x}{(x-1)^2}$

$$\begin{aligned} \therefore f'(x) &= \frac{4(x-1)^2 - 4x(2)(x-1)(1)}{[(x-1)^2]^2} && \{\text{quotient rule}\} \\ &= \frac{4(x-1)^2 - 8x(x-1)}{(x-1)^4} \\ &= \frac{4(x-1) - 8x}{(x-1)^3} \\ &= \frac{-4x-4}{(x-1)^3} \\ &= \frac{-4(x+1)}{(x-1)^3} \end{aligned}$$

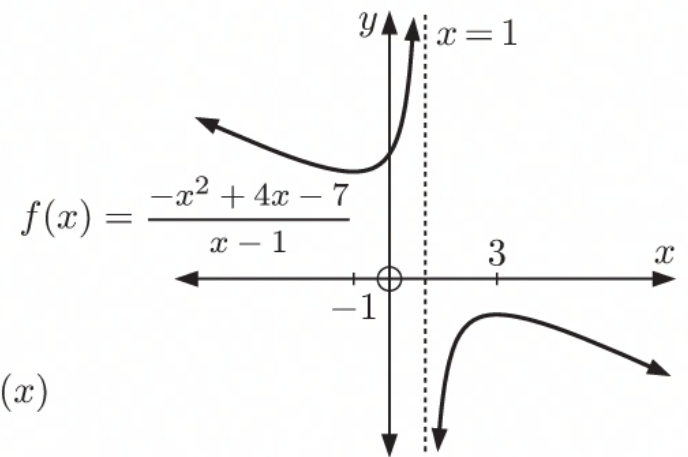


which has sign diagram:  $\begin{array}{ccccccc} & - & | & + & | & - & \\ & & -1 & & 1 & & \\ & \leftarrow & & & & & \rightarrow \end{array} \quad f'(x)$

$\therefore f(x)$  is increasing for  $-1 \leq x < 1$ , and is decreasing for  $x \leq -1$  and  $x > 1$ .

**c**  $f(x) = \frac{-x^2 + 4x - 7}{x-1}$

$$\begin{aligned} \therefore f'(x) &= \frac{(-2x+4)(x-1) - (-x^2+4x-7)(1)}{(x-1)^2} && \{\text{quotient rule}\} \\ &= \frac{-2x^2+2x+4x-4+x^2-4x+7}{(x-1)^2} \\ &= \frac{-x^2+2x+3}{(x-1)^2} \\ &= -\frac{(x-3)(x+1)}{(x-1)^2} \end{aligned}$$

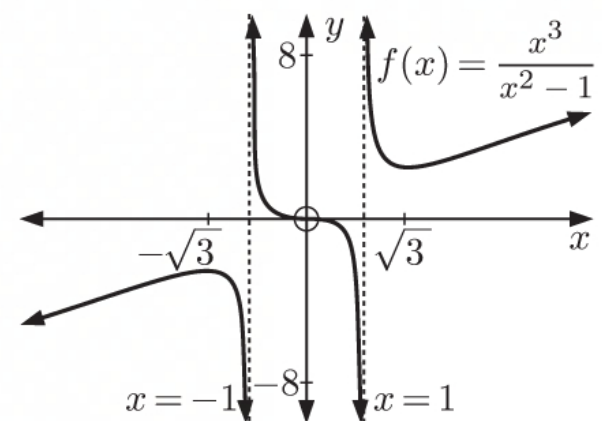


which has sign diagram:  $\begin{array}{ccccccc} & - & | & + & | & + & | & - \\ & & -1 & & 1 & & 3 & \\ & \leftarrow & & & & & & \rightarrow \end{array} \quad f'(x)$

$\therefore f(x)$  is increasing for  $-1 \leq x < 1$  and  $1 < x \leq 3$ , and is decreasing for  $x \leq -1$  and  $x \geq 3$ .

**d**  $f(x) = \frac{x^3}{x^2-1}$

$$\begin{aligned} \therefore f'(x) &= \frac{3x^2(x^2-1) - x^3(2x)}{(x^2-1)^2} && \{\text{quotient rule}\} \\ &= \frac{3x^4 - 3x^2 - 2x^4}{(x^2-1)^2} \\ &= \frac{x^4 - 3x^2}{(x^2-1)^2} \\ &= \frac{x^2(x^2-3)}{(x^2-1)^2} \end{aligned}$$

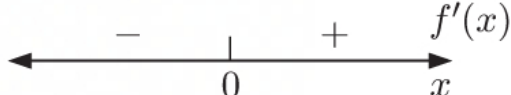


which has sign diagram:  $\begin{array}{ccccccc} & + & | & - & | & - & | & - & | & + \\ & & -\sqrt{3} & & -1 & & 0 & & 1 & & \sqrt{3} \\ & \leftarrow & & & & & & & & & \rightarrow \end{array} \quad f'(x)$

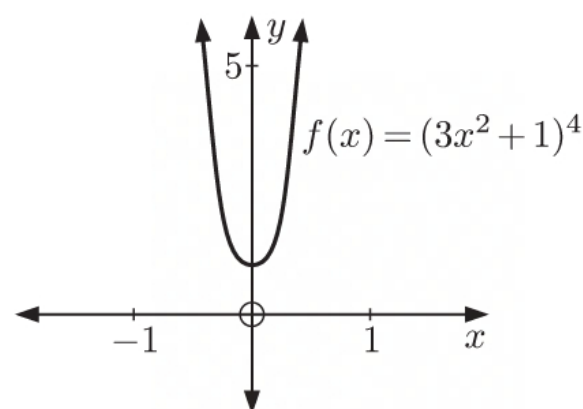
$\therefore f(x)$  is increasing for  $x \leq -\sqrt{3}$  and for  $x \geq \sqrt{3}$ , and decreasing for  $-\sqrt{3} \leq x < -1$ ,  $-1 < x < 1$ , and for  $1 < x \leq \sqrt{3}$ .

**e**  $f(x) = (3x^2 + 1)^4$   
 $\therefore f'(x) = 4(3x^2 + 1)^3(6x)$  {chain rule}  
 $= 24x(3x^2 + 1)^3$

which has sign diagram:



$\therefore f(x)$  is increasing for  $x \geq 0$ ,  
and decreasing for  $x \leq 0$ .



**f**  $f(x) = x^2 + \frac{4}{x-1} = x^2 + 4(x-1)^{-1}$   
 $\therefore f'(x) = 2x - 4(x-1)^{-2}(1)$  {chain rule}  
 $= 2x - \frac{4}{(x-1)^2}$


$f'(x) = 0$  when  $2x = \frac{4}{(x-1)^2}$

$\therefore x = \frac{2}{(x-1)^2}$

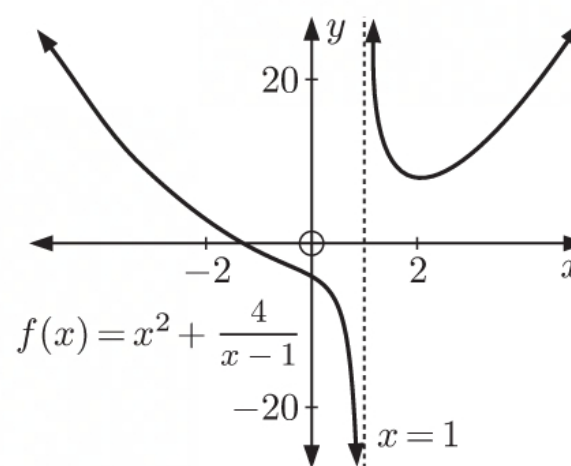
$\therefore x(x-1)^2 = 2$

$\therefore x = 2$  {using technology}

$\therefore f'(x)$  has sign diagram:



$\therefore f(x)$  is increasing for  $x \geq 2$ , and decreasing for  $x < 1$  and for  $1 < x \leq 2$ .

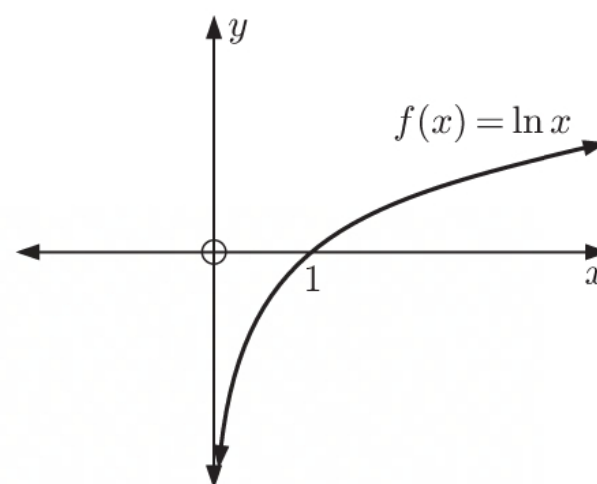


**9** Kenneth's conclusion that  $f(x) = \ln x$  is decreasing for  $x < 0$  is incorrect.

In this case,  $f(x)$  is only defined when  $x > 0$ .

$\therefore f'(x)$  is only defined when  $x > 0$ .

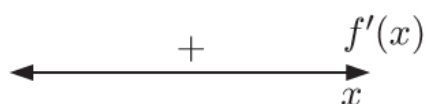
As we can see from the graph alongside,  $f(x) = \ln x$  is increasing for all  $x > 0$ , and never decreasing.



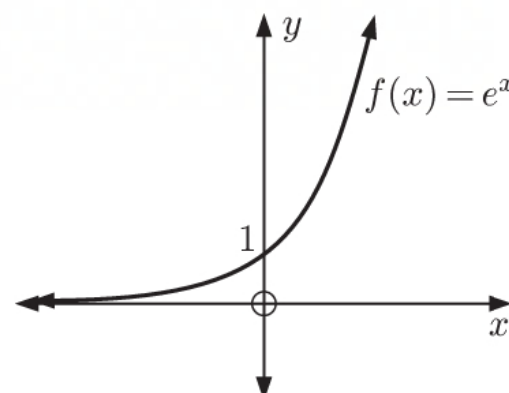
**10 a**  $f(x) = e^x$

$\therefore f'(x) = e^x$

which has sign diagram:



$\therefore f(x)$  is increasing for all  $x \in \mathbb{R}$ ,  
and never decreasing.

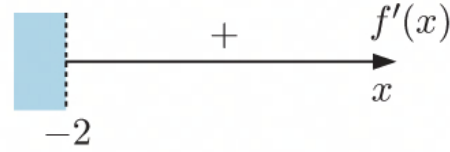




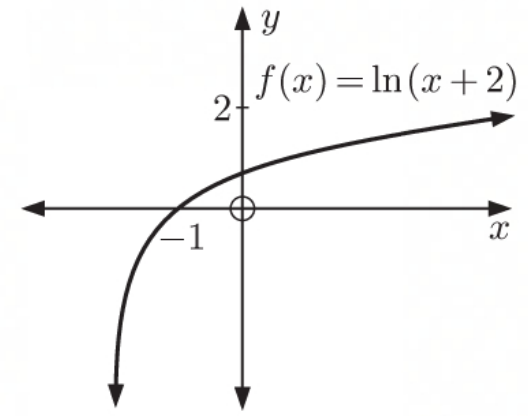
**b**  $f(x) = \ln(x+2)$

$$\therefore f'(x) = \frac{1}{x+2}$$

which has sign diagram:



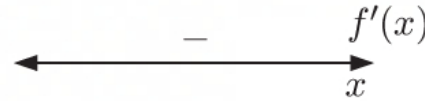
$\therefore f(x)$  is increasing for  $x > -2$ ,  
and never decreasing.



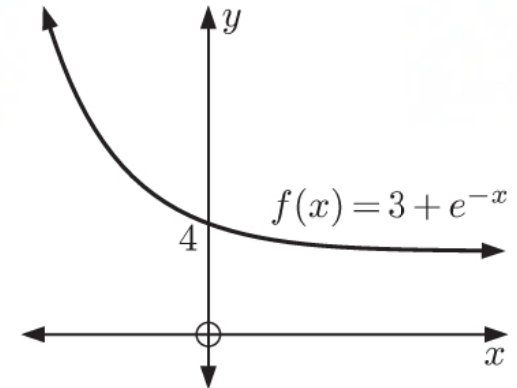
**c**  $f(x) = 3 + e^{-x}$

$$\therefore f'(x) = -e^{-x}$$

which has sign diagram:



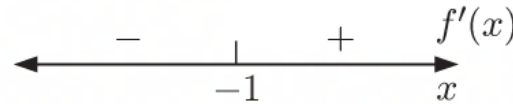
$\therefore f(x)$  is never increasing,  
and decreasing for all  $x \in \mathbb{R}$ .



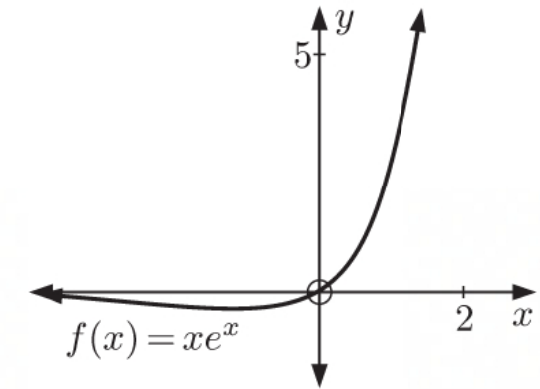
**d**  $f(x) = xe^x$

$$\begin{aligned}\therefore f'(x) &= e^x + xe^x \quad \{\text{product rule}\} \\ &= e^x(1+x)\end{aligned}$$

which has sign diagram:



$\therefore f(x)$  is increasing for  $x \geq -1$ ,  
and decreasing for  $x \leq -1$ .



**e**  $f(x) = x^3 \ln x$

$$\begin{aligned}\therefore f'(x) &= 3x^2 \ln x + x^3 \left(\frac{1}{x}\right) \quad \{\text{product rule}\} \\ &= 3x^2 \ln x + x^2 \\ &= x^2(3 \ln x + 1)\end{aligned}$$

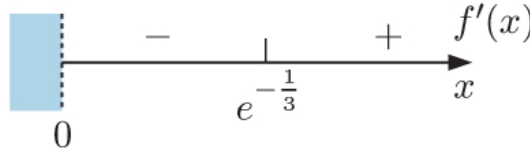
$$f'(x) = 0 \text{ when } x = 0 \text{ or } 3 \ln x + 1 = 0$$

$$\therefore 3 \ln x = -1$$

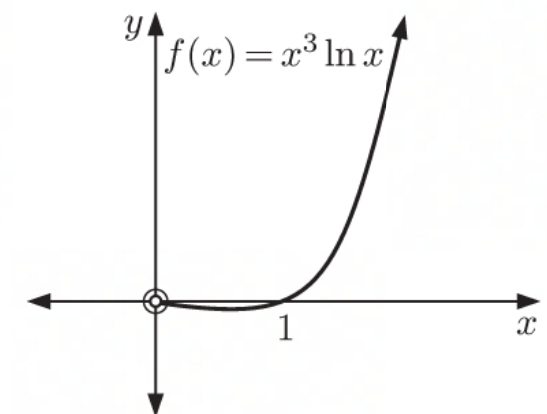
$$\therefore \ln x = -\frac{1}{3}$$

$$\therefore x = e^{-\frac{1}{3}}$$

$\therefore f'(x)$  has sign diagram:



$\therefore f(x)$  is increasing for  $x \geq e^{-\frac{1}{3}}$ , and decreasing for  $0 < x \leq e^{-\frac{1}{3}}$ .



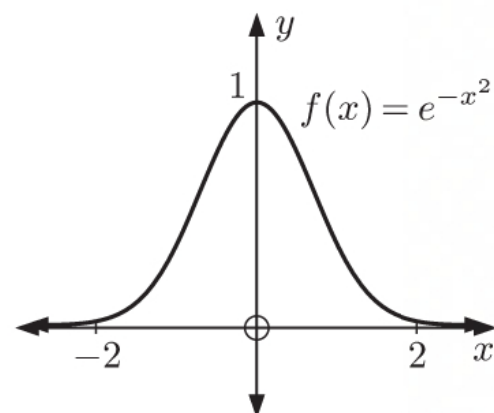
**f**  $f(x) = e^{-x^2}$

$$\therefore f'(x) = -2xe^{-x^2}$$

which has sign diagram:



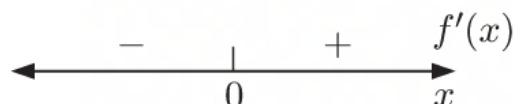
$\therefore f(x)$  is increasing for  $x \leq 0$ ,  
and decreasing for  $x \geq 0$ .



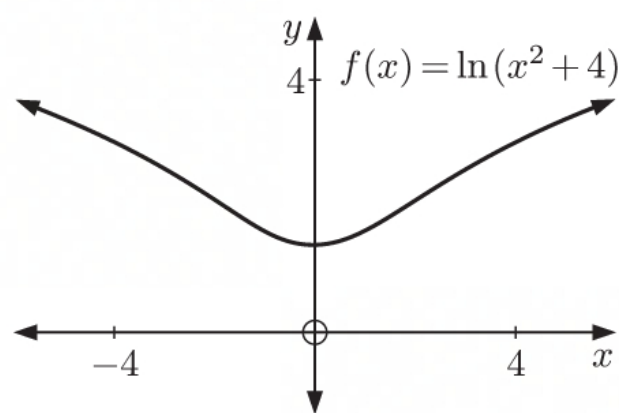
**9**  $f(x) = \ln(x^2 + 4)$

$$\therefore f'(x) = \frac{2x}{x^2 + 4}$$

which has sign diagram:



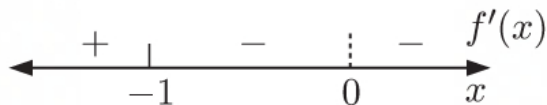
$\therefore f(x)$  is increasing for  $x \geq 0$   
and decreasing for  $x \leq 0$ .



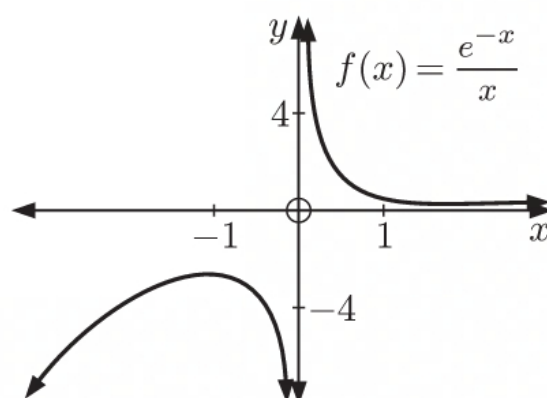
**h**  $f(x) = \frac{e^{-x}}{x}$

$$\begin{aligned} \therefore f'(x) &= \frac{(-e^{-x})x - e^{-x}(1)}{x^2} \quad \{\text{quotient rule}\} \\ &= \frac{-xe^{-x} - e^{-x}}{x^2} \\ &= \frac{-e^{-x}(x+1)}{x^2} \end{aligned}$$

which has sign diagram:



$\therefore f(x)$  is increasing for  $x \leq -1$  and decreasing for  $-1 \leq x < 0$  and for  $x > 0$ .

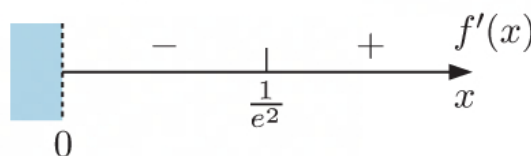


**i**  $f(x) = \sqrt{x} \ln x = x^{\frac{1}{2}} \ln x$

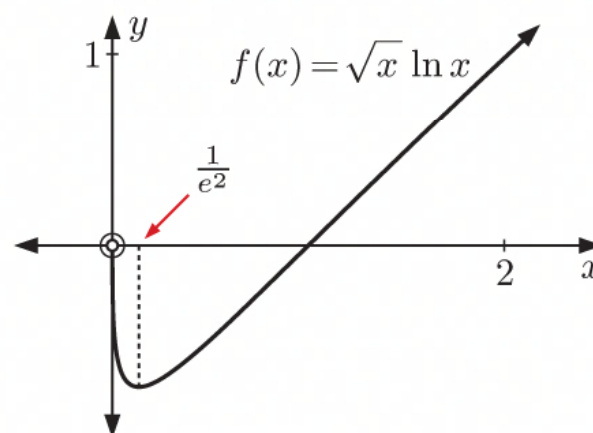
$$\begin{aligned} \therefore f'(x) &= \left(\frac{1}{2}x^{-\frac{1}{2}}\right) \ln x + x^{\frac{1}{2}} \left(\frac{1}{x}\right) \quad \{\text{product rule}\} \\ &= \frac{\ln x}{2\sqrt{x}} + \frac{1}{\sqrt{x}} \\ &= \frac{2 + \ln x}{2\sqrt{x}} \end{aligned}$$

$$\begin{aligned} f'(x) = 0 \quad \text{when} \quad 2 + \ln x &= 0 \\ \therefore \ln x &= -2 \\ \therefore x &= e^{-2} = \frac{1}{e^2} \end{aligned}$$

$\therefore f'(x)$  has sign diagram:



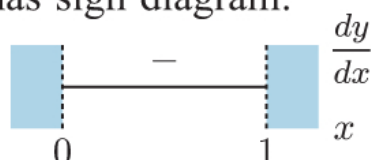
$\therefore f(x)$  is increasing for  $x \geq \frac{1}{e^2}$ , and is decreasing for  $0 < x \leq \frac{1}{e^2}$ .



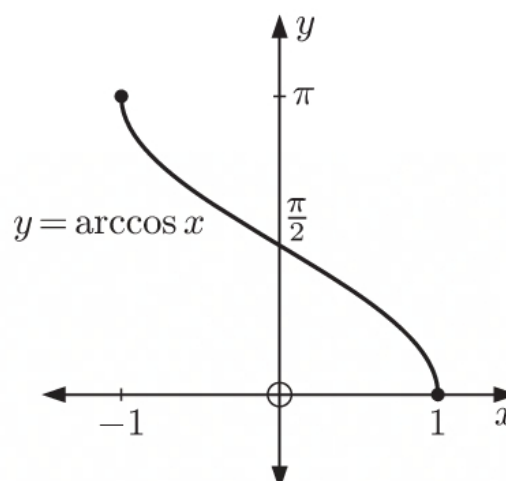
**11 a**  $y = \arccos x, \quad -1 \leq x \leq 1$

$$\therefore \frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}, \quad -1 < x < 1$$

which has sign diagram:

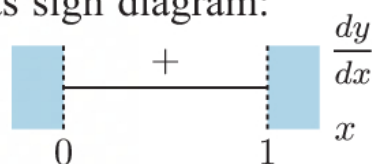


So  $y = \arccos x$  is never increasing, and is decreasing for  $-1 \leq x \leq 1$ .

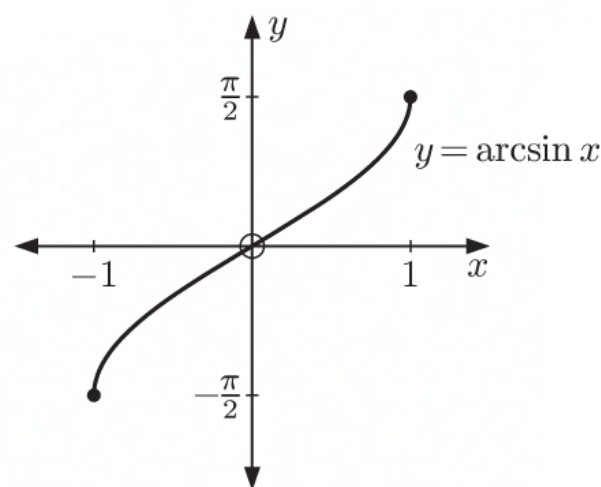


**b**  $y = \arcsin x, \quad -1 \leq x \leq 1$   
 $\therefore \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}, \quad -1 < x < 1$

which has sign diagram:

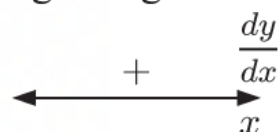


So  $y = \arcsin x$  is increasing for  $-1 \leq x \leq 1$ , and is never decreasing.

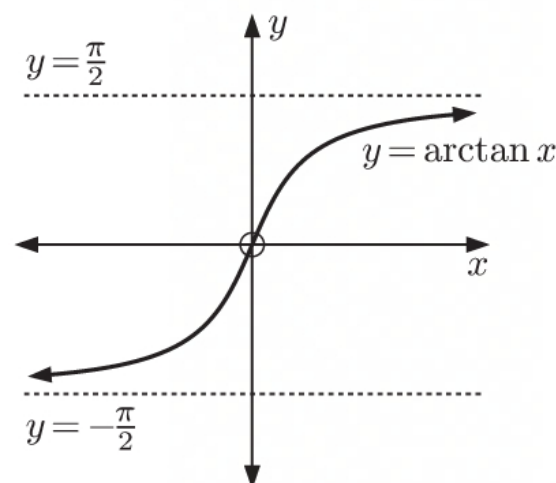


**c**  $y = \arctan x$   
 $\therefore \frac{dy}{dx} = \frac{1}{1+x^2}, \quad x \in \mathbb{R}$

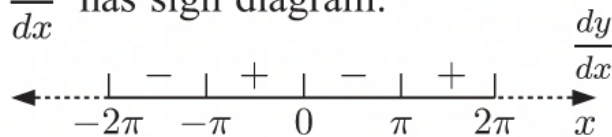
which has sign diagram:



So  $y = \arctan x$  is always increasing, and is never decreasing.



**d**  $y = \cos x$   
 $\therefore \frac{dy}{dx} = -\sin x, \quad x \in \mathbb{R}$   
 $-\sin x = 0$  when  $x = k\pi, \quad k \in \mathbb{Z}$ ,  
 $\therefore \frac{dy}{dx}$  has sign diagram:



So  $y = \cos x$  is increasing for  $\pi \leq x \leq 2\pi$ , but this is repeated periodically every  $2\pi$  units.

$\therefore y = \cos x$  is increasing for  $\pi + 2k\pi \leq x \leq 2\pi + 2k\pi, \quad k \in \mathbb{Z}$

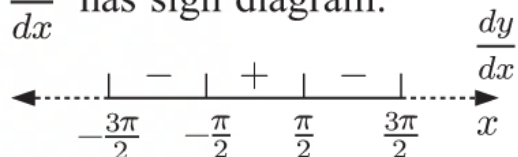
$\therefore (2k+1)\pi \leq x \leq (2k+2)\pi, \quad k \in \mathbb{Z}$

$y = \cos x$  is decreasing for  $0 \leq x \leq \pi$ , but this is repeated periodically every  $2\pi$  units.

$\therefore y = \cos x$  is decreasing for  $0 + 2k\pi \leq x \leq \pi + 2k\pi, \quad k \in \mathbb{Z}$

$\therefore 2k\pi \leq x \leq (2k+1)\pi, \quad k \in \mathbb{Z}$

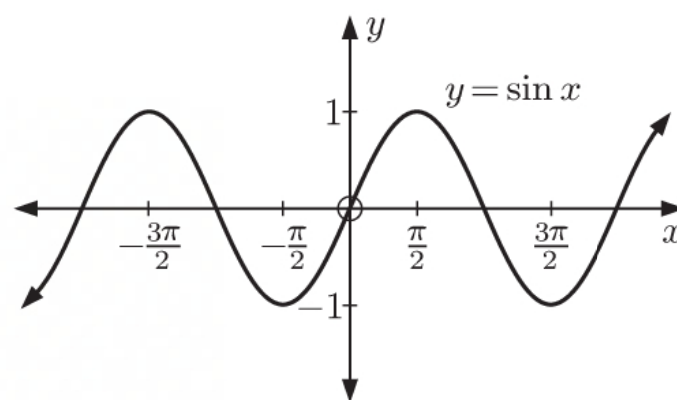
**e**  $y = \sin x$   
 $\therefore \frac{dy}{dx} = \cos x, \quad x \in \mathbb{R}$   
 $\cos x = 0$  when  $x = \frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z}$ ,  
 $\therefore \frac{dy}{dx}$  has sign diagram:



So  $y = \sin x$  is increasing for  $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ , but this is repeated periodically every  $2\pi$  units.

$\therefore y = \sin x$  is increasing for  $-\frac{\pi}{2} + 2k\pi \leq x \leq \frac{\pi}{2} + 2k\pi, \quad k \in \mathbb{Z}$

$\therefore \frac{(4k-1)\pi}{2} \leq x \leq \frac{(4k+1)\pi}{2}, \quad k \in \mathbb{Z}$





$y = \sin x$  is decreasing for  $\frac{\pi}{2} \leq x \leq \frac{3\pi}{2}$ , but this is repeated periodically every  $2\pi$  units.

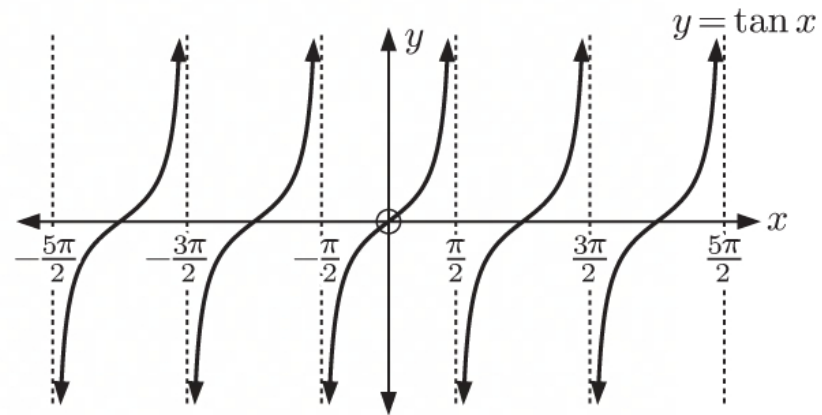
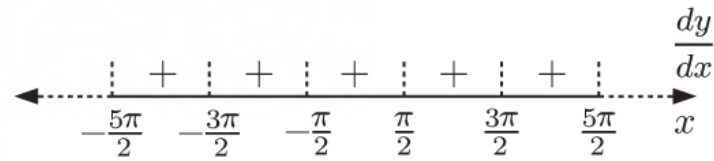
$\therefore y = \sin x$  is decreasing for  $\frac{\pi}{2} + 2k\pi \leq x \leq \frac{3\pi}{2} + 2k\pi, \quad k \in \mathbb{Z}$

$$\therefore \frac{(4k+1)\pi}{2} \leq x \leq \frac{(4k+3)\pi}{2}, \quad k \in \mathbb{Z}$$

**f**  $y = \tan x$

$$\therefore \frac{dy}{dx} = \sec^2 x, \quad x \neq \frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z}$$

which has sign diagram:



So  $y = \tan x$  is increasing for  $-\frac{\pi}{2} < x < \frac{\pi}{2}$ , but this is repeated periodically every  $\pi$  units.

$\therefore y = \tan x$  is increasing for  $x \neq \frac{(2k+1)\pi}{2}, \quad k \in \mathbb{Z}$

$y = \tan x$  is never decreasing.

**9**  $y = x - 2 \sin x$

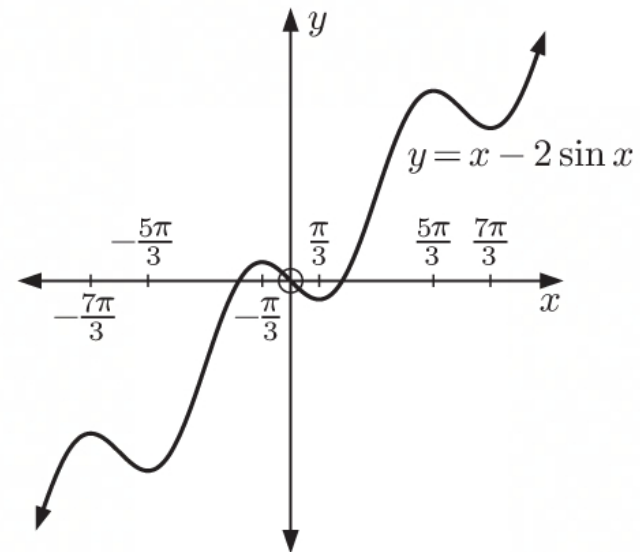
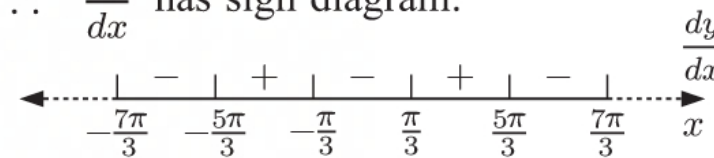
$$\therefore \frac{dy}{dx} = 1 - 2 \cos x, \quad x \in \mathbb{R}$$

$$1 - 2 \cos x = 0 \quad \text{when} \quad 2 \cos x = 1$$

$$\therefore \cos x = \frac{1}{2}$$

$$\therefore x = \pm \frac{\pi}{3} + 2k\pi, \quad k \in \mathbb{Z}$$

$\therefore \frac{dy}{dx}$  has sign diagram:



So  $y = x - 2 \sin x$  is increasing for  $\frac{\pi}{3} \leq x \leq \frac{5\pi}{3}$ , but this interval is repeated periodically every  $2\pi$  units.

$\therefore y = x - 2 \sin x$  is increasing for  $\frac{\pi}{3} + 2k\pi \leq x \leq \frac{5\pi}{3} + 2k\pi, \quad k \in \mathbb{Z}$

$$\therefore \frac{(6k+1)\pi}{3} \leq x \leq \frac{(6k+5)\pi}{3}, \quad k \in \mathbb{Z}$$

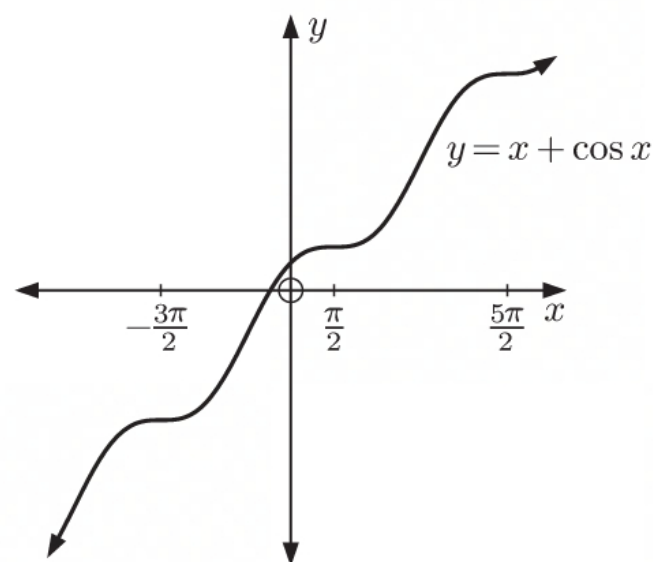
$y = x - 2 \sin x$  is decreasing for  $-\frac{\pi}{3} \leq x \leq \frac{\pi}{3}$ , but this interval is repeated periodically every  $2\pi$  units.

$\therefore y = x - 2 \sin x$  is decreasing for  $-\frac{\pi}{3} + 2k\pi \leq x \leq \frac{\pi}{3} + 2k\pi, \quad k \in \mathbb{Z}$

$$\therefore \frac{(6k-1)\pi}{3} \leq x \leq \frac{(6k+1)\pi}{3}, \quad k \in \mathbb{Z}$$



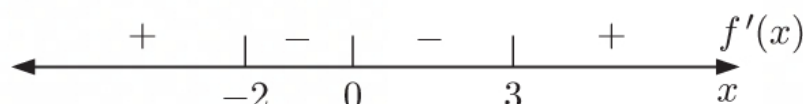
- h**  $y = x + \cos x$   
 $\therefore \frac{dy}{dx} = 1 - \sin x, \quad x \in \mathbb{R}$   
 $1 - \sin x = 0$  when  $\sin x = 1$   
 $\therefore x = \frac{\pi}{2} + 2k\pi$   
since  $-1 \leq \sin x \leq 1$ ,  $1 - \sin x \geq 0$ .  
 $\therefore \frac{dy}{dx}$  has sign diagram:
- $$\begin{array}{ccccccc} & + & & + & & + & \\ \leftarrow & | & | & | & | & | & \rightarrow \\ & -\frac{7\pi}{2} & -\frac{3\pi}{2} & \frac{\pi}{2} & \frac{5\pi}{2} & & x \end{array}$$
- $\therefore y = x + \cos x$  is always increasing, and is never decreasing.



## EXERCISE 18D

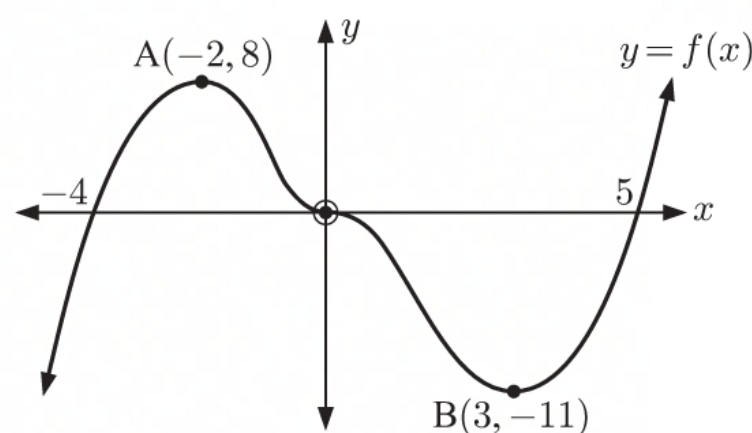
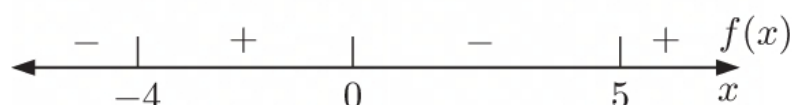
- 1 a** A is a local maximum, O is a stationary inflection, B is a local minimum.

- b**  $f'(x)$  has sign diagram:

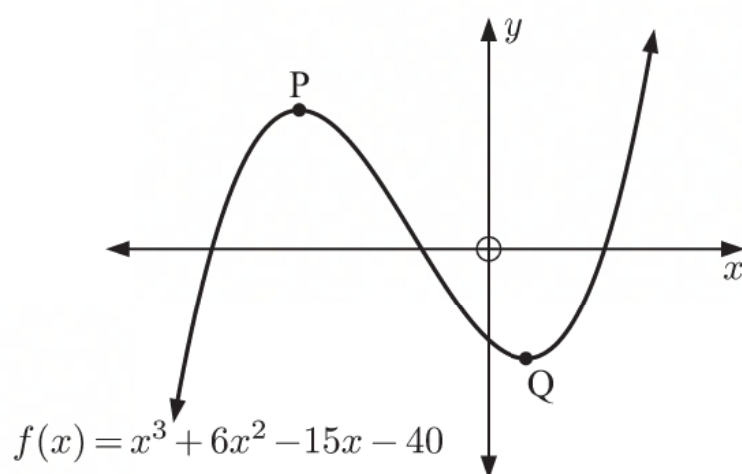


- c i**  $f(x)$  is increasing for  $x \leq -2$  and  $x \geq 3$ .  
**ii**  $f(x)$  is decreasing for  $-2 \leq x \leq 3$ .

- d**  $f(x)$  has sign diagram:



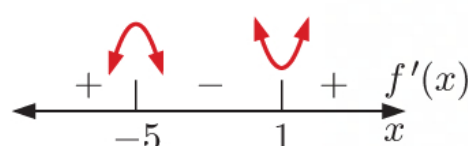
2



- a** P is a local maximum.  
Q is a local minimum.

**b**  $f(x) = x^3 + 6x^2 - 15x - 40$   
 $\therefore f'(x) = 3x^2 + 12x - 15$   
 $= 3(x^2 + 4x - 5)$   
 $= 3(x - 1)(x + 5)$

- c**  $f'(x)$  has sign diagram:



- $\therefore$  there is a local maximum at  $x = -5$   
and a local minimum at  $x = 1$ .

So, P has  $x$ -coordinate  $-5$ , and Q has  $x$ -coordinate  $1$ .

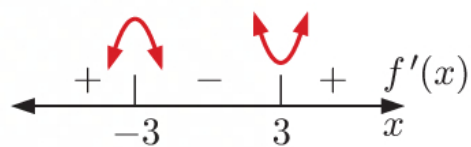
$$\begin{aligned} f(-5) &= (-5)^3 + 6(-5)^2 - 15(-5) - 40 \\ &= 60 \end{aligned}$$

$$\begin{aligned} f(1) &= 1^3 + 6(1)^2 - 15(1) - 40 \\ &= -48 \end{aligned}$$

So, P is  $(-5, 60)$  and Q is  $(1, -48)$ .

**3 a**  $f(x) = \frac{1}{3}x^3 - 9x + 4$   
 $\therefore f'(x) = x^2 - 9$   
 $= (x + 3)(x - 3)$

which has sign diagram:



**b**  $f(x)$  is increasing for  $x \leq -3$  and  $x \geq 3$ , and decreasing for  $-3 \leq x \leq 3$ .

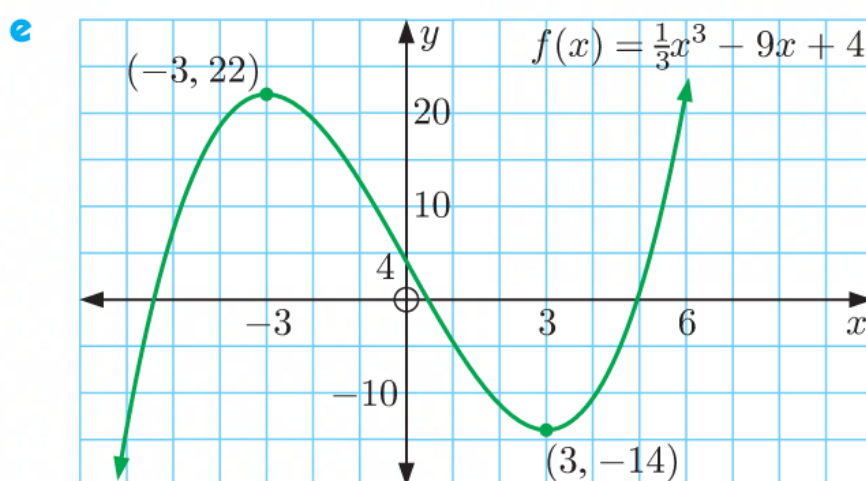
**c** From the sign diagram in **a**, there is a local maximum at  $x = -3$ , and a local minimum at  $x = 3$ .

$$f(-3) = \frac{1}{3}(-3)^3 - 9(-3) + 4 = 22$$

$$f(3) = \frac{1}{3}(3)^3 - 9(3) + 4 = -14$$

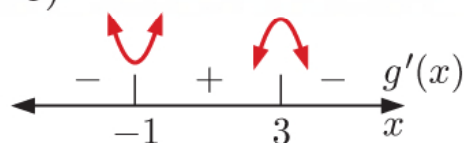
So, there is a local maximum at  $(-3, 22)$ , and a local minimum at  $(3, -14)$ .

**d** As  $x \rightarrow \infty$ ,  $f(x) \rightarrow \infty$ ,  
as  $x \rightarrow -\infty$ ,  $f(x) \rightarrow -\infty$ .



**4 a**  $g(x) = -2x^3 + 6x^2 + 18x - 7$   
 $\therefore g'(x) = -6x^2 + 12x + 18$   
 $= -6(x^2 - 2x - 3)$   
 $= -6(x + 1)(x - 3)$

which has sign diagram:



**b**  $g(x)$  is increasing for  $-1 \leq x \leq 3$ , and decreasing for  $x \leq -1$  and  $x \geq 3$ .

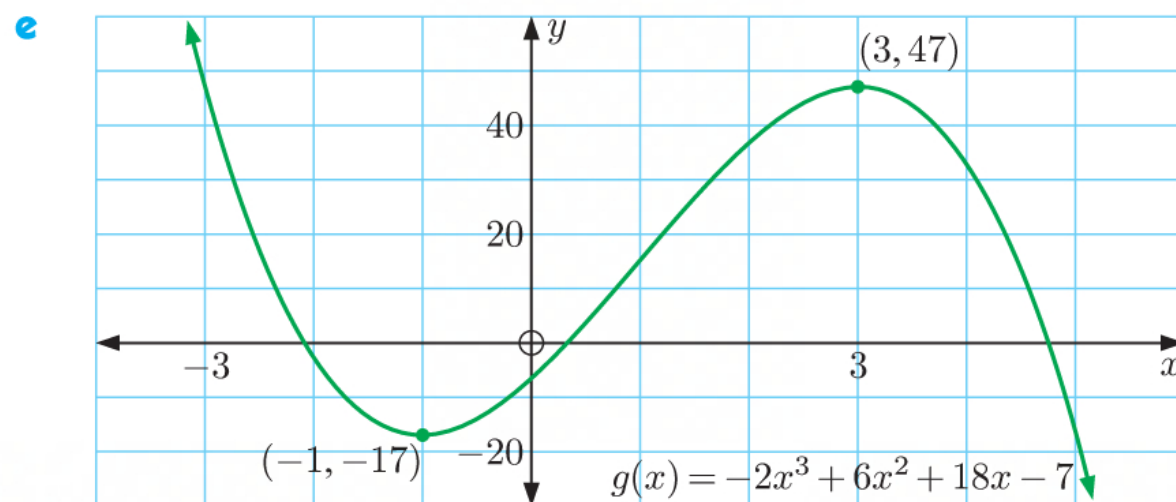
**c** From the sign diagram, there is a local minimum at  $x = -1$ , and a local maximum at  $x = 3$ .

$$g(-1) = -2(-1)^3 + 6(-1)^2 + 18(-1) - 7 = -17$$

$$g(3) = -2(3)^3 + 6(3)^2 + 18(3) - 7 = 47$$

So, there is a local minimum at  $(-1, -17)$ , and a local maximum at  $(3, 47)$ .

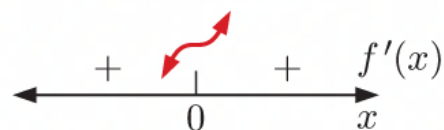
**d** As  $x \rightarrow \infty$ ,  $g(x) \rightarrow -\infty$ , as  $x \rightarrow -\infty$ ,  $g(x) \rightarrow \infty$ .



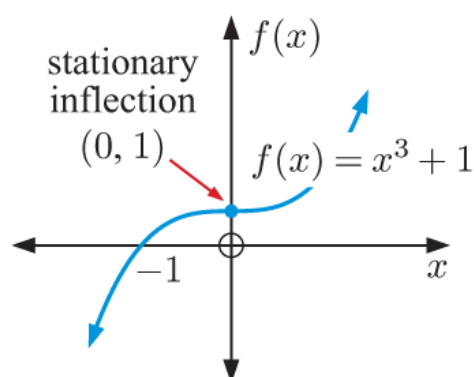
**5 a**  $f(x) = x^3 + 1$

$\therefore f'(x) = 3x^2$

which has sign diagram:



Now  $f(0) = 1$ , so there is a stationary inflection at  $(0, 1)$ .



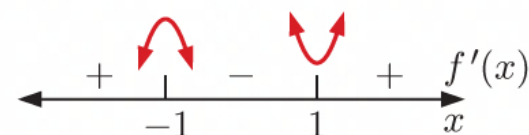
**b**  $f(x) = x^3 - 3x + 2$

$\therefore f'(x) = 3x^2 - 3$

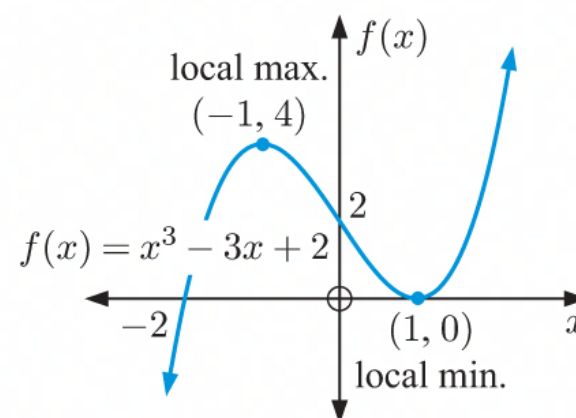
$= 3(x^2 - 1)$

$= 3(x + 1)(x - 1)$

which has sign diagram:



Now  $f(-1) = 4$ ,  $f(1) = 0$ , so there is a local maximum at  $(-1, 4)$ , and a local minimum at  $(1, 0)$ .



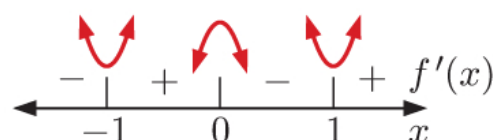
**c**  $f(x) = x^4 - 2x^2$

$\therefore f'(x) = 4x^3 - 4x$

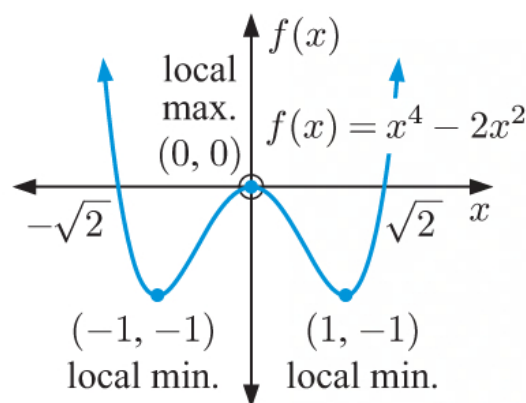
$= 4x(x^2 - 1)$

$= 4x(x + 1)(x - 1)$

which has sign diagram:



Now  $f(-1) = -1$ ,  $f(1) = -1$ ,  $f(0) = 0$ , so there are local minima at  $(-1, -1)$  and  $(1, -1)$ , and a local maximum at  $(0, 0)$ .



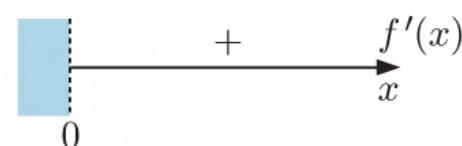
**d**  $f(x) = \sqrt{x} + 2$

$= x^{\frac{1}{2}} + 2$

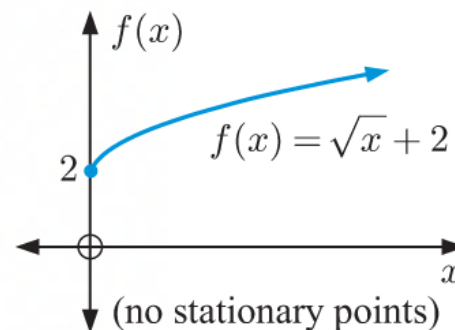
$\therefore f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$

$= \frac{1}{2\sqrt{x}} \neq 0$

which has sign diagram:



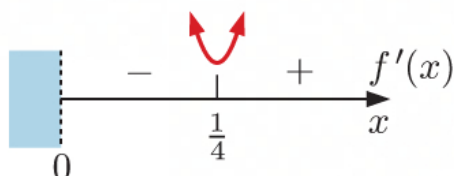
$\therefore$  there are no stationary points.





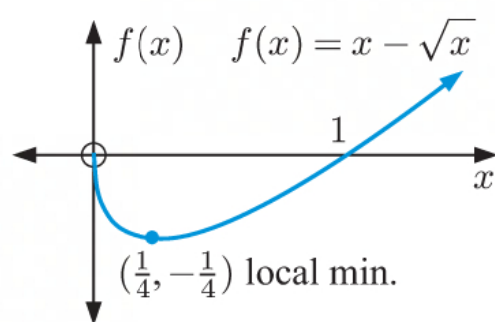
$$\begin{aligned}
 \text{e} \quad f(x) &= x - \sqrt{x} \\
 &= x - x^{\frac{1}{2}} \\
 \therefore f'(x) &= 1 - \frac{1}{2}x^{-\frac{1}{2}} \\
 &= 1 - \frac{1}{2\sqrt{x}} \\
 &= \frac{2\sqrt{x} - 1}{2\sqrt{x}}
 \end{aligned}$$

which has sign diagram:



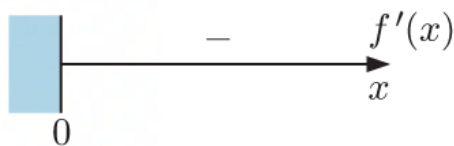
$f(x)$  is defined for all  $x \geq 0$

Now  $f(\frac{1}{4}) = -\frac{1}{4}$ , so there is a local minimum at  $(\frac{1}{4}, -\frac{1}{4})$ .



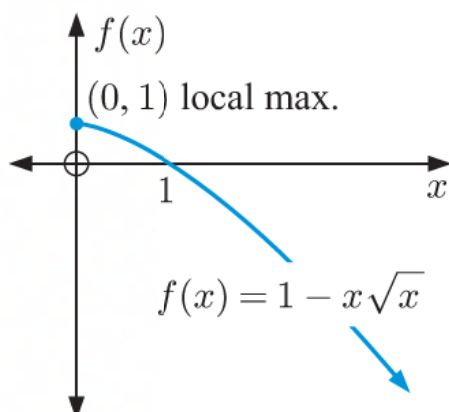
$$\begin{aligned}
 \text{g} \quad f(x) &= 1 - x\sqrt{x} \\
 &= 1 - x^{\frac{3}{2}} \\
 \therefore f'(x) &= -\frac{3}{2}x^{\frac{1}{2}} \\
 &= -\frac{3\sqrt{x}}{2}
 \end{aligned}$$

which has sign diagram:



$f(x)$  is only defined when  $x \geq 0$ .

Now  $f(0) = 1$ , so there is a local maximum at  $(0, 1)$ .

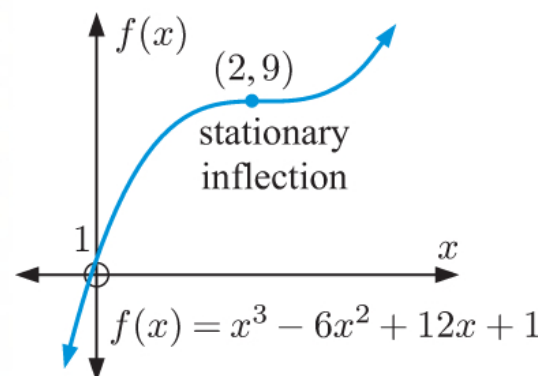


$$\begin{aligned}
 \text{f} \quad f(x) &= x^3 - 6x^2 + 12x + 1 \\
 \therefore f'(x) &= 3x^2 - 12x + 12 \\
 &= 3(x^2 - 4x + 4) \\
 &= 3(x - 2)^2
 \end{aligned}$$

which has sign diagram:

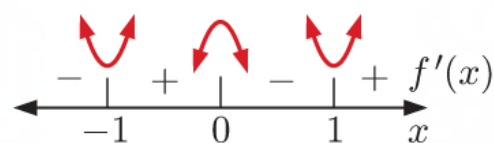


Now  $f(2) = 9$ , so there is a stationary inflection at  $(2, 9)$ .

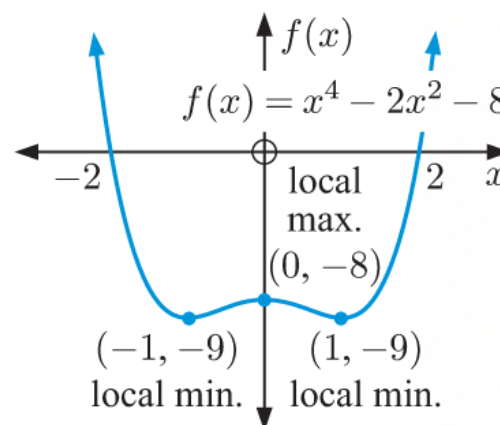


$$\begin{aligned}
 \text{h} \quad f(x) &= x^4 - 2x^2 - 8 \\
 \therefore f'(x) &= 4x^3 - 4x \\
 &= 4x(x^2 - 1) \\
 &= 4x(x + 1)(x - 1)
 \end{aligned}$$

which has sign diagram:



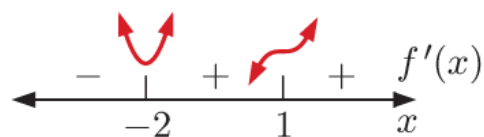
Now  $f(-1) = -9$ ,  $f(1) = -9$ ,  $f(0) = -8$ , so there are local minima at  $(-1, -9)$  and  $(1, -9)$ , and a local maximum at  $(0, -8)$ .



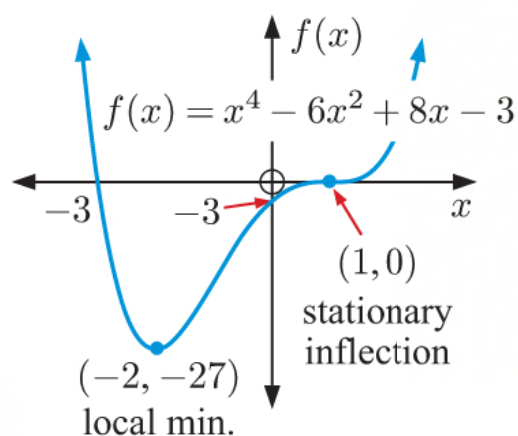


$$\begin{aligned}
 \text{i} \quad f(x) &= x^4 - 6x^2 + 8x - 3 \\
 \therefore f'(x) &= 4x^3 - 12x + 8 \\
 &= 4(x^3 - 3x + 2) \\
 &= 4(x-1)(x^2 + x - 2) \\
 &= 4(x-1)(x+2)(x-1)
 \end{aligned}$$

which has sign diagram:



Now  $f(-2) = -27$ ,  $f(1) = 0$ , so there is a local minimum at  $(-2, -27)$ , and a stationary inflection at  $(1, 0)$ .



$$6 \quad \text{a} \quad f(x) = ax^2 + bx + c, \quad a \neq 0$$

$$\therefore f'(x) = 2ax + b$$

$f(x)$  has a stationary point when  $f'(x) = 0$

$$\therefore 2ax + b = 0$$

$$\therefore x = -\frac{b}{2a}$$

b When  $a < 0$ ,  $f(x)$  is concave down , so there is a local maximum when  $a < 0$ .

When  $a > 0$ ,  $f(x)$  is concave up , so there is a local minimum when  $a > 0$ .

$$7 \quad \text{a} \quad f(x) = \sqrt{x^2 - 4x + 5} = (x^2 - 4x + 5)^{\frac{1}{2}}$$

$$\begin{aligned}
 \therefore f'(x) &= \frac{1}{2}(x^2 - 4x + 5)^{-\frac{1}{2}}(2x - 4) \quad \{\text{chain rule}\} \\
 &= \frac{x - 2}{\sqrt{x^2 - 4x + 5}}
 \end{aligned}$$

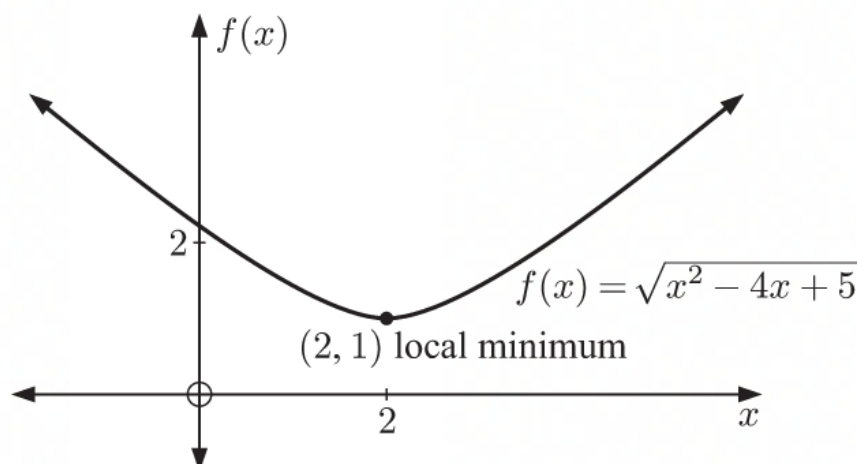
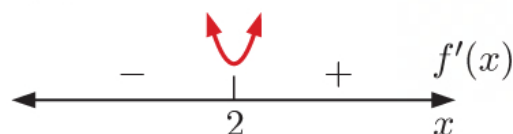
Now  $x^2 - 4x + 5$  has no real zeros as

$$\Delta = (-4)^2 - 4(1)(5) = -4 < 0$$

$$\therefore x^2 - 4x + 5 > 0 \text{ for all } x$$

$$\therefore f'(x) = 0 \text{ when } x = 2$$

So,  $f'(x)$  has sign diagram:



$$\begin{aligned}
 f(2) &= \sqrt{2^2 - 4(2) + 5} \\
 &= \sqrt{4 - 8 + 5} \\
 &= 1
 \end{aligned}$$

$\therefore$  there is a local minimum at  $(2, 1)$ .

**b**

$$f(x) = \frac{x^2}{x-1}$$

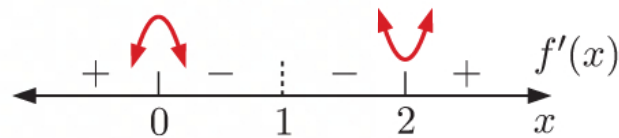
$$\therefore f'(x) = \frac{2x(x-1) - x^2(1)}{(x-1)^2} \quad \{\text{quotient rule}\}$$

$$= \frac{2x^2 - 2x - x^2}{(x-1)^2}$$

$$= \frac{x^2 - 2x}{(x-1)^2}$$

$$= \frac{x(x-2)}{(x-1)^2}$$

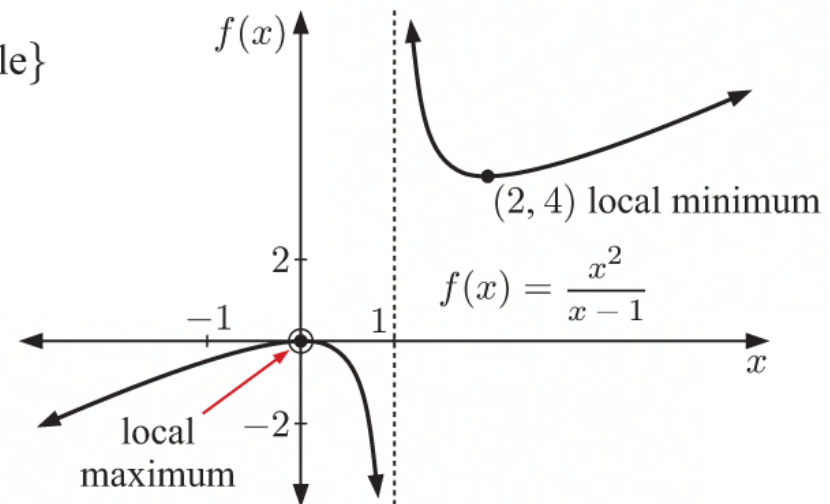
which has sign diagram:



$$f(0) = \frac{0^2}{0-1} = 0$$

$$f(2) = \frac{2^2}{2-1} = 4$$

$\therefore$  there is a local maximum at  $(0, 0)$ , and a local minimum at  $(2, 4)$ .



**c**

$$f(x) = \frac{x}{x^2+1}$$

$$\therefore f'(x) = \frac{(1)(x^2+1) - x(2x)}{(x^2+1)^2} \quad \{\text{quotient rule}\}$$

$$= \frac{x^2+1-2x^2}{(x^2+1)^2}$$

$$= \frac{1-x^2}{(x^2+1)^2}$$

$$= \frac{(1+x)(1-x)}{(x^2+1)^2}$$

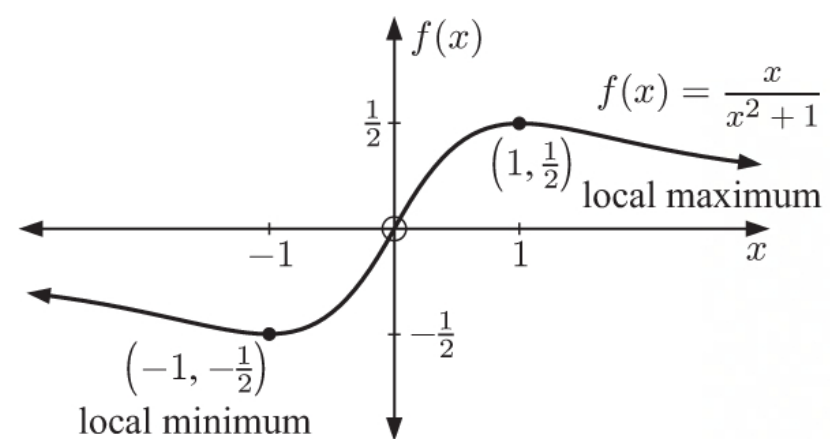
which has sign diagram:



$$f(-1) = \frac{-1}{(-1)^2+1} = -\frac{1}{2}$$

$$f(1) = \frac{1}{1^2+1} = \frac{1}{2}$$

$\therefore$  there is a local minimum at  $(-1, -\frac{1}{2})$ , and a local maximum at  $(1, \frac{1}{2})$ .




**8 a**  $y = xe^{-x}$

$$\begin{aligned}\therefore \frac{dy}{dx} &= (1)e^{-x} + x(e^{-x})(-1) \quad \{\text{product rule}\} \\ &= e^{-x} - xe^{-x} \\ &= e^{-x}(1 - x) \quad \text{where } e^{-x} \text{ is positive for all } x\end{aligned}$$

So,  $\frac{dy}{dx} = 0$  when  $x = 1$ .

The sign diagram of  $\frac{dy}{dx}$  is:



When  $x = 1$ ,  $y = (1)e^{-1} = \frac{1}{e}$


$\therefore$  there is a local maximum at  $\left(1, \frac{1}{e}\right)$ .

**b**  $y = x^2e^x$

$$\begin{aligned}\therefore \frac{dy}{dx} &= (2x)e^x + x^2(e^x) \quad \{\text{product rule}\} \\ &= 2xe^x + x^2e^x \\ &= xe^x(2 + x) \quad \text{where } e^x \text{ is positive for all } x\end{aligned}$$

So,  $\frac{dy}{dx} = 0$  when  $x = 0$  or  $-2$ .

The sign diagram of  $\frac{dy}{dx}$  is:



When  $x = -2$ ,  $y = (-2)^2e^{-2} = \frac{4}{e^2}$

When  $x = 0$ ,  $y = 0^2e^0 = 0$


$\therefore$  there is a local maximum at  $\left(-2, \frac{4}{e^2}\right)$  and a local minimum at  $(0, 0)$ .

**c**  $y = \frac{e^x}{x}$

$$\begin{aligned}\therefore \frac{dy}{dx} &= \frac{e^x x - e^x(1)}{x^2} \quad \{\text{quotient rule}\} \\ &= \frac{e^x(x - 1)}{x^2} \quad \text{where } e^x \text{ is positive for all } x\end{aligned}$$

So,  $\frac{dy}{dx} = 0$  when  $x = 1$ .

The sign diagram of  $\frac{dy}{dx}$  is:



When  $x = 1$ ,  $y = \frac{e^1}{1} = e$

$\therefore$  there is a local minimum at  $(1, e)$ .

**d**  $y = e^{-x}(x+2)$

$$\therefore \frac{dy}{dx} = e^{-x}(-1)(x+2) + e^{-x}(1) \quad \{\text{product rule}\}$$

$$= -e^{-x}(x+2-1)$$

$$= -e^{-x}(x+1) \quad \text{where } -e^{-x} \text{ is negative for all } x$$

So,  $\frac{dy}{dx} = 0$  when  $x = -1$ .

The sign diagram of  $\frac{dy}{dx}$  is: 

When  $x = -1$ ,  $y = e^{-(-1)}(-1+2) = e$

$\therefore$  there is a local maximum at  $(-1, e)$ .

**9 a**  $f(x) = 2x^3 + ax^2 - 24x + 1$

$$\therefore f'(x) = 6x^2 + 2ax - 24$$

But  $f'(-4) = 0$ , so  $6(-4)^2 + 2a(-4) - 24 = 0$

$$\therefore 96 - 8a - 24 = 0$$

$$\therefore 72 = 8a$$

$$\therefore a = 9$$

**b** Since  $a = 9$ , then  $f(x) = 2x^3 + 9x^2 - 24x + 1$

$$\begin{aligned} \therefore f(-4) &= 2(-4)^3 + 9(-4)^2 - 24(-4) + 1 \\ &= 113 \end{aligned}$$

$\therefore$  the local maximum is at  $(-4, 113)$ .

**10 a**  $f(x) = x^3 + ax + b$

$$\therefore f'(x) = 3x^2 + a$$

But  $f'(-2) = 0$

$$\therefore 3(-2)^2 + a = 0$$

$$\therefore 12 + a = 0$$

$$\therefore a = -12$$

Also,  $f(-2) = 3$

$$\therefore (-2)^3 - 12(-2) + b = 3$$

$$\therefore -8 + 24 + b = 3$$

$$\therefore b = -13$$

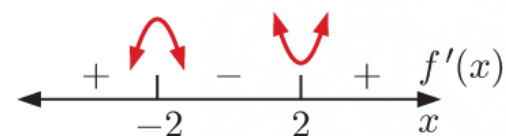
**b** Now  $f(x) = x^3 - 12x - 13$

$$\therefore f'(x) = 3x^2 - 12$$

$$= 3(x^2 - 4)$$

$$= 3(x+2)(x-2)$$

which has sign diagram:



Now  $f(2) = -29$ ,  $f(-2) = 3$ , so there is a local minimum at  $(2, -29)$  and a local maximum at  $(-2, 3)$ .



$$\begin{aligned}
 11 \quad y &= \frac{e^{ax}}{bx} \\
 \therefore \frac{dy}{dx} &= \frac{e^{ax}(a)(bx) - e^{ax}(b)}{(bx)^2} \quad \{\text{quotient rule}\} \\
 &= \frac{abxe^{ax} - be^{ax}}{b^2x^2} \\
 &= \frac{be^{ax}(ax - 1)}{b^2x^2} \\
 &= \frac{e^{ax}(ax - 1)}{bx^2} \quad \dots (*)
 \end{aligned}$$

Since  $\left(\frac{1}{3}, \frac{e}{2}\right)$  is a stationary point, then

$$\text{when } x = \frac{1}{3}, \quad \frac{dy}{dx} = 0$$

Substituting  $x = \frac{1}{3}$  into (\*) gives:

$$\begin{aligned}
 \therefore \frac{e^{\frac{a}{3}}\left(\frac{a}{3} - 1\right)}{b\left(\frac{1}{3}\right)^2} &= 0 \\
 \therefore e^{\frac{a}{3}}\left(\frac{a}{3} - 1\right) &= 0 \quad \{\text{as } b \neq 0\} \\
 \therefore \frac{a}{3} - 1 &= 0 \quad \{\text{as } e^{\frac{a}{3}} > 0\} \\
 \therefore \frac{a}{3} &= 1 \\
 \therefore a &= 3
 \end{aligned}$$

$$\text{when } x = \frac{1}{3}, \quad y = \frac{e}{2}$$

$$\therefore \frac{e}{2} = \frac{e^{3\left(\frac{1}{3}\right)}}{b\left(\frac{1}{3}\right)}$$

$$\therefore \frac{e}{2} = \frac{e}{\frac{b}{3}}$$

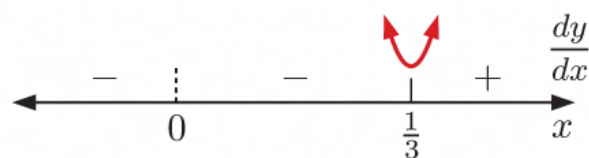
$$\therefore \frac{b}{3} = 2$$

$$\therefore b = 6$$

So,  $a = 3$  and  $b = 6$ .

$$\therefore \frac{dy}{dx} = \frac{e^{3x}(3x - 1)}{6x^2}$$

which has sign diagram:



$\therefore$  there is a local minimum at  $\left(\frac{1}{3}, \frac{e}{2}\right)$ .

- 12 a**  $x$  is defined for all  $x \in \mathbb{R}$ , but  $\ln x$  is only defined for  $x > 0$ .  
 $\therefore f(x) = x \ln x$  is only defined for  $x > 0$ .

**b**  $f'(x) = (1) \ln x + x \left( \frac{1}{x} \right)$  {product rule}  
 $= \ln x + 1$

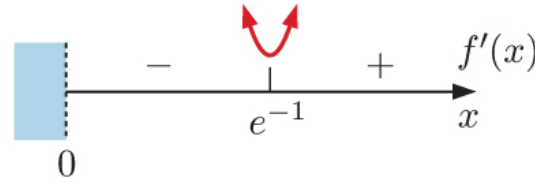
$f'(x) = 0$  when  $\ln x + 1 = 0$   
 $\therefore \ln x = -1$   
 $\therefore x = e^{-1}$

$\therefore f'(x)$  has sign diagram:

$f(e^{-1}) = e^{-1} \ln(e^{-1})$   
 $= -\frac{1}{e}$

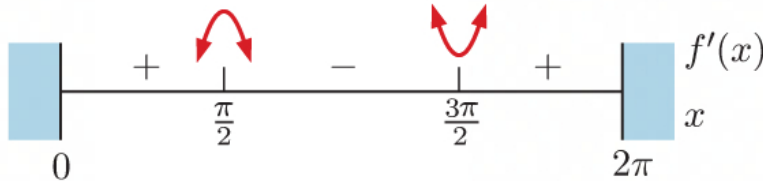
$\therefore$  there is a local minimum at  $\left( \frac{1}{e}, -\frac{1}{e} \right)$ .

$\therefore$  the minimum value of  $f(x)$  is  $-\frac{1}{e}$ .



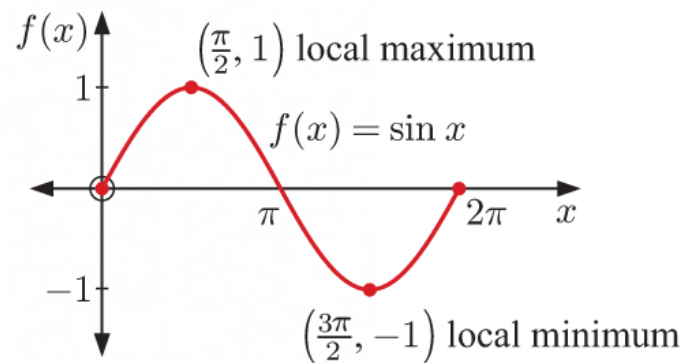
- 13 a**  $f(x) = \sin x$ ,  $0 \leq x \leq 2\pi$   
 $\therefore f'(x) = \cos x$

which has sign diagram:



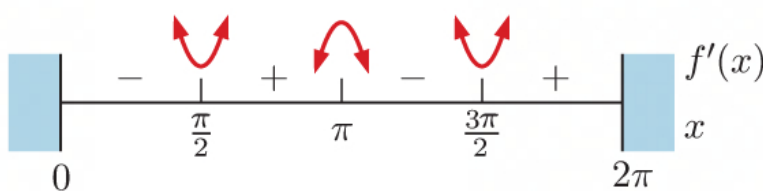
$f\left(\frac{\pi}{2}\right) = \sin \frac{\pi}{2} = 1$   
 $f\left(\frac{3\pi}{2}\right) = \sin \frac{3\pi}{2} = -1$

$\therefore$  there is a local maximum at  $\left(\frac{\pi}{2}, 1\right)$ ,  
 and a local minimum at  $\left(\frac{3\pi}{2}, -1\right)$ .



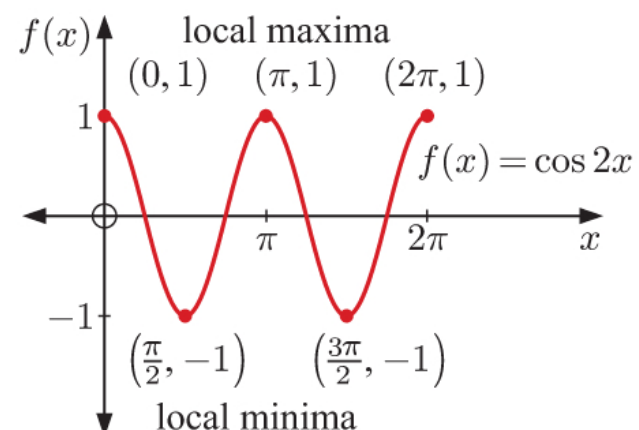
- b**  $f(x) = \cos 2x$ ,  $0 \leq x \leq 2\pi$   
 $\therefore f'(x) = -2 \sin 2x$   
 $f'(x) = 0$  when  $\sin 2x = 0$   
 $\therefore 2x = 0, \pi, 2\pi, 3\pi, 4\pi$   
 $\therefore x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$

So,  $f'(x)$  has sign diagram:



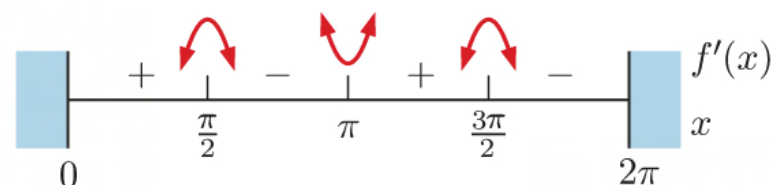
$f(0) = \cos 0 = 1$ ,  $f\left(\frac{\pi}{2}\right) = \cos \pi = -1$ ,  
 $f(\pi) = \cos 2\pi = 1$ ,  $f\left(\frac{3\pi}{2}\right) = \cos 3\pi = -1$ ,  
 $f(2\pi) = \cos 4\pi = 1$

$\therefore$  there are local maxima at  $(0, 1)$ ,  $(\pi, 1)$ , and  $(2\pi, 1)$ , and local minima at  $\left(\frac{\pi}{2}, -1\right)$  and  $\left(\frac{3\pi}{2}, -1\right)$ .

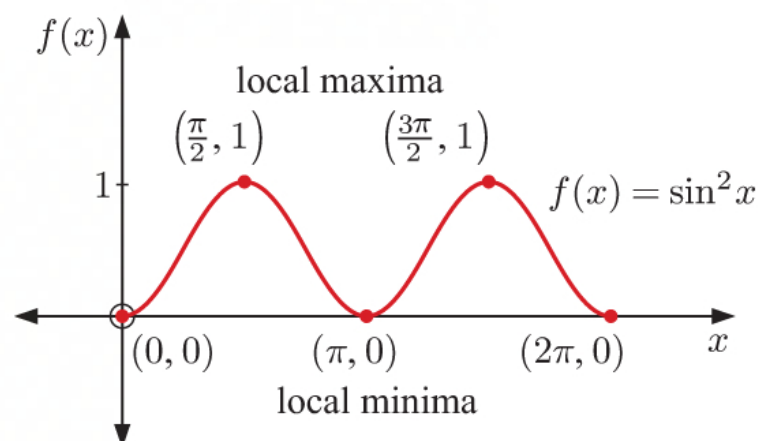


$$\begin{aligned}
 \text{c} \quad & f(x) = \sin^2 x = (\sin x)^2, \quad 0 \leq x \leq 2\pi \\
 \therefore & f'(x) = 2 \sin x (\cos x) \quad \{\text{chain rule}\} \\
 & = \sin 2x \quad \{\sin 2x = 2 \sin x \cos x\} \\
 & f'(x) = 0 \quad \text{when} \quad \sin 2x = 0 \\
 & \quad \therefore 2x = 0, \pi, 2\pi, 3\pi, 4\pi \\
 & \quad \therefore x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi
 \end{aligned}$$

So,  $f'(x)$  has sign diagram:



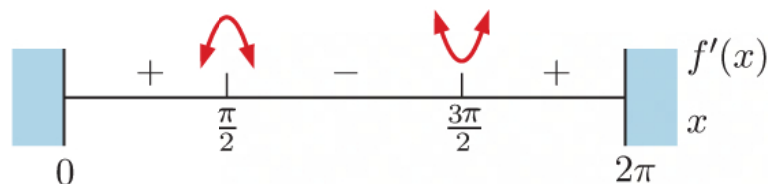
$$\begin{aligned}
 f(0) &= (\sin 0)^2 = 0^2 = 0, \\
 f\left(\frac{\pi}{2}\right) &= \left(\sin \frac{\pi}{2}\right)^2 = 1^2 = 1, \\
 f(\pi) &= (\sin \pi)^2 = 0^2 = 0, \\
 f\left(\frac{3\pi}{2}\right) &= \left(\sin \frac{3\pi}{2}\right)^2 = (-1)^2 = 1, \\
 f(2\pi) &= (\sin 2\pi)^2 = 0^2 = 0
 \end{aligned}$$



$\therefore$  there are local minima at  $(0, 0)$ ,  $(\pi, 0)$ , and  $(2\pi, 0)$ , and local maxima at  $(\frac{\pi}{2}, 1)$  and  $(\frac{3\pi}{2}, 1)$ .

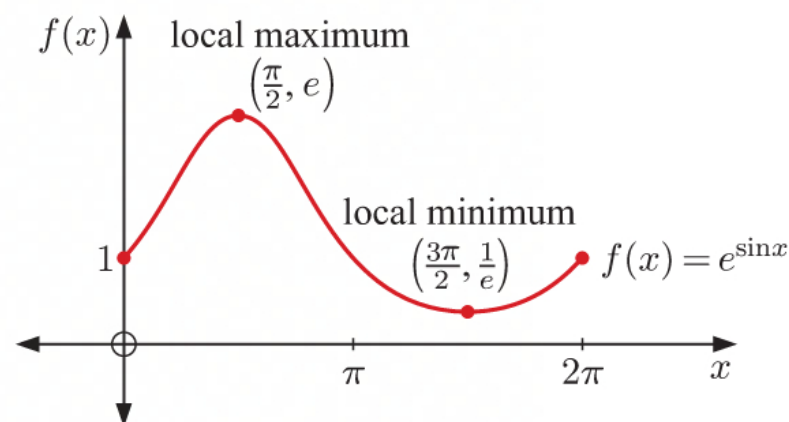
$$\begin{aligned}
 \text{d} \quad & f(x) = e^{\sin x}, \quad 0 \leq x \leq 2\pi \\
 \therefore & f'(x) = e^{\sin x} \cos x \quad \text{where } e^{\sin x} \text{ is positive for all } x \\
 \text{So, } & f'(x) = 0 \quad \text{when} \quad \cos x = 0 \\
 & \quad \therefore x = \frac{\pi}{2}, \frac{3\pi}{2}
 \end{aligned}$$

So,  $f'(x)$  has sign diagram:



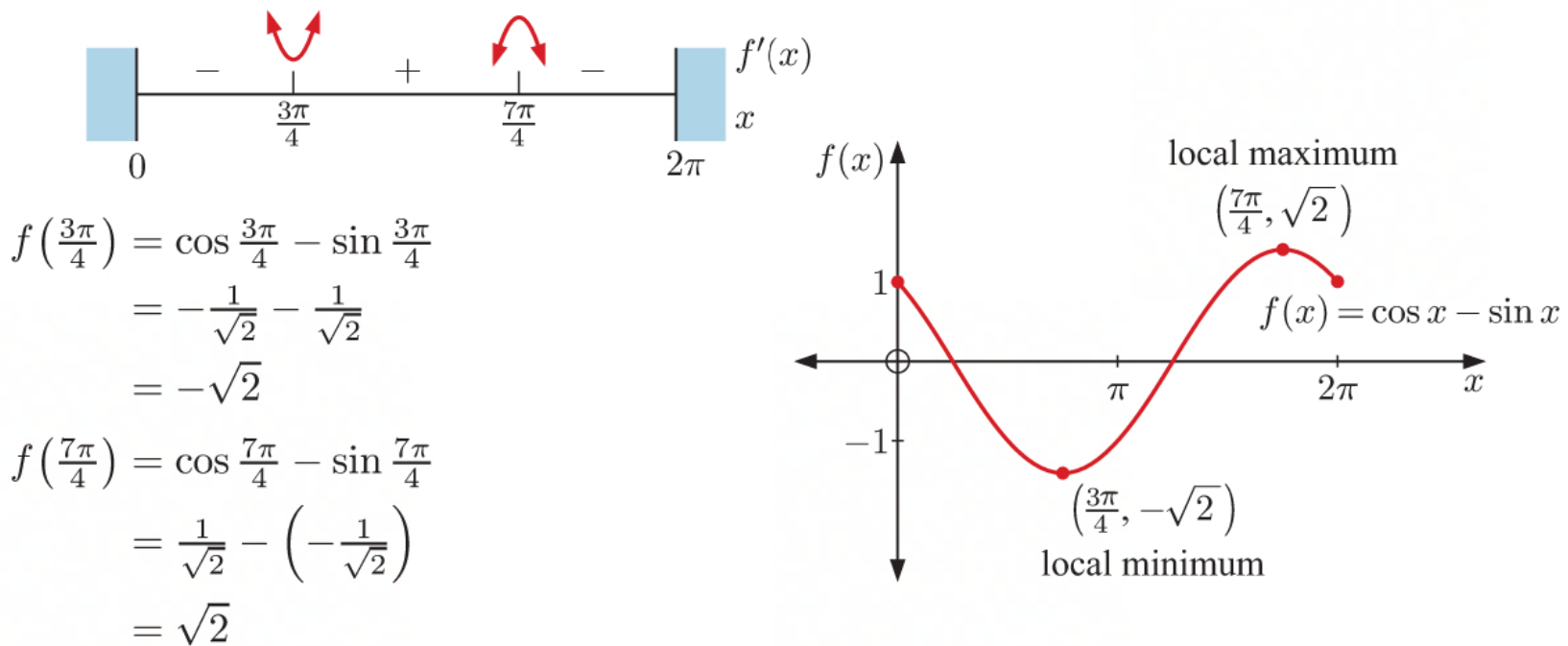
$$\begin{aligned}
 f\left(\frac{\pi}{2}\right) &= e^{\sin \frac{\pi}{2}} = e^1 = e, \\
 f\left(\frac{3\pi}{2}\right) &= e^{\sin \frac{3\pi}{2}} = e^{-1} = \frac{1}{e}
 \end{aligned}$$

$\therefore$  there is a local maximum at  $(\frac{\pi}{2}, e)$ ,  
and a local minimum at  $(\frac{3\pi}{2}, \frac{1}{e})$ .



$$\begin{aligned}
 \text{e} \quad f(x) &= \cos x - \sin x, \quad 0 \leq x \leq 2\pi \\
 \therefore f'(x) &= -\sin x - \cos x \\
 &= -(\sin x + \cos x) \\
 f'(x) = 0 &\text{ when } \sin x + \cos x = 0 \\
 &\therefore \sin x = -\cos x \\
 &\therefore \tan x = -1 \\
 &\therefore x = \frac{3\pi}{4}, \frac{7\pi}{4}
 \end{aligned}$$

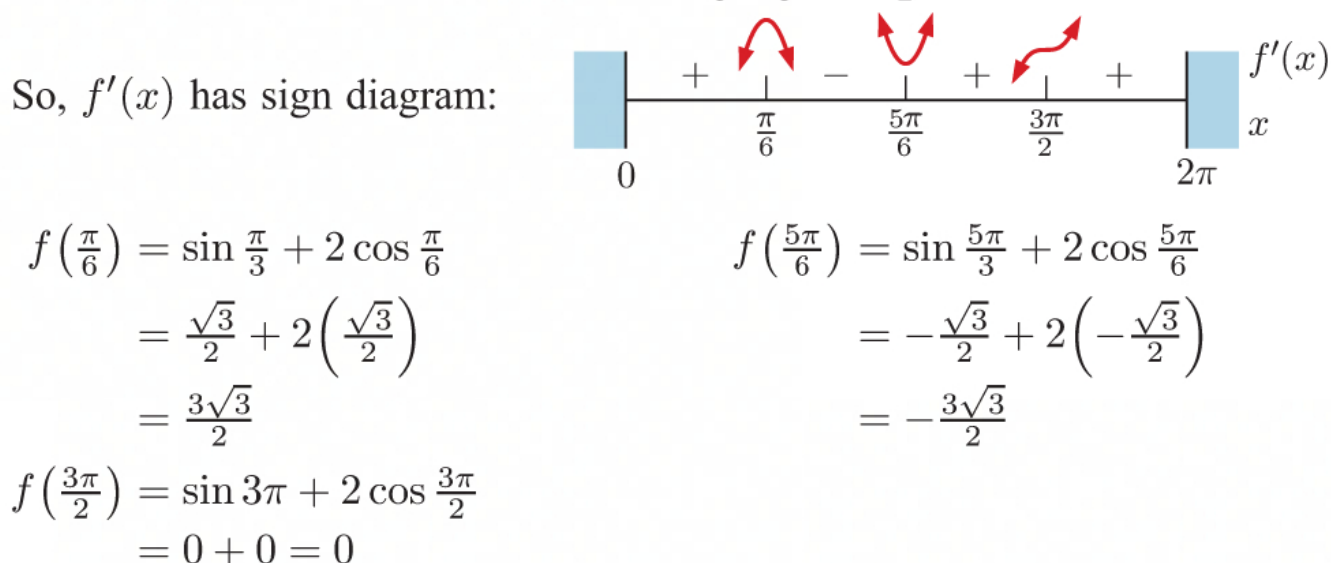
So,  $f'(x)$  has sign diagram:



$\therefore$  there is a local minimum at  $(\frac{3\pi}{4}, -\sqrt{2})$ , and a local maximum at  $(\frac{7\pi}{4}, \sqrt{2})$ .

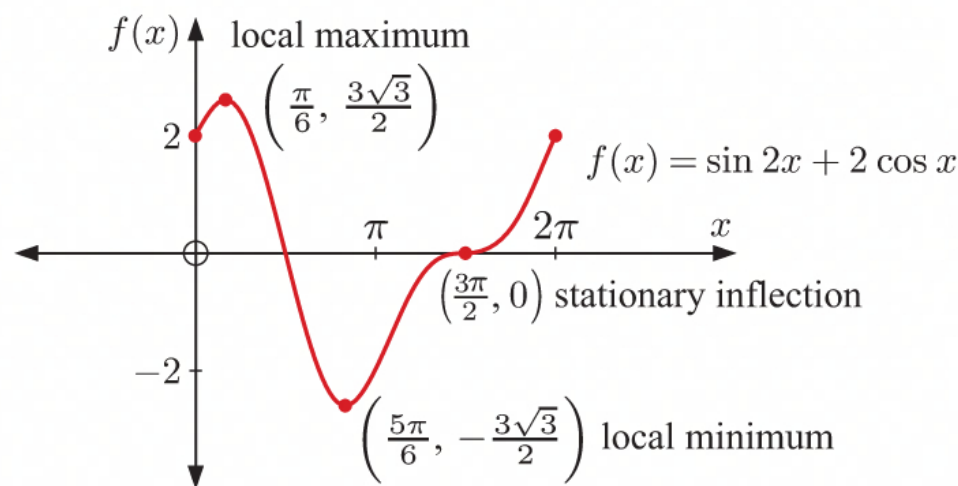
$$\begin{aligned}
 \text{f} \quad f(x) &= \sin 2x + 2 \cos x, \quad 0 \leq x \leq 2\pi \\
 \therefore f'(x) &= 2 \cos 2x - 2 \sin x \\
 f'(x) = 0 &\text{ when } 2 \cos 2x - 2 \sin x = 0 \\
 &\therefore 2 \sin x = 2 \cos 2x \\
 &\therefore \sin x = \cos 2x \\
 &\therefore \sin x = 1 - 2 \sin^2 x \quad \{\text{double angle formula}\} \\
 &\therefore 2 \sin^2 x + \sin x - 1 = 0 \\
 &\therefore (2 \sin x - 1)(\sin x + 1) = 0 \quad \{\text{compare } 2a^2 + a - 1 = (2a - 1)(a + 1)\} \\
 &\therefore \sin x = \frac{1}{2} \text{ or } -1 \\
 &\therefore x = \frac{\pi}{6}, \frac{5\pi}{6}, \text{ or } \frac{3\pi}{2}
 \end{aligned}$$

So,  $f'(x)$  has sign diagram:



$\therefore$  there is a local maximum at  $(\frac{\pi}{6}, \frac{3\sqrt{3}}{2})$ , a local minimum at  $(\frac{5\pi}{6}, -\frac{3\sqrt{3}}{2})$ , and a stationary inflection at  $(\frac{3\pi}{2}, 0)$ .





**14**  $P(x) = ax^3 + bx^2 + cx + d$   
 $\therefore P'(x) = 3ax^2 + 2bx + c \quad \dots (1)$

Now  $(0, 2)$  lies on  $P(x)$ , so  $P(0) = 2$   
 $\therefore a(0) + b(0) + c(0) + d = 2$   
 $\therefore d = 2$

The tangent at  $(0, 2)$  is  $y = 9x + 2$ , so  $P'(0) = 9$   
 $\therefore 3a(0) + 2b(0) + c = 9$   
 $\therefore c = 9 \quad \dots (2)$

There is a stationary point at  $(-1, -7)$ , so  $P'(-1) = 0$ .

$\therefore 3a(-1)^2 + 2b(-1) + c = 0 \quad \{\text{using (1)}\}$   
 $\therefore 3a - 2b + c = 0$

So, using (2),  $3a - 2b = -9 \quad \dots (3)$

Finally,  $(-1, -7)$  lies on  $P(x)$

$\therefore a(-1)^3 + b(-1)^2 + c(-1) + d = -7$   
 $\therefore -a + b - 9 + 2 = -7$   
 $\therefore b - a = 0$   
 $\therefore a = b$

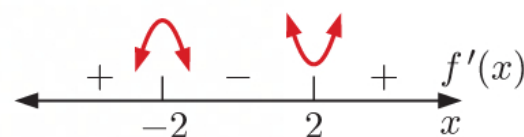
So, using (3),  $3a - 2a = -9$   
 $\therefore a = -9$   
 $\therefore a = b = -9$

$\therefore P(x) = -9x^3 - 9x^2 + 9x + 2$

**15 a** Let  $f(x) = x^3 - 12x - 2$ , for  $-3 \leq x \leq 5$   
 $\therefore f'(x) = 3x^2 - 12$   
 $= 3(x^2 - 4)$   
 $= 3(x + 2)(x - 2)$

which is 0 when  $x = -2$  or  $2$

The sign diagram of  $f'(x)$  is:



$\therefore$  there is a local maximum at  $x = -2$ , and a local minimum at  $x = 2$ .

Critical value ( $x$ )	$f(x)$
-3 (end point)	7
-2 (local maximum)	14
2 (local minimum)	-18
5 (end point)	63

The greatest of these values is 63 when  $x = 5$ .

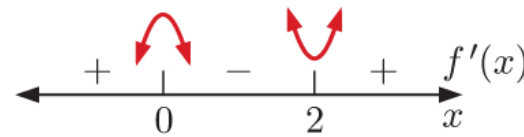
The least of these values is -18 when  $x = 2$ .

**b** Let  $f(x) = 4 - 3x^2 + x^3$ , for  $-2 \leq x \leq 3$

$$\begin{aligned}\therefore f'(x) &= -6x + 3x^2 \\ &= 3x(x - 2)\end{aligned}$$

which is 0 when  $x = 0$  or 2

The sign diagram of  $f'(x)$  is:



$\therefore$  there is a local maximum at  $x = 0$ , and a local minimum at  $x = 2$ .

Critical value ( $x$ )	$f(x)$
-2 (end point)	-16
0 (local maximum)	4
2 (local minimum)	0
3 (end point)	4

The greatest of these values is 4 when  $x = 0$  or  $x = 3$ .

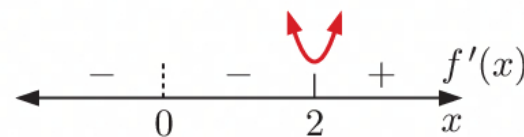
The least of these values is -16 when  $x = -2$ .

**c** Let  $f(x) = x^2 + \frac{16}{x} = x^2 + 16x^{-1}$ , for  $1 \leq x \leq 4$

$$\begin{aligned}\therefore f'(x) &= 2x - 16x^{-2} \\ &= 2x - \frac{16}{x^2} \\ &= \frac{2x^3 - 16}{x^2} \\ &= \frac{2(x^3 - 8)}{x^2}\end{aligned}$$

which is 0 when  $x = 2$

The sign diagram of  $f'(x)$  is:



$\therefore$  there is a local minimum at  $x = 2$ .

Critical value ( $x$ )	$f(x)$
1 (end point)	17
2 (local minimum)	12
4 (end point)	20

The greatest of these values is 20 when  $x = 4$ .

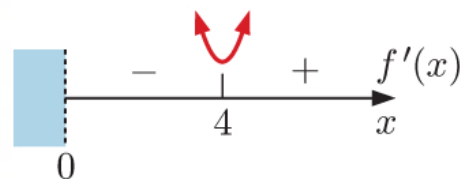
The least of these values is 12 when  $x = 2$ .

**d** Let  $f(x) = x - 4\sqrt{x} = x - 4x^{\frac{1}{2}}$ , for  $0 \leq x \leq 5$

$$\begin{aligned}\therefore f'(x) &= 1 - 2x^{-\frac{1}{2}} \\ &= 1 - \frac{2}{\sqrt{x}} \\ &= \frac{\sqrt{x} - 2}{\sqrt{x}}\end{aligned}$$

which is 0 when  $x = 4$

The sign diagram of  $f'(x)$  is:



$\therefore$  there is a local minimum at  $x = 4$ .

Critical value ( $x$ )	$f(x)$
0 (end point)	0
4 (local minimum)	-4
5 (end point)	$\approx -3.94$

The greatest of these values is 0 when  $x = 0$ .

The least of these values is -4 when  $x = 4$ .

**16**

$$y = 4e^{-x} \sin x$$

$$\begin{aligned}\therefore \frac{dy}{dx} &= -4e^{-x} \sin x + 4e^{-x} \cos x \quad \{\text{product rule}\} \\ &= 4e^{-x}(\cos x - \sin x) \quad \text{where } 4e^{-x} \text{ is positive for all } x\end{aligned}$$

$$\text{So, } \frac{dy}{dx} = 0 \text{ when } \cos x - \sin x = 0$$

$$\therefore \cos x = \sin x$$

$$\therefore \tan x = 1$$

$$\therefore x = \frac{\pi}{4} + k\pi, \quad k \in \mathbb{Z}$$



$\therefore y = 4e^{-x} \sin x$  has a local maximum when  $x = \frac{\pi}{4}$ .

**17**

**a**  $f(x) = \sin x \cos 2x$ , for  $0 \leq x \leq \pi$

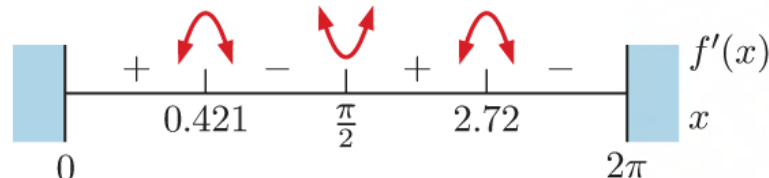
$$\begin{aligned}\therefore f'(x) &= \cos x \cos 2x + \sin x(-2 \sin 2x) \quad \{\text{product rule}\} \\ &= \cos x \cos 2x - 2 \sin x \sin 2x \\ &= \cos x(2 \cos^2 x - 1) - 2 \sin x(2 \sin x \cos x) \\ &= 2 \cos^3 x - \cos x - 4 \sin^2 x \cos x \\ &= 2 \cos^3 x - \cos x - 4(1 - \cos^2 x) \cos x \\ &= 2 \cos^3 x - \cos x - 4 \cos x + 4 \cos^3 x \\ &= 6 \cos^3 x - 5 \cos x\end{aligned}$$

$$\begin{aligned} \text{b } f'(x) &= 6 \cos^3 x - 5 \cos x \\ &= \cos x(6 \cos^2 x - 5) \end{aligned}$$

$$\begin{aligned} \text{So } f'(x) = 0 \text{ when } \cos x = 0 \text{ or } 6 \cos^2 x - 5 &= 0 \\ \therefore 6 \cos^2 x &= 5 \\ \therefore \cos^2 x &= \frac{5}{6} \\ \therefore \cos x &= \pm \sqrt{\frac{5}{6}} \end{aligned}$$

$$\begin{aligned} \text{c } \cos x &= 0 \text{ or } \pm \sqrt{\frac{5}{6}} \\ \therefore x &= \frac{\pi}{2}, x \approx 0.421, 2.72 \end{aligned}$$

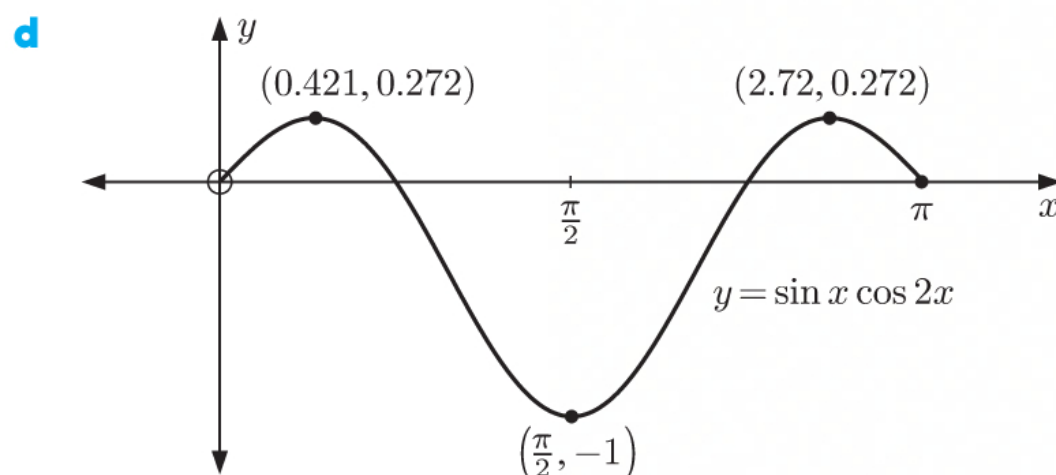
The sign diagram of  $f'(x)$  is:



$$\begin{aligned} f\left(\frac{\pi}{2}\right) &= \sin \frac{\pi}{2} \cos\left(2\left(\frac{\pi}{2}\right)\right) \\ &= \sin \frac{\pi}{2} \cos \pi \\ &= 1 \times (-1) \\ &= -1 \end{aligned}$$

$$\begin{aligned} f(0.421) &= \sin 0.421 \cos(2 \times 0.421) \\ &\approx 0.272 \\ f(2.72) &= \sin 2.72 \cos(2 \times 2.72) \\ &\approx 0.272 \end{aligned}$$

$\therefore$  there are local maxima at  $(0.421, 0.272)$ ,  $(2.72, 0.272)$ , and a local minimum at  $(\frac{\pi}{2}, -1)$ .



$$\begin{aligned} \text{18 } f(t) &= ate^{bt^2} \\ \therefore f'(t) &= ae^{bt^2} + ate^{bt^2}(2bt) \quad \{\text{product rule}\} \\ &= ae^{bt^2} + 2abt^2e^{bt^2} \\ &= ae^{bt^2}(1 + 2bt^2) \end{aligned}$$

If  $f(t)$  has maximum value 1 when  $t = 2$ , then

$$\begin{aligned} f(2) &= 1 & \text{and} & & f'(2) &= 0 \\ \therefore a(2)e^{b(2)^2} &= 1 & \therefore & & ae^{b(2)^2}(1 + 2b(2)^2) &= 0 \\ \therefore 2ae^{4b} &= 1 & \therefore & & ae^{4b}(1 + 8b) &= 0 \quad \dots (2) \\ \therefore ae^{4b} &= \frac{1}{2} \quad \dots (1) \end{aligned}$$

$$\begin{aligned} \text{Substituting (1) into (2) gives:} & \quad \frac{1}{2}(1 + 8b) = 0 \\ \therefore 1 + 8b &= 0 \\ \therefore 8b &= -1 \\ \therefore b &= -\frac{1}{8} \end{aligned}$$



Substituting  $b = -\frac{1}{8}$  into (1) gives:  $ae^{4(-\frac{1}{8})} = \frac{1}{2}$   
 $\therefore ae^{-\frac{1}{2}} = \frac{1}{2}$   
 $\therefore a = \frac{1}{2}e^{\frac{1}{2}} = \frac{\sqrt{e}}{2}$

So,  $a = \frac{\sqrt{e}}{2}$  and  $b = -\frac{1}{8}$ .

**19** Let  $f(x) = \frac{\ln x}{x}$

$$\therefore f'(x) = \frac{\left(\frac{1}{x}\right)x - \ln x(1)}{x^2} \quad \{\text{quotient rule}\}$$

$$= \frac{1 - \ln x}{x^2}$$

$$f'(x) = 0 \quad \text{when} \quad 1 - \ln x = 0$$

$$\therefore \ln x = 1$$

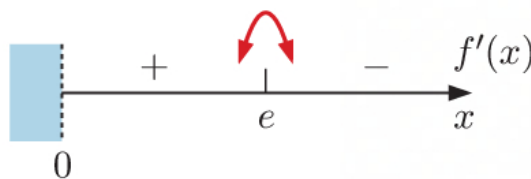
$$\therefore x = e$$

So, the sign diagram of  $f'(x)$  is:

$$f(e) = \frac{\ln e}{e} = \frac{1}{e}$$

$\therefore$  there is a local maximum at  $\left(e, \frac{1}{e}\right)$ .

$$\therefore f(x) \leq \frac{1}{e} \quad \text{for all } x > 0, \quad \text{and so} \quad \frac{\ln x}{x} \leq \frac{1}{e} \quad \text{for all } x > 0.$$



**20 a**  $f(x) = x - \ln x$

$$\therefore f'(x) = 1 - \frac{1}{x} = \frac{x-1}{x}$$

So, the sign diagram of  $f'(x)$  is:

$$f(1) = 1 - \ln 1$$

$$= 1 - 0$$

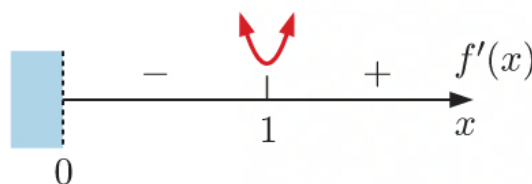
$$= 1$$

$\therefore f(x)$  has a local minimum at  $(1, 1)$ . This is the only turning point.

**b**  $f(x) \geq 1$  for all  $x > 0$

$$\therefore x - \ln x \geq 1$$

$$\therefore \ln x \leq x - 1 \quad \text{for all } x > 0$$



**21 a**  $f(x) = \sec x = \frac{1}{\cos x}$

$f(x)$  is undefined when  $\cos x = 0$

$$\therefore f(x) \text{ is undefined when } x = \frac{\pi}{2} \text{ or } \frac{3\pi}{2} \quad \{0 \leq x \leq 2\pi\}$$

**b**  $f'(x) = \sec x \tan x$

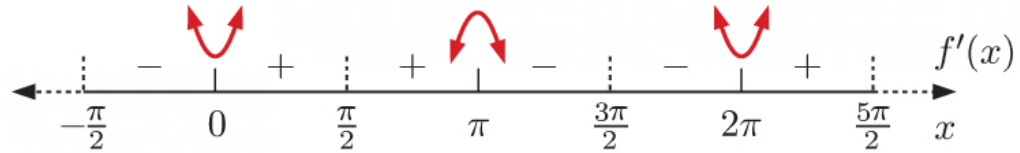
$$= \frac{1}{\cos x} \times \frac{\sin x}{\cos x} = \frac{\sin x}{\cos^2 x}$$

$$\therefore f'(x) = 0 \quad \text{when} \quad \sin x = 0$$

$$\therefore x = 0, \pi, \text{ or } 2\pi \quad \{0 \leq x \leq 2\pi\}$$

$$\cos^2 x \geq 0 \quad \text{for all } x \in \mathbb{R}$$

$\therefore$  the sign diagram of  $f'(x)$  is:



$$f(0) = \frac{1}{\cos 0} = 1$$

$$f(\pi) = \frac{1}{\cos \pi} = -1$$

$$f(2\pi) = \frac{1}{\cos 2\pi} = 1$$

$\therefore$  there is a local maximum at  $(\pi, -1)$ , and local minima at  $(0, 1)$  and  $(2\pi, 1)$ .

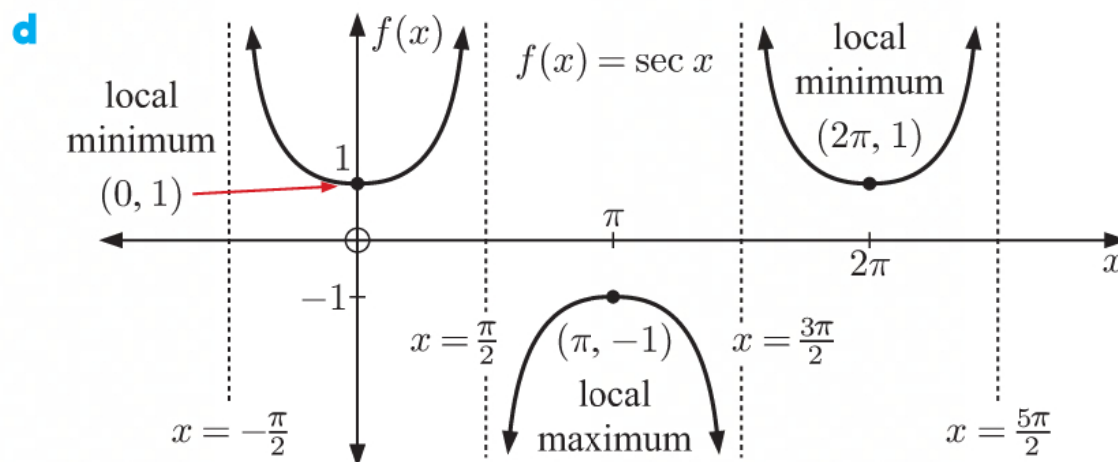
**c**  $f(x) = \sec x = \frac{1}{\cos x}, \quad x \neq \frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z}$

$\therefore f(x)$  and  $f(x + 2\pi)$  are defined for the same values.

$$\begin{aligned} \therefore f(x + 2\pi) &= \frac{1}{\cos(x + 2\pi)} \\ &= \frac{1}{\cos x \cos 2\pi - \sin x \sin 2\pi} \\ &= \frac{1}{\cos x \times 1 - \sin x \times 0} \\ &= \frac{1}{\cos x} \end{aligned}$$

$$\therefore f(x) = f(x + 2\pi) \quad \text{wherever } f(x) \text{ is defined}$$

$\therefore f(x) = \sec x$  has a period of  $2\pi$ .



**22 a**

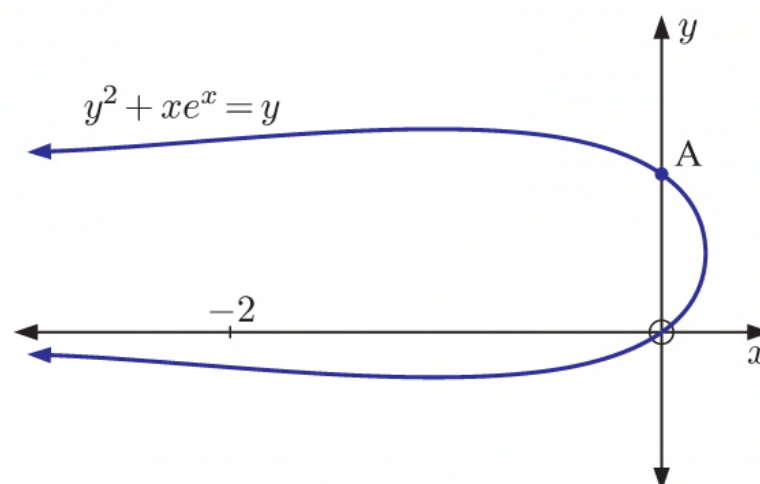
$$y^2 + xe^x = y$$

$$\therefore 2y \frac{dy}{dx} + (e^x + xe^x) = \frac{dy}{dx}$$

{product rule}

$$\therefore (1 - 2y) \frac{dy}{dx} = e^x + xe^x$$

$$\therefore \frac{dy}{dx} = \frac{e^x(x+1)}{1-2y}$$



**b** When  $x = 0$ ,  $y^2 + 0 = y$   
 $\therefore y^2 - y = 0$   
 $\therefore y(y - 1) = 0$   
 $\therefore y = 0 \text{ or } 1$

$\therefore$  A has coordinates  $(0, 1)$ .

At the point  $A(0, 1)$ ,  $\frac{dy}{dx} = \frac{e^0(0+1)}{1-2(1)} = -1$

$\therefore$  the gradient of the normal to the graph at A is 1.

$\therefore$  the equation of the normal at A is  $y = 1(x - 0) + 1$   
 $\therefore y = x + 1$

**c** The stationary points of the curve occur when  $\frac{dy}{dx} = 0$   
 $\therefore e^x(x+1) = 0$   
 $\therefore x = -1 \quad \{e^x > 0 \text{ for all } x\}$

When  $x = -1$ ,  $y^2 - e^{-1} = y$

$\therefore y^2 - y - \frac{1}{e} = 0$

$\therefore y = \frac{1 \pm \sqrt{(-1)^2 - 4(1)\left(-\frac{1}{e}\right)}}{2(1)} \quad \{\text{quadratic formula}\}$   
 $= \frac{1 \pm \sqrt{1 + \frac{4}{e}}}{2}$

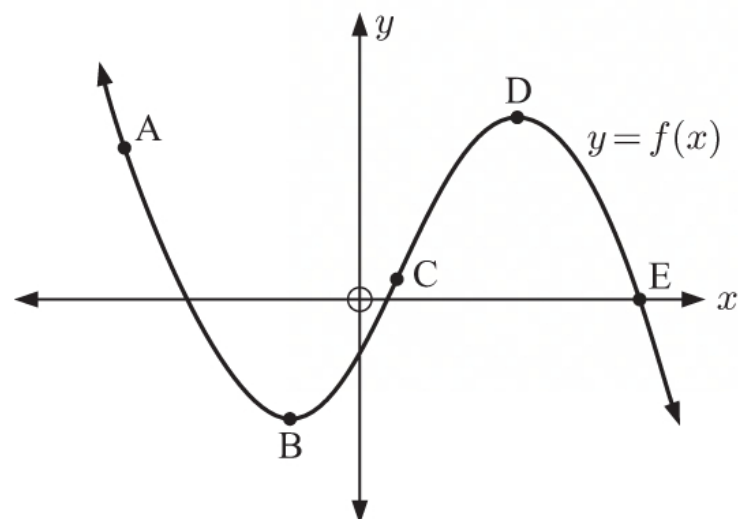
The exact coordinates of the stationary points are

$\left(-1, \frac{1 + \sqrt{1 + \frac{4}{e}}}{2}\right)$  and  $\left(-1, \frac{1 - \sqrt{1 + \frac{4}{e}}}{2}\right)$ .

## EXERCISE 18E

**1 a**

Point	$f(x)$	$f'(x)$	$f''(x)$
A	+	−	+
B	−	0	+
C	+	+	0
D	+	0	−
E	0	−	−



**b** B is a local minimum, D is a local maximum.

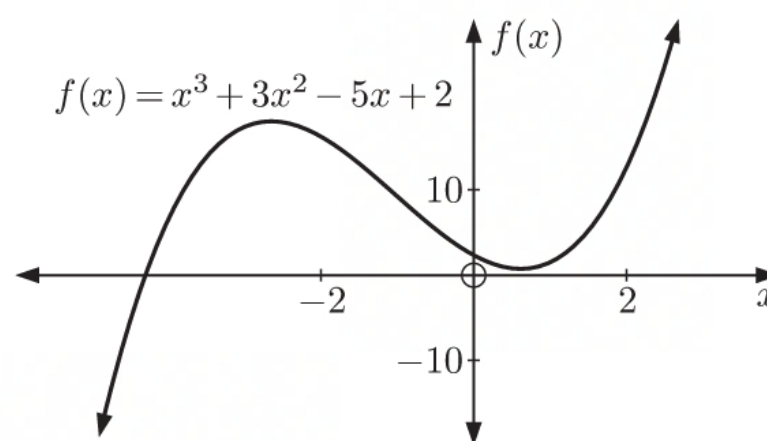
**c** The shape of  $y = f(x)$  changes at C.

**2 a**  $f(x) = x^3 + 3x^2 - 5x + 2$   
 $\therefore f'(x) = 3x^2 + 6x - 5$   
 $\therefore f''(x) = 6x + 6$


**b**  $f''(x)$  has sign diagram:




- c i** The curve is concave up for  $x \geq -1$ .  
**ii** The curve is concave down for  $x \leq -1$ .




**3 a**  $y = 2x^2 - 3x + 4$  is a quadratic with  $a = 2 > 0$ .

$\therefore$  the quadratic has shape   
 $\therefore$  the quadratic is concave up.


**c**  $y = -4 - x^2 + 6x$  is a quadratic with  $a = -1 < 0$ .

$\therefore$  the quadratic has shape   
 $\therefore$  the quadratic is concave down.

**b**  $y = -2(x-3)(x+1)$  is a quadratic with  $a = -2 < 0$ .

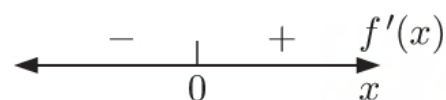
$\therefore$  the quadratic has shape   
 $\therefore$  the quadratic is concave down.

**d**  $y = (5-x)(1-2x)$   
 $= 5 - 10x - x + 2x^2$   
 $= 2x^2 - 11x + 5$

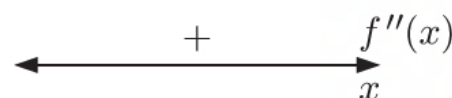
is a quadratic with  $a = 2 > 0$ .  
 $\therefore$  the quadratic has shape   
 $\therefore$  the quadratic is concave up.

**4 a**  $f(x) = x^2 + 1$

$\therefore f'(x) = 2x$  which has sign diagram:



$\therefore f''(x) = 2$  which has sign diagram:



**i**  $f(x)$  is increasing for  $x \geq 0$ .

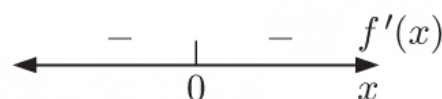
**ii**  $f(x)$  is decreasing for  $x \leq 0$ .

**iii**  $f(x)$  is concave upwards for all  $x \in \mathbb{R}$ .

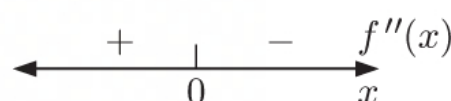
**iv**  $f(x)$  is never concave downwards.

**b**  $f(x) = -x^3$

$\therefore f'(x) = -3x^2$  which has sign diagram:



$\therefore f''(x) = -6x$  which has sign diagram:



**i**  $f(x)$  is never increasing.

**ii**  $f(x)$  is decreasing for all  $x \in \mathbb{R}$ .

**iii**  $f(x)$  is concave upwards for  $x \leq 0$ .

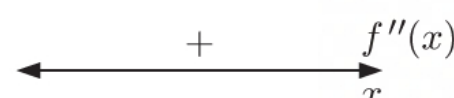
**iv**  $f(x)$  is concave downwards for  $x \geq 0$ .

**c**  $f(x) = e^x$

$\therefore f'(x) = e^x$  which has sign diagram:



$\therefore f''(x) = e^x$  which has sign diagram:



**i**  $f(x)$  is increasing for all  $x \in \mathbb{R}$ .

**ii**  $f(x)$  is never decreasing.

**iii**  $f(x)$  is concave upwards for all  $x \in \mathbb{R}$ .

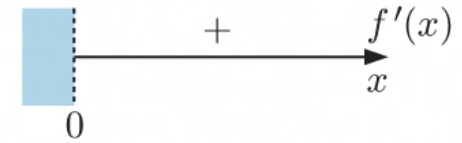
**iv**  $f(x)$  is never concave downwards.



**d**  $f(x) = \sqrt{x} - 2 = x^{\frac{1}{2}} - 2$

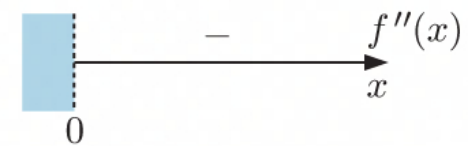
$\therefore f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$

which has sign diagram:



$\therefore f''(x) = -\frac{1}{4}x^{-\frac{3}{2}} = -\frac{1}{4x\sqrt{x}}$

which has sign diagram:



**i**  $f(x)$  is increasing for  $x > 0$ .

**ii**  $f(x)$  is never decreasing.

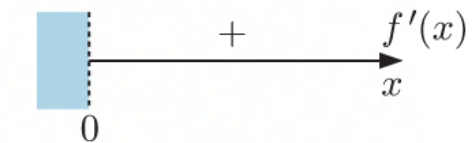
**iii**  $f(x)$  is never concave upwards.

**iv**  $f(x)$  is concave downwards for  $x > 0$ .

**e**  $f(x) = -\frac{1}{\sqrt{x}} = -x^{-\frac{1}{2}}$

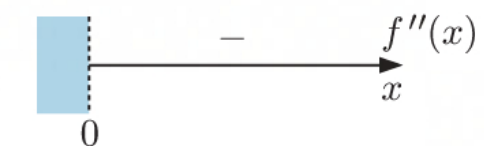
$\therefore f'(x) = \frac{1}{2}x^{-\frac{3}{2}} = \frac{1}{2x\sqrt{x}}$

which has sign diagram:



$\therefore f''(x) = -\frac{3}{4}x^{-\frac{5}{2}} = -\frac{3}{4x^2\sqrt{x}}$

which has sign diagram:



**i**  $f(x)$  is increasing for  $x > 0$ .

**ii**  $f(x)$  is never decreasing.

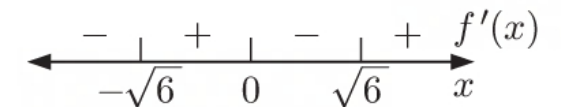
**iii**  $f(x)$  is never concave upwards.

**iv**  $f(x)$  is concave downwards for  $x > 0$ .

**f**  $f(x) = x^4 - 12x^2$

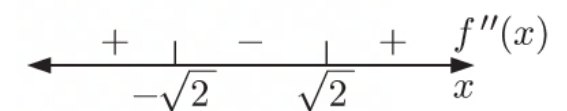
$\therefore f'(x) = 4x^3 - 24x$   
 $= 4x(x^2 - 6)$

$= 4x(x + \sqrt{6})(x - \sqrt{6})$  which has sign diagram:



$\therefore f''(x) = 12x^2 - 24$   
 $= 12(x^2 - 2)$

$= 12(x + \sqrt{2})(x - \sqrt{2})$  which has sign diagram:



**i**  $f(x)$  is increasing for  $-\sqrt{6} \leq x \leq 0$  and  $x \geq \sqrt{6}$ .

**ii**  $f(x)$  is decreasing for  $x \leq -\sqrt{6}$  and  $0 \leq x \leq \sqrt{6}$ .

**iii**  $f(x)$  is concave upwards for all  $x \leq -\sqrt{2}$  and  $x \geq \sqrt{2}$ .

**iv**  $f(x)$  is concave downwards for  $-\sqrt{2} \leq x \leq \sqrt{2}$ .

**5**  $f(x) = \ln(2x - 1) - 3$

**a** The  $x$ -intercept occurs when  $y = 0$

$\therefore \ln(2x - 1) - 3 = 0$

$\therefore \ln(2x - 1) = 3$

$\therefore 2x - 1 = e^3$

$\therefore 2x = e^3 + 1$

$\therefore x = \frac{e^3 + 1}{2} \approx 10.5$

$\therefore$  the  $x$ -intercept is  $\frac{e^3 + 1}{2}$ .

**b**  $f(0)$  cannot be found as  $\ln(-1)$  is not defined.  
 $\therefore$  there is no  $y$ -intercept.

**c**  $f(x) = \ln(2x - 1) - 3$  is defined when  $2x - 1 > 0$   
 $\therefore 2x > 1$   
 $\therefore x > \frac{1}{2}$

$\therefore$  domain of  $f = \{x \mid x > \frac{1}{2}\}$

**d**  $f'(x) = \frac{2}{2x - 1}$

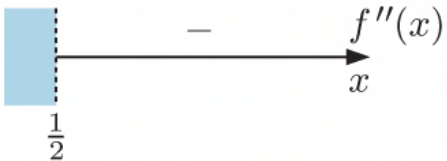
$\therefore f'(1) = \frac{2}{2(1) - 1} = 2$

$\therefore$  the tangent to the curve at  $x = 1$  has gradient 2.

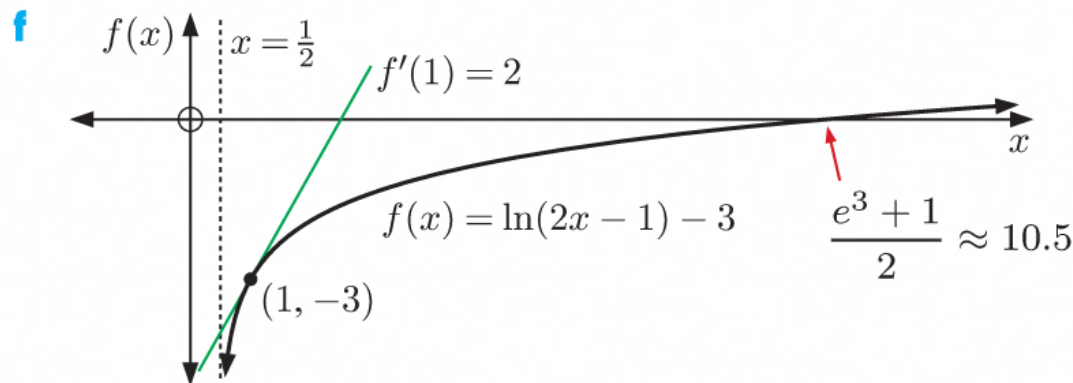
**e**  $f'(x) = 2(2x - 1)^{-1}$

$\therefore f''(x) = -2(2x - 1)^{-2}(2) \quad \{\text{chain rule}\}$   
 $= -\frac{4}{(2x - 1)^2}, \quad x > \frac{1}{2}$

$\therefore f''(x)$  has sign diagram:

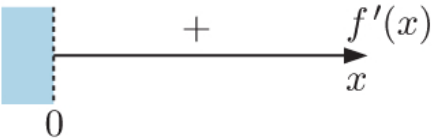


$\therefore f''(x) < 0$  for all  $x > \frac{1}{2}$ , so  $f(x)$  is concave down for all  $x$  in the domain of  $f$ .



**6 a**  $f(x) = \ln x$  is defined when  $x > 0$ .

**b**  $f'(x) = \frac{1}{x}$  which has sign diagram:



$\therefore f(x)$  is increasing for  $x > 0$ .

$f'(x) = x^{-1}$

$\therefore f''(x) = -x^{-2}$

$= -\frac{1}{x^2}$  which has sign diagram:



$\therefore f(x)$  is concave down for  $x > 0$ .

**c** When  $y = 1$ ,  $\ln x = 1$   
 $\therefore x = e$

So, the point of contact is  $(e, 1)$ .

Now  $f'(e) = \frac{1}{e}$ .

$\therefore$  the normal at  $(e, 1)$  has gradient  $-e$ .

$\therefore$  the normal has equation  $y = -e(x - e) + 1$

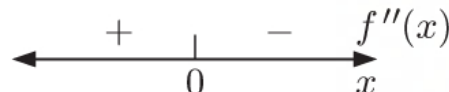
$$\therefore y = -ex + e^2 + 1$$

$$\therefore ex + y = e^2 + 1$$

**7 a**  $f(x) = -x^3 + 3x - 2$

$$\therefore f'(x) = -3x^2 + 3$$

$$\therefore f''(x) = -6x \quad \text{which has sign diagram:}$$

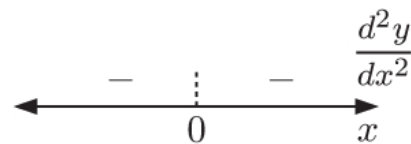


The curve is concave up for  $x \leq 0$ , and concave down for  $x \geq 0$ .

**b**  $y = -\frac{4}{x^2} = -4x^{-2}$

$$\therefore \frac{dy}{dx} = 8x^{-3}$$

$$\therefore \frac{d^2y}{dx^2} = -24x^{-4} \quad \text{which has sign diagram:}$$



The curve is never concave up, and is concave down for all  $x \neq 0$ .

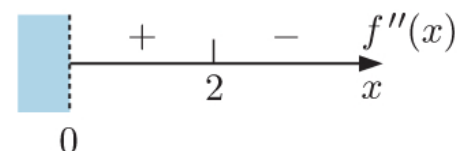
**c**  $f(x) = \frac{1}{x} + \ln x = x^{-1} + \ln x, \quad x > 0$

$$\therefore f'(x) = -x^{-2} + \frac{1}{x} = \frac{x-1}{x^2}$$

$$\therefore f''(x) = \frac{(1)(x^2) - (x-1)(2x)}{(x^2)^2} \quad \{\text{quotient rule}\}$$

$$= \frac{x^2 - 2x^2 + 2x}{x^4}$$

$$= \frac{2-x}{x^3}, \quad x > 0 \quad \text{which has sign diagram:}$$



The curve is concave up for  $0 < x \leq 2$ , and concave down for  $x \geq 2$ .

**d**  $f(x) = \frac{3-x}{x+2}$

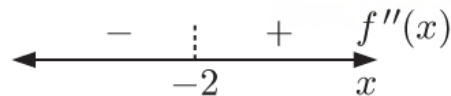
$$\therefore f'(x) = \frac{(-1)(x+2) - (3-x)(1)}{(x+2)^2} \quad \{\text{quotient rule}\}$$

$$= \frac{-x-2-3+x}{(x+2)^2}$$

$$= -\frac{5}{(x+2)^2} = -5(x+2)^{-2}$$

$$\therefore f''(x) = -5(-2)(x+2)^{-3}(1) \quad \{\text{chain rule}\}$$

$$= \frac{10}{(x+2)^3} \quad \text{which has sign diagram:}$$



The curve is concave up for  $x > -2$ , and concave down for  $x < -2$ .

**e**  $y = \frac{x^2 + x - 3}{x + 1}$

$$\therefore \frac{dy}{dx} = \frac{(2x + 1)(x + 1) - (x^2 + x - 3)(1)}{(x + 1)^2} \quad \{\text{quotient rule}\}$$

$$= \frac{2x^2 + 2x + x + 1 - x^2 - x + 3}{(x + 1)^2}$$

$$= \frac{x^2 + 2x + 4}{(x + 1)^2}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{(2x + 2)(x + 1)^2 - (x^2 + 2x + 4)(2)(x + 1)(1)}{(x + 1)^4} \quad \{\text{quotient rule}\}$$

$$= \frac{(2x + 2)(x + 1) - 2(x^2 + 2x + 4)}{(x + 1)^3}$$

$$= \frac{2x^2 + 2x + 2x + 2 - 2x^2 - 4x - 8}{(x + 1)^3}$$

$$= -\frac{6}{(x + 1)^3} \quad \text{which has sign diagram:}$$

$\begin{array}{c} + \quad \quad - \\ \leftarrow \quad \quad \rightarrow \\ \quad \quad -1 \quad \quad x \end{array}$ 
 $\frac{d^2y}{dx^2}$

The curve is concave up for  $x < -1$ , and concave down for  $x > -1$ .

**f**  $f(x) = \frac{\ln x}{x - 2}, \quad x > 0, \quad x \neq 2$

$$\therefore f'(x) = \frac{\left(\frac{1}{x}\right)(x - 2) - (\ln x)(1)}{(x - 2)^2} \quad \{\text{quotient rule}\}$$

$$= \frac{1 - \frac{2}{x} - \ln x}{(x - 2)^2}$$

$$\therefore f''(x) = \frac{\left(\frac{2}{x^2} - \frac{1}{x}\right)(x - 2)^2 - \left(1 - \frac{2}{x} - \ln x\right)(2)(x - 2)(1)}{(x - 2)^4} \quad \{\text{quotient rule}\}$$

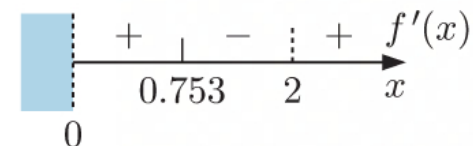
$$= \frac{\left(\frac{2}{x^2} - \frac{1}{x}\right)(x - 2) - 2\left(1 - \frac{2}{x} - \ln x\right)}{(x - 2)^3}$$

$$= \frac{\frac{2}{x} - \frac{4}{x^2} - 1 + \frac{2}{x} - 2 + \frac{4}{x} + 2 \ln x}{(x - 2)^3}$$

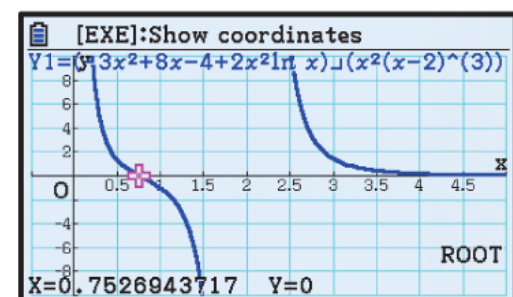
$$= \frac{-3 + \frac{8}{x} - \frac{4}{x^2} + 2 \ln x}{(x - 2)^3} \quad \times \frac{x^2}{x^2}$$

$$= \frac{-3x^2 + 8x - 4 + 2x^2 \ln x}{x^2(x - 2)^3}, \quad x > 0, \quad x \neq 2$$

We obtain the sign diagram of  $f''(x)$  using technology:

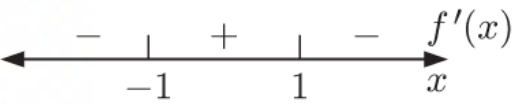


$\therefore f(x)$  is concave up for  $0 < x \leq 0.753$  and  $x > 2$ ,  
and concave down for  $0.753 \leq x < 2$ .

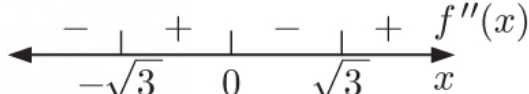




**8**  $f(x) = \frac{x}{x^2 + 1}$

$$\begin{aligned} \therefore f'(x) &= \frac{(1)(x^2 + 1) - (x)(2x)}{(x^2 + 1)^2} \quad \{\text{quotient rule}\} \\ &= \frac{x^2 + 1 - 2x^2}{(x^2 + 1)^2} \\ &= \frac{-x^2 + 1}{(x^2 + 1)^2} \\ &= \frac{-(x+1)(x-1)}{(x^2 + 1)^2} \quad \text{which has sign diagram:} \end{aligned}$$


$\therefore f''(x) = \frac{(-2x)(x^2 + 1)^2 - (-x^2 + 1)(2)(x^2 + 1)(2x)}{(x^2 + 1)^4} \quad \{\text{quotient rule}\}$

$$\begin{aligned} &= \frac{-2x(x^2 + 1) - 4x(-x^2 + 1)}{(x^2 + 1)^3} \\ &= \frac{-2x^3 - 2x + 4x^3 - 4x}{(x^2 + 1)^3} \\ &= \frac{2x(x^2 - 3)}{(x^2 + 1)^3} \quad \text{which has sign diagram:} \end{aligned}$$


From the sign diagram of  $f'(x)$ :

- a**  $f(x)$  is increasing for  $-1 \leq x \leq 1$ .
- b**  $f(x)$  is decreasing for  $x \leq -1$  and  $x \geq 1$ .

From the sign diagram of  $f''(x)$ :

- c**  $f(x)$  is concave up for  $-\sqrt{3} \leq x \leq 0$  and  $x \geq \sqrt{3}$ .
- d**  $f(x)$  is concave down for  $x \leq -\sqrt{3}$  and  $0 \leq x \leq \sqrt{3}$ .

**9 a**  $f(x) = \frac{e^x}{x} \neq 0$  since  $e^x \neq 0$  for all  $x$   $\therefore$  there is no  $x$ -intercept.

Also,  $f(x)$  is not defined when  $x = 0$   $\therefore$  there is no  $y$ -intercept.

$\therefore$  the graph of  $y = f(x)$  does not have any  $x$  or  $y$ -intercepts.

**b** As  $x \rightarrow \infty$ ,  $e^x \rightarrow \infty$  (at a much faster rate than  $x$ )

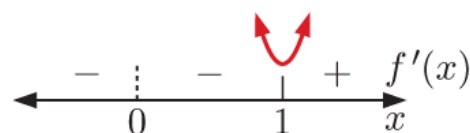
$$\therefore \text{ as } x \rightarrow \infty, f(x) = \frac{e^x}{x} \rightarrow \infty$$

As  $x \rightarrow -\infty$ ,  $e^x \rightarrow 0^+$

$$\therefore \text{ as } x \rightarrow -\infty, f(x) = \frac{e^x}{x} \rightarrow 0^- \quad \{f(x) < 0 \text{ for } x < 0\}$$

**c**  $f'(x) = \frac{e^x x - e^x(1)}{x^2} \quad \{\text{quotient rule}\}$

$$= \frac{e^x(x-1)}{x^2} \quad \text{which has sign diagram:}$$



$$\text{Now } f(1) = \frac{e^1}{1} = e$$

$\therefore (1, e)$  is a local minimum.

$$\begin{aligned}
 \text{d} \quad f'(x) &= \frac{e^x x - e^x}{x^2} \\
 &= \frac{e^x}{x} - \frac{e^x}{x^2} \\
 \therefore f''(x) &= \frac{e^x x - e^x(1)}{x^2} - \left( \frac{e^x x^2 - e^x(2x)}{(x^2)^2} \right) \quad \{\text{quotient rule twice}\} \\
 &= \frac{e^x x - e^x}{x^2} - \left( \frac{e^x x^2 - 2e^x x}{x^4} \right) \\
 &= \frac{e^x x - e^x}{x^2} - \left( \frac{e^x x - 2e^x}{x^3} \right) \\
 &= \frac{e^x x^2 - e^x x - (e^x x - 2e^x)}{x^3} \\
 &= \frac{e^x x^2 - e^x x - e^x x + 2e^x}{x^3} \\
 &= \frac{e^x x^2 - 2e^x x + 2e^x}{x^3} \\
 &= \frac{e^x(x^2 - 2x + 2)}{x^3}
 \end{aligned}$$

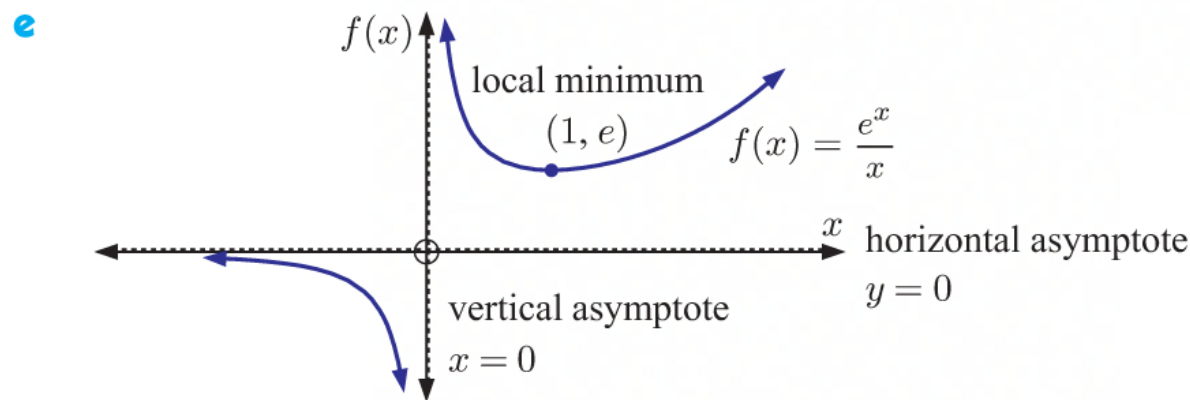
Now consider  $x^2 - 2x + 2$ :  $\Delta = (-2)^2 - 4(1)(2)$   
 $= -4 < 0$

$\therefore x^2 - 2x + 2$  has no real roots.

$\therefore f''(x)$  has sign diagram:  $\begin{array}{ccc} & - & + \\ & \vdots & \\ & 0 & \\ \leftarrow & & \rightarrow f''(x) \\ & x & \end{array}$

i  $f(x)$  is concave up for  $x > 0$ .

ii  $f(x)$  is concave down for  $x < 0$ .



f  $f(-1) = \frac{e^{-1}}{-1} = -\frac{1}{e}$

$\therefore$  the point of contact is  $\left(-1, -\frac{1}{e}\right)$ .

$$\begin{aligned}
 \text{Now } f'(-1) &= \frac{e^{-1}(-1-1)}{(-1)^2} \\
 &= -\frac{2}{e}
 \end{aligned}$$

So, the tangent has equation  $y = -\frac{2}{e}(x+1) - \frac{1}{e}$

$$\therefore ey = -2(x+1) - 1$$

$$\therefore ey = -2x - 2 - 1$$

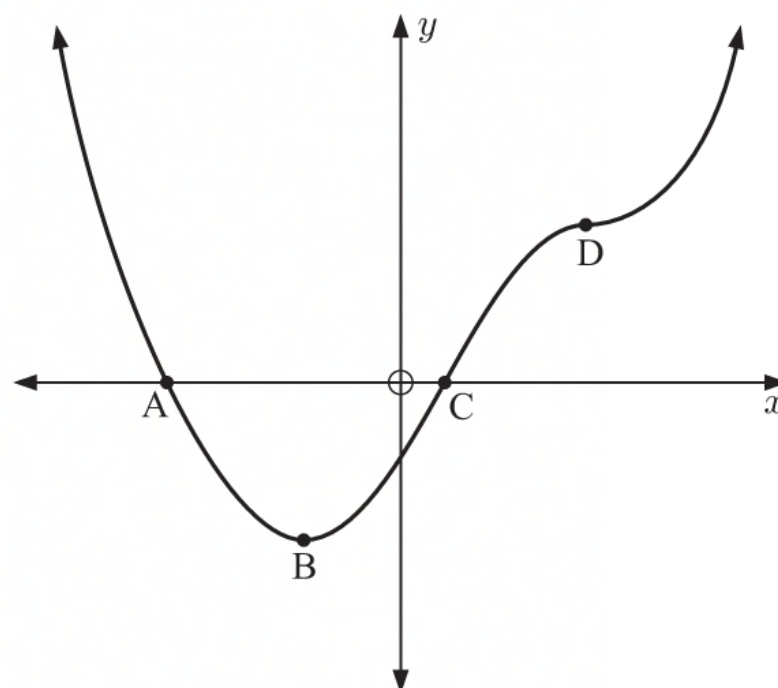
$$\therefore 2x + ey = -3$$

## EXERCISE 18F

1 a

Point	$f(x)$	$f'(x)$	$f''(x)$
A	0	−	+
B	−	0	+
C	0	+	0
D	+	0	0

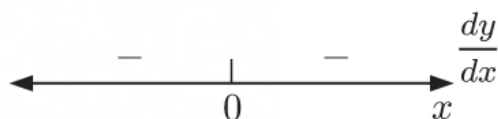
- b The turning point B is a local minimum.  
 c C is a non-stationary inflection point, and D is a stationary inflection point.



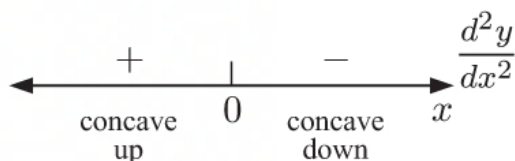
2 a

$$y = 2 - x^3$$

$$\therefore \frac{dy}{dx} = -3x^2$$



$$\therefore \frac{d^2y}{dx^2} = -6x$$



Since the sign of  $\frac{d^2y}{dx^2}$  changes at  $x = 0$ , this is a point of inflection.

When  $x = 0$ ,  $y = 2$  and  $\frac{dy}{dx} = 0$

$\therefore (0, 2)$  is a stationary inflection.

b

$$y = x^3 - 6x^2 + 9x + 1$$

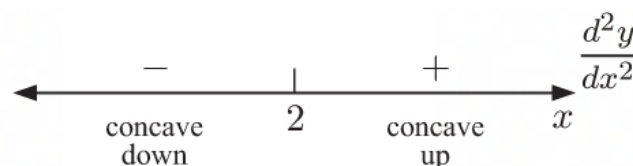
$$\therefore \frac{dy}{dx} = 3x^2 - 12x + 9$$

$$= 3(x^2 - 4x + 3)$$

$$= 3(x - 1)(x - 3)$$

$$\therefore \frac{d^2y}{dx^2} = 6x - 12$$

$$= 6(x - 2)$$



Since the sign of  $\frac{d^2y}{dx^2}$  changes about  $x = 2$ , this is a point of inflection.

When  $x = 2$ ,  $y = 2^3 - 6(2)^2 + 9(2) + 1$  and  $\frac{dy}{dx} = 3(1)(-1) \neq 0$   
 $= 8 - 24 + 18 + 1$   
 $= 3$

$\therefore (2, 3)$  is a non-stationary inflection.

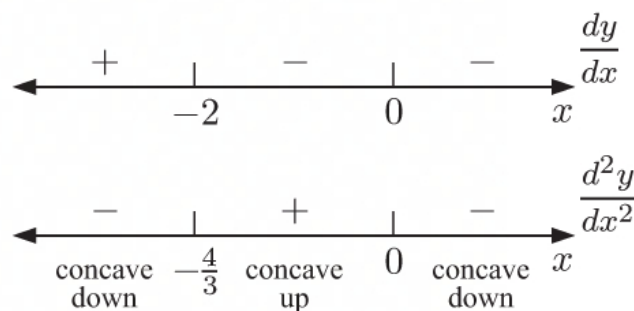
**c**  $y = -3x^4 - 8x^3 + 2$

$$\therefore \frac{dy}{dx} = -12x^3 - 24x^2$$

$$= -12x^2(x + 2)$$

$$\therefore \frac{d^2y}{dx^2} = -36x^2 - 48x$$

$$= -12x(3x + 4)$$



Since the signs of  $\frac{d^2y}{dx^2}$  change about  $x = -\frac{4}{3}$  and  $x = 0$ , both of these points are points of inflection.

When  $x = -\frac{4}{3}$ ,  $y = -3\left(-\frac{4}{3}\right)^4 - 8\left(-\frac{4}{3}\right)^3 + 2 = \frac{310}{27}$

and  $\frac{dy}{dx} = -12\left(-\frac{4}{3}\right)^3 - 24\left(-\frac{4}{3}\right)^2 \neq 0$

$\therefore \left(-1\frac{1}{3}, 11\frac{13}{27}\right)$  is a non-stationary inflection.

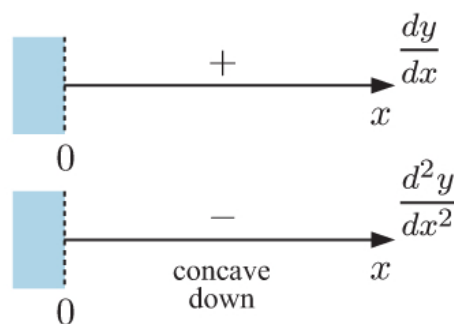
When  $x = 0$ ,  $y = 2$  and  $\frac{dy}{dx} = 0$

$\therefore (0, 2)$  is a stationary inflection.

**d**  $y = 3 - \frac{1}{\sqrt{x}} = 3 - x^{-\frac{1}{2}}$

$$\therefore \frac{dy}{dx} = \frac{1}{2}x^{-\frac{3}{2}} = \frac{1}{2x\sqrt{x}}$$

$$\therefore \frac{d^2y}{dx^2} = -\frac{3}{4}x^{-\frac{5}{2}} = -\frac{3}{4x^2\sqrt{x}}$$



$\frac{d^2y}{dx^2} \neq 0$ , so there are no points of inflection.

**e**  $y = x^3 + 6x^2 + 12x + 5$

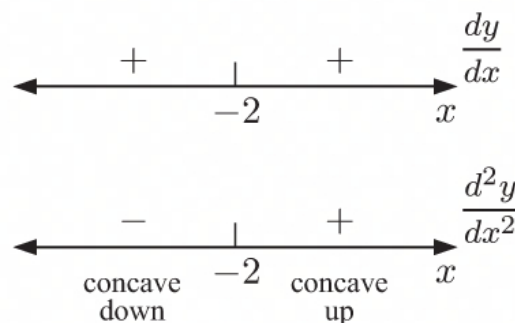
$$\therefore \frac{dy}{dx} = 3x^2 + 12x + 12$$

$$= 3(x^2 + 4x + 4)$$

$$= 3(x + 2)^2$$

$$\therefore \frac{d^2y}{dx^2} = 6x + 12$$

$$= 6(x + 2)$$



Since the sign of  $\frac{d^2y}{dx^2}$  changes at  $x = -2$ , this is a point of inflection.

When  $x = -2$ ,  $y = (-2)^3 + 6(-2)^2 + 12(-2) + 5$  and  $\frac{dy}{dx} = 3(0)^2 = 0$

$$= -8 + 24 - 24 + 5$$

$$= -3$$

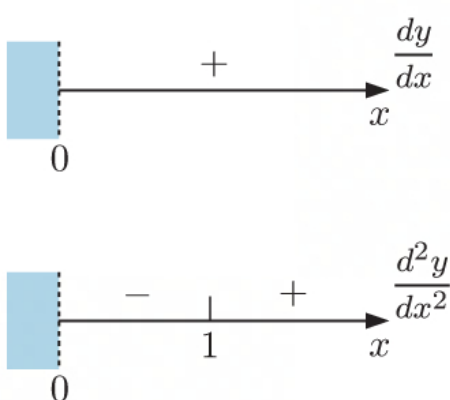
$\therefore (-2, -3)$  is a stationary inflection.



**f**  $y = x^2 + 8\sqrt{x} = x^2 + 8x^{\frac{1}{2}}$

$$\therefore \frac{dy}{dx} = 2x + 4x^{-\frac{1}{2}} = 2x + \frac{4}{\sqrt{x}}$$

$$\therefore \frac{d^2y}{dx^2} = 2 - 2x^{-\frac{3}{2}} = 2 - \frac{2}{x\sqrt{x}}$$

$$= 2\left(1 - \frac{1}{x\sqrt{x}}\right)$$


The first diagram shows the sign of  $\frac{dy}{dx}$  for  $x > 0$ . A vertical dashed line at  $x=0$  is followed by a blue shaded region for  $x < 0$  and an unshaded region for  $x > 0$ . The sign is '+' for all  $x > 0$ . The second diagram shows the sign of  $\frac{d^2y}{dx^2}$  for  $x > 0$ . A vertical dashed line at  $x=0$  is followed by a blue shaded region for  $x < 0$  and an unshaded region for  $x > 0$ . The sign is '-' for  $0 < x < 1$  and '+' for  $x > 1$ .

Since the sign of  $\frac{d^2y}{dx^2}$  changes at  $x = 1$ , this is a point of inflection.

When  $x = 1$ ,  $y = 1^2 + 8\sqrt{1}$  and  $\frac{dy}{dx} = 2(1) + \frac{4}{\sqrt{1}} \neq 0$

$$= 1 + 8$$

$$= 9$$

$\therefore (1, 9)$  is a non-stationary inflection.

**g**  $y = x^4 - 6x^2 + 10$

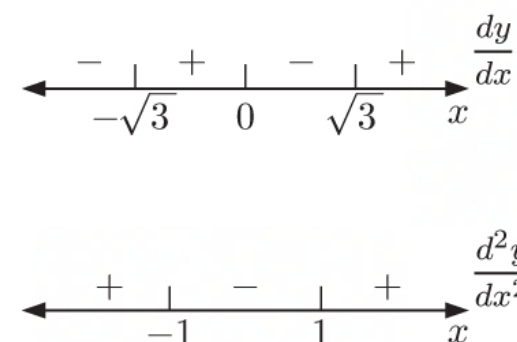
$$\therefore \frac{dy}{dx} = 4x^3 - 12x$$

$$= 4x(x^2 - 3)$$

$$= 4x(x + \sqrt{3})(x - \sqrt{3})$$

$$\therefore \frac{d^2y}{dx^2} = 12x^2 - 12$$

$$= 12(x^2 - 1)$$

$$= 12(x + 1)(x - 1)$$


The first diagram shows the sign of  $\frac{dy}{dx}$  for  $x \in \mathbb{R}$ . Critical points are at  $x = -\sqrt{3}$ ,  $x = 0$ , and  $x = \sqrt{3}$ . The sign is '-' for  $x < -\sqrt{3}$ , '+' for  $-\sqrt{3} < x < 0$ , '-' for  $0 < x < \sqrt{3}$ , and '+' for  $x > \sqrt{3}$ . The second diagram shows the sign of  $\frac{d^2y}{dx^2}$  for  $x \in \mathbb{R}$ . Critical points are at  $x = -1$  and  $x = 1$ . The sign is '+' for  $x < -1$ , '-' for  $-1 < x < 1$ , and '+' for  $x > 1$ .

Since the sign of  $\frac{d^2y}{dx^2}$  changes at  $x = -1$  and  $x = 1$ , both of these points are points of inflection.

When  $x = -1$ ,  $y = (-1)^4 - 6(-1)^2 + 10$

$$= 1 - 6 + 10$$

$$= 5$$

and  $\frac{dy}{dx} = 4(-1)^3 - 12(-1) \neq 0$

When  $x = 1$ ,  $y = 1^4 - 6(1)^2 + 10$

$$= 1 - 6 + 10$$

$$= 5$$

and  $\frac{dy}{dx} = 4(1)^3 - 12(1) \neq 0$

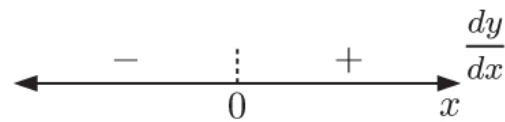
$\therefore (-1, 5)$  and  $(1, 5)$  are non-stationary inflection points.

**h**

$$y = 2x^2 - \frac{6}{x^2}$$

$$\therefore \frac{dy}{dx} = 4x + \frac{12}{x^3}$$

$$= \frac{4x^4 + 12}{x^3}$$

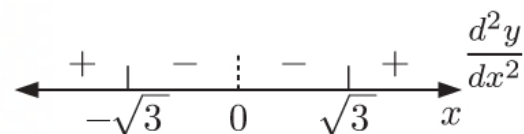


$$\therefore \frac{d^2y}{dx^2} = 4 - \frac{36}{x^4}$$

$$= \frac{4(x^4 - 9)}{x^4}$$

$$= \frac{4(x^2 + 3)(x^2 - 3)}{x^4}$$

$$= \frac{4(x^2 + 3)(x + \sqrt{3})(x - \sqrt{3})}{x^4}$$



Since the sign of  $\frac{d^2y}{dx^2}$  changes at  $x = -\sqrt{3}$  and  $x = \sqrt{3}$ , both of these points are points of inflection.

$$\text{When } x = -\sqrt{3}, \quad y = 2(-\sqrt{3})^2 - \frac{6}{(-\sqrt{3})^2}$$

$$= 6 - 2$$

$$= 4$$

$$\text{and } \frac{dy}{dx} = \frac{4(-\sqrt{3})^4 + 12}{(-\sqrt{3})^3}$$

$$= \frac{48}{-3\sqrt{3}} \neq 0$$

$$\text{When } x = \sqrt{3}, \quad y = 2(\sqrt{3})^2 - \frac{6}{(\sqrt{3})^2}$$

$$= 6 - 2$$

$$= 4$$

$$\text{and } \frac{dy}{dx} = \frac{4(\sqrt{3})^4 + 12}{(\sqrt{3})^3}$$

$$= \frac{48}{3\sqrt{3}} \neq 0$$

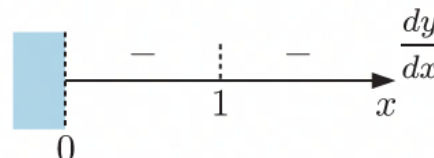
$\therefore (-\sqrt{3}, 4)$  and  $(\sqrt{3}, 4)$  are non-stationary inflection points.

**i**

$$y = \frac{1}{\ln x} = (\ln x)^{-1}, \quad x > 0, \quad x \neq 1$$

$$\therefore \frac{dy}{dx} = (-1)(\ln x)^{-2} \left( \frac{1}{x} \right) \quad \{\text{chain rule}\}$$

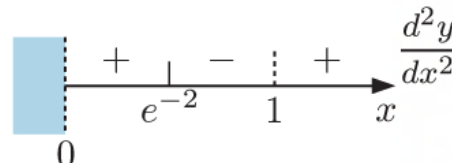
$$= -\frac{1}{x(\ln x)^2} = -x^{-1}(\ln x)^{-2}$$



$$\therefore \frac{d^2y}{dx^2} = -(-1)x^{-2}(\ln x)^{-2} - x^{-1}(-2)(\ln x)^{-3}(x^{-1}) \quad \{\text{product rule, chain rule}\}$$

$$= \frac{1}{x^2(\ln x)^2} + \frac{2}{x^2(\ln x)^3}$$

$$= \frac{\ln x + 2}{x^2(\ln x)^3}$$



Since the sign of  $\frac{d^2y}{dx^2}$  changes at  $x = e^{-2}$ , this is a point of inflection.

$$\text{When } x = e^{-2}, \quad y = \frac{1}{\ln(e^{-2})} \quad \text{and} \quad \frac{dy}{dx} = -\frac{1}{e^{-2}[\ln(e^{-2})]^2}$$

$$= -\frac{1}{2}$$

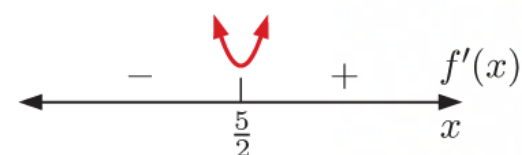
$$= -\frac{1}{e^{-2}(-2)^2}$$

$$= -\frac{e^2}{4} \neq 0$$

$\therefore \left( \frac{1}{e^2}, -\frac{1}{2} \right)$  is a non-stationary inflection point.

**3 a i**  $f(x) = x^2 - 5x + 4$

$\therefore f'(x) = 2x - 5 \quad \therefore f'(x) \text{ has sign diagram:}$



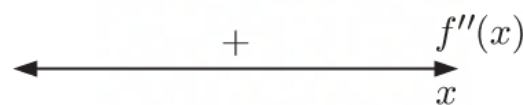
Now  $f\left(\frac{5}{2}\right) = \left(\frac{5}{2}\right)^2 - 5\left(\frac{5}{2}\right) + 4$

$$= \frac{25}{4} - \frac{25}{2} + 4$$

$$= -\frac{9}{4}$$

$\therefore \left(\frac{5}{2}, -\frac{9}{4}\right)$  is a local minimum.

**ii**  $f''(x) = 2 \quad \therefore f''(x) \text{ has sign diagram:}$

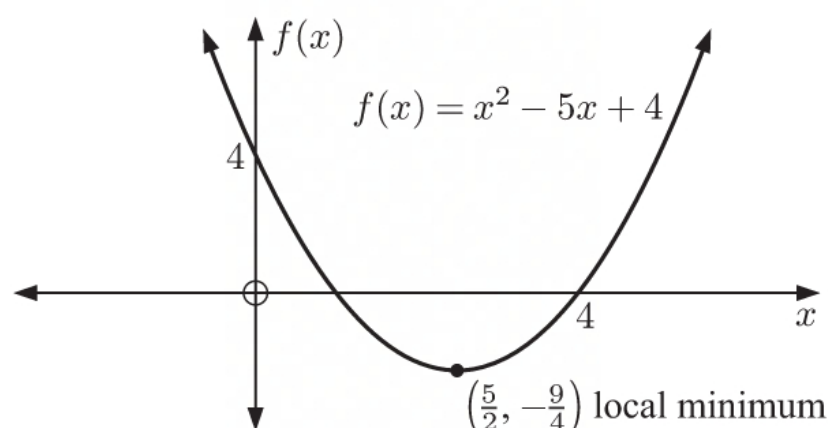


$f''(x) \neq 0$ , so there are no points of inflection.

**iii**  $f(x)$  is increasing for  $x \geq \frac{5}{2}$ , and decreasing for  $x \leq \frac{5}{2}$ .

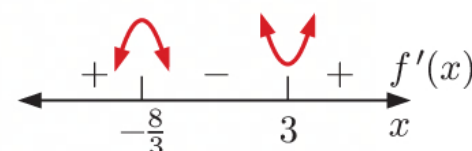
**iv**  $f(x)$  is concave up for all  $x \in \mathbb{R}$ , and never concave down.

**v**



**b i**  $f(x) = x^3 + 4x^2$

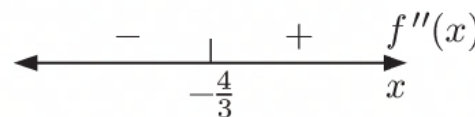
$\therefore f'(x) = 3x^2 + 8x \quad \therefore f'(x) \text{ has sign diagram:}$   
 $= x(3x + 8)$



Now  $f(0) = 0, \quad f\left(-\frac{8}{3}\right) = \left(-\frac{8}{3}\right)^3 + 4\left(-\frac{8}{3}\right)^2$   
 $= \frac{256}{27}$

$\therefore \left(-\frac{8}{3}, \frac{256}{27}\right)$  is a local maximum,  $(0, 0)$  is a local minimum.

**ii**  $f''(x) = 6x + 8 \quad \therefore f''(x) \text{ has sign diagram:}$

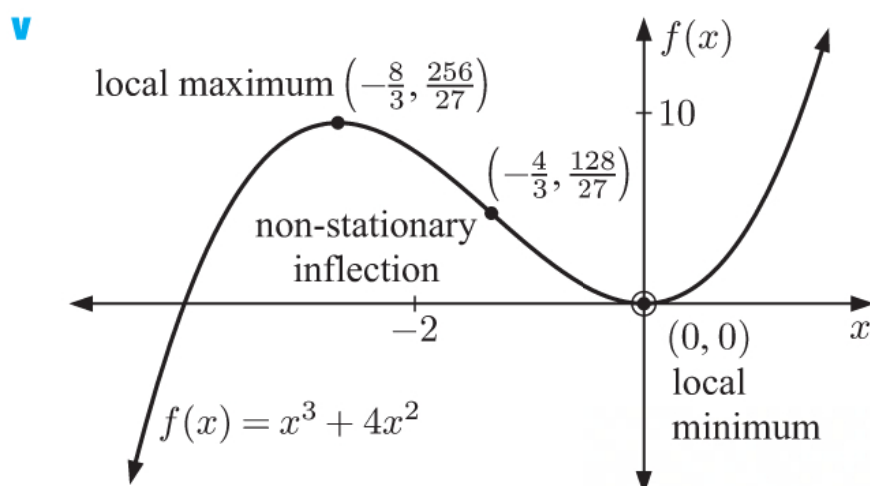


Now  $f\left(-\frac{4}{3}\right) = \left(-\frac{4}{3}\right)^3 + 4\left(-\frac{4}{3}\right)^2$  and  $f'\left(-\frac{4}{3}\right) \neq 0$   
 $= -\frac{64}{27} + \frac{64}{9}$   
 $= \frac{128}{27}$

$\therefore \left(-\frac{4}{3}, \frac{128}{27}\right)$  is a non-stationary inflection.

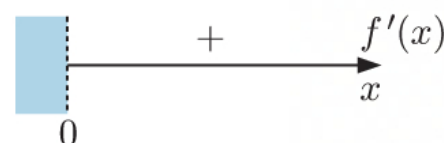
**iii**  $f(x)$  is increasing for all  $x \leq -\frac{8}{3}$  and  $x \geq 0$ , and decreasing for  $-\frac{8}{3} \leq x \leq 0$ .

**iv**  $f(x)$  is concave up for all  $x \geq -\frac{4}{3}$ , and concave down for  $x \leq -\frac{4}{3}$ .



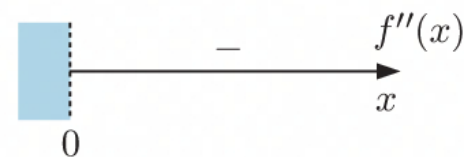
**c i**  $f(x) = \sqrt{x} = x^{\frac{1}{2}}$

$\therefore f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$   $\therefore f'(x)$  has sign diagram:



$\therefore$  there are no turning points.

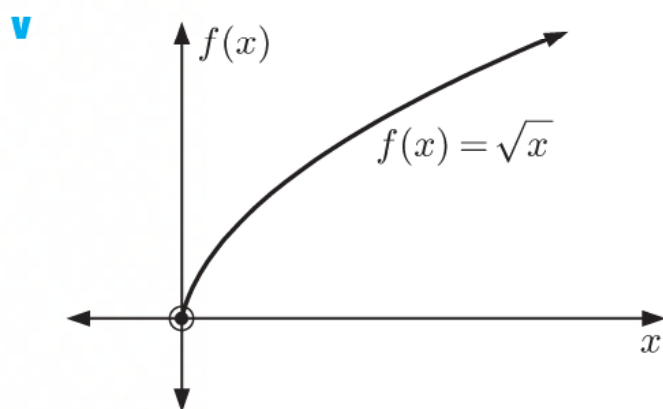
**ii**  $f''(x) = -\frac{1}{4}x^{-\frac{3}{2}} = -\frac{1}{4x\sqrt{x}}$   $\therefore f''(x)$  has sign diagram:



$f''(x) \neq 0$ , so there are no points of inflection.

**iii**  $f(x)$  is increasing for  $x > 0$ , and never decreasing.

**iv**  $f(x)$  is concave down for  $x > 0$ , and never concave up.



**d i**  $f(x) = x^3 - 3x^2 - 24x + 1$

$\therefore f'(x) = 3x^2 - 6x - 24 = 3(x^2 - 2x - 8) = 3(x+2)(x-4)$   $\therefore f'(x)$  has sign diagram:



Now  $f(-2) = (-2)^3 - 3(-2)^2 - 24(-2) + 1 = -8 - 12 + 48 + 1 = 29$

and  $f(4) = 4^3 - 3(4)^2 - 24(4) + 1 = 64 - 48 - 96 + 1 = -79$

$\therefore (-2, 29)$  is a local maximum, and  $(4, -79)$  is a local minimum.



ii  $f''(x) = 6x - 6 \quad \therefore f''(x) \text{ has sign diagram:}$

$$= 6(x - 1)$$

$$f(1) = 1^3 - 3(1)^2 - 24(1) + 1 \quad \text{and} \quad f'(1) \neq 0$$

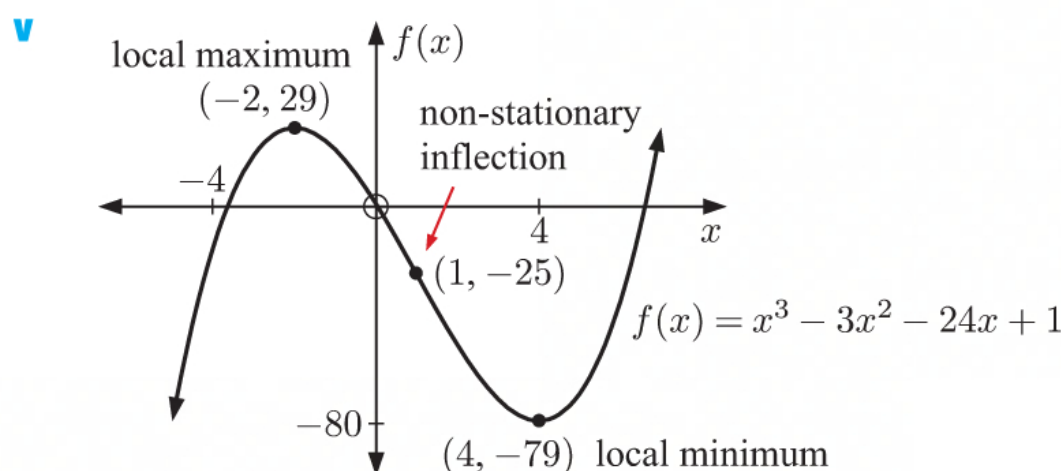
$$= 1 - 3 - 24 + 1$$

$$= -25$$

$\therefore (1, -25)$  is a non-stationary point of inflection.

iii  $f(x)$  is increasing for  $x \leq -2$  and  $x \geq 4$ , and decreasing for  $-2 \leq x \leq 4$ .

iv  $f(x)$  is concave down for  $x \leq 1$ , and concave up for  $x \geq 1$ .



e i  $f(x) = 3x^4 + 4x^3 - 2$

$$\therefore f'(x) = 12x^3 + 12x^2$$

$$= 12x^2(x + 1)$$

$\therefore f'(x) \text{ has sign diagram:}$

Now  $f(-1) = 3(-1)^4 + 4(-1)^3 - 2$  and  $f(0) = -2$

$$= 3 - 4 - 2$$

$$= -3$$

$\therefore (-1, -3)$  is a local minimum, and  $(0, -2)$  is a point of inflection but not a turning point.

ii  $f''(x) = 36x^2 + 24x \quad \therefore f''(x) \text{ has sign diagram:}$

$$= 12x(3x + 2)$$

Now  $f(-\frac{2}{3}) = 3(-\frac{2}{3})^4 + 4(-\frac{2}{3})^3 - 2$  and  $f'(-\frac{2}{3}) \neq 0$

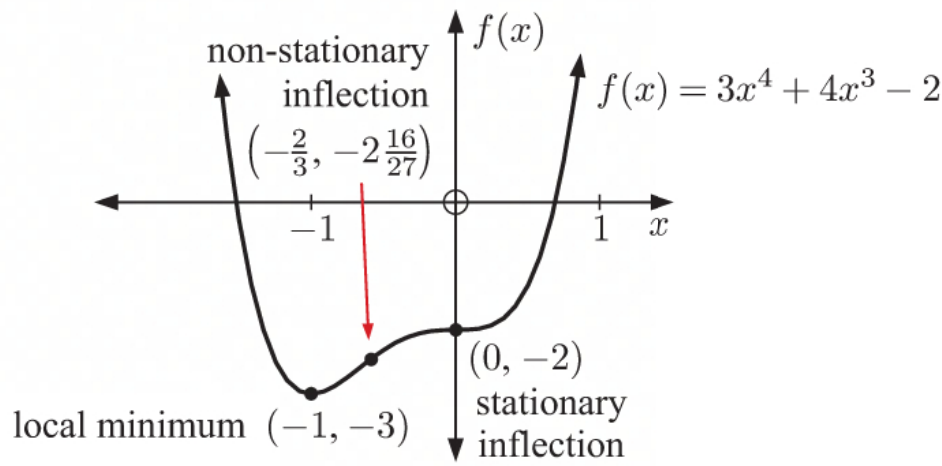
$$= -\frac{70}{27}$$

$\therefore (-\frac{2}{3}, -2\frac{16}{27})$  is a non-stationary inflection, and  $(0, -2)$  is a stationary inflection.

iii  $f(x)$  is increasing for  $x \geq -1$ , and decreasing for  $x \leq -1$ .

iv  $f(x)$  is concave down for  $-\frac{2}{3} \leq x \leq 0$ , and concave up for  $x \leq -\frac{2}{3}$  and  $x \geq 0$ .

v

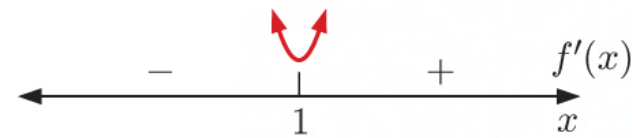


f i

$$f(x) = (x-1)^4$$

$$\therefore f'(x) = 4(x-1)^3(1) \quad \{\text{chain rule}\}$$

$$= 4(x-1)^3$$

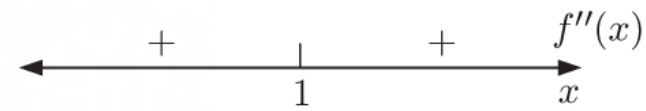
 $\therefore f'(x)$  has sign diagram:

$$\text{Now } f(1) = (1-1)^4 = 0$$

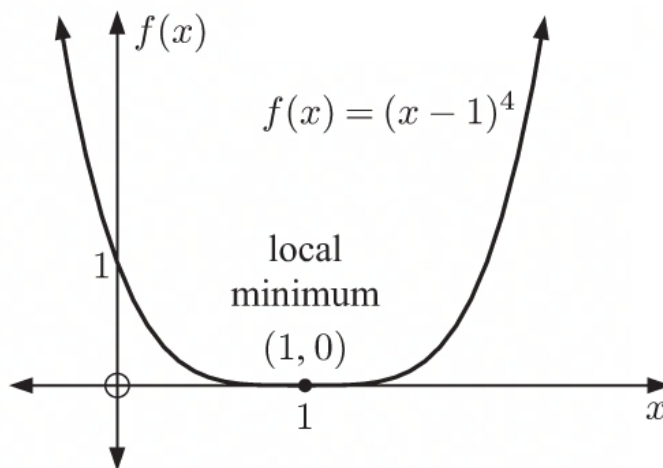
 $\therefore (1, 0)$  is a local minimum.

$$\text{ii } f''(x) = 12(x-1)^2(1) \quad \{\text{chain rule}\}$$

$$= 12(x-1)^2$$

 $\therefore f''(x)$  has sign diagram:There is no change in sign of  $f''(x)$ , so there are no points of inflection.iii  $f(x)$  is increasing for  $x \geq 1$ , and decreasing for  $x \leq 1$ .iv  $f(x)$  is concave up for all  $x \in \mathbb{R}$ , and never concave down.

v



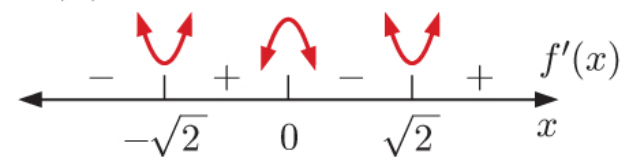
9 i

$$f(x) = x^4 - 4x^2 + 3$$

$$\therefore f'(x) = 4x^3 - 8x$$

$$= 4x(x^2 - 2)$$

$$= 4x(x + \sqrt{2})(x - \sqrt{2})$$

 $\therefore f'(x)$  has sign diagram:

$$\text{Now } f(-\sqrt{2}) = (-\sqrt{2})^4 - 4(-\sqrt{2})^2 + 3$$

$$= 4 - 8 + 3$$

$$= -1$$

$$f(0) = 3$$

$$\text{and } f(\sqrt{2}) = (\sqrt{2})^4 - 4(\sqrt{2})^2 + 3$$

$$= -1$$

 $\therefore (-\sqrt{2}, -1)$  and  $(\sqrt{2}, -1)$  are local minima, and  $(0, 3)$  is a local maximum.

ii  $f''(x) = 12x^2 - 8$

$$= 4(3x^2 - 2)$$

$$= 4(\sqrt{3}x + \sqrt{2})(\sqrt{3}x - \sqrt{2})$$

$\therefore f''(x)$  has sign diagram:



Now  $f\left(-\sqrt{\frac{2}{3}}\right) = \left(-\sqrt{\frac{2}{3}}\right)^4 - 4\left(-\sqrt{\frac{2}{3}}\right)^2 + 3$  and  $f'\left(-\sqrt{\frac{2}{3}}\right) \neq 0$

$$= \frac{4}{9} - \frac{8}{3} + 3$$

$$= \frac{7}{9}$$

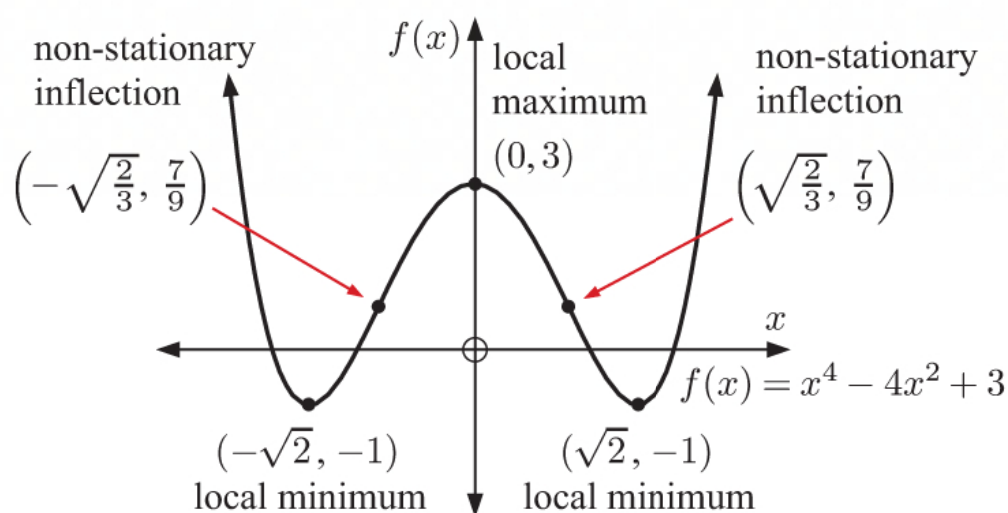
$$f\left(\sqrt{\frac{2}{3}}\right) = \frac{7}{9} \text{ and } f'\left(\sqrt{\frac{2}{3}}\right) \neq 0$$

$\therefore \left(\sqrt{\frac{2}{3}}, \frac{7}{9}\right)$  and  $\left(-\sqrt{\frac{2}{3}}, \frac{7}{9}\right)$  are non-stationary inflections.

iii  $f(x)$  is increasing for  $-\sqrt{2} \leq x \leq 0$  and  $x \geq \sqrt{2}$ , and decreasing for  $x \leq -\sqrt{2}$  and  $0 \leq x \leq \sqrt{2}$ .

iv  $f(x)$  is concave down for  $-\sqrt{\frac{2}{3}} \leq x \leq \sqrt{\frac{2}{3}}$ , and concave up for  $x \leq -\sqrt{\frac{2}{3}}$  and  $x \geq \sqrt{\frac{2}{3}}$ .

v



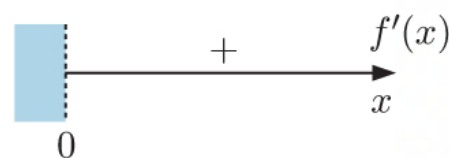
h i  $f(x) = 3 - \frac{4}{\sqrt{x}} = 3 - 4x^{-\frac{1}{2}}$

$$\therefore f'(x) = -4\left(-\frac{1}{2}\right)x^{-\frac{3}{2}}$$

$$= 2x^{-\frac{3}{2}}$$

$$= \frac{2}{x\sqrt{x}}$$

$\therefore f'(x)$  has sign diagram:



$f'(x) \neq 0$ , so there are no turning points.

ii  $f''(x) = 2\left(-\frac{3}{2}\right)x^{-\frac{5}{2}}$

$\therefore f''(x)$  has sign diagram:

$$= -3x^{-\frac{5}{2}}$$

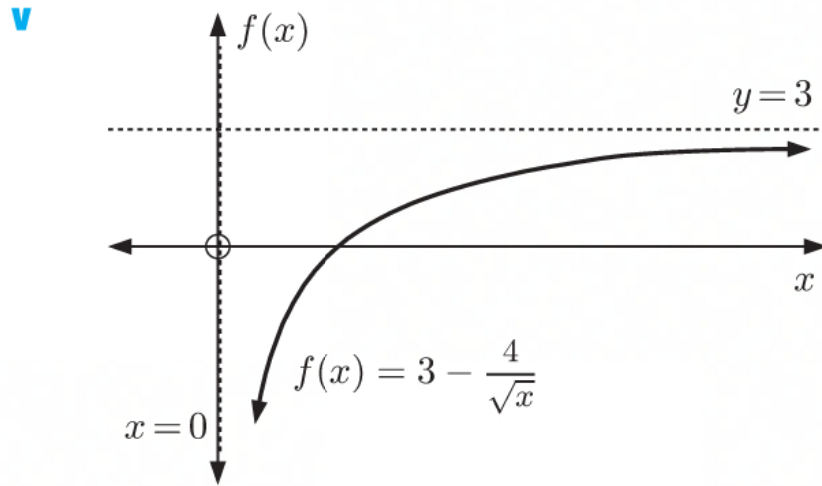
$$= -\frac{3}{x^2\sqrt{x}}$$



$f''(0) \neq 0$ , so there are no points of inflection.

iii  $f(x)$  is increasing for  $x > 0$ , and never decreasing.

**iv**  $f(x)$  is concave down for  $x > 0$ , and never concave up.



**4 a**  $f(x) = e^{2x} - 3$

The  $x$ -intercept occurs when  $f(x) = 0$

$$\therefore e^{2x} - 3 = 0$$

$$\therefore e^{2x} = 3$$

$$\therefore 2x = \ln 3$$

$$\therefore x = \frac{\ln 3}{2}$$

$$= \frac{1}{2} \ln 3$$

$$= \ln 3^{\frac{1}{2}}$$

$$= \ln \sqrt{3}$$

$\therefore$  the  $x$ -intercept is  $\ln \sqrt{3}$  and the  $y$ -intercept is  $-2$ .

**b**  $f'(x) = 2e^{2x}$

Now  $e^{2x} > 0$  for all  $x$ ,

so  $f'(x) > 0$  for all  $x$ .

$\therefore$  the function is increasing for all  $x$ .

**d** As  $x \rightarrow -\infty$ ,  $e^{2x} \rightarrow 0^+$   $\therefore e^{2x} - 3 \rightarrow -3^+$

So, the horizontal asymptote is  $y = -3$ .

The  $y$ -intercept occurs when  $x = 0$

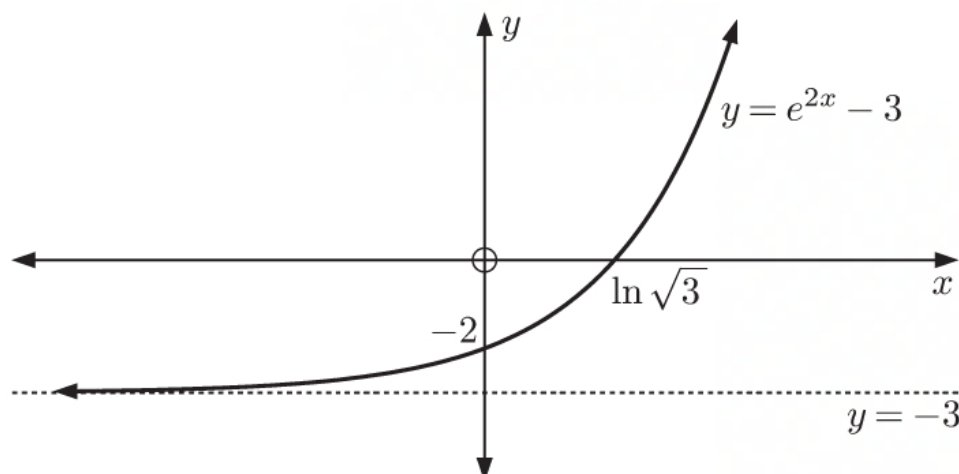
$$f(0) = e^0 - 3 = -2$$

**c**  $f''(x) = 2e^{2x}(2)$

$$= 4e^{2x} \text{ which is } > 0 \text{ for all } x.$$

$\therefore f(x)$  is concave up for all  $x$ .

**e**





**5 a**  $f(x) = e^x - 3$

The  $x$ -intercept occurs when  $f(x) = 0$

$$\therefore e^x - 3 = 0$$

$$\therefore e^x = 3$$

$$\therefore x = \ln 3$$

The  $y$ -intercept occurs when  $x = 0$

$$f(0) = e^0 - 3$$

$$= -2$$

$\therefore$  the  $x$ -intercept of  $f(x)$  is  $\ln 3$ , and the  $y$ -intercept is  $-2$ .

$$g(x) = 3 - 5e^{-x}$$

The  $x$ -intercept occurs when  $g(x) = 0$

$$\therefore 3 - 5e^{-x} = 0$$

$$\therefore 5e^{-x} = 3$$

$$\therefore e^{-x} = \frac{3}{5}$$

$$\therefore e^x = \frac{5}{3}$$

$$\therefore x = \ln\left(\frac{5}{3}\right)$$

The  $y$ -intercept occurs when  $x = 0$

$$g(0) = 3 - 5e^0$$

$$= -2$$

$\therefore$  the  $x$ -intercept of  $g(x)$  is  $\ln\left(\frac{5}{3}\right)$ , and the  $y$ -intercept is  $-2$ .

**b**  $f(x)$ : as  $x \rightarrow \infty$ ,  $e^x \rightarrow \infty$   
 $\therefore f(x) = e^x - 3 \rightarrow \infty$   
 as  $x \rightarrow -\infty$ ,  $e^x \rightarrow 0^+$   
 $\therefore f(x) = e^x - 3 \rightarrow -3^+$

$g(x)$ : as  $x \rightarrow \infty$ ,  $-5e^{-x} \rightarrow 0^-$   
 $\therefore g(x) = 3 - 5e^{-x} \rightarrow 3^-$   
 as  $x \rightarrow -\infty$ ,  $-5e^{-x} \rightarrow -\infty$   
 $\therefore g(x) = 3 - 5e^{-x} \rightarrow -\infty$

**c**  $f'(x) = e^x$ ,  $f''(x) = e^x$

$$\begin{array}{c} + \\ \leftarrow \text{-----} \rightarrow \\ x \end{array} \quad f'(x)$$

$$\begin{array}{c} + \\ \leftarrow \text{-----} \rightarrow \\ x \end{array} \quad f''(x)$$

$\therefore f(x)$  is increasing and concave up for all  $x \in \mathbb{R}$ .

$$g'(x) = 5e^{-x}, \quad g''(x) = -5e^{-x}$$

$$\begin{array}{c} + \\ \leftarrow \text{-----} \rightarrow \\ x \end{array} \quad g'(x)$$

$$\begin{array}{c} - \\ \leftarrow \text{-----} \rightarrow \\ x \end{array} \quad g''(x)$$

$\therefore g(x)$  is increasing and concave down for all  $x \in \mathbb{R}$ .

**d** The functions intersect when  $f(x) = g(x)$

$$\therefore e^x - 3 = 3 - 5e^{-x}$$

$$\therefore e^x - 6 + 5e^{-x} = 0$$

$$\therefore e^{2x} - 6e^x + 5 = 0 \quad \{\text{multiplying both sides by } e^x\}$$

$$\therefore (e^x - 1)(e^x - 5) = 0 \quad \{\text{compare } a^2 - 6a + 5 = (a - 1)(a - 5)\}$$

$$\therefore e^x = 1 \text{ or } 5$$

$$\therefore x = 0 \text{ or } \ln 5$$

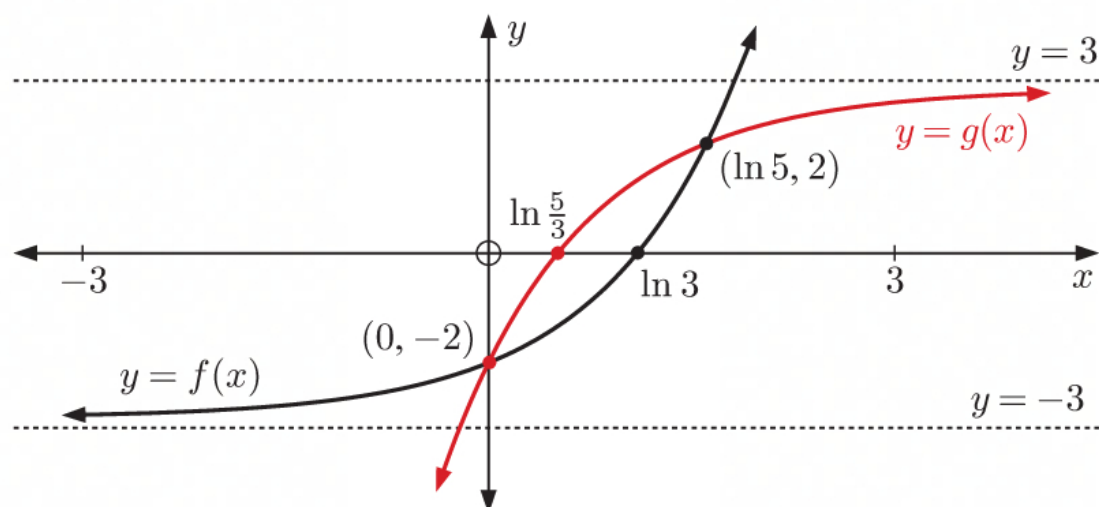
$$\text{Now } f(0) = e^0 - 3 = -2 \quad \text{and} \quad f(\ln 5) = e^{\ln 5} - 3$$

$$= 5 - 3$$

$$= 2$$

$\therefore$  the points of intersection are  $(0, -2)$  and  $(\ln 5, 2)$ .

e



6 a  $y = e^x - 3e^{-x}$

The  $x$ -intercept occurs when  $y = 0$

$$\therefore e^x - 3e^{-x} = 0$$

$$\therefore e^{2x} - 3 = 0 \quad \{\text{multiplying both sides by } e^x\}$$

$$\therefore e^{2x} = 3$$

$$\therefore 2x = \ln 3$$

$$\therefore x = \frac{1}{2} \ln 3 = \ln(3^{\frac{1}{2}}) = \ln \sqrt{3}$$

The  $y$ -intercept occurs when  $x = 0$

$$\therefore y = e^0 - 3e^0$$

$$= 1 - 3$$

$$= -2$$

$\therefore$  the  $x$ -intercept is  $\ln \sqrt{3}$  and the  $y$ -intercept is  $-2$ .

b

$$\frac{dy}{dx} = e^x + 3e^{-x}$$

$$= e^x + \frac{3}{e^x}$$

Now  $e^x > 0$  for all  $x$ ,

so  $\frac{dy}{dx} > 0$  for all  $x$ .

$\therefore$  the function is increasing for all  $x$ .

c

$$\frac{dy}{dx} = e^x + 3e^{-x}$$

$$\therefore \frac{d^2y}{dx^2} = e^x - 3e^{-x}$$

$$= y$$

Above the  $x$ -axis,  $y > 0 \therefore \frac{d^2y}{dx^2} > 0$

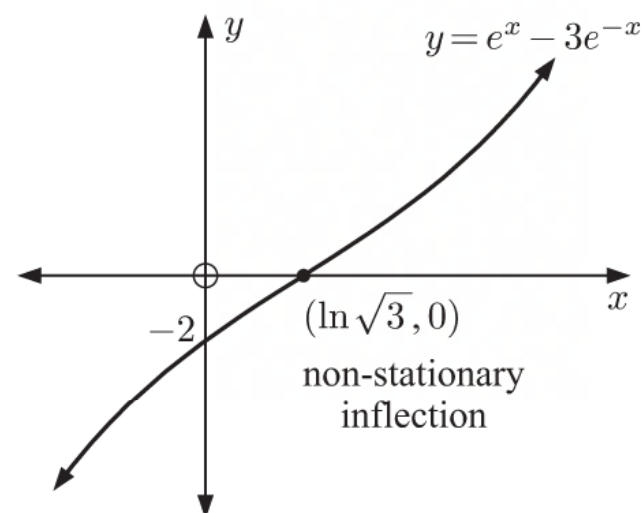
$\therefore$  the function is concave up.

Below the  $x$ -axis,  $y < 0 \therefore \frac{d^2y}{dx^2} < 0$

$\therefore$  the function is concave down.

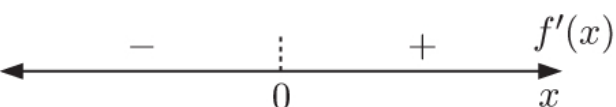
$\therefore y$  is concave down below the  $x$ -axis and concave up above the  $x$ -axis.

d



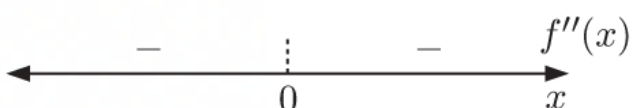
**7 a**  $f(x) = \ln(3x^2)$  is defined when  $3x^2 > 0$   
 $\therefore x^2 > 0$   
 $\therefore x \neq 0$

**b**  $f'(x) = \frac{6x}{3x^2}$   
 $= \frac{2}{x}$  which has sign diagram:

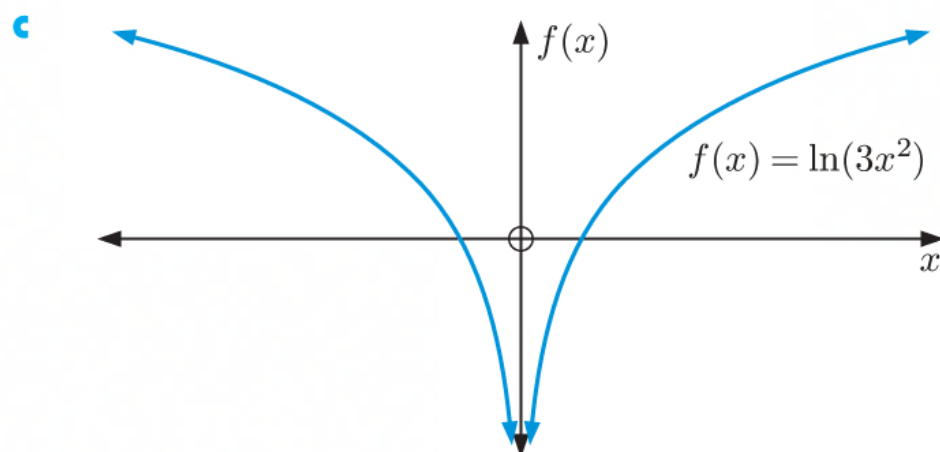


$\therefore f(x)$  is increasing for  $x > 0$  and decreasing for  $x < 0$ .

$f'(x) = 2x^{-1}$   
 $\therefore f''(x) = -2x^{-2}$   
 $= -\frac{2}{x^2}$  which has sign diagram:

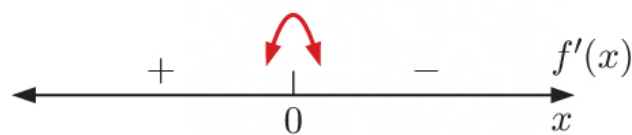


$\therefore f(x)$  is concave down for all  $x \neq 0$ .



**8 a**  $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$   
 $\therefore f'(x) = \frac{1}{\sqrt{2\pi}} (-x) e^{-\frac{1}{2}x^2}$   
 $= -\frac{1}{\sqrt{2\pi}} x e^{-\frac{1}{2}x^2}$  where  $e^{-\frac{1}{2}x^2}$  is positive for all  $x$

So,  $f'(x)$  has sign diagram:



Now  $f(0) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(0)^2}$   
 $= \frac{1}{\sqrt{2\pi}}$

$\therefore \left(0, \frac{1}{\sqrt{2\pi}}\right)$  is a local maximum.

$f(x)$  is increasing for  $x \leq 0$ , and decreasing for  $x \geq 0$ .

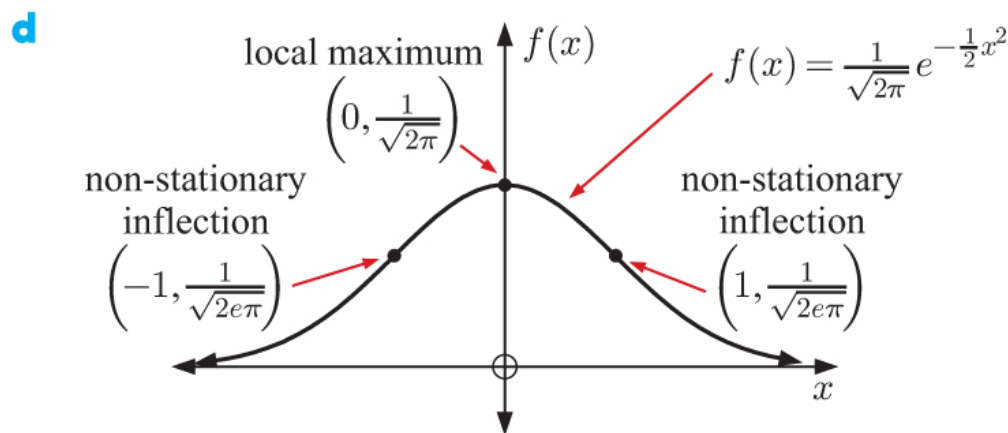
$$\begin{aligned}
 \text{b } f''(x) &= -\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} + \left( -\frac{1}{\sqrt{2\pi}} x(-x)e^{-\frac{1}{2}x^2} \right) \quad \{\text{product rule}\} \\
 &= -\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} + \frac{1}{\sqrt{2\pi}} x^2 e^{-\frac{1}{2}x^2} \\
 &= -\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} (1 - x^2) \\
 &= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} (x^2 - 1) \quad \text{which has sign diagram:}
 \end{aligned}$$

$\begin{array}{ccccccc} & + & & - & & + & \\ & | & & | & & | & \\ \leftarrow & -1 & & & & 1 & \rightarrow x \end{array}$

$$\begin{aligned}
 \text{Now } f(-1) &= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(-1)^2} & \text{and} & & f(1) &= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(1)^2} \\
 &= \frac{1}{\sqrt{2\pi}\sqrt{e}} & & & &= \frac{1}{\sqrt{2\pi}\sqrt{e}} \\
 &= \frac{1}{\sqrt{2e\pi}} & & & &= \frac{1}{\sqrt{2e\pi}}
 \end{aligned}$$

Since  $f'(-1) \neq 0$  and  $f'(1) \neq 0$ , then  $\left(-1, \frac{1}{\sqrt{2e\pi}}\right)$  and  $\left(1, \frac{1}{\sqrt{2e\pi}}\right)$  are non-stationary inflections.

$$\begin{aligned}
 \text{c } \text{As } x \rightarrow \infty, e^{-\frac{1}{2}x^2} &\rightarrow 0^+ \\
 \therefore \text{ as } x \rightarrow \infty, f(x) &= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \rightarrow 0^+ \\
 \text{As } x \rightarrow -\infty, e^{-\frac{1}{2}x^2} &\rightarrow 0^+ \\
 \therefore \text{ as } x \rightarrow -\infty, f(x) &= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \rightarrow 0^+
 \end{aligned}$$



$$\begin{aligned}
 \text{9 a } f(x) &= \cos x \\
 \therefore f'(x) &= -\sin x \\
 \therefore f''(x) &= -\cos x = -f(x) \\
 \therefore \text{ the inflection points coincide with the } x\text{-intercepts.}
 \end{aligned}$$

$$\begin{aligned}
 \text{b } f''(x) &= -\cos x, \quad 0 \leq x \leq 2\pi \\
 \text{The sign diagram of } f''(x) \text{ is:}
 \end{aligned}$$

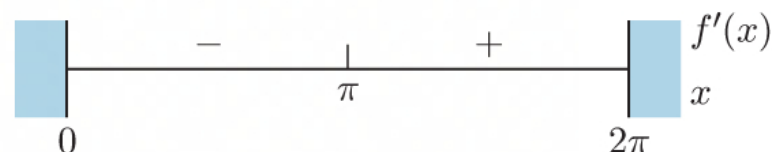
$\begin{array}{ccccccc} & & - & & + & & - \\ & | & & | & & | & \\ \boxed{\phantom{0}} & 0 & & \frac{\pi}{2} & & \frac{3\pi}{2} & \boxed{\phantom{0}} \\ & & & & & & x \end{array}$

Since  $f'(\frac{\pi}{2}) \neq 0$  and  $f'(\frac{3\pi}{2}) \neq 0$ , then there are non-stationary inflection points at  $(\frac{\pi}{2}, 0)$  and  $(\frac{3\pi}{2}, 0)$ .

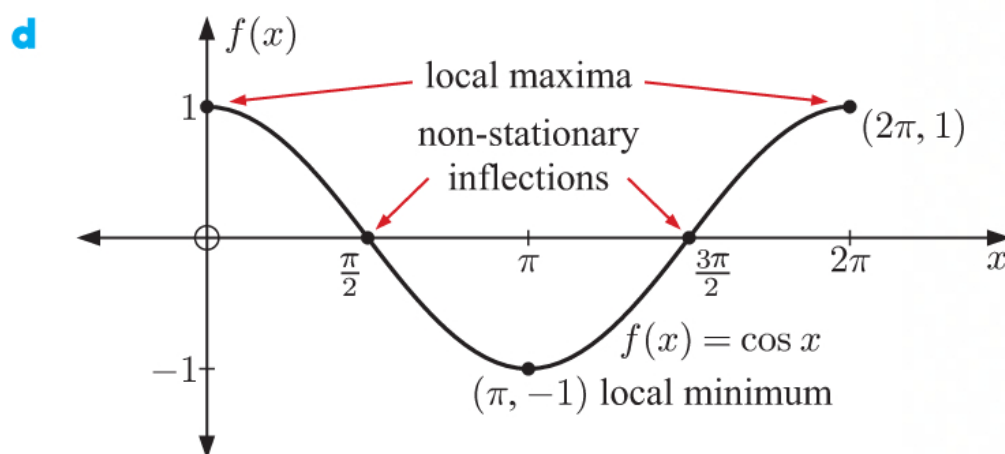


**c**  $f'(x) = -\sin x, \quad 0 \leq x \leq 2\pi$

The sign diagram of  $f'(x)$  is:



- i**  $f(x)$  is increasing for  $\pi \leq x \leq 2\pi$
- ii**  $f(x)$  is decreasing for  $0 \leq x \leq \pi$
- iii**  $f(x)$  is concave up for  $\frac{\pi}{2} \leq x \leq \frac{3\pi}{2}$
- iv**  $f(x)$  is concave down for  $0 \leq x \leq \frac{\pi}{2}$  and  $\frac{3\pi}{2} \leq x \leq 2\pi$



**10**  $y = 4^x - 2^x$

**a** When  $x = 0$ ,  $y = 4^0 - 2^0$   
 $= 1 - 1 = 0$

$\therefore$  the  $y$ -intercept is 0.

When  $y = 0$ ,  $4^x - 2^x = 0$

$$\therefore (2^x)^2 - 2^x = 0$$

$$\therefore 2^x(2^x - 1) = 0$$

$$\therefore 2^x - 1 = 0 \quad \{2^x > 0 \text{ for } x \in \mathbb{R}\}$$

$$\therefore 2^x = 1$$

$$\therefore x = 0$$

$\therefore$  the  $x$ -intercept is 0.

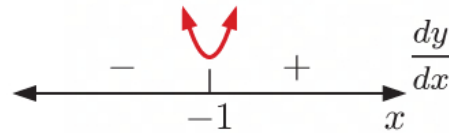
**b** As  $x \rightarrow \infty$ ,  $4^x \rightarrow \infty$  (at a faster rate than  $2^x$ )  $\therefore y \rightarrow \infty$ .

As  $x \rightarrow -\infty$ ,  $y \rightarrow 0^-$ .  $\{4^x - 2^x < 0 \text{ for } x < 0\}$

**c**  $\frac{dy}{dx} = 4^x \ln 4 - 2^x \ln 2$   
 $= (2^x)^2 \times 2 \ln 2 - 2^x \ln 2$   
 $= 2^x \ln 2 (2^x \times 2 - 1)$   
 $= 2^x \ln 2 (2^{x+1} - 1)$

So,  $\frac{dy}{dx} = 0$  when  $2^{x+1} - 1 = 0$   $\{2^x > 0 \text{ for } x \in \mathbb{R}\}$   
 $\therefore 2^{x+1} = 1$   
 $\therefore x + 1 = 0$   
 $\therefore x = -1$

The sign diagram of  $\frac{dy}{dx}$  is:



$$\begin{aligned}\text{When } x = -1, \quad y &= 4^{-1} - 2^{-1} \\ &= \frac{1}{4} - \frac{1}{2} = -\frac{1}{4}\end{aligned}$$

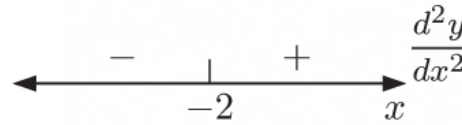
$\therefore$  there is a local minimum at  $(-1, -\frac{1}{4})$ .

**d**  $\frac{dy}{dx} = 4^x \ln 4 - 2^x \ln 2$  {from **c**}

$$\begin{aligned}\therefore \frac{d^2y}{dx^2} &= (4^x \ln 4) \ln 4 - (2^x \ln 2) \ln 2 \\ &= (2^x)^2 (2 \ln 2)^2 - 2^x (\ln 2)^2 \\ &= 2^x (\ln 2)^2 (2^x (4) - 1) \\ &= 2^x (\ln 2)^2 (2^{x+2} - 1)\end{aligned}$$

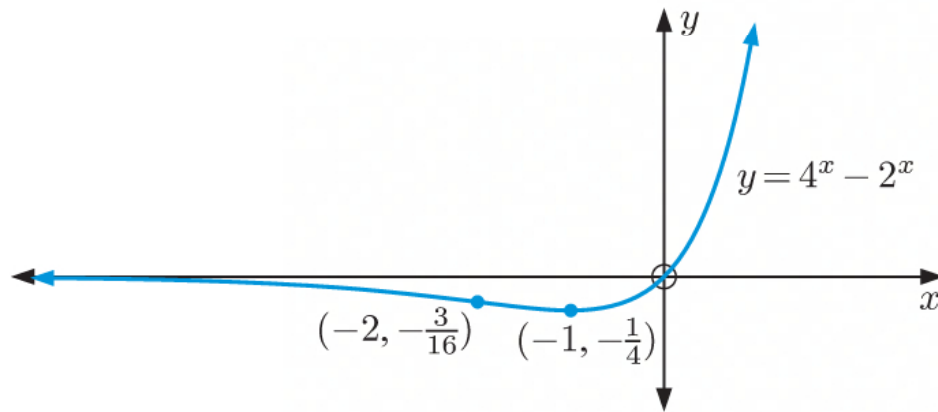
$$\begin{aligned}\text{So, } \frac{d^2y}{dx^2} = 0 \quad \text{when } 2^{x+2} &= 1 \quad \{2^x > 0 \text{ for } x \in \mathbb{R}\} \\ \therefore x &= -2\end{aligned}$$

The sign diagram of  $\frac{d^2y}{dx^2}$  is:



$\therefore$  the curve is concave up for  $x \geq -2$ , and is concave down for  $x \leq -2$ .

**e**



**11**  $f(t) = Ate^{-bt}$ ,  $t \geq 0$ ,  $A, b > 0$

**a i**  $f'(t) = Ae^{-bt} + Ate^{-bt}(-b)$  {product rule}  
 $= Ae^{-bt} - Abte^{-bt}$   
 $= Ae^{-bt}(1 - bt)$

$$f'(t) = 0 \quad \text{when } Ae^{-bt}(1 - bt) = 0 \quad \text{but } A > 0 \text{ and } e^{-bt} > 0$$

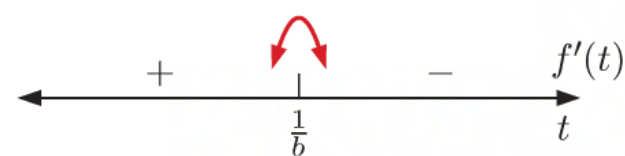
$$\therefore 1 - bt = 0$$

$$\therefore -bt = -1$$

$$\therefore t = \frac{1}{b} \quad \{\text{where } b > 0\}$$

$\therefore$  there is a local maximum at  $t = \frac{1}{b}$ .

$\therefore$  the sign diagram of  $f'(t)$  is:



$$\begin{aligned}
 \text{ii } f''(t) &= Ae^{-bt}(-b) - [Abe^{-bt} + Abte^{-bt}(-b)] \quad \{\text{product rule}\} \\
 &= -Abe^{-bt} - (Abe^{-bt} - Ab^2te^{-bt}) \\
 &= -2Abe^{-bt} + Ab^2te^{-bt} \\
 &= Abe^{-bt}(bt - 2)
 \end{aligned}$$

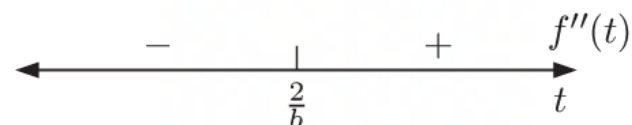
$$f''(t) = 0 \text{ when } Abe^{-bt}(bt - 2) = 0 \text{ but } A, b > 0 \text{ and } e^{-bt} > 0$$

$$\therefore bt - 2 = 0$$

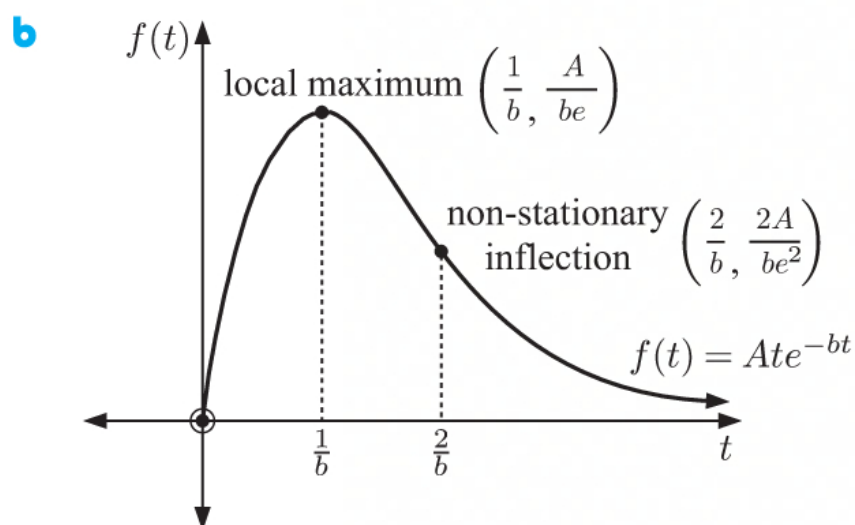
$$\therefore bt = 2$$

$$\therefore t = \frac{2}{b} \quad \{\text{where } b > 0\}$$

$\therefore$  the sign diagram of  $f''(t)$  is:



Since  $f'(\frac{2}{b}) \neq 0$ , there is a non-stationary point of inflection at  $t = \frac{2}{b}$ .



**12**  $f(t) = \frac{C}{1 + Ae^{-bt}}, \quad t \geq 0, \quad A, b, C > 0$

**a** The  $y$ -intercept occurs when  $t = 0$ .

$$\text{Now } f(0) = \frac{C}{1 + Ae^{-b(0)}} = \frac{C}{1 + A}$$

So, the  $y$ -intercept is  $\frac{C}{1 + A}$ .

**b** As  $t \rightarrow \infty$ ,  $Ae^{-bt} \rightarrow 0^+ \quad \{A, b > 0\}$

$$\therefore f(t) = \frac{C}{1 + Ae^{-bt}} \rightarrow \frac{C}{1 + 0^+} = C^-$$

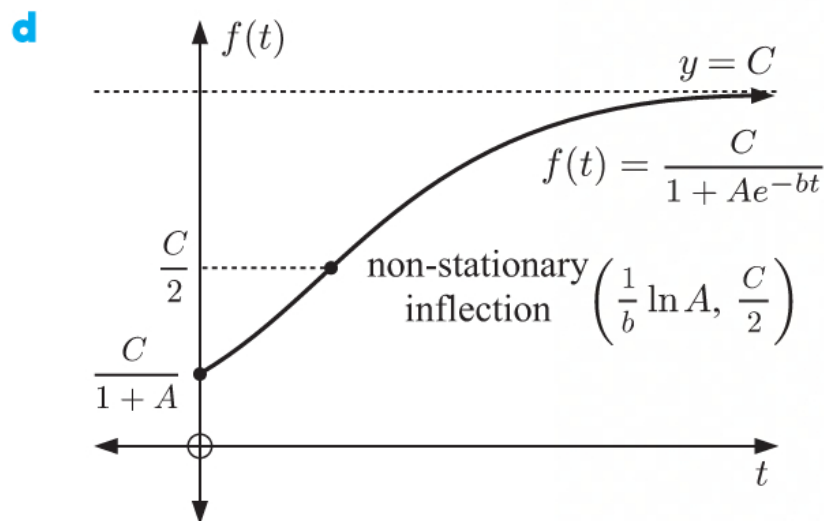
$\therefore y = C$  is a horizontal asymptote.

$$\begin{aligned}
 \text{c} \quad f(t) &= C(1 + Ae^{-bt})^{-1} \\
 \therefore f'(t) &= C(-1)(1 + Ae^{-bt})^{-2}(-bAe^{-bt}) \quad \{\text{chain rule}\} \\
 &= AbCe^{-bt}(1 + Ae^{-bt})^{-2} \\
 \therefore f''(t) &= (-b)AbCe^{-bt}(1 + Ae^{-bt})^{-2} + AbCe^{-bt}(-2)(1 + Ae^{-bt})^{-3}(-bAe^{-bt}) \\
 &\quad \{\text{product rule and chain rule}\} \\
 &= -\frac{Ab^2C}{e^{bt}(1 + Ae^{-bt})^2} + \frac{2A^2b^2C}{e^{2bt}(1 + Ae^{-bt})^3} \\
 f''(t) = 0 \quad \text{when} \quad &\frac{2A^2b^2C}{e^{2bt}(1 + Ae^{-bt})^3} = \frac{Ab^2C}{e^{bt}(1 + Ae^{-bt})^2} \\
 \therefore \frac{2A(Ab^2C)}{e^{bt}e^{bt}(1 + Ae^{-bt})^3} &= \frac{Ab^2C}{e^{bt}(1 + Ae^{-bt})^2} \\
 \therefore \frac{2A}{e^{bt}(1 + Ae^{-bt})} &= 1 \\
 \therefore 2A &= e^{bt} + Ae^{-bt}e^{bt} \\
 \therefore 2A &= e^{bt} + A \\
 \therefore A &= e^{bt} \\
 \therefore \ln A &= bt \\
 \therefore t &= \frac{\ln A}{b}
 \end{aligned}$$

Since  $t \geq 0$  and  $b > 0$ , then  $\frac{\ln A}{b} \geq 0 \therefore A > 1$

$$\begin{aligned}
 f\left(\frac{\ln A}{b}\right) &= \frac{C}{1 + Ae^{-b\left(\frac{\ln A}{b}\right)}} \\
 &= \frac{C}{1 + Ae^{-\ln A}} \\
 &= \frac{C}{1 + Ae^{\ln A^{-1}}} \\
 &= \frac{C}{1 + AA^{-1}} \\
 &= \frac{C}{1 + 1} = \frac{C}{2}
 \end{aligned}$$

So, if  $A > 1$ , there is a point of inflection with  $y$ -coordinate  $\frac{C}{2}$ .





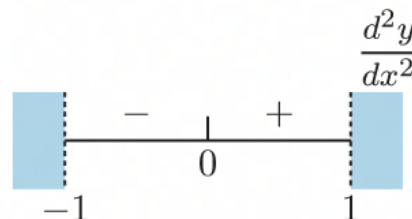
**13 a**  $y = \arcsin x, \quad -1 \leq x \leq 1$

$$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}, \quad -1 < x < 1$$

$$= (1-x^2)^{-\frac{1}{2}}$$

$$\therefore \frac{d^2y}{dx^2} = -\frac{1}{2}(1-x^2)^{-\frac{3}{2}}(-2x) \quad \{\text{chain rule}\}$$

$$= \frac{x}{(1-x^2)^{\frac{3}{2}}} \quad \text{which has sign diagram:}$$



$\frac{d^2y}{dx^2}$  changes sign when  $x = 0$ , so this point is a point of inflection.

When  $x = 0$ ,  $y = \arcsin 0$  and  $\frac{dy}{dx} = \frac{1}{(1-0^2)^{\frac{3}{2}}} = 1 \neq 0$   
 $\therefore y = 0$

$\therefore (0, 0)$  is a non-stationary inflection point.

**b**  $y = \arctan(x - \frac{1}{2})$

Now,  $\frac{d}{dx}(\arctan[f(x)]) = \frac{f'(x)}{1+[f(x)]^2}$  from **Exercise 17H** question **3 b**.

$$\therefore \frac{dy}{dx} = \frac{1}{1+(x-\frac{1}{2})^2}$$

$$= \frac{1}{1+x^2-x+\frac{1}{4}}$$

$$= \frac{1}{x^2-x+\frac{5}{4}}$$

$$= (x^2-x+\frac{5}{4})^{-1}$$

$$\therefore \frac{d^2y}{dx^2} = -(x^2-x+\frac{5}{4})^{-2}(2x-1) \quad \{\text{chain rule}\}$$

$$= -\frac{2x-1}{(x^2-x+\frac{5}{4})^2}$$

For  $x^2-x+\frac{5}{4}$ ,  $\Delta = (-1)^2 - 4(1)(\frac{5}{4})$   
 $= 1 - 5 = -4 < 0$

$$\therefore x^2-x+\frac{5}{4} \neq 0$$

$$\therefore (x^2-x+\frac{5}{4})^2 > 0$$

$\therefore \frac{d^2y}{dx^2}$  has sign diagram:

Since the sign of  $\frac{d^2y}{dx^2}$  changes when  $x = \frac{1}{2}$ , this is a point of inflection.

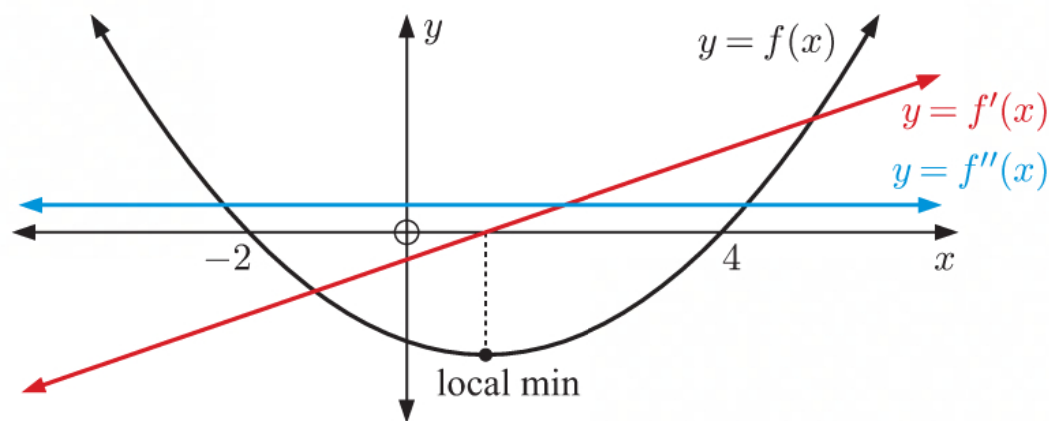
When  $x = \frac{1}{2}$ ,  $y = \arctan(\frac{1}{2} - \frac{1}{2})$  and  $\frac{dy}{dx} = \frac{1}{1+(\frac{1}{2}-\frac{1}{2})^2} = 1 \neq 0$   
 $\therefore y = \arctan 0$

$$\therefore y = 0$$

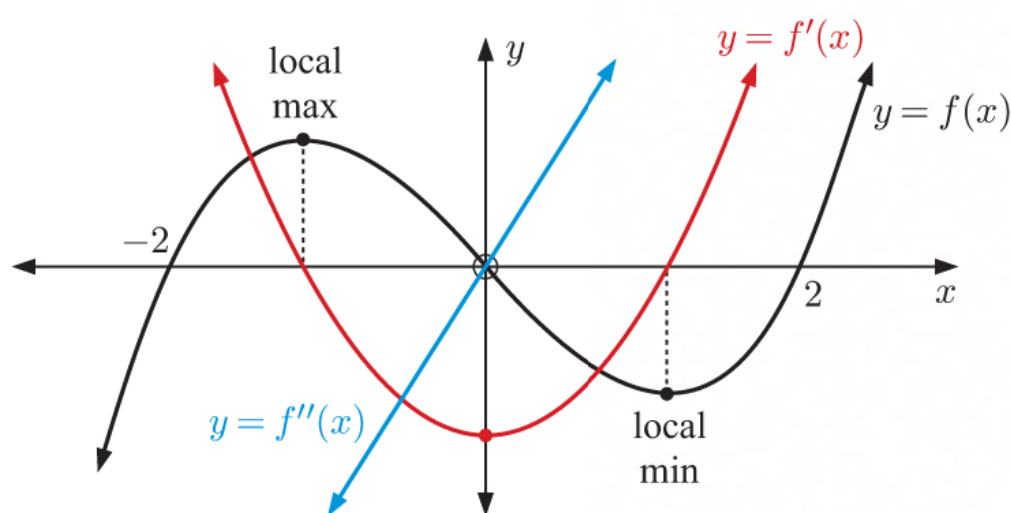
$\therefore (\frac{1}{2}, 0)$  is a non-stationary inflection point.

## EXERCISE 18G

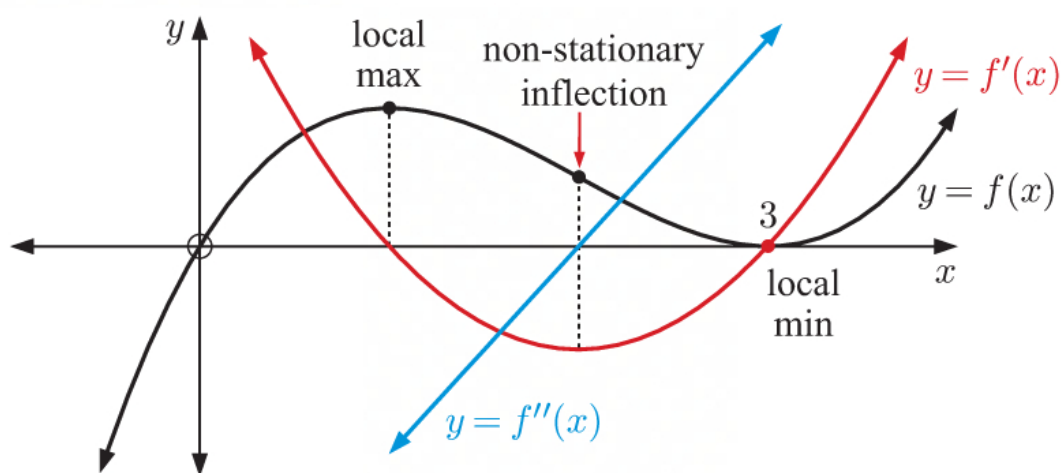
- 1 a** At the local minimum,  
 $f'(x) = 0$  and  $f''(x) > 0$ .  
 The graph is concave up for  
 all  $x$ , so  $f''(x) > 0$ .



- b** At the local maximum,  
 $f'(x) = 0$  and  $f''(x) < 0$ .  
 At the local minimum,  
 $f'(x) = 0$  and  $f''(x) > 0$ .  
 At the non-stationary point of inflection,  
 $f'(x) \neq 0$   
 and  $f''(x) = 0$ .



- c** At the local maximum,  $f'(x) = 0$  and  $f''(x) < 0$ .  
 At the local minimum,  $f'(x) = 0$  and  $f''(x) > 0$ .  
 At the non-stationary point of inflection,  $f'(x) \neq 0$  and  $f''(x) = 0$ .



**2 Note:** Other solutions are possible.

**a** The sign diagram of  $f'(x)$  is:

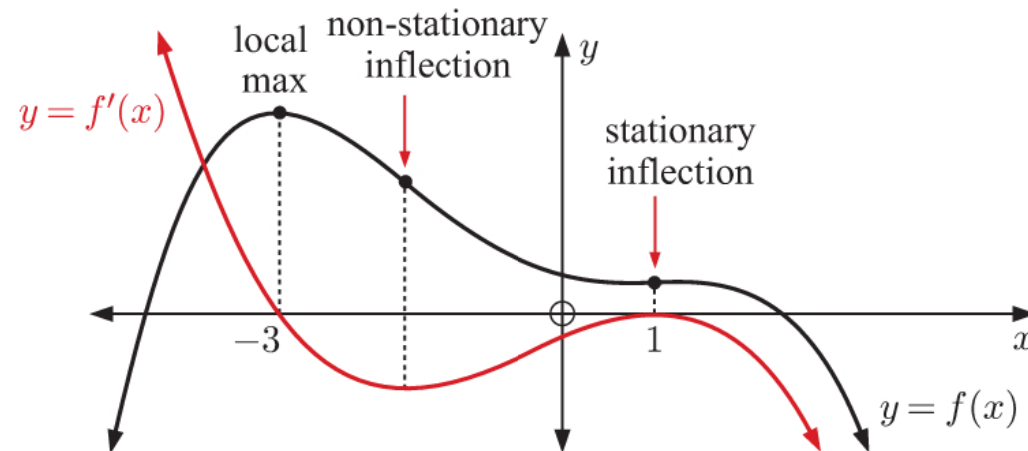


$\therefore y = f(x)$  has a local maximum at  $x = -3$ , and an inflection point at  $x = 1$ .

$f'(x)$  is a minimum when  $x \approx -\frac{3}{2}$  and a maximum when  $x = 1$ .

Now  $f''(-\frac{3}{2}) = 0$  but  $f'(-\frac{3}{2}) \neq 0$ , so this point corresponds to a non-stationary point of inflection.

Also  $f''(1) = 0$  and  $f'(1) = 0$ , so this point corresponds to a stationary point of inflection.



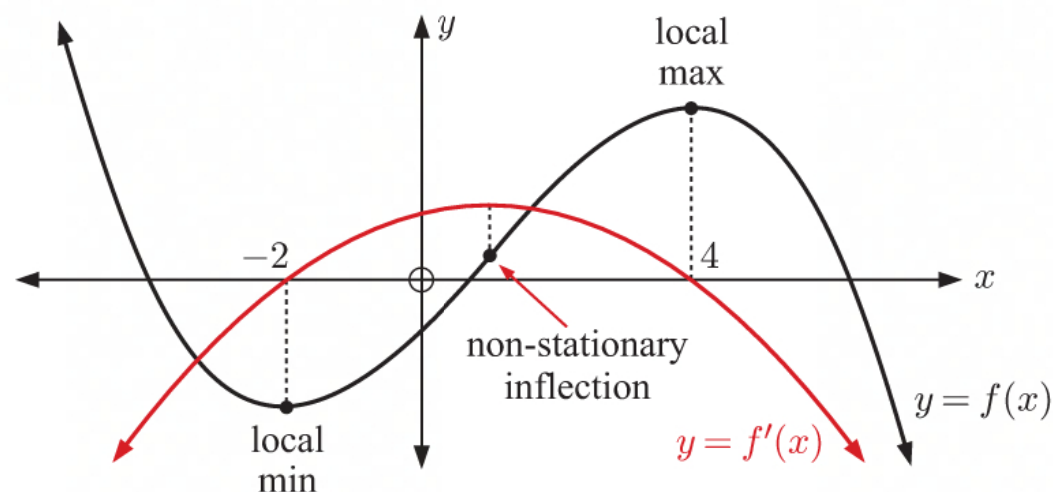
**b** The sign diagram of  $f'(x)$  is:



$\therefore y = f(x)$  has a local minimum at  $x = -2$ , and a local maximum at  $x = 4$ .

$f'(x)$  is a maximum when  $x \approx 1$ .

At this point,  $f''(1) = 0$  but  $f'(1) \neq 0$ , so it corresponds to a non-stationary point of inflection.



**3**  $g'(0) = 0$  and  $g''(0) = 0$

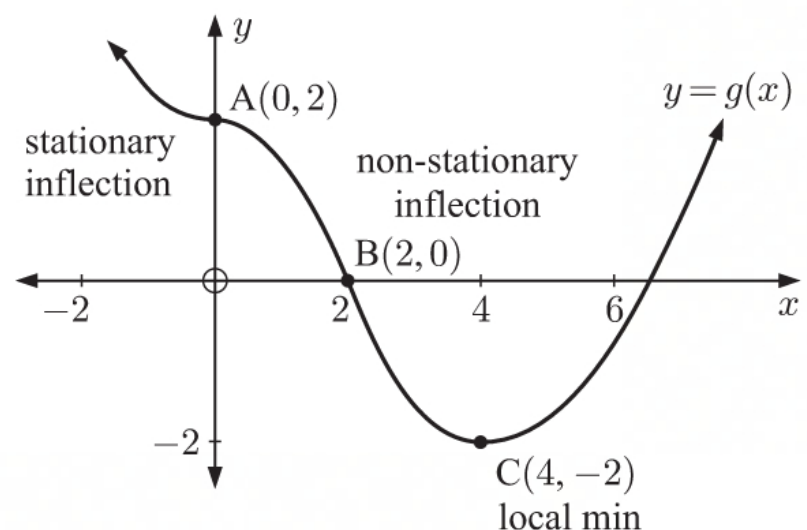
$\therefore$  there is a stationary inflection point at  $A(0, 2)$ .

$g''(2) = 0$  and  $g'(2) \neq 0$

$\therefore$  there is a non-stationary inflection point at  $B(2, 0)$ .

$g'(4) = 0$  and  $g''(4) > 0$

$\therefore$  there is a local minimum at  $C(4, -2)$ .



**EXERCISE 18H**

- 1 a**  $\lim_{x \rightarrow 0} (1 - \cos x) = 0$  and  $\lim_{x \rightarrow 0} x^2 = 0$ , so we can use l'Hôpital's rule.

$$\begin{aligned} \therefore \lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} &= \lim_{x \rightarrow 0} \frac{\frac{d}{dx}(1 - \cos x)}{\frac{d}{dx}(x^2)} \quad \{\text{l'Hôpital's rule}\} \\ &= \lim_{x \rightarrow 0} \frac{\sin x}{2x} \\ &= \frac{1}{2} \lim_{x \rightarrow 0} \frac{\sin x}{x} \\ &= \frac{1}{2} \quad \{\text{Fundamental Trigonometric Limit}\} \end{aligned}$$

- b**  $\lim_{x \rightarrow 0} (e^x - 1 - x) = 1 - 1 - 0 = 0$  and  $\lim_{x \rightarrow 0} x^2 = 0$ , so we can use l'Hôpital's rule.

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{e^x - 1 - x}{x^2} &= \lim_{x \rightarrow 0} \frac{\frac{d}{dx}(e^x - 1 - x)}{\frac{d}{dx}(x^2)} \quad \{\text{l'Hôpital's rule}\} \\ &= \lim_{x \rightarrow 0} \frac{e^x - 1}{2x} \end{aligned}$$

As  $x \rightarrow 0$ ,  $e^x - 1 \rightarrow 0$  and  $2x \rightarrow 0$ , so we can use l'Hôpital's rule again.

$$\begin{aligned} \therefore \lim_{x \rightarrow 0} \frac{e^x - 1}{2x} &= \frac{\lim_{x \rightarrow 0} \frac{d}{dx}(e^x - 1)}{\lim_{x \rightarrow 0} \frac{d}{dx}(2x)} \quad \{\text{l'Hôpital's rule}\} \\ &= \frac{\lim_{x \rightarrow 0} e^x}{\lim_{x \rightarrow 0} 2} \\ &= \frac{1}{2} \\ \therefore \lim_{x \rightarrow 0} \frac{e^x - 1 - x}{x^2} &= \frac{1}{2} \end{aligned}$$

- c**  $\lim_{x \rightarrow 1} \ln x = 0$  and  $\lim_{x \rightarrow 1} (x - 1) = 0$ , so we can use l'Hôpital's rule.

$$\begin{aligned} \therefore \lim_{x \rightarrow 1} \frac{\ln x}{x - 1} &= \lim_{x \rightarrow 1} \frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(x - 1)} \quad \{\text{l'Hôpital's rule}\} \\ &= \frac{\lim_{x \rightarrow 1} \frac{1}{x}}{\lim_{x \rightarrow 1} 1} \\ &= \frac{1}{1} = 1 \end{aligned}$$



**d** As  $x \rightarrow \infty$ ,  $e^x \rightarrow \infty$ , so we can use l'Hôpital's rule.

$$\begin{aligned}\therefore \lim_{x \rightarrow \infty} \frac{e^x}{x} &= \lim_{x \rightarrow \infty} \frac{\frac{d}{dx}(e^x)}{\frac{d}{dx}(x)} \quad \{\text{l'Hôpital's rule}\} \\ &= \frac{\lim_{x \rightarrow \infty} e^x}{\lim_{x \rightarrow \infty} 1} \\ &= \lim_{x \rightarrow \infty} e^x\end{aligned}$$

As  $x \rightarrow \infty$ ,  $e^x \rightarrow \infty$

$\therefore \lim_{x \rightarrow \infty} \frac{e^x}{x}$  does not exist.

**e** As  $x \rightarrow 0^+$ ,  $\ln x \rightarrow -\infty$ , so we can use l'Hôpital's rule.

$$\begin{aligned}\therefore \lim_{x \rightarrow 0^+} x \ln x &= \lim_{x \rightarrow 0^+} \frac{\ln x}{x^{-1}} \quad \{\text{converting to the form } \frac{\infty}{\infty}\} \\ &= \lim_{x \rightarrow 0^+} \frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(x^{-1})} \quad \{\text{l'Hôpital's rule}\} \\ &= \lim_{x \rightarrow 0^+} \frac{(\frac{1}{x})}{-x^{-2}} \\ &= \lim_{x \rightarrow 0^+} (-x) \\ &= 0\end{aligned}$$

**f**  $\lim_{x \rightarrow 0} \arctan x = 0$  and  $\lim_{x \rightarrow 0} x = 0$ , so we can use l'Hôpital's rule.

$$\begin{aligned}\therefore \lim_{x \rightarrow 0} \frac{\arctan x}{x} &= \lim_{x \rightarrow 0} \frac{\frac{d}{dx}(\arctan x)}{\frac{d}{dx}(x)} \quad \{\text{l'Hôpital's rule}\} \\ &= \frac{\lim_{x \rightarrow 0} \frac{1}{1+x^2}}{\lim_{x \rightarrow 0} 1} \\ &= \frac{\frac{1}{1+0}}{1} = 1\end{aligned}$$

**g**  $\lim_{x \rightarrow 0} (x^2 + x) = 0$  and  $\lim_{x \rightarrow 0} \sin 2x = 0$ , so we can use l'Hôpital's rule.

$$\begin{aligned}\therefore \lim_{x \rightarrow 0} \frac{x^2 + x}{\sin 2x} &= \lim_{x \rightarrow 0} \frac{\frac{d}{dx}(x^2 + x)}{\frac{d}{dx}(\sin 2x)} \quad \{\text{l'Hôpital's rule}\} \\ &= \frac{\lim_{x \rightarrow 0} (2x + 1)}{\lim_{x \rightarrow 0} 2 \cos 2x} \\ &= \frac{1}{2}\end{aligned}$$

**h**  $\lim_{x \rightarrow 0^+} \sin x = 0$  and  $\lim_{x \rightarrow 0^+} \sqrt{x} = 0$ , so we can use l'Hôpital's rule.

$$\begin{aligned} \therefore \lim_{x \rightarrow 0^+} \frac{\sin x}{\sqrt{x}} &= \lim_{x \rightarrow 0^+} \frac{\frac{d}{dx}(\sin x)}{\frac{d}{dx}(\sqrt{x})} \quad \{\text{l'Hôpital's rule}\} \\ &= \lim_{x \rightarrow 0^+} \frac{\cos x}{\left(\frac{1}{2\sqrt{x}}\right)} \\ &= \lim_{x \rightarrow 0^+} 2\sqrt{x} \cos x \\ &= 2 \times 0 \times 1 = 0 \end{aligned}$$

**i**  $\lim_{x \rightarrow 0} (x + \sin x) = 0$  and  $\lim_{x \rightarrow 0} (x - \sin x) = 0$ , so we can use l'Hôpital's rule.

$$\begin{aligned} \therefore \lim_{x \rightarrow 0} \frac{x + \sin x}{x - \sin x} &= \lim_{x \rightarrow 0} \frac{\frac{d}{dx}(x + \sin x)}{\frac{d}{dx}(x - \sin x)} \quad \{\text{l'Hôpital's rule}\} \\ &= \frac{\lim_{x \rightarrow 0} (1 + \cos x)}{\lim_{x \rightarrow 0} (1 - \cos x)} \end{aligned}$$

Now  $\lim_{x \rightarrow 0} (1 + \cos x) = 1 + 1 = 2$  but  $\lim_{x \rightarrow 0} (1 - \cos x) = 1 - 1 = 0$

$\therefore \lim_{x \rightarrow 0} \frac{x + \sin x}{x - \sin x}$  does not exist.

**j** As  $x \rightarrow 0^+$ ,  $x^2 \rightarrow 0$  and  $\ln x \rightarrow -\infty$ , so we can use l'Hôpital's rule.

$$\begin{aligned} \therefore \lim_{x \rightarrow 0^+} x^2 \ln x &= \lim_{x \rightarrow 0^+} \frac{\ln x}{x^{-2}} \quad \{\text{converting to the form } \frac{\infty}{\infty}\} \\ &= \lim_{x \rightarrow 0^+} \frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(x^{-2})} \quad \{\text{l'Hôpital's rule}\} \\ &= \lim_{x \rightarrow 0^+} \frac{\left(\frac{1}{x}\right)}{-2x^{-3}} \\ &= \lim_{x \rightarrow 0^+} \left(-\frac{x^2}{2}\right) \\ &= 0 \end{aligned}$$

**k**  $\lim_{x \rightarrow 0} (a^x - b^x) = 1 - 1 = 0$  and  $\lim_{x \rightarrow 0} \sin x = 0$ , so we can use l'Hôpital's rule.

$$\begin{aligned} \therefore \lim_{x \rightarrow 0} \frac{a^x - b^x}{\sin x} &= \lim_{x \rightarrow 0} \frac{\frac{d}{dx}(a^x - b^x)}{\frac{d}{dx}(\sin x)} && \{\text{l'Hôpital's rule}\} \\ &= \lim_{x \rightarrow 0} \frac{a^x \ln a - b^x \ln b}{\cos x} && \{a, b > 0\} \\ &= \frac{\lim_{x \rightarrow 0} (a^x \ln a - b^x \ln b)}{\lim_{x \rightarrow 0} \cos x} \\ &= \frac{\ln a - \ln b}{1} \\ &= \ln\left(\frac{a}{b}\right) \end{aligned}$$

**2** As  $x \rightarrow \infty$ ,  $x^2 \rightarrow \infty$  and  $e^{-x} \rightarrow 0$ , so we can use l'Hôpital's rule.

$$\begin{aligned} \lim_{x \rightarrow \infty} x^2 e^{-x} &= \lim_{x \rightarrow \infty} \frac{x^2}{e^x} && \{\text{converting to the form } \frac{\infty}{\infty}\} \\ &= \lim_{x \rightarrow \infty} \frac{\frac{d}{dx}(x^2)}{\frac{d}{dx}(e^x)} && \{\text{l'Hôpital's rule}\} \\ &= \lim_{x \rightarrow \infty} \frac{2x}{e^x} \end{aligned}$$

As  $x \rightarrow \infty$ ,  $2x \rightarrow \infty$  and  $e^x \rightarrow \infty$ , so we can use l'Hôpital's rule again.

$$\begin{aligned} \therefore \lim_{x \rightarrow \infty} \frac{2x}{e^x} &= \lim_{x \rightarrow \infty} \frac{\frac{d}{dx}(2x)}{\frac{d}{dx}(e^x)} && \{\text{l'Hôpital's rule}\} \\ &= \lim_{x \rightarrow \infty} \frac{2}{e^x} \\ &= 0 \end{aligned}$$

$f(x) = x^2 e^{-x} \geq 0$  for all  $x \in \mathbb{R}$ , so as  $x \rightarrow \infty$ ,  $f(x) \rightarrow 0^+$ .

**3** As  $x \rightarrow \frac{\pi}{2}^-$ ,  $\tan x \rightarrow \infty$  and  $\sec x \rightarrow \infty$ , so we can use l'Hôpital's rule.

$$\begin{aligned} \therefore \lim_{x \rightarrow \frac{\pi}{2}^-} \frac{\tan x}{\sec x} &= \lim_{x \rightarrow \frac{\pi}{2}^-} \frac{\frac{d}{dx}(\tan x)}{\frac{d}{dx}(\sec x)} && \{\text{l'Hôpital's rule}\} \\ &= \lim_{x \rightarrow \frac{\pi}{2}^-} \frac{\sec^2 x}{\sec x \tan x} \\ &= \lim_{x \rightarrow \frac{\pi}{2}^-} \frac{\sec x}{\tan x} && \text{so we can use l'Hôpital's rule again.} \\ \therefore \lim_{x \rightarrow \frac{\pi}{2}^-} \frac{\sec x}{\tan x} &= \lim_{x \rightarrow \frac{\pi}{2}^-} \frac{\frac{d}{dx}(\sec x)}{\frac{d}{dx}(\tan x)} && \{\text{l'Hôpital's rule}\} \\ &= \lim_{x \rightarrow \frac{\pi}{2}^-} \frac{\sec x \tan x}{\sec^2 x} \\ &= \lim_{x \rightarrow \frac{\pi}{2}^-} \frac{\tan x}{\sec x} && \text{which is where we started.} \end{aligned}$$

So, l'Hôpital's rule does not help us here.

- 4**  $\lim_{x \rightarrow 0} \left( \frac{\pi}{2} - \arccos x - x \right) = \frac{\pi}{2} - \frac{\pi}{2} - 0 = 0$  and  $\lim_{x \rightarrow 0} x^3 = 0$ , so we can use l'Hôpital's rule to determine the behaviour of  $f(x)$  as  $x \rightarrow 0$ .

$$\begin{aligned}
 \therefore \lim_{x \rightarrow 0} \frac{\frac{\pi}{2} - \arccos x - x}{x^3} &= \lim_{x \rightarrow 0} \frac{\frac{d}{dx} \left( \frac{\pi}{2} - \arccos x - x \right)}{\frac{d}{dx} (x^3)} \quad \{\text{l'Hôpital's rule}\} \\
 &= \lim_{x \rightarrow 0} \frac{0 - \left( \frac{-1}{\sqrt{1-x^2}} \right) - 1}{3x^2} \\
 &= \lim_{x \rightarrow 0} \frac{\frac{1}{\sqrt{1-x^2}} - 1}{3x^2} \\
 &= \lim_{x \rightarrow 0} \frac{1 - \sqrt{1-x^2}}{3x^2 \sqrt{1-x^2}} \times \left( \frac{1 + \sqrt{1-x^2}}{1 + \sqrt{1-x^2}} \right) \\
 &= \lim_{x \rightarrow 0} \frac{1 - (1-x^2)}{3x^2 \sqrt{1-x^2} (1 + \sqrt{1-x^2})} \\
 &= \lim_{x \rightarrow 0} \frac{1}{3\sqrt{1-x^2} (1 + \sqrt{1-x^2})} \quad \{\text{as } x \neq 0\} \\
 &= \frac{1}{3(1)(2)} = \frac{1}{6}
 \end{aligned}$$

If there were a vertical asymptote at  $x = 0$ , then the function would approach  $\pm\infty$  as  $x$  approaches 0.

$$\therefore y = \frac{\frac{\pi}{2} - \arccos x - x}{x^3} \quad \text{does not have a vertical asymptote when } x = 0.$$

**5 a**  $f(x) = \frac{e^x - 1}{x^2 + x}$   
 $= \frac{e^x - 1}{x(x+1)}$

So, the domain of  $f(x)$  is  $\{x \mid x \neq 0, -1\}$ .

**b**  $f'(x) = \frac{e^x(x^2 + x) - (e^x - 1)(2x + 1)}{(x^2 + x)^2} \quad \{\text{quotient rule}\}$   
 $= \frac{x^2 e^x + x e^x - 2x e^x - e^x + 2x + 1}{(x^2 + x)^2}$   
 $= \frac{e^x(x^2 - x - 1) + 2x + 1}{(x(x+1))^2}$

So, the domain of  $f'(x)$  is also  $\{x \mid x \neq 0, -1\}$ .

**c**  $f'(x) = 0$  when  $e^x(x^2 - x - 1) + 2x + 1 = 0$   
 $\therefore x = 0$  or  $x \approx 0.774$  {using technology}

But  $x \neq 0$ ,  $\therefore$  the turning point is approximately  $(0.774, 0.851)$ .

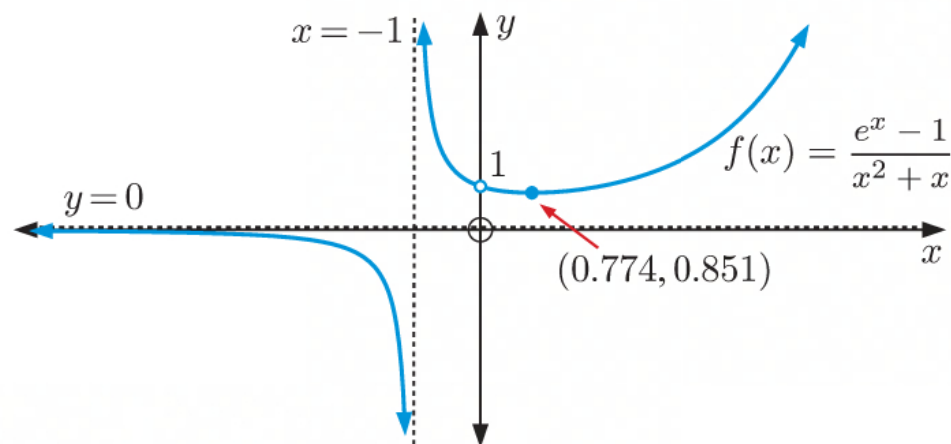
**d i** As  $x \rightarrow -\infty$ ,  $e^x - 1 \rightarrow -1$  and  $x^2 + x \rightarrow \infty$ , so  $\frac{e^x - 1}{x^2 + x} \rightarrow 0^-$ .



ii As  $x \rightarrow 0$ ,  $e^x - 1 \rightarrow 0$  and  $x^2 + x \rightarrow 0$ , so we can use l'Hôpital's rule.

$$\begin{aligned}
 \therefore \lim_{x \rightarrow 0} \frac{e^x - 1}{x^2 + x} &= \lim_{x \rightarrow 0} \frac{\frac{d}{dx}(e^x - 1)}{\frac{d}{dx}(x^2 + x)} \quad \{\text{l'Hôpital's rule}\} \\
 &= \lim_{x \rightarrow 0} \frac{e^x}{2x + 1} \\
 &= \frac{\lim_{x \rightarrow 0} e^x}{\lim_{x \rightarrow 0} (2x + 1)} \\
 &= \frac{1}{1} = 1
 \end{aligned}$$

e



6 a  $\lim_{x \rightarrow 0^+} \ln(\cos 5x) = 0$  and  $\lim_{x \rightarrow 0^+} \ln(\cos 3x) = 0$ , so we can use l'Hôpital's rule.

$$\begin{aligned}
 \therefore \lim_{x \rightarrow 0^+} \frac{\ln(\cos 5x)}{\ln(\cos 3x)} &= \lim_{x \rightarrow 0^+} \frac{\frac{d}{dx}(\ln(\cos 5x))}{\frac{d}{dx}(\ln(\cos 3x))} \quad \{\text{l'Hôpital's rule}\} \\
 &= \lim_{x \rightarrow 0^+} \frac{\left( \frac{-5 \sin 5x}{\cos 5x} \right)}{\left( \frac{-3 \sin 3x}{\cos 3x} \right)} \\
 &= \lim_{x \rightarrow 0^+} \frac{5 \sin 5x \cos 3x}{3 \sin 3x \cos 5x} \\
 &= \left( \lim_{x \rightarrow 0^+} \frac{\sin 5x}{\sin 3x} \right) \times \left( \lim_{x \rightarrow 0^+} \frac{5 \cos 3x}{3 \cos 5x} \right) \\
 &= \left( \lim_{x \rightarrow 0^+} \frac{\sin 5x}{\sin 3x} \right) \times \frac{5}{3} \\
 &= \lim_{x \rightarrow 0} \frac{\left( \frac{\sin 5x}{5x} \right)}{\left( \frac{\sin 3x}{3x} \right)} \times \frac{25}{9} \\
 &= \frac{\lim_{x \rightarrow 0} \frac{\sin 5x}{5x}}{\lim_{x \rightarrow 0} \frac{\sin 3x}{3x}} \times \frac{25}{9} \\
 &= \frac{25}{9} \quad \{\text{Fundamental Trigonometric Limit}\}
 \end{aligned}$$

**b**  $\frac{\ln(\sin 2x)}{\ln(\sin 3x)}$  is not defined when  $\sin 3x \leq 0$

that is, when  $\pi \leq 3x \leq 2\pi$

$$\therefore \frac{\pi}{3} \leq x \leq \frac{2\pi}{3}$$

Since  $\frac{\pi}{3} < \frac{\pi}{2} < \frac{2\pi}{3}$ ,  $\frac{\ln(\sin 2x)}{\ln(\sin 3x)}$  is undefined in the region around  $x = \frac{\pi}{2}$ .

$\therefore \lim_{x \rightarrow \frac{\pi}{2}^-} \frac{\ln(\sin 2x)}{\ln(\sin 3x)}$  does not exist.

**7** As  $x \rightarrow 0^+$ ,  $\frac{1}{x} \rightarrow \infty$  and  $\frac{1}{\sin x} \rightarrow \infty$ .

Now  $\frac{1}{x} - \frac{1}{\sin x} = \frac{\sin x - x}{x \sin x}$  {converting to a quotient}

$$\therefore \lim_{x \rightarrow 0^+} \left( \frac{1}{x} - \frac{1}{\sin x} \right) = \lim_{x \rightarrow 0^+} \frac{\sin x - x}{x \sin x}$$

But  $\lim_{x \rightarrow 0^+} (\sin x - x) = 0$  and  $\lim_{x \rightarrow 0^+} x \sin x = 0$

$$\therefore \lim_{x \rightarrow 0^+} \left( \frac{1}{x} - \frac{1}{\sin x} \right) = \lim_{x \rightarrow 0^+} \frac{\cos x - 1}{\sin x + x \cos x} \quad \{\text{l'Hôpital's rule}\}$$

But  $\lim_{x \rightarrow 0^+} (\cos x - 1) = 1 - 1 = 0$  and  $\lim_{x \rightarrow 0^+} (\sin x + x \cos x) = 0$

$$\begin{aligned} \therefore \lim_{x \rightarrow 0^+} \left( \frac{1}{x} - \frac{1}{\sin x} \right) &= \lim_{x \rightarrow 0^+} \frac{-\sin x}{\cos x + \cos x - x \sin x} \quad \{\text{l'Hôpital's rule}\} \\ &= \frac{\lim_{x \rightarrow 0^+} (-\sin x)}{\lim_{x \rightarrow 0^+} (2 \cos x - x \sin x)} \\ &= \frac{0}{2 - 0} = 0 \end{aligned}$$

**8 a** As  $x \rightarrow 0^+$ ,  $\frac{1}{x} \rightarrow \infty$  and  $\frac{1}{\sin 2x} \rightarrow \infty$ .

Now  $\frac{1}{x} - \frac{1}{\sin 2x} = \frac{\sin 2x - x}{x \sin 2x}$  {converting to a quotient}

$$\therefore \lim_{x \rightarrow 0^+} \left( \frac{1}{x} - \frac{1}{\sin 2x} \right) = \lim_{x \rightarrow 0^+} \frac{\sin 2x - x}{x \sin 2x}$$

But  $\lim_{x \rightarrow 0^+} (\sin 2x - x) = 0$  and  $\lim_{x \rightarrow 0^+} x \sin 2x = 0$

$$\begin{aligned} \therefore \lim_{x \rightarrow 0^+} \left( \frac{1}{x} - \frac{1}{\sin 2x} \right) &= \lim_{x \rightarrow 0^+} \frac{2 \cos 2x - 1}{\sin 2x + 2x \cos 2x} \quad \{\text{l'Hôpital's rule}\} \\ &= \frac{\lim_{x \rightarrow 0^+} (2 \cos 2x - 1)}{\lim_{x \rightarrow 0^+} (\sin 2x + 2x \cos 2x)} \end{aligned}$$

Now  $\lim_{x \rightarrow 0^+} (2 \cos 2x - 1) = 2 - 1 = 1$  but  $\lim_{x \rightarrow 0^+} (\sin 2x + 2x \cos 2x) = 0 + 0 = 0$

$\therefore \lim_{x \rightarrow 0^+} \left( \frac{1}{x} - \frac{1}{\sin 2x} \right)$  does not exist.

$$\begin{aligned}
 \text{b} \quad \sec^2 x - \tan x &= \frac{1}{\cos^2 x} - \frac{\sin x}{\cos x} \\
 &= \frac{1 - \sin x \cos x}{\cos^2 x} \quad \{\text{converting to a quotient}\} \\
 &= \frac{1 - \frac{1}{2} \sin 2x}{\cos^2 x}
 \end{aligned}$$

$$\text{Now } \lim_{x \rightarrow \frac{\pi}{2}^-} (1 - \frac{1}{2} \sin 2x) = 1 \quad \text{but} \quad \lim_{x \rightarrow \frac{\pi}{2}^-} \cos^2 x = 0$$

$$\therefore \lim_{x \rightarrow \frac{\pi}{2}^-} (\sec^2 x - \tan x) \text{ does not exist.}$$

9 a For all  $k \in \mathbb{Z}^+$  as  $x \rightarrow \infty$ ,  $x^k \rightarrow \infty$  and  $e^x \rightarrow \infty$ , so we can use l'Hôpital's rule.

$$\begin{aligned}
 \therefore \lim_{x \rightarrow \infty} \frac{x^k}{e^x} &= \lim_{x \rightarrow \infty} \frac{kx^{k-1}}{e^x} \\
 &= \lim_{x \rightarrow \infty} \frac{k(k-1)x^{k-2}}{e^x} \\
 &\quad \vdots \\
 &= \lim_{x \rightarrow \infty} \frac{k! x^0}{e^x} \\
 &= k! \lim_{x \rightarrow \infty} e^{-x} \quad \{k \text{ is fixed}\} \\
 &= 0
 \end{aligned}
 \left. \vphantom{\lim_{x \rightarrow \infty} \frac{x^k}{e^x}} \right\} \text{ \{applying l'Hôpital's rule } k \text{ times}\}$$

b The result in a implies that for any power  $k \in \mathbb{Z}^+$ ,  $e^x$  is much greater than  $x^k$  as  $x$  becomes large.

This is in fact true for any  $k > 0$ , since  $x^n < x^k < x^{n+1}$  for any  $n < k < n+1$ ,  $n \in \mathbb{Z}^+$ ,  $x > 0$ .

$\therefore e^x$  increases more rapidly than any fixed positive power of  $x$ .

10 Consider  $\lim_{x \rightarrow \infty} \frac{\ln x}{x^k}$ ,  $k > 0$

For any fixed positive  $k$ , as  $x \rightarrow \infty$ ,  $\ln x \rightarrow \infty$  and  $x^k \rightarrow \infty$ , so we can use l'Hôpital's rule.

$$\begin{aligned}
 \therefore \lim_{x \rightarrow \infty} \frac{\ln x}{x^k} &= \lim_{x \rightarrow \infty} \frac{\left(\frac{1}{x}\right)}{kx^{k-1}} \quad \{\text{l'Hôpital's rule}\} \\
 &= \lim_{x \rightarrow \infty} \frac{1}{kx^k} \\
 &= 0
 \end{aligned}$$

$\therefore$  as  $x \rightarrow \infty$ ,  $x^k$  becomes much larger than  $\ln x$ .

$\therefore \ln x$  increases more slowly than any fixed positive power of  $x$ .

$$\mathbf{11} \quad \mathbf{a} \quad \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right) = 1 + 0 = 1$$

$$\begin{aligned} \therefore \lim_{x \rightarrow \infty} \ln \left(1 + \frac{1}{x}\right) &= \ln \left( \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right) \right) && \{\ln x \text{ is continuous for all } x > 0\} \\ &= \ln 1 \\ &= 0 \end{aligned}$$

Now consider  $x \ln \left(1 + \frac{1}{x}\right) = \frac{\ln(1 + x^{-1})}{x^{-1}}$ .

$\lim_{x \rightarrow \infty} \ln(1 + x^{-1}) = 0$  and  $\lim_{x \rightarrow \infty} x^{-1} = 0$ , so we can use l'Hôpital's rule.

$$\begin{aligned} \therefore \lim_{x \rightarrow \infty} x \ln \left(1 + \frac{1}{x}\right) &= \lim_{x \rightarrow \infty} \frac{\left(\frac{-x^{-2}}{1+x^{-1}}\right)}{-x^{-2}} && \{\text{l'Hôpital's rule}\} \\ &= \lim_{x \rightarrow \infty} \frac{1}{1+x^{-1}} \\ &= \frac{1}{1+0} \\ &= 1 \quad \text{as required} \end{aligned}$$

$$\mathbf{b} \quad \left(1 + \frac{1}{x}\right) = e^{\ln \left(1 + \frac{1}{x}\right)} \quad \{e^{\ln a} = a\}$$

$$\therefore \left(1 + \frac{1}{x}\right)^x = e^{x \ln \left(1 + \frac{1}{x}\right)}$$

$$\begin{aligned} \therefore \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x &= \lim_{x \rightarrow \infty} e^{x \ln \left(1 + \frac{1}{x}\right)} \\ &= e^{\left[ \lim_{x \rightarrow \infty} x \ln \left(1 + \frac{1}{x}\right) \right]} && \{e^x \text{ is continuous for all } x \in \mathbb{R}\} \\ &= e^1 && \{\text{from } \mathbf{a}\} \\ &= e \quad \text{as required} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad \lim_{x \rightarrow \infty} \left(1 + \frac{a}{x}\right)^x &= \lim_{x \rightarrow \infty} \left(1 + \frac{1}{\left(\frac{x}{a}\right)}\right)^x && \{a \neq 0\} \\ &= \lim_{x \rightarrow \infty} \left[ \left(1 + \frac{1}{\left(\frac{x}{a}\right)}\right)^{\frac{x}{a}} \right]^a \\ &= e^a && \{\text{using } \mathbf{b}\} \end{aligned}$$



**12**  $x = e^{\ln x}$  for  $x > 0$

$$\begin{aligned}\therefore x^{\sin x} &= (e^{\ln x})^{\sin x} \\ &= e^{\sin x \ln x} \\ &= e^{\frac{\ln x}{(\sin x)^{-1}}}\end{aligned}$$

As  $x \rightarrow 0^+$ ,  $\ln x \rightarrow -\infty$  and  $(\sin x)^{-1} \rightarrow \infty$ , so we can use l'Hôpital's rule.

$$\begin{aligned}\therefore \lim_{x \rightarrow 0^+} \frac{\ln x}{(\sin x)^{-1}} &= \lim_{x \rightarrow 0^+} \frac{\left(\frac{1}{x}\right)}{-(\sin x)^{-2} \cos x} \quad \{\text{l'Hôpital's rule}\} \\ &= \lim_{x \rightarrow 0^+} -\frac{\sin^2 x}{x \cos x}\end{aligned}$$

As  $x \rightarrow 0^+$ ,  $\sin^2 x \rightarrow 0$  and  $x \cos x \rightarrow 0$ , so we can use l'Hôpital's rule.

$$\begin{aligned}\therefore \lim_{x \rightarrow 0^+} \frac{\ln x}{(\sin x)^{-1}} &= \lim_{x \rightarrow 0^+} \frac{-2 \sin x \cos x}{\cos x + x(-\sin x)} \\ &= \frac{-2(0)(1)}{1 - 0} \\ &= 0 \quad \dots (*)\end{aligned}$$

$$\begin{aligned}\therefore \lim_{x \rightarrow 0^+} x^{\sin x} &= \lim_{x \rightarrow 0^+} e^{\frac{\ln x}{(\sin x)^{-1}}} \\ &= e^{\lim_{x \rightarrow 0^+} \frac{\ln x}{(\sin x)^{-1}}} \quad \{e^x \text{ is continuous for all } x \in \mathbb{R}\} \\ &= e^0 \quad \{\text{using } (*)\} \\ &= 1 \quad \text{as required}\end{aligned}$$

**13**  $x = e^{\ln x}$  for  $x > 0$

$$\begin{aligned}\therefore x^{\frac{1}{x}} &= (e^{\ln x})^{\frac{1}{x}} \\ &= e^{\frac{\ln x}{x}}\end{aligned}$$

Consider  $\lim_{x \rightarrow \infty} \frac{\ln x}{x}$ .

As  $x \rightarrow \infty$ ,  $\ln x \rightarrow \infty$ , so we can use l'Hôpital's rule.

$$\begin{aligned}\therefore \lim_{x \rightarrow \infty} \frac{\ln x}{x} &= \lim_{x \rightarrow \infty} \frac{\left(\frac{1}{x}\right)}{1} \quad \{\text{l'Hôpital's rule}\} \\ &= \lim_{x \rightarrow \infty} \frac{1}{x} \\ &= 0\end{aligned}$$

$$\begin{aligned}\therefore \lim_{x \rightarrow \infty} x^{\frac{1}{x}} &= e^{\lim_{x \rightarrow \infty} \frac{\ln x}{x}} \quad \{e^x \text{ is continuous for all } x \in \mathbb{R}\} \\ &= e^0 \\ &= 1\end{aligned}$$

So as  $x \rightarrow \infty$ ,  $y = x^{\frac{1}{x}} \rightarrow 1$ .

## REVIEW SET 18A

1 a  $y = -2x^2$

When  $x = -1$ ,  $y = -2(-1)^2 = -2$

$\therefore$  the point of contact is  $(-1, -2)$ .

Now  $\frac{dy}{dx} = -4x$

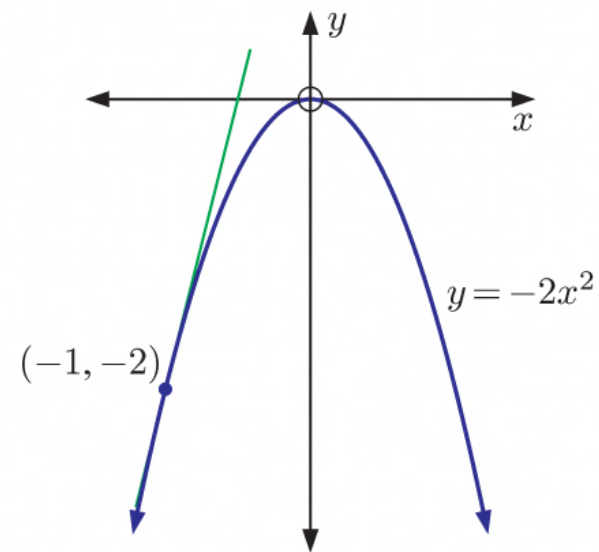
When  $x = -1$ ,  $\frac{dy}{dx} = -4(-1) = 4$

So, the tangent has equation  $y = 4(x - (-1)) - 2$

$$\therefore y = 4(x + 1) - 2$$

$$\therefore y = 4x + 4 - 2$$

$$\therefore y = 4x + 2$$



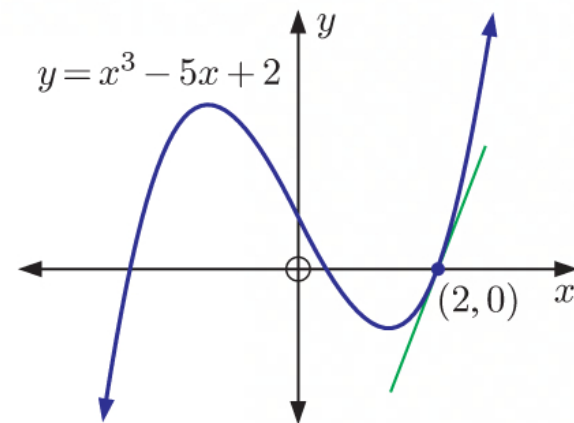
b  $y = x^3 - 5x + 2$

$$\therefore \frac{dy}{dx} = 3x^2 - 5$$

When  $x = 2$ ,  $\frac{dy}{dx} = 3(2)^2 - 5$   
 $= 12 - 5$   
 $= 7$

So, the tangent has equation  $y = 7(x - 2) + 0$

$$\therefore y = 7x - 14$$



c  $y = \frac{1 - 2x}{x^2}$

$$\therefore \frac{dy}{dx} = \frac{(-2)x^2 - (1 - 2x)(2x)}{x^4} \quad \{\text{quotient rule}\}$$

$$= \frac{-2x^2 - 2x + 4x^2}{x^4}$$

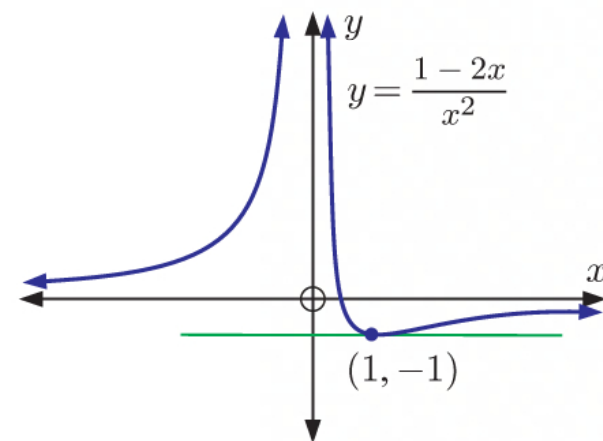
$$= \frac{2x(x - 1)}{x^4}$$

$$= \frac{2(x - 1)}{x^3}$$

When  $x = 1$ ,  $\frac{dy}{dx} = \frac{2(1 - 1)}{1^3} = 0$

So, the tangent has equation  $y = 0(x - 1) - 1$

$$\therefore y = -1$$



**d**  $f(x) = e^{3x-1}$

$$\begin{aligned}\therefore f(0) &= e^{3(0)-1} \\ &= e^{-1} \\ &= \frac{1}{e}\end{aligned}$$

$\therefore$  the point of contact is  $(0, \frac{1}{e})$ .

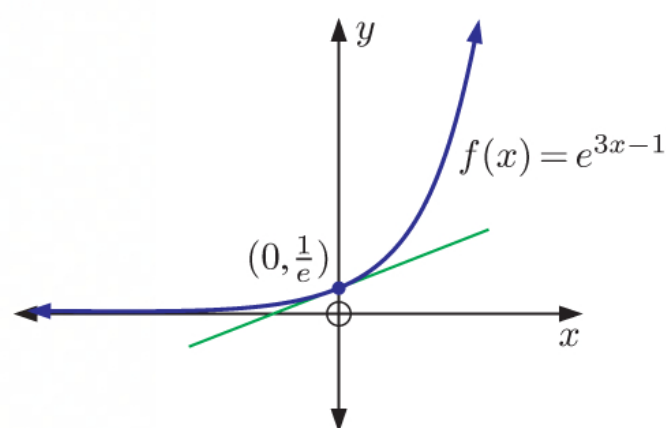
Now  $f(x) = e^{3x-1}$

$$\therefore f'(x) = 3e^{3x-1}$$

$$\begin{aligned}\therefore f'(0) &= 3e^{3(0)-1} \\ &= 3e^{-1} \\ &= \frac{3}{e}\end{aligned}$$

So, the tangent has equation  $3x - ey = 3(0) - e\left(\frac{1}{e}\right)$

$$\therefore 3x - ey = -1$$



**e**  $f(x) = \ln(x^2)$

$$\begin{aligned}\therefore f(e) &= \ln(e^2) \\ &= 2\end{aligned}$$

$\therefore$  the point of contact is  $(e, 2)$ .

Now  $f(x) = \ln(x^2)$   
 $= 2 \ln x$

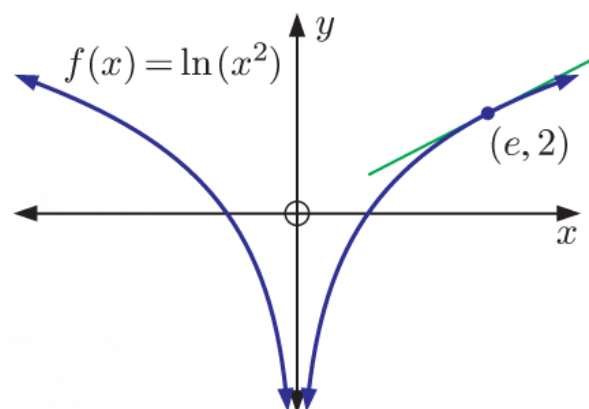
$$\therefore f'(x) = \frac{2}{x}$$

$$\therefore f'(e) = \frac{2}{e}$$

So, the tangent has equation  $y = \frac{2}{e}(x - e) + 2$

$$\therefore y = \frac{2}{e}x - 2 + 2$$

$$\therefore y = \frac{2}{e}x$$



**2 a**  $y = \sqrt{3x+4} = (3x+4)^{\frac{1}{2}}$

$$\therefore \frac{dy}{dx} = \frac{1}{2}(3x+4)^{-\frac{1}{2}} \times 3 \quad \{\text{chain rule}\}$$

$$= \frac{3}{2\sqrt{3x+4}}, \quad \text{so at } (4, 4)$$

$$\frac{dy}{dx} = \frac{3}{2\sqrt{3(4)+4}}$$

$$= \frac{3}{2\sqrt{16}}$$

$$= \frac{3}{8}$$

$\therefore$  the normal at  $(4, 4)$  has gradient  $-\frac{8}{3}$ .

$\therefore$  the equation of the normal is  $8x + 3y = 8(4) + 3(4)$

$$\therefore 8x + 3y = 44$$

**b**  $y = 3e^{2x}$

When  $x = 1$ ,  $y = 3e^{2(1)} = 3e^2$

So, the point of contact is  $(1, 3e^2)$ .

Now  $\frac{dy}{dx} = 3e^{2x} \times 2$  {chain rule}  
 $= 6e^{2x}$

So at  $x = 1$ ,  $\frac{dy}{dx} = 6e^{2(1)}$   
 $= 6e^2$

$\therefore$  the normal at  $(1, 3e^2)$  has gradient  $-\frac{1}{6e^2}$ .

$\therefore$  the equation of the normal is  $y = -\frac{1}{6e^2}(x - 1) + 3e^2$

$$\therefore y = -\frac{1}{6e^2}x + \frac{1}{6e^2} + 3e^2 \times \frac{6e^2}{6e^2}$$

$$\therefore y = -\frac{1}{6e^2}x + \frac{1}{6e^2} + \frac{18e^4}{6e^2}$$

$$\therefore y = -\frac{1}{6e^2}x + \frac{18e^4 + 1}{6e^2}$$

**c**  $y = \frac{x+1}{x^2-2}$

When  $x = 1$ ,  $y = \frac{1+1}{1-2} = -2$

So, the point of contact is  $(1, -2)$ .

Now  $\frac{dy}{dx} = \frac{1(x^2-2) - (x+1)(2x)}{(x^2-2)^2}$  {quotient rule}

$$= \frac{x^2 - 2 - 2x^2 - 2x}{(x^2-2)^2}$$

$$= \frac{-x^2 - 2x - 2}{(x^2-2)^2}$$

So at  $x = 1$ ,  $\frac{dy}{dx} = \frac{-(1)^2 - 2(1) - 2}{(1^2-2)^2}$

$$= \frac{-5}{1}$$

$$= -5$$

$\therefore$  the normal at  $(1, -2)$  has gradient  $\frac{1}{5}$ .

$\therefore$  the equation of the normal is  $y = \frac{1}{5}(x - 1) - 2$

$$\therefore 5y = x - 1 - 10$$

$$\therefore x - 5y = 11$$



**d**  $y = 3x^e - e^x$

When  $x = e$ ,  $y = 3e^e - e^e$   
 $= 2e^e$

So, the point of contact is  $(e, 2e^e)$ .

Now  $\frac{dy}{dx} = (e)3x^{e-1} - e^x$   
 $= 3ex^{e-1} - e^x$

So at  $x = e$ ,  $\frac{dy}{dx} = 3ee^{e-1} - e^e$   
 $= 3e^e - e^e$   
 $= 2e^e$

$\therefore$  the normal at  $(e, 2e^e)$  has gradient  $-\frac{1}{2e^e}$ .

$\therefore$  the equation of the normal is  $y = -\frac{1}{2e^e}(x - e) + 2e^e$

$$\therefore 2e^e y = -x + e + 4e^{2e}$$

$$\therefore x + 2e^e y = e + 4e^{2e}$$

**3**  $f(x) = e^{4x} + px + q$

$\therefore f(0) = e^{4(0)} + p(0) + q$   
 $= 1 + q$

So, the point of contact is  $(0, 1 + q)$ .

Now  $f'(x) = 4e^{4x} + p$   
 $\therefore f'(0) = 4e^{4(0)} + p$   
 $= 4 + p$

So, the tangent has equation  $y = (4 + p)(x - 0) + 1 + q$   
 $\therefore y = (4 + p)x + 1 + q$

But we know the tangent has equation  $y = 5x - 7$ .

$\therefore 4 + p = 5$  and  $1 + q = -7$   
 $\therefore p = 1$  and  $q = -8$

**4 Note:** The first print of this book erroneously repeats **Review set 17A** question **8**.

$y = 4x^3 + 6x^2 - 13x + 1$   
 $\therefore \frac{dy}{dx} = 12x^2 + 12x - 13$

The gradient of the tangent is 11 when  $12x^2 + 12x - 13 = 11$

$$\therefore 12x^2 + 12x - 24 = 0$$

$$\therefore 12(x^2 + x - 2) = 0$$

$$\therefore 12(x + 2)(x - 1) = 0$$

$$\therefore x = -2 \text{ or } 1$$

When  $x = -2$ ,

$$y = 4(-2)^3 + 6(-2)^2 - 13(-2) + 1$$

$$= 19$$

When  $x = 1$ ,

$$y = 4(1)^3 + 6(1)^2 - 13(1) + 1$$

$$= -2$$

So, the gradient of the tangent to  $y = 4x^3 + 6x^2 - 13x + 1$  is 11 at the points  $(-2, 19)$  and  $(1, -2)$ .

$$5 \quad y = \frac{a}{(x+2)^2} = a(x+2)^{-2}, \quad A(2, 4), \quad B(0, 8)$$

$$\text{The gradient of the line (AB)} = \frac{8-4}{0-2} = \frac{4}{-2} = -2$$

$$\therefore \text{ the equation of the tangent is } \frac{y-8}{x-0} = -2 \text{ or } y = -2x + 8$$

$$\text{Now } \frac{dy}{dx} = -2a(x+2)^{-3}, \text{ so for the given tangent, } -2a(x+2)^{-3} = -2$$

$$\begin{aligned} \therefore \frac{a}{(x+2)^3} &= 1 \\ \therefore a &= (x+2)^3 \quad \dots (*) \end{aligned}$$

$$\text{The line (AB) meets the curve where } -2x + 8 = \frac{a}{(x+2)^2}$$

$$\therefore -2x + 8 = \frac{(x+2)^3}{(x+2)^2} \quad \{\text{using } (*)\}$$

$$\therefore -2x + 8 = x + 2$$

$$\therefore -3x = -6$$

$$\therefore x = 2$$

$$\text{and so } a = (2+2)^3 = 64$$

$$6 \quad y = \frac{x}{\sqrt{1-x}} = x(1-x)^{-\frac{1}{2}}$$

$$\text{When } x = -3, \quad y = \frac{-3}{\sqrt{1-(-3)}} = -\frac{3}{2}$$

So, the point of contact is  $(-3, -\frac{3}{2})$ .

$$\text{Now } \frac{dy}{dx} = 1(1-x)^{-\frac{1}{2}} + x(-\frac{1}{2})(1-x)^{-\frac{3}{2}}(-1) \quad \{\text{product rule}\}$$

$$= (1-x)^{-\frac{1}{2}} + \frac{x}{2}(1-x)^{-\frac{3}{2}}$$

$$= \frac{1}{\sqrt{1-x}} + \frac{x}{2(1-x)\sqrt{1-x}}$$

$$\text{When } x = -3, \quad \frac{dy}{dx} = \frac{1}{\sqrt{1-(-3)}} + \frac{-3}{2(1-(-3))\sqrt{1-(-3)}}$$

$$= \frac{1}{2} - \frac{3}{2(4)(2)}$$

$$= \frac{8}{16} - \frac{3}{16}$$

$$= \frac{5}{16}$$

$$\therefore \text{ the equation of the tangent at } (-3, -\frac{3}{2}) \text{ is } y = \frac{5}{16}(x - (-3)) - \frac{3}{2}$$

$$\therefore 16y = 5x + 15 - 24$$

$$\therefore 5x - 16y = 9$$

Equating coefficients with  $5x + by = a$ ,  $a = 9$  and  $b = -16$ .

**7** Let  $f(x) = 2x^3 + 4x - 1$

$$\therefore f'(x) = 6x^2 + 4$$

$$\begin{aligned}\therefore f'(1) &= 6(1)^2 + 4 \\ &= 10\end{aligned}$$

$$\begin{aligned}\therefore \text{the equation of the tangent at } (1, 5) \text{ is } y &= 10(x - 1) + 5 \\ \text{which is } y &= 10x - 5\end{aligned}$$

The curve meets the tangent when  $2x^3 + 4x - 1 = 10x - 5$

$$\therefore 2x^3 - 6x + 4 = 0$$

$$\therefore x^3 - 3x + 2 = 0$$

$$\therefore (x - 1)^2(x + 2) = 0 \quad \{\text{tangent at } x = 1\}$$

$$\begin{aligned}f(-2) &= 2(-2)^3 + 4(-2) - 1 \\ &= -16 - 8 - 1 \\ &= -25\end{aligned}$$

$\therefore$  the tangent meets the curve again at  $(-2, -25)$ .

**8 a**  $y = e^{2x}$

When  $x = a$ ,  $y = e^{2a}$

So, the point of contact is  $(a, e^{2a})$ .

Now  $\frac{dy}{dx} = 2e^{2x}$

When  $x = a$ ,  $\frac{dy}{dx} = 2e^{2a}$

$\therefore$  the normal at  $x = a$  has gradient  $-\frac{1}{2e^{2a}}$ .

$\therefore$  the equation of the normal is  $y = -\frac{1}{2e^{2a}}(x - a) + e^{2a}$

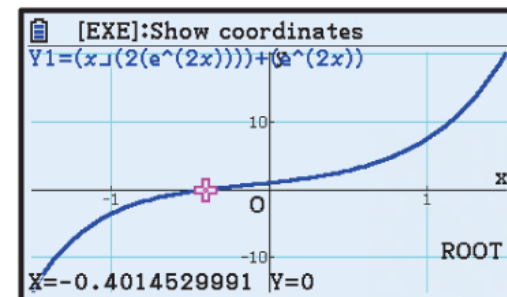
$$\therefore y = -\frac{1}{2e^{2a}}x + \frac{a}{2e^{2a}} + e^{2a}$$

**b**  $y = -\frac{1}{2e^{2a}}x + \frac{a}{2e^{2a}} + e^{2a}$  passes through the origin when  $x = 0$ ,  $y = 0$

$$\therefore 0 = -\frac{1}{2e^{2a}}(0) + \frac{a}{2e^{2a}} + e^{2a}$$

$$\therefore \frac{a}{2e^{2a}} + e^{2a} = 0$$

$$\therefore a \approx -0.40145$$



$\therefore$  the normal to  $y = e^{2x}$  which passes through the origin is

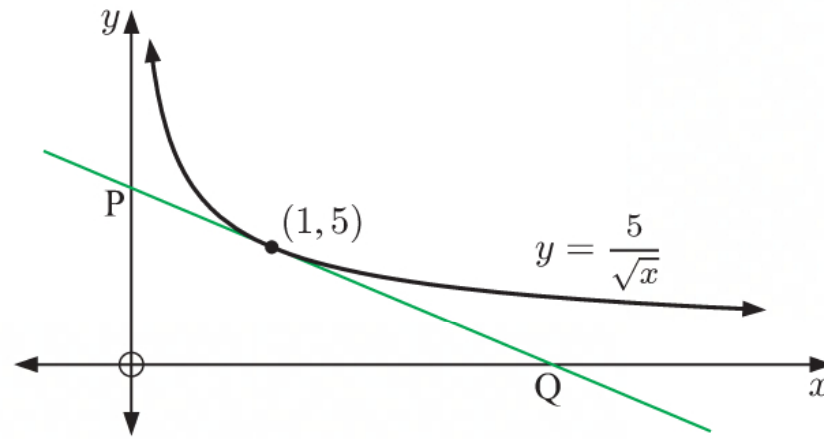
$$y = -\frac{1}{2e^{2(-0.40145)}}x \quad \left\{ \frac{a}{2e^{2a}} + e^{2a} = 0 \right\}$$

$$\therefore y \approx -1.12x$$

$$9 \quad y = \frac{5}{\sqrt{x}} = 5x^{-\frac{1}{2}}$$

$$\therefore \frac{dy}{dx} = -\frac{5}{2}x^{-\frac{3}{2}} \\ = -\frac{5}{2x\sqrt{x}}$$

$$\text{At the point } (1, 5), \quad \frac{dy}{dx} = -\frac{5}{2(1)\sqrt{1}} \\ = -\frac{5}{2}$$



$\therefore$  the gradient of the tangent at  $(1, 5)$  is  $-\frac{5}{2}$ .

$\therefore$  the tangent has equation  $y = -\frac{5}{2}(x - 1) + 5$

$$\text{which is } y = -\frac{5}{2}x + \frac{15}{2}$$

$$\text{At point P, } x = 0 \quad \therefore y = -\frac{5}{2}(0) + \frac{15}{2} = \frac{15}{2}$$

$$\text{At point Q, } y = 0 \quad \therefore 0 = -\frac{5}{2}x + \frac{15}{2}$$

$$\therefore \frac{5}{2}x = \frac{15}{2}$$

$$\therefore x = 3$$

So, P is  $(0, \frac{15}{2})$  or  $(0, 7.5)$ , and Q is  $(3, 0)$ .

$$10 \quad y = x^2\sqrt{1-x}$$

$$\text{When } x = -3, \quad y = (-3)^2\sqrt{1-(-3)} = 18$$

$\therefore$  the point of contact is  $(-3, 18)$ .

$$\text{Now } y = x^2\sqrt{1-x} = x^2(1-x)^{\frac{1}{2}}$$

$$\therefore \frac{dy}{dx} = 2x(1-x)^{\frac{1}{2}} + x^2\left(\frac{1}{2}\right)(1-x)^{-\frac{1}{2}}(-1) \quad \{\text{product rule and chain rule}\}$$

$$= 2x\sqrt{1-x} - \frac{x^2}{2\sqrt{1-x}}$$

$$\text{When } x = -3, \quad \frac{dy}{dx} = 2(-3)\sqrt{1-(-3)} - \frac{(-3)^2}{2\sqrt{1-(-3)}} \\ = -12 - \frac{9}{4} = -\frac{57}{4}$$

So, the tangent has equation  $y = -\frac{57}{4}(x + 3) + 18$

$$\therefore y = -\frac{57}{4}x - \frac{99}{4}$$

$$\text{When } y = 0, \quad -\frac{57}{4}x - \frac{99}{4} = 0$$

$$\therefore 57x = -99$$

$$\therefore x = -\frac{99}{57}$$

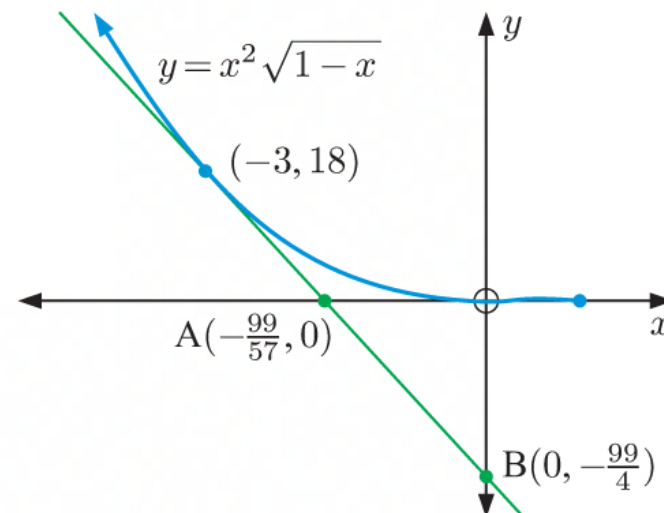
$\therefore$  the  $x$ -intercept of the tangent is  $-\frac{99}{57}$ .

$$\text{When } x = 0, \quad y = -\frac{99}{4}$$

$\therefore$  the  $y$ -intercept of the tangent is  $-\frac{99}{4}$ .

$\therefore$  A is  $(-\frac{99}{57}, 0)$  and B is  $(0, -\frac{99}{4})$ .

$$\therefore \text{area of triangle OAB} = \frac{1}{2} \times \frac{99}{57} \times \frac{99}{4} \\ = \frac{3267}{152} \approx 21.5 \text{ units}^2$$





$$\begin{aligned}
 11 \quad f(x) &= \frac{3x}{1+x} \\
 \therefore f'(x) &= \frac{3(1+x) - 3x(1)}{(1+x)^2} \quad \{\text{quotient rule}\} \\
 &= \frac{3 + 3x - 3x}{(1+x)^2} \\
 &= \frac{3}{(1+x)^2} \\
 \therefore f'(2) &= \frac{3}{(1+2)^2} \\
 &= \frac{1}{3}
 \end{aligned}$$

$\therefore$  the normal at  $(2, 2)$  has gradient  $-3$ .

$\therefore$  the equation of the normal is  $y = -3(x - 2) + 2$   
 $\therefore y = -3x + 8$

When  $x = 0$ ,  $y = 8$ , so the  $y$ -intercept of the normal is 8.

When  $y = 0$ ,  $-3x + 8 = 0$

$\therefore x = \frac{8}{3}$ , so the  $x$ -intercept of the normal is  $\frac{8}{3}$ .

So, B is  $(0, 8)$  and C is  $(\frac{8}{3}, 0)$ .

$$\begin{aligned}
 BC &= \sqrt{(\frac{8}{3} - 0)^2 + (0 - 8)^2} \\
 &= \sqrt{\frac{64}{9} + 64} \\
 &= \sqrt{\frac{640}{9}} \\
 &= \frac{8\sqrt{10}}{3} \text{ units}
 \end{aligned}$$

$$12 \quad a \quad y = \sec x = \frac{1}{\cos x}$$

$$\text{When } x = \frac{\pi}{3}, \quad y = \frac{1}{(\frac{1}{2})} = 2$$

So, the point of contact is  $(\frac{\pi}{3}, 2)$ .

$$\begin{aligned}
 \text{Now } \frac{dy}{dx} &= \tan x \sec x \\
 &= \frac{\sin x}{\cos^2 x}
 \end{aligned}$$

$$\begin{aligned}
 \text{When } x = \frac{\pi}{3}, \quad \frac{dy}{dx} &= \frac{(\frac{\sqrt{3}}{2})}{(\frac{1}{2})^2} \\
 &= 2\sqrt{3}
 \end{aligned}$$

$\therefore$  the tangent at  $(\frac{\pi}{3}, 2)$  has equation  $y = 2\sqrt{3}(x - \frac{\pi}{3}) + 2$

$$\therefore y = 2\sqrt{3}x - \frac{2\sqrt{3}\pi}{3} + 2$$

$$\therefore 2\sqrt{3}x - y = \frac{2\sqrt{3}\pi}{3} - 2$$

**b**  $y = \arctan x$

$$\therefore \tan y = x$$

When  $x = \sqrt{3}$ ,  $\tan y = \sqrt{3}$

$$\therefore y = \frac{\pi}{3} \quad \left\{ -\frac{\pi}{2} < y < \frac{\pi}{2} \right\}$$

So, the point of contact is  $(\sqrt{3}, \frac{\pi}{3})$ .

Now  $\frac{dy}{dx} = \frac{1}{1+x^2}$

When  $x = \sqrt{3}$ ,  $\frac{dy}{dx} = \frac{1}{1+(\sqrt{3})^2} = \frac{1}{4}$

$\therefore$  the normal at  $(\sqrt{3}, \frac{\pi}{3})$  has gradient  $-4$ .

$\therefore$  the equation of the normal is  $y = -4(x - \sqrt{3}) + \frac{\pi}{3}$

$$\therefore y = -4x + 4\sqrt{3} + \frac{\pi}{3}$$

$$\therefore 4x + y = 4\sqrt{3} + \frac{\pi}{3}$$

**13**  $x^2 + y^2 = 1$

$$\therefore y^2 = 1 - x^2$$

$$\therefore y = \pm\sqrt{1-x^2}, \quad -1 \leq x \leq 1$$

Consider a point on the circle with  $x$ -coordinate  $a$ .

When  $x = a$ ,  $y = \pm\sqrt{1-a^2}$ ,  $-1 \leq a \leq 1$ .

So, the point of contact is  $(a, \pm\sqrt{1-a^2})$ .

Now  $2x + 2y \frac{dy}{dx} = 0$

$$\therefore 2y \frac{dy}{dx} = -2x$$

$$\therefore \frac{dy}{dx} = -\frac{x}{y}$$

When  $x = a$ ,  $\frac{dy}{dx} = -\frac{a}{\pm\sqrt{1-a^2}}$

$$= \mp \frac{a}{\sqrt{1-a^2}}$$

$\therefore$  the gradient of the normal at  $x = a$  is  $\pm \frac{\sqrt{1-a^2}}{a}$

$\therefore$  the equation of the normal is  $y = \pm \frac{\sqrt{1-a^2}}{a} (x - a) \pm \sqrt{1-a^2}$

$$\therefore y = \pm \frac{\sqrt{1-a^2}}{a} x \mp \sqrt{1-a^2} \pm \sqrt{1-a^2}$$

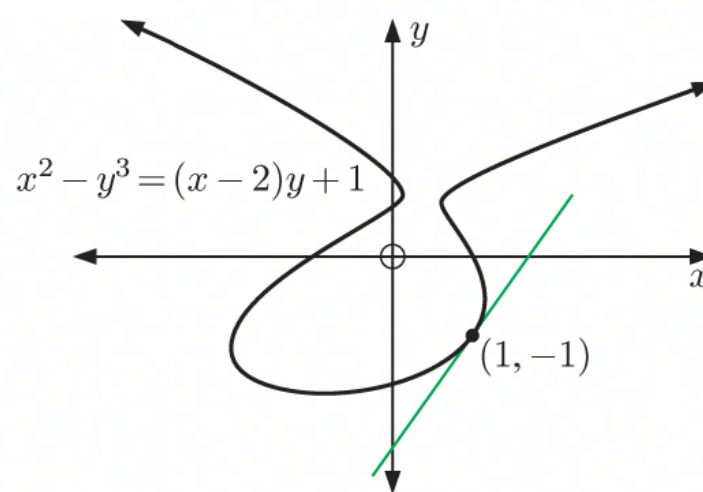
$$\therefore y = \pm \frac{\sqrt{1-a^2}}{a} x$$

This is of the form  $y = mx$ , and so always passes through the origin.

$$\begin{aligned}
 \mathbf{14} \quad \mathbf{a} \quad & x^2 - y^3 = (x-2)y + 1 \\
 & \therefore x^2 - y^3 = xy - 2y + 1 \\
 & \therefore \frac{d}{dx}(x^2 - y^3) = \frac{d}{dx}(xy - 2y + 1) \\
 & \therefore 2x - 3y^2 \frac{dy}{dx} = y + x \frac{dy}{dx} - 2 \frac{dy}{dx} \quad \{\text{chain rule, product rule}\} \\
 & \therefore 2x - y = \frac{dy}{dx}(3y^2 + x - 2) \\
 & \therefore \frac{dy}{dx} = \frac{2x - y}{3y^2 + x - 2}
 \end{aligned}$$

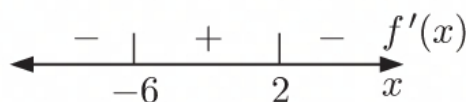
$$\begin{aligned}
 \mathbf{b} \quad \text{At } (1, -1), \quad & \frac{dy}{dx} = \frac{2(1) - (-1)}{3(-1)^2 + 1 - 2} \quad \{\text{using } \mathbf{a}\} \\
 & = \frac{3}{2}
 \end{aligned}$$

$$\begin{aligned}
 \therefore \text{the tangent at } (1, -1) \text{ has equation} \\
 y &= \frac{3}{2}(x - 1) - 1 \\
 \therefore y &= \frac{3}{2}x - \frac{5}{2}
 \end{aligned}$$



$$\begin{aligned}
 \mathbf{15} \quad & f(x) = -x^3 - 6x^2 + 36x - 17 \\
 \therefore f'(x) &= -3x^2 - 12x + 36 \\
 &= -3(x^2 + 4x - 12) \\
 &= -3(x+6)(x-2)
 \end{aligned}$$

which has sign diagram:

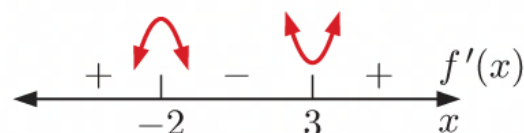


- a**  $f(x)$  is increasing for  $-6 \leq x \leq 2$ .
- b**  $f(x)$  is decreasing for  $x \leq -6$  or  $x \geq 2$ .

$$\mathbf{16} \quad f(x) = 2x^3 - 3x^2 - 36x + 7$$

$$\begin{aligned}
 \mathbf{a} \quad f'(x) &= 6x^2 - 6x - 36 \\
 &= 6(x^2 - x - 6) \\
 &= 6(x+2)(x-3)
 \end{aligned}$$

which has sign diagram:



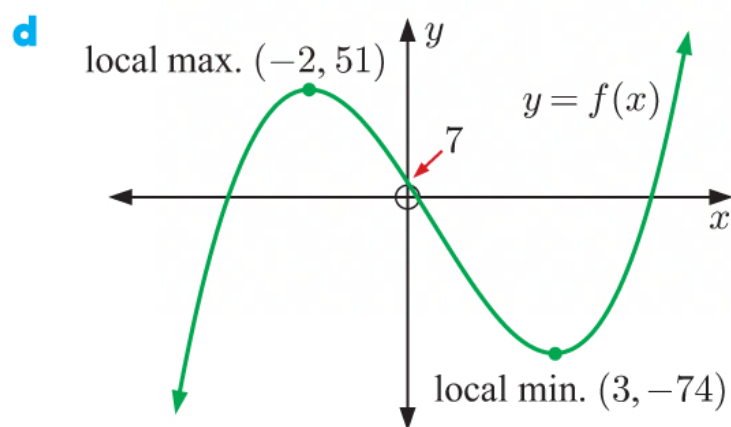
So, there is a local maximum at  $x = -2$  and a local minimum at  $x = 3$ .

$$\begin{aligned}
 f(-2) &= 2(-2)^3 - 3(-2)^2 - 36(-2) + 7 \\
 &= 51
 \end{aligned}$$

$$\begin{aligned}
 f(3) &= 2(3)^3 - 3(3)^2 - 36(3) + 7 \\
 &= -74
 \end{aligned}$$

There is a local maximum at  $(-2, 51)$  and a local minimum at  $(3, -74)$ .

- b**  $f(x)$  is increasing for  $x \leq -2$  and  $x \geq 3$ .  
 $f(x)$  is decreasing for  $-2 \leq x \leq 3$ .
- c** As  $x \rightarrow \infty$ ,  $f(x) \rightarrow \infty$   
 As  $x \rightarrow -\infty$ ,  $f(x) \rightarrow -\infty$



**17 a**  $f(x) = \frac{3x-2}{x+3}$  is defined when  $x+3 \neq 0$   
 $\therefore x \neq -3$

$\therefore f(x)$  has domain  $\{x \mid x \neq -3\}$ .

**b**  $f(x) = 0$  when  $3x - 2 = 0$   $f(0) = \frac{3(0)-2}{0+3}$   
 $\therefore 3x = 2$   $= -\frac{2}{3}$   
 $\therefore x = \frac{2}{3}$   $\therefore$  the  $y$ -intercept is  $-\frac{2}{3}$ .  
 $\therefore$  the  $x$ -intercept is  $\frac{2}{3}$ .

**c**  $f'(x) = \frac{3(x+3) - (3x-2)(1)}{(x+3)^2}$  {quotient rule}  
 $= \frac{\cancel{3x} + 9 - \cancel{3x} + 2}{(x+3)^2}$   
 $= \frac{11}{(x+3)^2}$  which has sign diagram:

**d** There are no values of  $x$  such that  $f'(x) = 0$ .  
 $\therefore f(x)$  does not have any stationary points.

**18** Let  $y = x + \frac{32}{x^2} = x + 32x^{-2}$ ,  $2 \leq x \leq 10$

$\therefore \frac{dy}{dx} = 1 - 64x^{-3} = 1 - \frac{64}{x^3}$

$\frac{dy}{dx} = 0$  when  $1 - \frac{64}{x^3} = 0$

$\therefore \frac{64}{x^3} = 1$

$\therefore 64 = x^3$

$\therefore x = \sqrt[3]{64} = 4$

$\frac{dy}{dx}$  has sign diagram:

$\therefore$  there is a local minimum at  $x = 4$ .

Critical value ( $x$ )	$y$
2 (end point)	10
4 (local minimum)	6
10 (end point)	$\approx 10.32$

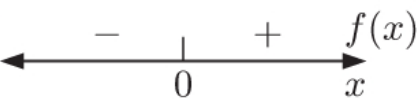
The greatest of these values is about 10.3 when  $x = 10$ .

The least of these values is 6 when  $x = 4$ .



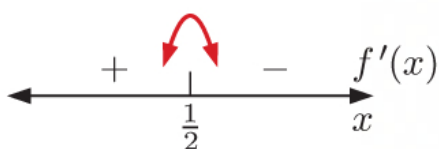
**19 a**  $f(x) = xe^{1-2x}$   
 $\therefore f'(x) = (1)e^{1-2x} + xe^{1-2x}(-2)$  {product rule, chain rule}  
 $= e^{1-2x} - 2xe^{1-2x}$   
 $= e^{1-2x}(1 - 2x)$

**b i**  $f(x) = 0$  when  $x = 0$   $\{e^{1-2x} > 0\}$

$\therefore f(x)$  has sign diagram: 

$\therefore f(x) > 0$  when  $x > 0$ .

**ii**  $f'(x) = e^{1-2x}(1 - 2x)$   
 $f'(x) = 0$  when  $1 - 2x = 0$   $\{e^{1-2x} > 0\}$   
 $\therefore x = \frac{1}{2}$

$\therefore f'(x)$  has sign diagram: 

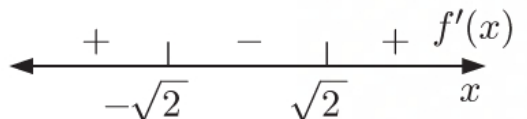
$\therefore f'(x) > 0$  when  $x < \frac{1}{2}$ .

**c** Stationary points corresponds to where  $f'(x) = 0$ .

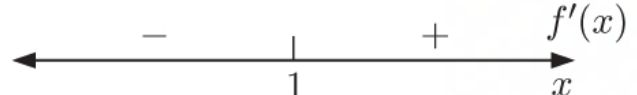
$f'(x) = 0$  when  $x = \frac{1}{2}$

and  $f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)e^{1-2\left(\frac{1}{2}\right)}$   
 $= \frac{1}{2}e^0$   
 $= \frac{1}{2}$

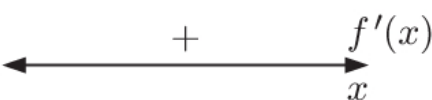
$\therefore \left(\frac{1}{2}, \frac{1}{2}\right)$  is a local maximum.

**20 a**  $f(x) = x^3 - 6x$   
 $\therefore f'(x) = 3x^2 - 6$   
 $= 3(x^2 - 2)$   
 $= 3(x + \sqrt{2})(x - \sqrt{2})$  which has sign diagram: 

$f(x)$  is increasing for  $x \leq -\sqrt{2}$  and  $x \geq \sqrt{2}$ , and decreasing for  $-\sqrt{2} \leq x \leq \sqrt{2}$ .

**b**  $f(x) = e^x(x - 2)$   
 $\therefore f'(x) = e^x(x - 2) + e^x(1)$  {product rule}  
 $= e^x(x - 1)$  which has sign diagram: 

$f(x)$  is increasing for  $x \geq 1$ , and decreasing for  $x \leq 1$ .

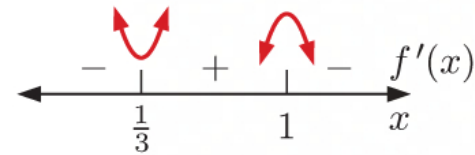
**c**  $f(x) = 2x - \sin x$   
 $\therefore f'(x) = 2 - \cos x$  which has sign diagram: 

$f(x)$  is increasing for all  $x \in \mathbb{R}$ .

**21 a**  $f(x) = -x^3 + 2x^2 - x + 3$

$$\therefore f'(x) = -3x^2 + 4x - 1$$

$$= (1 - 3x)(x - 1) \quad \text{which has sign diagram:}$$



$$\begin{aligned} \text{Now } f\left(\frac{1}{3}\right) &= -\left(\frac{1}{3}\right)^3 + 2\left(\frac{1}{3}\right)^2 - \left(\frac{1}{3}\right) + 3 & \text{and} & \quad f(1) = -(1)^3 + 2(1)^2 - (1) + 3 \\ &= -\frac{1}{27} + \frac{2}{9} - \frac{1}{3} + 3 & & \quad = -1 + 2 - 1 + 3 \\ &= \frac{77}{27} & & \quad = 3 \end{aligned}$$

$\therefore \left(\frac{1}{3}, \frac{77}{27}\right)$  is a local minimum, and  $(1, 3)$  is a local maximum.

**b**  $f(x) = \frac{x^2}{x+3}$

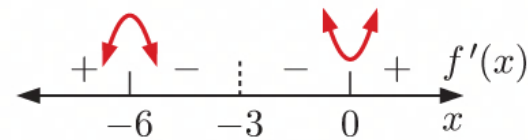
$$\therefore f'(x) = \frac{2x(x+3) - x^2(1)}{(x+3)^2} \quad \{\text{quotient rule}\}$$

$$= \frac{2x^2 + 6x - x^2}{(x+3)^2}$$

$$= \frac{x^2 + 6x}{(x+3)^2}$$

$$= \frac{x(x+6)}{(x+3)^2}$$

which has sign diagram:



$$\begin{aligned} f(-6) &= \frac{(-6)^2}{(-6+3)} & \text{and} & \quad f(0) = \frac{(0)^2}{0+3} \\ &= \frac{36}{-3} & & \quad = 0 \\ &= -12 \end{aligned}$$

$\therefore (-6, -12)$  is a local maximum, and  $(0, 0)$  is a local minimum.

**22 a**  $y = \sin \frac{x}{2}, \quad -\pi \leq x \leq \pi$

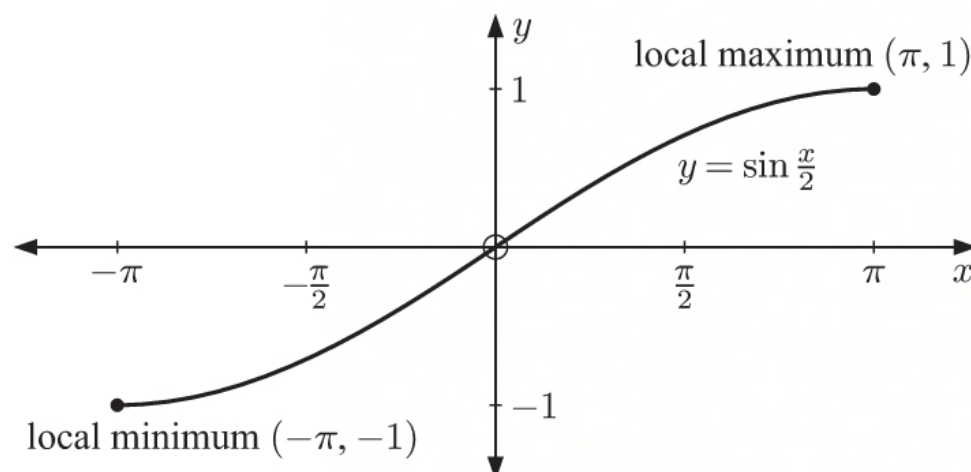
$$\therefore \frac{dy}{dx} = \frac{1}{2} \cos \frac{x}{2}$$

$$= 0 \quad \text{when } x = -\pi \text{ or } \pi$$

$$\begin{aligned} \text{When } x = -\pi, \quad y &= \sin\left(-\frac{\pi}{2}\right) \\ &= -1 \end{aligned}$$

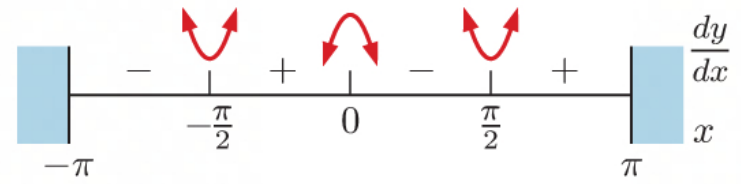
$$\begin{aligned} \text{When } x = \pi, \quad y &= \sin \frac{\pi}{2} \\ &= 1 \end{aligned}$$

$\therefore (-\pi, -1)$  is a local minimum,  $(\pi, 1)$  is a local maximum.



**b**  $y = \cos^2 x, \quad -\pi \leq x \leq \pi$

$$\begin{aligned}\therefore \frac{dy}{dx} &= 2 \cos x (-\sin x) \\ &= -2 \sin x \cos x \\ &= -\sin 2x \quad \text{which has sign diagram:}\end{aligned}$$



When  $x = -\pi$ ,  $y = \cos^2(-\pi) = 1$

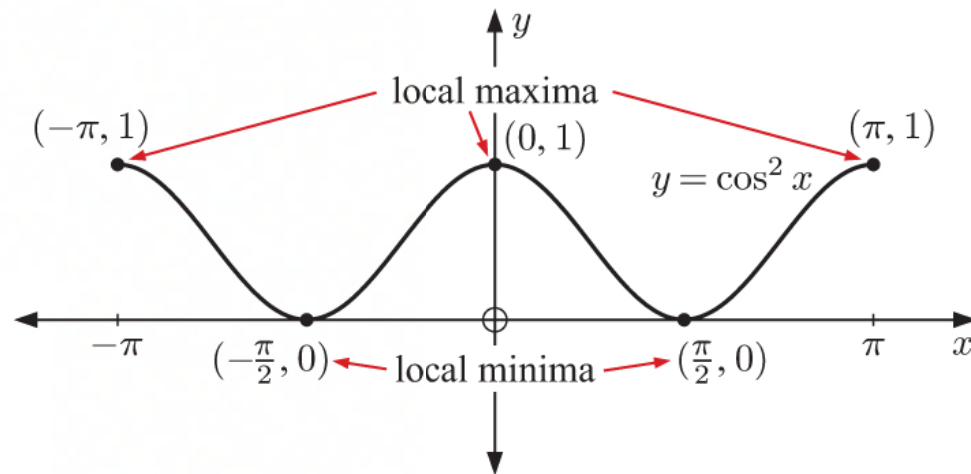
When  $x = -\frac{\pi}{2}$ ,  $y = \cos^2(-\frac{\pi}{2}) = 0$

When  $x = 0$ ,  $y = \cos^2 0 = 1$

When  $x = \frac{\pi}{2}$ ,  $y = \cos^2(\frac{\pi}{2}) = 0$

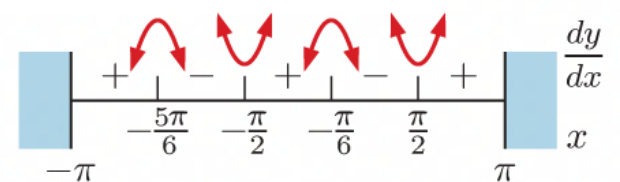
When  $x = \pi$ ,  $y = \cos^2 \pi = 1$

$\therefore (-\pi, 1)$ ,  $(0, 1)$ , and  $(\pi, 1)$  are local maxima,  $(-\frac{\pi}{2}, 0)$  and  $(\frac{\pi}{2}, 0)$  are local minima.



**c**  $y = \cos 2x - 2 \sin x$

$$\begin{aligned}\therefore \frac{dy}{dx} &= -2 \sin 2x - 2 \cos x \\ &= -2(2 \sin x \cos x) - 2 \cos x \\ &= -4 \sin x \cos x - 2 \cos x \\ &= -2 \cos x(2 \sin x + 1) \quad \text{which has sign diagram:}\end{aligned}$$



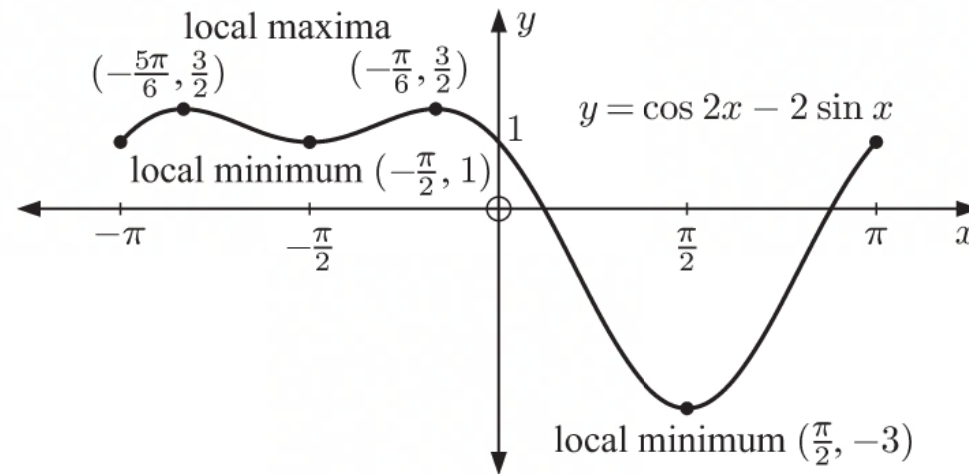
When  $x = -\frac{5\pi}{6}$ ,  $y = \cos(2 \times -\frac{5\pi}{6}) - 2 \sin(-\frac{5\pi}{6})$   
 $= \cos(-\frac{5\pi}{3}) - 2 \sin(-\frac{5\pi}{6})$   
 $= \frac{1}{2} - 2 \times (-\frac{1}{2})$   
 $= \frac{1}{2} + 1$   
 $= \frac{3}{2}$

When  $x = -\frac{\pi}{2}$ ,  $y = \cos(2 \times -\frac{\pi}{2}) - 2 \sin(-\frac{\pi}{2})$   
 $= \cos(-\pi) - 2 \sin(-\frac{\pi}{2})$   
 $= -1 - 2 \times -1$   
 $= -1 + 2$   
 $= 1$

$$\begin{aligned}
 \text{When } x = -\frac{\pi}{6}, \quad y &= \cos\left(2 \times -\frac{\pi}{6}\right) - 2 \sin\left(-\frac{\pi}{6}\right) \\
 &= \cos\left(-\frac{\pi}{3}\right) - 2 \sin\left(-\frac{\pi}{6}\right) \\
 &= \frac{1}{2} - 2 \times \left(-\frac{1}{2}\right) \\
 &= \frac{1}{2} + 1 \\
 &= \frac{3}{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{When } x = \frac{\pi}{2}, \quad y &= \cos\left(2 \times \frac{\pi}{2}\right) - 2 \sin \frac{\pi}{2} \\
 &= \cos \pi - 2 \sin \frac{\pi}{2} \\
 &= -1 - 2 \\
 &= -3
 \end{aligned}$$

$\therefore \left(-\frac{5\pi}{6}, \frac{3}{2}\right)$  and  $\left(-\frac{\pi}{6}, \frac{3}{2}\right)$  are local maxima,  $\left(-\frac{\pi}{2}, 1\right)$  and  $\left(\frac{\pi}{2}, -3\right)$  are local minima.

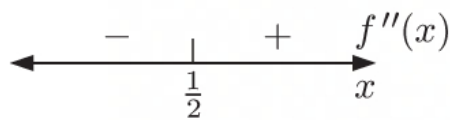


**23 a**  $f(x) = 2x^3 - 3x^2 + x - 12$

$$\therefore f'(x) = 6x^2 - 6x + 1$$

$$\therefore f''(x) = 12x - 6$$

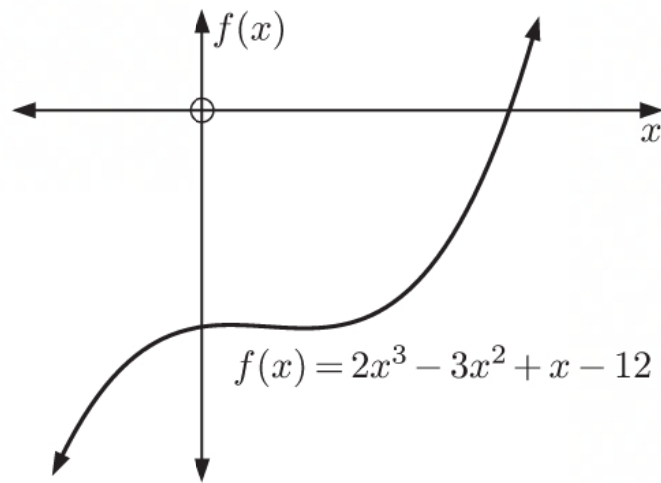
**b**  $f''(x) = 6(2x - 1)$  has sign diagram:



**c**  $f(x)$  is concave down for  $x \leq \frac{1}{2}$ .

$$\begin{aligned}
 \text{d } f\left(\frac{1}{2}\right) &= 2\left(\frac{1}{2}\right)^3 - 3\left(\frac{1}{2}\right)^2 + \frac{1}{2} - 12 \\
 &= \frac{1}{4} - \frac{3}{4} + \frac{1}{2} - 12 \\
 &= -12
 \end{aligned}$$

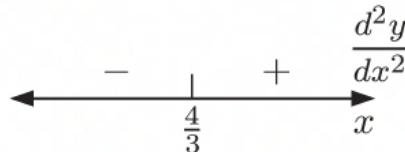
$\therefore$  the shape of  $f(x)$  changes at  $\left(\frac{1}{2}, -12\right)$ .



**24 a**  $y = x^3 - 4x^2 + 11$

$$\therefore \frac{dy}{dx} = 3x^2 - 8x$$

$$\therefore \frac{d^2y}{dx^2} = 6x - 8 \quad \text{which has sign diagram:}$$



The curve is concave up for  $x \geq \frac{4}{3}$ , and concave down for  $x \leq \frac{4}{3}$ .



**b**

$$y = -\frac{x+1}{x^2}$$

$$= \frac{x+1}{-x^2}$$

$$\therefore \frac{dy}{dx} = \frac{(1)(-x^2) - (x+1)(-2x)}{(-x^2)^2} \quad \{\text{quotient rule}\}$$

$$= \frac{-x^2 + 2x^2 + 2x}{x^4}$$

$$= \frac{x^2 + 2x}{x^4}$$

$$= \frac{x(x+2)}{x^4}$$

$$= \frac{x+2}{x^3}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{(1)x^3 - (x+2)(3x^2)}{(x^3)^2}$$

$$= \frac{x^3 - 3x^3 - 6x^2}{x^6}$$

$$= \frac{-2x^3 - 6x^2}{x^6}$$

$$= \frac{-x^2(2x+6)}{x^6}$$

$$= -\frac{2x+6}{x^4} \quad \text{which has sign diagram:}$$

The curve is concave up for  $x \leq -3$ , and concave down for  $-3 \leq x < 0$  and  $x > 0$ .

**c**

$$y = \frac{x+2}{x(x+4)}$$

$$= \frac{x+2}{x^2+4x}$$

$$\therefore \frac{dy}{dx} = \frac{(1)(x^2+4x) - (x+2)(2x+4)}{(x^2+4x)^2} \quad \{\text{quotient rule}\}$$

$$= \frac{x^2+4x - (2x^2+8x+8)}{(x^2+4x)^2}$$

$$= \frac{x^2+4x-2x^2-8x-8}{(x^2+4x)^2}$$

$$= \frac{-x^2-4x-8}{(x^2+4x)^2}$$

$$= \frac{-(x^2+4x)}{(x^2+4x)^2} - \frac{8}{(x^2+4x)^2}$$

$$= -(x^2+4x)^{-1} - 8(x^2+4x)^{-2}$$

$$\therefore \frac{d^2y}{dx^2} = (x^2+4x)^{-2}(2x+4) + 16(x^2+4x)^{-3}(2x+4)$$

$$= (2x+4) \left[ \frac{1}{(x^2+4x)^2} + \frac{16}{(x^2+4x)^3} \right]$$

$$= (2x+4) \left[ \frac{x^2+4x+16}{(x^2+4x)^3} \right]$$

Now  $x^2 + 4x + 16$  has  $\Delta = 4^2 - 4(1)(16)$   
 $= -48 < 0$

and so does not have real roots.

So,  $\frac{d^2y}{dx^2} = 0$  when  $x = -2$  and is undefined when  $x = 0$  or  $-4$

So,  $\frac{d^2y}{dx^2}$  has sign diagram:

$$\begin{array}{ccccccc} & - & + & - & + & & \\ & \vdots & | & \vdots & | & & \\ \leftarrow & -4 & -2 & 0 & x & \rightarrow & \frac{d^2y}{dx^2} \end{array}$$

The curve is concave up for  $-4 < x \leq -2$  and  $x > 0$ , and concave down for  $x < -4$  and  $-2 \leq x < 0$ .

**25 a**  $f(x) = x + \ln x$  is defined for  $x > 0$ .

**b**  $f'(x) = 1 + \frac{1}{x}$  which has sign diagram:

$$\begin{array}{ccc} & + & \\ & \longrightarrow & f'(x) \\ \leftarrow & 0 & x \end{array}$$

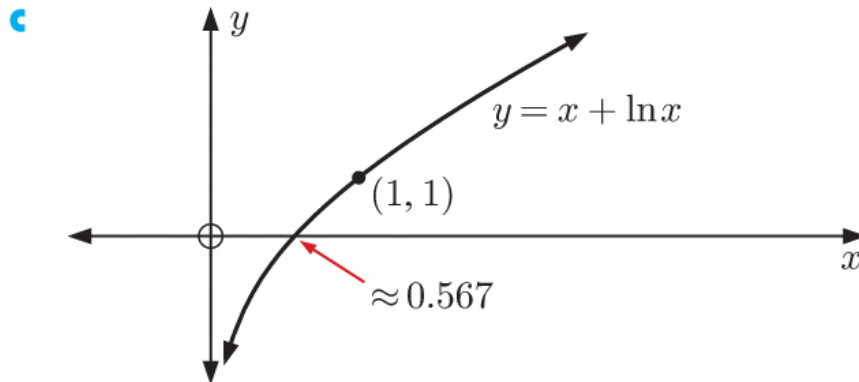
$$f'(x) = 1 + x^{-1}$$

$$\therefore f''(x) = -x^{-2}$$

$$= -\frac{1}{x^2} \text{ which has sign diagram:}$$

$$\begin{array}{ccc} & - & \\ & \longrightarrow & f''(x) \\ \leftarrow & 0 & x \end{array}$$

So,  $f(x)$  is increasing for all  $x > 0$  and is concave downwards for all  $x > 0$ .



**d**  $f(1) = 1 + \ln 1 = 1$  and  $f'(1) = 1 + \frac{1}{1} = 2$

$\therefore$  the normal at  $(1, 1)$  has gradient  $-\frac{1}{2}$ .

$\therefore$  the normal has equation  $y = -\frac{1}{2}(x - 1) + 1$

$$\therefore 2y = -x + 1 + 2$$

$$\therefore x + 2y = 3$$

**26 a**  $f(x) = e^{x\sqrt{3}} \sin x$

$\therefore f'(x) = e^{x\sqrt{3}}(\sqrt{3}) \sin x + e^{x\sqrt{3}} \cos x$  {chain rule, product rule}

$$= e^{x\sqrt{3}}(\cos x + \sqrt{3} \sin x)$$

**b**

$$f'(x) = 0$$

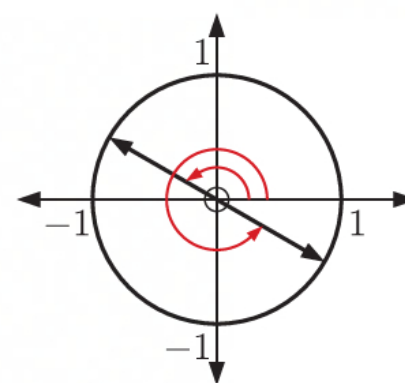
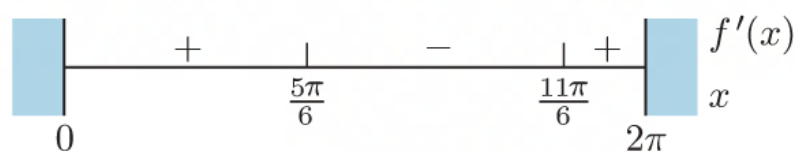
$$\text{when } e^{x\sqrt{3}}(\cos x + \sqrt{3}\sin x) = 0$$

$$\therefore \cos x + \sqrt{3}\sin x = 0 \quad \{\text{as } e^{x\sqrt{3}} > 0 \text{ for all } x\}$$

$$\therefore \sqrt{3}\sin x = -\cos x$$

$$\therefore \tan x = -\frac{1}{\sqrt{3}}$$

$$\therefore x = \frac{5\pi}{6} \text{ or } \frac{11\pi}{6}$$

**c****d**

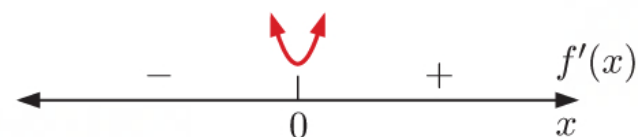
**i**  $f(x)$  is increasing for  $0 \leq x \leq \frac{5\pi}{6}$  and  $\frac{11\pi}{6} \leq x \leq 2\pi$ .

**ii**  $f(x)$  is decreasing for  $\frac{5\pi}{6} \leq x \leq \frac{11\pi}{6}$ .

**27**

**a**  $f(x) = \ln(x^2 + 5)$

$$\therefore f'(x) = \frac{2x}{x^2 + 5} \quad \text{which has sign diagram:}$$



$$\begin{aligned} \text{Now } f(0) &= \ln(0^2 + 5) \\ &= \ln 5 \end{aligned}$$

$\therefore (0, \ln 5)$  is a local minimum.

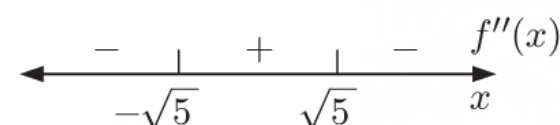
**b**  $f''(x) = \frac{2(x^2 + 5) - 2x(2x)}{(x^2 + 5)^2} \quad \{\text{quotient rule}\}$

$$= \frac{2x^2 + 10 - 4x^2}{(x^2 + 5)^2}$$

$$= \frac{-2(x^2 - 5)}{(x^2 + 5)^2}$$

$$= \frac{-2(x + \sqrt{5})(x - \sqrt{5})}{(x^2 + 5)^2}$$

which has sign diagram:



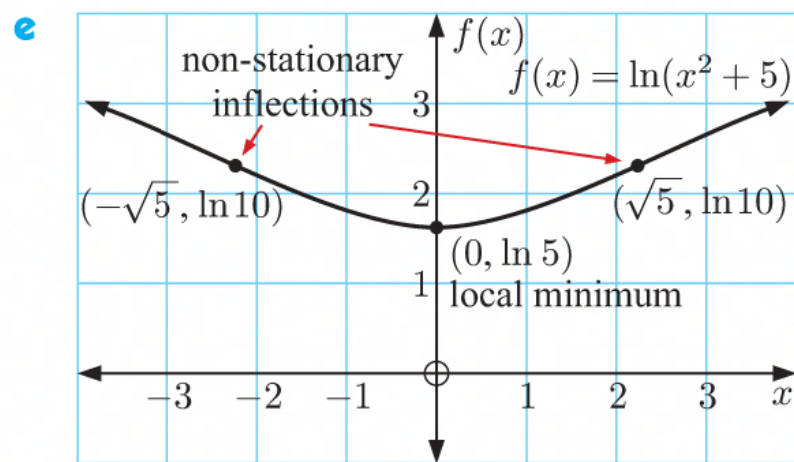
$$\begin{aligned} \text{Now } f(-\sqrt{5}) &= \ln((- \sqrt{5})^2 + 5) & \text{and} & & f(\sqrt{5}) &= \ln((\sqrt{5})^2 + 5) \\ &= \ln 10 & & & &= \ln 10 \end{aligned}$$

$$\begin{aligned} \text{also } f'(-\sqrt{5}) &= \frac{2(-\sqrt{5})}{(-\sqrt{5})^2 + 5} & f'(\sqrt{5}) &= \frac{2\sqrt{5}}{(\sqrt{5})^2 + 5} \\ &= -\frac{2\sqrt{5}}{10} \neq 0 & & & &= \frac{2\sqrt{5}}{10} \neq 0 \end{aligned}$$

$\therefore (-\sqrt{5}, \ln 10)$  and  $(\sqrt{5}, \ln 10)$  are non-stationary inflections.

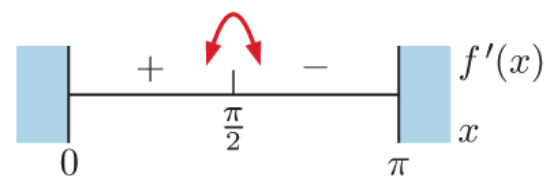
**c**  $f(x)$  is increasing for  $x \geq 0$ , and decreasing for  $x \leq 0$ .

**d**  $f(x)$  is concave up for  $-\sqrt{5} \leq x \leq \sqrt{5}$ , and concave down for  $x \leq -\sqrt{5}$  and  $x \geq \sqrt{5}$ .



**28**  $f(x) = e^{\sin^2 x}$ ,  $0 \leq x \leq \pi$

**a**  $f'(x) = e^{\sin^2 x} \times 2 \sin x \cos x$  {chain rule}  
 $= e^{\sin^2 x} \sin 2x$  which has sign diagram:

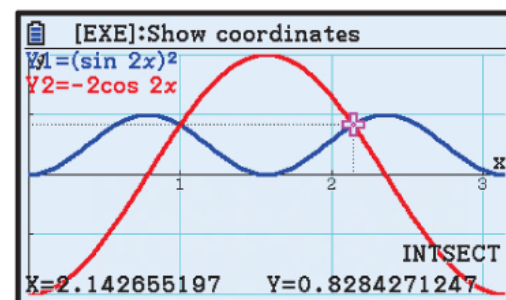


$f'(x) = 0$  when  $\sin 2x = 0$   
 $\therefore 2x = 0 + k\pi, \quad k \in \mathbb{Z}$   
 $\therefore x = 0 + \frac{k\pi}{2}$   
 $\therefore x = 0, \frac{\pi}{2}, \text{ or } \pi$

$\therefore f(x)$  has a maximum turning point when  $x = \frac{\pi}{2}$ , and this maximum value is  $e$ .

**b**  $f''(x) = e^{\sin^2 x} \times \sin 2x \sin 2x + e^{\sin^2 x} \times 2 \cos 2x$   
 $= e^{\sin^2 x} (\sin^2 2x + 2 \cos 2x)$

**c**  $f''(x) = 0$  when  $\sin^2 2x + 2 \cos 2x = 0$  on  $0 \leq x \leq \pi$   
 $\therefore \sin^2 2x = -2 \cos 2x$   
 $\therefore x \approx 0.999 \text{ or } 2.14$   
 {using technology}



So,  $f(0.999) \approx 2.03$  and  $f(2.14) \approx 2.03$   
 $f'(0.999) \neq 0$   $f'(2.14) \neq 0$

$\therefore (0.999, 2.03)$  and  $(2.14, 2.03)$  are non-stationary points of inflection.

**29**  $f(x)$  has a turning point at  $x = 0$

$\therefore f'(0) = 0$

$f(x)$  is increasing for  $x \geq 0$ , except at the asymptote.

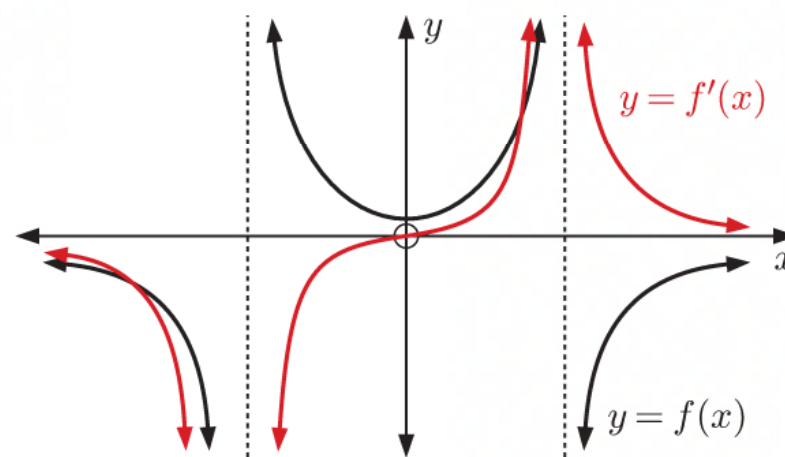
$\therefore f'(x)$  is positive for  $x \geq 0$ .

$f(x)$  is decreasing for  $x \leq 0$ , except at the asymptote.

$\therefore f'(x)$  is negative for  $x \leq 0$ .

As  $x \rightarrow \infty$ ,  $f'(x) \rightarrow 0^+$ .

As  $x \rightarrow -\infty$ ,  $f'(x) \rightarrow 0^-$ .





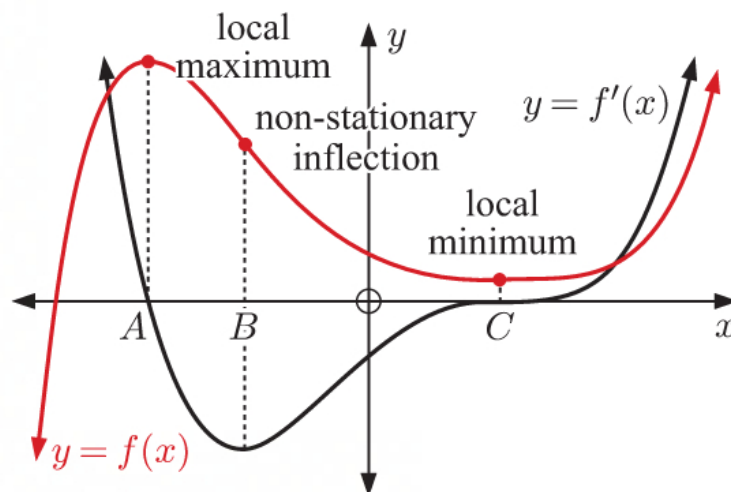
**30** At  $x = B$ ,  $f''(x) = 0$  but  $f'(x) \neq 0$

$\therefore f(x)$  has a non-stationary inflection point at  $x = B$ .

$f'(x)$  is above the  $x$ -axis for  $x \leq A$  and  $x \geq C$ , and below the  $x$ -axis for  $A \leq x \leq C$

$\therefore f(x)$  is increasing for  $x \leq A$ , decreasing for  $A \leq x \leq C$ , then increasing for  $x \geq C$

$\therefore f(x)$  has a local maximum at  $x = A$  and a local minimum at  $x = C$ .



**31 a** As  $x \rightarrow \infty$ ,  $e^x \rightarrow \infty$ , so we can use l'Hôpital's rule.

$$\begin{aligned} \therefore \lim_{x \rightarrow \infty} \frac{x}{e^x} &= \lim_{x \rightarrow \infty} \frac{1}{e^x} \quad \{\text{l'Hôpital's rule}\} \\ &= 0 \end{aligned}$$

**b**  $\lim_{x \rightarrow 0} \arcsin x = 0$  and  $\lim_{x \rightarrow 0} x = 0$ , so we can use l'Hôpital's rule.

$$\begin{aligned} \therefore \lim_{x \rightarrow 0} \frac{\arcsin x}{x} &= \lim_{x \rightarrow 0} \frac{\left(\frac{1}{\sqrt{1-x^2}}\right)}{1} \quad \{\text{l'Hôpital's rule}\} \\ &= \lim_{x \rightarrow 0} \frac{1}{\sqrt{1-x^2}} \\ &= 1 \end{aligned}$$

**c**  $\lim_{x \rightarrow 0} \sin x = 0$  and  $\lim_{x \rightarrow 0} (2^x - 1) = 1 - 1 = 0$ , so we can use l'Hôpital's rule.

$$\begin{aligned} \therefore \lim_{x \rightarrow 0} \frac{\sin x}{2^x - 1} &= \lim_{x \rightarrow 0} \frac{\cos x}{2^x \ln 2} \quad \{\text{l'Hôpital's rule}\} \\ &= \frac{\lim_{x \rightarrow 0} \cos x}{\lim_{x \rightarrow 0} 2^x \ln 2} \\ &= \frac{1}{2^0 \ln 2} \\ &= \frac{1}{\ln 2} \end{aligned}$$

**32 a**  $f(x) = xe^{-x^2}$

$$\begin{aligned} \therefore f'(x) &= (1)e^{-x^2} + xe^{-x^2}(-2x) \quad \{\text{product rule, chain rule}\} \\ &= e^{-x^2}(1 - 2x^2) \end{aligned}$$

$$\begin{aligned} \therefore f''(x) &= e^{-x^2}(-2x)(1 - 2x^2) + e^{-x^2}(-4x) \quad \{\text{product rule, chain rule}\} \\ &= -2xe^{-x^2} + 4x^3e^{-x^2} - 4xe^{-x^2} \\ &= 4x^3e^{-x^2} - 6xe^{-x^2} \\ &= 2xe^{-x^2}(2x^2 - 3) \end{aligned}$$

**b**  $f'(x) = 0$  when  $1 - 2x^2 = 0$ , as  $e^{-x^2} > 0$  for all  $x$  {using **a**}

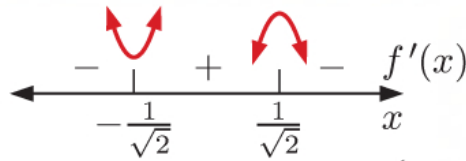
$$\therefore x^2 = \frac{1}{2}$$

$$\therefore x = \pm \frac{1}{\sqrt{2}}$$

$$f\left(-\frac{1}{\sqrt{2}}\right) = -\frac{1}{\sqrt{2}} e^{-\left(\frac{-1}{\sqrt{2}}\right)^2} = \frac{-1}{\sqrt{2}e}$$

$$f\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}} e^{-\left(\frac{1}{\sqrt{2}}\right)^2} = \frac{1}{\sqrt{2}e}$$

$f'(x)$  has sign diagram:



$\therefore \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}e}\right)$  is a local minimum, and  $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}e}\right)$  is a local maximum.

**c** As  $x \rightarrow \pm\infty$ ,  $e^{-x^2} \rightarrow 0$ , so we can use l'Hôpital's rule.

$$\therefore \lim_{x \rightarrow \pm\infty} x e^{-x^2} = \lim_{x \rightarrow \pm\infty} \frac{x}{e^{x^2}} \quad \left\{ \text{converting to the form } \frac{\infty}{\infty} \right\}$$

$$= \lim_{x \rightarrow \pm\infty} \frac{1}{e^{x^2}(2x)} \quad \left\{ \text{l'Hôpital's rule} \right\}$$

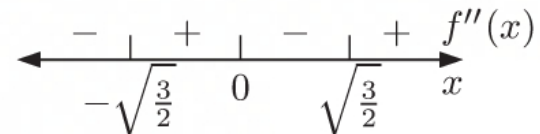
$$= \lim_{x \rightarrow \pm\infty} \frac{x^{-1}}{2e^{x^2}}, \quad \text{which has the form } \frac{0}{\infty}$$

$$= 0$$

**d**  $f''(x) = 0$  when  $x = 0$  or  $2x^2 - 3 = 0$ , as  $e^{-x^2} > 0$  for all  $x$  {using **a**}

$$\therefore x^2 = \frac{3}{2}$$

$$\therefore x = \pm \sqrt{\frac{3}{2}}$$



$$f(0) = 0$$

and

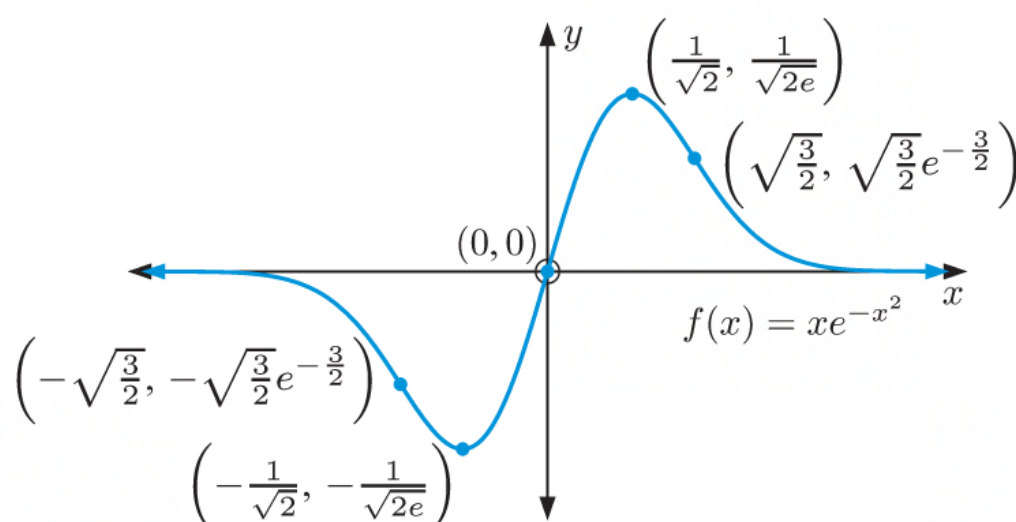
$$f'(0) = e^0(1 - 2(0)) = 1 \neq 0$$

$$f\left(-\sqrt{\frac{3}{2}}\right) = -\sqrt{\frac{3}{2}} e^{-\frac{3}{2}} \quad \text{and} \quad f'\left(-\sqrt{\frac{3}{2}}\right) = e^{-\frac{3}{2}}(1 - 2(\frac{3}{2})) \neq 0$$

$$f\left(\sqrt{\frac{3}{2}}\right) = \sqrt{\frac{3}{2}} e^{-\frac{3}{2}} \quad \text{and} \quad f'\left(\sqrt{\frac{3}{2}}\right) = e^{-\frac{3}{2}}(1 - 2(\frac{3}{2})) \neq 0$$

$\therefore (0, 0)$ ,  $\left(-\sqrt{\frac{3}{2}}, -\sqrt{\frac{3}{2}} e^{-\frac{3}{2}}\right)$ , and  $\left(\sqrt{\frac{3}{2}}, \sqrt{\frac{3}{2}} e^{-\frac{3}{2}}\right)$  are non-stationary inflections.

**e**



**33**  $\lim_{x \rightarrow 0} (2 \sin x - \sin 2x) = 0 - 0 = 0$  and  $\lim_{x \rightarrow 0} (x - \sin x) = 0 - 0 = 0$ , so we can use l'Hôpital's rule.

$$\therefore \lim_{x \rightarrow 0} \frac{2 \sin x - \sin 2x}{x - \sin x} = \lim_{x \rightarrow 0} \frac{2 \cos x - 2 \cos 2x}{1 - \cos x} \quad \{\text{l'Hôpital's rule}\}$$

This also has the form  $\frac{0}{0}$ , so we can use l'Hôpital's rule again.

$$\begin{aligned} \therefore \lim_{x \rightarrow 0} \frac{2 \sin x - \sin 2x}{x - \sin x} &= \lim_{x \rightarrow 0} \frac{-2 \sin x + 4 \sin 2x}{\sin x} \quad \{\text{l'Hôpital's rule}\} \\ &= \lim_{x \rightarrow 0} \left( -2 + \frac{4 \sin 2x}{\sin x} \right) \\ &= -2 + \lim_{x \rightarrow 0} \frac{4 \sin 2x}{\sin x} \\ &= -2 + \lim_{x \rightarrow 0} \frac{8 \sin x \cos x}{\sin x} \\ &= -2 + \lim_{x \rightarrow 0} 8 \cos x \\ &= -2 + 8 \\ &= 6 \end{aligned}$$

**34 a** As  $x \rightarrow \infty$ ,  $e^x + e^{-x} \rightarrow \infty$  and  $e^x - e^{-x} \rightarrow \infty$ , so we can use l'Hôpital's rule.

$$\therefore \lim_{x \rightarrow \infty} \frac{e^x + e^{-x}}{e^x - e^{-x}} = \lim_{x \rightarrow \infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} \quad \{\text{l'Hôpital's rule}\}$$

This also has form  $\frac{\infty}{\infty}$ , so we could use l'Hôpital's rule again, but doing so would give us

$$\lim_{x \rightarrow \infty} \frac{e^x + e^{-x}}{e^x - e^{-x}} \quad \text{which is where we started, and is therefore not helpful.}$$

$$\begin{aligned} \text{b} \quad \frac{e^x + e^{-x}}{e^x - e^{-x}} &\times \frac{e^x}{e^x} = \frac{e^{2x} + 1}{e^{2x} - 1} \\ \therefore \lim_{x \rightarrow \infty} \frac{e^x + e^{-x}}{e^x - e^{-x}} &= \lim_{x \rightarrow \infty} \frac{e^{2x} + 1}{e^{2x} - 1} \quad \text{which is of the form } \frac{\infty}{\infty} \\ &= \lim_{x \rightarrow \infty} \frac{2e^{2x}}{2e^{2x}} \quad \{\text{l'Hôpital's rule}\} \\ &= \lim_{x \rightarrow \infty} 1 \\ &= 1 \end{aligned}$$

**35 a**  $p(x) = ax^2 + bx + c$

$$\begin{aligned} (1) \quad &= a(z+r)^2 + b(z+r) + c && \{\text{letting } x = z + r\} \\ (2) \quad &= az^2 + 2azr + ar^2 + bz + br + c && \{\text{expanding brackets}\} \\ (3) \quad &= a(x-r)^2 + 2a(x-r)r + ar^2 + b(x-r) + br + c && \{x = z + r \quad \therefore \quad z = x - r\} \\ &= a(x-r)^2 + (2ar+b)(x-r) + ar^2 + br + c && \{\text{factorising}\} \end{aligned}$$

**b**  $p(x) = (ar^2 + br + c) + (2ar + b)(x - r) + a(x - r)^2$   
 and  $p(r) = ar^2 + br + c$   
 $\therefore p'(r) = 2ar + b$   
 $\therefore p''(r) = 2a$   
 $\therefore p(x) = p(r) + p'(r)(x - r) + \frac{p''(r)}{2}(x - r)^2$  as required

**c**  $p(x) = a(x - r)^2 + (2ar + b)(x - r) + ar^2 + br + c$

The tangent to  $p(x)$  at  $x = r$  is  $p(x) = p(r) + p'(r)(x - r)$ .

We can use the form in **a** to find  $p(r)$ ,  $p'(r)$ , and hence a formula for the tangent to  $p(x)$  at  $x = r$ .

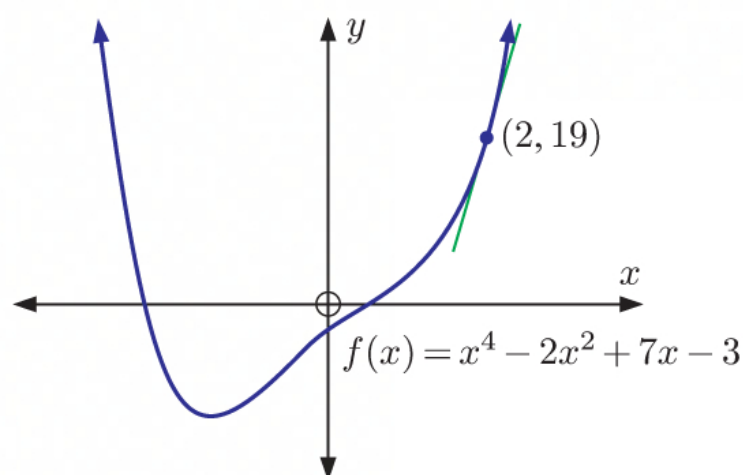
## REVIEW SET 18B

**1 a**  $f(x) = x^4 - 2x^2 + 7x - 3$   
 $\therefore f'(x) = 4x^3 - 4x + 7$   
 $\therefore f'(2) = 4(2)^3 - 4(2) + 7$   
 $= 32 - 8 + 7$   
 $= 31$

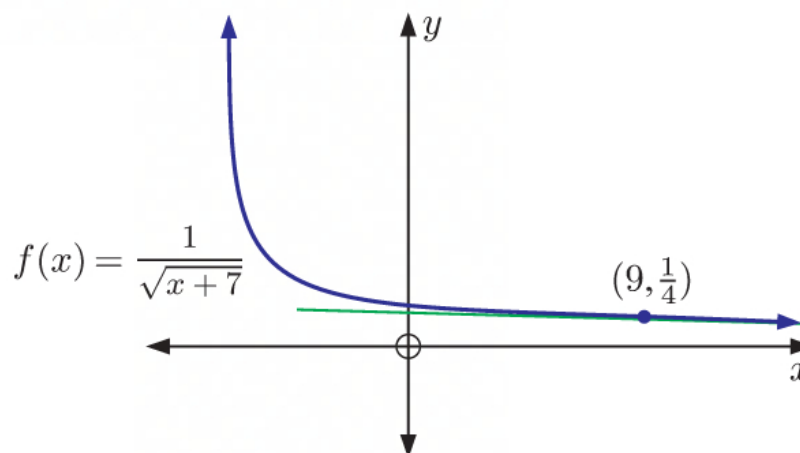
So, the tangent has equation

$$y = 31(x - 2) + 19$$

$$\therefore y = 31x - 43$$



**b**  $f(x) = \frac{1}{\sqrt{x+7}} = (x+7)^{-\frac{1}{2}}$   
 $\therefore f'(x) = -\frac{1}{2}(x+7)^{-\frac{3}{2}}$   
 $= -\frac{1}{2(x+7)^{\frac{3}{2}}}$   
 $\therefore f'(9) = -\frac{1}{2(9+7)^{\frac{3}{2}}}$   
 $= -\frac{1}{2(16)^{\frac{3}{2}}}$   
 $= -\frac{1}{128}$



So, the tangent has equation  $x + 128y = 9 + 128(\frac{1}{4})$

$$\therefore x + 128y = 41$$



**c**  $f(x) = 3 \sin 2x$

$$\therefore f\left(\frac{\pi}{6}\right) = 3 \sin \frac{\pi}{3}$$

$$= 3\left(\frac{\sqrt{3}}{2}\right)$$

$$= \frac{3\sqrt{3}}{2}$$

$$\therefore \text{the point of contact is } \left(\frac{\pi}{6}, \frac{3\sqrt{3}}{2}\right).$$

Now  $f(x) = 3 \sin 2x$

$$\therefore f'(x) = 6 \cos 2x$$

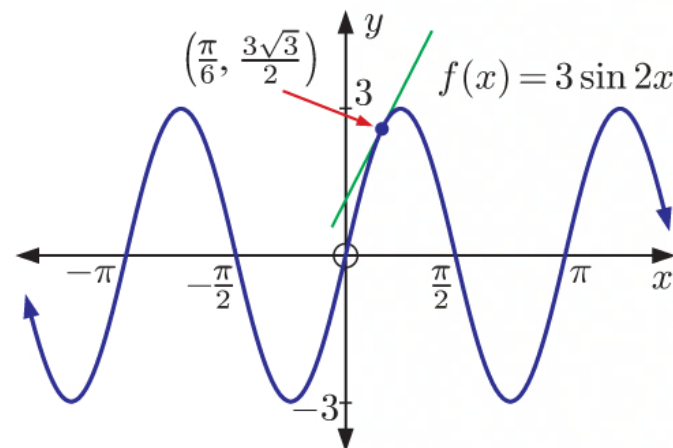
$$\therefore f'\left(\frac{\pi}{6}\right) = 6 \cos \frac{\pi}{3}$$

$$= 6\left(\frac{1}{2}\right)$$

$$= 3$$

So, the tangent has equation  $y = 3\left(x - \frac{\pi}{6}\right) + \frac{3\sqrt{3}}{2}$

$$= 3x + \frac{3\sqrt{3}}{2} - \frac{\pi}{2}$$



**d**  $f(x) = \frac{e^x}{2-x}$

$$\therefore f(0) = \frac{e^0}{2-0} = \frac{1}{2}$$

$$\therefore \text{the point of contact is } \left(0, \frac{1}{2}\right).$$

Now  $f(x) = \frac{e^x}{2-x}$

$$\therefore f'(x) = \frac{e^x(2-x) - e^x(-1)}{(2-x)^2} \quad \{\text{quotient rule}\}$$

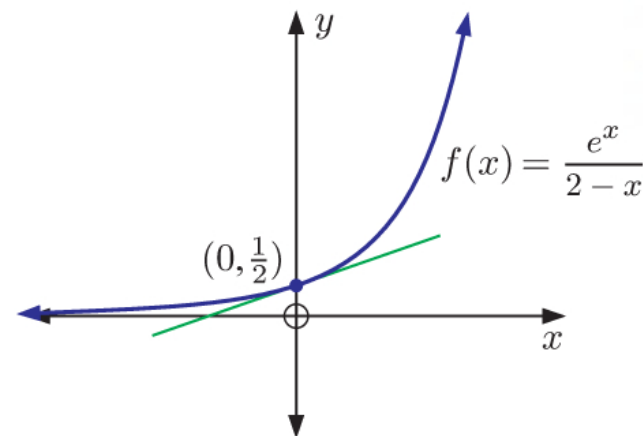
$$= \frac{e^x(3-x)}{(2-x)^2}$$

$$\therefore f'(0) = \frac{e^0(3-0)}{(2-0)^2}$$

$$= \frac{3}{4}$$

So, the tangent has equation  $y = \frac{3}{4}(x-0) + \frac{1}{2}$

$$\therefore y = \frac{3}{4}x + \frac{1}{2}$$



**e**  $y = \ln(x^2 + 3)$

When  $x = 0$ ,  $y = \ln(0^2 + 3)$   
 $= \ln 3$

$$\therefore \text{the point of contact is } (0, \ln 3).$$

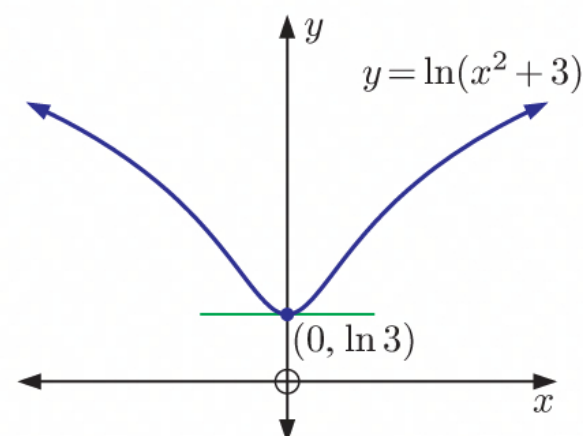
Now  $\frac{dy}{dx} = \frac{2x}{x^2 + 3}$

When  $x = 0$ ,  $\frac{dy}{dx} = \frac{2(0)}{0^2 + 3} = 0$

$$\therefore \text{the tangent to the curve at } (0, \ln 3) \text{ has gradient } 0.$$

$$\therefore \text{the equation of the tangent is } y = 0(x-0) + \ln 3$$

$$\therefore y = \ln 3$$



**2 a**  $y = \frac{1}{x^2} - \frac{2}{x} = x^{-2} - 2x^{-1}$

When  $x = 1$ ,  $y = \frac{1}{1^2} - \frac{2}{1} = 1 - 2 = -1$

So, the point of contact is  $(1, -1)$ .

Now as  $y = x^{-2} - 2x^{-1}$ ,  $\frac{dy}{dx} = -2x^{-3} + 2x^{-2}$

$$= -\frac{2}{x^3} + \frac{2}{x^2}$$

$\therefore$  when  $x = 1$ ,  $\frac{dy}{dx} = -\frac{2}{1^3} + \frac{2}{1^2} = -2 + 2 = 0$

$\therefore$  the normal at  $(1, -1)$  has gradient which is undefined.

So, the normal must be a vertical line.

Since the normal passes through  $(1, -1)$ , then the equation of the normal is  $x = 1$ .

**b**  $y = x \sin x$

$\therefore \frac{dy}{dx} = (1) \sin x + x(\cos x) \quad \{\text{product rule}\}$

$$= \sin x + x \cos x$$

$\therefore$  when  $x = 0$ ,  $\frac{dy}{dx} = \sin 0 + 0 \times \cos x$

$$= 0$$

$\therefore$  the normal at  $(0, 0)$  has gradient which is undefined.

So, the normal must be a vertical line.

Since the normal passes through  $(0, 0)$ , then the equation of the normal is  $x = 0$ .

**c**  $y = e^{-x^2}$

When  $x = 1$ ,  $y = e^{-1} = \frac{1}{e}$

So, the point of contact is  $\left(1, \frac{1}{e}\right)$ .

Now  $\frac{dy}{dx} = e^{-x^2}(-2x) \quad \{\text{chain rule}\}$

$$= -2xe^{-x^2}$$

When  $x = 1$ ,  $\frac{dy}{dx} = -2e^{-1}$

$$= -\frac{2}{e}$$

$\therefore$  the normal at  $\left(1, \frac{1}{e}\right)$  has gradient  $\frac{e}{2}$ .

$\therefore$  the normal has equation  $y = \frac{e}{2}(x - 1) + \frac{1}{e}$

$$= \frac{e}{2}x + \frac{1}{e} - \frac{e}{2}$$

**3**  $y = x \tan x$

When  $x = \frac{\pi}{4}$ ,  $y = \frac{\pi}{4}(1)$   
 $= \frac{\pi}{4}$

So, the point of contact is  $(\frac{\pi}{4}, \frac{\pi}{4})$ .

Now  $\frac{dy}{dx} = (1) \tan x + x \sec^2 x$   
 $= \tan x + \frac{x}{\cos^2 x}$

When  $x = \frac{\pi}{4}$ ,  $\frac{dy}{dx} = 1 + \frac{\frac{\pi}{4}}{\left(\frac{1}{\sqrt{2}}\right)^2}$   
 $= 1 + \frac{\frac{\pi}{4}}{\frac{1}{2}}$   
 $= 1 + \frac{\pi}{2}$

$\therefore$  the equation of the tangent at  $(\frac{\pi}{4}, \frac{\pi}{4})$  is  $y = (1 + \frac{\pi}{2})(x - \frac{\pi}{4}) + \frac{\pi}{4}$   
 $\therefore y = x - \frac{\pi}{4} + \frac{\pi}{2}x - \frac{\pi^2}{8} + \frac{\pi}{4}$   
 $\therefore 2y = 2x + \pi x - \frac{\pi^2}{4}$   
 $\therefore (2 + \pi)x - 2y = \frac{\pi^2}{4}$  as required

**4**  $y = 2x^3 + ax + b$

$\therefore \frac{dy}{dx} = 6x^2 + a$

Since the gradient of the tangent at  $(-2, 33)$  is 10, then  $6(-2)^2 + a = 10$   
 $\therefore 24 + a = 10$   
 $\therefore a = -14$   
 $\therefore y = 2x^3 - 14x + b$

Since the curve passes through  $(-2, 33)$ , then  $33 = 2(-2)^3 - 14(-2) + b$   
 $= -16 + 28 + b$   
 $\therefore b = 21$

**5 a** The tangent shown on the graph passes through  $(0, 5)$  and  $(5, 0)$ .

$\therefore$  the gradient of the tangent is  $\frac{0 - 5}{5 - 0} = -1$ , so  
 $f'(3) = -1$ .

Also, since the tangent passes through  $(0, 5)$ , it has

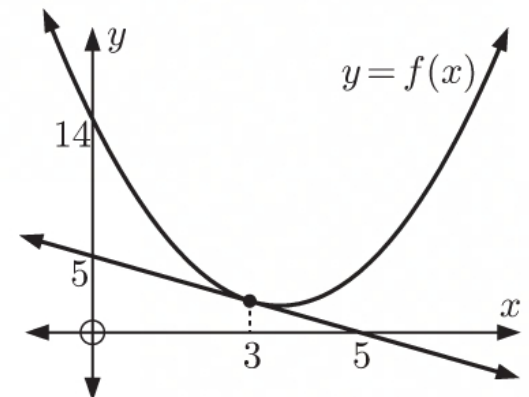
equation  $\frac{y - 5}{x - 0} = -1$

$\therefore y - 5 = -x$

$\therefore y = -x + 5$

So when  $x = 3$ ,  $y = -3 + 5 = 2$

$\therefore$  the point of contact is  $(3, 2)$ , and hence  $f(3) = 2$ .



**b**  $f(x)$  has the form  $f(x) = ax^2 + bx + c$

The  $y$ -intercept is 14  $\therefore f(0) = 14$

$$\therefore a(0)^2 + b(0) + c = 14$$

$$\therefore c = 14$$

Now  $f(3) = 2$  {from **a**}

$$\therefore a(3)^2 + b(3) + 14 = 2$$

$$\therefore 9a + 3b = -12 \quad \dots (1)$$

Also  $f'(3) = -1$

and  $f'(x) = 2ax + b$

$$\therefore 2a(3) + b = -1$$

$$\therefore 6a + b = -1$$

$$\therefore b = -6a - 1 \quad \dots (2)$$

Substituting (2) into (1) gives  $9a + 3(-6a - 1) = -12$

$$\therefore 9a - 18a - 3 = -12$$

$$\therefore -9a = -9$$

$$\therefore a = 1$$

Using (2),  $b = -6(1) - 1$

$$\therefore b = -7$$

So,  $f(x) = x^2 - 7x + 14$

$$\mathbf{6} \quad y = 2 - \frac{7}{1+2x} = 2 - 7(1+2x)^{-1}$$

$$\therefore \frac{dy}{dx} = 7(1+2x)^{-2} \times 2 \quad \{\text{chain rule}\}$$

$$= \frac{14}{(1+2x)^2}$$

The tangent is horizontal when the gradient  $\frac{dy}{dx} = 0$ .

But  $\frac{14}{(1+2x)^2}$  is never 0, so  $y = 2 - \frac{7}{1+2x}$  has no horizontal tangents.

$$\mathbf{7} \quad y = x^3 + ax^2 - 4x + 3$$

$$\mathbf{a} \quad \frac{dy}{dx} = 3x^2 + 2ax - 4$$

The tangent at  $x = 1$  is parallel to the line  $y = 3x$ , and  $y = 3x$  has gradient 3.

$\therefore$  the tangent at  $x = 1$  has gradient 3.

$$\therefore 3(1)^2 + 2a(1) - 4 = 3$$

$$\therefore 3 + 2a - 4 = 3$$

$$\therefore 2a = 4$$

$$\therefore a = 2$$



- b** Since  $a = 2$ ,  $y = x^3 + 2x^2 - 4x + 3$  and  $\frac{dy}{dx} = 3x^2 + 4x - 4$

When  $x = 1$ ,  $y = 1^3 + 2(1)^2 - 4(1) + 3 = 2$

and  $\frac{dy}{dx} = 3(1)^2 + 4(1) - 4 = 3$

So, the point of contact is  $(1, 2)$ , and the tangent at  $(1, 2)$  has gradient 3.

$\therefore$  the tangent has equation  $y = 3(x - 1) + 2$

which is  $y = 3x - 1$

- c** The tangent meets the curve again when  $x^3 + 2x^2 - 4x + 3 = 3x - 1$

$$\therefore x^3 + 2x^2 - 7x + 4 = 0$$

$$\therefore (x - 1)^2(x + 4) = 0$$

$\{(x - 1)^2$  is a factor since the tangent to the curve is at  $x = 1\}$

When  $x = -4$ ,  $y = (-4)^3 + 2(-4)^2 - 4(-4) + 3 = -13$

$\therefore$  the tangent meets the curve again at  $(-4, -13)$ .

**8**  $y = x^2 - 4x + 2$

$$\therefore \frac{dy}{dx} = 2x - 4$$

When  $x = 3$ ,  $y = (3)^2 - 4(3) + 2$  and  $\frac{dy}{dx} = 2(3) - 4$   
 $\phantom{\text{When } x = 3, } = 9 - 12 + 2$   $\phantom{\text{When } x = 3, } = 6 - 4$   
 $\phantom{\text{When } x = 3, } = -1$   $\phantom{\text{When } x = 3, } = 2$

So, the point of contact is  $(3, -1)$  and the normal at  $(3, -1)$  has gradient  $-\frac{1}{2}$ .

$\therefore$  the normal has equation  $y = -\frac{1}{2}(x - 3) - 1$

$$\therefore y = -\frac{1}{2}x + \frac{3}{2} - 1$$

$$\therefore y = -\frac{1}{2}x + \frac{1}{2}$$

The normal meets the curve again when  $-\frac{1}{2}x + \frac{1}{2} = x^2 - 4x + 2$

$$\therefore -x + 1 = 2x^2 - 8x + 4$$

$$\therefore 2x^2 - 7x + 3 = 0$$

$$\therefore (2x - 1)(x - 3) = 0$$

When  $x = \frac{1}{2}$ ,  $y = \left(\frac{1}{2}\right)^2 - 4\left(\frac{1}{2}\right) + 2$

$$= \frac{1}{4} - 2 + 2$$

$$= \frac{1}{4}$$

$\therefore$  the normal meets the curve again at  $\left(\frac{1}{2}, \frac{1}{4}\right)$ .

9  $y = \ln(x^4 + 3)$

When  $x = 1$ ,  $y = \ln(1^4 + 3)$   
 $= \ln 4$

So, the point of contact is  $(1, \ln 4)$ .

Now  $\frac{dy}{dx} = \frac{4x^3}{x^4 + 3}$

When  $x = 1$ ,  $\frac{dy}{dx} = \frac{4(1)^3}{1^4 + 3}$   
 $= \frac{4}{4} = 1$

$\therefore$  the tangent at  $(1, \ln 4)$  has equation  $y = (x - 1) + \ln 4$   
 $\therefore y = x + 2 \ln 2 - 1$

When  $x = 0$ ,  $y = 2 \ln 2 - 1$ , so  $y = \ln(x^4 + 3)$  cuts the  $y$ -axis at  $(0, 2 \ln 2 - 1)$ .

10 a  $y = \operatorname{cosec} x = \frac{1}{\sin x}$

When  $x = \frac{\pi}{3}$ ,  $y = \frac{1}{\sin \frac{\pi}{3}}$   
 $= \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}}$

$\therefore$  the point of contact is  $\left(\frac{\pi}{3}, \frac{2}{\sqrt{3}}\right)$ .

Now  $y = \frac{1}{\sin x} = (\sin x)^{-1}$

$\therefore \frac{dy}{dx} = -(\sin x)^{-2} \cos x$  {chain rule}  
 $= -\frac{\cos x}{\sin^2 x}$

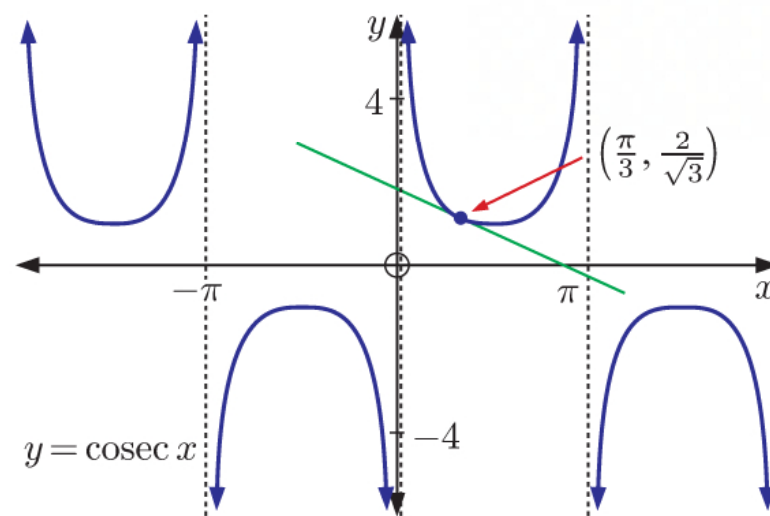
When  $x = \frac{\pi}{3}$ ,  $\frac{dy}{dx} = -\frac{\cos \frac{\pi}{3}}{\sin^2(\frac{\pi}{3})}$   
 $= -\frac{\frac{1}{2}}{\left(\frac{\sqrt{3}}{2}\right)^2}$   
 $= -\frac{\frac{1}{2}}{\frac{3}{4}} = -\frac{2}{3}$

So, the tangent has equation  $y = -\frac{2}{3}\left(x - \frac{\pi}{3}\right) + \frac{2}{\sqrt{3}}$

$\therefore 3y = -2\left(x - \frac{\pi}{3}\right) + 2\sqrt{3}$

$= -2x + \frac{2\pi}{3} + 2\sqrt{3}$

$\therefore 2x + 3y = \frac{2\pi}{3} + 2\sqrt{3}$



**b**

$$y = \cos \frac{x}{2}$$

$$\text{When } x = \frac{\pi}{2}, \quad y = \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$\therefore \text{ the point of contact is } \left( \frac{\pi}{2}, \frac{1}{\sqrt{2}} \right).$$

$$\text{Now } y = \cos \frac{x}{2}$$

$$\therefore \frac{dy}{dx} = -\frac{1}{2} \sin \frac{x}{2}$$

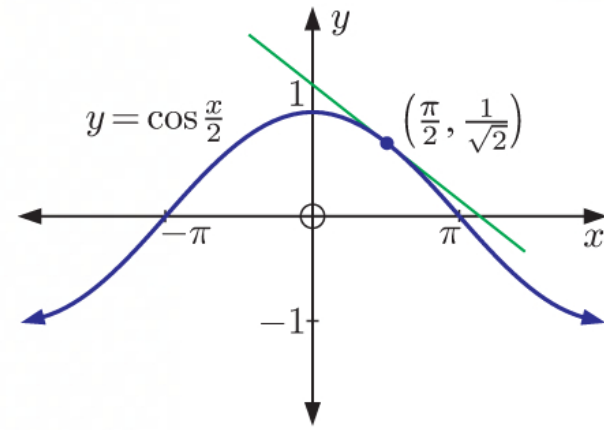
$$\begin{aligned} \text{When } x = \frac{\pi}{2}, \quad \frac{dy}{dx} &= -\frac{1}{2} \sin \frac{\pi}{4} \\ &= -\frac{1}{2\sqrt{2}} \end{aligned}$$

$$\text{So, the tangent has equation } y = -\frac{1}{2\sqrt{2}} \left( x - \frac{\pi}{2} \right) + \frac{1}{\sqrt{2}}$$

$$\therefore 2\sqrt{2}y = -\left(x - \frac{\pi}{2}\right) + 2$$

$$= -x + \frac{\pi}{2} + 2$$

$$\therefore x + 2\sqrt{2}y = \frac{\pi}{2} + 2$$

**11**

$$\text{a } y = \sec 2x = \frac{1}{\cos 2x}$$

$$\begin{aligned} \text{When } x = \frac{\pi}{3}, \quad y &= \frac{1}{\cos \frac{2\pi}{3}} \\ &= \frac{1}{(-\frac{1}{2})} \\ &= -2 \end{aligned}$$

$$\text{So, the point of contact is } \left( \frac{\pi}{3}, -2 \right).$$

$$\begin{aligned} \text{Now } \frac{dy}{dx} &= \sec 2x \tan 2x (2) \quad \{\text{chain rule}\} \\ &= \frac{2 \sin 2x}{\cos^2(2x)} \end{aligned}$$

$$\begin{aligned} \text{When } x = \frac{\pi}{3}, \quad \frac{dy}{dx} &= \frac{2 \sin \frac{2\pi}{3}}{\cos^2(\frac{2\pi}{3})} \\ &= \frac{2(\frac{\sqrt{3}}{2})}{(\frac{1}{2})^2} \\ &= \frac{\sqrt{3}}{\frac{1}{4}} \\ &= 4\sqrt{3} \end{aligned}$$

$$\therefore \text{ the normal at } \left( \frac{\pi}{3}, -2 \right) \text{ has gradient } -\frac{1}{4\sqrt{3}}.$$

$$\therefore \text{ the equation of the normal is } y = -\frac{1}{4\sqrt{3}} \left( x - \frac{\pi}{3} \right) - 2$$

$$\therefore 4\sqrt{3}y = -x + \frac{\pi}{3} - 8\sqrt{3}$$

$$\therefore x + 4\sqrt{3}y = \frac{\pi}{3} - 8\sqrt{3}$$

$$\text{b } y = \operatorname{cosec} \frac{x}{2} = \frac{1}{\sin \frac{x}{2}}$$

$$\begin{aligned} \text{When } x = \frac{\pi}{2}, \quad y &= \frac{1}{\sin \frac{\pi}{4}} \\ &= \frac{1}{(\frac{1}{\sqrt{2}})} = \sqrt{2} \end{aligned}$$

So, the point of contact is  $(\frac{\pi}{2}, \sqrt{2})$ .

$$\begin{aligned} \text{Now } \frac{dy}{dx} &= -\operatorname{cosec} \frac{x}{2} \cot \frac{x}{2} \left(\frac{1}{2}\right) \quad \{\text{chain rule}\} \\ &= -\frac{\cos \frac{x}{2}}{2 \sin^2(\frac{x}{2})} \end{aligned}$$

$$\begin{aligned} \text{When } x = \frac{\pi}{2}, \quad y &= -\frac{\cos \frac{\pi}{4}}{2 \sin^2(\frac{\pi}{4})} \\ &= -\frac{\frac{1}{\sqrt{2}}}{2(\frac{1}{\sqrt{2}})^2} \\ &= -\frac{\frac{1}{\sqrt{2}}}{1} \\ &= -\frac{1}{\sqrt{2}} \end{aligned}$$

$\therefore$  the normal at  $(\frac{\pi}{2}, \sqrt{2})$  has gradient  $\sqrt{2}$ .

$\therefore$  the equation of the normal is  $y = \sqrt{2}(x - \frac{\pi}{2}) + \sqrt{2}$

$$\therefore \sqrt{2}y = 2x - \pi + 2$$

$$\therefore 2x - \sqrt{2}y = \pi - 2$$

**12** The curves  $y = \sqrt{3x+1}$  and  $y = \sqrt{5x-x^2}$  meet when

$$\sqrt{3x+1} = \sqrt{5x-x^2}$$

Squaring both sides,  $3x+1 = 5x-x^2$

$$\therefore x^2 - 2x + 1 = 0$$

$$\therefore (x-1)^2 = 0$$

$$\therefore x = 1$$

When  $x = 1$ ,  $y = \sqrt{3+1} = 2$ , so the curves meet at  $(1, 2)$ .

For  $y = \sqrt{3x+1} = (3x+1)^{\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{1}{2}(3x+1)^{-\frac{1}{2}}(3) \quad \{\text{chain rule}\}$$

$$= \frac{3}{2\sqrt{3x+1}}$$

$$\therefore \text{ at } (1, 2), \quad \frac{dy}{dx} = \frac{3}{2\sqrt{3+1}} = \frac{3}{4}$$



For  $y = \sqrt{5x - x^2} = (5x - x^2)^{\frac{1}{2}}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{2}(5x - x^2)^{-\frac{1}{2}}(5 - 2x) \quad \{\text{chain rule}\} \\ &= \frac{5 - 2x}{2\sqrt{5x - x^2}}\end{aligned}$$

$$\therefore \text{ at } (1, 2), \quad \frac{dy}{dx} = \frac{5 - 2}{2\sqrt{5 - 1}} = \frac{3}{4}$$

$\therefore$  the curves have a common tangent at their point of intersection.

The common tangent has equation  $y = \frac{3}{4}(x - 1) + 2$

$$\therefore y = \frac{3}{4}x + \frac{5}{4}$$

13

$$\begin{aligned}y &= \frac{ax + b}{\sqrt{x}} \\ &= a\sqrt{x} + \frac{b}{\sqrt{x}} = ax^{\frac{1}{2}} + bx^{-\frac{1}{2}} \\ \therefore \frac{dy}{dx} &= \frac{a}{2}x^{-\frac{1}{2}} - \frac{b}{2}x^{-\frac{3}{2}} \\ &= \frac{a}{2\sqrt{x}} - \frac{b}{2x\sqrt{x}}\end{aligned}$$

The equation of the tangent at  $x = 1$  is  $2x - y = 1$   
which is  $y = 2x - 1$

so the gradient of the tangent is 2

$$\begin{aligned}\therefore \text{ at } x = 1, \quad \frac{dy}{dx} &= \frac{a}{2} - \frac{b}{2} = 2 \\ \therefore a - b &= 4 \\ \therefore a &= b + 4 \quad \dots (*)\end{aligned}$$

Also at  $x = 1$ , the tangent touches the curve

$$\begin{aligned}\therefore \frac{a(1) + b}{\sqrt{1}} &= 2(1) - 1 \\ \therefore a + b &= 1 \\ \therefore b + 4 + b &= 1 \quad \{\text{using } (*)\} \\ \therefore 2b &= -3 \\ \therefore b &= -\frac{3}{2} \quad \text{and} \quad a = -\frac{3}{2} + 4 = \frac{5}{2}\end{aligned}$$

**14 a**  $x^2 + y^2 = 4$

When  $y = 1$ ,  $x^2 + 1 = 4$

$$\therefore x^2 = 3$$

$$\therefore x = \pm\sqrt{3}$$

So, the points of contact are  $(\sqrt{3}, 1)$  and  $(-\sqrt{3}, 1)$ .

Now  $x^2 + y^2 = 4$

$$\therefore 2x + 2y \frac{dy}{dx} = 0$$

$$\therefore 2y \frac{dy}{dx} = -2x$$

$$\therefore \frac{dy}{dx} = -\frac{x}{y}$$

When  $x = \sqrt{3}$ ,  $\frac{dy}{dx} = -\frac{\sqrt{3}}{1}$   
 $= -\sqrt{3}$

$\therefore$  the tangent at  $(\sqrt{3}, 1)$  has equation  $y = -\sqrt{3}(x - \sqrt{3}) + 1$

$$\therefore y = -\sqrt{3}x + 3 + 1$$

$$\therefore \sqrt{3}x + y = 4$$

When  $x = -\sqrt{3}$ ,  $\frac{dy}{dx} = -\frac{(-\sqrt{3})}{1}$   
 $= \sqrt{3}$

$\therefore$  the tangent at  $(-\sqrt{3}, 1)$  has equation  $y = \sqrt{3}(x - (-\sqrt{3})) + 1$

$$\therefore y = \sqrt{3}x + 3 + 1$$

$$\therefore \sqrt{3}x - y = -4$$

**b** The tangents intersect when  $\sqrt{3}x + y - 4 = \sqrt{3}x - y - (-4)$

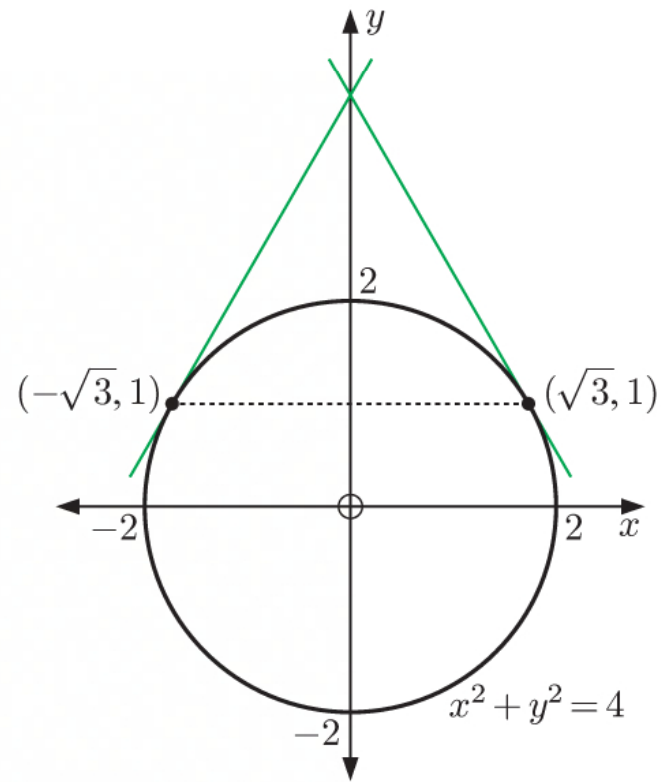
$$\therefore 2y = 8$$

$$\therefore y = 4$$

When  $y = 4$ ,  $\sqrt{3}x + 4 = 4$

$$\therefore x = 0$$

$\therefore$  the tangents intersect at  $(0, 4)$ .



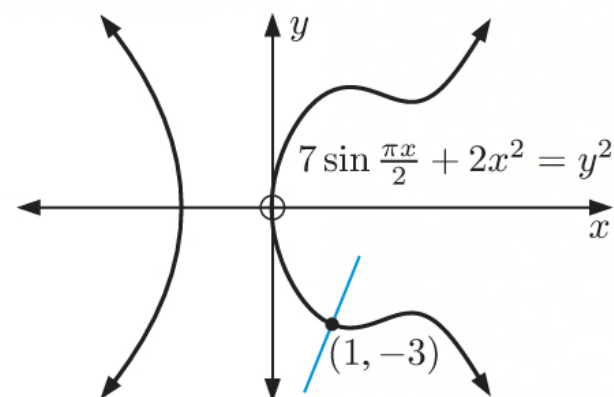
**15 a**  $7 \sin \frac{\pi x}{2} + 2x^2 = y^2$

$$\therefore \frac{d}{dx} (7 \sin \frac{\pi x}{2} + 2x^2) = \frac{d}{dx} (y^2)$$

$$\therefore \frac{7\pi}{2} \cos \frac{\pi x}{2} + 4x = 2y \frac{dy}{dx} \quad \{\text{chain rule}\}$$

$$\therefore \frac{dy}{dx} = \frac{\frac{7\pi}{2} \cos \frac{\pi x}{2} + 4x}{2y}$$

$$= \frac{7\pi \cos \frac{\pi x}{2} + 8x}{4y}$$



$$\begin{aligned}
 \text{b At } (1, -3), \quad \frac{dy}{dx} &= \frac{7\pi \cos \frac{\pi}{2} + 8}{-12} \\
 &= \frac{0 + 8}{-12} \\
 &= -\frac{2}{3}
 \end{aligned}$$

$\therefore$  the normal at  $(1, -3)$  has gradient  $\frac{3}{2}$ .

$$\begin{aligned}
 \therefore \text{ the normal has equation } y &= \frac{3}{2}(x - 1) - 3 \\
 &= \frac{3}{2}x - \frac{3}{2} - 3 \\
 &= \frac{3}{2}x - \frac{9}{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{16} \quad f(x) &= x^4 - 4x^3 - 8x^2 + 5 \\
 \therefore f'(x) &= 4x^3 - 12x^2 - 16x \\
 &= 4x(x^2 - 3x - 4) \\
 &= 4x(x + 1)(x - 4)
 \end{aligned}$$

which has sign diagram: 

**a**  $f(x)$  is increasing for  $-1 \leq x \leq 0$  and  $x \geq 4$ .

**b**  $f(x)$  is decreasing for  $x \leq -1$  and  $0 \leq x \leq 4$ .

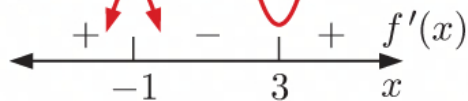
$$\begin{aligned}
 \text{17} \quad f(x) &= x^3 - 3x^2 + ax + 50 \\
 \therefore f'(x) &= 3x^2 - 6x + a
 \end{aligned}$$

**a**  $f(x)$  has a stationary point at  $x = 3$

$$\begin{aligned}
 \therefore f'(3) &= 0 \\
 \therefore 3(3)^2 - 6(3) + a &= 0 \\
 \therefore 27 - 18 + a &= 0 \\
 \therefore a &= -9
 \end{aligned}$$

**b** Since  $a = -9$ , then  $f(x) = x^3 - 3x^2 - 9x + 50$   
and  $f'(x) = 3x^2 - 6x - 9$

$$\begin{aligned}
 &= 3(x^2 - 2x - 3) \\
 &= 3(x + 1)(x - 3)
 \end{aligned}$$

which has sign diagram: 

$$\text{When } x = -1, \quad f(-1) = (-1)^3 - 3(-1)^2 - 9(-1) + 50 = 55$$

$$\text{When } x = 3, \quad f(3) = 3^3 - 3(3)^2 - 9(3) + 50 = 23$$

So, there is a local maximum at  $(-1, 55)$  and a local minimum at  $(3, 23)$ .

**18**  $f(x) = x^3 - 4x^2 + 4x$

**a**  $f(0) = 0$ , so the  $y$ -intercept is 0.

When  $f(x) = 0$ ,  $x^3 - 4x^2 + 4x = 0$

$$\therefore x(x^2 - 4x + 4) = 0$$

$$\therefore x(x - 2)^2 = 0$$

$$\therefore x = 0 \text{ or } 2$$

So, the  $x$ -intercepts are 0 and 2.

**b**  $f'(x) = 3x^2 - 8x + 4$

$f'(x) = 0$  when  $3x^2 - 8x + 4 = 0$

$$\therefore (3x - 2)(x - 2) = 0$$

$f'(x)$  has sign diagram:

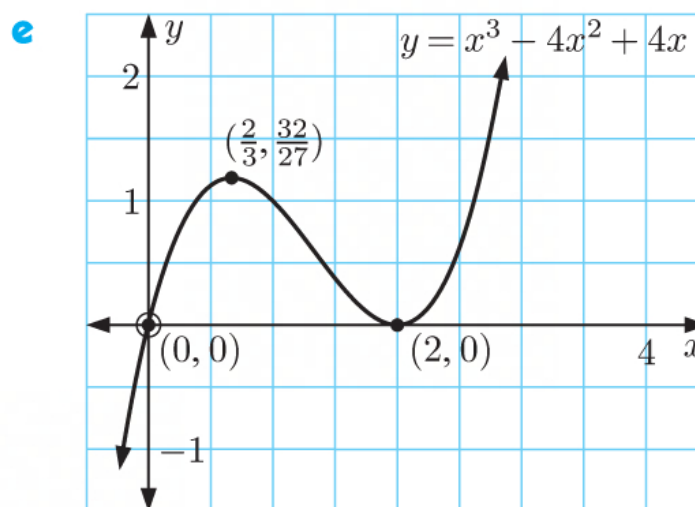
When  $x = \frac{2}{3}$ ,  $f(\frac{2}{3}) = (\frac{2}{3})^3 - 4(\frac{2}{3})^2 + 4(\frac{2}{3}) = \frac{8}{27} - \frac{16}{9} + \frac{8}{3} = \frac{32}{27}$

When  $x = 2$ ,  $f(2) = 2^3 - 4(2)^2 + 4(2) = 8 - 16 + 8 = 0$

So, there is a local maximum at  $(\frac{2}{3}, \frac{32}{27})$ , and a local minimum at  $(2, 0)$ .

**c** From the sign diagram of  $f'(x)$  in **b**,  $f(x)$  is increasing for  $x \leq \frac{2}{3}$  and  $x \geq 2$ , and  $f(x)$  is decreasing for  $\frac{2}{3} \leq x \leq 2$ .

**d** As  $x \rightarrow \infty$ ,  $y \rightarrow \infty$ ,  
as  $x \rightarrow -\infty$ ,  $y \rightarrow -\infty$ .



**19 a**  $f(x) = \frac{x+1}{x^2 - 2x - 8}$

$$\therefore f'(x) = \frac{(1)(x^2 - 2x - 8) - (x+1)(2x-2)}{(x^2 - 2x - 8)^2} \quad \{\text{quotient rule}\}$$

$$= \frac{x^2 - 2x - 8 - (2x^2 - 2)}{(x^2 - 2x - 8)^2}$$

$$= \frac{x^2 - 2x - 8 - 2x^2 + 2}{(x^2 - 2x - 8)^2}$$

$$= \frac{-x^2 - 2x - 6}{(x^2 - 2x - 8)^2}$$

$$= -\frac{x^2 + 2x + 6}{(x^2 - 2x - 8)^2} \quad \text{which has sign diagram:}$$

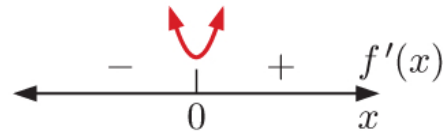


$$\begin{aligned} \text{b } f'(x) &= -\frac{x^2 + 2x + 1 + 5}{(x^2 - 2x - 8)^2} \\ &= -\frac{(x+1)^2 + 5}{(x^2 - 2x - 8)^2} < 0 \quad \text{for all } x \in \mathbb{R}, \quad x \neq -2, 4 \end{aligned}$$

$\therefore f(x)$  is decreasing for all  $x \in \mathbb{R}$ ,  $x \neq -2, 4$

$\therefore f(x)$  is never increasing.

**20 a**  $f(x) = e^x - x$   
 $\therefore f'(x) = e^x - 1$  which has sign diagram:



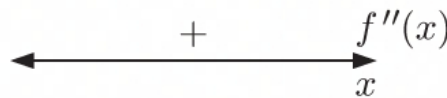
Now  $f(0) = e^0 - 0 = 1$

$\therefore y = f(x)$  has a local minimum at  $(0, 1)$ .

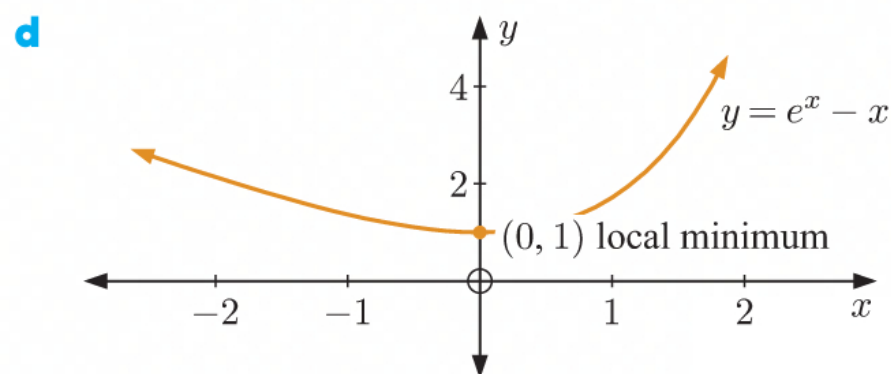
**b** As  $x \rightarrow \infty$ ,  $e^x \rightarrow \infty$  {at a much faster rate than  $x$ }

$\therefore$  as  $x \rightarrow \infty$ ,  $f(x) \rightarrow \infty$

**c**  $f''(x) = e^x$  which has sign diagram:



$\therefore f(x)$  is concave up for all  $x \in \mathbb{R}$ .



**e**  $y = f(x)$  has a local minimum at  $(0, 1)$

$$\therefore f(x) \geq 1$$

$$\therefore e^x - x \geq 1$$

$$\therefore e^x \geq x + 1 \quad \text{for all } x$$

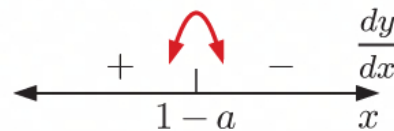
**21**  $y = \frac{x+a}{e^x}$

$$\therefore \frac{dy}{dx} = \frac{(1)e^x - (x+a)e^x}{(e^x)^2} \quad \{\text{quotient rule}\}$$

$$= \frac{e^x - xe^x - ae^x}{e^{2x}}$$

$$= \frac{e^x(1-x-a)}{e^{2x}}$$

$$= \frac{(1-a)-x}{e^x} \quad \text{which has sign diagram:}$$



$$\frac{dy}{dx} = 0 \quad \text{when } x = 1 - a$$

When  $x = 1 - a$ ,  $y = \frac{(1-a)+a}{e^{1-a}}$

$$= \frac{1}{e^{1-a}}$$

$$= e^{a-1}$$

$\therefore$  the stationary point of  $y = \frac{x+a}{e^x}$  where  $a$  is a constant, is a local maximum  $(1-a, e^{a-1})$ .

$$\begin{aligned}
 22 \quad f(x) &= \frac{\ln(ax)}{bx} \\
 &= \frac{\ln a + \ln x}{bx} \\
 \therefore f'(x) &= \frac{\left(\frac{1}{x}\right) \times bx - (\ln a + \ln x) \times b}{b^2 x^2} \quad \{\text{quotient rule}\} \\
 &= \frac{b - b(\ln a + \ln x)}{b^2 x^2} \\
 &= \frac{b(1 - \ln a - \ln x)}{b^2 x^2} \\
 &= \frac{1 - \ln a - \ln x}{bx^2}
 \end{aligned}$$

$$f'(x) = 0 \quad \text{when} \quad \ln x = 1 - \ln a$$

$$\therefore \ln\left(\frac{e}{2}\right) = 1 - \ln a \quad \left\{ \left(\frac{e}{2}, \frac{2}{3e}\right) \text{ is a stationary point} \right\}$$

$$\therefore \ln e - \ln 2 = 1 - \ln a$$

$$\therefore 1 - \ln 2 = 1 - \ln a$$

$$\therefore a = 2$$

$$\text{Now } f\left(\frac{e}{2}\right) = \frac{2}{3e}$$

$$\therefore \frac{\ln(2 \times \frac{e}{2})}{b(\frac{e}{2})} = \frac{2}{3e}$$

$$\therefore \frac{\ln e}{\frac{be}{2}} = \frac{2}{3e}$$

$$\therefore 1 \times \frac{2}{be} = \frac{2}{3e}$$

$$\therefore b = 3$$

$$\begin{aligned}
 23 \quad f(x) &= -\frac{1}{2}x^4 + x^3 + 6x^2 - 3x + 2 \\
 \therefore f'(x) &= -2x^3 + 3x^2 + 12x - 3 \quad \text{which has sign diagram:} \quad \begin{array}{ccccccc} & + & | & - & | & + & | & - \\ \leftarrow & -1.96 & & 0.238 & & 3.22 & & \rightarrow x \end{array} \\
 \therefore f''(x) &= -6x^2 + 6x + 12 \\
 &= -6(x^2 - x - 2) \\
 &= -6(x+1)(x-2) \quad \text{which has sign diagram:} \quad \begin{array}{ccccccc} & - & | & + & | & - \\ \leftarrow & -1 & & 2 & & & \rightarrow x \end{array}
 \end{aligned}$$

**a**  $f(x)$  is increasing for  $x \leq -1.96$  and  $0.238 \leq x \leq 3.22$ .

**b**  $f(x)$  is decreasing for  $-1.96 \leq x \leq 0.238$  and  $x \geq 3.22$ .

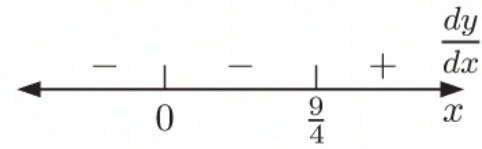
**c**  $f(x)$  is concave upwards for  $-1 \leq x \leq 2$ .

**d**  $f(x)$  is concave downwards for  $x \leq -1$  and  $x \geq 2$ .

**24 a**  $y = x^4 - 3x^3 + 9$

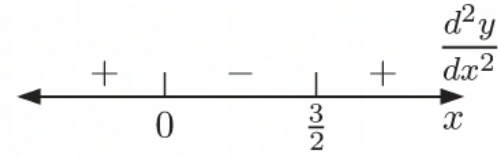
$$\therefore \frac{dy}{dx} = 4x^3 - 9x^2$$

$$= x^2(4x - 9) \quad \text{which has sign diagram:}$$



$$\therefore \frac{d^2y}{dx^2} = 12x^2 - 18x$$

$$= 6x(2x - 3) \quad \text{which has sign diagram:}$$



Since the sign of  $\frac{d^2y}{dx^2}$  changes at  $x = 0$  and  $x = \frac{3}{2}$ , both of these points are points of inflection.

When  $x = 0$ ,  $y = (0)^4 - 3(0)^3 + 9$   
 $= 9$

and  $\frac{dy}{dx} = 0$

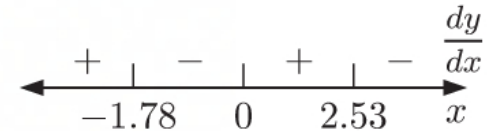
When  $x = \frac{3}{2}$ ,  $y = \left(\frac{3}{2}\right)^4 - 3\left(\frac{3}{2}\right)^3 + 9$   
 $= \frac{81}{16} - \frac{81}{8} + 9$   
 $= \frac{63}{16}$

and  $\frac{dy}{dx} \neq 0$

$\therefore (0, 9)$  is a stationary inflection point, and  $\left(\frac{3}{2}, \frac{63}{16}\right)$  is a non-stationary inflection point.

**b**  $y = -x^4 + x^3 + 9x^2 + 1$

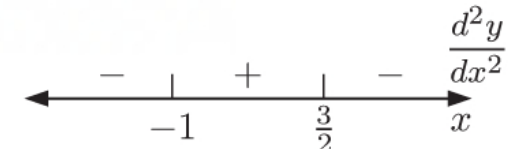
$$\therefore \frac{dy}{dx} = -4x^3 + 3x^2 + 18x \quad \text{which has sign diagram:}$$



$$\therefore \frac{d^2y}{dx^2} = -12x^2 + 6x + 18$$

$$= -6(2x^2 - x - 3)$$

$$= -6(2x - 3)(x + 1) \quad \text{which has sign diagram:}$$



Since the sign of  $\frac{d^2y}{dx^2}$  changes at  $x = -1$  and  $x = \frac{3}{2}$ , both of these points are points of inflection.

When  $x = -1$ ,

$$y = -(-1)^4 + (-1)^3 + 9(-1)^2 + 1$$

$$= -1 - 1 + 9 + 1$$

$$= 8$$

and  $\frac{dy}{dx} \neq 0$

When  $x = \frac{3}{2}$ ,

$$y = -\left(\frac{3}{2}\right)^4 + \left(\frac{3}{2}\right)^3 + 9\left(\frac{3}{2}\right)^2 + 1$$

$$= -\frac{81}{16} + \frac{27}{8} + \frac{81}{4} + 1$$

$$= \frac{313}{16}$$

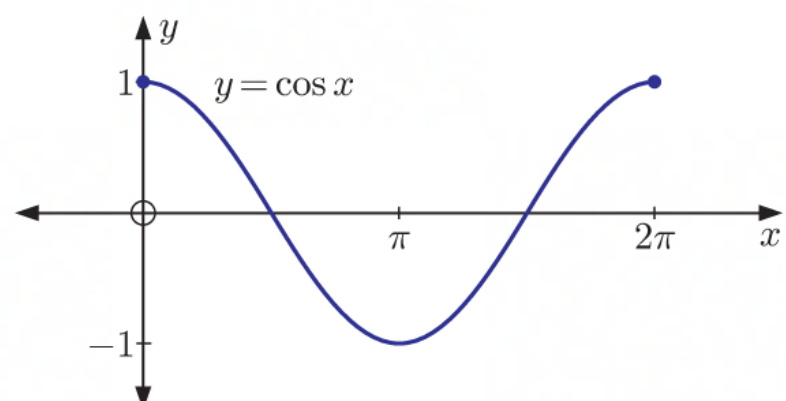
and  $\frac{dy}{dx} \neq 0$

$\therefore (-1, 8)$  and  $\left(\frac{3}{2}, \frac{313}{16}\right)$  are non-stationary inflection points.

**25 a**  $f(x) = \sqrt{\cos x}$ ,  $0 \leq x \leq 2\pi$  is defined

when  $\cos x \geq 0$

$$\therefore 0 \leq x \leq \frac{\pi}{2} \quad \text{and} \quad \frac{3\pi}{2} \leq x \leq 2\pi$$



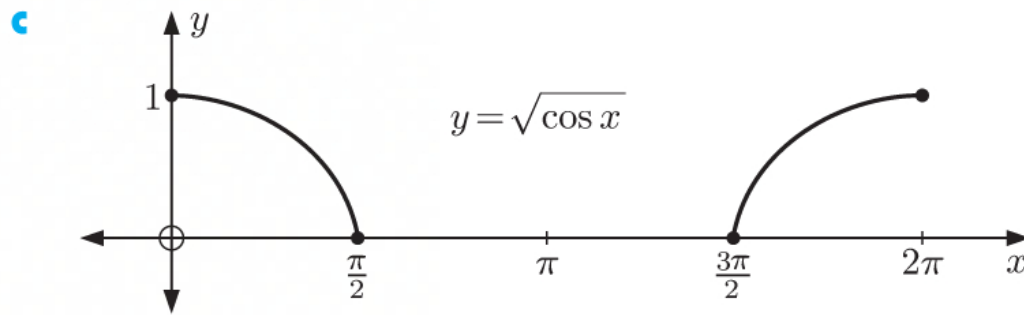
$$\begin{aligned} \mathbf{b} \quad f(x) &= \sqrt{\cos x} = (\cos x)^{\frac{1}{2}} \\ \therefore f'(x) &= \frac{1}{2}(\cos x)^{-\frac{1}{2}}(-\sin x) \\ &= -\frac{\sin x}{2\sqrt{\cos x}} \end{aligned}$$

From **a**, since  $f(x)$  is only defined when  $0 \leq x \leq \frac{\pi}{2}$  and  $\frac{3\pi}{2} \leq x \leq 2\pi$ , we only consider these values of  $x$ .

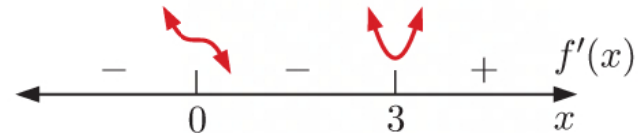
When  $0 \leq x < \frac{\pi}{2}$ ,  $f'(x) \leq 0$

When  $\frac{3\pi}{2} < x \leq 2\pi$ ,  $f'(x) \geq 0$

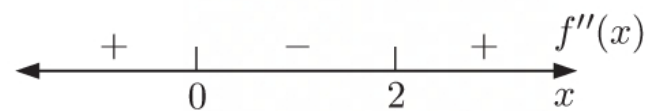
$\therefore f(x)$  is increasing for  $\frac{3\pi}{2} \leq x \leq 2\pi$ , and decreasing for  $0 \leq x \leq \frac{\pi}{2}$ .



$$\begin{aligned} \mathbf{26} \quad \mathbf{a} \quad f(x) &= x^4 - 4x^3 + 7 \\ \therefore f'(x) &= 4x^3 - 12x^2 \\ &= 4x^2(x - 3) \quad \text{which has sign diagram:} \end{aligned}$$



$$\begin{aligned} \therefore f''(x) &= 12x^2 - 24x \\ &= 12x(x - 2) \quad \text{which has sign diagram:} \end{aligned}$$



$$\begin{aligned} \mathbf{b} \quad f(3) &= 3^4 - 4(3)^3 + 7 \\ &= 81 - 108 + 7 \\ &= -20 \end{aligned}$$

$\therefore (3, -20)$  is a local minimum.

**c** Since the signs of  $f''(x)$  change about  $x = 0$  and  $x = 2$ , these are points of inflection.

$$f(0) = 7 \quad \text{and} \quad f'(0) = 0$$

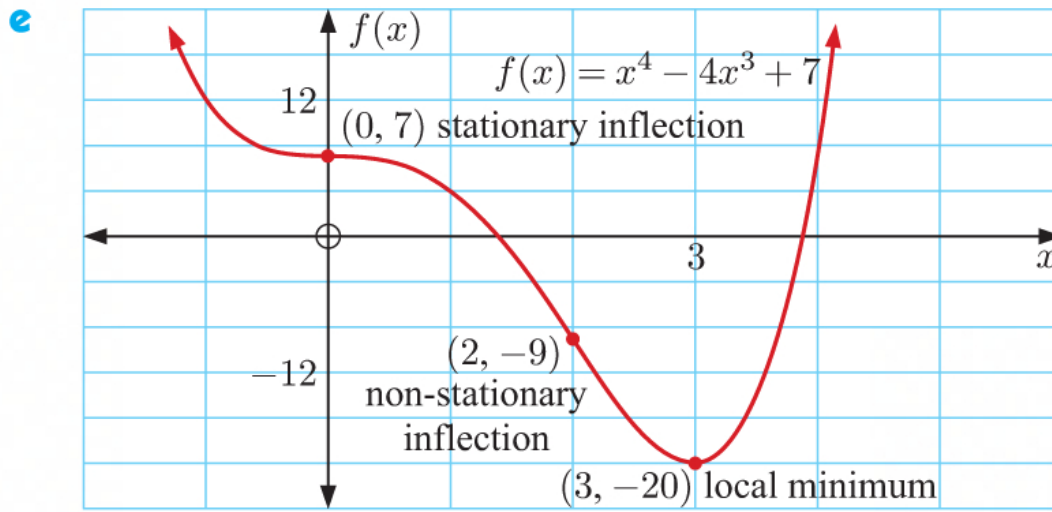
$\therefore (0, 7)$  is a stationary inflection.

$$\begin{aligned} f(2) &= 2^4 - 4(2)^3 + 7 & \text{and} & \quad f'(2) = 4(2)^3 - 12(2)^2 \\ &= 16 - 32 + 7 & & \quad = 32 - 48 \neq 0 \\ &= -9 \end{aligned}$$

$\therefore (2, -9)$  is a non-stationary inflection.

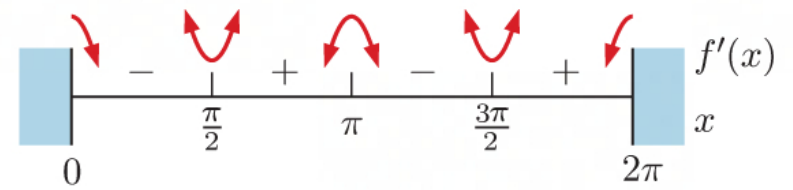
- d**
- i**  $f(x)$  is increasing for  $x \geq 3$ .
  - ii**  $f(x)$  is decreasing for  $x \leq 3$ .
  - iii**  $f(x)$  is concave up for  $x \leq 0$  and  $x \geq 2$ .
  - iv**  $f(x)$  is concave down for  $0 \leq x \leq 2$ .





**27 a**  $f(x) = \cos^2 x, \quad 0 \leq x \leq 2\pi$

$$\begin{aligned} \therefore f'(x) &= 2 \cos x (-\sin x) \\ &= -2 \sin x \cos x \\ &= -\sin 2x \quad \text{which has sign diagram:} \end{aligned}$$



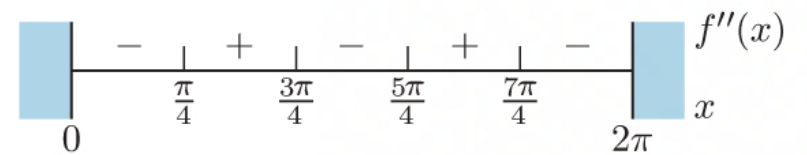
$$\begin{aligned} f'(x) = 0 \quad \text{when} \quad -\sin 2x &= 0 \\ \therefore 2x &= 0 + k\pi, \quad k \in \mathbb{Z} \\ \therefore x &= 0 + \frac{k\pi}{2} \\ \therefore x &= 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \text{ or } 2\pi \end{aligned}$$

$$\begin{aligned} f(0) &= \cos^2 0 & f\left(\frac{\pi}{2}\right) &= \cos^2\left(\frac{\pi}{2}\right) & f(\pi) &= \cos^2 \pi \\ &= 1 & &= 0 & &= 1 \\ f\left(\frac{3\pi}{2}\right) &= \cos^2\left(\frac{3\pi}{2}\right) & f(2\pi) &= \cos^2 2\pi \\ &= 0 & &= 1 \end{aligned}$$

$\therefore$  there are local maxima at  $(0, 1)$ ,  $(\pi, 1)$ ,  $(2\pi, 1)$ , and local minima at  $(\frac{\pi}{2}, 0)$ ,  $(\frac{3\pi}{2}, 0)$ .

**b**  $f''(x) = -\cos 2x \times 2$

$$= -2 \cos 2x \quad \text{which has sign diagram:}$$

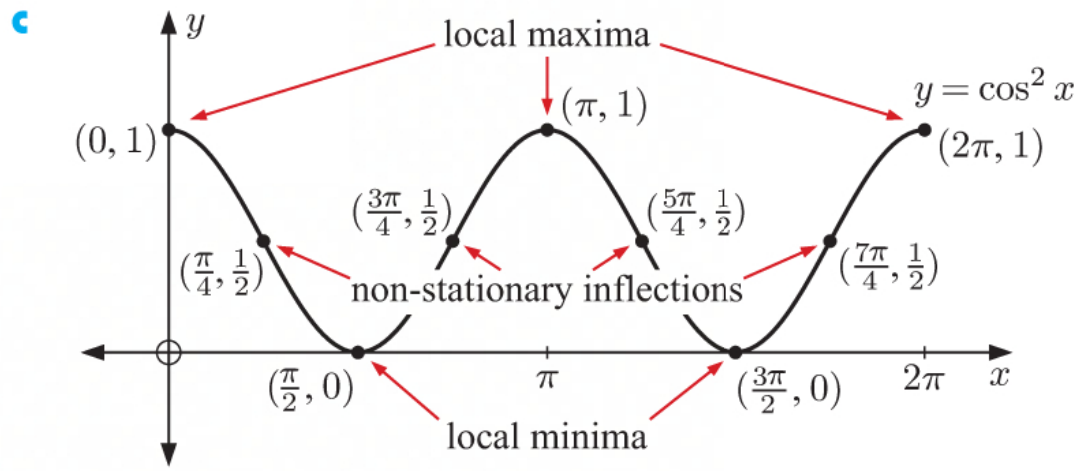


$$\begin{aligned} f''(x) = 0 \quad \text{when} \quad -2 \cos 2x &= 0 \\ \therefore 2x &= \frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z} \\ \therefore x &= \frac{\pi}{4} + \frac{k\pi}{2} \\ \therefore x &= \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \text{ or } \frac{7\pi}{4} \end{aligned}$$

$$\begin{aligned} f\left(\frac{\pi}{4}\right) &= \cos^2\left(\frac{\pi}{4}\right) & f\left(\frac{3\pi}{4}\right) &= \cos^2\left(\frac{3\pi}{4}\right) & f\left(\frac{5\pi}{4}\right) &= \cos^2\left(\frac{5\pi}{4}\right) & f\left(\frac{7\pi}{4}\right) &= \cos^2\left(\frac{7\pi}{4}\right) \\ &= \left(\frac{1}{\sqrt{2}}\right)^2 & &= \left(-\frac{1}{\sqrt{2}}\right)^2 & &= \left(-\frac{1}{\sqrt{2}}\right)^2 & &= \left(\frac{1}{\sqrt{2}}\right)^2 \\ &= \frac{1}{2} & &= \frac{1}{2} & &= \frac{1}{2} & &= \frac{1}{2} \end{aligned}$$

and  $f'(x) \neq 0$  for  $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \text{ or } \frac{7\pi}{4}$

$\therefore$  there are non-stationary points of inflection at  $(\frac{\pi}{4}, \frac{1}{2})$ ,  $(\frac{3\pi}{4}, \frac{1}{2})$ ,  $(\frac{5\pi}{4}, \frac{1}{2})$ ,  $(\frac{7\pi}{4}, \frac{1}{2})$ .



**28 a**  $f(x) = \frac{e^x}{x-1}$

Now  $f(0) = \frac{e^0}{-1} = -1$  so the  $y$ -intercept is  $-1$ .

**b**  $f(x)$  is defined when  $x - 1 \neq 0$   
 $\therefore x \neq 1$

**c**  $f'(x) = \frac{e^x(x-1) - e^x(1)}{(x-1)^2}$  {quotient rule}

$$= \frac{e^x(x-2)}{(x-1)^2}$$

which has sign diagram:  $\begin{array}{c} - & - & + \\ \vdots & & \uparrow \\ 1 & & 2 \end{array} \quad \begin{array}{c} f'(x) \\ x \end{array}$

$\therefore f'(x) \leq 0$  for  $x < 1$  and  $1 < x \leq 2$  and  $f'(x) \geq 0$  for  $x \geq 2$

$\therefore f(x)$  is decreasing for all defined values of  $x \leq 2$ , and increasing for  $x \geq 2$ .

$$f''(x) = \frac{[e^x(x-2) + e^x(1)](x-1)^2 - e^x(x-2)[2(x-1)^1(1)]}{(x-1)^4}$$
 {product and quotient rules}
$$= \frac{[e^x(x-2+1)(x-1)^2] - 2e^x(x-2)(x-1)}{(x-1)^4}$$

$$= \frac{e^x(x-1)(x-1)^2 - 2e^x(x-2)(x-1)}{(x-1)^4}$$

$$= \frac{e^x(x-1)[(x-1)^2 - 2(x-2)]}{(x-1)^4}$$

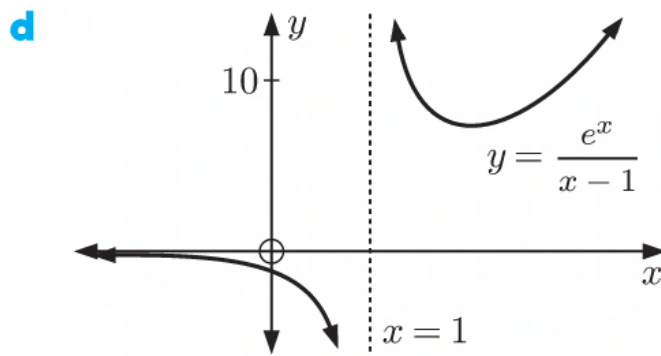
$$= \frac{e^x(x-1)[x^2 - 2x + 1 - 2x + 4]}{(x-1)^4}$$

$$= \frac{e^x(x^2 - 4x + 5)}{(x-1)^3}$$

where the quadratic term has  $\Delta = (-4)^2 - 4(1)(5)$   
 $= 16 - 20$   
 $= -4 < 0$

The sign diagram of  $f''(x)$  is:  $\begin{array}{c} - & + \\ \vdots & \\ 1 & \end{array} \quad \begin{array}{c} f''(x) \\ x \end{array}$

$\therefore f(x)$  is concave down for all  $x < 1$   
and concave up for all  $x > 1$ .



**e**  $f(2) = \frac{e^2}{2-1} = e^2$

Using **c**, we have a local minimum at  $(2, e^2)$

$\therefore$  the tangent at  $x = 2$  is horizontal  
with equation  $y = e^2$ .

**29** At  $x = A$ ,  $f'(x) = 0$  and  $f''(x) = 0$

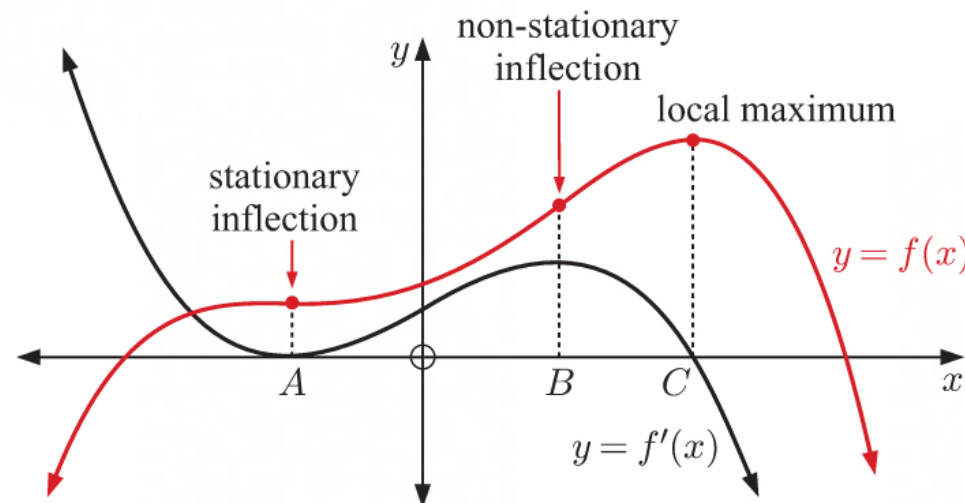
$\therefore f(x)$  has a stationary inflection point at  $x = A$ .

At  $x = B$ ,  $f''(x) = 0$  but  $f'(x) \neq 0$

$\therefore f(x)$  has a non-stationary inflection point at  $x = B$ .

$f'(x)$  is above the  $x$ -axis for  $x \leq C$ , and below the  $x$ -axis for  $x \geq C$

$\therefore f(x)$  is increasing for  $x \leq C$  and decreasing for  $x \geq C$ , so  $f(x)$  has a local maximum at  $x = C$ .



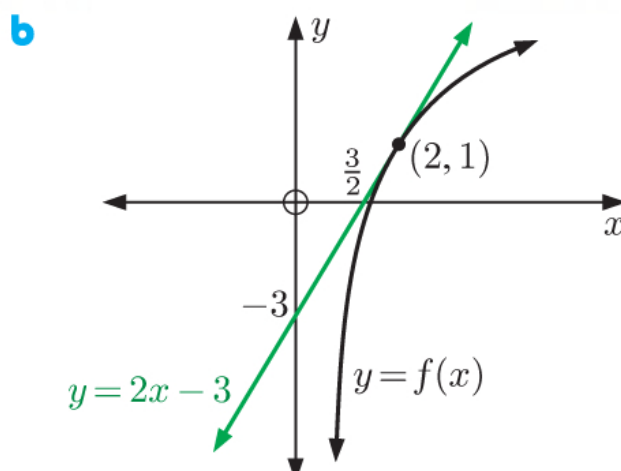
**30**  $f'(x) > 0$  and  $f''(x) < 0$  for all  $x$

$\therefore f(x)$  is increasing and concave downwards for all  $x$ .

**a**  $f(2) = 1$  and  $f'(2) = 2$

$\therefore (2, 1)$  lies on the curve and the tangent at this point has gradient 2

$\therefore$  the tangent has equation  $y = 2(x - 2) + 1$   
 $= 2x - 4 + 1$   
 $= 2x - 3$



**c**  $f(x)$  is always increasing with  $f'(x) > 0$  so it has *at most one* zero.

$f(x)$  is also concave downwards for all  $x$ , so it always lies below the tangent shown.

So, for  $x < \frac{3}{2}$ , the tangent and  $f(x)$  lie below the  $x$ -axis.

$\therefore f(x)$  has exactly one zero.

**d** From the graph, the  $x$ -intercept of  $y = f(x)$  lies in the interval  $\frac{3}{2} < x < 2$ .

**31 a**  $\lim_{x \rightarrow 0} x^2 = 0$  and  $\lim_{x \rightarrow 0} (x + e^{-x} - 1) = 0 + 1 - 1 = 0$ , so we can use l'Hôpital's rule.

$$\begin{aligned} \therefore \lim_{x \rightarrow 0} \frac{x^2}{x + e^{-x} - 1} &= \lim_{x \rightarrow 0} \frac{2x}{1 - e^{-x} - 0} \quad \{\text{l'Hôpital's rule}\} \\ &= \lim_{x \rightarrow 0} \frac{2x}{1 - e^{-x}} \end{aligned}$$

This is also of the form  $\frac{0}{0}$ , so we can use l'Hôpital's rule again.

$$\begin{aligned} \therefore \lim_{x \rightarrow 0} \frac{x^2}{x + e^{-x} - 1} &= \lim_{x \rightarrow 0} \frac{2}{e^{-x}} \quad \{\text{l'Hôpital's rule}\} \\ &= \frac{2}{1} \\ &= 2 \end{aligned}$$

**b** As  $x \rightarrow \pi^-$ ,  $\sin 3x \rightarrow 0^+$  and  $\sin 5x \rightarrow 0^+$ .

As  $x \rightarrow 0^+$ ,  $\ln x \rightarrow -\infty$ .

$\therefore$  as  $x \rightarrow \pi^-$ ,  $\ln(\sin 3x) \rightarrow -\infty$  and  $\ln(\sin 5x) \rightarrow -\infty$ , we can use l'Hôpital's rule.

$$\begin{aligned} \therefore \lim_{x \rightarrow \pi^-} \frac{\ln(\sin 3x)}{\ln(\sin 5x)} &= \lim_{x \rightarrow \pi^-} \frac{\left(\frac{3 \cos 3x}{\sin 3x}\right)}{\left(\frac{5 \cos 5x}{\sin 5x}\right)} \quad \{\text{l'Hôpital's rule}\} \\ &= \lim_{x \rightarrow \pi^-} \frac{3 \sin 5x \cos 3x}{5 \sin 3x \cos 5x} \\ &= \left( \lim_{x \rightarrow \pi^-} \frac{\sin 5x}{\sin 3x} \right) \times \left( \lim_{x \rightarrow \pi^-} \frac{3 \cos 3x}{5 \cos 5x} \right) \quad \{\text{limit laws}\} \\ &= \left( \lim_{x \rightarrow \pi^-} \frac{\sin 5x}{\sin 3x} \right) \times \frac{3}{5} \end{aligned}$$

This is of the form  $\frac{0}{0}$ , so we can use l'Hôpital's rule again.

$$\begin{aligned} \therefore \lim_{x \rightarrow \pi^-} \frac{\ln(\sin 3x)}{\ln(\sin 5x)} &= \frac{3}{5} \times \lim_{x \rightarrow \pi^-} \frac{5 \cos 5x}{3 \cos 3x} \quad \{\text{l'Hôpital's rule}\} \\ &= \frac{3}{5} \times \frac{5}{3} \\ &= 1 \end{aligned}$$

**32** 
$$\begin{aligned} \frac{x}{x-1} - \frac{1}{\ln x} &= \frac{x \ln x - (x-1)}{(x-1) \ln x} \\ &= \frac{x \ln x + 1 - x}{(x-1) \ln x} \end{aligned}$$

$$\begin{aligned} \lim_{x \rightarrow 1} (x \ln x + 1 - x) &= 1 \times \ln 1 + 1 - 1 \\ &= 0 \end{aligned}$$

$$\begin{aligned} \text{and } \lim_{x \rightarrow 1} ((x-1) \ln x) &= (1-1) \ln 1 \\ &= 0 \end{aligned}$$

so we can use l'Hôpital's rule.



$$\begin{aligned}\therefore \lim_{x \rightarrow 1} \left( \frac{x}{x-1} - \frac{1}{\ln x} \right) &= \lim_{x \rightarrow 1} \frac{\ln x + x \left( \frac{1}{x} \right) + 0 - 1}{\ln x + (x-1) \left( \frac{1}{x} \right)} \quad \{\text{l'Hôpital's rule}\} \\ &= \lim_{x \rightarrow 1} \frac{\ln x}{\ln x + 1 - \frac{1}{x}}\end{aligned}$$

$\lim_{x \rightarrow 1} \ln x = 0$  and  $\lim_{x \rightarrow 1} \left( \ln x + 1 - \frac{1}{x} \right) = 0 + 1 - 1 = 0$ , so we can use l'Hôpital's rule again.

$$\begin{aligned}\therefore \lim_{x \rightarrow 1} \left( \frac{x}{x-1} - \frac{1}{\ln x} \right) &= \lim_{x \rightarrow 1} \frac{\frac{1}{x}}{\frac{1}{x} + \frac{1}{x^2}} \quad \{\text{l'Hôpital's rule}\} \\ &= \frac{\lim_{x \rightarrow 1} \frac{1}{x}}{\lim_{x \rightarrow 1} \left( \frac{1}{x} + \frac{1}{x^2} \right)} \\ &= \frac{1}{2} \quad \text{as required}\end{aligned}$$

**33** To differentiate  $(x+h)^n$  with respect to  $h$ , we would need to use the rule that we are trying to prove.

**34 a**  $\lim_{x \rightarrow 0} (x - \sin x) = 0 - 0 = 0$  and  $\lim_{x \rightarrow 0} x^3 = 0$ , so we can use l'Hôpital's rule.

$$\therefore \lim_{x \rightarrow 0} \frac{x - \sin x}{x^3} = \lim_{x \rightarrow 0} \frac{1 - \cos x}{3x^2} \quad \{\text{l'Hôpital's rule}\}$$

This is also of the form  $\frac{0}{0}$ , so we can use l'Hôpital's rule again.

$$\begin{aligned}\therefore \lim_{x \rightarrow 0} \frac{x - \sin x}{x^3} &= \lim_{x \rightarrow 0} \frac{\sin x}{6x} \quad \{\text{l'Hôpital's rule}\} \\ &= \frac{1}{6} \times \lim_{x \rightarrow 0} \frac{\sin x}{x} \\ &= \frac{1}{6} \quad \{\text{Fundamental Trigonometric Limit}\}\end{aligned}$$

**b** As  $x \rightarrow \infty$ ,  $e^{2x} + x^2 \rightarrow \infty$  and  $e^x + 4x \rightarrow \infty$ , so we can use l'Hôpital's rule.

$$\therefore \lim_{x \rightarrow \infty} \frac{e^{2x} + x^2}{e^x + 4x} = \lim_{x \rightarrow \infty} \frac{2e^{2x} + 2x}{e^x + 4} \quad \{\text{l'Hôpital's rule}\}$$

This is also of the form  $\frac{\infty}{\infty}$ , so we can use l'Hôpital's rule again.

$$\begin{aligned}\therefore \lim_{x \rightarrow \infty} \frac{e^{2x} + x^2}{e^x + 4x} &= \lim_{x \rightarrow \infty} \frac{4e^{2x} + 2}{e^x} \quad \{\text{l'Hôpital's rule}\} \\ &= \lim_{x \rightarrow \infty} \left( 4e^x + \frac{2}{e^x} \right)\end{aligned}$$

As  $x \rightarrow \infty$ ,  $4e^x \rightarrow \infty$ , so  $\lim_{x \rightarrow \infty} \frac{e^{2x} + x^2}{e^x + 4x}$  does not exist.

**35 a**  $f(x) = \frac{1}{4b} x^2$

$$\therefore f'(x) = \frac{1}{2b} x$$

The tangent to  $y = f(x)$  at  $P(a, f(a))$  has equation  $y = f'(a)(x - a) + f(a)$

$$\begin{aligned} &= \frac{a}{2b}(x - a) + \frac{a^2}{4b} \\ &= \frac{a}{2b}x - \frac{a^2}{2b} + \frac{a^2}{4b} \\ &= \frac{a}{2b}x - \frac{a^2}{4b} \end{aligned}$$

The tangent meets the  $x$ -axis when  $y = 0$

$$\begin{aligned} \therefore 0 &= \frac{a}{2b}x - \frac{a^2}{4b} \\ \therefore \frac{a}{2b}x &= \frac{a^2}{4b} \\ \therefore x &= \frac{a^2}{4b} \times \frac{2b}{a} \\ &= \frac{a}{2} \end{aligned}$$

$\therefore$  the tangent meets the  $x$ -axis at  $P'\left(\frac{a}{2}, 0\right)$ .

**b i** The perpendicular to the tangent at  $P'\left(\frac{a}{2}, 0\right)$  has gradient  $-\frac{2b}{a}$ .

$$\begin{aligned} \therefore \text{the perpendicular has equation } y &= -\frac{2b}{a}\left(x - \frac{a}{2}\right) + 0 \\ &= -\frac{2b}{a}x + b \end{aligned}$$

**ii** When  $x = 0$ ,  $y = -\frac{2b}{a}(0) + b$   
 $= b$

$\therefore$  the perpendicular line has  $y$ -intercept  $b$ .

**iii** We first find the distance between  $P(a, f(a))$  and  $P''(a, -b)$ .

$$\begin{aligned} PP'' &= \sqrt{(a - a)^2 + (-b - f(a))^2} \\ &= \sqrt{0 + \left(b + \frac{a^2}{4b}\right)^2} \end{aligned}$$

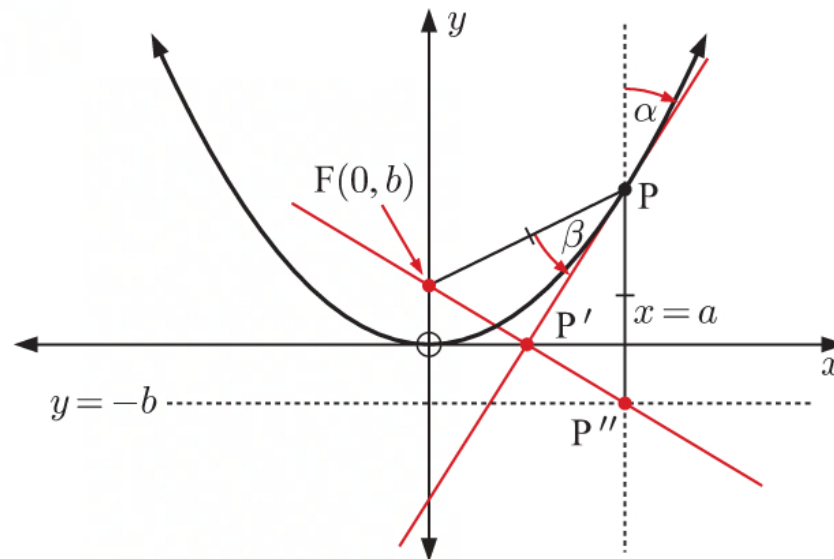
$$\therefore PP'' = b + \frac{a^2}{4b}$$

The distance between  $F(0, b)$  and  $P(a, f(a))$ ,  $FP = \sqrt{(a-0)^2 + \left(\frac{a^2}{4b} - b\right)^2}$

$$\begin{aligned}
 &= \sqrt{a^2 + \left(\frac{a^2 - 4b^2}{4b}\right)^2} \\
 &= \sqrt{a^2 + \frac{a^4 - 8a^2b^2 + 16b^4}{16b^2}} \\
 &= \sqrt{\frac{16a^2b^2 + a^4 - 8a^2b^2 + 16b^4}{16b^2}} \\
 &= \sqrt{\frac{a^4 + 8a^2b^2 + 16b^4}{16b^2}} \\
 &= \frac{a^2 + 4b^2}{4b} \\
 \therefore FP &= b + \frac{a^2}{4b}
 \end{aligned}$$

$\therefore$  the distance  $FP$  is the same as the distance between  $P$  and the line  $y = -b$ , as required.

**c i**



We have  $F(0, b)$  and  $P' \left( \frac{a}{2}, 0 \right)$ .

$$\begin{aligned}
 \therefore FP' &= \sqrt{\left(\frac{a}{2} - 0\right)^2 + (0 - b)^2} \\
 &= \sqrt{\frac{a^2}{4} + b^2}
 \end{aligned}$$

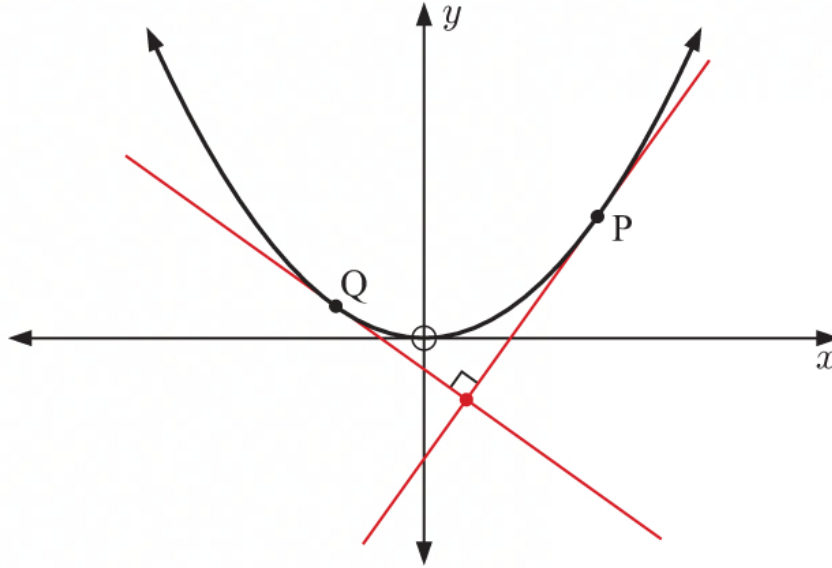
We also have  $P' \left( \frac{a}{2}, 0 \right)$  and  $P''(a, -b)$ .

$$\begin{aligned}
 \therefore P'P'' &= \sqrt{\left(a - \frac{a}{2}\right)^2 + (-b - 0)^2} \\
 &= \sqrt{\frac{a^2}{4} + b^2}
 \end{aligned}$$

So,  $FP' = P'P''$  and  $FP = PP''$  {using **b iii**}

$\therefore \triangle FPP'$  and  $\triangle P''PP'$  are congruent. {SSS}  
 $\therefore \widehat{P'PF} = \widehat{P'PP''} = \beta$   
 $\therefore \alpha = \beta$  {opposite angles}  
 $\therefore$  any vertical ray is *reflected* in the normal to the parabola at its point of contact P.  
 $\therefore$  the reflection of any vertical ray passes through the focus F.

ii



The equation of the tangent at  $Q(c, f(c))$  is  $y = \frac{c}{2b}(x - c) + \frac{c^2}{4b}$   
 $= \frac{c}{2b}x - \frac{c^2}{4b}$

The tangents at P and Q intersect when  $\frac{c}{2b}x - \frac{c^2}{4b} = \frac{a}{2b}x - \frac{a^2}{4b}$   
 $\therefore 2cx - c^2 = 2ax - a^2$   
 $\therefore 2(a - c)x = a^2 - c^2$   
 $\therefore 2(a - c)x = (a - c)(a + c)$   
 $\therefore x = \frac{a + c}{2}$   
 $\therefore y = \frac{c}{2b} \left( \frac{a + c}{2} - c \right) - \frac{c^2}{4b}$   
 $= \frac{ac}{4b} + \frac{c^2}{4b} - \frac{c^2}{4b}$   
 $= \frac{ac}{4b}$

Now, the tangents at P and Q are perpendicular.

$\therefore f'(c) = -\frac{1}{f'(a)}$   
 $\therefore \frac{c}{2b} = -\frac{2b}{a}$   
 $\therefore c = -\frac{4b^2}{a}$

$\therefore$  the tangents intersect at  $y = \frac{a \left( -\frac{4b^2}{a} \right)}{4b} = -b$ , which is the directrix.



# Chapter 19

## APPLICATIONS OF DIFFERENTIATION

### EXERCISE 19A

1 a  $P(t) = 2t^2 - 12t + 118$  thousand dollars

$$\begin{aligned} P(0) &= 2(0)^2 - 12(0) + 118 \\ &= 118 \end{aligned}$$

$\therefore$  the current annual profit is \$118 000.

b  $P = 2t^2 - 12t + 118$

$$\therefore \frac{dP}{dt} = 4t - 12 \text{ thousand dollars per year}$$

c When  $t = 8$ ,  $\frac{dP}{dt} = 4(8) - 12$   
 $= 32 - 12$   
 $= 20$

This means that in 8 years from now, profits will be increasing at a rate of \$20 000 per year.

2 a  $V = 2(50 - t)^2 \text{ m}^3$

$$\begin{aligned} \text{When } t = 0, \quad V &= 2(50)^2 \\ &= 2 \times 2500 \\ &= 5000 \text{ m}^3 \end{aligned}$$

$$\begin{aligned} \text{When } t = 5, \quad V &= 2(50 - 5)^2 \\ &= 2(45)^2 \\ &= 2 \times 2025 \\ &= 4050 \text{ m}^3 \end{aligned}$$

$\therefore$  the average rate at which the water evaporates in the first 5 days is

$$\frac{5000 - 4050}{5} = \frac{950}{5} = 190 \text{ m}^3 \text{ per day.}$$

b  $V = 2(50 - t)^2$

$$\begin{aligned} \therefore \frac{dV}{dt} &= 2 \times 2(50 - t)(-1) \quad \{\text{chain rule}\} \\ &= -4(50 - t) \\ &= 4t - 200 \end{aligned}$$

$$\begin{aligned} \text{When } t = 5, \quad \frac{dV}{dt} &= 4(5) - 200 \\ &= 20 - 200 \\ &= -180 \end{aligned}$$

$\therefore$  the instantaneous rate at which the water is evaporating at  $t = 5$  days is 180 m<sup>3</sup> per day.

3 a  $Q(t) = 100 - 10\sqrt{t}$

i  $Q(0) = 100 - 10\sqrt{0}$   
 $= 100$

ii  $Q(25) = 100 - 10\sqrt{25}$   
 $= 50$

iii  $Q(100) = 100 - 10\sqrt{100}$   
 $= 0$

**b**  $Q(t) = 100 - 10\sqrt{t} = 100 - 10t^{\frac{1}{2}}$  units,  $t \geq 0$

$$\begin{aligned}\therefore Q'(t) &= -5t^{-\frac{1}{2}} \\ &= -\frac{5}{\sqrt{t}} \text{ units per year}\end{aligned}$$

**i**  $Q'(25) = -\frac{5}{\sqrt{25}} = -1$

$\therefore$  when the person is aged 25 years, the quantity of the chemical is decreasing by 1 unit per year.

**ii**  $Q'(50) = -\frac{5}{\sqrt{50}}$   
 $= -\frac{5}{5\sqrt{2}}$   
 $= -\frac{1}{\sqrt{2}}$

$\therefore$  when the person is aged 50 years, the quantity of the chemical is decreasing by  $\frac{1}{\sqrt{2}}$  units per year.

**c**  $Q'(t) = -\frac{5}{\sqrt{t}} < 0$  for all  $t > 0$

$\therefore$  the quantity of the chemical is decreasing for all  $t > 0$ .

**4 a**  $H = 35 - \frac{172.5}{t+5}$  metres

When  $t = 0$ ,  $H = 35 - \frac{172.5}{5}$   
 $= 0.5$

$\therefore$  the tree was 0.5 m tall when it was planted.

**b i** When  $t = 4$ ,  $H = 35 - \frac{172.5}{9}$   
 $\approx 15.8$

$\therefore$  after 4 years, the tree is about 15.8 m tall.

**ii** When  $t = 8$ ,  $H = 35 - \frac{172.5}{13}$   
 $\approx 21.7$

$\therefore$  after 8 years, the tree is about 21.7 m tall.

**iii** When  $t = 12$ ,  $H = 35 - \frac{172.5}{17}$   
 $\approx 24.9$

$\therefore$  after 12 years, the tree is about 24.9 m tall.

**c**  $H = 35 - \frac{172.5}{t+5}$  m,  $t \geq 0$

$$= 35 - 172.5(t+5)^{-1}$$

$$\begin{aligned}\therefore \frac{dH}{dt} &= 172.5(t+5)^{-2} \\ &= \frac{172.5}{(t+5)^2} \text{ m per year}\end{aligned}$$

When  $t = 0$ ,  $\frac{dH}{dt} = \frac{172.5}{(0+5)^2}$   
 $= 6.9$

$\therefore$  the tree is initially growing at a rate of 6.9 m per year.

When  $t = 5$ ,  $\frac{dH}{dt} = \frac{172.5}{10^2}$   
 $= 1.725$

$\therefore$  the tree is growing at a rate of 1.725 m per year after 5 years.

$$\begin{aligned}\text{When } t = 10, \quad \frac{dH}{dt} &= \frac{172.5}{15^2} \\ &\approx 0.767\end{aligned}$$

$\therefore$  the tree is growing at a rate of about 0.767 m per year after 10 years.

$$\text{d} \quad \frac{dH}{dt} = \frac{172.5}{(t+5)^2} > 0 \quad \text{for all } t \geq 0$$

This model predicts that the tree will continue to grow forever.

$$\text{5} \quad C(x) = 7800 + 6x + 12x^{0.7} \quad \text{dollars}$$

$$\begin{aligned}\text{a} \quad \text{The marginal cost function is } C'(x) &= 6 + 12(0.7x^{-0.3}) \\ &= 6 + 8.4x^{-0.3} \quad \text{dollars per pair}\end{aligned}$$

$$\begin{aligned}\text{b} \quad C'(220) &= 6 + 8.4(220)^{-0.3} \\ &\approx \$7.67\end{aligned}$$

This estimates the cost of making the 221st pair of jeans if 220 pairs are currently being made.

$$\begin{aligned}\text{c} \quad C(221) - C(220) &= 7800 + 6(221) + 12(221)^{0.7} - (7800 + 6(220) + 12(220)^{0.7}) \\ &\approx \$7.66\end{aligned}$$

This is the actual cost of making the 221st pair of jeans.

The answer in **b** is a very good estimate.

$$\text{6} \quad \text{a} \quad C(v) = \frac{1}{5}v^2 + \frac{200\,000}{v} \quad \text{euros}$$

$$\begin{aligned}\text{i} \quad C(50) &= \frac{1}{5}(50)^2 + \frac{200\,000}{50} \\ &= 500 + 4000 \\ &= 4500\end{aligned}$$

$\therefore$  if the average speed is  $50 \text{ km h}^{-1}$ , the total cost of the journey is 4500 euros.

$$\begin{aligned}\text{ii} \quad C(100) &= \frac{1}{5}(100)^2 + \frac{200\,000}{100} \\ &= 2000 + 2000 \\ &= 4000\end{aligned}$$

$\therefore$  if the average speed is  $100 \text{ km h}^{-1}$ , the total cost of the journey is 4000 euros.

$$\begin{aligned}\text{b} \quad C(v) &= \frac{1}{5}v^2 + \frac{200\,000}{v} \quad \text{euros, } v > 0 \\ &= \frac{1}{5}v^2 + 200\,000v^{-1} \\ \therefore C'(v) &= \frac{2}{5}v - 200\,000v^{-2} \\ &= \frac{2}{5}v - \frac{200\,000}{v^2} \quad \text{euros per km h}^{-1}\end{aligned}$$

$$\begin{aligned}\text{i} \quad C'(30) &= \frac{2}{5}(30) - \frac{200\,000}{30^2} \\ &= 12 - \frac{2000}{9} \\ &\approx -210.22\end{aligned}$$

$\therefore$  if the average speed is  $30 \text{ km h}^{-1}$ , the rate of change in the cost of running the train is decreasing at about 210.22 euros per  $\text{km h}^{-1}$ .

$$\begin{aligned}
 \text{ii } C'(90) &= \frac{2}{5}(90) - \frac{200\,000}{90^2} \\
 &= 36 - \frac{2000}{81} \\
 &\approx 11.31
 \end{aligned}$$

$\therefore$  if the average speed is  $90 \text{ km h}^{-1}$ , the rate of change in the cost of running the train is increasing at about  $11.31 \text{ euros per km h}^{-1}$ .

c  $C(v)$  is a minimum when  $C'(v) = 0$

$$\therefore \frac{2}{5}v - \frac{200\,000}{v^2} = 0$$

$$\therefore \frac{2}{5}v^3 - 200\,000 = 0$$

$$\therefore \frac{2}{5}v^3 = 200\,000$$

$$\therefore v^3 = 500\,000$$

$$\therefore v \approx 79.4 \text{ km h}^{-1}$$

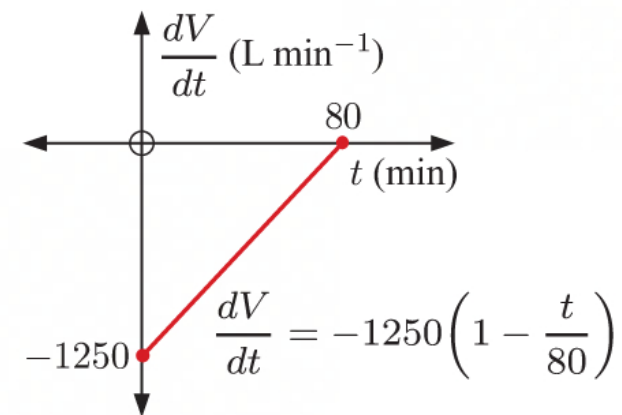
$C'(v)$  has sign diagram:



$\therefore$  the cost of running the train is a minimum when the average speed of the train is about  $79.4 \text{ km h}^{-1}$ .

**7 a**  $V = 50\,000 \left(1 - \frac{t}{80}\right)^2 \text{ L}, \quad 0 \leq t \leq 80$

$$\begin{aligned}
 \therefore \frac{dV}{dt} &= 100\,000 \left(1 - \frac{t}{80}\right) \left(-\frac{1}{80}\right) \quad \{\text{chain rule}\} \\
 &= -1250 \left(1 - \frac{t}{80}\right) \text{ L min}^{-1}
 \end{aligned}$$



b The outflow is fastest when  $\frac{dV}{dt} = -1250 \left(1 - \frac{t}{80}\right)$  is smallest.

Looking at the graph in **a**, the minimum value of  $\frac{dV}{dt}$  is  $-1250 \text{ L min}^{-1}$  which occurs when  $t = 0$ .

So, the outflow is fastest at  $t = 0$ , when the tap was first opened.

c 
$$\begin{aligned}
 \frac{dV}{dt} &= -1250 \left(1 - \frac{t}{80}\right) \\
 &= -1250 + \frac{125}{8}t
 \end{aligned}$$

$$\therefore \frac{d^2V}{dt^2} = \frac{125}{8} > 0$$

This shows that the rate of change of  $V$  is constantly increasing, so the outflow is increasing at a constant rate.



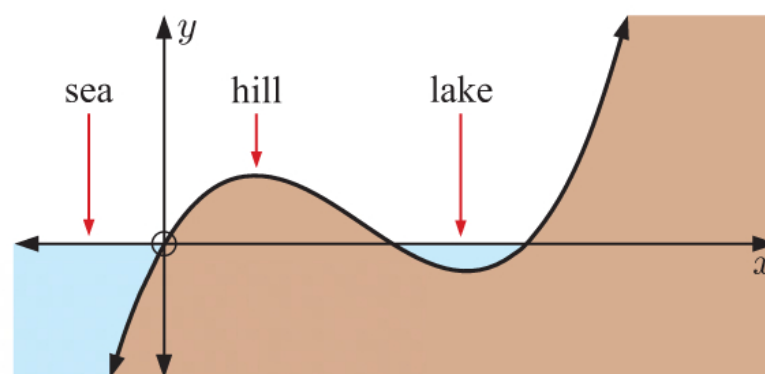
**8 a**  $y = \frac{1}{10}x(x-2)(x-3)$

The edges of the lake correspond to values of  $x$  such that  $y = 0$ .

$$\therefore \frac{1}{10}x(x-2)(x-3) = 0$$

$$\therefore x = 0, 2, \text{ or } 3$$

From the graph, we can see that the near part of the lake is 2 km from the sea, and the furthest part is 3 km.



**b**

$$y = \frac{1}{10}x(x-2)(x-3)$$

$$= \frac{1}{10}x(x^2 - 5x + 6)$$

$$= \frac{1}{10}x^3 - \frac{1}{2}x^2 + \frac{3}{5}x$$

$$\therefore \frac{dy}{dx} = \frac{3}{10}x^2 - x + \frac{3}{5}$$

When  $x = \frac{1}{2}$ ,  $\frac{dy}{dx} = \frac{3}{10}\left(\frac{1}{2}\right)^2 - \frac{1}{2} + \frac{3}{5}$

$$= 0.175$$

$\therefore$  the height of the hill is increasing when  $x = \frac{1}{2}$  km, as the gradient is positive.  
So the land is sloping upwards at this point.

When  $x = 1\frac{1}{2} = \frac{3}{2}$ ,  $\frac{dy}{dx} = \frac{3}{10}\left(\frac{3}{2}\right)^2 - \frac{3}{2} + \frac{3}{5}$

$$= -0.225$$

$\therefore$  the height of the hill is decreasing when  $x = 1\frac{1}{2}$  km, as the gradient is negative.  
So the land is sloping downwards at this point.

This means the top of the hill is between  $x = \frac{1}{2}$  km and  $x = 1\frac{1}{2}$  km.

**c** The deepest point of the lake occurs when the slope of the land is 0, which is when  $\frac{dy}{dx} = 0$ .

$$\therefore \frac{3}{10}x^2 - x + \frac{3}{5} = 0$$

$$\therefore x \approx 0.785 \text{ or } 2.55 \quad \{\text{using technology}\}$$

The deepest point of the lake is a turning point which lies between the edges of the lake at  $x = 2$  and  $x = 3$ . So we only consider the value of  $x$  which lies between 2 and 3.

When  $x = 2.55$ ,  $y \approx -0.0631$  km

$$\approx -63.1 \text{ m}$$

So, the deepest point of the lake is about 2.55 km from the sea, and about 63.1 m deep.

**9 a**  $\frac{dP}{dt} = aP\left(1 - \frac{P}{b}\right) - \left(\frac{c}{100}\right)P$  and when  $\frac{dP}{dt} = 0$ , the rate of change of population is zero, so the population is not changing and is stable.

**b** If  $a = 0.06$ ,  $b = 24\,000$ ,  $c = 5$  then

$$\frac{dP}{dt} = 0.06P\left(1 - \frac{P}{24\,000}\right) - \frac{5}{100}P$$

$$= 0.06P - 0.05P - \frac{0.06P^2}{24\,000}$$

$$= P\left(0.01 - \frac{P}{400\,000}\right)$$

Now for a stable population,  $\frac{dP}{dt} = 0$

$$\therefore P \left( 0.01 - \frac{P}{400\,000} \right) = 0$$

$$\therefore P = 0 \quad \text{or} \quad \frac{P}{400\,000} = 0.01$$

$$\therefore P = 4000 \quad \{\text{if } P = 0, \text{ there are no fish}\}$$

$\therefore$  the stable population is 4000 fish.

• If the harvest rate is reduced to 4%, then  $\frac{dP}{dt} = 0.06P \left( 1 - \frac{P}{24\,000} \right) - \frac{4}{100} P$

$$= 0.06P - 0.04P - \frac{0.06P^2}{24\,000}$$

$$= P \left( 0.02 - \frac{0.06P}{24\,000} \right)$$

Now for a stable population,  $\frac{dP}{dt} = 0$

$$\therefore P \left( 0.02 - \frac{0.06P}{24\,000} \right) = 0$$

$$\therefore P = 0 \quad \text{or} \quad \frac{0.06P}{24\,000} = 0.02$$

$$\therefore P = \frac{0.02 \times 24\,000}{0.06} \quad \{\text{if } P = 0, \text{ there are no fish}\}$$

$$\therefore P = 8000$$

$\therefore$  the stable population will increase to 8000 fish.

• In general,  $\frac{dP}{dt} = aP - \frac{a}{b} P^2 - \frac{c}{100} P$

$$= -\frac{a}{b} P^2 + \left( a - \frac{c}{100} \right) P$$

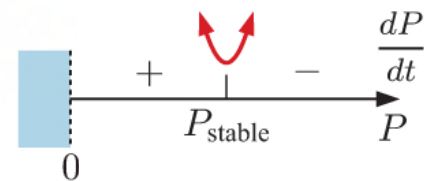
$$= P \left( a - \frac{c}{100} - \frac{a}{b} P \right)$$

$$\therefore \frac{dP}{dt} = 0 \quad \text{when} \quad P = 0 \quad \text{or} \quad P = \frac{a - \frac{c}{100}}{\frac{a}{b}}$$

$$\therefore P = \frac{b \left( a - \frac{c}{100} \right)}{a} \quad \{\text{if } P = 0, \text{ there are no fish}\}$$

So, the stable population  $P_{\text{stable}} = \frac{b \left( a - \frac{c}{100} \right)}{a}$ .

Since  $\frac{dP}{dt}$  is a concave quadratic, its sign diagram is



$\therefore P$  is increasing for  $0 < P \leq P_{\text{stable}}$  and decreasing for  $P \geq P_{\text{stable}}$ .

$\therefore$  for any initial population  $P_0$ ,  $P$  tends to  $P_{\text{stable}}$ .

**10 a**  $W = 20e^{-kt}$  grams,  $t \geq 0$

When  $t = 50$ ,  $W = 10$

$$\therefore 20e^{-50k} = 10$$

$$\therefore e^{-50k} = \frac{1}{2}$$

$$\therefore -50k = \ln\left(\frac{1}{2}\right)$$

$$\therefore k = -\frac{1}{50} \ln\left(\frac{1}{2}\right)$$

$$\therefore k = \frac{1}{50} \ln 2 \approx 0.0139$$

**b i** When  $t = 0$ ,  $W = 20e^{-\frac{0}{50} \ln 2}$   
 $= 20e^0$   
 $= 20$

$\therefore$  there are initially 20 grams of radioactive substance present.

**ii** When  $t = 24$ ,  $W = 20e^{-\frac{24}{50} \ln 2}$   
 $\approx 14.3$

$\therefore$  after 24 hours, there are about 14.3 grams of radioactive substance present.

**iii** 1 week  $= 7 \times 24 = 168$  hours

When  $t = 168$ ,  $W = 20e^{-\frac{168}{50} \ln 2}$   
 $\approx 1.95$

$\therefore$  after 1 week, there are about 1.95 grams of radioactive substance present.

**c** When  $W = 1$ ,  $20e^{-\frac{t}{50} \ln 2} = 1$

$$\therefore e^{-\frac{t}{50} \ln 2} = \frac{1}{20}$$

$$\therefore -\frac{t}{50} \ln 2 = \ln\left(\frac{1}{20}\right)$$

$$\therefore t \ln 2 = 50 \ln 20$$

$$\therefore t = \frac{50 \ln 20}{\ln 2}$$

$$\approx 216$$

$\therefore$  it will take about 216 hours or about 9 days for the weight of the radioactive substance to reach 1 gram.

**d**  $W = 20e^{-\frac{t}{50} \ln 2}$

$$\therefore \frac{dW}{dt} = -\frac{1}{50} \ln 2 \times 20e^{-\frac{t}{50} \ln 2} \quad \{\text{chain rule}\}$$

**i** When  $t = 100$ ,  $\frac{dW}{dt} = -\frac{1}{50} \ln 2 \times 20e^{-2 \ln 2}$   
 $\approx -0.0693$

$\therefore$  after 100 hours, the rate of radioactive decay is about  $-0.0693 \text{ gh}^{-1}$ .

**ii** When  $t = 1000$ ,  $\frac{dW}{dt} = -\frac{1}{50} \ln 2 \times 20e^{-20 \ln 2}$   
 $\approx -2.64 \times 10^{-7}$

$\therefore$  after 1000 hours, the rate of radioactive decay is about  $-2.64 \times 10^{-7} \text{ gh}^{-1}$ .



$$\begin{aligned} \text{e } \frac{dW}{dt} &= -\frac{1}{50} \ln 2 \times 20e^{-\frac{t}{50} \ln 2} \\ &= bW \quad \{\text{where } b = -\frac{1}{50} \ln 2 \text{ is constant}\} \end{aligned}$$

$$11 \quad \text{a } T = 5 + 95e^{-kt} \text{ } ^\circ\text{C}, \quad t \geq 0$$

$$\text{When } t = 15, \quad T = 20$$

$$\therefore 5 + 95e^{-15k} = 20$$

$$\therefore 95e^{-15k} = 15$$

$$\therefore e^{-15k} = \frac{3}{19}$$

$$\therefore -15k = \ln\left(\frac{3}{19}\right)$$

$$\therefore k = \frac{1}{15} \ln\left(\frac{19}{3}\right) \approx 0.123$$

$$\begin{aligned} \text{b } \text{When } t = 0, \quad T &= 5 + 95e^{-\frac{0}{15} \ln\left(\frac{19}{3}\right)} \\ &= 5 + 95e^0 \\ &= 100 \end{aligned}$$

$\therefore$  the temperature of the liquid when it was first placed in the refrigerator was  $100^\circ\text{C}$ .

$$\begin{aligned} \text{c } \quad T &= 5 + 95e^{-\frac{t}{15} \ln\left(\frac{19}{3}\right)}, \quad T - 5 = 95e^{-\frac{t}{15} \ln\left(\frac{19}{3}\right)} \\ \therefore \frac{dT}{dt} &= -\frac{1}{15} \ln\left(\frac{19}{3}\right) \times 95e^{-\frac{t}{15} \ln\left(\frac{19}{3}\right)} \\ &= -\frac{1}{15} \ln\left(\frac{19}{3}\right) \times (T - 5) \\ &= c(T - 5) \quad \{\text{where } c = -\frac{1}{15} \ln\left(\frac{19}{3}\right) = -k \text{ is constant}\} \end{aligned}$$

$$\text{d } \frac{dT}{dt} = -\frac{1}{15} \ln\left(\frac{19}{3}\right) \times 95e^{-\frac{t}{15} \ln\left(\frac{19}{3}\right)} \text{ } ^\circ\text{C per minute} \quad \{\text{using c}\}$$

$$\begin{aligned} \text{i } \text{When } t = 0, \quad \frac{dT}{dt} &= -\frac{1}{15} \ln\left(\frac{19}{3}\right) \times 95e^{-\frac{0}{15} \ln\left(\frac{19}{3}\right)} \\ &= -\frac{19}{3} \ln\left(\frac{19}{3}\right) \\ &\approx -11.7 \end{aligned}$$

$\therefore$  the temperature is initially decreasing at about  $11.7^\circ\text{C}$  per minute.

$$\begin{aligned} \text{ii } \text{When } t = 10, \quad \frac{dT}{dt} &= -\frac{1}{15} \ln\left(\frac{19}{3}\right) \times 95e^{-\frac{2}{3} \ln\left(\frac{19}{3}\right)} \\ &\approx -3.42 \end{aligned}$$

$\therefore$  after 10 minutes, the temperature is decreasing at about  $3.42^\circ\text{C}$  per minute.

$$\begin{aligned} \text{iii } \text{When } t = 20, \quad \frac{dT}{dt} &= -\frac{1}{15} \ln\left(\frac{19}{3}\right) \times 95e^{-\frac{4}{3} \ln\left(\frac{19}{3}\right)} \\ &\approx -0.998 \end{aligned}$$

$\therefore$  after 20 minutes, the temperature is decreasing at about  $0.998^\circ\text{C}$  per minute.

$$12 \quad \text{a } H(t) = 20 \ln(3t + 2) + 30 \text{ cm}, \quad t \geq 0$$

$$\therefore H(0) = 20 \ln 2 + 30$$

$$\approx 43.9$$

$\therefore$  the shrub was about 43.9 cm high when it was planted.



**b** When  $H(t) = 100$  cm,

$$20 \ln(3t + 2) + 30 = 100$$

$$\therefore 20 \ln(3t + 2) = 70$$

$$\therefore \ln(3t + 2) = \frac{7}{2}$$

$$\therefore 3t + 2 = e^{\frac{7}{2}}$$

$$\therefore 3t = e^{\frac{7}{2}} - 2$$

$$\therefore t = \frac{e^{\frac{7}{2}} - 2}{3}$$

$$\approx 10.4$$

$\therefore$  it will take about 10.4 years for the shrub to reach a height of 1 m.

**c**  $H(t) = 20 \ln(3t + 2) + 30$  cm,  $t \geq 0$

$$\therefore H'(t) = 20 \times \frac{3}{3t + 2}$$

$$= \frac{60}{3t + 2}$$

**i**  $H'(3) = \frac{60}{3(3) + 2}$

$$= \frac{60}{11}$$

$$\approx 5.45$$

$\therefore$  3 years after being planted, the shrub is growing at about 5.45 cm per year.

**ii**  $H'(10) = \frac{60}{3(10) + 2}$

$$= \frac{60}{32}$$

$$= 1.875$$

$\therefore$  10 years after being planted, the shrub is growing at 1.875 cm per year.

**13 a**  $A = s(1 - e^{-kt})$  litres

When  $t = 0$ ,  $A = s(1 - e^{-k(0)})$

$$= s(1 - e^0)$$

$$= 0 \text{ litres}$$

**b i**  $s = 10$ , and when  $t = 3$ ,  $A = 5$

$$\therefore 5 = 10(1 - e^{-3k})$$

$$\therefore 1 - e^{-3k} = \frac{1}{2}$$

$$\therefore e^{-3k} = \frac{1}{2}$$

$$\therefore -3k = \ln \frac{1}{2}$$

$$\therefore k = \frac{\ln 2}{3} \approx 0.231$$

$$\text{ii} \quad A = 10 \left( 1 - e^{-\frac{\ln 2}{3}t} \right) \text{ L}$$

$$\therefore \frac{dA}{dt} = 10 \left( \frac{\ln 2}{3} e^{-\frac{\ln 2}{3}t} \right) \text{ L h}^{-1}$$

$$\text{When } t = 5, \quad \frac{dA}{dt} = 10 \left( \frac{\ln 2}{3} e^{-\frac{5 \ln 2}{3}} \right) \\ \approx 0.728$$

$\therefore$  after 5 hours, the speed of the reaction is about 0.728 litres of alcohol produced per hour.

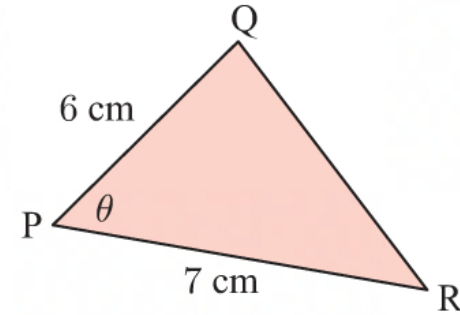
**14** Triangle PQR has area  $A = \frac{1}{2} \times 6 \times 7 \times \sin \theta$

$$\therefore A = 21 \sin \theta \text{ cm}^2$$

$$\therefore \frac{dA}{d\theta} = 21 \cos \theta \text{ cm}^2 \text{ per radian}$$

$$\text{When } \theta = 45^\circ = \frac{\pi}{4}, \quad \frac{dA}{d\theta} = 21 \cos \frac{\pi}{4} \\ = 21 \times \frac{1}{\sqrt{2}} \\ = \frac{21}{\sqrt{2}} \text{ cm}^2 \text{ per radian} \approx 0.259 \text{ cm}^2 \text{ per degree}$$

$\therefore$  the area of triangle PQR is changing at a rate of  $\frac{21}{\sqrt{2}} \text{ cm}^2$  per radian or about 0.259  $\text{cm}^2$  per degree at the time when  $\theta = 45^\circ$ .



**15 a** Using the cosine rule,

$$l^2 = 20^2 + 20^2 - 2 \times 20 \times 20 \times \cos \theta$$

$$\therefore l^2 = 400 + 400 - 800 \cos \theta$$

$$\therefore l^2 = 800 - 800 \cos \theta$$

$$\therefore l = \sqrt{800 - 800 \cos \theta} \text{ cm} \quad \{l > 0\}$$

$$\text{b} \quad l = (800 - 800 \cos \theta)^{\frac{1}{2}}$$

$$\therefore \frac{dl}{d\theta} = \frac{1}{2} (800 - 800 \cos \theta)^{-\frac{1}{2}} (800 \sin \theta)$$

$$= \frac{400 \sin \theta}{\sqrt{800 - 800 \cos \theta}} \text{ cm per radian}$$

$$\text{When } \theta = 120^\circ = \frac{2\pi}{3}, \quad \frac{dl}{d\theta} = \frac{400 \sin \frac{2\pi}{3}}{\sqrt{800 - 800 \cos \frac{2\pi}{3}}}$$

$$= \frac{400 \left( \frac{\sqrt{3}}{2} \right)}{\sqrt{800 - 800 \left( -\frac{1}{2} \right)}}$$

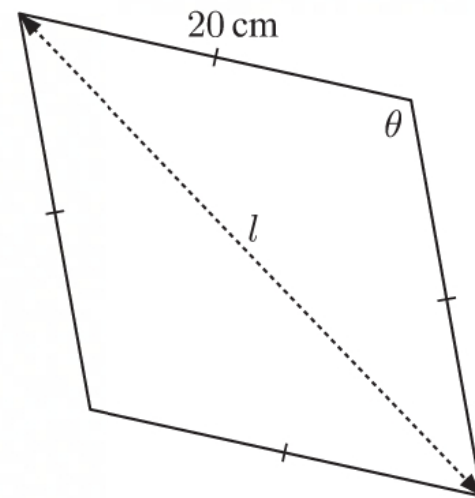
$$= \frac{200\sqrt{3}}{\sqrt{800 + 400}}$$

$$= \frac{200\sqrt{3}}{\sqrt{1200}}$$

$$= \frac{200\sqrt{3}}{20\sqrt{3}}$$

$$= 10 \text{ cm per radian} \approx 0.175 \text{ cm per degree}$$

$\therefore l$  is changing at a rate of 10 cm per radian or about 0.175 cm per degree at the time when  $\theta = 120^\circ$ .



**16 a**  $V(t) = 340 \sin(100\pi t)$  volts

**i**  $V(0) = 340 \sin(100\pi(0))$   
 $= 340 \sin 0$   
 $= 0$

$\therefore$  there are initially 0 volts in the circuit.

**ii**  $V(0.125) = 340 \sin(100\pi(0.125))$   
 $= 340 \sin(12.5\pi)$   
 $= 340$

$\therefore$  there are 340 volts in the circuit after 0.125 seconds.

**b**  $V(t) = 340 \sin(100\pi t)$  volts

$\therefore V'(t) = 34000\pi \cos(100\pi t)$  volts per second

**i**  $V'(0.01) = 34000\pi \cos(100\pi(0.01))$   
 $= 34000\pi \cos \pi$   
 $= -34000\pi$

$\therefore$  after 0.01 seconds, the voltage is changing at  $-34000\pi$  volts per second.

**ii**  $V(t)$  is a maximum when  $V'(t) = 0$ .

So the voltage is changing at 0 volts per second.

**17**  $B(t) = \frac{3000}{1 + 0.5e^{-1.73t}}$

**a**  $B(0) = \frac{3000}{1 + 0.5e^{-1.73(0)}}$   
 $= \frac{3000}{1 + 0.5e^0}$   
 $= \frac{3000}{1.5}$   
 $= 2000$

$\therefore$  the initial bee population is 2000.

**b**  $B(1) = \frac{3000}{1 + 0.5e^{-1.73(1)}}$   
 $\approx 2756$

The percentage increase after 1 month  $\approx \frac{2756 - 2000}{2000} \times 100\%$   
 $\approx 37.8\%$

$\therefore$  the percentage increase in the population after 1 month is about 37.8%.

**c** As  $t \rightarrow \infty$ ,  $e^{-1.73t} \rightarrow 0$

$\therefore \frac{3000}{1 + 0.5e^{-1.73t}} \rightarrow \frac{3000}{1} = 3000$

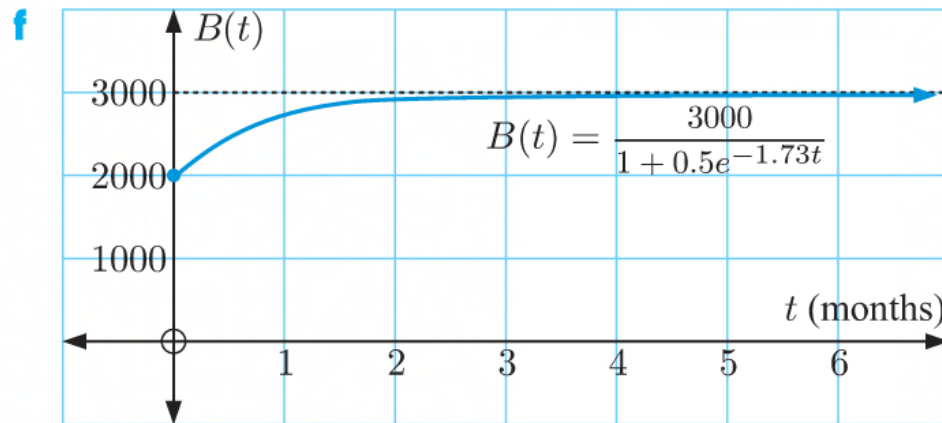
$\therefore$  the population is limited to 3000 bees.

$$\begin{aligned}
 \text{d} \quad B(t) &= \frac{3000}{1 + 0.5e^{-1.73t}} = 3000(1 + 0.5e^{-1.73t})^{-1} \text{ bees} \\
 \therefore B'(t) &= -3000(1 + 0.5e^{-1.73t})^{-2}(0.5(-1.73)e^{-1.73t}) \quad \{\text{chain rule}\} \\
 &= \frac{2595}{e^{1.73t}(1 + 0.5e^{-1.73t})^2} \text{ bees per month} \\
 &> 0 \text{ for all } t \geq 0
 \end{aligned}$$

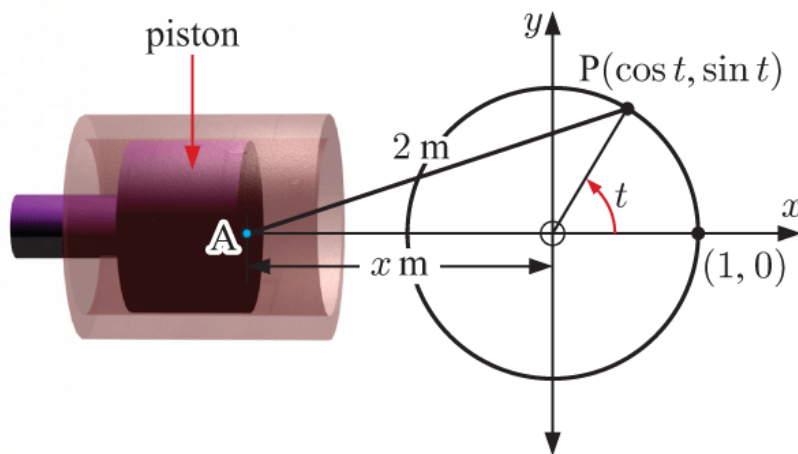
$\therefore$  the population is increasing over time.

$$\begin{aligned}
 \text{e} \quad B'(6) &= \frac{2595}{e^{1.73(6)}(1 + 0.5e^{-1.73(6)})^2} \\
 &\approx 0.0806
 \end{aligned}$$

$\therefore$  after 6 months, the population is increasing at about 0.0806 bees per month.



**18**



**a** The distance from  $A(-x, 0)$  to  $P(\cos t, \sin t)$  is fixed at 2 m.

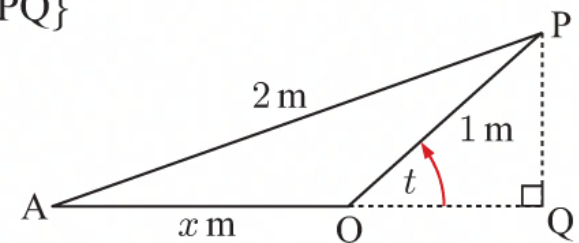
$$\cos t = \frac{OQ}{1} = OQ$$

Now,  $(x + \cos t)^2 + \sin^2 t = 2^2$  {Pythagoras in triangle APQ}

$$\therefore (x + \cos t)^2 = 4 - \sin^2 t$$

$$\therefore x + \cos t = \pm \sqrt{4 - \sin^2 t}$$

$$\therefore \text{since } x > 0, \quad x = \sqrt{4 - \sin^2 t} - \cos t$$



$$\begin{aligned}
 \text{b} \quad \text{Now} \quad \frac{dx}{dt} &= \frac{1}{2}(4 - \sin^2 t)^{-\frac{1}{2}}(-2 \sin t \cos t) + \sin t \quad \{\text{chain rule}\} \\
 &= \frac{-\sin t \cos t}{\sqrt{4 - \sin^2 t}} + \sin t
 \end{aligned}$$

**i** When  $t = 0$ ,  $\sin t = 0$  and  $\cos t = 1$

$$\begin{aligned}
 \therefore \frac{dx}{dt} &= 0 + 0 \\
 &= 0 \text{ metres per radian}
 \end{aligned}$$

**ii** When  $t = \frac{\pi}{2}$ ,  $\sin t = 1$  and  $\cos t = 0$

$$\begin{aligned}
 \therefore \frac{dx}{dt} &= 0 + 1 \\
 &= 1 \text{ metre per radian}
 \end{aligned}$$



iii When  $t = \frac{2\pi}{3}$ ,  $\sin t = \frac{\sqrt{3}}{2}$  and  $\cos t = -\frac{1}{2}$

$$\therefore \frac{dx}{dt} = \frac{-\frac{\sqrt{3}}{2}(-\frac{1}{2})}{\sqrt{4 - \frac{3}{4}}} + \frac{\sqrt{3}}{2}$$

$$\approx 1.11 \text{ metres per radian}$$

## EXERCISE 19B

1  $P(x) = -0.022x^2 + 11x - 720$

$$\therefore P'(x) = -0.044x + 11$$

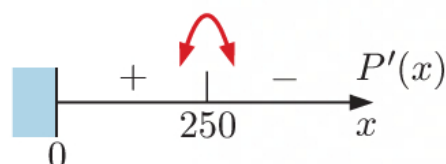
Now,  $P'(x) = 0$  when  $-0.044x + 11 = 0$

$$\therefore 11 = 0.044x$$

$$\therefore x = \frac{11}{0.044}$$

$$= 250$$

So,  $P'(x)$  has sign diagram:



So, the profit is maximised when 250 items are made per day.

2 a Let the remaining fence have length  $y$  m.

The total length of the fence is 60 m

$$\therefore 2x + y = 60$$

$$\therefore y = 60 - 2x$$

The area of the enclosure  $A = \text{width} \times \text{length}$

$$= xy$$

$$= x(60 - 2x) \text{ m}^2$$

$\therefore$  the area of the enclosure is given by  $A(x) = x(60 - 2x) \text{ m}^2$ .

b  $A(x) = x(60 - 2x)$

$$= 60x - 2x^2$$

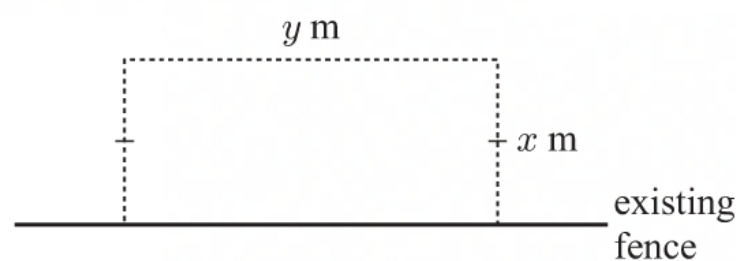
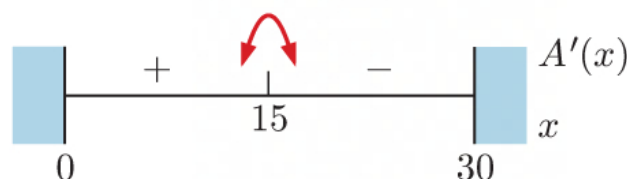
$$\therefore A'(x) = 60 - 4x$$

So,  $A'(x) = 0$  when  $60 - 4x = 0$

$$\therefore 4x = 60$$

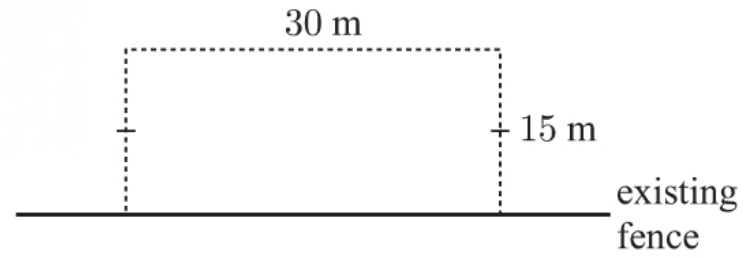
$$\therefore x = 15$$

$A'(x)$  has sign diagram:



The area is maximised when  $x = 15$   
and  $y = 60 - 2(15)$   
 $= 30$

The area of the enclosure is maximised by constructing a fence with width 15 m and length 30 m.



**3 a** Now  $A = 100$

$$\therefore xy = 100$$

$$\therefore y = \frac{100}{x}$$

$$\text{So, } L = 2x + y$$

$$\therefore L = 2x + \frac{100}{x}$$

**b**  $L = 2x + 100x^{-1}$

$$\therefore \frac{dL}{dx} = 2 - 100x^{-2} = 2 - \frac{100}{x^2}$$

which is 0 when  $\frac{100}{x^2} = 2$

$$\therefore x^2 = 50$$

$$\therefore x = \sqrt{50} \quad \{x > 0\}$$

So,  $\frac{dL}{dx}$  has sign diagram:

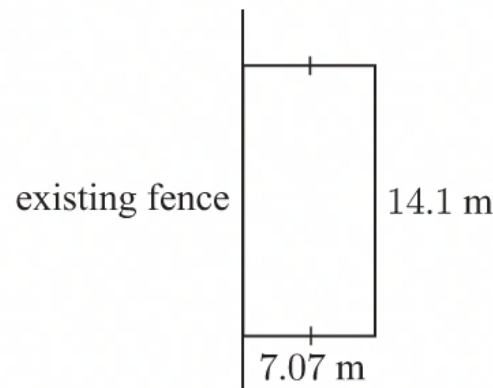
So, the length is a minimum when  $x = \sqrt{50}$ .

$$\text{When } x = \sqrt{50}, \quad L = 2\sqrt{50} + \frac{100}{\sqrt{50}} \\ \approx 28.3$$

So, the minimum value of  $L$  is about 28.3 m which occurs when  $x = \sqrt{50} \approx 7.07$ .

**c**  $x = \sqrt{50} \approx 7.07$  and  $y = \frac{100}{\sqrt{50}} \approx 14.1$

So, the optimal situation is:



**4** Production cost  $C(x) = \frac{1}{4}x^2 + 8x + 20$  pounds

Selling price  $p(x) = 23 - \frac{1}{2}x$  pounds per blanket

Revenue  $R(x) = xp(x) = 23x - \frac{1}{2}x^2$  pounds

Profit  $P(x) = \text{revenue} - \text{cost}$

$$= (23x - \frac{1}{2}x^2) - (\frac{1}{4}x^2 + 8x + 20)$$

$$= -\frac{3}{4}x^2 + 15x - 20$$

$$\therefore P'(x) = -\frac{3}{2}x + 15$$

Now  $P'(x) = 0$  when  $-\frac{3}{2}x + 15 = 0$

$$\therefore x = \frac{15}{\frac{3}{2}} = 10$$

$P'(x)$  has sign diagram:

So, the profit is maximised when 10 blankets are produced per day.

**5 a** The base has dimensions in the ratio 2 : 1.

$\therefore$  if one side is  $x$  cm, then the other side must be  $2x$  cm.

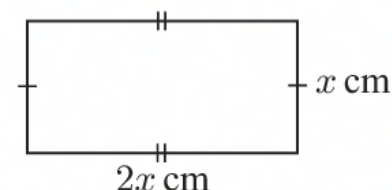
$V = \text{area of base} \times \text{height}$

$$= 2x \times x \times h$$

$$= 2x^2h$$

but  $V = 200 \text{ cm}^3$ ,  $\therefore 2x^2h = 200$

$$\therefore x^2h = 100$$



**b**  $x^2h = 100$  {from **a**}

$$\therefore h = \frac{100}{x^2}$$

Inner surface area

$$A(x) = 2(2x \times x) + 2\left(x \times \frac{100}{x^2}\right) + 2\left(2x \times \frac{100}{x^2}\right)$$

$$= 2\left(2x^2 + \frac{100}{x} + \frac{200}{x}\right)$$

$$= 2\left(2x^2 + \frac{300}{x}\right)$$

$$\therefore A(x) = 4x^2 + \frac{600}{x} \text{ cm}^2$$

**c**  $A(x) = 4x^2 + 600x^{-1}$

$$\therefore A'(x) = 8x - 600x^{-2}$$

$$= 8x - \frac{600}{x^2}$$

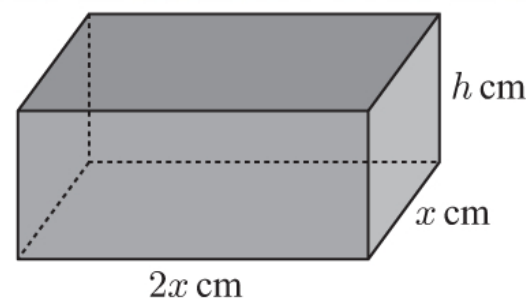
$$A'(x) = 0 \text{ when } 8x - \frac{600}{x^2} = 0$$

$$\therefore 8x = \frac{600}{x^2}$$

$$\therefore 8x^3 = 600$$

$$\therefore x^3 = 75$$

$$\therefore x = \sqrt[3]{75} \approx 4.22$$



$A'(x)$  has sign diagram:

So, the inner surface area of the box is a minimum when  $x = \sqrt[3]{75} \approx 4.22$ .

$$A(\sqrt[3]{75}) = 4(\sqrt[3]{75})^2 + \frac{600}{\sqrt[3]{75}} \\ \approx 213$$

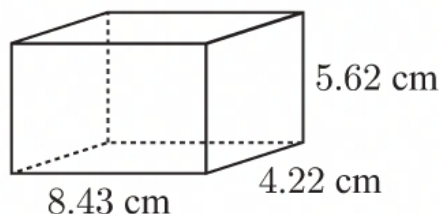
So, the minimum inner surface area is about  $213 \text{ cm}^2$ , when  $x = \sqrt[3]{75} \approx 4.22$ .

**d**  $x = \sqrt[3]{75} \approx 4.22$

$$\therefore 2x = 2\sqrt[3]{75} \approx 8.43$$

$$h = \frac{100}{x^2} = \frac{100}{(\sqrt[3]{75})^2} \approx 5.62$$

So, the optimal box shape is:



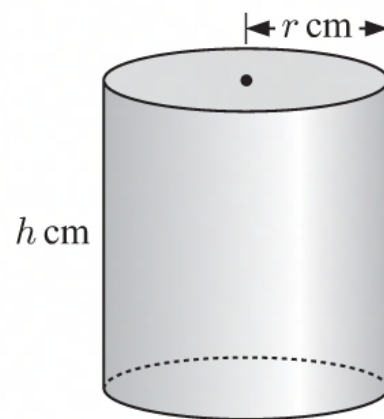
**6 a** volume of cylindrical can  $= \pi r^2 h$

Now, capacity is 1 litre which is equivalent to  $1000 \text{ cm}^3$ .

So, the volume  $= 1000 \text{ cm}^3$

$$\therefore \pi r^2 h = 1000$$

$$\therefore h = \frac{1000}{\pi r^2} \text{ cm}$$



**b** Total surface area of cylindrical can  $A = 2\pi r^2 + 2\pi r h$

$$\therefore A = 2\pi r^2 + 2\pi r \left( \frac{1000}{\pi r^2} \right) \quad \{\text{using a}\}$$

$$\therefore A = 2\pi r^2 + \frac{2000}{r} \text{ cm}^2$$

**c**  $A = 2\pi r^2 + 2000r^{-1}$

$$\therefore \frac{dA}{dr} = 4\pi r - 2000r^{-2}$$

$$= 4\pi r - \frac{2000}{r^2}$$

$$\frac{dA}{dr} = 0 \text{ when } 4\pi r - \frac{2000}{r^2} = 0$$

$$\therefore 4\pi r = \frac{2000}{r^2}$$

$$\therefore 4\pi r^3 = 2000$$

$$\therefore r^3 = \frac{500}{\pi}$$

$$\therefore r = \sqrt[3]{\frac{500}{\pi}} \approx 5.42$$

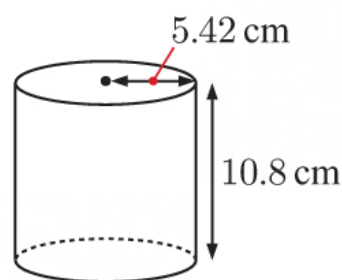
$\frac{dA}{dr}$  has sign diagram:

So, the total surface area is a minimum when  $r = \sqrt[3]{\frac{500}{\pi}} \approx 5.42$ .

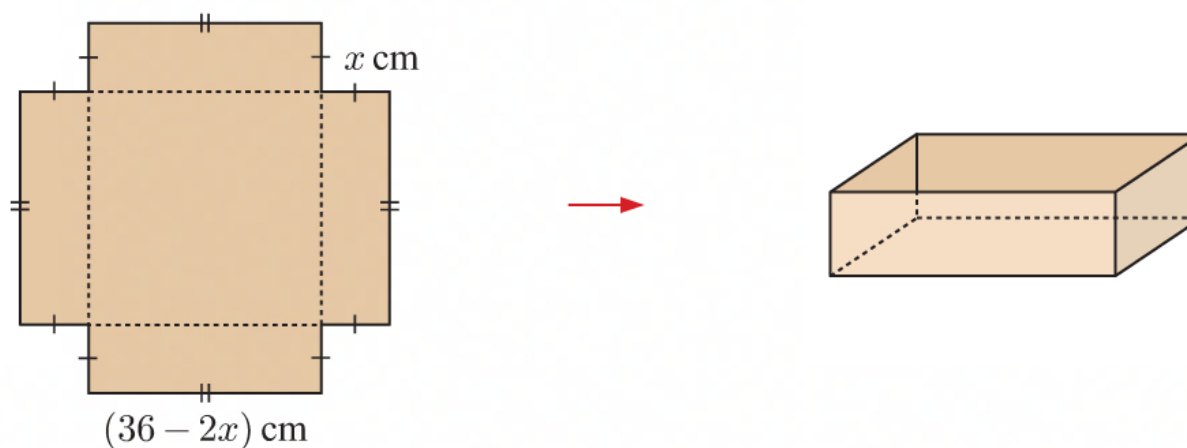


When  $r = \sqrt[3]{\frac{500}{\pi}}$ ,  $h = \frac{1000}{\pi \left(\sqrt[3]{\frac{500}{\pi}}\right)^2} \approx 10.8$

So, the can should have dimensions:



**7 a**



The volume of the container is  $V(x) = \text{area of base} \times \text{height}$

$$\therefore V(x) = (36 - 2x)^2 \times x$$

$$\therefore V(x) = x(36 - 2x)^2 \text{ cm}^3$$

**b**

$$\begin{aligned} V(x) &= x(36 - 2x)^2 \\ &= x(1296 - 144x + 4x^2) \\ &= 1296x - 144x^2 + 4x^3 \end{aligned}$$

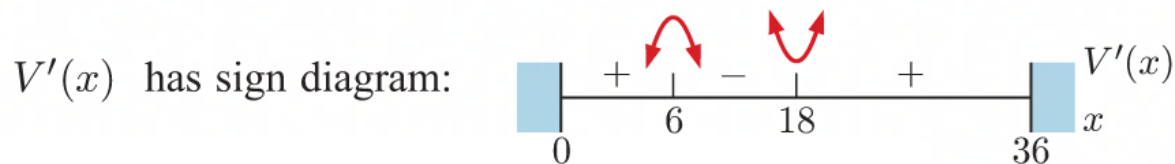
$$\therefore V'(x) = 1296 - 288x + 12x^2$$

$$V'(x) = 0 \text{ when } 1296 - 288x + 12x^2 = 0$$

$$\therefore 12(108 - 24x + x^2) = 0$$

$$\therefore 12(x^2 - 24x + 108) = 0$$

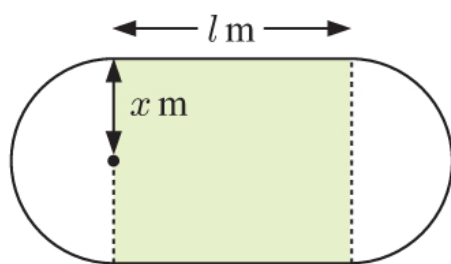
$$\therefore 12(x - 6)(x - 18) = 0$$



The volume is a maximum when  $x = 6$ .

So,  $6 \text{ cm} \times 6 \text{ cm}$  squares should be cut out to produce the container of greatest capacity.

8 a



$$\text{Perimeter} = 2l + 2\pi x$$

$$\therefore 400 = 2l + 2\pi x$$

$$\therefore 2l = 400 - 2\pi x$$

$$\therefore l = 200 - \pi x$$

$x > 0$  and  $l > 0$  for the track to exist

$$\therefore 200 - \pi x > 0$$

$$\therefore \pi x < 200$$

$$\therefore x < \frac{200}{\pi}$$

$$\text{So, } 0 < x < \frac{200}{\pi} \approx 63.7$$

b Area of rectangle  $A = 2x \times l$

$$= 2x \times (200 - \pi x)$$

$$\therefore A = 400x - 2\pi x^2$$

$$\therefore \frac{dA}{dx} = 400 - 4\pi x$$

$$\frac{dA}{dx} = 0 \text{ when } 400 - 4\pi x = 0$$

$$\therefore 4\pi x = 400$$

$$\therefore x = \frac{100}{\pi}$$

$\frac{dA}{dx}$  has sign diagram:

The area is a maximum when  $x = \frac{100}{\pi} \approx 31.8$

$$\therefore l = 200 - \pi \left( \frac{100}{\pi} \right) = 100$$

$$\text{When } x = \frac{100}{\pi}, \quad A = 400 \left( \frac{100}{\pi} \right) - 2\pi \left( \frac{100}{\pi} \right)^2$$

$$= \frac{40\,000}{\pi} - \frac{20\,000}{\pi}$$

$$= \frac{20\,000}{\pi}$$

$$\approx 6370$$

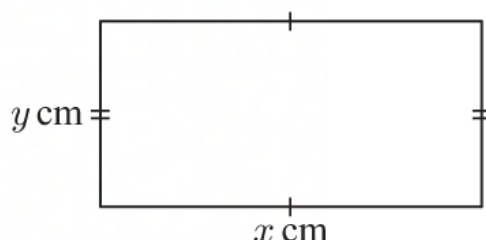
So,  $l = 100$  and  $x = \frac{100}{\pi} \approx 31.8$  give the maximum area  $A = \frac{20\,000}{\pi} \approx 6370 \text{ m}^2$ .

9 a Perimeter = 60 cm

$$\therefore 2x + 2y = 60$$

$$\therefore x + y = 30$$

$$\therefore y = 30 - x$$



b Area of rectangle  $A = xy$

$$\therefore A(x) = x(30 - x) \text{ cm}^2 \quad \{\text{using a}\}$$

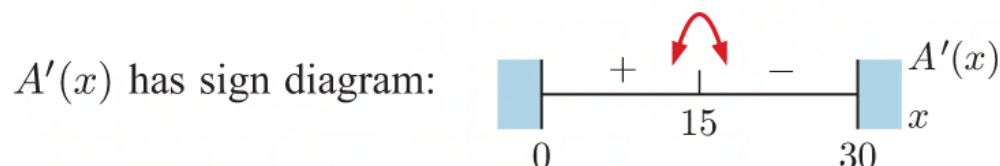
**c**  $A(x) = 30x - x^2$

$\therefore A'(x) = 30 - 2x$

**d**  $A'(x) = 0$  when  $30 - 2x = 0$

$\therefore 2x = 30$

$\therefore x = 15$



The area is a maximum when  $x = 15$

$\therefore y = 30 - 15$  {using **a**}

$= 15$

$\therefore$  the dimensions of the rectangle with maximum area are  $15 \text{ cm} \times 15 \text{ cm}$ .

**10 a** Let  $CN = y \text{ cm}$

Now  $x^2 + y^2 = 5^2$  {Pythagoras}

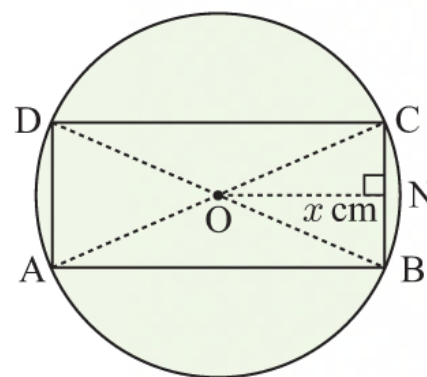
$\therefore y = \sqrt{25 - x^2}$   $\{y > 0\}$

Rectangle ABCD has area  $A = \text{length} \times \text{width}$

$= 2x \times 2y$

$= 4xy$

$= 4x\sqrt{25 - x^2} \text{ cm}^2$



**b**  $A = 4x\sqrt{25 - x^2} = 4x(25 - x^2)^{\frac{1}{2}}$

$\therefore \frac{dA}{dx} = 4(25 - x^2)^{\frac{1}{2}} + 2x(25 - x^2)^{-\frac{1}{2}}(-2x)$  {product rule and chain rule}

$= 4\sqrt{25 - x^2} - \frac{4x^2}{\sqrt{25 - x^2}}$

$= \frac{4(25 - x^2) - 4x^2}{\sqrt{25 - x^2}}$

$= \frac{100 - 4x^2 - 4x^2}{\sqrt{25 - x^2}}$

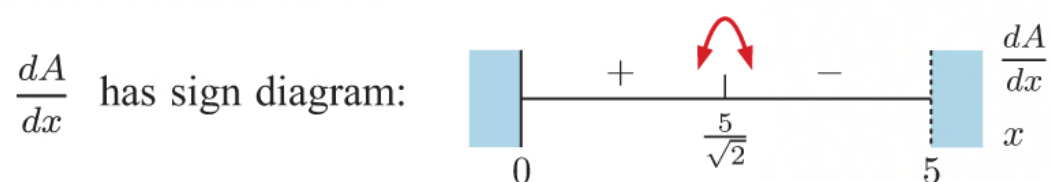
$= \frac{100 - 8x^2}{\sqrt{25 - x^2}}$

So,  $\frac{dA}{dx} = 0$  when  $100 - 8x^2 = 0$

$\therefore 8x^2 = 100$

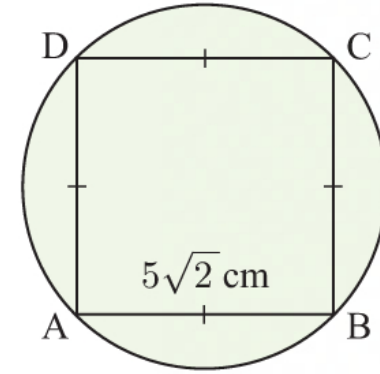
$\therefore x^2 = \frac{25}{2}$

$\therefore x = \frac{5}{\sqrt{2}}$  {as  $x > 0$ }



$$\begin{aligned}
 \text{Area ABCD is maximised when } x &= \frac{5}{\sqrt{2}} \text{ and } y = \sqrt{25 - \left(\frac{5}{\sqrt{2}}\right)^2} \\
 &= \sqrt{25 - \frac{25}{2}} \\
 &= \sqrt{\frac{25}{2}} \\
 &= \frac{5}{\sqrt{2}}
 \end{aligned}$$

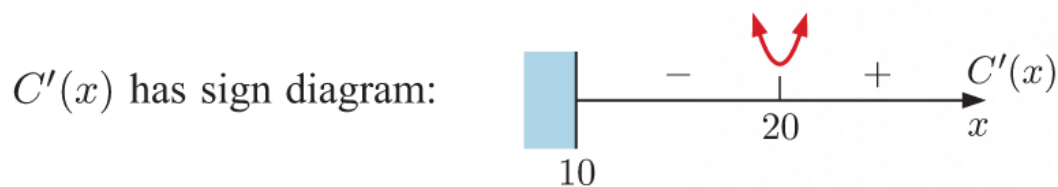
Area ABCD is maximised when it is a square with side lengths  $5\sqrt{2} \text{ cm} \times 5\sqrt{2} \text{ cm}$ .



**11**  $C(x) = 4 \ln x + \left(\frac{30-x}{10}\right)^2$  pounds,  $x \geq 10$

$$\begin{aligned}
 \therefore C'(x) &= \frac{4}{x} + 2\left(\frac{30-x}{10}\right)\left(-\frac{1}{10}\right) \\
 &= \frac{4}{x} - \left(\frac{30-x}{50}\right) \\
 &= \frac{200 - x(30-x)}{50x} \\
 &= \frac{x^2 - 30x + 200}{50x}
 \end{aligned}$$

$$\begin{aligned}
 \text{So, } C'(x) = 0 \text{ when } x^2 - 30x + 200 &= 0 \\
 \therefore (x-10)(x-20) &= 0 \\
 \therefore x &= 10 \text{ or } 20
 \end{aligned}$$

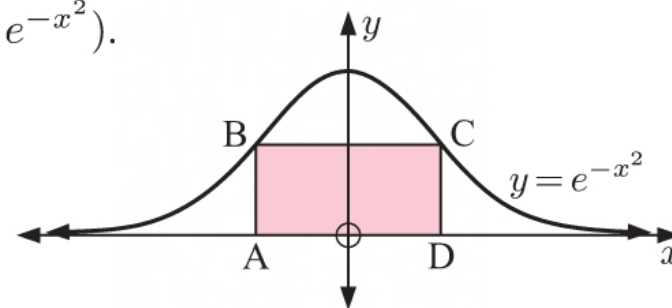


The cost per kettle is minimised when 20 kettles are manufactured per day.

**12** Let D have  $x$ -coordinate  $x$ , so C has coordinates  $(x, e^{-x^2})$ .

$$\therefore CD = e^{-x^2} \text{ and } AD = 2x$$

$$\begin{aligned}
 \therefore \text{rectangle ABCD has area } A &= \text{length} \times \text{width} \\
 &= 2xe^{-x^2}
 \end{aligned}$$



$$\begin{aligned}
 A &= 2xe^{-x^2} \\
 \therefore \frac{dA}{dx} &= 2e^{-x^2} + 2xe^{-x^2}(-2x) \quad \{\text{product rule and chain rule}\} \\
 &= 2e^{-x^2} - 4x^2e^{-x^2} \\
 &= 2e^{-x^2}(1 - 2x^2)
 \end{aligned}$$



$$\begin{aligned}\text{So, } \frac{dA}{dx} = 0 \quad &\text{when } 1 - 2x^2 = 0 \quad \{\text{as } e^{-x^2} > 0\} \\ \therefore 2x^2 &= 1 \\ \therefore x^2 &= \frac{1}{2} \\ \therefore x &= \frac{1}{\sqrt{2}} \quad \{\text{as } x > 0\}\end{aligned}$$

$\frac{dA}{dx}$  has sign diagram:

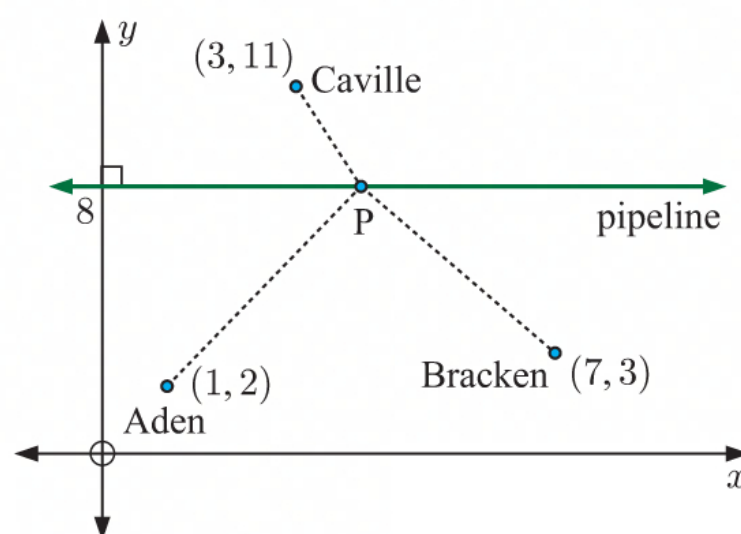
Area ABCD is maximised when  $x = \frac{1}{\sqrt{2}}$ .

$$\begin{aligned}\text{When } x = \frac{1}{\sqrt{2}}, \quad y &= e^{-\left(\frac{1}{\sqrt{2}}\right)^2} \\ &= e^{-\frac{1}{2}}\end{aligned}$$

$\therefore$  ABCD has maximum area when C is at  $\left(\frac{1}{\sqrt{2}}, e^{-\frac{1}{2}}\right)$ .

**13** Suppose P has coordinates  $(x, 8)$ .

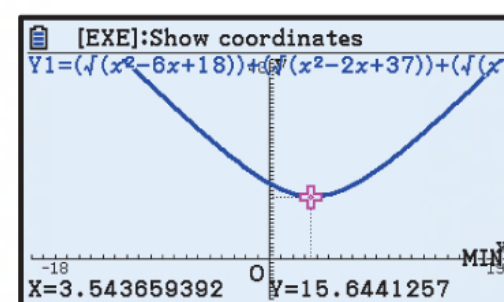
$$\begin{aligned}\therefore AP &= \sqrt{(x-1)^2 + (8-2)^2} \\ &= \sqrt{x^2 - 2x + 1 + 36} \\ &= \sqrt{x^2 - 2x + 37} \\ BP &= \sqrt{(x-7)^2 + (8-3)^2} \\ &= \sqrt{x^2 - 14x + 49 + 25} \\ &= \sqrt{x^2 - 14x + 74} \\ CP &= \sqrt{(x-3)^2 + (8-11)^2} \\ &= \sqrt{x^2 - 6x + 9 + 9} \\ &= \sqrt{x^2 - 6x + 18}\end{aligned}$$



$$\therefore \text{the length of pipeline } L = \sqrt{x^2 - 2x + 37} + \sqrt{x^2 - 14x + 74} + \sqrt{x^2 - 6x + 18}$$

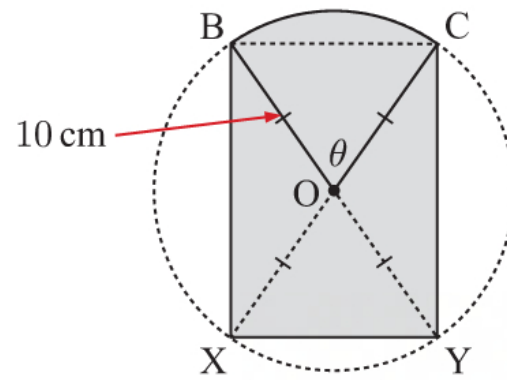
Using technology, the minimum value of  $L$  is approximately 15.64, occurring when  $x \approx 3.544$ .

$\therefore$  P is at  $(3.544, 8)$  and the shortest length of pipe required is about 15.64 km.



$$\begin{aligned}
 \text{14 a Area of sector BOC} &= \frac{1}{2}\theta r^2 \\
 &= \frac{1}{2}\theta(10)^2 \\
 &= 50\theta \text{ cm}^2
 \end{aligned}$$

$$\begin{aligned}
 \text{Now } \widehat{\text{BOX}} &= \frac{\pi}{2} - \theta \quad \{\text{angles on a line}\} \\
 \therefore \text{ area of } \triangle \text{BOX} &= \frac{1}{2}ab \sin\left(\frac{\pi}{2} - \theta\right) \\
 &= \frac{1}{2} \times 10 \times 10 \times \sin \theta \\
 &\quad \{\sin(\frac{\pi}{2} - \theta) = \sin \theta\} \\
 &= 50 \sin \theta \text{ cm}^2
 \end{aligned}$$



$$\text{Similarly, } \widehat{\text{COY}} = \frac{\pi}{2} - \theta \quad \text{and} \quad \text{area of } \triangle \text{COY} = 50 \sin \theta \text{ cm}^2$$

$$\begin{aligned}
 \text{Also, } \widehat{\text{XOY}} &= \widehat{\text{BOC}} = \theta \quad \{\text{vertically opposite angles}\} \\
 \therefore \text{ area of } \triangle \text{XOY} &= \frac{1}{2}ab \sin \theta \\
 &= \frac{1}{2} \times 10 \times 10 \times \sin \theta \\
 &= 50 \sin \theta \text{ cm}^2
 \end{aligned}$$

$$\begin{aligned}
 \text{Shaded area } A &= \text{area of sector BOC} + \text{area of } \triangle \text{BOX} \\
 &\quad + \text{area of } \triangle \text{COY} + \text{area of } \triangle \text{XOY} \\
 &= 50\theta + 50 \sin \theta + 50 \sin \theta + 50 \sin \theta \\
 &= 50\theta + 150 \sin \theta \\
 &= 50(\theta + 3 \sin \theta) \text{ cm}^2
 \end{aligned}$$

$$\text{b } A = 50(\theta + 3 \sin \theta)$$

$$\therefore \frac{dA}{d\theta} = 50(1 + 3 \cos \theta)$$

$$\begin{aligned}
 \text{So, } \frac{dA}{d\theta} = 0 \quad \text{when} \quad 1 + 3 \cos \theta &= 0 \\
 \therefore 3 \cos \theta &= -1 \\
 \therefore \cos \theta &= -\frac{1}{3} \\
 \therefore \theta &\approx 1.91 \quad \{0 < \theta < \pi\}
 \end{aligned}$$

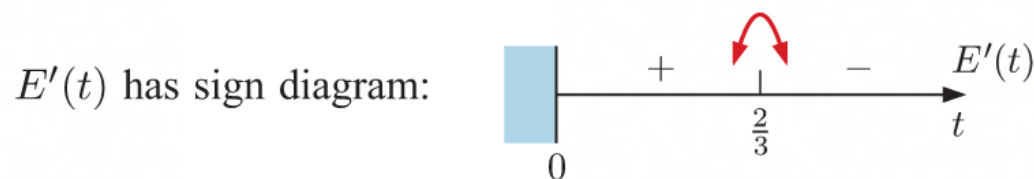
$$\begin{aligned}
 \text{When } \theta &\approx 1.91, \quad A \approx 50(1.91 + 3 \sin 1.91) \\
 &\approx 237
 \end{aligned}$$

The area  $A$  has a maximum of about  $237 \text{ cm}^2$  when  $\theta \approx 1.91$ .

$$\text{15 a } E(t) = 750te^{-1.5t} \text{ units, } t \geq 0$$

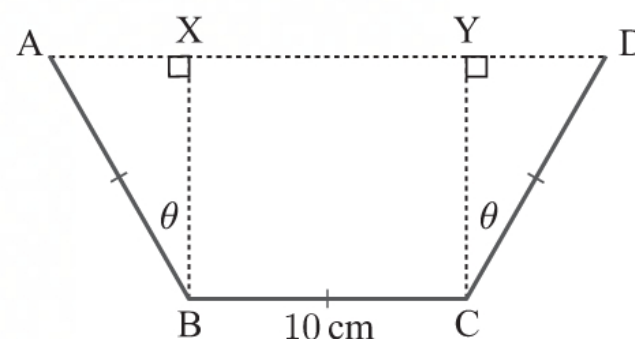
$$\begin{aligned}
 \therefore E'(t) &= 750e^{-1.5t} + 750te^{-1.5t}(-1.5) \quad \{\text{product rule}\} \\
 &= 750e^{-1.5t}(1 - 1.5t)
 \end{aligned}$$

**b**  $E'(t) = 0$  when  $750e^{-1.5t}(1 - 1.5t) = 0$   
 $\therefore 1 - 1.5t = 0$  {as  $e^{-1.5t} > 0$ }  
 $\therefore 1.5t = 1$   
 $\therefore t = \frac{2}{3}$  hours (= 40 minutes)



The anaesthetic is most effective 40 minutes after the injection.

**16 a** In  $\triangle ABX$ ,  $\cos \theta = \frac{BX}{10}$   
 $\therefore BX = CY = 10 \cos \theta$  cm  
 $\sin \theta = \frac{AX}{10}$   
 $\therefore AX = DY = 10 \sin \theta$  cm

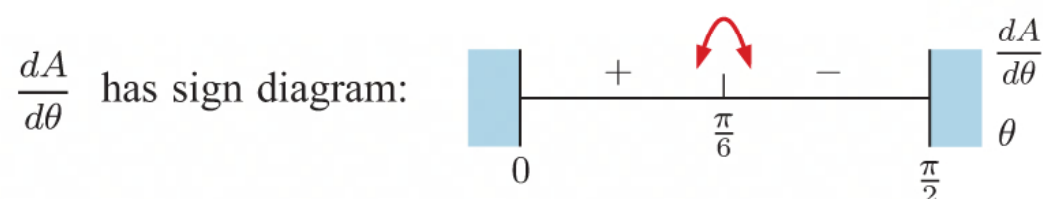


Now, the cross-sectional area  $A$   
 $=$  area  $BCYX$  + area of  $\triangle ABX$  + area of  $\triangle CDY$   
 $= 10 \times 10 \cos \theta + \frac{1}{2} \times 10 \sin \theta \times 10 \cos \theta + \frac{1}{2} \times 10 \sin \theta \times 10 \cos \theta$   
 $= 100 \cos \theta + 100 \sin \theta \cos \theta$   
 $= 100 \cos \theta (1 + \sin \theta)$  cm<sup>2</sup>

**b**  $A = 100 \cos \theta (1 + \sin \theta)$   
 $\therefore \frac{dA}{d\theta} = 100(-\sin \theta (1 + \sin \theta) + \cos \theta \times \cos \theta)$  {product rule}  
 $= 100(-\sin \theta - \sin^2 \theta + \cos^2 \theta)$   
 $= 100(-\sin \theta - \sin^2 \theta + 1 - \sin^2 \theta)$   
 $= -100(2 \sin^2 \theta + \sin \theta - 1)$   
 $= -100(2 \sin \theta - 1)(\sin \theta + 1)$   
 $\therefore \frac{dA}{d\theta} = 0$  when  $2 \sin \theta - 1 = 0$  or  $\sin \theta + 1 = 0$   
 $\therefore \sin \theta = \frac{1}{2}$  or  $\sin \theta = -1$

**c** The gutter has maximum carrying capacity when the cross-sectional area  $A$  is maximised.

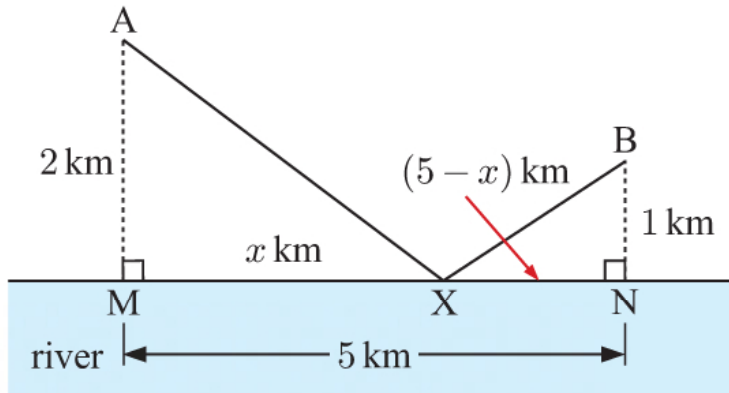
$\frac{dA}{d\theta} = 0$  when  $\sin \theta = \frac{1}{2}$  or  $-1$   
 $\therefore \theta = \frac{\pi}{6}$  { $0 \leq \theta \leq \frac{\pi}{2}$ }



The carrying capacity is maximised when  $\theta = \frac{\pi}{6}$ .

$$\begin{aligned}
 \text{When } \theta = \frac{\pi}{6}, \quad A &= 100 \cos \frac{\pi}{6} \left(1 + \sin \frac{\pi}{6}\right) \\
 &= 100 \times \frac{\sqrt{3}}{2} \times \frac{3}{2} \\
 &= 75\sqrt{3} \\
 &\approx 130
 \end{aligned}$$

$\therefore$  the cross-sectional area for the maximum carrying capacity is about 130 cm<sup>2</sup>.

**17**

Let  $MX = x$  km, so  $XN = 5 - x$  km

$$\therefore AX = \sqrt{2^2 + x^2} \text{ km and } XB = \sqrt{1^2 + (5-x)^2} \text{ km} \quad \{\text{Pythagoras}\}$$

Let the total length of pipeline required be  $P$  km.

Now  $P = AX + XB$

$$= (4 + x^2)^{\frac{1}{2}} + (26 - 10x + x^2)^{\frac{1}{2}}$$

$$\therefore \frac{dP}{dx} = \frac{1}{2}(4 + x^2)^{-\frac{1}{2}}(2x) + \frac{1}{2}(26 - 10x + x^2)^{-\frac{1}{2}}(-10 + 2x) \quad \{\text{chain rule}\}$$

$$= \frac{x}{\sqrt{4 + x^2}} + \frac{x - 5}{\sqrt{x^2 - 10x + 26}}$$

$$\text{Now } \frac{dP}{dx} = 0 \text{ when } \frac{x}{\sqrt{4 + x^2}} + \frac{x - 5}{\sqrt{x^2 - 10x + 26}} = 0$$

$$\therefore \frac{x}{\sqrt{4 + x^2}} = \frac{5 - x}{\sqrt{x^2 - 10x + 26}}$$

$$\therefore \frac{x^2}{4 + x^2} = \frac{(5 - x)^2}{x^2 - 10x + 26}$$

$$\therefore x^2(x^2 - 10x + 26) = (4 + x^2)(25 - 10x + x^2)$$

$$\therefore \cancel{x^4} - \cancel{10x^3} + 26x^2 = 100 - 40x + 4x^2 + 25x^2 - \cancel{10x^3} + \cancel{x^4}$$

$$\therefore -3x^2 + 40x - 100 = 0$$

$$\therefore -(3x - 10)(x - 10) = 0$$

$$\therefore x = \frac{10}{3} = 3\frac{1}{3} \quad \{\text{as } 0 \leq x \leq 5\}$$

$\frac{dP}{dx}$  has sign diagram:

	-	$\updownarrow$ $3\frac{1}{3}$	+	
0				5
				$x$

The minimum length pipeline occurs when  $x = 3\frac{1}{3}$  km.

$\therefore$  X should be  $3\frac{1}{3}$  km from M to minimise the total length of pipeline.



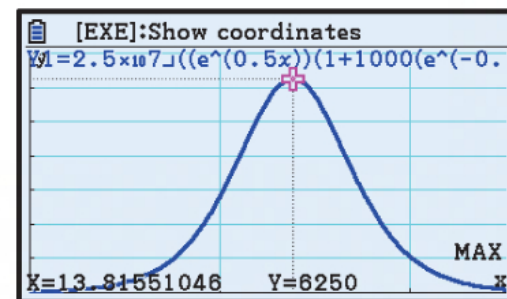
$$\begin{aligned}
 18 \quad P(t) &= \frac{50\,000}{1 + 1000e^{-0.5t}}, \quad 0 \leq t \leq 25 \\
 &= 50\,000(1 + 1000e^{-0.5t})^{-1} \\
 \therefore P'(t) &= -50\,000(1 + 1000e^{-0.5t})^{-2}(-500e^{-0.5t}) \\
 &= \frac{2.5 \times 10^7}{e^{0.5t}(1 + 1000e^{-0.5t})^2}
 \end{aligned}$$

The wasp population is growing the fastest when  $\frac{dP}{dt}$  is a maximum.

We use technology to draw the graph of  $P'(t)$  and find where it is maximised.

The maximum occurs when  $t \approx 13.8$ .

$\therefore$  the wasp population is growing fastest after about 13.8 weeks.



- 19 a Consider each boat's position  $t$  hours after 1:00 pm.

$$PA = 12t \quad \text{and} \quad QB = 8t$$

$$\therefore PB = 100 - 8t$$

Using the cosine rule in  $\triangle PAB$ ,

$$\begin{aligned}
 [D(t)]^2 &= PA^2 + PB^2 - 2PA \times PB \cos 60^\circ \\
 &= (12t)^2 + (100 - 8t)^2 - 2(12t)(100 - 8t)\left(\frac{1}{2}\right) \\
 &= 144t^2 + 10\,000 - 1600t + 64t^2 - 12t(100 - 8t) \\
 &= 144t^2 + 10\,000 - 1600t + 64t^2 - 1200t + 96t^2 \\
 &= 304t^2 - 2800t + 10\,000
 \end{aligned}$$

$$\therefore D(t) = \sqrt{304t^2 - 2800t + 10\,000} \quad \{D(t) > 0\}$$

b Now  $\frac{d[D(t)]^2}{dt} = 608t - 2800$

$$\therefore \frac{d[D(t)]^2}{dt} = 0 \quad \text{when} \quad t = \frac{2800}{608} \approx 4.605$$

$\frac{d[D(t)]^2}{dt}$  has sign diagram:

$\therefore D(t)$  is a minimum when  $t \approx 4.605$  hours after 1:00 pm

$$\begin{aligned}
 \text{When } t \approx 4.605, \quad [D(t)]^2 &\approx 304(4.605)^2 - 2800(4.605) + 10\,000 \\
 &\approx 3550
 \end{aligned}$$

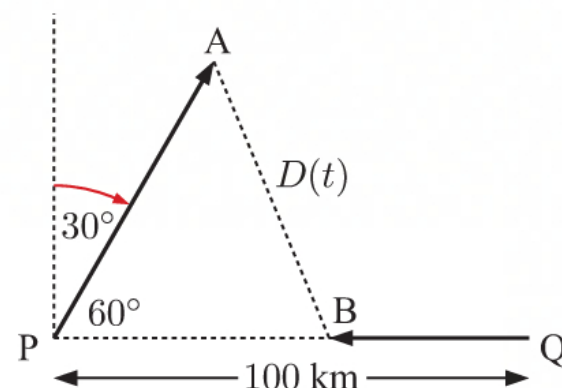
$\therefore$  the minimum value of  $D^2$  is about 3550.

- c The ships are closest when  $t \approx 4.605$  hours

$$0.605 \text{ hours} \approx 0.605 \times 60$$

$$\approx 36 \text{ minutes}$$

$\therefore$  the ships are closest together 4 hours and 36 minutes after 1:00 pm, which is 5:36 pm.



**20**  $r^2 + h^2 = s^2$  {Pythagoras}

$$\therefore h^2 = s^2 - r^2$$

$$\therefore h = \sqrt{s^2 - r^2} \quad \{\text{as } h > 0\}$$

But  $V = \frac{1}{3}\pi r^2 h$

$$= \frac{1}{3}\pi r^2 \sqrt{s^2 - r^2}$$

$$= \frac{1}{3}\pi r^2 (s^2 - r^2)^{\frac{1}{2}}$$

$$\therefore \frac{dV}{dr} = \frac{2}{3}\pi r (s^2 - r^2)^{\frac{1}{2}} + \frac{1}{3}\pi r^2 \times \frac{1}{2}(s^2 - r^2)^{-\frac{1}{2}}(-2r) \quad \{\text{product rule}\}$$

$$= \frac{2}{3}\pi r \sqrt{s^2 - r^2} - \frac{1}{3}\pi r^3 \frac{1}{\sqrt{s^2 - r^2}}$$

$$= \frac{1}{3}\pi r \left( 2\sqrt{s^2 - r^2} - \frac{r^2}{\sqrt{s^2 - r^2}} \right)$$

$$= \frac{1}{3}\pi r \left( \frac{2(s^2 - r^2) - r^2}{\sqrt{s^2 - r^2}} \right)$$

$$= \frac{1}{3}\pi r \left( \frac{2s^2 - 3r^2}{\sqrt{s^2 - r^2}} \right)$$

So,  $\frac{dV}{dr} = 0$  when  $2s^2 - 3r^2 = 0$  {as  $r > 0$ }

$$\therefore 2s^2 = 3r^2$$

$$\therefore \frac{s^2}{r^2} = \frac{3}{2}$$

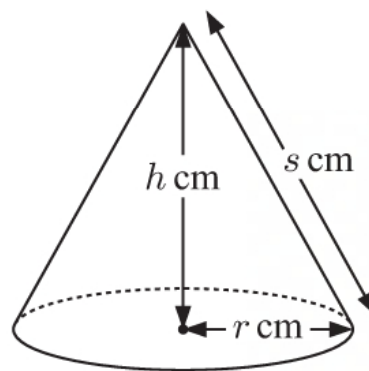
$$\therefore \frac{s}{r} = \frac{\sqrt{3}}{\sqrt{2}} \quad \{\text{as } s, r > 0\}$$

$$\therefore r = \frac{\sqrt{2}}{\sqrt{3}} s$$

$\frac{dV}{dr}$  has sign diagram:

0	+	$\frac{\sqrt{2}}{\sqrt{3}} s$	-	s
		$\frac{dV}{dr}$		

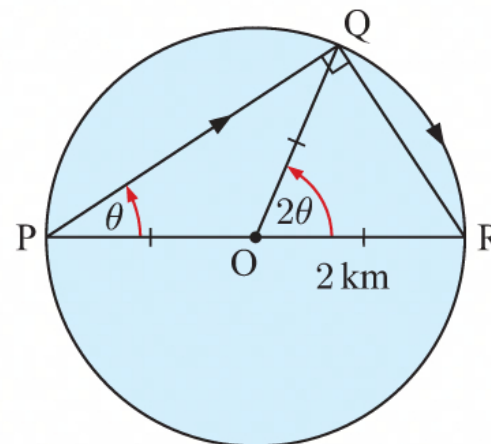
$\therefore V$  is a maximum when  $s : r = \sqrt{3} : \sqrt{2}$



**21 a**  $\widehat{PQR} = 90^\circ$  {angle in a semi-circle theorem}

$$\therefore \cos \theta = \frac{PQ}{4}$$

$$\therefore PQ = 4 \cos \theta \text{ km}$$



- b** Hieu can row at  $3 \text{ km h}^{-1}$ .

$$\begin{aligned}\therefore \text{time taken to row from P to Q} &= \frac{\text{distance}}{\text{speed}} \\ &= \frac{4 \cos \theta}{3} \\ &= \frac{4}{3} \cos \theta \text{ hours}\end{aligned}$$

Now  $\widehat{\text{QOR}} = 2\theta$  {angle at the centre theorem}

$$\begin{aligned}\therefore \text{arc QR} &= 2\theta \times r \\ &= 2\theta \times 2 \\ &= 4\theta \text{ km}\end{aligned}$$

Hieu can walk at  $6 \text{ km h}^{-1}$ .

$$\begin{aligned}\therefore \text{time taken to walk along arc QR} &= \frac{\text{distance}}{\text{speed}} \\ &= \frac{4\theta}{6} \\ &= \frac{2\theta}{3} \text{ hours}\end{aligned}$$

$\therefore$  the time taken for Hieu's journey  $T = \frac{4}{3} \cos \theta + \frac{2\theta}{3}$  hours where  $0 \leq \theta \leq \frac{\pi}{2}$ .

**c**  $T = \frac{4}{3} \cos \theta + \frac{2\theta}{3}, \quad 0 \leq \theta \leq \frac{\pi}{2}$

$$\therefore \frac{dT}{d\theta} = -\frac{4}{3} \sin \theta + \frac{2}{3}$$

$$\therefore \frac{dT}{d\theta} = 0 \text{ when } -\frac{4}{3} \sin \theta + \frac{2}{3} = 0$$

$$\therefore \frac{4}{3} \sin \theta = \frac{2}{3}$$

$$\therefore \sin \theta = \frac{1}{2}$$

$$\therefore \theta = \frac{\pi}{6} \quad \{0 \leq \theta \leq \frac{\pi}{2}\}$$

**d**  $\frac{dT}{d\theta}$  has sign diagram:

- e i** From the sign diagram in **d**, the time  $T$  is a maximum when  $\theta = \frac{\pi}{6}$ . So the longest time taken involves Hieu rowing from P to Q at an angle of  $\frac{\pi}{6}$  to the diameter of the lake, then walking from Q to R.

- ii** The maximum value of  $T$  occurs when  $\theta = \frac{\pi}{6}$ .

$\therefore$  the minimum value of  $T$  must occur at one of the end points, that is, when  $\theta = 0$  or  $\theta = \frac{\pi}{2}$ .

When  $\theta = 0$ , this means Hieu must row directly from P to R.

When  $\theta = \frac{\pi}{2}$ , this means Hieu must walk the whole way around the shore from P to R.

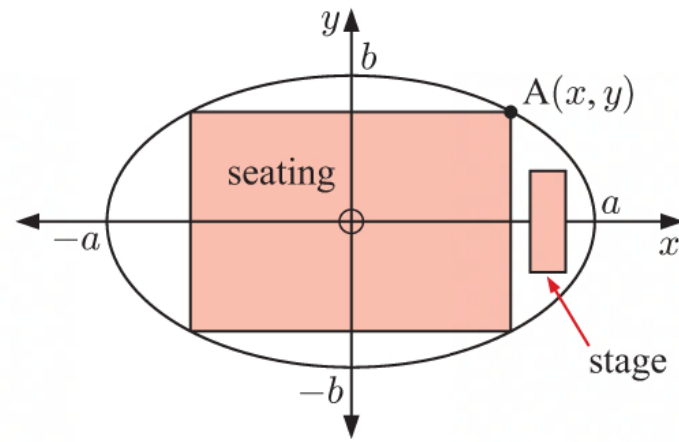
Firstly, when  $\theta = 0$ ,  $T = \frac{4}{3} \cos 0 + \frac{2(0)}{3} = \frac{4}{3} \approx 1.33$  hours.

Secondly, when  $\theta = \frac{\pi}{2}$ ,  $T = \frac{4}{3} \cos \frac{\pi}{2} + \frac{2(\frac{\pi}{2})}{3} = \frac{\pi}{3} \approx 1.05$  hours.

So, the shortest time taken involves Hieu walking the whole way around the shore from P to R.

**22 a**

$$\begin{aligned}\frac{x^2}{a^2} + \frac{y^2}{b^2} &= 1 \\ \therefore x^2 b^2 + y^2 a^2 &= a^2 b^2 \\ \therefore a^2 y^2 &= a^2 b^2 - x^2 b^2 \\ \therefore y^2 &= \frac{a^2 b^2 - x^2 b^2}{a^2} \\ \therefore y &= \pm \sqrt{\frac{a^2 b^2 - x^2 b^2}{a^2}}\end{aligned}$$



Since A lies in quadrant 1, then  $y > 0$

$$\begin{aligned}\therefore y &= \sqrt{\left(\frac{b^2}{a^2}\right)(a^2 - x^2)} \\ \therefore y &= \frac{b}{a} \sqrt{a^2 - x^2}\end{aligned}$$

**b** The seating area is  $A = 2x \times 2y$

$$\begin{aligned}&= 4xy \\ &= 4x \left( \frac{b}{a} \sqrt{a^2 - x^2} \right) \quad \{\text{from a}\} \\ \therefore A(x) &= \frac{4bx}{a} \sqrt{a^2 - x^2} \quad \text{as required}\end{aligned}$$

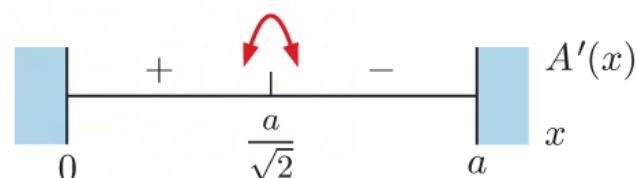
**c**

$$\begin{aligned}A(x) &= \frac{4b}{a} x (a^2 - x^2)^{\frac{1}{2}} \\ \therefore A'(x) &= \frac{4b}{a} (1) (a^2 - x^2)^{\frac{1}{2}} + \frac{4b}{a} x \times \frac{1}{2} (a^2 - x^2)^{-\frac{1}{2}} (-2x) \quad \{\text{product rule}\} \\ &= \frac{4b}{a} \sqrt{a^2 - x^2} - \frac{4bx^2}{a\sqrt{a^2 - x^2}} \\ &= \frac{4b}{a} \left( \sqrt{a^2 - x^2} - \frac{x^2}{\sqrt{a^2 - x^2}} \right) \\ &= \frac{4b}{a} \left( \frac{a^2 - x^2 - x^2}{\sqrt{a^2 - x^2}} \right) \\ &= \frac{4b}{a} \left( \frac{a^2 - 2x^2}{\sqrt{a^2 - x^2}} \right)\end{aligned}$$

So,  $A'(x) = 0$  when  $a^2 - 2x^2 = 0$

$$\begin{aligned}\therefore 2x^2 &= a^2 \\ \therefore x &= \pm \frac{a}{\sqrt{2}} \\ \therefore x &= \frac{a}{\sqrt{2}} \quad \{\text{as } x \text{ is in quadrant 1 and } a > 0\}\end{aligned}$$

$A'(x)$  has sign diagram:



The area is maximised when  $x = \frac{a}{\sqrt{2}}$ .



**d** Maximum seating area  $= A \left( \frac{a}{\sqrt{2}} \right)$  {from **c**}

$$= \frac{4b}{a} \times \frac{a}{\sqrt{2}} \sqrt{a^2 - \left( \frac{a}{\sqrt{2}} \right)^2}$$

$$= \frac{4b}{\sqrt{2}} \times \sqrt{\frac{a^2}{2}}$$

$$= \frac{4b}{\sqrt{2}} \times \frac{a}{\sqrt{2}}$$

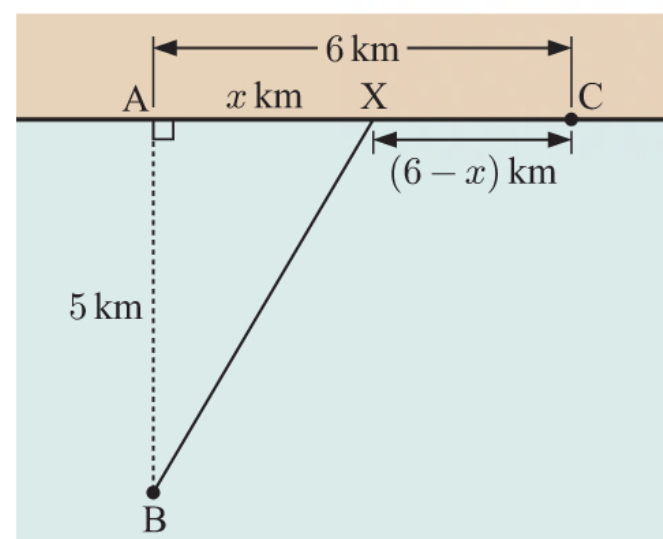
$$= \frac{4ab}{2} = 2ab$$

$$\% \text{ occupied by seats} = \frac{\text{maximum seating area}}{\text{total area of ellipse}} \times 100\%$$

$$= \frac{2ab}{\pi ab} \times 100\%$$

$$\approx 63.7\%$$

- 23 a** AC has length 6 km and X lies between A and C.  
 $\therefore 0 \leq x \leq 6$



- b** Now  $XC = 6 - x$  and  $BX = \sqrt{x^2 + 5^2}$  {Pythagoras}

$$\therefore \text{the time taken to row from B to X} = \frac{\text{distance}}{\text{speed}} = \frac{BX}{8} = \frac{\sqrt{x^2 + 5^2}}{8} \text{ hours}$$

$$\text{and the time taken to run from X to C} = \frac{\text{distance}}{\text{speed}} = \frac{XC}{17} = \frac{6 - x}{17} \text{ hours}$$

$$\therefore \text{the total time } T = \frac{\sqrt{x^2 + 25}}{8} + \frac{6 - x}{17} \text{ hours, } 0 \leq x \leq 6$$

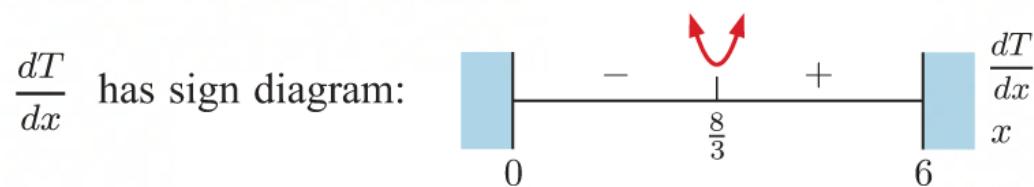
**c** 
$$T = \frac{\sqrt{x^2 + 25}}{8} + \frac{6 - x}{17}$$

$$= \frac{1}{8}(x^2 + 25)^{\frac{1}{2}} + \frac{6}{17} - \frac{x}{17}$$

$$\therefore \frac{dT}{dx} = \frac{1}{16}(x^2 + 25)^{-\frac{1}{2}}(2x) - \frac{1}{17} \quad \{\text{chain rule}\}$$

$$= \frac{x}{8\sqrt{x^2 + 25}} - \frac{1}{17}$$

$$\begin{aligned}
 \text{So, } \frac{dT}{dx} = 0 \text{ when } \frac{x}{8\sqrt{x^2 + 25}} &= \frac{1}{17} \\
 \therefore 17x &= 8\sqrt{x^2 + 25} \\
 \therefore 289x^2 &= 64(x^2 + 25) \\
 \therefore 289x^2 &= 64x^2 + 1600 \\
 \therefore 225x^2 &= 1600 \\
 \therefore x^2 &= \frac{1600}{225} \\
 \therefore x &= \frac{40}{15} \quad \{x > 0\} \\
 &= \frac{8}{3} \approx 2.67
 \end{aligned}$$



$\therefore x \approx 2.67$  is the distance in km from A to X which minimises the time taken for Peter to travel from B to C.

**24 a** Arc AC =  $\theta r_{\text{sector}}$   
 $= \theta \times 10$   
 $= 10\theta$

Now, arc AC forms the circumference of the base of the cone.

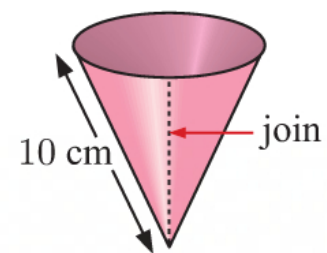
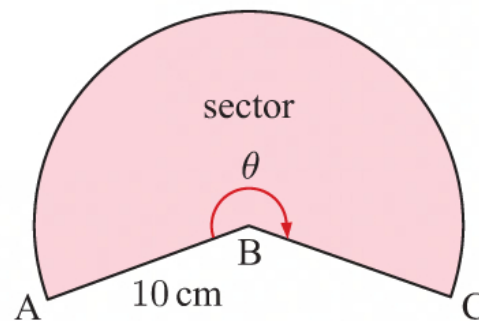
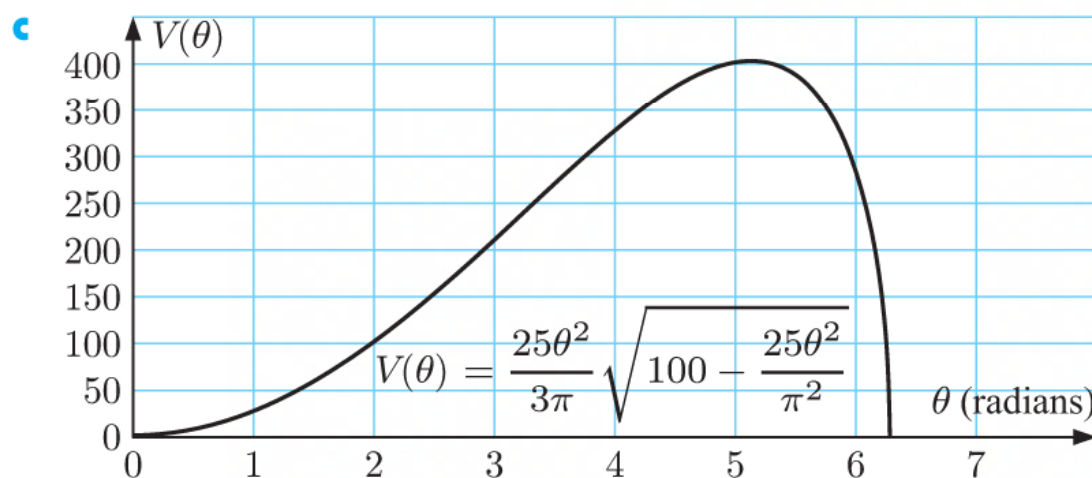
$$\therefore 2\pi r = 10\theta$$

$$\therefore r = \frac{5\theta}{\pi}$$

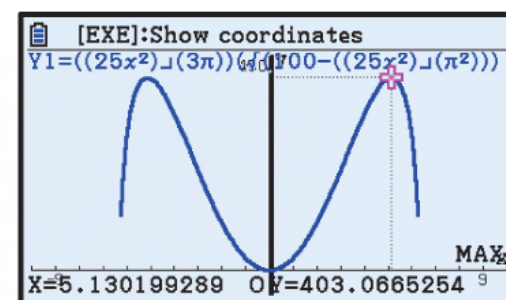
Height of cone =  $\sqrt{10^2 - r^2}$  {Pythagoras}

$$\therefore h = \sqrt{100 - \left(\frac{5\theta}{\pi}\right)^2}$$

**b**  $V = \frac{1}{3}\pi r^2 h$   
 $= \frac{1}{3}\pi \times \left(\frac{5\theta}{\pi}\right)^2 \times \sqrt{100 - \left(\frac{5\theta}{\pi}\right)^2}$  {from a}  
 $= \frac{25\theta^2}{3\pi} \sqrt{100 - \frac{25\theta^2}{\pi^2}}$



- d** Using technology, the maximum value is  $V \approx 403$ , when  $\theta \approx 5.13$ .



- 25 a** At time  $t$ , the mosquito is at  $(3 - t^2, 2 + \sqrt{t}, 2 - \sqrt{t})$ .  
 $\therefore$  the mosquito's distance from the origin

$$\begin{aligned} D &= \sqrt{(3 - t^2)^2 + (2 + \sqrt{t})^2 + (2 - \sqrt{t})^2} \\ &= \sqrt{9 - 6t^2 + t^4 + 4 + 4\sqrt{t} + t + 4 - 4\sqrt{t} + t} \\ &= \sqrt{t^4 - 6t^2 + 2t + 17} \end{aligned}$$

$$\therefore D^2 = t^4 - 6t^2 + 2t + 17$$

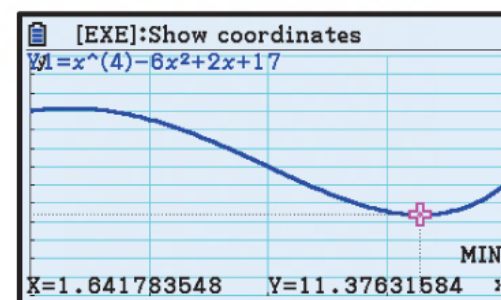
- b** Using technology, the minimum value of  $D^2 = t^4 - 6t^2 + 2t + 17$ ,  $t \geq 0$ , is about 11.376 when  $t \approx 1.64$ .

$$\text{If } D^2 \approx 11.376$$

$$\text{then } D \approx \sqrt{11.376}$$

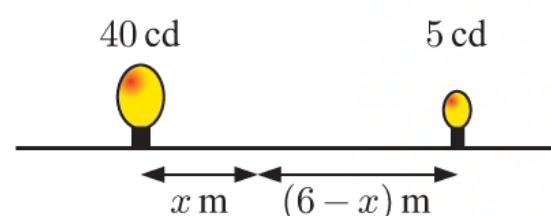
$$\approx 3.37 \quad \{D > 0\}$$

$\therefore$  the closest the mosquito came to the source of the repellent was about 3.37 m.



- 26** The intensity of illumination  $I \propto \frac{s}{d^2}$  where  $s$  is the power of the source and  $d$  is the distance from it.

$$\therefore I = \frac{ks}{d^2} \quad \text{where } k \text{ is a constant.}$$



$$\text{The intensity from the 40 cd bulb} = \frac{40k}{x^2}$$

$$\text{The intensity from the 5 cd bulb} = \frac{5k}{(6 - x)^2}$$

$$\text{The total intensity at a point on the line } I = \frac{40k}{x^2} + \frac{5k}{(6 - x)^2}$$

$$= k[40x^{-2} + 5(6 - x)^{-2}]$$

$$\therefore \frac{dI}{dx} = k[-80x^{-3} - 10(6 - x)^{-3} \times (-1)]$$

$$= k \left[ -\frac{80}{x^3} + \frac{10}{(6 - x)^3} \right]$$

$$\frac{dI}{dx} = 0 \quad \text{when} \quad \frac{80}{x^3} = \frac{10}{(6-x)^3}$$

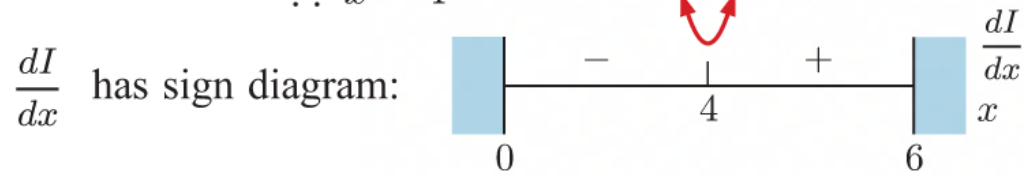
$$\therefore 8(6-x)^3 = x^3$$

$$\therefore 2(6-x) = x$$

$$\therefore 12 - 2x = x$$

$$\therefore 3x = 12$$

$$\therefore x = 4$$



$\therefore I$  is a minimum when  $x = 4$ .

$\therefore$  the darkest point on the line segment joining the two lamps is 4 m from the 40 cd lamp.

**27 a**  $\cos \theta = \frac{3}{x}$

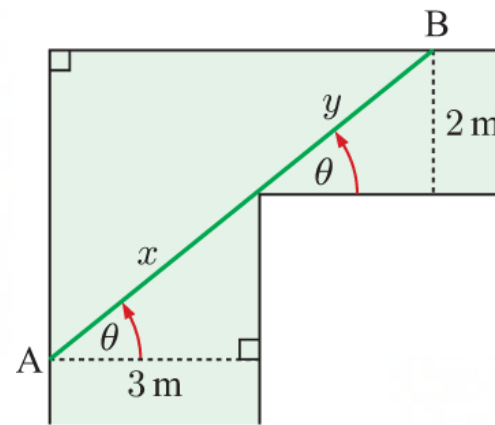
$$\therefore x = \frac{3}{\cos \theta} = 3 \sec \theta$$

$$\sin \theta = \frac{2}{y}$$

$$\therefore y = \frac{2}{\sin \theta} = 2 \operatorname{cosec} \theta$$

$\therefore$  the length of AB is given by  $L = x + y$

$$\therefore L = 3 \sec \theta + 2 \operatorname{cosec} \theta$$



**b**  $L = 3 \sec \theta + 2 \operatorname{cosec} \theta$

$$\therefore \frac{dL}{d\theta} = 3 \sec \theta \tan \theta - 2 \operatorname{cosec} \theta \cot \theta$$

$$= \frac{3 \sin \theta}{\cos^2 \theta} - \frac{2 \cos \theta}{\sin^2 \theta}$$

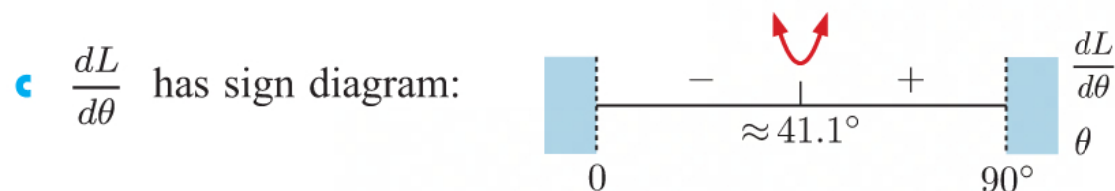
So,  $\frac{dL}{d\theta} = 0$  when  $\frac{3 \sin \theta}{\cos^2 \theta} = \frac{2 \cos \theta}{\sin^2 \theta}$

$$\therefore 3 \sin^3 \theta = 2 \cos^3 \theta$$

$$\therefore \tan^3 \theta = \frac{2}{3}$$

$$\therefore \tan \theta = \sqrt[3]{\frac{2}{3}}$$

$$\therefore \theta = \arctan \left( \sqrt[3]{\frac{2}{3}} \right) \approx 41.1^\circ$$



So, AB is a minimum when  $\theta \approx 41.1^\circ$ .

$$\text{When } \theta = \arctan \left( \sqrt[3]{\frac{2}{3}} \right) \approx 41.1^\circ, \quad L \approx 3 \sec 41.1^\circ + 2 \operatorname{cosec} 41.1^\circ \approx 7.02$$

So,  $L \approx 7.02$  m. This is the greatest length of metal tube which can be carried horizontally around the corner from one corridor to the other without bending.



**28 a**  $\tan \alpha = \frac{2}{x}, \quad \tan(\alpha + \theta) = \frac{3}{x}$

**b**  $\theta = (\alpha + \theta) - \alpha$   
 $= \arctan \frac{3}{x} - \arctan \frac{2}{x} \quad \{\text{from a}\}$

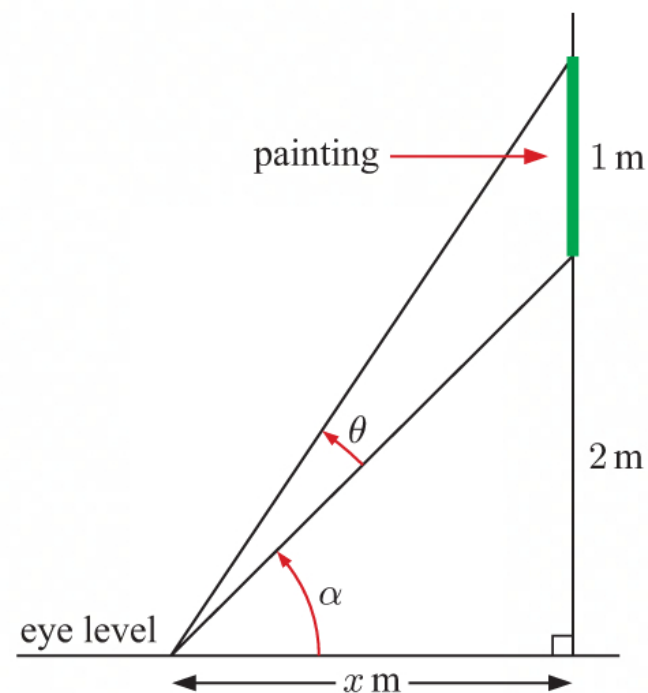
**c**  $\frac{d\theta}{dx} = \frac{d}{dx} \left( \arctan \frac{3}{x} - \arctan \frac{2}{x} \right)$   
 $= \left( -\frac{3}{x^2} \right) \times \frac{1}{1 + \left( \frac{3}{x} \right)^2} - \left( -\frac{2}{x^2} \right) \times \frac{1}{1 + \left( \frac{2}{x} \right)^2}$   
 $= \frac{-3}{x^2 \left( 1 + \frac{9}{x^2} \right)} - \frac{-2}{x^2 \left( 1 + \frac{4}{x^2} \right)}$   
 $= \frac{2}{x^2 + 4} - \frac{3}{x^2 + 9}$

**d**  $\frac{d\theta}{dx} = \frac{2}{x^2 + 4} - \frac{3}{x^2 + 9}$   
 $= \frac{2(x^2 + 9) - 3(x^2 + 4)}{(x^2 + 4)(x^2 + 9)}$

$\therefore \frac{d\theta}{dx} = 0 \quad \text{when} \quad 2(x^2 + 9) - 3(x^2 + 4) = 0$   
 $\therefore 2x^2 + 18 - 3x^2 - 12 = 0$   
 $\therefore x^2 = 6$   
 $\therefore x = \sqrt{6} \quad \{x > 0\}$

$\frac{d\theta}{dx}$  has sign diagram:

$\therefore$  the maximum viewing angle occurs when  $x = \sqrt{6}$ , which is when Sonia is  $\sqrt{6}$  m from the wall.



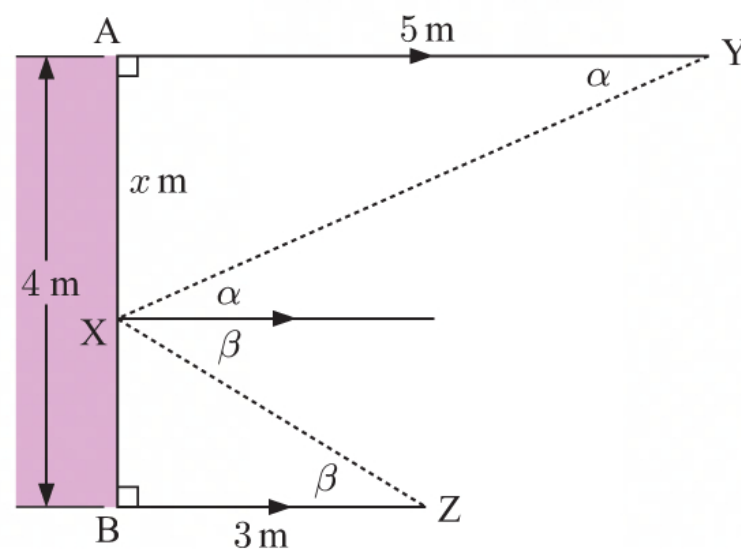
**29** Let  $AX = x$  m

$\therefore XB = (4 - x)$  m

$\therefore \tan \alpha = \frac{x}{5} \quad \text{and} \quad \tan \beta = \frac{4-x}{3}$

Now  $\theta = \alpha + \beta$   
 $= \arctan \frac{x}{5} + \arctan \left( \frac{4-x}{3} \right)$

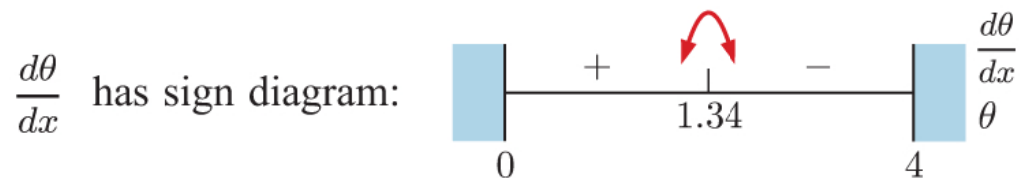
$\therefore \frac{d\theta}{dx} = \frac{1}{5} \frac{1}{1 + \left( \frac{x}{5} \right)^2} + \left( -\frac{1}{3} \right) \left( \frac{1}{1 + \left( \frac{4-x}{3} \right)^2} \right)$   
 $= \frac{5}{25 + x^2} - \frac{3}{9 + (4-x)^2}$   
 $= \frac{5(9 + (4-x)^2) - 3(25 + x^2)}{(25 + x^2)(9 + (4-x)^2)}$   
 $= \frac{45 + 80 - 40x + 5x^2 - 75 - 3x^2}{(25 + x^2)(9 + (4-x)^2)}$   
 $= \frac{2x^2 - 40x + 50}{(25 + x^2)(9 + (4-x)^2)}$



$$\therefore \frac{d\theta}{dx} = 0 \quad \text{when} \quad 2x^2 - 40x + 50 = 0$$

$$\therefore x^2 - 20x + 25 = 0$$

Using technology,  $x \approx 1.3397$  or  $18.660$   
 $\therefore x \approx 1.34 \quad \{0 \leq x \leq 4\}$



$\therefore \theta$  is a maximum when  $x$  is about 1.34 m from A.

## EXERCISE 19C

- 1 a Differentiating both sides of  $ab^3 = 40$  with respect to  $t$ :

$$b^3 \frac{da}{dt} + 3ab^2 \frac{db}{dt} = 0 \quad \{\text{product rule}\}$$

- b When  $a = 5$ ,  $\frac{db}{dt} = 1$  unit per second and  $5b^3 = 40$   
 $\therefore b^3 = 8$   
 $\therefore b = 2$

$$\therefore 2^3 \frac{da}{dt} + 3(5)(2)^2(1) = 0$$

$$\therefore 8 \frac{da}{dt} + 60 = 0$$

$$\therefore \frac{da}{dt} = -\frac{60}{8} = -7.5$$

$\therefore a$  is decreasing at 7.5 units per second.

- 2 a  $A = x^2$

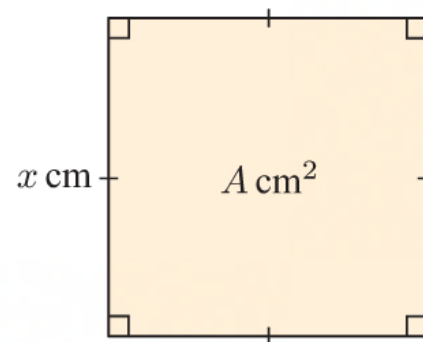
- b Differentiating both sides of  $A = x^2$  with respect to  $t$ :

$$\frac{dA}{dt} = 2x \frac{dx}{dt}$$

- c When  $x = 6$ ,  $\frac{dx}{dt} = 2$  cm per second

$$\therefore \frac{dA}{dt} = 2(6)(2) = 24 \text{ cm}^2 \text{ per second}$$

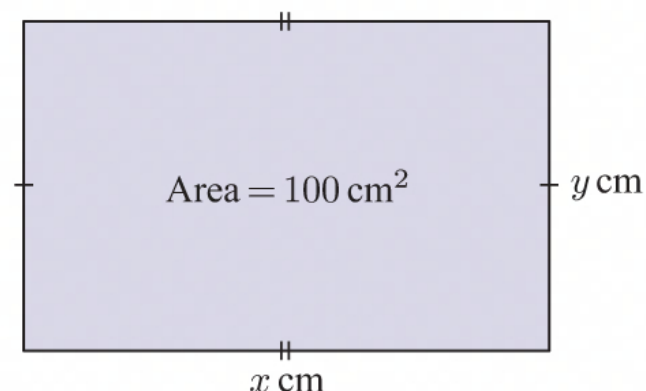
$\therefore$  the area is increasing at  $24 \text{ cm}^2$  per second.



**3 a**  $xy = 100$

**b** Differentiating both sides of  $xy = 100$  with respect to  $t$ :

$$y \frac{dx}{dt} + x \frac{dy}{dt} = 0 \quad \{\text{product rule}\}$$



**c** For all values of  $t$ ,  $\frac{dx}{dt} = -1$  cm per minute

**i** When  $x = 20$ ,  $y = 5$

$$\therefore 5(-1) + 20 \frac{dy}{dt} = 0$$

$$\therefore 20 \frac{dy}{dt} = 5$$

$$\therefore \frac{dy}{dt} = \frac{1}{4}$$

$= 0.25$  cm per minute

$\therefore$  the width of the rectangle is increasing at 0.25 cm per minute.

**ii** Since the rectangle is a square,  $x = y = 10$

$$\therefore 10(-1) + 10 \frac{dy}{dt} = 0$$

$$\therefore 10 \frac{dy}{dt} = 10$$

$$\therefore \frac{dy}{dt} = 1 \text{ cm per minute}$$

$\therefore$  the width of the rectangle is increasing at 1 cm per minute.

**4** The area of the circular ripple is  $A = \pi r^2$ .

Differentiating both sides of  $A = \pi r^2$  with respect to  $t$ :

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

Since the ripple moves out at a constant speed of  $1 \text{ m s}^{-1}$ ,

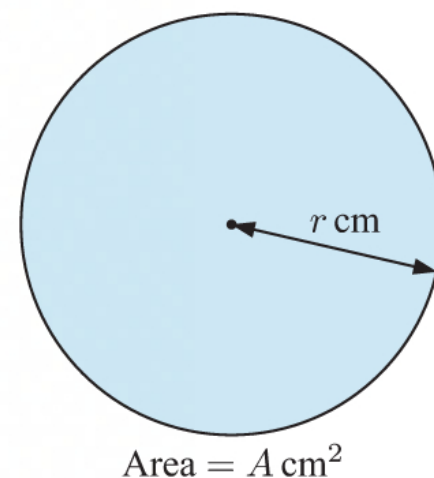
$$\frac{dr}{dt} = 1 \text{ m s}^{-1}.$$

**a** When  $r = t = 2$ ,  $\frac{dA}{dt} = 2\pi \times 2 \times 1 = 4\pi \text{ m}^2 \text{ s}^{-1}$

$\therefore$  the circle's area is increasing at  $4\pi \text{ m}^2$  per second.

**b** When  $r = t = 4$ ,  $\frac{dA}{dt} = 2\pi \times 4 \times 1 = 8\pi \text{ m}^2 \text{ s}^{-1}$

$\therefore$  the circle's area is increasing at  $8\pi \text{ m}^2$  per second.



**5** The volume of a spherical balloon is  $V = \frac{4}{3}\pi r^3$ .

Differentiating both sides of  $V = \frac{4}{3}\pi r^3$  with respect to  $t$ :

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

*Particular case:*

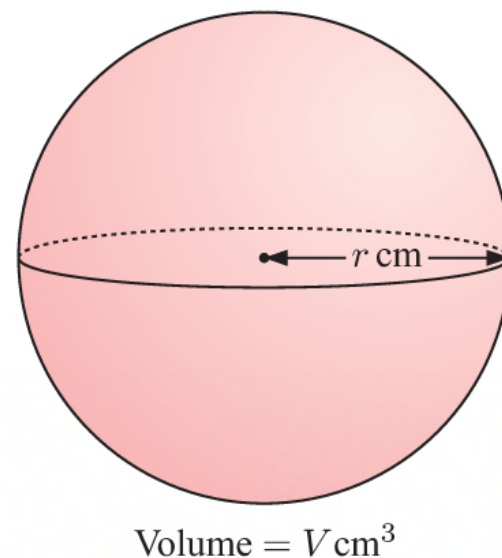
When  $r = 2$  and  $\frac{dV}{dt} = 6\pi$ ,  $6\pi = 4\pi \times 2^2 \times \frac{dr}{dt}$

$$\therefore \frac{dr}{dt} = \frac{6\pi}{16\pi}$$

$$= \frac{3}{8}$$

$= 0.375$  m per minute

$\therefore$  the balloon's radius is increasing at 0.375 m per minute.



- 6 Differentiating both sides of  $pV^{\frac{3}{2}} = 400$  with respect to  $t$ :

$$\frac{dp}{dt} V^{\frac{3}{2}} + \frac{3}{2} p V^{\frac{1}{2}} \frac{dV}{dt} = 0 \quad \{\text{product rule}\}$$

Particular case:

$$\text{When } p = 50 \text{ and } \frac{dp}{dt} = 3, \quad 50V^{\frac{3}{2}} = 400$$

$$\therefore V^{\frac{3}{2}} = 8$$

$$\therefore V = 4$$

$$\therefore 3 \times 4^{\frac{3}{2}} + \frac{3}{2} \times 50 \times 4^{\frac{1}{2}} \frac{dV}{dt} = 0$$

$$\therefore 24 + 150 \frac{dV}{dt} = 0$$

$$\therefore \frac{dV}{dt} = \frac{-24}{150} = -0.16 \text{ m}^3 \text{ per minute}$$

$\therefore$  the volume is decreasing by  $0.16 \text{ m}^3$  per minute when the pressure is  $50 \text{ N m}^{-2}$ .

- 7  $V = \frac{1}{3}\pi r^2 h$  and  $r = 3h$

$$\therefore V = \frac{1}{3}\pi(3h)^2 h = 3\pi h^3$$

Differentiating both sides of  $V = 3\pi h^3$  with respect to  $t$ :

$$\frac{dV}{dt} = 9\pi h^2 \frac{dh}{dt}$$

Particular case:

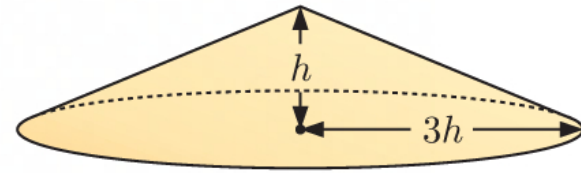
$$\text{After 1 minute, } h = 20 \text{ cm and the volume } V = 3\pi(20)^3 = 24\,000\pi \text{ cm}^3$$

$$\therefore \frac{dV}{dt} = 24\,000\pi \text{ cm}^3 \text{ per minute}$$

$$\therefore \text{when } h = 20 \text{ cm, } 24\,000\pi = 9\pi \times 20^2 \times \frac{dh}{dt}$$

$$\therefore \frac{dh}{dt} = \frac{24\,000\pi}{400 \times 9\pi} = \frac{20}{3} \text{ cm per minute}$$

$\therefore$  the height is rising at  $\frac{20}{3} \text{ cm per minute}$ .



- 8 Let  $P_1$  in the diagram be the faster jet and  $P_2$  be the slower jet. Let  $y \text{ m}$  be the distance that  $P_2$  is ahead of  $P_1$ , and  $x \text{ m}$  be the distance between them.

$$\text{Now } x^2 = y^2 + (12\,000)^2 \quad \{\text{Pythagoras}\}$$

$$\text{Differentiating with respect to } t: \quad 2x \frac{dx}{dt} = 2y \frac{dy}{dt}$$

Particular case:

As  $P_1$  is behind  $P_2$ , it is catching up at a rate of  $50 \text{ m s}^{-1}$ .

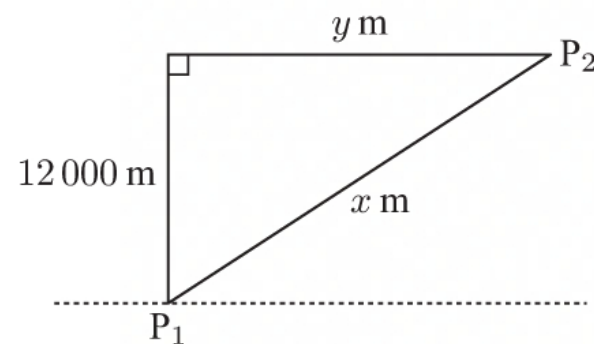
$$\therefore \frac{dy}{dt} = -50 \text{ m s}^{-1}$$

$$\text{When } y = 5000, \quad x = 13\,000 \quad \{\text{Pythagoras}\}$$

$$\therefore 26\,000 \times \frac{dx}{dt} = 10\,000 \times (-50)$$

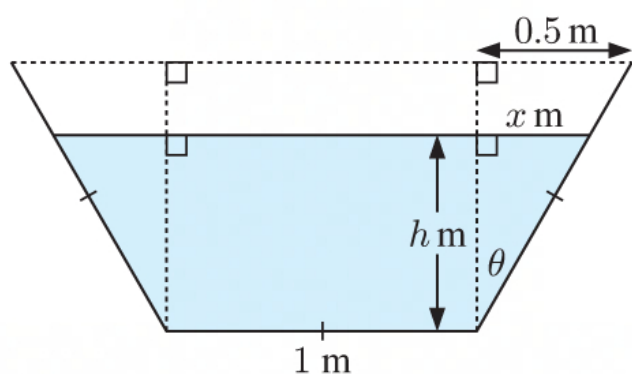
$$\frac{dx}{dt} = \frac{10}{26} \times (-50) = -\frac{250}{13} \text{ m s}^{-1}$$

$\therefore$  the distance between the jets is decreasing at  $\frac{250}{13} \approx 19.2 \text{ m s}^{-1}$ .





9



$$\sin \theta = \frac{0.5}{1} = 0.5$$

$$\therefore \theta = 30^\circ$$

Let the height of the water be  $h$  m.

$$\therefore \tan 30^\circ = \frac{x}{h}$$

$$\therefore x = h \tan 30^\circ$$

$$\therefore x = \frac{h}{\sqrt{3}}$$

$$\therefore \text{the water in the trough has volume } V = \frac{h}{2} [1 + (1 + 2x)] \times 6$$

$$\therefore V = \frac{h}{2} \left( 2 + \frac{2h}{\sqrt{3}} \right) \times 6$$

$$\therefore V = 6h + 2\sqrt{3}h^2$$

Differentiating with respect to  $t$ :

$$\frac{dV}{dt} = 6 \frac{dh}{dt} + 4\sqrt{3}h \frac{dh}{dt}$$

Particular case:

When  $h = 0.2$  and  $\frac{dV}{dt} = -0.1 \text{ m}^3 \text{ per minute}$ ,

$$-0.1 = 6 \frac{dh}{dt} + 4\sqrt{3}(0.2) \frac{dh}{dt}$$

$$\therefore \frac{dh}{dt} = \frac{-0.1}{6 + 0.8\sqrt{3}} \approx -0.0135 \text{ m per minute}$$

$\therefore$  the water level is falling at about 1.35 cm per minute when the water is 20 cm deep.

- 10** Let  $S$  m be the height of the person's shadow and  $x$  m be the person's distance from the building.

Now triangles ABC and AXY are similar.

$$\therefore \frac{AB}{AX} = \frac{BC}{XY}$$

$$\therefore \frac{40 - x}{40} = \frac{2}{S}$$

$$\therefore S = \frac{80}{40 - x} = 80(40 - x)^{-1}$$

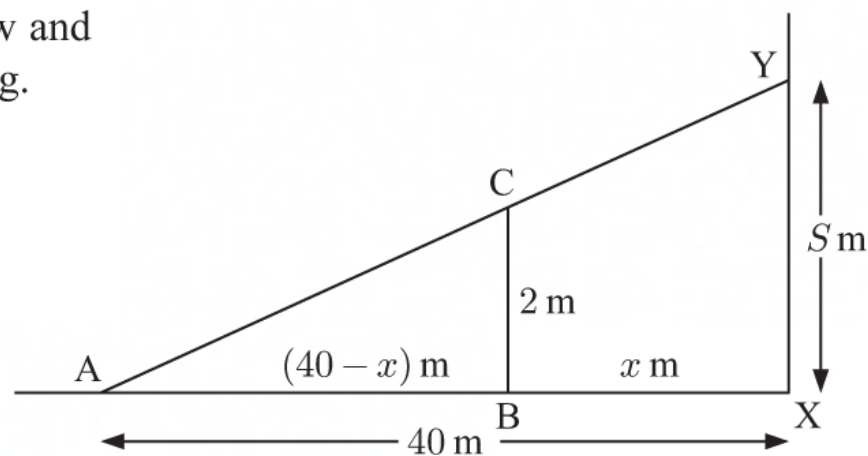
Differentiating with respect to  $t$ :

$$\begin{aligned} \frac{dS}{dt} &= -80(40 - x)^{-2}(-1) \frac{dx}{dt} \\ &= \frac{80}{(40 - x)^2} \frac{dx}{dt} \end{aligned}$$

But  $\frac{dx}{dt} = -1 \text{ m s}^{-1}$  for all values of  $t$ , so  $\frac{dS}{dt} = -\frac{80}{(40 - x)^2}$ .

**a** When  $x = 20$ ,  $\frac{dS}{dt} = -\frac{80}{(40 - 20)^2} = -\frac{80}{400} = -0.2 \text{ m s}^{-1}$

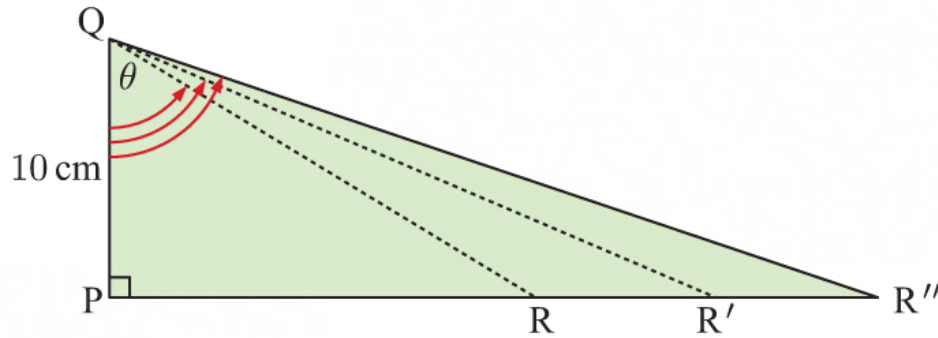
$\therefore$  the person's shadow is shortening at  $0.2 \text{ m s}^{-1}$ .



**b** When  $x = 10$ ,  $\frac{dS}{dt} = -\frac{80}{(40-10)^2} = -\frac{80}{900} = -\frac{4}{45} \text{ m s}^{-1}$

$\therefore$  the person's shadow is shortening at  $\frac{4}{45} \text{ m s}^{-1}$ .

**11 a**



Since triangle PQR is right angled and QP remains constant at 10 cm, QR will increase as  $\widehat{PQR}$  increases.

**b** Let  $QR = x \text{ cm}$  and  $\widehat{PQR} = \theta$ .

Now  $\cos \theta = \frac{10}{x} = 10x^{-1}$

Differentiating with respect to  $t$ :

$$-\sin \theta \frac{d\theta}{dt} = -10x^{-2} \frac{dx}{dt}$$

Particular case:

When  $\theta = 60^\circ$ ,  $\cos 60^\circ = \frac{10}{x}$

$$\therefore \frac{1}{2} = \frac{10}{x}$$

$$\therefore x = 20$$

Also,  $\frac{d\theta}{dt} = 2^\circ \text{ per minute}$   
 $= \frac{\pi}{90} \text{ radians per minute}$

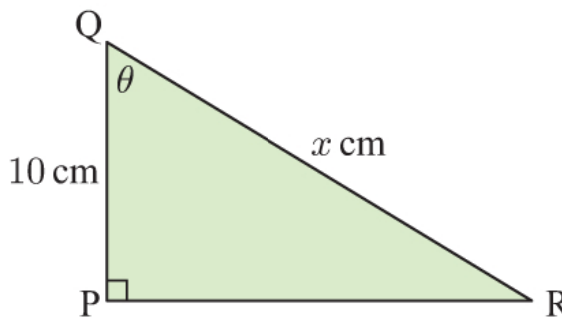
Thus  $-\sin 60^\circ \times \frac{\pi}{90} = -10 \times \frac{1}{400} \times \frac{dx}{dt}$

$$\therefore -\frac{\sqrt{3}}{2} \times \frac{\pi}{90} = -\frac{1}{40} \frac{dx}{dt}$$

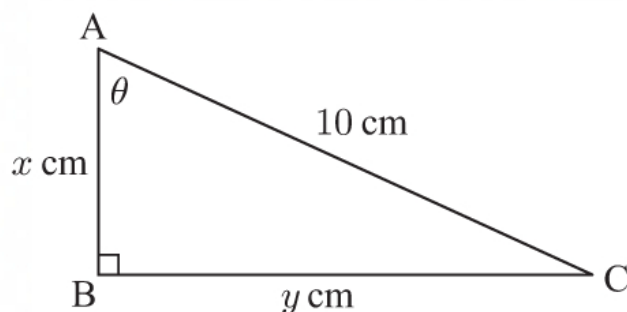
$$\therefore \frac{dx}{dt} = -\frac{\pi\sqrt{3}}{180} \times -\frac{40}{1} \text{ cm per minute}$$

$$= \frac{2\pi}{3\sqrt{3}} \approx 1.21 \text{ cm per minute}$$

$\therefore$  QR is increasing at approximately 1.21 cm per minute when  $\widehat{PQR} = 60^\circ$ .



**12**



Let  $AB = x \text{ cm}$  and  $BC = y \text{ cm}$ .

Now  $\cos \theta = \frac{x}{10}$

Differentiating with respect to  $t$ :

$$-\sin \theta \frac{d\theta}{dt} = \frac{1}{10} \frac{dx}{dt}$$

If AB increases at  $0.1 \text{ cm s}^{-1}$ ,  $\frac{dx}{dt} = 0.1 \text{ cm s}^{-1}$

Particular case:

When triangle ABC is isosceles,  $\theta = 45^\circ$

$$\therefore -\sin 45^\circ \frac{d\theta}{dt} = \frac{1}{10}(0.1)$$

$$\therefore -\frac{1}{\sqrt{2}} \frac{d\theta}{dt} = \frac{1}{100}$$

$$\therefore \frac{d\theta}{dt} = -\frac{\sqrt{2}}{100} \text{ radians per second}$$

$\therefore \widehat{CAB}$  is decreasing at  $\frac{\sqrt{2}}{100}$  radians per second.

- 13** Let the angle of elevation be  $E$ , and the horizontal distance between the observer and aeroplane be  $x$  m.

$$\text{Now } \tan E = \frac{5000}{x} = 5000x^{-1}$$

Differentiating with respect to  $t$ :

$$\sec^2 E \frac{dE}{dt} = -5000x^{-2} \frac{dx}{dt}$$

$$\text{Now } \frac{dx}{dt} = 200 \text{ m s}^{-1},$$

$$\therefore \frac{1}{\cos^2 E} \frac{dE}{dt} = -5000 \times \frac{1}{x^2} \times 200$$

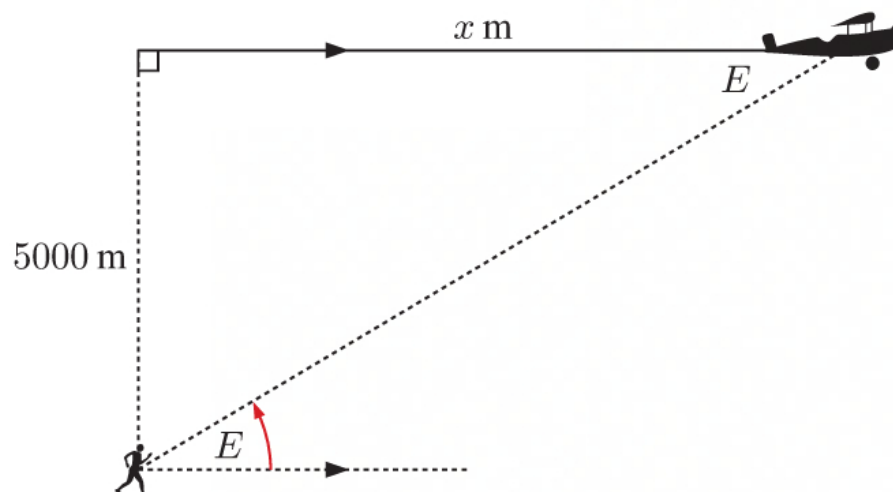
$$\therefore \frac{dE}{dt} = -1\,000\,000 \times \frac{\cos^2 E}{x^2}$$

**a** When  $E = 60^\circ$ ,  $\cos 60^\circ = \frac{1}{2}$   
and  $\tan 60^\circ = \sqrt{3} = \frac{5000}{x}$   
 $\therefore x = \frac{5000}{\sqrt{3}}$

$$\therefore \frac{dE}{dt} = -1\,000\,000 \times \frac{\left(\frac{1}{2}\right)^2}{\left(\frac{5000}{\sqrt{3}}\right)^2}$$

$$= -\frac{3}{100} \text{ radians per second}$$

$\therefore$  the angle of elevation is decreasing at  $\frac{3}{100}$  radians per second.



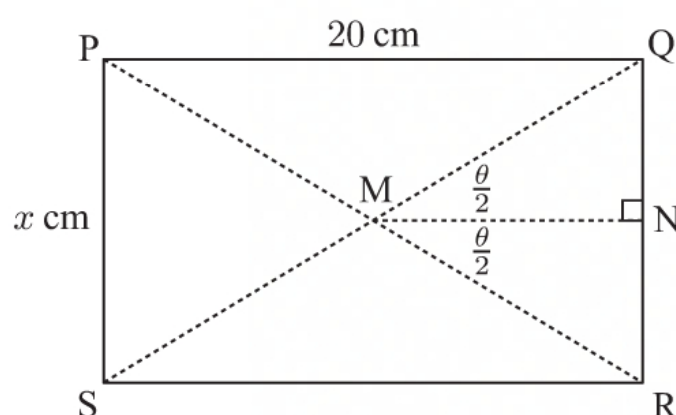
**b** When  $E = 30^\circ$ ,  $\cos 30^\circ = \frac{\sqrt{3}}{2}$   
and  $\tan 30^\circ = \frac{1}{\sqrt{3}} = \frac{5000}{x}$   
 $\therefore x = 5000\sqrt{3}$

$$\therefore \frac{dE}{dt} = -1\,000\,000 \times \frac{\left(\frac{\sqrt{3}}{2}\right)^2}{(5000\sqrt{3})^2}$$

$$= -\frac{1}{100} \text{ radians per second}$$

$\therefore$  the angle of elevation is decreasing at  $\frac{1}{100}$  radians per second.

**14**



Let N be the midpoint of [QR] in isosceles triangle QMR.

$$\therefore MN = 10 \text{ cm}$$

Let  $QR = x$  cm and let  $\widehat{QMR} = \theta$ .

$$\text{In triangle MNQ, } \tan \frac{\theta}{2} = \frac{\frac{x}{2}}{10} = \frac{x}{20}$$

Differentiating with respect to  $t$ :

$$\frac{1}{2} \sec^2\left(\frac{\theta}{2}\right) \frac{d\theta}{dt} = \frac{1}{20} \frac{dx}{dt}$$

$$\therefore \frac{d\theta}{dt} = \frac{1}{10} \cos^2\left(\frac{\theta}{2}\right) \frac{dx}{dt}$$

where  $\frac{dx}{dt} = 2 \text{ cm s}^{-1}$

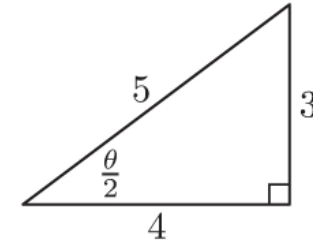
*Particular case:*

When  $x = 15 \text{ cm}$ ,  $\tan \frac{\theta}{2} = \frac{15}{20} = \frac{3}{4}$

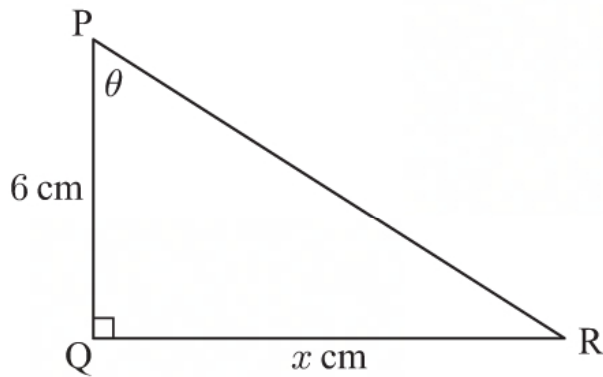
$$\therefore \cos \frac{\theta}{2} = \frac{4}{5}$$

$$\therefore \frac{d\theta}{dt} = \frac{1}{10} \left(\frac{4}{5}\right)^2 2 = 0.128 \text{ radians per second}$$

$\therefore \theta$  is increasing at 0.128 radians per second.



**15**



Let  $QR = x \text{ cm}$  and the angle at P be  $\theta$ .

Then  $\tan \theta = \frac{x}{6}$

Differentiating with respect to  $t$ :

$$\sec^2 \theta \frac{d\theta}{dt} = \frac{1}{6} \frac{dx}{dt}$$

$$\therefore \frac{d\theta}{dt} = \frac{\cos^2 \theta}{6} \frac{dx}{dt}$$

where  $\frac{dx}{dt} = 2 \text{ cm per minute}$

*Particular case:*

When  $x = 8 \text{ cm}$ ,  $PR = 10 \text{ cm}$  {Pythagoras}

Now  $\cos \theta = \frac{6}{10}$ , so  $\frac{d\theta}{dt} = \left(\frac{6}{10}\right)^2 \times \frac{1}{6} \times 2 = 0.12 \text{ radians per minute}$

$\therefore \widehat{QPR}$  is increasing at a rate of 0.12 radians per minute.

**16** Let  $x$  and  $y$  be the distances the cyclists A and B have travelled respectively at time  $t$ , and let  $z$  be the distance between them.

So,  $z^2 = x^2 + y^2 - 2xy \cos 120^\circ$  {cosine rule}

$$\therefore z^2 = x^2 + y^2 + xy \dots (1)$$

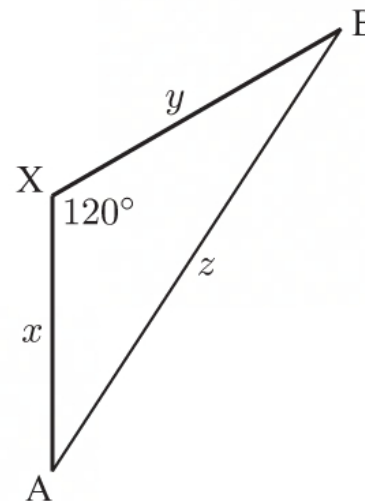
Differentiating with respect to  $t$ :

$$\therefore 2z \frac{dz}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt} + \frac{dx}{dt} y + x \frac{dy}{dt} \dots (2)$$

*Particular case:*

After 2 minutes,  $t = 120 \text{ s}$ ,  $x = 12 \times 120 = 1440 \text{ m}$ ,  $y = 16 \times 120 = 1920 \text{ m}$ ,

$\frac{dx}{dt} = 12 \text{ m s}^{-1}$ , and  $\frac{dy}{dt} = 16 \text{ m s}^{-1}$ .





Using (1),  $z^2 = 1440^2 + 1920^2 + 1440 \times 1920 = 8\,524\,800$

$$\therefore z = \sqrt{8\,524\,800} = 480\sqrt{37} \quad \{z \geq 0\}$$

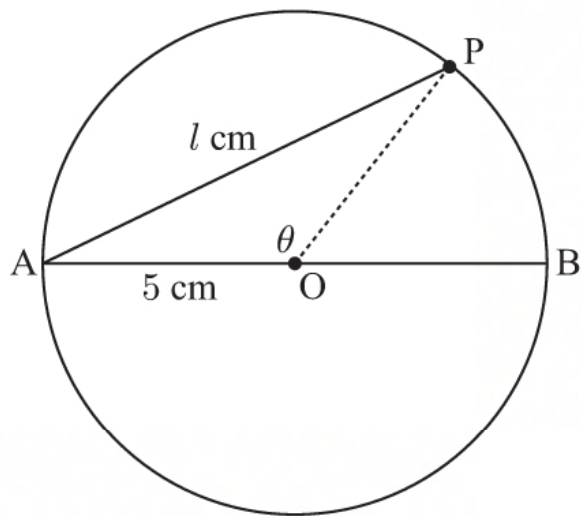
$$\therefore 2(480\sqrt{37}) \frac{dz}{dt} = 2880(12) + 3840(16) + (12)1920 + 1440(16) \quad \{\text{using (2)}\}$$

$$\therefore 960\sqrt{37} \frac{dz}{dt} = 142\,080$$

$$\therefore \frac{dz}{dt} = \frac{148}{\sqrt{37}} = 4\sqrt{37} \approx 24.3 \text{ m s}^{-1}$$

$\therefore$  the distance between the cyclists is increasing at approximately  $24.3 \text{ m s}^{-1}$ .

17



Let  $AP = l \text{ cm}$  and let  $\widehat{AOP} = \theta$

$$\therefore l^2 = 5^2 + 5^2 - 2 \times 5 \times 5 \cos \theta \quad \{\text{cosine rule}\}$$

$$\therefore l^2 = 50 - 50 \cos \theta$$

Differentiating with respect to  $t$ :

$$2l \frac{dl}{dt} = 50 \sin \theta \frac{d\theta}{dt}$$

$$\therefore \frac{dl}{dt} = \frac{25 \sin \theta}{l} \frac{d\theta}{dt}$$

Now the point moves at one revolution every 10 seconds.

$$\therefore \frac{d\theta}{dt} = \frac{2\pi}{10} = \frac{\pi}{5} \text{ radians per second}$$

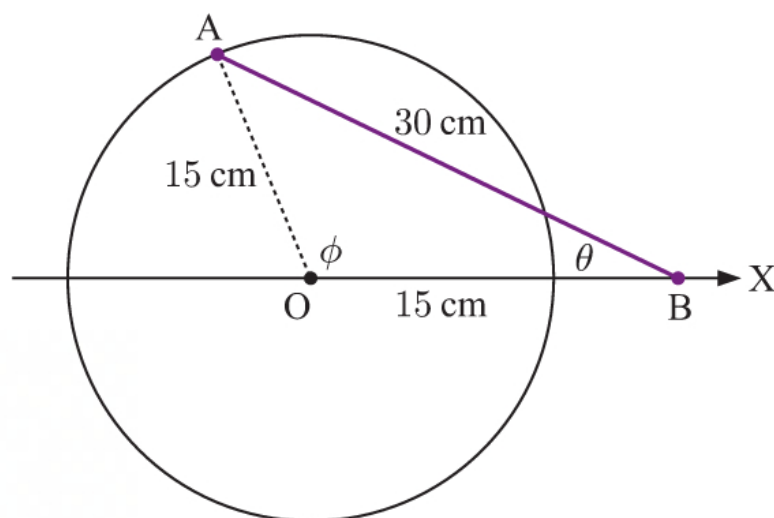
**a** If  $AP = l = 5 \text{ cm}$ ,  $\frac{dl}{dt} > 0$ ,  
then  $\theta = \frac{\pi}{3}$   $\{\triangle APO \text{ is equilateral}\}$

$$\begin{aligned} \therefore \frac{dl}{dt} &= \frac{25 \sin \frac{\pi}{3}}{5} \times \frac{\pi}{5} \\ &= \frac{\sqrt{3}}{2} \pi \text{ cm s}^{-1} \end{aligned}$$

**b** If P is at B, then  $l = 10 \text{ cm}$   
and  $\theta = \pi$

$$\begin{aligned} \therefore \frac{dl}{dt} &= \frac{25 \sin \pi}{10} \times \frac{\pi}{5} \\ &= 0 \text{ cm s}^{-1} \end{aligned}$$

18



Let  $\widehat{AOB} = \phi$  and  $\widehat{ABO} = \theta$

Now  $\frac{d\phi}{dt} = -100$  revolutions per second  
 $\{\text{negative for clockwise rotation}\}$

$$\therefore \frac{d\phi}{dt} = -200\pi \text{ radians per second}$$

$$\text{Also, } \frac{30}{\sin \phi} = \frac{15}{\sin \theta} \quad \{\text{sine rule}\}$$

$$\therefore \sin \phi = 2 \sin \theta$$

Differentiating with respect to  $t$ :

$$\cos \phi \frac{d\phi}{dt} = 2 \cos \theta \frac{d\theta}{dt}$$

$$\therefore \frac{d\theta}{dt} = \frac{1}{2} \frac{\cos \phi}{\cos \theta} \frac{d\phi}{dt}$$

**a** When  $\widehat{AOX} = 120^\circ$ ,  $\phi = \frac{2\pi}{3}$

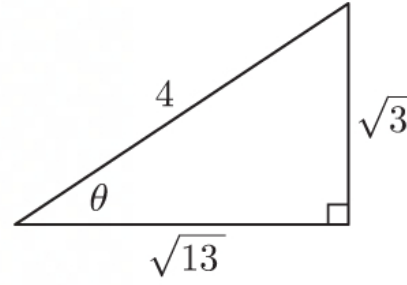
$$\therefore \cos \phi = -\frac{1}{2} \quad \text{and} \quad \sin \phi = \frac{\sqrt{3}}{2}$$

$$\begin{aligned} \therefore \sin \theta &= \frac{1}{2} \sin \phi \\ &= \frac{1}{2} \left( \frac{\sqrt{3}}{2} \right) \\ &= \frac{\sqrt{3}}{4} \end{aligned}$$

$$\therefore \cos \theta = \frac{\sqrt{13}}{4}$$

$$\therefore \frac{d\theta}{dt} = \frac{1}{2} \times \frac{-\frac{1}{2}}{\frac{\sqrt{13}}{4}} \times (-200\pi) = \frac{200\pi}{\sqrt{13}} \text{ radians per second}$$

$\therefore \widehat{ABO}$  is increasing at  $\frac{200\pi}{\sqrt{13}}$  radians per second.



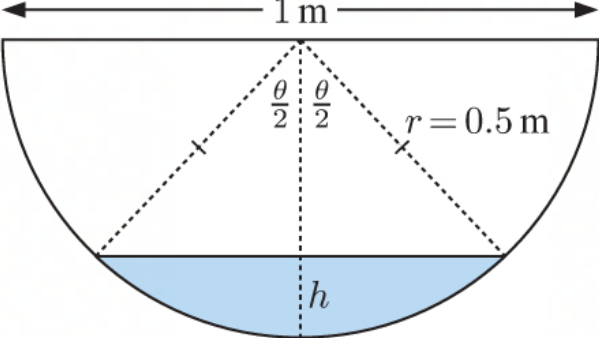
**b** When  $\widehat{AOX} = 180^\circ$ ,  $\phi = \pi$

$$\therefore \cos \phi = -1 \quad \text{and} \quad \sin \phi = 0$$

$$\therefore \sin \theta = 0 \quad \text{and} \quad \cos \theta = 1$$

$$\begin{aligned} \therefore \frac{d\theta}{dt} &= \frac{1}{2} \times \frac{-1}{1} \times (-200\pi) \\ &= 100\pi \text{ radians per second} \end{aligned}$$

$\therefore \widehat{ABO}$  is increasing at  $100\pi$  radians per second.

**19**  Let  $h$  be the depth and  $V$  be the volume of water in the trough at time  $t$ .

**a** The cross-sectional area of water in the trough = area of sector – area of triangle

$$= \frac{1}{2}r^2\theta - \frac{1}{2}r^2 \sin \theta$$

$$= \frac{1}{2}r^2(\theta - \sin \theta)$$

$$= \frac{1}{8}(\theta - \sin \theta)$$

$$\therefore V = \text{area of water} \times \text{length of trough}$$

$$\therefore V = \frac{1}{8}(\theta - \sin \theta) \times 8$$

$$\therefore V = \theta - \sin \theta$$

**b** Differentiating the equation in **a** with respect to  $t$ :

$$\frac{dV}{dt} = \frac{d\theta}{dt} - \cos \theta \frac{d\theta}{dt}$$

$$\therefore \frac{dV}{dt} = \frac{d\theta}{dt} (1 - \cos \theta)$$

$$\text{But } \frac{dV}{dt} = 0.1 \text{ m}^3 \text{ min}^{-1}$$

$$\therefore \frac{d\theta}{dt} = \frac{0.1}{1 - \cos \theta} \quad \dots (1)$$

$$\text{Also, } \cos \frac{\theta}{2} = \frac{\frac{1}{2} - h}{\frac{1}{2}} = 1 - 2h$$

Differentiating this equation with respect to  $t$ :

$$\begin{aligned} -\sin \frac{\theta}{2} \times \frac{1}{2} \frac{d\theta}{dt} &= -2 \frac{dh}{dt} \\ \therefore \frac{dh}{dt} &= \frac{1}{4} \sin \frac{\theta}{2} \frac{d\theta}{dt} \quad \dots (2) \end{aligned}$$

*Particular case:*

$$\text{When } h = 0.25 \text{ m, } \cos \frac{\theta}{2} = \frac{\frac{1}{2} - \frac{1}{4}}{\frac{1}{2}} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$$

$$\therefore \sin \frac{\theta}{2} = \frac{\sqrt{3}}{2} \quad \text{and} \quad \cos \theta = 2 \cos^2 \left( \frac{\theta}{2} \right) - 1 = 2 \left( \frac{1}{2} \right)^2 - 1 = -\frac{1}{2}$$

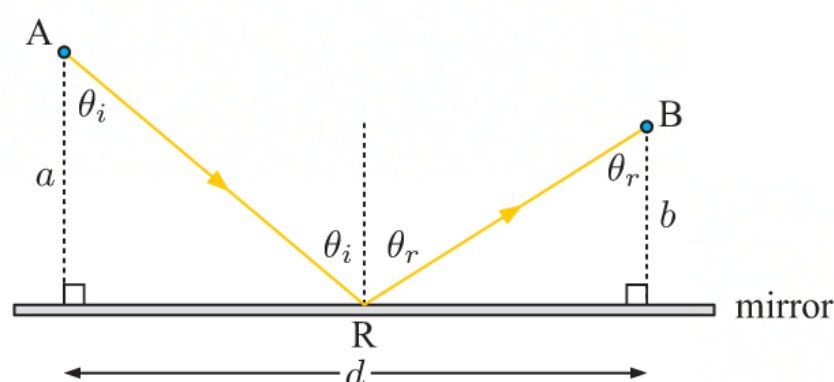
$$\text{Using (1), } \frac{d\theta}{dt} = \frac{0.1}{1 - (-\frac{1}{2})} = \frac{1}{15} \text{ radians per minute}$$

$\therefore \theta$  is increasing at  $\frac{1}{15}$  radians per minute.

$$\begin{aligned} \text{Using (2), } \frac{dh}{dt} &= \frac{1}{4} \sin \frac{\theta}{2} \frac{d\theta}{dt} \\ &= \frac{1}{4} \times \frac{\sqrt{3}}{2} \times \frac{1}{15} \\ &= \frac{\sqrt{3}}{120} \text{ metres per minute} \end{aligned}$$

$\therefore h$  is increasing at  $\frac{\sqrt{3}}{120}$  metres per minute.

**20 a**



Suppose a light is turned on at point A and its reflection in the mirror is observed at point B. Let the point of reflection be R, and label distances  $a$ ,  $b$ , and  $d$  as shown.

$$\text{Now } d = a \tan \theta_i + b \tan \theta_r$$

$$\therefore a \tan \theta_i = d - b \tan \theta_r$$

$$\therefore a \sec^2 \theta_i = -b \sec^2 \theta_r \frac{d\theta_r}{d\theta_i} \quad \{\text{implicit differentiation with respect to } \theta_i\}$$

$$\therefore \frac{d\theta_r}{d\theta_i} = -\frac{a \cos^2 \theta_r}{b \cos^2 \theta_i}$$

The total distance travelled by the light is  $D = AR + RB$

$$= \frac{a}{\cos \theta_i} + \frac{b}{\cos \theta_r}$$

Since the light travels at constant speed, we only need to minimise  $D$ .

$$\frac{dD}{d\theta_i} = \frac{-a(-\sin \theta_i)}{\cos^2 \theta_i} + \frac{-b(-\sin \theta_r)}{\cos^2 \theta_r} \frac{d\theta_r}{d\theta_i} \quad \{\text{chain rule}\}$$

$$= \frac{a \sin \theta_i}{\cos^2 \theta_i} + \frac{\cancel{b} \sin \theta_r}{\cancel{\cos^2 \theta_r}} \times \frac{-a \cancel{\cos^2 \theta_r}}{\cancel{b} \cos^2 \theta_i}$$

$$= \frac{a}{\cos^2 \theta_i} (\sin \theta_i - \sin \theta_r)$$

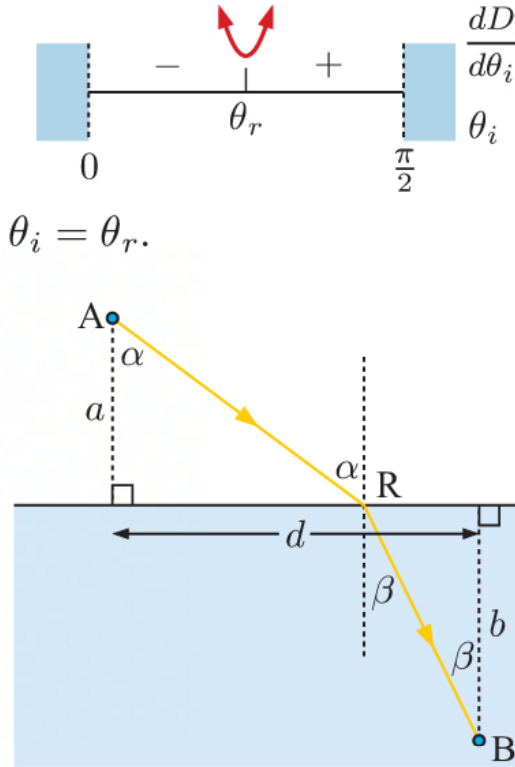
$$\therefore \frac{dD}{d\theta_i} = 0 \quad \text{only when} \quad \sin \theta_i = \sin \theta_r$$

Since  $\theta_i$  and  $\theta_r$  are both acute, we require  $\theta_i = \theta_r$ .

- b** Suppose a light is turned on at point A. A ray is refracted at point R, and is observed at point B. We label distances  $a$ ,  $b$ , and  $d$  as shown.

Now  $d = a \tan \alpha + b \tan \beta$

$$\therefore \frac{d\beta}{d\alpha} = -\frac{a \cos^2 \beta}{b \cos^2 \alpha} \quad \{\text{using a}\}$$



The light travels distance  $AR = \frac{a}{\cos \alpha}$  at speed  $\frac{c}{n_a}$ ,

then distance  $BR = \frac{b}{\cos \beta}$  at speed  $\frac{c}{n_b}$ .

The total time taken is  $T = \frac{\frac{a}{\cos \alpha}}{\frac{c}{n_a}} + \frac{\frac{b}{\cos \beta}}{\frac{c}{n_b}}$

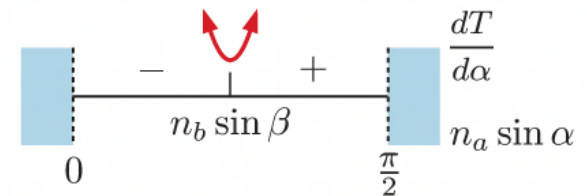
$$= \frac{1}{c} \left( \frac{an_a}{\cos \alpha} + \frac{bn_b}{\cos \beta} \right)$$

$$\therefore \frac{dT}{d\alpha} = \frac{1}{c} \left( \frac{-an_a(-\sin \alpha)}{\cos^2 \alpha} + \frac{-bn_b(-\sin \beta)}{\cos^2 \beta} \frac{d\beta}{d\alpha} \right) \quad \{\text{chain rule}\}$$

$$= \frac{1}{c} \left( \frac{an_a \sin \alpha}{\cos^2 \alpha} + \frac{\cancel{b} n_b \sin \beta}{\cancel{\cos^2 \beta}} \times \frac{-a \cancel{\cos^2 \beta}}{\cancel{b} \cos^2 \alpha} \right)$$

$$= \frac{a}{c \cos^2 \alpha} (n_a \sin \alpha - n_b \sin \beta)$$

$$\therefore \frac{dT}{d\alpha} = 0 \quad \text{only when} \quad n_a \sin \alpha = n_b \sin \beta$$





## ACTIVITY

## CUBIC SPLINES

$$\begin{aligned}
 \mathbf{2} \quad \mathbf{a} \quad \mathbf{i} \quad C_i(x) &= a_i(x - x_i)^3 + b_i(x - x_i)^2 + c_i(x - x_i) + d_i \\
 \therefore C_i'(x) &= a_i \times 3(x - x_i)^2(1) + b_i \times 2(x - x_i)(1) + c_i(1) \quad \{\text{chain rule}\} \\
 \therefore C_i'(x) &= 3a_i(x - x_i)^2 + 2b_i(x - x_i) + c_i
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{ii} \quad C_i''(x) &= 3a_i \times 2(x - x_i)(1) + 2b_i(1) \quad \{\text{chain rule}\} \\
 \therefore C_i''(x) &= 6a_i(x - x_i) + 2b_i
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad C_i(x_i) &= a_i(x_i - x_i)^3 + b_i(x_i - x_i)^2 + c_i(x_i - x_i) + d_i \\
 &= d_i \\
 C_i'(x_i) &= 3a_i(x_i - x_i)^2 + 2b_i(x_i - x_i) + c_i \\
 &= c_i \\
 C_i''(x_i) &= 6a_i(x_i - x_i) + 2b_i \\
 &= 2b_i
 \end{aligned}$$

- $\mathbf{3}$
- $C_i(x_i) = C_{i-1}(x_i)$  ensures that as we transition from the  $(i-1)$ th cubic to the  $i$ th cubic at  $x_i$ , the curve is continuous. This requirement gives us the data point at the left end of  $C_i(x)$ .
  - $C_i'(x_i) = C_{i-1}'(x_i)$  ensures that as we transition from the  $(i-1)$ th cubic to the  $i$ th cubic at  $x_i$ , the gradients are the same. This requirement gives us the gradient at the left end of  $C_i(x)$ .
  - $C_i''(x_i) = C_{i-1}''(x_i)$  ensures that as we transition from the  $(i-1)$ th cubic to the  $i$ th cubic at  $x_i$ , the cubics have the same shape. This requirement gives us the curvature at the left end of  $C_i(x)$ .

$$\begin{aligned}
 \mathbf{4} \quad \mathbf{a} \quad \mathbf{i} \quad Q(x) &= -x^2 + 8x + 4 \\
 Q(0) &= 4 \quad \checkmark \\
 Q(2) &= -(2)^2 + 8(2) + 4 \\
 &= -4 + 16 + 4 \\
 &= 16 \quad \checkmark \\
 Q(4) &= -(4)^2 + 8(4) + 4 \\
 &= -16 + 32 + 4 \\
 &= 20 \quad \checkmark \\
 \therefore (0, 4), (2, 16), \text{ and } (4, 20) &\text{ all lie} \\
 &\text{on } Q(x) = -x^2 + 8x + 4.
 \end{aligned}$$

$$\mathbf{ii} \quad C_0''(x_0) = 2b_0 \quad \{\text{from } \mathbf{2} \mathbf{b}\}$$

$$\text{Now } Q(x) = -x^2 + 8x + 4$$

$$\therefore Q'(x) = -2x + 8$$

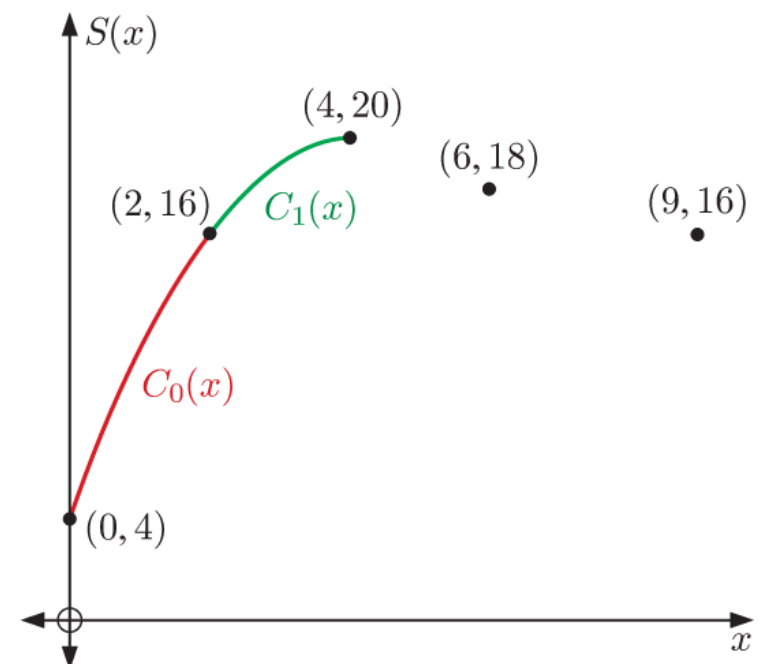
$$\therefore Q''(x) = -2$$

$$\therefore Q''(x_0) = -2$$

$$\text{If } C_0''(x_0) = Q''(x_0)$$

$$\text{then } 2b_0 = -2$$

$$\therefore b_0 = -1$$



$$\text{iii } C_0'(x_0) = c_0 \quad \{\text{from 2 b}\}$$

$$Q'(x) = -2x + 8$$

$$\begin{aligned} \therefore Q'(x_0) &= -2x_0 + 8 \\ &= 8 \quad \{x_0 = 0\} \end{aligned}$$

$$\text{If } C_0'(x_0) = Q'(x_0)$$

$$\text{then } c_0 = 8$$

$$\text{iv } C_0(x_0) = d_0 \quad \{\text{from 2 b}\}$$

$$\text{If } C_0(x_0) = y_0$$

$$\text{then } d_0 = 4 \quad \{y_0 = 4\}$$

$$\begin{aligned} \text{v } C_0(x) &= a_0(x - x_0)^3 + b_0(x - x_0)^2 + c_0(x - x_0) + d_0 \\ \therefore C_0(x_1) &= a_0(x_1 - x_0)^3 + b_0(x_1 - x_0)^2 + c_0(x_1 - x_0) + d_0 \\ &= a_0(2 - 0)^3 + b_0(2 - 0)^2 + 8(2 - 0) + 4 \quad \{x_1 = 2\} \\ &= 8a_0 - 4 + 16 + 4 \\ &= 8a_0 + 16 \end{aligned}$$

$$\text{If } C_0(x_1) = y_1$$

$$\text{then } 8a_0 + 16 = 16 \quad \{y_1 = 16\}$$

$$\therefore 8a_0 = 0$$

$$\therefore a_0 = 0$$

$$\text{So, } C_0(x) = -x^2 + 8x + 4, \quad 0 \leq x \leq 2$$

$$\text{b i } C_1''(x_1) = 2b_1 \quad \{\text{from 2 b}\}$$

$$C_0(x) = -x^2 + 8x + 4$$

$$\therefore C_0'(x) = -2x + 8$$

$$\therefore C_0''(x) = -2$$

$$\therefore C_0''(x_1) = -2$$

$$\text{If } C_1''(x_1) = C_0''(x_1)$$

$$\text{then } 2b_1 = -2$$

$$\therefore b_1 = -1$$

$$\text{ii } C_1'(x_1) = c_1 \quad \{\text{from 2 b}\}$$

$$C_0'(x) = -2x + 8$$

$$\begin{aligned} \therefore C_0'(x_1) &= -2x_1 + 8 \\ &= -2(2) + 8 \quad \{x_1 = 2\} \\ &= -4 + 8 \\ &= 4 \end{aligned}$$

$$\text{If } C_1'(x_1) = C_0'(x_1)$$

$$\text{then } c_1 = 4$$

$$\text{iii } C_1(x_1) = d_1 \quad \{\text{from 2 b}\}$$

$$\text{If } C_1(x_1) = y_1$$

$$\text{then } d_1 = 16 \quad \{y_1 = 16\}$$

$$\begin{aligned}
 \text{iv} \quad C_1(x) &= a_1(x - x_1)^3 + b_1(x - x_1)^2 + c_1(x - x_1) + d_1 \\
 \therefore C_1(x_2) &= a_1(x_2 - x_1)^3 + b_1(x_2 - x_1)^2 + c_1(x_2 - x_1) + d_1 \\
 &= a_1(4 - 2)^3 - (4 - 2)^2 + 4(4 - 2) + 16 \quad \{x_2 = 4\} \\
 &= 8a_1 - 4 + 8 + 16 \\
 &= 8a_1 + 20
 \end{aligned}$$

$$\begin{aligned}
 \text{If } C_1(x_2) &= y_2 \\
 \text{then } 8a_1 + 20 &= 20 \quad \{y_2 = 20\} \\
 \therefore 8a_1 &= 0 \\
 \therefore a_1 &= 0
 \end{aligned}$$

$$\text{So, } C_1(x) = -(x - 2)^2 + 4(x - 2) + 16, \quad 2 < x \leq 4$$

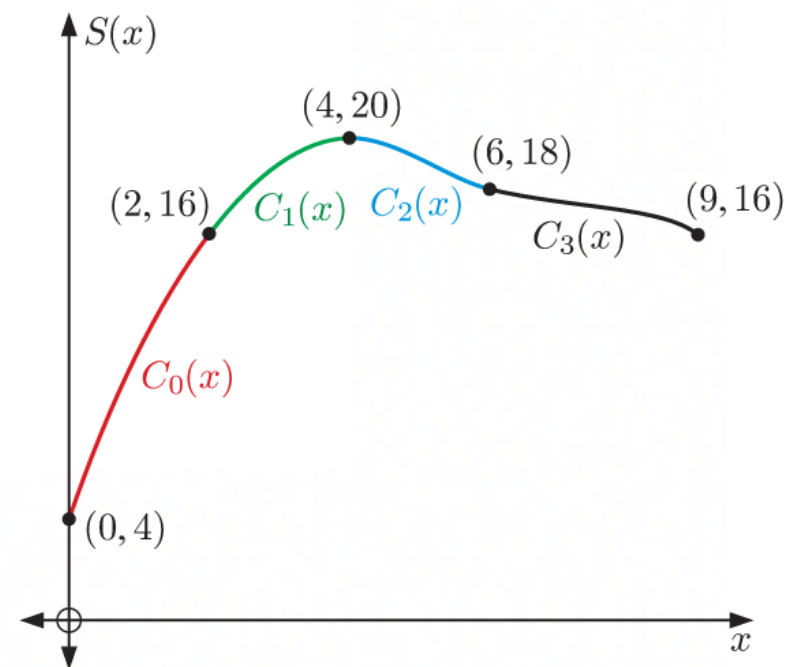
$$\begin{aligned}
 \text{c} \quad \text{i} \quad \text{From the spreadsheet, } a_2 &= \frac{1}{4}, \quad b_2 = -1, \quad c_2 = 0, \quad d_2 = 20 \\
 a_3 &= -\frac{7}{54}, \quad b_3 = \frac{1}{2}, \quad c_3 = -1, \quad d_3 = 18
 \end{aligned}$$

$$\text{Now } x_2 = 4, \quad x_3 = 6$$

$$\therefore C_2(x) = \frac{1}{4}(x - 4)^3 - (x - 4)^2 + 20,$$

$$C_3(x) = -\frac{7}{54}(x - 6)^3 + \frac{1}{2}(x - 6)^2 - (x - 6) + 18$$

iii The cubic spline passes through each of the data points, and appears to be a good representation of the data.



5 a The data points  $(0, 1)$ ,  $(2, 7.39)$ ,  $(4, 54.6)$ ,  $(6, 403.4)$ ,  $(8, 2981)$ , and  $(10, 22\,026)$  are of the form  $(x, e^x)$  where  $x = 0, 2, 4, 6, 8, 10$ .

$x$	1	3.5	4.25	5.25	7.5	9
$e^x$	2.718	33.135	70.105	190.566	1808.042	8103.084
$S(x)$	-0.907	38.971	63.881	168.500	1895.985	7724.231

b For the  $x$ -values 3.5, 4.25, and 7.5, which are close to our original  $x$ -values,  $S(x)$  approximates  $e^x$  reasonably well. For the  $x$ -values 1, 5.25, and 9, which are not as close to our original  $x$ -values,  $S(x)$  is a less accurate approximation of  $e^x$ .

c In a,  $S(x)$  predicts that  $e^1 = -0.907$ , which we know is absurd since  $e^x > 0$  for all  $x$ . If we add more data values, such as  $(1, 2.718)$ , to the spreadsheet, the accuracy of the approximation  $S(x)$  is greatly improved.



## REVIEW SET 19A

**1**  $H(t) = 60 + 40 \ln(2t + 1)$  cm,  $t \geq 0$

**a**  $H(0) = 60 + 40 \ln(2(0) + 1)$   
 $= 60$

$\therefore$  the tree was 60 cm tall when it was planted.

**b i** When  $H = 150$ ,  
 $60 + 40 \ln(2t + 1) = 150$

$$\therefore 40 \ln(2t + 1) = 90$$

$$\therefore \ln(2t + 1) = \frac{9}{4}$$

$$\therefore 2t + 1 = e^{\frac{9}{4}}$$

$$\therefore 2t = e^{\frac{9}{4}} - 1$$

$$\therefore t = \frac{e^{\frac{9}{4}} - 1}{2} \approx 4.24$$

$\therefore$  it will take about 4.24 years for the tree to reach a height of 150 cm.

**c**  $H(t) = 60 + 40 \ln(2t + 1)$  cm,  $t \geq 0$

$$\therefore H'(t) = 40 \left( \frac{2}{2t + 1} \right)$$

$$= \frac{80}{2t + 1} \text{ cm per year}$$

**i**  $H'(2) = \frac{80}{2(2) + 1} = 16$

$\therefore$  after 2 years, the tree's height is increasing at a rate of 16 cm per year.

**ii**  $H'(20) = \frac{80}{2(20) + 1}$   
 $= \frac{80}{41} \approx 1.95$

$\therefore$  after 20 years, the tree's height is increasing at a rate of about 1.95 cm per year.

**2 a**  $V = 20\,000e^{-0.4t}$  pounds

When  $t = 0$ ,  $V = 20\,000e^{-0.4(0)}$   
 $= 20\,000$

$\therefore$  the purchase price of the car is £20 000.

**b**  $V = 20\,000e^{-0.4t}$  pounds

$$\therefore \frac{dV}{dt} = 20\,000e^{-0.4t}(-0.4)$$

$$= -8000e^{-0.4t} \text{ pounds per year}$$

When  $t = 10$ ,  $\frac{dV}{dt} = -8000e^{-0.4(10)}$   
 $= -8000e^{-4}$   
 $\approx -146.53$

$\therefore$  after 10 years, the value of the car is decreasing at about £146.53 per year.

**ii** When  $H = 300$ ,

$$60 + 40 \ln(2t + 1) = 300$$

$$\therefore 40 \ln(2t + 1) = 240$$

$$\therefore \ln(2t + 1) = 6$$

$$\therefore 2t + 1 = e^6$$

$$\therefore 2t = e^6 - 1$$

$$\therefore t = \frac{e^6 - 1}{2} \approx 201$$

$\therefore$  it will take about 201 years for the tree to reach a height of 300 cm.



**3 a**  $C(v) = \frac{v^2}{20} + \frac{50\,000}{v}$  dollars

$$\begin{aligned}\therefore C(64) &= \frac{64^2}{20} + \frac{50\,000}{64} \\ &= 204.8 + 781.25 \\ &= 986.05\end{aligned}$$

$\therefore$  the cost of running the train for 1 hour at  $64 \text{ km h}^{-1}$  is \$986.05

$\therefore$  the cost of running the train for 5 hours at  $64 \text{ km h}^{-1}$  is \$4930.25.

**b**  $C(v) = \frac{v^2}{20} + \frac{50\,000}{v}$  euros,  $v > 0$

$$= \frac{1}{20}v^2 + 50\,000v^{-1}$$

$$\therefore C'(v) = \frac{1}{10}v - 50\,000v^{-2}$$

$$= \frac{1}{10}v - \frac{50\,000}{v^2}$$

**i**  $C'(75) = \frac{1}{10}(75) - \frac{50\,000}{75^2}$

$$\begin{aligned}&= \frac{75}{10} - \frac{80}{9} \\ &= -\frac{25}{18} \\ &\approx -1.39\end{aligned}$$

$\therefore$  if the average speed is  $75 \text{ km h}^{-1}$ , the rate of change in the cost of running the train is decreasing at about \$1.39 per  $\text{km h}^{-1}$ .

**ii**  $C'(90) = \frac{1}{10}(90) - \frac{50\,000}{90^2}$

$$\begin{aligned}&= 9 - \frac{500}{81} \\ &= \frac{229}{81} \\ &\approx 2.83\end{aligned}$$

$\therefore$  if the average speed is  $90 \text{ km h}^{-1}$ , the rate of change in the cost of running the train is increasing at about \$2.83 per  $\text{km h}^{-1}$ .

**c**  $C(v)$  is a minimum when  $C'(v) = 0$

$$\therefore \frac{1}{10}v - \frac{50\,000}{v^2} = 0$$

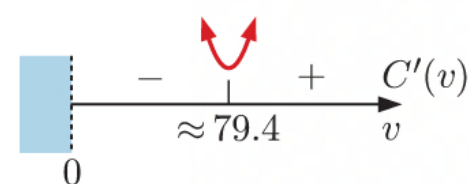
$$\therefore \frac{1}{10}v^3 - 50\,000 = 0$$

$$\therefore \frac{1}{10}v^3 = 50\,000$$

$$\therefore v^3 = 500\,000$$

$$\therefore v \approx 79.4 \text{ km h}^{-1}$$

$C'(v)$  has sign diagram:

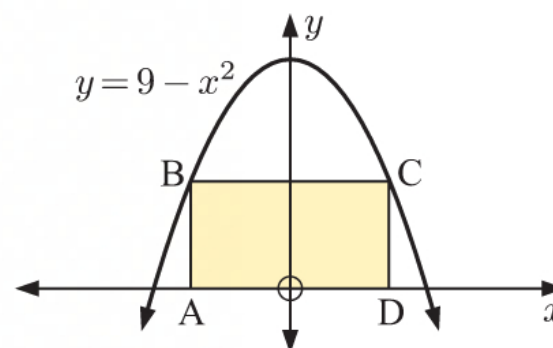


$\therefore$  the cost of running the train is a minimum when the average speed of the train is about  $79.4 \text{ km h}^{-1}$ .

**4 a** Let  $OD = x$ , so C has coordinates  $(x, 9 - x^2)$ .

Area of rectangle ABCD = length  $\times$  width

$$\begin{aligned}\therefore A(x) &= 2x \times (9 - x^2) \\ &= 18x - 2x^3\end{aligned}$$



$$\begin{aligned}
 \text{b} \quad A(x) &= 18x - 2x^3 \\
 \therefore A'(x) &= 18 - 6x^2 \\
 A'(x) = 0 \quad &\text{when} \quad 18 - 6x^2 = 0 \\
 &\therefore 6x^2 = 18 \\
 &\therefore x^2 = 3 \\
 &\therefore x = \sqrt{3} \quad \{x > 0\}
 \end{aligned}$$

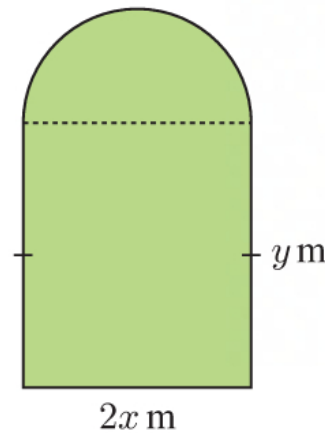
which has sign diagram:

So, the area is a maximum when  $x = \sqrt{3}$ .

When  $x = \sqrt{3}$ ,  $y = 9 - (\sqrt{3})^2 = 6$

So, C has coordinates  $(\sqrt{3}, 6)$ .

$$\begin{aligned}
 \text{5 a} \quad \text{perimeter} &= 2x + 2y + \pi x \\
 \therefore 200 &= 2x + 2y + \pi x \\
 \therefore 2y &= 200 - 2x - \pi x \\
 \therefore y &= 100 - x - \frac{\pi}{2}x
 \end{aligned}$$



$$\begin{aligned}
 \text{b} \quad \text{area of lawn } A &= \text{area of rectangle} + \text{area of semi-circle} \\
 &= 2x \times y + \frac{1}{2}\pi x^2 \\
 &= 2x(100 - x - \frac{\pi}{2}x) + \frac{\pi}{2}x^2 \quad \{\text{using a}\} \\
 &= 200x - 2x^2 - \pi x^2 + \frac{\pi}{2}x^2 \\
 \therefore A &= 200x - 2x^2 - \frac{\pi}{2}x^2 \text{ m}^2
 \end{aligned}$$

$$\text{c} \quad \frac{dA}{dx} = 200 - 4x - \pi x$$

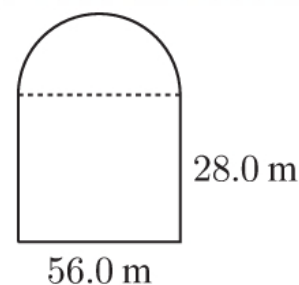
$$\begin{aligned}
 \text{Now } \frac{dA}{dx} = 0 \quad &\text{when} \quad 200 - 4x - \pi x = 0 \\
 &\therefore 4x + \pi x = 200 \\
 &\therefore x(4 + \pi) = 200 \\
 &\therefore x = \frac{200}{4 + \pi} \\
 &\therefore x \approx 28.0
 \end{aligned}$$

$\frac{dA}{dx}$  has sign diagram:

The area of the lawn is maximised when  $x = \frac{200}{4 + \pi} \approx 28.0$

$$\text{and } y = 100 - \frac{200}{4 + \pi} - \frac{\pi}{2} \left( \frac{200}{4 + \pi} \right) \approx 28.0$$

The dimensions of the lawn of maximum area are:

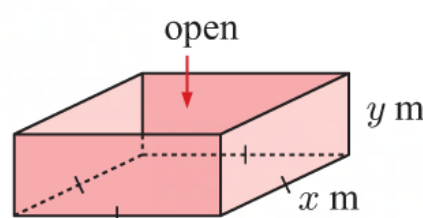


**6 a** capacity = 1 kL  $\equiv$  1 m<sup>3</sup>

volume of box = area of base  $\times$  height

$$\therefore 1 = x^2 y$$

$$\therefore y = \frac{1}{x^2}, \quad x > 0$$



**b** area of steel needed = area of base + area of 4 sides

$$= x^2 + 4xy$$

$$= x^2 + 4x \left( \frac{1}{x^2} \right) \quad \{\text{from a}\}$$

$$= x^2 + \frac{4}{x}$$

Steel costs £2 per m<sup>2</sup>, so total cost of steel =  $\left( x^2 + \frac{4}{x} \right) \times 2$

$$\therefore C(x) = 2x^2 + \frac{8}{x} \text{ pounds}$$

**c**  $C(x) = 2x^2 + \frac{8}{x} = 2x^2 + 8x^{-1}$

$$\therefore C'(x) = 4x - 8x^{-2} = 4x - \frac{8}{x^2}$$

$$C'(x) = 0 \text{ when } 4x - \frac{8}{x^2} = 0$$

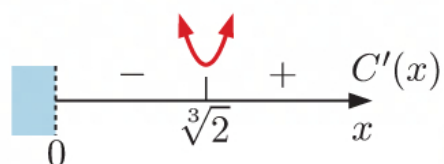
$$\therefore 4x = \frac{8}{x^2}$$

$$\therefore 4x^3 = 8$$

$$\therefore x^3 = 2$$

$$\therefore x = \sqrt[3]{2}$$

$C'(x)$  has sign diagram:



So, the cost is a minimum when  $x = \sqrt[3]{2} \approx 1.26$

$$\text{When } x = \sqrt[3]{2}, \quad y = \frac{1}{(\sqrt[3]{2})^2} \approx 0.630$$

So, the box which would cost the least to make would have square base with sides about 1.26 m and height about 0.630 m.

**7 a**  $D = 9.3 + 6.8 \cos(0.507t) \text{ m}$   
 $\therefore \frac{dD}{dt} = 6.8(-\sin(0.507t))(0.507)$   
 $= -3.4476 \sin(0.507t)$

This tells us the rate at which the depth of water is increasing or decreasing  $t$  hours after midnight.

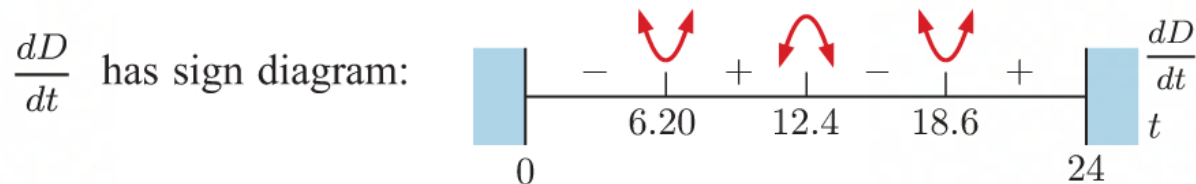
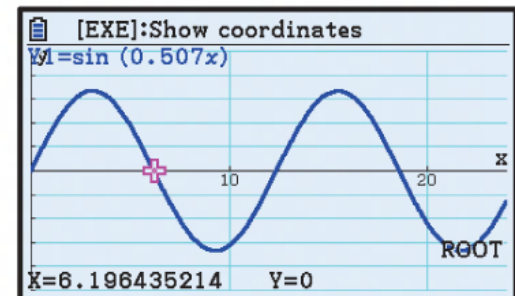
**b** When  $t = 8$ ,  $D = 9.3 + 6.8 \cos(0.507 \times 8)$   
 $\approx 5.15$

$\therefore$  the depth of the water at 8 am is about 5.15 m.

**c** When  $t = 8$ ,  $\frac{dD}{dt} = -3.4476 \sin(0.507 \times 8)$   
 $\approx 2.73 > 0$

$\therefore$  the tide is rising at 8 am at a rate of about 2.73 m per hour.

**d**  $\frac{dD}{dt} = 0$  when  $-3.4476 \sin(0.507t) = 0$   
 $\therefore \sin(0.507t) = 0$   
 $\therefore t = 0,$   
 or  $\approx 6.20, 12.4, 18.6$   
 $\{\text{on } 0 \leq t \leq 24\}$



So, the tide is highest 0 hours and about 12.4 hours after midnight.

0.4 hours  $= 0.4 \times 60$   
 $= 24 \text{ minutes}$

$\therefore$  the tide will be highest at midnight and at about 12:24 pm.

When  $t = 0$ ,  $D = 9.3 + 6.8 \cos 0$   
 $= 16.1 \text{ m}$

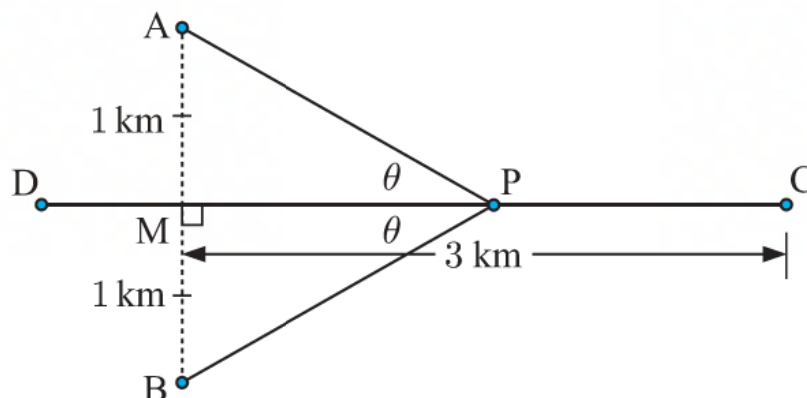
$\therefore$  the maximum depth is 16.1 m.

**8 a i** Length of cable required  $= 1 + 1 + 3 = 5 \text{ km}$

**ii**  $AC = BC = \sqrt{1^2 + 3^2} \text{ \{Pythagoras\}}$   
 $= \sqrt{10} \text{ km}$

$\therefore$  length of cable required  $= 2\sqrt{10} \text{ km}$

**b**  $\sin \theta = \frac{1}{AP} = \frac{1}{BP}$   
 $\therefore AP = BP = \frac{1}{\sin \theta} \text{ km}$   
 $\tan \theta = \frac{1}{MP}$   
 $\therefore MP = \frac{1}{\tan \theta} \text{ km}$





The length of cable required,  $L(\theta) = AP + BP + PC$

$$\begin{aligned}
 &= AP + BP + (3 - MP) \\
 &= \frac{1}{\sin \theta} + \frac{1}{\sin \theta} + 3 - \frac{1}{\tan \theta} \\
 &= 3 + \frac{2}{\sin \theta} - \frac{\cos \theta}{\sin \theta} \\
 &= 3 + \frac{2 - \cos \theta}{\sin \theta} \text{ km}
 \end{aligned}$$

**c**  $L = 3 + \frac{2 - \cos \theta}{\sin \theta}$

$$\begin{aligned}
 \therefore \frac{dL}{d\theta} &= \frac{(\sin \theta) \sin \theta - (2 - \cos \theta) \cos \theta}{\sin^2 \theta} && \{\text{quotient rule}\} \\
 &= \frac{\sin^2 \theta - 2 \cos \theta + \cos^2 \theta}{\sin^2 \theta} \\
 &= \frac{1 - 2 \cos \theta}{\sin^2 \theta} && \{\sin^2 \theta + \cos^2 \theta = 1\}
 \end{aligned}$$

$\frac{dL}{d\theta} = 0$  when  $1 - 2 \cos \theta = 0$

$$\begin{aligned}
 \therefore 2 \cos \theta &= 1 \\
 \therefore \cos \theta &= \frac{1}{2} \\
 \therefore \theta &= \frac{\pi}{3} && \{0 < \theta < \frac{\pi}{2}\}
 \end{aligned}$$

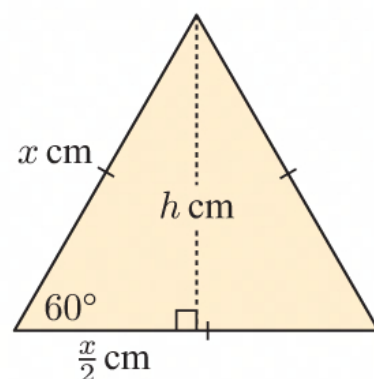
$\frac{dL}{d\theta}$  has sign diagram:

So, the required cable length is a minimum when  $\theta = \frac{\pi}{3}$ .

$$\begin{aligned}
 L\left(\frac{\pi}{3}\right) &= 3 + \frac{2 - \cos \frac{\pi}{3}}{\sin \frac{\pi}{3}} \\
 &= 3 + \frac{2 - \frac{1}{2}}{\frac{\sqrt{3}}{2}} \\
 &= 3 + \frac{\frac{3}{2}}{\frac{\sqrt{3}}{2}} \\
 &= 3 + \frac{3}{\sqrt{3}} \\
 &= 3 + \sqrt{3}
 \end{aligned}$$

$\therefore$  the minimum length of cable required is  $(3 + \sqrt{3})$  km.

**9 a**



$$\sin 60^\circ = \frac{h}{x}$$

$$\therefore h = x \sin 60^\circ = \frac{\sqrt{3}}{2}x$$

Now  $A = \frac{1}{2}xh$

$$\therefore A = \frac{1}{2}x\left(\frac{\sqrt{3}}{2}x\right) = \frac{\sqrt{3}}{4}x^2 \text{ cm}^2$$

- b** Differentiating both sides of  $A = \frac{\sqrt{3}}{4}x^2$  with respect to  $t$ :  $\frac{dA}{dt} = \frac{\sqrt{3}}{2}x \frac{dx}{dt}$

Particular case:

When  $x = 15$  cm and  $\frac{dx}{dt} = 3$  cm per minute,

$$\frac{dA}{dt} = \frac{\sqrt{3}}{2} \times 15 \times 3 = \frac{45\sqrt{3}}{2} \text{ cm}^2 \text{ per minute.}$$

$\therefore$  the area is increasing at  $\frac{45\sqrt{3}}{2}$  cm<sup>2</sup> per minute when the side length is 15 cm.

- 10 a**  $AC = 2x$  m

ABC is an isosceles triangle, so

$$XC = x$$

$$\text{But } BC^2 = BX^2 + XC^2 \quad \{\text{Pythagoras}\}$$

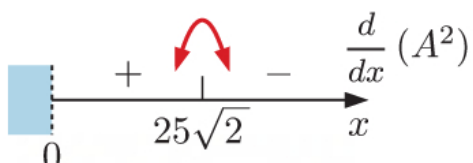
$$\therefore 2500 = BX^2 + x^2$$

$$\therefore BX = \sqrt{2500 - x^2} \quad \{\text{as } BX > 0\}$$

$$\begin{aligned} \text{So, the area } A &= \frac{1}{2} \times 2x \times \sqrt{2500 - x^2} \\ &= x\sqrt{2500 - x^2} \end{aligned}$$

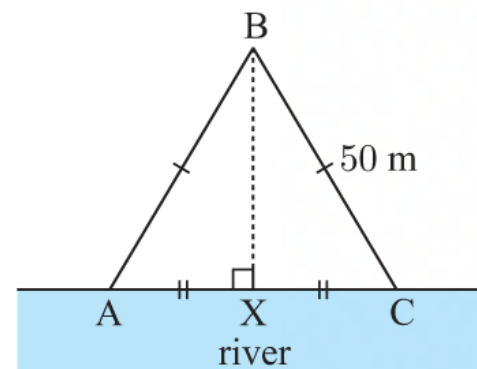
- b** Now  $A^2 = x^2(2500 - x^2)$   
 $= 2500x^2 - x^4$

$$\therefore \frac{d}{dx}(A^2) = 5000x - 4x^3$$

- c**  $\frac{d}{dx}(A^2)$  has sign diagram:
- 

$\therefore$  the maximum area occurs when  $x = 25\sqrt{2} \approx 35.4$

The corresponding maximum area  $= \sqrt{1250} \times \sqrt{1250} = 1250$  m<sup>2</sup>.



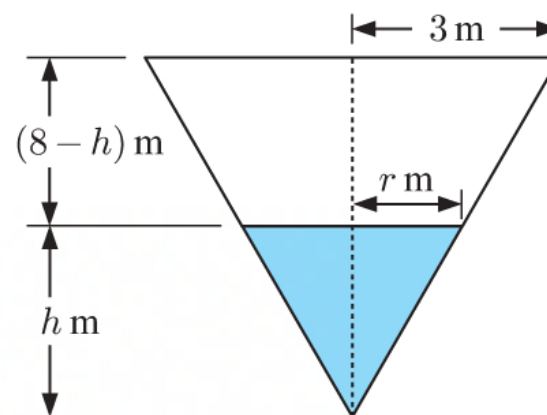
- 11 a** Volume  $V = \frac{1}{3}\pi r^2 h$

Using similar triangles,  $\frac{h}{r} = \frac{8}{3}$

$$\therefore h = \frac{8r}{3}$$

$$\therefore V = \frac{1}{3}\pi r^2 \left(\frac{8r}{3}\right)$$

$$\therefore V(r) = \frac{8}{9}\pi r^3 \text{ m}^3$$



- b** Differentiating both sides of  $V = \frac{8}{9}\pi r^3$  with respect to  $t$ :  $\frac{dV}{dt} = \frac{8}{3}\pi r^2 \frac{dr}{dt}$

*Particular case:*

When  $h = 5$ ,  $r = \frac{3h}{8} = \frac{15}{8}$  and  $\frac{dV}{dt} = -0.2 = -\frac{1}{5}$  m<sup>3</sup> per minute

$$-\frac{1}{5} = \frac{8}{3}\pi \left(\frac{15}{8}\right)^2 \frac{dr}{dt}$$

$$\therefore -\frac{1}{5} = \frac{225}{24}\pi \frac{dr}{dt}$$

$$\therefore \frac{dr}{dt} = -\frac{8}{375\pi} \approx -0.00679 \text{ m per minute}$$

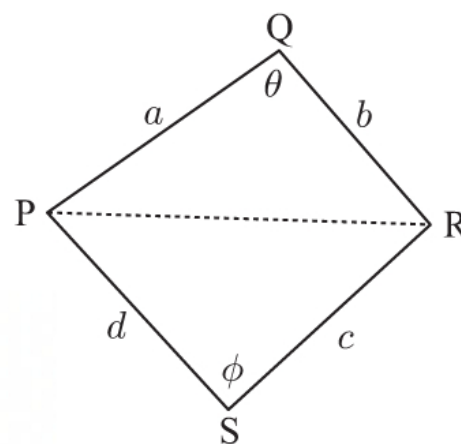
$\therefore$  the radius is decreasing at approximately 0.00679 m per minute.

- 12 a** Using the cosine rule:

$$\text{in } \triangle PQR, \quad PR^2 = a^2 + b^2 - 2ab \cos \theta$$

$$\text{in } \triangle PSR, \quad PR^2 = c^2 + d^2 - 2cd \cos \phi$$

$$\therefore a^2 + b^2 - 2ab \cos \theta = c^2 + d^2 - 2cd \cos \phi$$



- b** Now  $a$ ,  $b$ ,  $c$ , and  $d$  are constants, so differentiating both sides with respect to  $\phi$ ,

$$2ab \sin \theta \frac{d\theta}{d\phi} = 2cd \sin \phi$$

$$\begin{aligned} \therefore \frac{d\theta}{d\phi} &= \frac{2cd \sin \phi}{2ab \sin \theta} \\ &= \frac{cd \sin \phi}{ab \sin \theta} \quad \text{as required} \end{aligned}$$

- c** Area of quadrilateral,  $A = \text{area of } \triangle PQR + \text{area of } \triangle PSR$

$$= \frac{1}{2}ab \sin \theta + \frac{1}{2}cd \sin \phi$$

$$\begin{aligned} \therefore \frac{dA}{d\phi} &= \frac{1}{2}ab \cos \theta \frac{d\theta}{d\phi} + \frac{1}{2}cd \cos \phi \\ &= \frac{1}{2}ab \cos \theta \left( \frac{cd \sin \phi}{ab \sin \theta} \right) + \frac{1}{2}cd \cos \phi \quad \{\text{using b}\} \\ &= \frac{1}{2}cd \left[ \frac{\cos \theta \sin \phi}{\sin \theta} + \cos \phi \right] \\ &= \frac{cd}{2 \sin \theta} (\sin \phi \cos \theta + \cos \phi \sin \theta) \\ &= \frac{cd}{2 \sin \theta} \sin(\phi + \theta) \end{aligned}$$

$$\therefore \frac{dA}{d\phi} = 0 \quad \text{when} \quad \sin(\phi + \theta) = 0, \quad \text{which is when} \quad \phi + \theta = \pi$$

$\therefore$  the area of PQRS is a maximum when the opposite angles are supplementary, which occurs when PQRS is a cyclic quadrilateral.

## REVIEW SET 19B

$$\begin{aligned}
 \text{1 a } C(x) &= 850 + 3.3x^{0.85} + 2.8x^{0.5} \text{ euros} \\
 \therefore C'(x) &= 3.3(0.85x^{-0.15}) + 2.8(0.5x^{-0.5}) \\
 &= 2.805x^{-0.15} + 1.4x^{-0.5} \text{ euros per item}
 \end{aligned}$$

$$\begin{aligned}
 \text{b } C'(1000) &= 2.805(1000)^{-0.15} + 1.4(1000)^{-0.5} \\
 &\approx \text{€}1.04
 \end{aligned}$$

This estimates the cost of making the 1001st item each day.

$$\begin{aligned}
 \text{c } C(1001) - C(1000) \\
 &= 850 + 3.3(1001)^{0.85} + 2.8(1001)^{0.5} - (850 + 3.3(1000)^{0.85} + 2.8(1000)^{0.5}) \\
 &\approx \text{€}1.04
 \end{aligned}$$

This is the actual cost of making the 1001st item each day. The answer in **b** is a very good estimate.

$$\begin{aligned}
 \text{2 } P(t) &= 60\,000 \left(1 + 2e^{-\frac{t}{4}}\right)^{-1} \\
 &= \frac{60\,000}{1 + 2e^{-\frac{t}{4}}}, \quad t \geq 0
 \end{aligned}$$

$$\begin{aligned}
 \text{a } P(0) &= \frac{60\,000}{1 + 2e^0} \\
 &= \frac{60\,000}{3} \\
 &= 20\,000
 \end{aligned}$$

$\therefore$  the initial population is 20 000.

$$\begin{aligned}
 \text{b } P'(t) &= -60\,000 \left(1 + 2e^{-\frac{t}{4}}\right)^{-2} \times 2e^{-\frac{t}{4}} \left(-\frac{1}{4}\right) \quad \{\text{chain rule}\} \\
 &= \frac{30\,000e^{-\frac{t}{4}}}{\left(1 + 2e^{-\frac{t}{4}}\right)^2}
 \end{aligned}$$

$$\text{c Since } e^{-\frac{t}{4}} > 0 \text{ for all } t \text{ and } \left(1 + 2e^{-\frac{t}{4}}\right)^2 > 0 \text{ for all } t, \quad P'(t) > 0 \text{ for all } t \geq 0.$$

This means  $P(t)$  is increasing for all  $t \geq 0$ .



$$\begin{aligned}
 \text{d } P''(t) &= \frac{30\,000e^{-\frac{t}{4}}\left(-\frac{1}{4}\right)\left(1+2e^{-\frac{t}{4}}\right)^2 - 30\,000e^{-\frac{t}{4}} \times 2\left(1+2e^{-\frac{t}{4}}\right)^1 \times 2e^{-\frac{t}{4}}\left(-\frac{1}{4}\right)}{\left(1+2e^{-\frac{t}{4}}\right)^4} \\
 &\quad \text{\{quotient rule\}} \\
 &= \frac{-7500e^{-\frac{t}{4}}\left(1+2e^{-\frac{t}{4}}\right)^2 + 30\,000e^{-\frac{t}{2}}\left(1+2e^{-\frac{t}{4}}\right)}{\left(1+2e^{-\frac{t}{4}}\right)^4} \\
 &= \frac{-7500e^{-\frac{t}{4}}\left(1+2e^{-\frac{t}{4}}\right) + 30\,000e^{-\frac{t}{2}}}{\left(1+2e^{-\frac{t}{4}}\right)^3} \\
 &= \frac{-7500e^{-\frac{t}{4}} - 15\,000e^{-\frac{t}{2}} + 30\,000e^{-\frac{t}{2}}}{\left(1+2e^{-\frac{t}{4}}\right)^3} \\
 &= \frac{15\,000e^{-\frac{t}{2}} - 7500e^{-\frac{t}{4}}}{\left(1+2e^{-\frac{t}{4}}\right)^3} \\
 &= \frac{7500e^{-\frac{t}{4}}\left(2e^{-\frac{t}{4}} - 1\right)}{\left(1+2e^{-\frac{t}{4}}\right)^3}
 \end{aligned}$$

e The growth is given by  $P'(t)$  which is maximised when  $P''(t) = 0$ .

$$P''(t) = 0 \quad \text{when} \quad 2e^{-\frac{t}{4}} - 1 = 0 \quad \{e^{-\frac{t}{4}} > 0\}$$

$$\therefore 2e^{-\frac{t}{4}} = 1$$

$$\therefore e^{-\frac{t}{4}} = \frac{1}{2}$$

$$\therefore -\frac{t}{4} = \ln\left(\frac{1}{2}\right)$$

$$\therefore t = -4\ln(2^{-1})$$

$$\therefore t = 4\ln 2$$

$$\text{When } t = 4\ln 2, \quad e^{-\frac{t}{4}} = e^{-\ln 2}$$

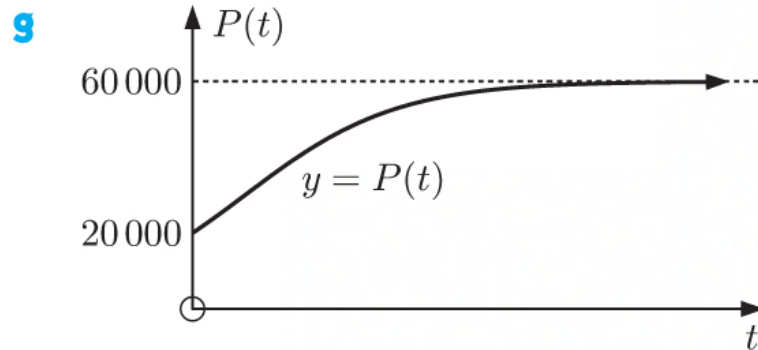
$$= e^{\ln(\frac{1}{2})}$$

$$= \frac{1}{2}$$

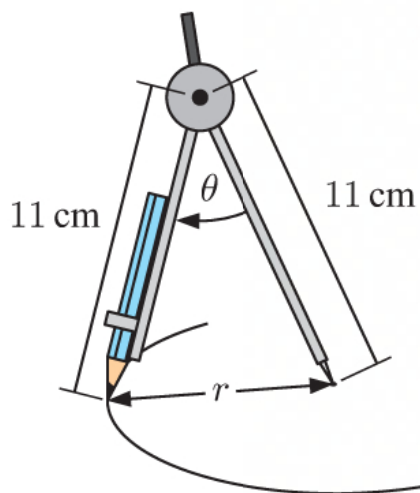
$$\text{and so } P'(t) = \frac{30\,000(\frac{1}{2})}{\left(1+2(\frac{1}{2})\right)^2} = \frac{15\,000}{4} = 3750$$

$\therefore$  the maximum growth rate is 3750 per year when  $t = 4\ln 2$  years.

**f** As  $t \rightarrow \infty$ ,  $e^{-\frac{t}{4}} \rightarrow 0^+$   
 $\therefore P(t) \rightarrow \frac{60\,000}{1+2(0)}$   
 $\therefore P(t) \rightarrow 60\,000^-$



**3 a**



Using the cosine rule,  $r^2 = 11^2 + 11^2 - 2 \times 11 \times 11 \times \cos \theta$   
 $= 121 + 121 - 242 \cos \theta$   
 $= 242 - 242 \cos \theta$   
 $= 242(1 - \cos \theta)$

Now, the area of the circle  $A = \pi r^2$   
 $= \pi \times 242(1 - \cos \theta)$   
 $= 242\pi(1 - \cos \theta) \text{ cm}^2$

**b**  $A = 242\pi(1 - \cos \theta) \text{ cm}^2$   
 $\therefore \frac{dA}{d\theta} = 242\pi(\sin \theta)$   
 $= 242\pi \sin \theta \text{ cm}^2 \text{ per radian}$

When  $\theta = \frac{\pi}{4}$ ,  $\frac{dA}{d\theta} = 242\pi \sin \frac{\pi}{4}$   
 $= 242\pi \times \frac{1}{\sqrt{2}}$   
 $= 121\sqrt{2}\pi \approx 538 \text{ cm}^2 \text{ per radian}$

$\therefore$  the area of the circle is changing at a rate of  $121\sqrt{2}\pi \approx 538 \text{ cm}^2 \text{ per radian}$  when  $\theta = \frac{\pi}{4}$ .

**4** Suppose the sheet is bent  $x$  cm from each end. To maximise the water carried we need to maximise the area of the cross-section.

$$A = x(24 - 2x), \quad 0 \leq x \leq 12$$

$$= 24x - 2x^2$$

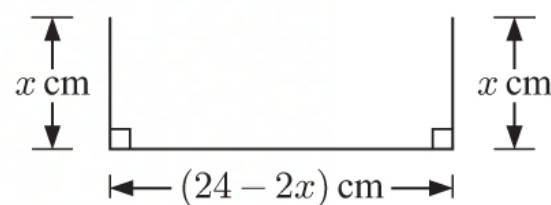
$$\therefore \frac{dA}{dx} = 24 - 4x$$

So,  $\frac{dA}{dx} = 0$  when  $24 - 4x = 0$   
 $\therefore x = 6$

$\frac{dA}{dx}$  has sign diagram:

The maximum water is held when  $x = 6$

$\therefore$  the bends must be made 6 cm from each end.



5 Let  $OA = x \quad \therefore AP = ae^{-x}$

$\therefore$  rectangle OAPB has perimeter  $P = 2x + 2ae^{-x}$

$$\therefore \frac{dP}{dx} = 2 - 2ae^{-x}$$

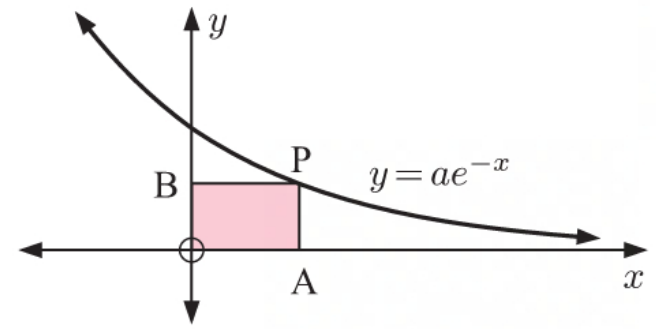
$$\frac{dP}{dx} = 0 \quad \text{when} \quad 2 - 2ae^{-x} = 0$$

$$\therefore 2ae^{-x} = 2$$

$$\therefore ae^{-x} = 1$$

$$\therefore a = e^x$$

$$\therefore x = \ln a$$



$\frac{dP}{dx}$  has sign diagram:

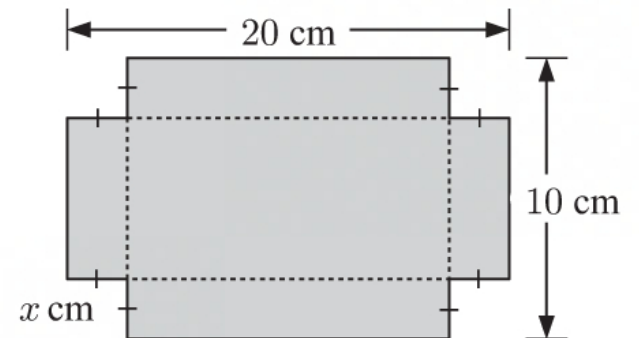
$$\begin{array}{c} \text{---} - \quad \quad \quad + \quad \text{---} \\ \quad \quad \quad \uparrow \\ \quad \quad \quad \ln a \end{array} \quad \frac{dP}{dx}$$

$\therefore$  the rectangle OAPB has minimum perimeter when the  $x$ -coordinate of P is  $\ln a$ .

6 4 squares with sides  $x$  cm are cut from the corners.

$\therefore$  the remaining sides have length  $(20 - 2x)$  cm and  $(10 - 2x)$  cm.

$$\begin{aligned} \text{Now, volume } V &= \text{length} \times \text{width} \times \text{depth} \\ &= (20 - 2x)(10 - 2x)x \\ &= (200 - 40x - 20x + 4x^2)x \\ &= (200 - 60x + 4x^2)x \\ &= 200x - 60x^2 + 4x^3 \text{ cm}^3 \end{aligned}$$



Since the side lengths must be positive,  $x > 0$  and  $10 - 2x > 0$

$$\therefore 2x < 10$$

$$\therefore 0 < x < 5$$

$$V = 4x^3 - 60x^2 + 200x$$

$$\therefore \frac{dV}{dx} = 12x^2 - 120x + 200$$

$$\text{So, } \frac{dV}{dx} = 0 \quad \text{when} \quad 12x^2 - 120x + 200 = 0$$

$$\therefore 3x^2 - 30x + 50 = 0$$

$$\therefore x \approx 2.11 \quad \{\text{using technology, } 0 < x < 5\}$$

$\frac{dV}{dx}$  has sign diagram:

$$\begin{array}{c} \text{---} + \quad \quad \quad - \quad \text{---} \\ \quad \quad \quad \uparrow \\ \quad \quad \quad \approx 2.11 \end{array} \quad \frac{dV}{dx}$$

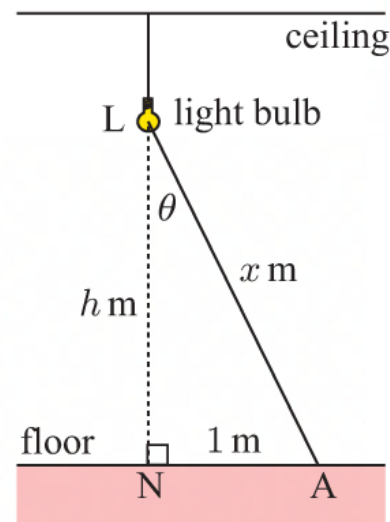
0 5

$\therefore$  the capacity of the container is maximised when  $x \approx 2.11$ .

**7**  $I = \frac{\sqrt{8} \cos \theta}{x^2}$  units

**a** If  $NA = 1$  metre,  $\sin \theta = \frac{NA}{x} = \frac{1}{x}$   
 $\therefore \frac{1}{x^2} = \sin^2 \theta$

$\therefore$  at A,  $I = \frac{\sqrt{8} \cos \theta}{x^2} = \sqrt{8} \cos \theta \sin^2 \theta$  units



**b**  $\frac{dI}{d\theta} = \sqrt{8}(-\sin \theta) \sin^2 \theta + \sqrt{8} \cos \theta (2 \sin \theta \cos \theta)$  {product rule}  
 $= \sqrt{8} \sin \theta [2 \cos^2 \theta - \sin^2 \theta]$   
 $= \sqrt{8} \sin \theta [2(1 - \sin^2 \theta) - \sin^2 \theta]$  { $\sin^2 \theta + \cos^2 \theta = 1$ }  
 $= \sqrt{8} \sin \theta [2 - 3 \sin^2 \theta]$

$\frac{dI}{d\theta} = 0$  when  $\sin \theta = \sqrt{\frac{2}{3}}$ ,  $0 < \theta < \frac{\pi}{2}$   
 $\therefore \theta = \sin^{-1} \left( \sqrt{\frac{2}{3}} \right)$

$\frac{dI}{d\theta}$  has sign diagram:

$\therefore$  the maximum illumination at A is obtained when  $\sin \theta = \sqrt{\frac{2}{3}}$

$\therefore x = \frac{1}{\sin \theta} = \sqrt{\frac{3}{2}}$

and  $h = \sqrt{x^2 - NA^2}$  {Pythagoras}

$= \sqrt{\frac{3}{2} - 1^2} = \frac{1}{\sqrt{2}}$

$\therefore$  the bulb should be  $\frac{1}{\sqrt{2}}$  m above the floor to provide the greatest illumination at A.

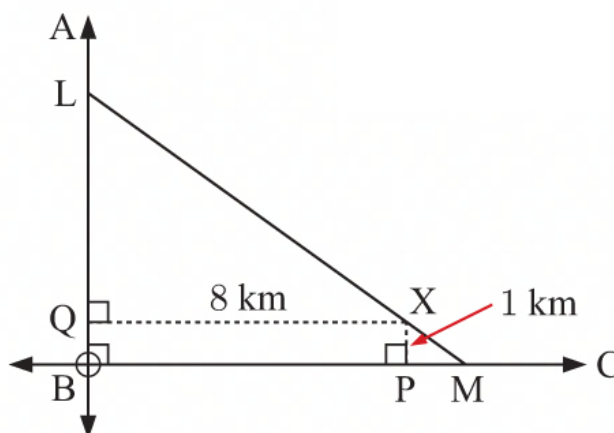
**8 a**  $\triangle LQX$  and  $\triangle XPM$  are similar.

$\therefore \frac{LQ}{XP} = \frac{QX}{PM}$

$\therefore \frac{LQ}{1} = \frac{8}{PM}$

$\therefore LQ = \frac{8}{PM}$

$\therefore LQ = \frac{8}{x}$  km





**b** The length of the pipeline [LM] is  $L$ .

$$\therefore L^2 = (\text{LQ} + \text{QB})^2 + (\text{BP} + \text{PM})^2 \quad \{\text{Pythagoras}\}$$

$$= \left(\frac{8}{x} + 1\right)^2 + (8 + x)^2 \quad \{\text{using a}\}$$

$$= \left(\frac{8}{x} + 1\right)^2 + \left[x \left(\frac{8}{x} + 1\right)\right]^2$$

$$= \left(\frac{8}{x} + 1\right)^2 + x^2 \left(\frac{8}{x} + 1\right)^2$$

$$= \left(\frac{8}{x} + 1\right)^2 (1 + x^2)$$

$$= (x^2 + 1) \left(\frac{8}{x} + 1\right)^2$$

$$\therefore L = \sqrt{x^2 + 1} \left(\frac{8}{x} + 1\right) \quad \{\text{as } L > 0\}$$

**c** 
$$L^2 = (x^2 + 1) \left(\frac{8}{x} + 1\right)^2$$

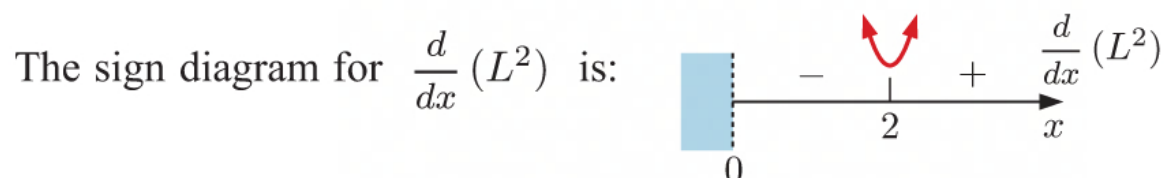
$$\therefore \frac{d}{dx}(L^2) = 2x \left(\frac{8}{x} + 1\right)^2 + (x^2 + 1)(2) \left(\frac{8}{x} + 1\right) \left(-\frac{8}{x^2}\right) \quad \{\text{product rule}\}$$

$$= 2 \left(\frac{8}{x} + 1\right) \left[ x \left(\frac{8}{x} + 1\right) - (x^2 + 1) \left(\frac{8}{x^2}\right) \right]$$

$$= 2 \left(\frac{8}{x} + 1\right) \left[ 8 + x - 8 - \frac{8}{x^2} \right]$$

$$= 2 \left(\frac{8+x}{x}\right) \left(\frac{x^3 - 8}{x^2}\right)$$

**d**  $\frac{d}{dx}(L^2) = 0$  when  $x = -8$  or  $x^3 = 8$ , but  $x > 0$  so  $\frac{d}{dx}(L^2) = 0$  when  $x = 2$

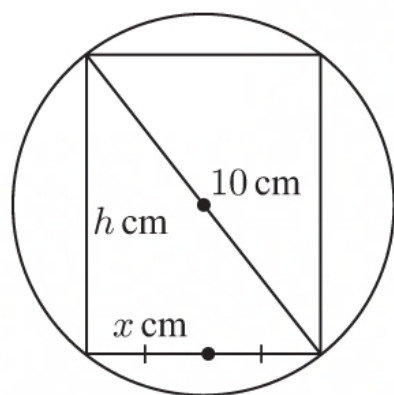


The minimum value of  $L$  occurs when  $x = 2$ .

The shortest possible length for the pipeline is

$$\begin{aligned} L(2) &= \sqrt{2^2 + 1} \left(\frac{8}{2} + 1\right) \\ &= 5\sqrt{5} \\ &\approx 11.2 \text{ km} \end{aligned}$$

**9 a**



$$(2x)^2 + h^2 = 10^2 \quad \{\text{Pythagoras}\}$$

$$\therefore h = \sqrt{100 - 4x^2} \quad \{\text{as } h > 0\}$$

$$\therefore V = \pi r^2 h$$

$$= \pi x^2 \times \sqrt{100 - 4x^2}$$

So,  $V(x) = \pi x^2 \sqrt{100 - 4x^2} \text{ cm}^3$



**12 a**  $(3 \cos \theta, 2 \sin \theta)$  lies on the curve

$$\begin{aligned}\therefore x &= 3 \cos \theta \quad \text{and} \quad y = 2 \sin \theta \\ \therefore x^2 &= 9 \cos^2 \theta \quad \text{and} \quad y^2 = 4 \sin^2 \theta\end{aligned}$$

$$\therefore \frac{x^2}{9} + \frac{y^2}{4} = \cos^2 \theta + \sin^2 \theta = 1$$

$$\therefore \text{the curve has equation } \frac{x^2}{9} + \frac{y^2}{4} = 1$$

**b**

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

$$\therefore \frac{2x}{9} + \frac{y}{2} \frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = \left( \frac{2}{y} \right) \left( -\frac{2x}{9} \right)$$

$$= -\frac{4x}{9y}$$

$$= -\frac{4 \times 3 \cos \theta}{9 \times 2 \sin \theta} \quad \{\text{from a}\}$$

$$= -\frac{2 \cos \theta}{3 \sin \theta}$$

**c** The tangent has gradient  $-\frac{2 \cos \theta}{3 \sin \theta}$  and passes through  $(3 \cos \theta, 2 \sin \theta)$ .

$$\therefore \text{the tangent has equation } \frac{y - 2 \sin \theta}{x - 3 \cos \theta} = -\frac{2 \cos \theta}{3 \sin \theta}$$

$$\therefore 3y \sin \theta - 6 \sin^2 \theta = -2x \cos \theta + 6 \cos^2 \theta$$

$$\therefore 2x \cos \theta + 3y \sin \theta = 6 \quad \{\sin^2 \theta + \cos^2 \theta = 1\}$$

The tangent meets the  $x$ -axis when  $y = 0$

$$\therefore 2x \cos \theta = 6$$

$$\therefore x = \frac{3}{\cos \theta}$$

$$\therefore \text{A is } \left( \frac{3}{\cos \theta}, 0 \right).$$

The tangent meets the  $y$ -axis when  $x = 0$

$$\therefore 3y \sin \theta = 6$$

$$\therefore y = \frac{2}{\sin \theta}$$

$$\therefore \text{B is at } \left( 0, \frac{2}{\sin \theta} \right).$$

$$\therefore \text{triangle OAB has area } A = \left| \frac{1}{2} \times \left( \frac{3}{\cos \theta} \right) \times \left( \frac{2}{\sin \theta} \right) \right|$$

$$= \left| \frac{6}{\sin 2\theta} \right| \quad \dots (*)$$

$$\therefore A^2 = 36(\sin 2\theta)^{-2}$$

$$\therefore \frac{d}{d\theta}(A^2) = -72(\sin 2\theta)^{-3} \times 2 \cos 2\theta$$

$$= -\frac{144 \cos 2\theta}{\sin^3 2\theta}$$

Since  $0 \leq \theta \leq 2\pi$ ,  $0 \leq 2\theta \leq 4\pi$

$$\therefore \frac{d}{d\theta}(A^2) = 0 \quad \text{when} \quad \cos 2\theta = 0$$

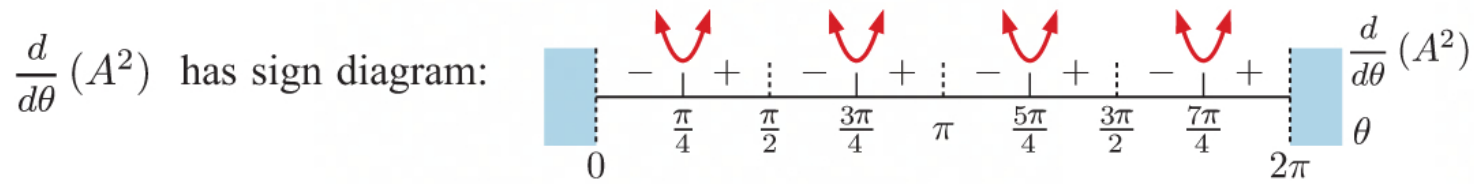
$$\therefore 2\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \text{ or } \frac{7\pi}{2}$$

$$\therefore \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \text{ or } \frac{7\pi}{4}$$

and is undefined when  $\sin 2\theta = 0$

$$\therefore 2\theta = 0, \pi, 2\pi, 3\pi, \text{ or } 4\pi$$

$$\therefore \theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \text{ or } 2\pi$$



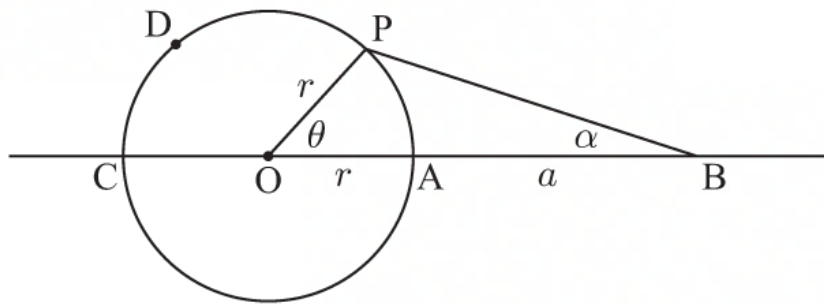
$\therefore$  there are local minima at  $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \text{ and } \frac{7\pi}{4}$ .

For all of these values of  $\theta$ ,  $\sin 2\theta = -1$  or  $1$

$$\therefore A = 6 \quad \{\text{using } (*)\}$$

$\therefore$  the smallest area of triangle OAB is 6 units<sup>2</sup>, which occurs when  $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \text{ or } \frac{7\pi}{4}$ .

**13 a**



Using the cosine rule in triangle BPO,

$$BP^2 = r^2 + (a + r)^2 - 2r(a + r)\cos\theta$$

$$\therefore BP = \sqrt{r^2 + (a + r)^2 - 2r(a + r)\cos\theta} \quad \{\text{as } BP > 0\}$$

$$\begin{aligned} \therefore \text{the time taken to travel from B to P} &= \frac{\text{distance}}{\text{speed}} \\ &= \frac{\sqrt{r^2 + (a + r)^2 - 2r(a + r)\cos\theta}}{v} \end{aligned}$$

Now, arc AP =  $r\theta$

$$\begin{aligned} \therefore \text{arc PC} &= (\text{perimeter of semi-circle}) - \text{arc AP} \\ &= \frac{1}{2} \times 2\pi r - r\theta \\ &= r(\pi - \theta) \end{aligned}$$

$$\therefore \text{the time taken to travel from P to C} = \frac{\text{distance}}{\text{speed}} = \frac{r(\pi - \theta)}{w}$$

$$\therefore \text{the total time for the journey } T = \frac{\sqrt{r^2 + (a + r)^2 - 2r(a + r)\cos\theta}}{v} + \frac{r(\pi - \theta)}{w}$$



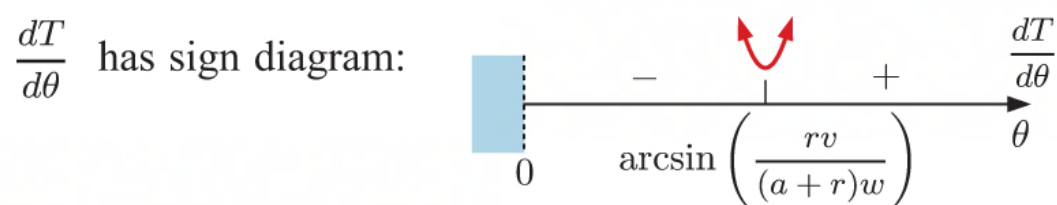
$$\begin{aligned}
 \text{b} \quad T &= \frac{w[r^2 + (a+r)^2 - 2r(a+r)\cos\theta]^{\frac{1}{2}} + rv(\pi - \theta)}{vw} \\
 \therefore \frac{dT}{d\theta} &= \frac{\frac{1}{2}w[r^2 + (a+r)^2 - 2r(a+r)\cos\theta]^{-\frac{1}{2}}(2r(a+r)\sin\theta) - rv}{vw} \quad \{\text{chain rule}\} \\
 &= \frac{2r(a+r)\sin\theta}{2v\sqrt{r^2 + (a+r)^2 - 2r(a+r)\cos\theta}} - \frac{rv}{vw} \\
 &= \frac{r(a+r)\sin\theta}{v \times \text{BP}} - \frac{rv}{vw} \quad \{\text{from a}\}
 \end{aligned}$$

$$\text{Now } \frac{\text{BP}}{\sin\theta} = \frac{r}{\sin\alpha} \quad \{\text{sine rule}\}$$

$$\therefore \text{BP} = \frac{r\sin\theta}{\sin\alpha}$$

$$\begin{aligned}
 \therefore \frac{dT}{d\theta} &= \frac{r(a+r)\sin\theta}{v \times \frac{r\sin\theta}{\sin\alpha}} - \frac{rv}{vw} \\
 &= \frac{a+r}{v} \sin\alpha - \frac{rv}{vw} \\
 &= \frac{a+r}{v} \left( \sin\alpha - \frac{rv}{(a+r)w} \right)
 \end{aligned}$$

$$\begin{aligned}
 \text{Now } a+r \neq 0 \quad \text{so} \quad \frac{dT}{d\theta} = 0 \quad \text{when} \quad \sin\alpha - \frac{rv}{(a+r)w} = 0 \\
 \therefore \sin\alpha = \frac{rv}{(a+r)w}
 \end{aligned}$$



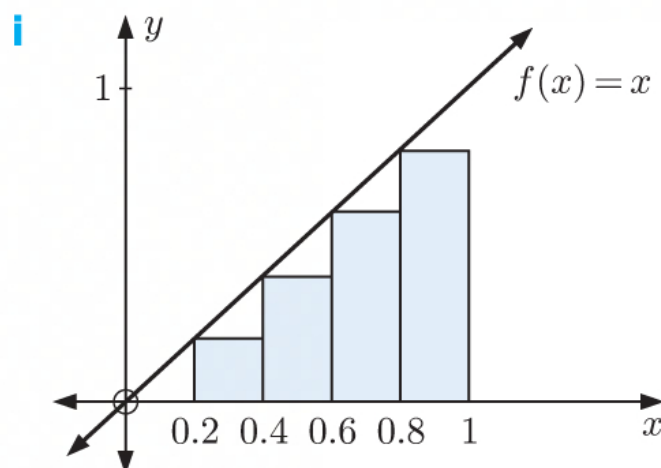
$$\therefore T \text{ is minimised when } \alpha = \arcsin\left(\frac{rv}{(a+r)w}\right).$$

# Chapter 20

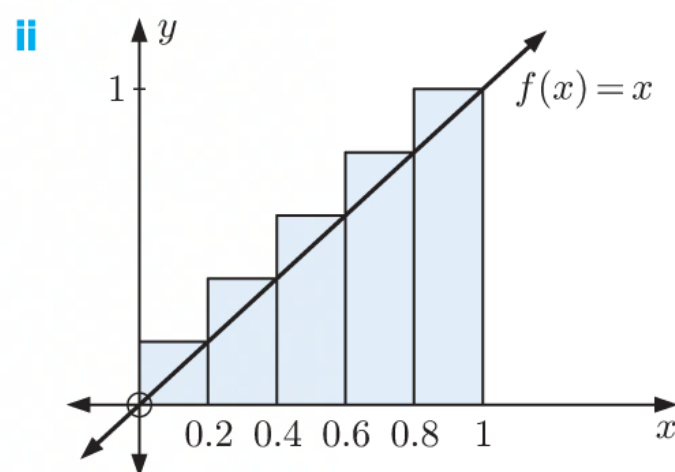
## INTRODUCTION TO INTEGRATION

### EXERCISE 20A

- 1 a The rectangles are  $\frac{1}{5} = 0.2$  units wide.



$$\begin{aligned} A_L &= 0.2 \times f(0) + 0.2 \times f(0.2) + 0.2 \times f(0.4) \\ &\quad + 0.2 \times f(0.6) + 0.2 \times f(0.8) \\ &= (0.2 \times 0) + (0.2 \times 0.2) + (0.2 \times 0.4) \\ &\quad + (0.2 \times 0.6) + (0.2 \times 0.8) \\ &= 0.4 \text{ units}^2 \end{aligned}$$



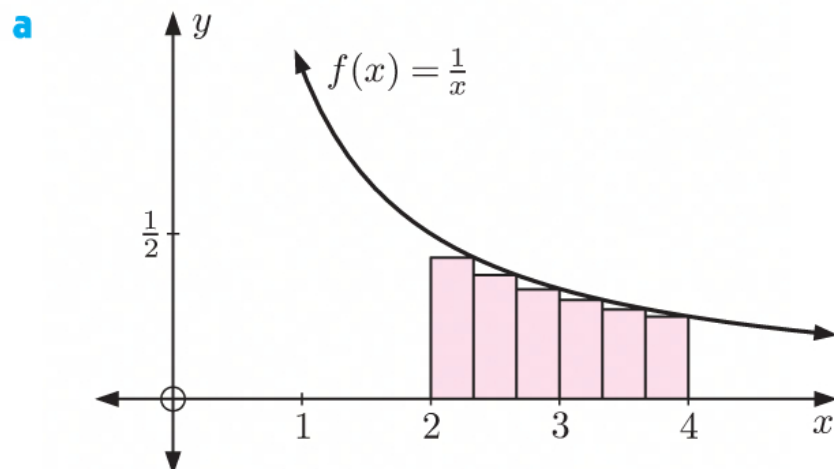
$$\begin{aligned} A_U &= 0.2 \times f(0.2) + 0.2 \times f(0.4) + 0.2 \times f(0.6) \\ &\quad + 0.2 \times f(0.8) + 0.2 \times f(1) \\ &= (0.2 \times 0.2) + (0.2 \times 0.4) + (0.2 \times 0.6) \\ &\quad + (0.2 \times 0.8) + (0.2 \times 1) \\ &= 0.6 \text{ units}^2 \end{aligned}$$

- b The area between  $y = x$  and the  $x$ -axis from  $x = 0$  to  $x = 1$  is a triangle.

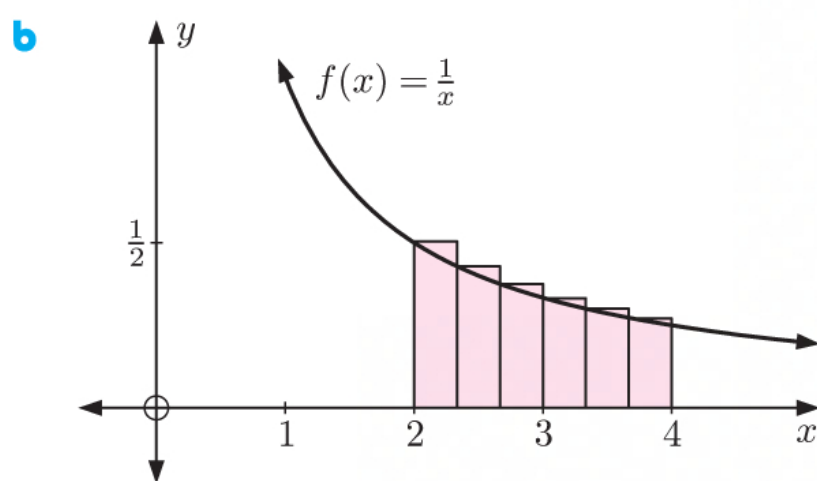
$$\begin{aligned} \therefore \text{area} &= \frac{1}{2} \times \text{base} \times \text{height} \\ &= \frac{1}{2} \times 1 \times 1 \\ &= 0.5 \text{ units}^2 \end{aligned}$$

$\therefore A_L < \text{area} < A_U$ , and both  $A_L$  and  $A_U$  are within  $0.1 \text{ units}^2$ , or 20%, of the actual area.

- 2 The rectangles are  $\frac{2}{6} = \frac{1}{3}$  units wide.



$$\begin{aligned} A_L &= \frac{1}{3} \times f\left(\frac{7}{3}\right) + \frac{1}{3} \times f\left(\frac{8}{3}\right) + \frac{1}{3} \times f(3) + \frac{1}{3} \times f\left(\frac{10}{3}\right) + \frac{1}{3} \times f\left(\frac{11}{3}\right) + \frac{1}{3} \times f(4) \\ &= \left(\frac{1}{3} \times \frac{3}{7}\right) + \left(\frac{1}{3} \times \frac{3}{8}\right) + \left(\frac{1}{3} \times \frac{1}{3}\right) + \left(\frac{1}{3} \times \frac{3}{10}\right) + \left(\frac{1}{3} \times \frac{3}{11}\right) + \left(\frac{1}{3} \times \frac{1}{4}\right) \\ &\approx 0.653 \text{ units}^2 \end{aligned}$$



$$\begin{aligned}
 A_U &= \frac{1}{3} \times f(2) + \frac{1}{3} \times f\left(\frac{7}{3}\right) + \frac{1}{3} \times f\left(\frac{8}{3}\right) + \frac{1}{3} \times f(3) + \frac{1}{3} \times f\left(\frac{10}{3}\right) + \frac{1}{3} \times f\left(\frac{11}{3}\right) \\
 &= \left(\frac{1}{3} \times \frac{1}{2}\right) + \left(\frac{1}{3} \times \frac{3}{7}\right) + \left(\frac{1}{3} \times \frac{3}{8}\right) + \left(\frac{1}{3} \times \frac{1}{3}\right) + \left(\frac{1}{3} \times \frac{3}{10}\right) + \left(\frac{1}{3} \times \frac{3}{11}\right) \\
 &\approx 0.737 \text{ units}^2
 \end{aligned}$$

**3** Using provided software,

$n$	$A_L$	$A_U$
10	2.1850	2.4850
25	2.2736	2.3936
50	2.3034	2.3634
100	2.3184	2.3484
500	2.3303	2.3363

$A_L$  and  $A_U$  converge to  $\frac{7}{3} = 2.\bar{3}$

**4 a i**

$n$	$A_L$	$A_U$
5	0.160 00	0.360 00
10	0.202 50	0.302 50
50	0.240 10	0.260 10
100	0.245 03	0.255 03
500	0.249 00	0.251 00
1000	0.249 50	0.250 50
10 000	0.249 95	0.250 05

**ii**

$n$	$A_L$	$A_U$
5	0.400 00	0.600 00
10	0.450 00	0.550 00
50	0.490 00	0.510 00
100	0.495 00	0.505 00
500	0.499 00	0.501 00
1000	0.499 50	0.500 50
10 000	0.499 95	0.500 05

**iii**

$n$	$A_L$	$A_U$
5	0.549 74	0.749 74
10	0.610 51	0.710 51
50	0.656 10	0.676 10
100	0.661 46	0.671 46
500	0.665 65	0.667 65
1000	0.666 16	0.667 16
10 000	0.666 62	0.666 72

**iv**

$n$	$A_L$	$A_U$
5	0.618 67	0.818 67
10	0.687 40	0.787 40
50	0.738 51	0.758 51
100	0.744 41	0.754 41
500	0.748 93	0.750 93
1000	0.749 47	0.750 47
10 000	0.749 95	0.750 05

**b i**  $A_L$  and  $A_U$  converge to  $0.25 = \frac{1}{4} = \frac{1}{3+1}$

**ii**  $A_L$  and  $A_U$  converge to  $0.5 = \frac{1}{2} = \frac{1}{1+1}$

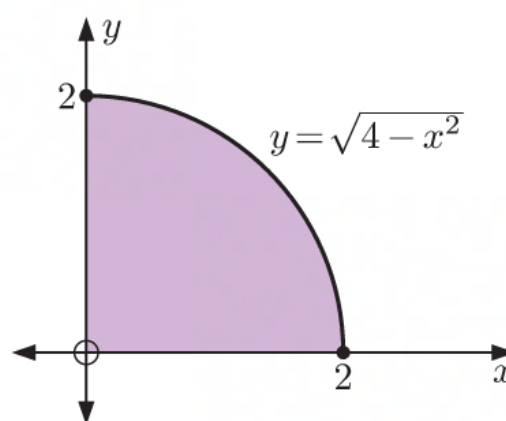
**iii**  $A_L$  and  $A_U$  converge to  $0.\bar{6} = \frac{2}{3} = \frac{1}{\frac{1}{2}+1}$

**iv**  $A_L$  and  $A_U$  converge to  $0.75 = \frac{3}{4} = \frac{1}{\frac{1}{3}+1}$

**c** From **b**, it appears that the area between the graph of  $y = x^a$  and the  $x$ -axis for  $0 \leq x \leq 1$  and any number  $a > 0$  is  $\frac{1}{a+1}$ .

5 a

$n$	Rational bounds for $\pi$
10	$2.9045 < \pi < 3.3045$
50	$3.0983 < \pi < 3.1783$
100	$3.1204 < \pi < 3.1604$
200	$3.1312 < \pi < 3.1512$
1000	$3.1396 < \pi < 3.1436$
10 000	$3.1414 < \pi < 3.1418$



b  $3\frac{10}{71} < \pi < 3\frac{1}{7}$  is approximately  $3.1408 < \pi < 3.1429$

This is a better approximation than our estimates in a using  $n = 10, 50, 100, 200$ , or  $1000$  rectangles. Only  $n = 10\,000$  gives us a better estimate than that of Archimedes.

## INVESTIGATION 1

## THE AREA UNDER $f(x) = x^2$

1 a  $f(x) = x^2$ ,  $0 \leq x \leq 1$  is divided into  $n$  subintervals.

Each lower rectangle has width  $\frac{1}{n}$ .

Since  $f(x)$  is increasing on  $0 \leq x \leq 1$ , the  $i$ th lower rectangle has height  $f\left(\frac{i-1}{n}\right)$ .

$$\begin{aligned} \therefore \text{the total area of lower rectangles } A_L &= \sum_{i=1}^n \left( \frac{1}{n} \times f\left(\frac{i-1}{n}\right) \right) \\ &= \frac{1}{n} \sum_{i=1}^n f\left(\frac{i-1}{n}\right) \end{aligned}$$

b

$$\begin{aligned} \frac{1}{n} \sum_{i=1}^n f\left(\frac{i-1}{n}\right) &= \frac{1}{n} \sum_{i=1}^n \left(\frac{i-1}{n}\right)^2 \\ &= \frac{1}{n} \sum_{i=1}^n \frac{i^2 - 2i + 1}{n^2} \\ &= \frac{1}{n^3} \sum_{i=1}^n (i^2 - 2i + 1) \\ &= \frac{1}{n^3} \left( \sum_{i=1}^n i^2 - 2 \sum_{i=1}^n i + \sum_{i=1}^n 1 \right) \\ &= \frac{1}{n^3} \left( \frac{n(n+1)(2n+1)}{6} - 2 \times \frac{n(n+1)}{2} + n \right) \\ &\quad \text{\{using provided formulae\}} \\ &= \frac{1}{n^3} \left( \frac{n(n+1)(2n+1)}{6} - n(n+1) + n \right) \\ &= \frac{1}{n^3} \left( \frac{2n^3 + 3n^2 + n}{6} - n^2 - n + n \right) \\ &= \frac{1}{n^3} \left( \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n - n^2 \right) \\ &= \frac{1}{n^3} \left( \frac{1}{3}n^3 - \frac{1}{2}n^2 + \frac{1}{6}n \right) \end{aligned}$$

$$\therefore A_L = \frac{1}{n} \sum_{i=1}^n f\left(\frac{i-1}{n}\right) = \frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2}$$



- c** As  $n \rightarrow \infty$ ,  $\frac{1}{2n} \rightarrow 0$  and  $\frac{1}{6n^2} \rightarrow 0$   
 $\therefore$  as  $n \rightarrow \infty$ ,  $A_L \rightarrow \frac{1}{3}$ .

- 2 a** Each upper rectangle has width  $\frac{1}{n}$ .

Since  $f(x)$  is increasing on  $0 \leq x \leq 1$ , the  $i$ th upper rectangle has height  $f\left(\frac{i}{n}\right)$ .

$$\begin{aligned} \therefore \text{the total area of upper rectangles } A_U &= \sum_{i=1}^n \left( \frac{1}{n} \times f\left(\frac{i}{n}\right) \right) \\ &= \frac{1}{n} \sum_{i=1}^n f\left(\frac{i}{n}\right) \end{aligned}$$

**b**

$$\begin{aligned} \frac{1}{n} \sum_{i=1}^n f\left(\frac{i}{n}\right) &= \frac{1}{n} \sum_{i=1}^n \left(\frac{i}{n}\right)^2 \\ &= \frac{1}{n} \sum_{i=1}^n \frac{i^2}{n^2} \\ &= \frac{1}{n^3} \sum_{i=1}^n i^2 \\ &= \frac{1}{n^3} \left( \frac{n(n+1)(2n+1)}{6} \right) \quad \left\{ \text{using } \sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6} \right\} \\ &= \frac{1}{n^3} \left( \frac{2n^3 + 3n^2 + n}{6} \right) \\ &= \frac{1}{n^3} \left( \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n \right) \end{aligned}$$

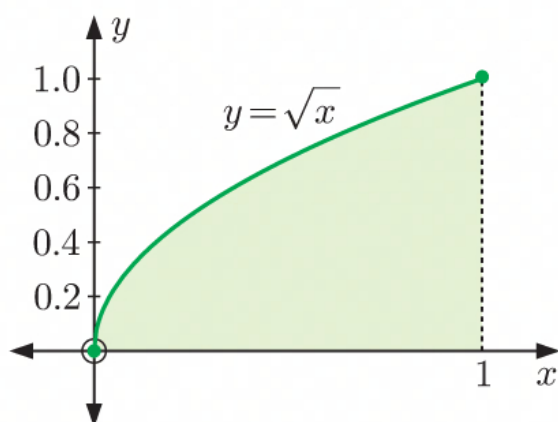
$$\therefore A_U = \frac{1}{n} \sum_{i=1}^n f\left(\frac{i}{n}\right) = \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}$$

- c** As  $n \rightarrow \infty$ ,  $\frac{1}{2n} \rightarrow 0$  and  $\frac{1}{6n^2} \rightarrow 0$   
 $\therefore$  as  $n \rightarrow \infty$ ,  $A_U \rightarrow \frac{1}{3}$ .

- 3** We know that  $A_L < A < A_U$ , and as  $n \rightarrow \infty$ ,  $A_L$  and  $A_U$  both converge to the value  $\frac{1}{3}$ .  
 $\therefore A = \frac{1}{3}$  units<sup>2</sup>.

## EXERCISE 20B

- 1 a**



- b**

$n$	$A_L$	$A_U$
5	0.5497	0.7497
10	0.6105	0.7105
50	0.6561	0.6761
100	0.6615	0.6715
500	0.6656	0.6676

**c**  $\int_0^1 \sqrt{x} \, dx \approx 0.67$

- 2 a** The rectangles will have width  $\frac{2-0}{n} = \frac{2}{n}$ .

Let  $x_i = \frac{2i}{n}$  for  $i = 0, \dots, n$ .

Since  $y = \sqrt{1+x^3}$  is increasing on  $0 \leq x \leq 2$ , the  $i$ th lower rectangle has height  $\sqrt{1+x_{i-1}^3}$  and the  $i$ th upper rectangle has height  $\sqrt{1+x_i^3}$ .

The lower rectangle sum will be

$$\begin{aligned} A_L &= \frac{2}{n} \times \sqrt{1+x_0^3} + \frac{2}{n} \times \sqrt{1+x_1^3} + \dots + \frac{2}{n} \times \sqrt{1+x_{n-1}^3} \\ &= \frac{2}{n} \sum_{i=0}^{n-1} \sqrt{1+x_i^3} \end{aligned}$$

and the upper rectangle sum will be

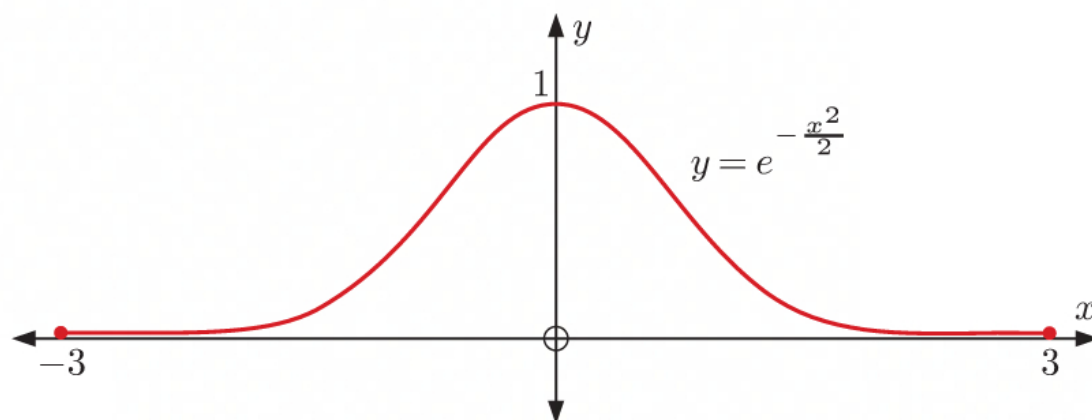
$$\begin{aligned} A_U &= \frac{2}{n} \sqrt{1+x_1^3} + \frac{2}{n} \sqrt{1+x_2^3} + \dots + \frac{2}{n} \sqrt{1+x_n^3} \\ &= \frac{2}{n} \sum_{i=1}^n \sqrt{1+x_i^3} \end{aligned}$$

**b**

$n$	$A_L$	$A_U$
50	3.2016	3.2816
100	3.2214	3.2614
500	3.2373	3.2453

**c**  $\int_0^2 \sqrt{1+x^3} dx \approx 3.24$

**3 a**



- b** Using provided software with the following settings:

From: 0

To: 3

Method: Upper/lower rectangles

Partitions: 2250

we find that  $A_L \approx 1.2493$  and  $A_U \approx 1.2506$

- c** Since  $y = e^{-\frac{x^2}{2}}$  is symmetrical about the  $y$ -axis, the lower and upper rectangle sums for  $-3 \leq x \leq 0$  with  $n = 2250$  is equivalent to the lower and upper rectangle sums for  $0 \leq x \leq 3$  with  $n = 2250$ .

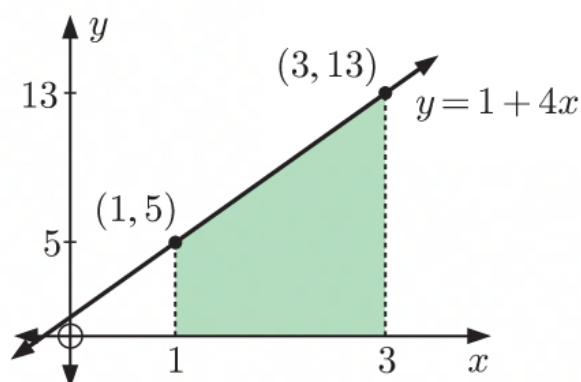
$\therefore A_L \approx 1.2493$  and  $A_U \approx 1.2506$

**d** Using lower rectangles,  $\int_{-3}^3 e^{-\frac{x^2}{2}} dx \approx 2 \times 1.2493 \approx 2.4986$

Using upper rectangles,  $\int_{-3}^3 e^{-\frac{x^2}{2}} dx \approx 2 \times 1.2506 \approx 2.5012$

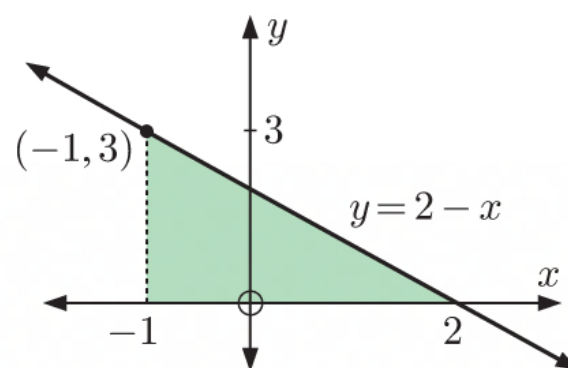
So,  $\int_{-3}^3 e^{-\frac{x^2}{2}} dx \approx \frac{2.4986 + 2.5012}{2} \approx 2.4999$ ,  $\sqrt{2\pi} \approx 2.5066$

**4 a**



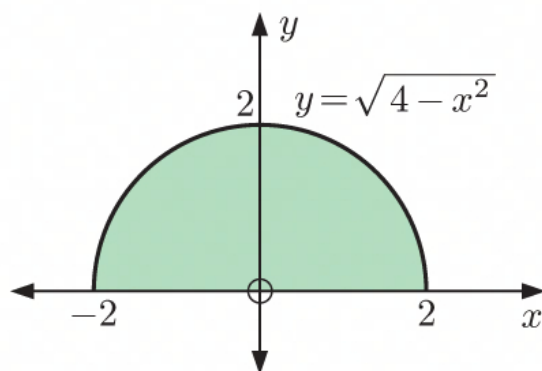
$$\begin{aligned} \int_1^3 (1 + 4x) dx &= \text{shaded area} \\ &= \left( \frac{5 + 13}{2} \right) \times 2 \\ &= 18 \end{aligned}$$

**b**



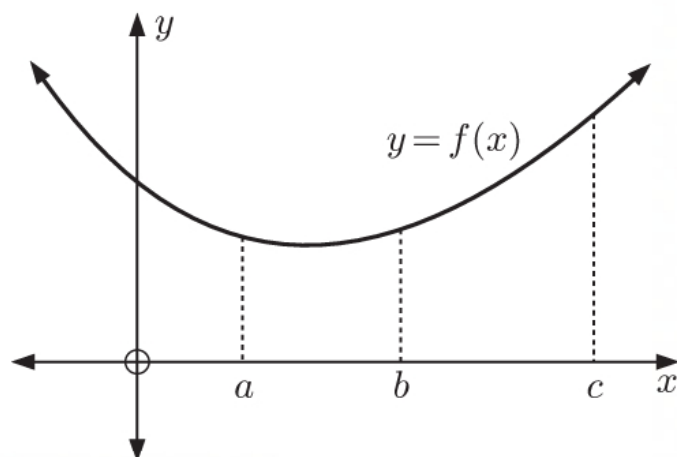
$$\begin{aligned} \int_{-1}^2 (2 - x) dx &= \text{shaded area} \\ &= \frac{1}{2}(3 \times 3) \\ &= 4.5 \end{aligned}$$

**c**



$$\begin{aligned} \int_{-2}^2 \sqrt{4 - x^2} dx &= \text{shaded area} \\ &= \frac{1}{2}(\pi \times 2^2) \\ &= 2\pi \end{aligned}$$

**5 a**



**i** From the diagram, we can see that there is no area under the curve for  $a \leq x \leq a$ .

$$\therefore \int_a^a f(x) dx = 0 \text{ for any positive function } f(x).$$

**ii** From the diagram, we can see that the area under the curve for  $a \leq x \leq c$  is made up of the area under the curve for  $a \leq x \leq b$  and  $b \leq x \leq c$ .

$$\begin{aligned} \therefore \int_a^b f(x) dx + \int_b^c f(x) dx &= \int_a^c f(x) dx \text{ for any positive function } f(x), \\ &\text{provided that } a \leq b \leq c. \end{aligned}$$

**b i**  $\int_5^5 f(x) dx = 0 \quad \left\{ \int_a^a f(x) dx = 0 \right\}$

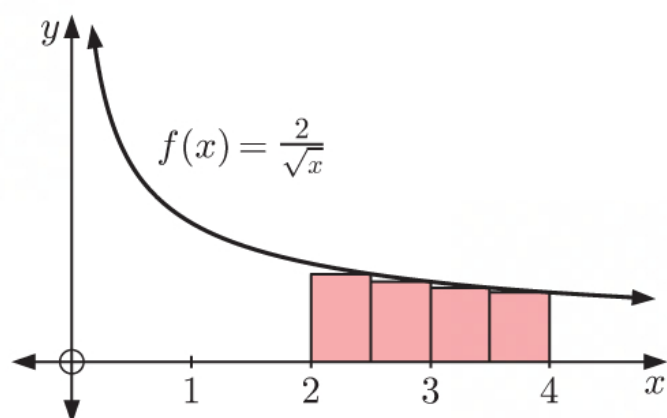
$$\begin{aligned}
 \text{ii} \quad & \int_2^9 f(x) \, dx \\
 &= \int_2^5 f(x) \, dx + \int_5^9 f(x) \, dx \quad \left\{ \int_a^b f(x) \, dx + \int_b^c f(x) \, dx = \int_a^c f(x) \, dx \right\} \\
 &= 10 + 12 \\
 &= 22
 \end{aligned}$$

## ACTIVITY

## THE TRAPEZOIDAL RULE

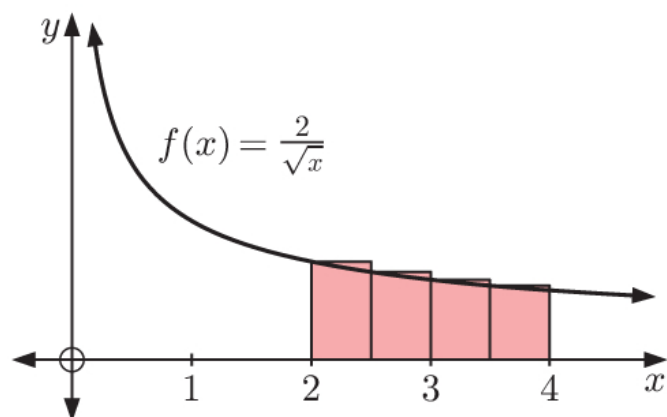
1 The rectangles are  $\frac{4-2}{4} = \frac{1}{2}$  units wide.

a i



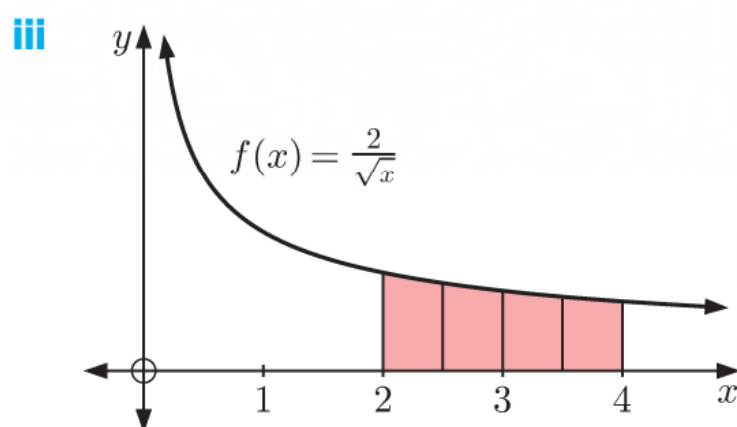
$$\begin{aligned}
 A_L &= \frac{1}{2} \times f\left(\frac{5}{2}\right) + \frac{1}{2} \times f(3) + \frac{1}{2} \times f\left(\frac{7}{2}\right) + \frac{1}{2} \times f(4) \\
 &= \left( \frac{1}{2} \times \frac{2}{\sqrt{\frac{5}{2}}} \right) + \left( \frac{1}{2} \times \frac{2}{\sqrt{3}} \right) + \left( \frac{1}{2} \times \frac{2}{\sqrt{\frac{7}{2}}} \right) + \left( \frac{1}{2} \times \frac{2}{\sqrt{4}} \right) \\
 &\approx 2.244
 \end{aligned}$$

ii



$$\begin{aligned}
 A_U &= \frac{1}{2} \times f(2) + \frac{1}{2} \times f\left(\frac{5}{2}\right) + \frac{1}{2} \times f(3) + \frac{1}{2} \times f\left(\frac{7}{2}\right) \\
 &= \left( \frac{1}{2} \times \frac{2}{\sqrt{2}} \right) + \left( \frac{1}{2} \times \frac{2}{\sqrt{\frac{5}{2}}} \right) + \left( \frac{1}{2} \times \frac{2}{\sqrt{3}} \right) + \left( \frac{1}{2} \times \frac{2}{\sqrt{\frac{7}{2}}} \right) \\
 &\approx 2.451
 \end{aligned}$$





$A_{\text{trapezoidal}}$

$$\begin{aligned}
 &= \frac{1}{2} \left( \frac{f(2) + f(\frac{5}{2})}{2} \right) + \frac{1}{2} \left( \frac{f(\frac{5}{2}) + f(3)}{2} \right) + \frac{1}{2} \left( \frac{f(3) + f(\frac{7}{2})}{2} \right) + \frac{1}{2} \left( \frac{f(\frac{7}{2}) + f(4)}{2} \right) \\
 &= \frac{1}{4} (f(2) + 2f(\frac{5}{2}) + 2f(3) + 2f(\frac{7}{2}) + f(4)) \\
 &= \frac{1}{4} \left( \frac{2}{\sqrt{2}} + 2 \times \frac{2}{\sqrt{\frac{5}{2}}} + 2 \times \frac{2}{\sqrt{3}} + 2 \times \frac{2}{\sqrt{\frac{7}{2}}} + \frac{2}{\sqrt{4}} \right) \\
 &\approx 2.348
 \end{aligned}$$

b

$$\begin{aligned}
 \frac{A_L + A_U}{2} &\approx \frac{2.244 \times 2.451}{2} \\
 &\approx 2.348 \\
 &\approx A_{\text{trapezoidal}}
 \end{aligned}$$

In fact,  $\frac{A_L + A_U}{2} = A_{\text{trapezoidal}}$

- 2 The trapezoidal rule will give the exact area between  $f(x)$  and the  $x$ -axis when  $f(x)$  is a straight line of the form  $y = mx + b$ .

3 a

$n$	Area estimate
8	3.089 82
40	3.136 95
100	3.140 42
1000	3.141 56

b

$$3\frac{10}{71} < \pi < 3\frac{1}{7} \quad \text{is approximately} \\ 3.140\,85 < \pi < 3.142\,86$$

Our estimate of  $\pi$  is more accurate than Archimedes' when  $n = 1000$ .

## INVESTIGATION 2

$$\int_0^b x^k dx$$

- 1 The  $x$ -coordinate of the right base point of the  $i$ th rectangle is  $r^{i-1} \times b$ .

$\therefore$  the height of the  $i$ th rectangle is  $(r^{i-1} \times b)^k$

The width of the first rectangle is  $b - rb = b(1 - r)$

The width of the second rectangle is  $rb - r^2b = br(1 - r)$

The width of the third rectangle is  $r^2b - r^3b = br^2(1 - r)$  and so on

$\therefore$  the width of the  $i$ th rectangle is  $br^{i-1}(1 - r)$

$$\begin{aligned}
\therefore \text{ the } i\text{th rectangle has area} &= \text{height} \times \text{width} \\
&= (r^{i-1} \times b)^k \times br^{i-1}(1-r) \\
&= (r^{i-1})^k b^k \times br^{i-1}(1-r) \\
&= b^{k+1}(1-r)(r^{i-1})^{k+1} \\
&= b^{k+1}(1-r)(r^{k+1})^{i-1}
\end{aligned}$$

**2** The total area of the  $n$  rectangles

$$\begin{aligned}
&= \sum_{i=1}^n b^{k+1}(1-r)(r^{k+1})^{i-1} \\
&= b^{k+1}(1-r) \sum_{i=1}^n (r^{k+1})^{i-1} \\
&= b^{k+1}(1-r) \frac{1 - (r^{k+1})^n}{1 - r^{k+1}} \quad \left\{ \sum_{i=1}^n (r^{k+1})^{i-1} \text{ is a geometric series with } u_1 = 1, r = r^{k+1} \right\}
\end{aligned}$$

**3 a** The  $x$ -coordinate of the left base point of the  $n$ th rectangle is  $r^n b$ .

Since  $0 < r < 1$ , as  $n \rightarrow \infty$ ,  $r^n \rightarrow 0$   
 $\therefore$  as  $n \rightarrow \infty$ ,  $r^n b \rightarrow 0$ .

So, the left base point of the left-most rectangle approaches 0, and the right base point of the right-most rectangle is  $b$ .

$\therefore$  in the limit as  $n \rightarrow \infty$ , the rectangles include the entire area under the curve between  $x = 0$  and  $x = b$ .

**b** As  $n \rightarrow \infty$ ,  $(r^{k+1})^n \rightarrow 0$

$$\begin{aligned}
\therefore \text{ as } n \rightarrow \infty, \text{ the total area of the rectangles} &\rightarrow \frac{b^{k+1}(1-r)}{1-r^{k+1}} \\
&= \frac{b^{k+1} \cancel{(1-r)}}{\cancel{(1-r)}(1+r+r^2+\dots+r^k)} \\
&= \frac{b^{k+1}}{1+r+r^2+\dots+r^k}
\end{aligned}$$

**4 a** As  $r \rightarrow 1$ , the widths of the rectangles approach 0, and the area of the rectangles which lie above the curve approaches 0.

$\therefore$  in the limit as  $r \rightarrow 1$ , the rectangles form the area under the curve between  $x = 0$  and  $x = b$ .

**b** As  $n \rightarrow \infty$ , the total area of the rectangles  $\rightarrow \frac{b^{k+1}}{1+r+r^2+\dots+r^k}$  {from **3 b**}

$$\text{And as } r \rightarrow 1, \quad \frac{b^{k+1}}{1+r+r^2+\dots+r^k} \rightarrow \frac{b^{k+1}}{1+\underbrace{1+1+\dots+1}_{k \text{ of these}}} = \frac{b^{k+1}}{k+1}$$

So, the area under the curve  $y = x^k$  between  $x = 0$  and  $x = b$  is  $\frac{b^{k+1}}{k+1}$  or

$$\int_0^b x^k dx = \frac{b^{k+1}}{k+1}.$$

**5**  $\int_0^1 x^4 dx = \frac{1^4}{4+1} = \frac{1}{5}$  which agrees with the result of **Example 1**.

## EXERCISE 20C

$$1 \quad a \quad i \quad \frac{d}{dx}(x^2) = 2x$$

$$\therefore \frac{d}{dx}\left(\frac{1}{2}x^2\right) = x$$

$\therefore$  the antiderivative of  $x$  is

$$\frac{1}{2}x^2 \quad \text{or} \quad \frac{x^2}{2}.$$

$$iii \quad \frac{d}{dx}(x^6) = 6x^5$$

$$\therefore \frac{d}{dx}\left(\frac{1}{6}x^6\right) = x^5$$

$\therefore$  the antiderivative of  $x^5$  is

$$\frac{1}{6}x^6 \quad \text{or} \quad \frac{x^6}{6}.$$

$$v \quad \frac{d}{dx}(x^{-3}) = -3x^{-4}$$

$$\therefore \frac{d}{dx}\left(-\frac{1}{3}x^{-3}\right) = x^{-4}$$

$\therefore$  the antiderivative of  $x^{-4}$  is

$$-\frac{1}{3}x^{-3} = -\frac{1}{3x^3}.$$

$$vii \quad \frac{d}{dx}(x^{\frac{2}{3}}) = \frac{2}{3}x^{-\frac{1}{3}}$$

$$\therefore \frac{d}{dx}\left(\frac{3}{2}x^{\frac{2}{3}}\right) = x^{-\frac{1}{3}}$$

$\therefore$  the antiderivative of  $x^{-\frac{1}{3}}$  is  $\frac{3}{2}x^{\frac{2}{3}}$ .

$$ii \quad \frac{d}{dx}(x^3) = 3x^2$$

$$\therefore \frac{d}{dx}\left(\frac{1}{3}x^3\right) = x^2$$

$\therefore$  the antiderivative of  $x^2$  is

$$\frac{1}{3}x^3 \quad \text{or} \quad \frac{x^3}{3}.$$

$$iv \quad \frac{d}{dx}(x^{-1}) = -x^{-2}$$

$$\therefore \frac{d}{dx}(-x^{-1}) = x^{-2}$$

$\therefore$  the antiderivative of  $x^{-2}$  is

$$-x^{-1} \quad \text{or} \quad -\frac{1}{x}.$$

$$vi \quad \frac{d}{dx}(x^{\frac{4}{3}}) = \frac{4}{3}x^{\frac{1}{3}}$$

$$\therefore \frac{d}{dx}\left(\frac{3}{4}x^{\frac{4}{3}}\right) = x^{\frac{1}{3}}$$

$\therefore$  the antiderivative of  $x^{\frac{1}{3}}$  is  $\frac{3}{4}x^{\frac{4}{3}}$ .

$$viii \quad \frac{d}{dx}(x^{\frac{5}{3}}) = \frac{5}{3}x^{\frac{2}{3}}$$

$$\therefore \frac{d}{dx}\left(\frac{3}{5}x^{\frac{5}{3}}\right) = x^{\frac{2}{3}}$$

$\therefore$  the antiderivative of  $x^{\frac{2}{3}}$  is  $\frac{3}{5}x^{\frac{5}{3}}$ .

**b** The antiderivative of  $x^n$  is  $\frac{x^{n+1}}{n+1}$ , for  $n \neq -1$ .

$$2 \quad a \quad i \quad \frac{d}{dx}(e^{3x}) = 3e^{3x}$$

$$\therefore \frac{d}{dx}\left(\frac{1}{3}e^{3x}\right) = e^{3x}$$

$\therefore$  the antiderivative of  $e^{3x}$  is  $\frac{1}{3}e^{3x}$ .

$$iii \quad \frac{d}{dx}(e^{\frac{1}{2}x}) = \frac{1}{2}e^{\frac{1}{2}x}$$

$$\therefore \frac{d}{dx}(2e^{\frac{1}{2}x}) = e^{\frac{1}{2}x}$$

$\therefore$  the antiderivative of  $e^{\frac{1}{2}x}$  is  $2e^{\frac{1}{2}x}$ .

$$ii \quad \frac{d}{dx}(e^{5x}) = 5e^{5x}$$

$$\therefore \frac{d}{dx}\left(\frac{1}{5}e^{5x}\right) = e^{5x}$$

$\therefore$  the antiderivative of  $e^{5x}$  is  $\frac{1}{5}e^{5x}$ .

$$iv \quad \frac{d}{dx}(e^{0.01x}) = 0.01e^{0.01x}$$

$$\therefore \frac{d}{dx}(100e^{0.01x}) = e^{0.01x}$$

$\therefore$  the antiderivative of  $e^{0.01x}$  is  $100e^{0.01x}$ .



$$\text{v} \quad \frac{d}{dx}(e^{\pi x}) = \pi e^{\pi x}$$

$$\therefore \frac{d}{dx}\left(\frac{1}{\pi}e^{\pi x}\right) = e^{\pi x}$$

$\therefore$  the antiderivative of  $e^{\pi x}$  is  $\frac{1}{\pi}e^{\pi x}$ .

$$\text{vi} \quad \frac{d}{dx}(e^{\frac{x}{3}}) = \frac{1}{3}e^{\frac{x}{3}}$$

$$\therefore \frac{d}{dx}(3e^{\frac{x}{3}}) = e^{\frac{x}{3}}$$

$\therefore$  the antiderivative of  $e^{\frac{x}{3}}$  is  $3e^{\frac{x}{3}}$ .

**b** The antiderivative of  $e^{kx}$  is  $\frac{1}{k}e^{kx}$ , where  $k \neq 0$  is a constant.

$$\text{3 a i} \quad \frac{d}{dx}(2^x) = 2^x \ln 2$$

$$\therefore \frac{d}{dx}\left(\frac{2^x}{\ln 2}\right) = 2^x$$

$\therefore$  the antiderivative of  $2^x$  is  $\frac{2^x}{\ln 2}$ .

$$\text{ii} \quad \frac{d}{dx}(2^{3x}) = 2^{3x} \times 3 \ln 2$$

$$\therefore \frac{d}{dx}\left(\frac{2^{3x}}{3 \ln 2}\right) = 2^{3x}$$

$\therefore$  the antiderivative of  $2^{3x}$  is  $\frac{2^{3x}}{3 \ln 2}$ .

$$\text{iii} \quad \frac{d}{dx}(2^{-x}) = 2^{-x} \times -\ln 2$$

$$\therefore \frac{d}{dx}\left(\frac{2^{-x}}{-\ln 2}\right) = 2^{-x}$$

$\therefore$  the antiderivative of  $2^{-x}$  is  $\frac{2^{-x}}{-\ln 2}$ .

$$\text{iv} \quad \frac{d}{dx}(2^{\frac{x}{2}}) = 2^{\frac{x}{2}} \times \frac{1}{2} \ln 2$$

$$\therefore \frac{d}{dx}\left(\frac{2^{\frac{x}{2}}}{\frac{1}{2} \ln 2}\right) = 2^{\frac{x}{2}}$$

$\therefore$  the antiderivative of  $2^{\frac{x}{2}}$  is  $\frac{2^{\frac{x}{2}}}{\frac{1}{2} \ln 2}$ .

**b** The antiderivative of  $2^{kx}$  is  $\frac{2^{kx}}{k \ln 2}$ , where  $k \neq 0$  is a constant.

$$\text{4 a} \quad \frac{d}{dx}(x^3 + x^2) = 3x^2 + 2x$$

$$\therefore \frac{d}{dx}(2x^3 + 2x^2) = 6x^2 + 4x$$

$\therefore$  the antiderivative of  $6x^2 + 4x$  is  $2x^3 + 2x^2$ .

$$\text{b} \quad \frac{d}{dx}(x\sqrt{x}) = \frac{d}{dx}(x^{\frac{3}{2}}) = \frac{3}{2}x^{\frac{1}{2}}$$

$$= \frac{3}{2}\sqrt{x}$$

$$\therefore \frac{d}{dx}\left(\frac{2}{3}x\sqrt{x}\right) = \sqrt{x}$$

$\therefore$  the antiderivative of  $\sqrt{x}$  is  $\frac{2}{3}x\sqrt{x}$ .

$$\text{c} \quad \frac{d}{dx}\left(\frac{1}{\sqrt{x}}\right) = \frac{d}{dx}(x^{-\frac{1}{2}}) = -\frac{1}{2}x^{-\frac{3}{2}}$$

$$= -\frac{1}{2x\sqrt{x}}$$

$$\therefore \frac{d}{dx}\left(-\frac{2}{\sqrt{x}}\right) = \frac{1}{x\sqrt{x}}$$

$\therefore$  the antiderivative of  $\frac{1}{x\sqrt{x}}$  is  $-\frac{2}{\sqrt{x}}$ .

$$\text{d} \quad \frac{d}{dx}(\cos x) = -\sin x$$

$$\therefore \frac{d}{dx}(-\cos x) = \sin x$$

$\therefore$  the antiderivative of  $\sin x$  is  $-\cos x$ .

$$\text{e} \quad \frac{d}{dx}[(4x+1)^4] = 4(4x+1)^3 \times 4$$

{chain rule}

$$= 16(4x+1)^3$$

$$\therefore \frac{d}{dx}\left[\frac{1}{16}(4x+1)^4\right] = (4x+1)^3$$

$\therefore$  the antiderivative of  $(4x+1)^3$  is  $\frac{1}{16}(4x+1)^4$ .

$$\text{f} \quad \frac{d}{dx}\left(\tan \frac{x}{3}\right) = \frac{1}{3}\sec^2\left(\frac{x}{3}\right) \quad \{\text{chain rule}\}$$

$$\therefore \frac{d}{dx}(3 \tan \frac{x}{3}) = \sec^2\left(\frac{x}{3}\right)$$

$\therefore$  the antiderivative of  $\sec^2\left(\frac{x}{3}\right)$  is  $3 \tan \frac{x}{3}$ .

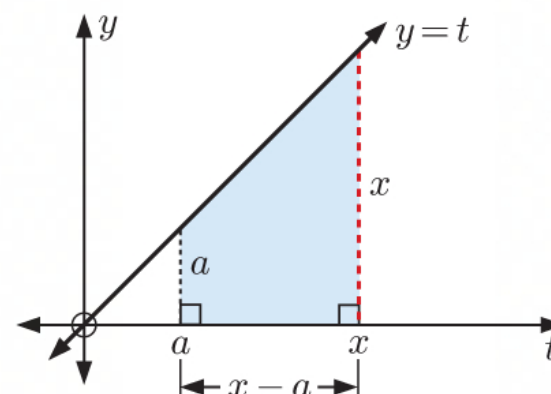


$$\begin{aligned} \mathbf{9} \quad & \frac{d}{dx} (\sin(3x - 1)) = 3 \cos(3x - 1) \quad \{\text{chain rule}\} \\ \therefore & \frac{d}{dx} \left( \frac{1}{3} \sin(3x - 1) \right) = \cos(3x - 1) \\ \therefore & \text{the antiderivative of } \cos(3x - 1) \text{ is } \frac{1}{3} \sin(3x - 1). \end{aligned}$$

**INVESTIGATION 3****THE AREA FUNCTION**

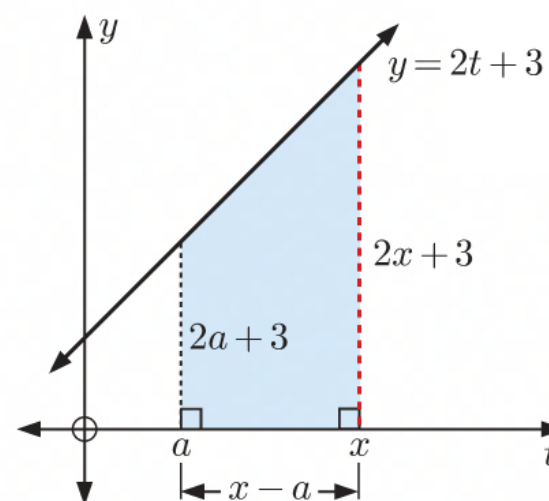
$$\begin{aligned} \mathbf{1} \quad & F(t) = 5t \\ \therefore & F'(t) = 5 = f(t) \\ \therefore & F(t) \text{ is the antiderivative of } f(t). \end{aligned}$$

$$\begin{aligned} \mathbf{2} \quad \mathbf{a} \quad A(x) &= \left( \frac{x+a}{2} \right) (x-a) \\ &= \frac{x^2 - a^2}{2} \\ &= \frac{x^2}{2} - \frac{a^2}{2} \\ &= F(x) - F(a) \quad \text{where } F(t) = \frac{t^2}{2} \end{aligned}$$



$$\begin{aligned} \mathbf{b} \quad & F(t) = \frac{t^2}{2} \\ \therefore & F'(t) = t = f(t) \\ \therefore & F(t) \text{ is the antiderivative of } f(t). \end{aligned}$$

$$\begin{aligned} \mathbf{3} \quad \mathbf{a} \quad A(x) &= \left( \frac{2x+3+2a+3}{2} \right) (x-a) \\ &= \left( \frac{2(x+a)+6}{2} \right) (x-a) \\ &= (x+a+3)(x-a) \\ &= (x+a)(x-a) + 3(x-a) \\ &= x^2 - a^2 + 3x - 3a \\ &= x^2 + 3x - (a^2 + 3a) \\ &= F(x) - F(a) \quad \text{where } F(t) = t^2 + 3t \end{aligned}$$

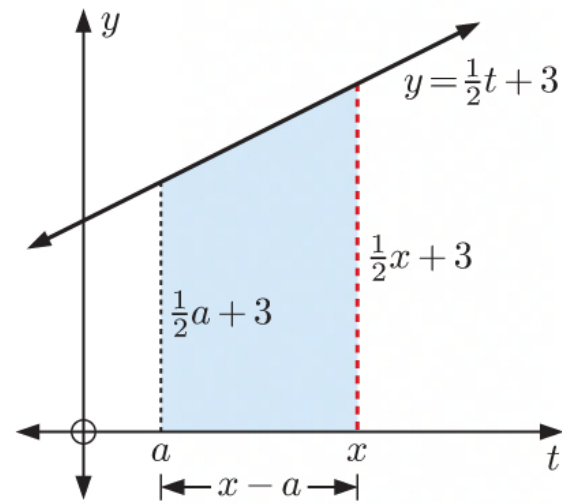


$$\begin{aligned} \mathbf{b} \quad & F(t) = t^2 + 3t \\ \therefore & F'(t) = 2t + 3 = f(t) \\ \therefore & F(t) \text{ is the antiderivative of } f(t). \end{aligned}$$

**4 a** Consider  $f(t) = \frac{1}{2}t + 3$ .

The corresponding area function is

$$\begin{aligned}
 A(x) &= \int_a^x \left(\frac{1}{2}t + 3\right) dt \\
 &= \text{shaded area} \\
 &= \left(\frac{\frac{1}{2}x + 3 + \frac{1}{2}a + 3}{2}\right)(x - a) \\
 &= \left(\frac{1}{4}x + \frac{3}{2} + \frac{1}{4}a + \frac{3}{2}\right)(x - a) \\
 &= \frac{1}{4}x^2 - \cancel{\frac{1}{4}ax} + \frac{3}{2}x - \frac{3}{2}a + \cancel{\frac{1}{4}ax} - \frac{1}{4}a^2 + \frac{3}{2}x - \frac{3}{2}a \\
 &= \frac{1}{4}x^2 + 3x - \frac{1}{4}a^2 - 3a \\
 &= \frac{1}{4}x^2 + 3x - \left(\frac{1}{4}a^2 + 3a\right) \\
 &= F(x) - F(a) \quad \text{where } F(t) = \frac{1}{4}t^2 + 3t
 \end{aligned}$$



Now  $F(t) = \frac{1}{4}t^2 + 3t$

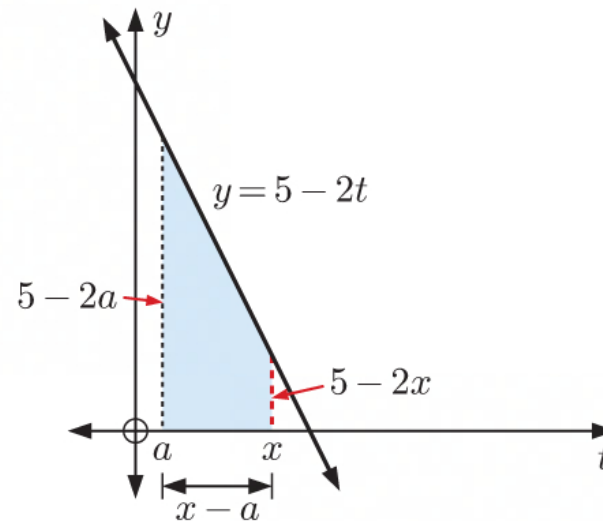
$$\therefore F'(t) = \frac{1}{2}t + 3 = f(t)$$

$\therefore F(t)$  is the antiderivative of  $f(t)$ .

**b** Consider  $f(t) = 5 - 2t$ .

The corresponding area function is

$$\begin{aligned}
 A(x) &= \int_a^x (5 - 2t) dt \\
 &= \text{shaded area} \\
 &= \left(\frac{5 - 2x + 5 - 2a}{2}\right)(x - a) \\
 &= \left(\frac{5}{2} - x + \frac{5}{2} - a\right)(x - a) \\
 &= \frac{5}{2}x - \frac{5}{2}a - x^2 + \cancel{ax} + \frac{5}{2}x - \frac{5}{2}a - \cancel{ax} + a^2 \\
 &= 5x - x^2 - 5a + a^2 \\
 &= 5x - x^2 - (5a - a^2) \\
 &= F(x) - F(a) \quad \text{where } F(t) = 5t - t^2
 \end{aligned}$$



Now  $F(t) = 5t - t^2$

$$\therefore F'(t) = 5 - 2t = f(t)$$

$\therefore F(t)$  is the antiderivative of  $f(t)$ .

**5**  $f(t) = 3t^2 + 4t + 5$

We predict that  $F(t)$  is the antiderivative of  $f(t)$ .

$$\frac{d}{dt}(t^3 + 2t^2 + 5t) = 3t^2 + 4t + 5 = f(t)$$

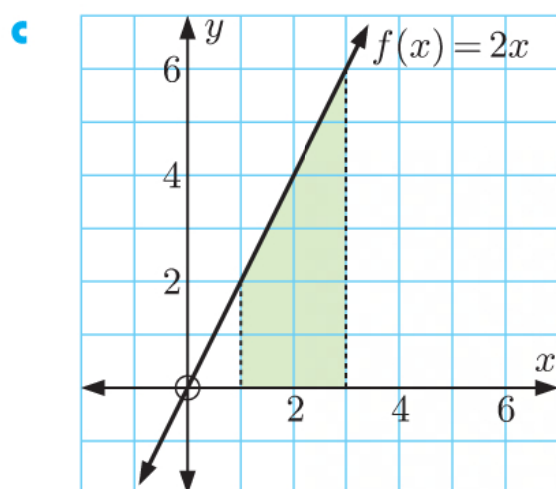
$$\therefore F(t) = t^3 + 2t^2 + 5t$$

**EXERCISE 20D**

**1 a**  $\frac{d}{dx}(x^2) = 2x$

$\therefore$  the antiderivative of  $f(x) = 2x$  is  $F(x) = x^2$ .

**b** 
$$\begin{aligned}\int_1^3 2x \, dx &= F(3) - F(1) \\ &= 3^2 - 1^2 \\ &= 8 \text{ units}^2\end{aligned}$$



$$\begin{aligned}\int_1^3 2x \, dx &= \text{shaded area} \\ &= \left(\frac{2+6}{2}\right) \times 2 \\ &= 8 \text{ units}^2\end{aligned}$$

**2 a** 
$$\begin{aligned}\frac{d}{dx}(x\sqrt{x}) &= \frac{d}{dx}(x^{\frac{3}{2}}) \\ &= \frac{3}{2}x^{\frac{1}{2}} \\ &= \frac{3}{2}\sqrt{x}\end{aligned}$$

$$\begin{aligned}\therefore \frac{d}{dx}\left(\frac{2}{3}x^{\frac{3}{2}}\right) &= x^{\frac{1}{2}} \\ &= \sqrt{x}\end{aligned}$$

$\therefore$  the antiderivative of  $f(x) = \sqrt{x}$  is  $F(x) = \frac{2}{3}x^{\frac{3}{2}}$ .

**b** 
$$\begin{aligned}\int_0^1 \sqrt{x} \, dx &= F(1) - F(0) \\ &= \frac{2}{3} - 0 \\ &= \frac{2}{3} \text{ units}^2\end{aligned}$$

**c**  $\frac{2}{3} \approx 0.67$  to 2 significant figures  
 $\therefore$  the answer is the same as **Exercise 20B question 1**.

**3 a**  $\frac{d}{dx}(x^4) = 4x^3$

$$\therefore \frac{d}{dx}\left(\frac{1}{4}x^4\right) = x^3$$

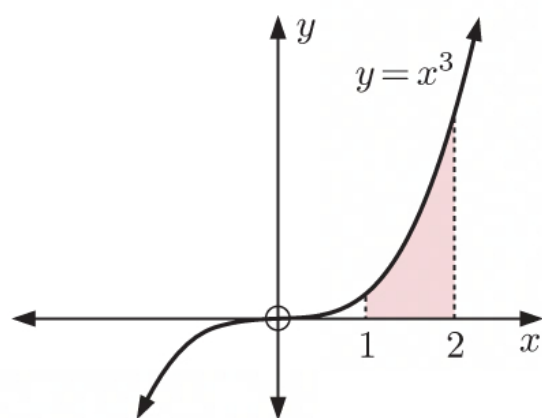
$\therefore$  the antiderivative of  $f(x) = x^3$  is  $F(x) = \frac{1}{4}x^4$ .

**i** 
$$\begin{aligned}\int_0^2 x^3 \, dx \\ &= F(2) - F(0) \\ &= 4 - 0 \\ &= 4 \text{ units}^2\end{aligned}$$

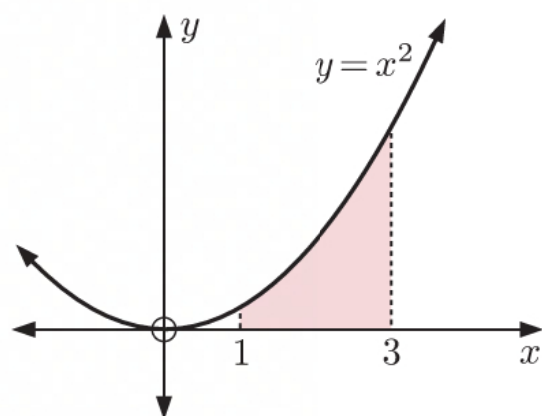
**ii** 
$$\begin{aligned}\int_2^3 x^3 \, dx \\ &= F(3) - F(2) \\ &= \frac{81}{4} - 4 \\ &= 16\frac{1}{4} \text{ units}^2\end{aligned}$$

**iii** 
$$\begin{aligned}\int_0^3 x^3 \, dx \\ &= F(3) - F(0) \\ &= \frac{81}{4} - 0 \\ &= 20\frac{1}{4} \text{ units}^2\end{aligned}$$

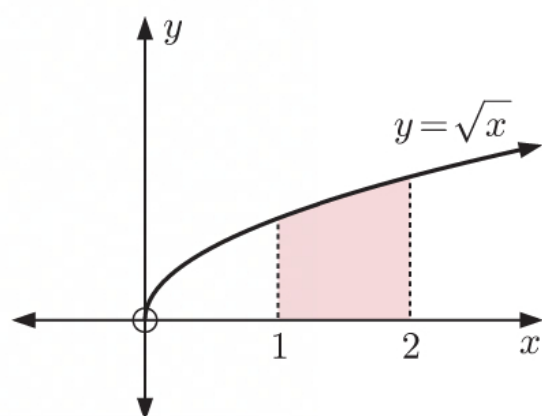
**b** 
$$\int_0^3 x^3 \, dx = \int_0^2 x^3 \, dx + \int_2^3 x^3 \, dx$$

**4 a**
 $f(x) = x^3$  has antiderivative  $F(x) = \frac{x^4}{4}$ 

$$\begin{aligned}\therefore \text{shaded area} &= \int_1^2 x^3 dx \\ &= F(2) - F(1) \\ &= 4 - \frac{1}{4} \\ &= 3\frac{3}{4} \text{ units}^2\end{aligned}$$

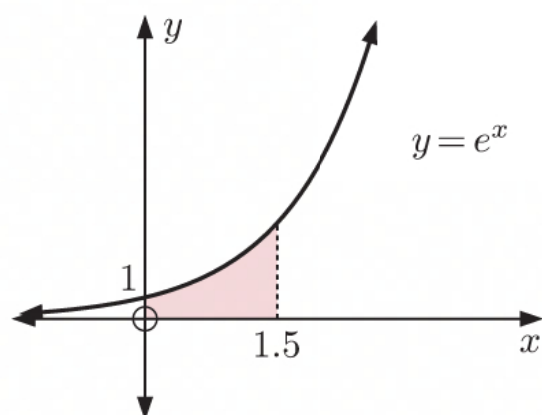
**b**
 $f(x) = x^2$  has antiderivative  $F(x) = \frac{x^3}{3}$ 

$$\begin{aligned}\therefore \text{shaded area} &= \int_1^3 x^2 dx \\ &= F(3) - F(1) \\ &= 9 - \frac{1}{3} \\ &= 8\frac{2}{3} \text{ units}^2\end{aligned}$$

**c**
 $f(x) = \sqrt{x} = x^{\frac{1}{2}}$  has antiderivative

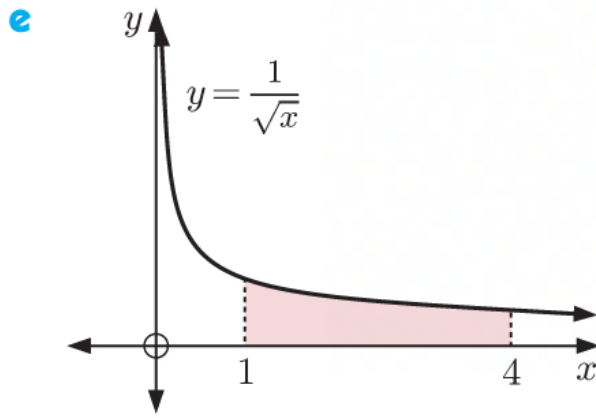
$$F(x) = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} = \frac{2}{3}x\sqrt{x}$$

$$\begin{aligned}\therefore \text{shaded area} &= \int_1^2 \sqrt{x} dx \\ &= F(2) - F(1) \\ &= \frac{2}{3}(2\sqrt{2}) - \frac{2}{3}(1\sqrt{1}) \\ &= \frac{4\sqrt{2}}{3} - \frac{2}{3} \\ &= \frac{4\sqrt{2} - 2}{3} \text{ units}^2\end{aligned}$$

**d**
 $f(x) = e^x$  has antiderivative  $F(x) = e^x$ 

$$\begin{aligned}\therefore \text{shaded area} &= \int_0^{1.5} e^x dx \\ &= F(1.5) - F(0) \\ &= e^{1.5} - e^0 \\ &= (e^{\frac{3}{2}} - 1) \text{ units}^2\end{aligned}$$





$$f(x) = \frac{1}{\sqrt{x}} = x^{-\frac{1}{2}} \text{ has antiderivative}$$

$$F(x) = \frac{x^{\frac{1}{2}}}{\frac{1}{2}} = 2\sqrt{x}$$

$$\begin{aligned} \therefore \text{shaded area} &= \int_1^4 \frac{1}{\sqrt{x}} dx \\ &= F(4) - F(1) \\ &= 2\sqrt{4} - 2\sqrt{1} \\ &= 2 \text{ units}^2 \end{aligned}$$

**5** Let  $F(x)$  be the antiderivative of  $f(x)$  and  $G(x)$  be the antiderivative of  $g(x)$ .

**a**  $\int_a^a f(x) dx = F(a) - F(a) = 0$

$\int_a^a f(x) dx = \text{area of the region under the curve } y = f(x) \text{ between } x = a \text{ and } x = a.$

This region has 0 width, so its area = 0.

**c** 
$$\begin{aligned} \int_b^a f(x) dx &= F(a) - F(b) \\ &= -[F(b) - F(a)] \\ &= -\int_a^b f(x) dx \end{aligned}$$

**b** The antiderivative of  $f(x) = k$  is  $F(x) = kx$ .

$$\begin{aligned} \therefore \int_a^b k dx &= F(b) - F(a) \\ &= kb - ka \\ &= k(b - a) \end{aligned}$$

**d**  $\frac{d}{dx} F(x) = f(x)$

$$\therefore \frac{d}{dx} (k F(x)) = k f(x)$$

$\therefore k F(x)$  is the antiderivative of  $k f(x)$ .

$$\begin{aligned} \text{So, } \int_a^b k f(x) dx &= k F(b) - k F(a) \\ &= k[F(b) - F(a)] \\ &= k \int_a^b f(x) dx \end{aligned}$$

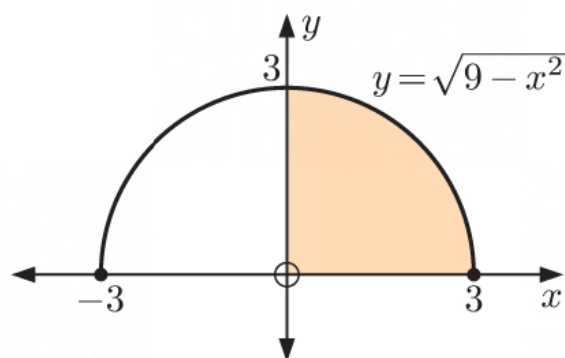
**e**  $\frac{d}{dx} F(x) = f(x) \quad \text{and} \quad \frac{d}{dx} G(x) = g(x)$

$$\therefore \frac{d}{dx} [F(x) + G(x)] = f(x) + g(x)$$

$\therefore F(x) + G(x)$  is the antiderivative of  $f(x) + g(x)$ .

$$\begin{aligned} \text{So, } \int_a^b [f(x) + g(x)] dx &= [F(b) + G(b)] - [F(a) + G(a)] \\ &= [F(b) - F(a)] + [G(b) - G(a)] \\ &= \int_a^b f(x) dx + \int_a^b g(x) dx \end{aligned}$$

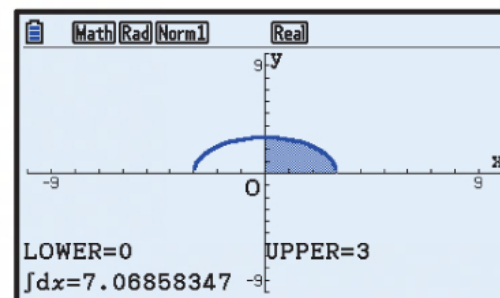
6



Using technology,  $\text{area} = \int_0^3 \sqrt{9 - x^2} \, dx \approx 7.07 \text{ units}^2$

*Check:* The area is a quarter circle with radius 3 units.

$$\begin{aligned} \therefore \text{area} &= \frac{1}{4} \pi r^2 \\ &= \frac{1}{4} \times \pi \times 3^2 \\ &= \frac{9\pi}{4} \\ &\approx 7.07 \text{ units}^2 \quad \checkmark \end{aligned}$$

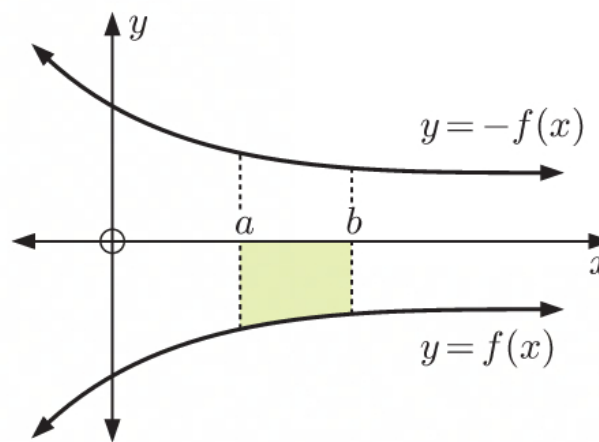


**7 a** If  $\frac{d}{dx} F(x) = f(x)$  then  $\frac{d}{dx} (-F(x)) = -f(x)$

$$\begin{aligned} \therefore \int_a^b (-f(x)) \, dx &= -F(b) - (-F(a)) \\ &= -(F(b) - F(a)) \\ &= -\int_a^b f(x) \, dx \end{aligned}$$

**b** Since  $y = -f(x)$  is a reflection of  $y = f(x)$  in the  $x$ -axis, then

$$\begin{aligned} &\text{shaded area} \\ &= \text{area between the } x\text{-axis and } y = -f(x) \\ &\text{from } x = a \text{ to } x = b \\ &= \int_a^b (-f(x)) \, dx \\ &= -\int_a^b f(x) \, dx \quad \{\text{using a}\} \end{aligned}$$

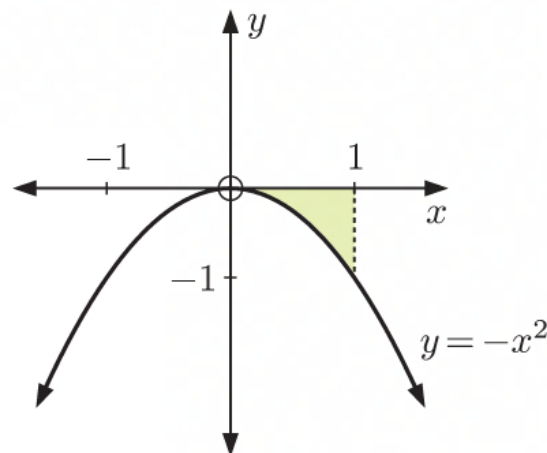


**c i**  $\int_0^1 (-x^2) \, dx = -\int_0^1 x^2 \, dx$

Now  $f(x) = x^2$  has antiderivative  $F(x) = \frac{1}{3}x^3$

$$\begin{aligned} \therefore \int_0^1 (-x^2) \, dx &= -(F(1) - F(0)) \\ &= -\left(\frac{1}{3} - 0\right) \\ &= -\frac{1}{3} \end{aligned}$$

The shaded region has area  $\frac{1}{3} \text{ units}^2$ .



$$\text{ii} \quad \int_0^1 (x^2 - x) dx = - \int_0^1 (x - x^2) dx$$

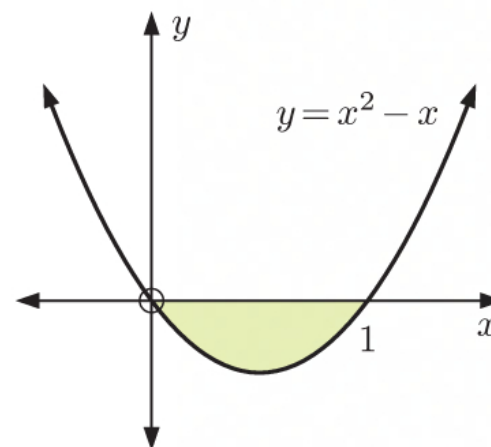
$$\{x^2 - x \leq 0 \text{ for all } 0 \leq x \leq 1\}$$

Now  $f(x) = x - x^2$  has antiderivative

$$F(x) = \frac{1}{2}x^2 - \frac{1}{3}x^3$$

$$\begin{aligned} \therefore \int_0^1 (x^2 - x) dx &= -(F(1) - F(0)) \\ &= -\left(\frac{1}{2} - \frac{1}{3} - (0 - 0)\right) \\ &= -\frac{1}{6} \end{aligned}$$

The shaded region has area  $\frac{1}{6}$  units<sup>2</sup>.



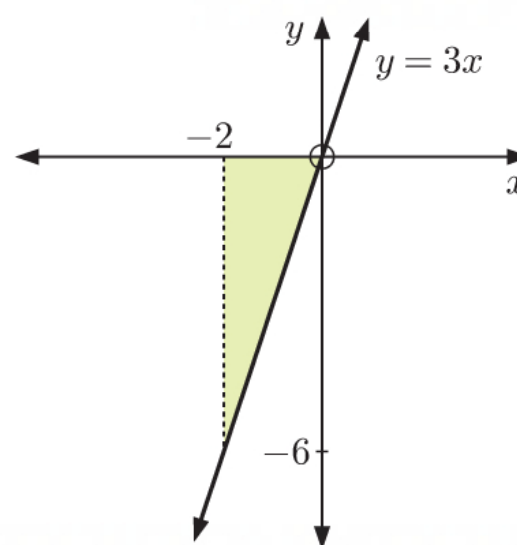
$$\text{iii} \quad \int_{-2}^0 3x dx = - \int_{-2}^0 -3x dx$$

Now  $f(x) = -3x$  has antiderivative

$$F(x) = -\frac{3}{2}x^2$$

$$\begin{aligned} \therefore \int_{-2}^0 3x dx &= -(F(0) - F(-2)) \\ &= -(0 - (-6)) \\ &= -6 \end{aligned}$$

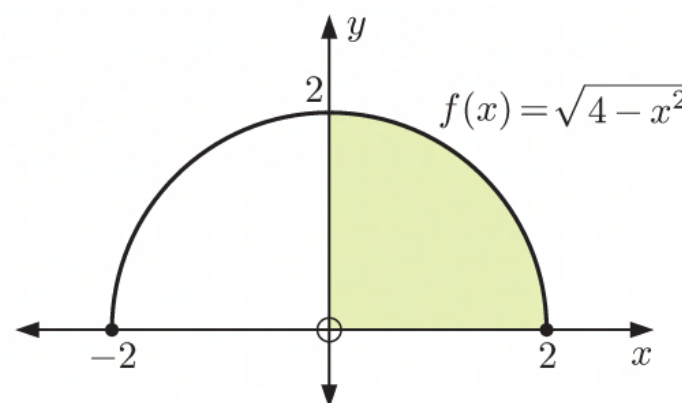
The shaded region has area 6 units<sup>2</sup>.



$$\text{d} \quad \int_0^2 \left(-\sqrt{4-x^2}\right) dx = - \int_0^2 \sqrt{4-x^2} dx$$

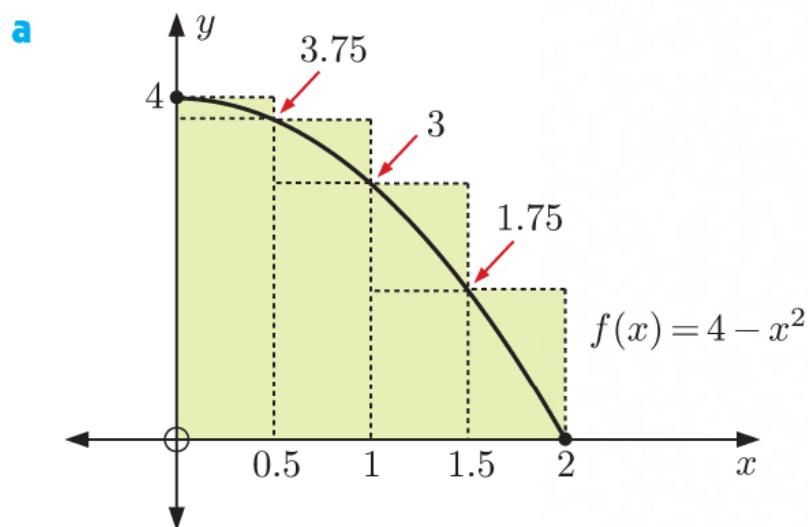
Now  $f(x) = \sqrt{4-x^2}$  is the top half of a circle with radius 2 units and centre (0, 0).

$$\begin{aligned} \therefore \int_0^2 \left(-\sqrt{4-x^2}\right) dx &= - \int_0^2 \sqrt{4-x^2} dx \\ &= -(\text{shaded area}) \\ &= -\frac{1}{4} \times \pi \times 2^2 \\ &= -\pi \end{aligned}$$



## REVIEW SET 20A

- 1 The rectangles are  $\frac{2}{4} = \frac{1}{2}$  units wide.



$$\begin{aligned} A_L &= \frac{1}{2} \left[ f\left(\frac{1}{2}\right) + f(1) + f\left(\frac{3}{2}\right) + f(2) \right] \\ &= \frac{1}{2} \left( \frac{15}{4} + 3 + \frac{7}{4} + 0 \right) \\ &= \frac{17}{4} \end{aligned}$$

$$\begin{aligned} A_U &= \frac{1}{2} \left[ f(0) + f\left(\frac{1}{2}\right) + f(1) + f\left(\frac{3}{2}\right) \right] \\ &= \frac{1}{2} \left( 4 + \frac{15}{4} + 3 + \frac{7}{4} \right) \\ &= \frac{25}{4} \end{aligned}$$

$$\therefore \frac{17}{4} < \int_0^2 (4 - x^2) dx < \frac{25}{4}$$

$$\therefore A = \frac{17}{4}, \quad B = \frac{25}{4}$$

**b**  $\int_0^2 (4 - x^2) dx \approx \frac{A+B}{2} \approx \frac{42}{8} \approx \frac{21}{4}$

**2 a**  $\frac{d}{dx}(x^5) = 5x^4$   
 $\therefore \frac{d}{dx}\left(\frac{1}{5}x^5\right) = x^4$   
 $\therefore$  the antiderivative of  $x^4$  is  $\frac{1}{5}x^5$  or  $\frac{x^5}{5}$ .

**c**  $\frac{d}{dx}(e^{-\frac{1}{2}x}) = -\frac{1}{2}e^{-\frac{1}{2}x}$   
 $\therefore \frac{d}{dx}(-2e^{-\frac{1}{2}x}) = e^{-\frac{1}{2}x}$   
 $\therefore$  the antiderivative of  $e^{-\frac{1}{2}x}$  is  $-2e^{-\frac{1}{2}x}$ .

**e**  $\frac{d}{dx}(3^x) = 3^x \ln 3$   
 $\therefore \frac{d}{dx}\left(\frac{3^x}{\ln 3}\right) = 3^x$   
 $\therefore$  the antiderivative of  $3^x$  is  $\frac{3^x}{\ln 3}$ .

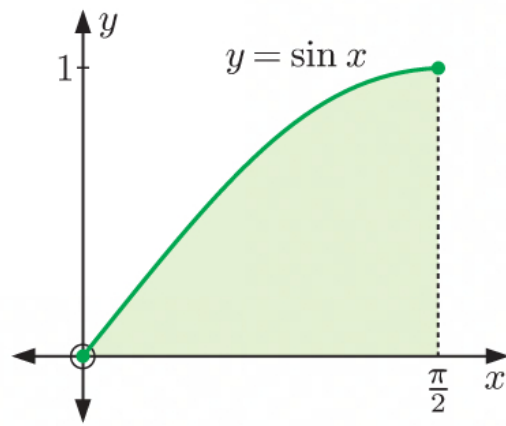
**b**  $\frac{1}{2x^2} = \frac{1}{2}x^{-2}$   
 Now,  $\frac{d}{dx}(x^{-1}) = -x^{-2}$

$$\therefore \frac{d}{dx}\left(-\frac{1}{2}x^{-1}\right) = \frac{1}{2}x^{-2}$$

$\therefore$  the antiderivative of  $\frac{1}{2x^2}$  is  $-\frac{1}{2}x^{-1} = -\frac{1}{2x}$ .

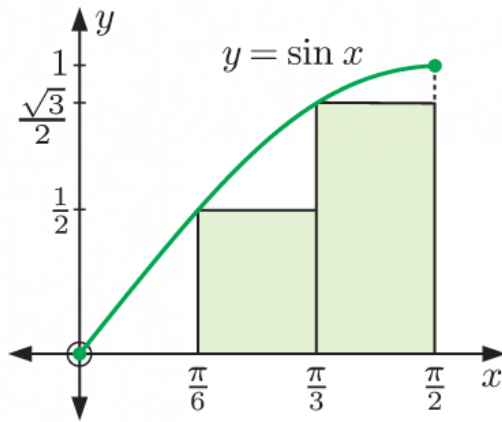
**d**  $\frac{d}{dx}(\sin x) = \cos x$   
 $\therefore$  the antiderivative of  $\cos x$  is  $\sin x$ .



**3 a**

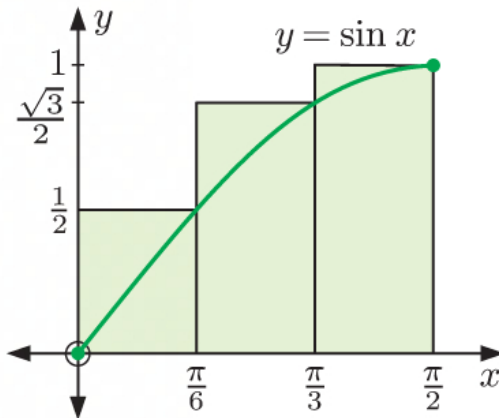
**b** The rectangles are  $\frac{\pi/2}{3} = \frac{\pi}{6}$  units wide.

Lower rectangles:



$$\begin{aligned} A_L &= \frac{\pi}{6} \left( \sin 0 + \sin \frac{\pi}{6} + \sin \frac{\pi}{3} \right) \\ &= \frac{\pi}{6} \left( 0 + \frac{1}{2} + \frac{\sqrt{3}}{2} \right) \\ &= \frac{\pi(1 + \sqrt{3})}{12} \end{aligned}$$

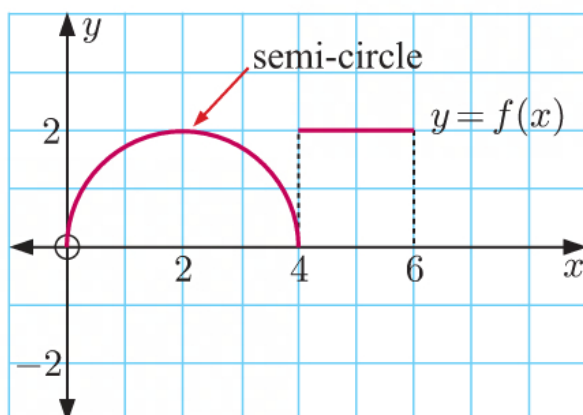
Upper rectangles:



$$\begin{aligned} A_U &= \frac{\pi}{6} \left( \sin \frac{\pi}{6} + \sin \frac{\pi}{3} + \sin \frac{\pi}{2} \right) \\ &= \frac{\pi}{6} \left( \frac{1}{2} + \frac{\sqrt{3}}{2} + 1 \right) \\ &= \frac{\pi}{6} \left( \frac{1 + \sqrt{3} + 2}{2} \right) \\ &= \frac{\pi}{6} \left( \frac{3 + \sqrt{3}}{2} \right) \\ &= \frac{\pi(3 + \sqrt{3})}{12} \end{aligned}$$

$$\frac{\pi(1 + \sqrt{3})}{12} < \int_0^{\pi/2} \sin x \, dx < \frac{\pi(3 + \sqrt{3})}{12}$$

$$\text{or } 0.715 < \int_0^{\pi/2} \sin x \, dx < 1.24$$

**4****a**

$$\begin{aligned} &\int_0^4 f(x) \, dx \\ &= \text{area of semi-circle} \\ &\quad \text{with radius 2} \\ &= \frac{1}{2} \times \pi(2)^2 \\ &= 2\pi \end{aligned}$$

**b**

$$\begin{aligned} &\int_4^6 f(x) \, dx \\ &= \text{area of square} \\ &= 2^2 \\ &= 4 \end{aligned}$$

**5 a**  $\frac{d}{dx}(x^3) = 3x^2$

$\therefore \frac{d}{dx}\left(\frac{1}{3}x^3\right) = x^2$

$\therefore$  the antiderivative of  $f(x) = x^2$  is  $F(x) = \frac{1}{3}x^3$ .

**i**  $\int_0^1 x^2 dx$

$= F(1) - F(0)$

$= \frac{1}{3} - 0$

$= \frac{1}{3} \text{ units}^2$

**ii**  $\int_1^2 x^2 dx$

$= F(2) - F(1)$

$= \frac{8}{3} - \frac{1}{3}$

$= \frac{7}{3} = 2\frac{1}{3} \text{ units}^2$

**iii**  $\int_0^2 x^2 dx$

$= F(2) - F(0)$

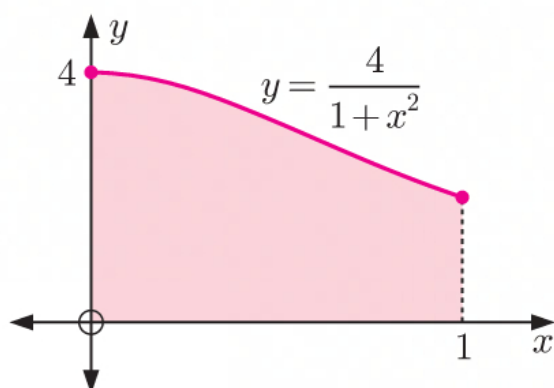
$= \frac{8}{3} - 0$

$= \frac{8}{3} = 2\frac{2}{3} \text{ units}^2$

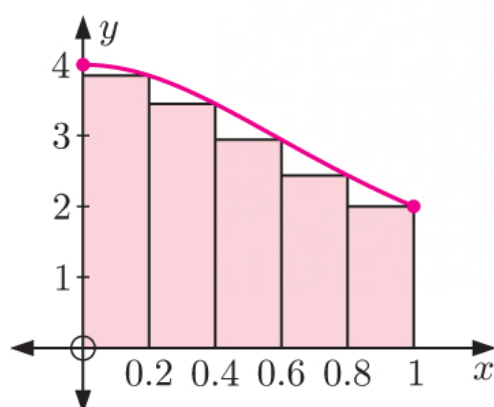
**b**  $\int_0^2 x^2 dx = \int_0^1 x^2 dx + \int_1^2 x^2 dx$

## REVIEW SET 20B

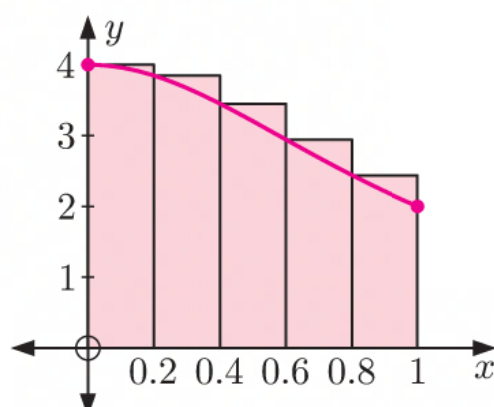
**1 a**



lower rectangles



upper rectangles

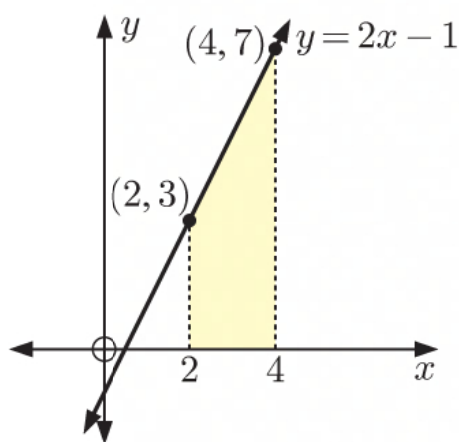


**b**

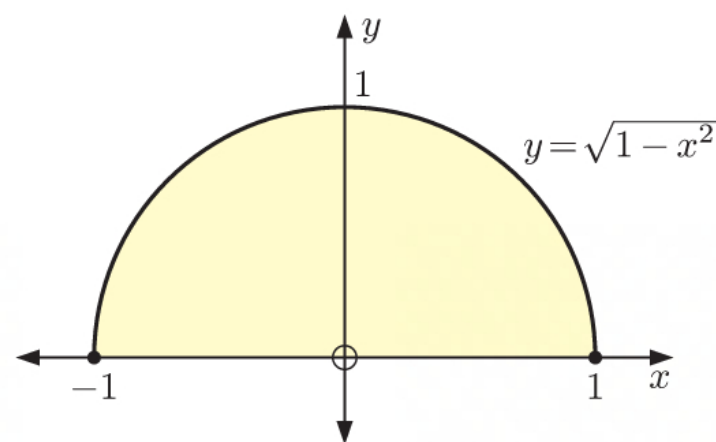
$n$	$A_L$	$A_U$
5	2.9349	3.3349
50	3.1215	3.1615
100	3.1316	3.1516
500	3.1396	3.1436

**c** Using  $n = 500$  rectangles,

$$\begin{aligned} \int_0^1 \frac{4}{1+x^2} dx &\approx \frac{A_L + A_U}{2} \\ &\approx \frac{3.1396 + 3.1436}{2} \\ &\approx 3.1416 \approx \pi \end{aligned}$$

**2 a**

$$\begin{aligned}\int_2^4 (2x - 1) dx &= \text{shaded area} \\ &= \left(\frac{3+7}{2}\right) \times 2 \\ &= 10\end{aligned}$$

**b**

$$\begin{aligned}\int_{-1}^1 \sqrt{1 - x^2} dx &= \text{shaded area} \\ &= \frac{1}{2} \pi r^2 \\ &= \frac{1}{2} (\pi \times 1^2) \\ &= \frac{\pi}{2}\end{aligned}$$

**3 a**

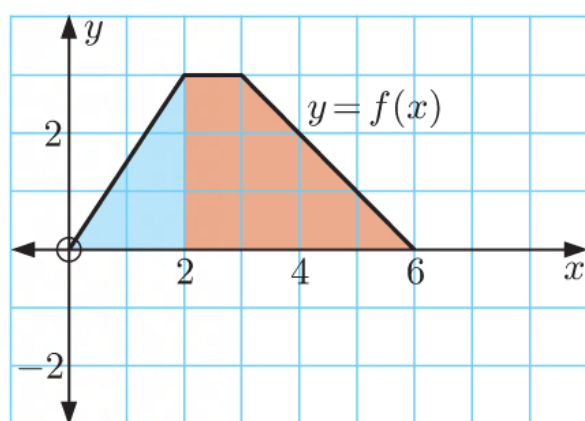
$$\begin{aligned}\frac{d}{dx} (x^3 - 2x) &= 3x^2 - 2 \\ \therefore \text{the antiderivative of } 3x^2 - 2 &\text{ is } \\ x^3 - 2x.\end{aligned}$$

**b**

$$\begin{aligned}\frac{d}{dx} (x^{\frac{4}{3}}) &= \frac{4}{3} x^{\frac{1}{3}} = \frac{4}{3} \sqrt[3]{x} \\ \therefore \frac{d}{dx} \left(\frac{3}{4} x^{\frac{4}{3}}\right) &= \sqrt[3]{x} \\ \therefore \text{the antiderivative of } \sqrt[3]{x} &\text{ is } \frac{3}{4} x^{\frac{4}{3}}.\end{aligned}$$

**c**

$$\begin{aligned}\frac{d}{dx} (\arccos x) &= -\frac{1}{\sqrt{1 - x^2}} \\ \therefore \frac{d}{dx} (-\arccos x) &= \frac{1}{\sqrt{1 - x^2}} \\ \therefore \text{the antiderivative of } \frac{1}{\sqrt{1 - x^2}} &\text{ is } -\arccos x.\end{aligned}$$

**4****a**

$$\begin{aligned}\int_0^2 f(x) dx &= \text{area of blue triangle} \\ &= \frac{1}{2} \times 2 \times 2 \\ &= 2\end{aligned}$$

**b**

$$\begin{aligned}\int_2^6 f(x) dx &= \text{area of red trapezium} \\ &= \left(\frac{2+0}{2}\right) \times 4 \\ &= 4\end{aligned}$$

**5 a**

$$\begin{aligned}\frac{d}{dx} (x^2) &= 2x \\ \therefore \frac{d}{dx} (2x^2) &= 4x \\ \therefore \text{the antiderivative of } f(x) = 4x &\text{ is } F(x) = 2x^2.\end{aligned}$$

$$\begin{aligned}\int_0^3 4x dx &= F(3) - F(0) \\ &= 2(3)^2 - 2(0)^2 \\ &= 18 \text{ units}^2\end{aligned}$$

$$\mathbf{b} \quad \frac{d}{dx} (x^{\frac{3}{2}}) = \frac{3}{2} x^{\frac{1}{2}} = \frac{3}{2} \sqrt{x}$$

$$\therefore \frac{d}{dx} \left( \frac{2}{3} x^{\frac{3}{2}} \right) = \sqrt{x}$$

$\therefore$  the antiderivative of  $f(x) = \sqrt{x}$  is  $F(x) = \frac{2}{3} x^{\frac{3}{2}}$ .

$$\begin{aligned} \int_0^9 \sqrt{x} \, dx &= F(9) - F(0) \\ &= \frac{2}{3} (9)^{\frac{3}{2}} - \frac{2}{3} (0)^{\frac{3}{2}} \\ &= \frac{2}{3} \times 9 \times 3 \\ &= 18 \text{ units}^2 \end{aligned}$$

$$\mathbf{6} \quad \mathbf{a} \quad \left( \frac{x}{a} \right)^2 + \left( \frac{y}{b} \right)^2 = 1$$

$$\therefore \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

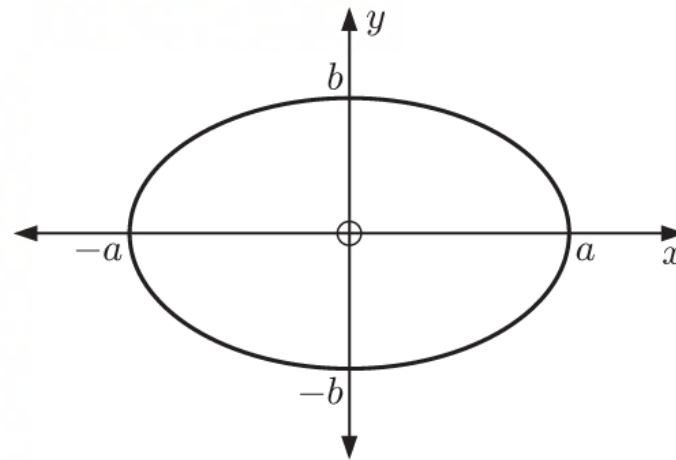
$$\therefore b^2 x^2 + a^2 y^2 = a^2 b^2$$

$$\therefore a^2 y^2 = a^2 b^2 - b^2 x^2$$

$$\therefore a^2 y^2 = b^2 (a^2 - x^2)$$

$$\therefore y^2 = \frac{b^2}{a^2} (a^2 - x^2)$$

$$\therefore y = \pm \frac{b}{a} \sqrt{a^2 - x^2}$$



The points on the top half of the ellipse have  $y$ -coordinate  $\geq 0$ .

$\therefore$  the top half of the ellipse is defined by the function  $y = \frac{b}{a} \sqrt{a^2 - x^2}$ .

$$\mathbf{b} \quad \text{Area of the top half of the ellipse} = \int_{-a}^a \frac{b}{a} \sqrt{a^2 - x^2} \, dx \text{ units}^2$$

$$\begin{aligned} \therefore \text{by symmetry, the total area of the ellipse} &= 2 \times \int_{-a}^a \frac{b}{a} \sqrt{a^2 - x^2} \, dx \text{ units}^2 \\ &= \frac{b}{a} \int_{-a}^a 2\sqrt{a^2 - x^2} \, dx \text{ units}^2 \end{aligned}$$

$$\mathbf{c} \quad \frac{d}{dx} \left( a^2 \arcsin \frac{x}{a} + x \sqrt{a^2 - x^2} \right)$$

$$= a^2 \times \frac{\frac{1}{a}}{\sqrt{1 - \left( \frac{x}{a} \right)^2}} + \sqrt{a^2 - x^2} + x \times \frac{1}{2} (a^2 - x^2)^{-\frac{1}{2}} \times (-2x)$$

$$= \frac{a}{\sqrt{1 - \frac{x^2}{a^2}}} \times \frac{\sqrt{a^2}}{\sqrt{a^2}} + \sqrt{a^2 - x^2} - \frac{x^2}{\sqrt{a^2 - x^2}}$$

$$= \frac{a^2}{\sqrt{a^2 - x^2}} + \frac{a^2 - x^2}{\sqrt{a^2 - x^2}} - \frac{x^2}{\sqrt{a^2 - x^2}}$$

$$= \frac{2(a^2 - x^2)}{\sqrt{a^2 - x^2}}$$

$$= 2\sqrt{a^2 - x^2}$$



$$\text{d} \quad \frac{d}{dx} \left( a^2 \arcsin \frac{x}{a} + x\sqrt{a^2 - x^2} \right) = 2\sqrt{a^2 - x^2} \quad \{\text{from c}\}$$

$\therefore$  the antiderivative of  $f(x) = 2\sqrt{a^2 - x^2}$  is  $F(x) = a^2 \arcsin \frac{x}{a} + x\sqrt{a^2 - x^2}$ .

$$\text{So, area of ellipse} = \frac{b}{a} \int_{-a}^a 2\sqrt{a^2 - x^2} dx \quad \{\text{from b}\}$$

$$= \frac{b}{a} (F(a) - F(-a))$$

$$= \frac{b}{a} \left[ \left( a^2 \arcsin 1 + \cancel{a\sqrt{a^2 - a^2}} \right) - \left( a^2 \arcsin(-1) - \cancel{a\sqrt{a^2 - (a^2)}} \right) \right]$$

$$= \frac{b}{a} [a^2 \arcsin 1 - a^2 \arcsin(-1)]$$

$$= ab \left[ \frac{\pi}{2} - \left( -\frac{\pi}{2} \right) \right]$$

$$= \pi ab \text{ units}^2$$

# Chapter 21

## TECHNIQUES FOR INTEGRATION

### EXERCISE 21A

1  $\frac{d}{dx}(kx) = k$

$$\therefore \int k \, dx = kx + c$$

2 a  $\frac{d}{dx}(x^7) = 7x^6$

$$\therefore \int 7x^6 \, dx = x^7 + c$$

$$\therefore 7 \int x^6 \, dx = x^7 + c$$

$$\therefore \int x^6 \, dx = \frac{1}{7}x^7 + c$$

c  $\frac{d}{dx}(x^{-\frac{1}{2}}) = -\frac{1}{2}x^{-\frac{3}{2}}$

$$\therefore \int -\frac{1}{2}x^{-\frac{3}{2}} \, dx = x^{-\frac{1}{2}} + c$$

$$\therefore -\frac{1}{2} \int x^{-\frac{3}{2}} \, dx = x^{-\frac{1}{2}} + c$$

$$\therefore \int x^{-\frac{3}{2}} \, dx = -2x^{-\frac{1}{2}} + c$$

3 a  $\frac{d}{dx}(e^{4x}) = 4e^{4x}$

$$\therefore \int 4e^{4x} \, dx = e^{4x} + c$$

$$\therefore 4 \int e^{4x} \, dx = e^{4x} + c$$

$$\therefore \int e^{4x} \, dx = \frac{1}{4}e^{4x} + c$$

c  $\frac{d}{dx}(e^{kx}) = ke^{kx}, \quad k \neq 0$

$$\therefore \int ke^{kx} \, dx = e^{kx} + c$$

$$\therefore k \int e^{kx} \, dx = e^{kx} + c$$

$$\therefore \int e^{kx} \, dx = \frac{1}{k}e^{kx} + c, \quad k \neq 0$$

b  $\frac{d}{dx}(x^{\frac{3}{2}}) = \frac{3}{2}x^{\frac{1}{2}} = \frac{3}{2}\sqrt{x}$

$$\therefore \int \frac{3}{2}\sqrt{x} \, dx = x^{\frac{3}{2}} + c$$

$$\therefore \frac{3}{2} \int \sqrt{x} \, dx = x^{\frac{3}{2}} + c$$

$$\therefore \int \sqrt{x} \, dx = \frac{2}{3}x^{\frac{3}{2}} + c$$

d  $\frac{d}{dx}(x^{n+1}) = (n+1)x^n, \quad n \neq -1$

$$\therefore \int (n+1)x^n \, dx = x^{n+1} + c$$

$$\therefore (n+1) \int x^n \, dx = x^{n+1} + c$$

$$\therefore \int x^n \, dx = \frac{x^{n+1}}{n+1} + c, \quad n \neq -1$$

b  $\frac{d}{dx}(e^{-\frac{x}{2}}) = -\frac{1}{2}e^{-\frac{x}{2}}$

$$\therefore \int -\frac{1}{2}e^{-\frac{x}{2}} \, dx = e^{-\frac{x}{2}} + c$$

$$\therefore -\frac{1}{2} \int e^{-\frac{x}{2}} \, dx = e^{-\frac{x}{2}} + c$$

$$\therefore \int e^{-\frac{x}{2}} \, dx = -2e^{-\frac{x}{2}} + c$$

$$4 \quad \mathbf{a} \quad \frac{d}{dx}(\sin x) = \cos x$$

$$\therefore \int \cos x \, dx = \sin x + c$$

$$\mathbf{c} \quad \frac{d}{dx}(\tan x) = \sec^2 x$$

$$\therefore \int \sec^2 x \, dx = \tan x + c$$

$$5 \quad \mathbf{a} \quad \frac{d}{dx}(x^3 + x^2) = 3x^2 + 2x$$

$$\therefore \int (3x^2 + 2x) \, dx = x^3 + x^2 + c$$

$$\mathbf{b} \quad \frac{d}{dx}(\cos x) = -\sin x$$

$$\therefore \int (-\sin x) \, dx = \cos x + c$$

$$\therefore -\int \sin x \, dx = \cos x + c$$

$$\therefore \int \sin x \, dx = -\cos x + c$$

$$\mathbf{b} \quad \frac{d}{dx}(3x^4 - 2x^2) = 12x^3 - 4x$$

$$\therefore \int (12x^3 - 4x) \, dx = 3x^4 - 2x^2 + c$$

$$\therefore 4 \int (3x^3 - x) \, dx = 3x^4 - 2x^2 + c$$

$$\therefore \int (3x^3 - x) \, dx = \frac{3}{4}x^4 - \frac{1}{2}x^2 + c$$

$$6 \quad \mathbf{a} \quad \frac{d}{dx}[F(x) + G(x)] = F'(x) + G'(x) \\ = f(x) + g(x)$$

$$\mathbf{b} \quad \text{Using } \mathbf{a}, \quad \int [f(x) + g(x)] \, dx = F(x) + G(x) + c \\ = \int f(x) \, dx + \int g(x) \, dx$$

$$7 \quad \mathbf{a} \quad \frac{d}{dx}(3^x) = 3^x \ln 3$$

$$\therefore \int 3^x \ln 3 \, dx = 3^x + c$$

$$\therefore \ln 3 \int 3^x \, dx = 3^x + c$$

$$\therefore \int 3^x \, dx = \frac{3^x}{\ln 3} + c$$

$$\mathbf{b} \quad \frac{d}{dx}(a^x) = a^x \ln a, \quad a > 0, \quad a \neq 1$$

$$\therefore \int a^x \ln a \, dx = a^x + c$$

$$\therefore \ln a \int a^x \, dx = a^x + c$$

$$\therefore \int a^x \, dx = \frac{a^x}{\ln a} + c, \quad a > 0, \quad a \neq 1$$

$$8 \quad \mathbf{a} \quad \frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$$

$$\therefore \int (-\operatorname{cosec} x \cot x) \, dx = \operatorname{cosec} x + c$$

$$\mathbf{b} \quad \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\therefore \int \sec x \tan x \, dx = \sec x + c$$

$$\mathbf{c} \quad \frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$$

$$\therefore \int (-\operatorname{cosec}^2 x) \, dx = \cot x + c$$

$$\mathbf{9} \quad \mathbf{a} \quad \frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$$

$$\therefore \int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + c$$

$$\mathbf{c} \quad \frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$$

$$\therefore \int \frac{1}{1+x^2} dx = \arctan x + c$$

$$\mathbf{b} \quad \frac{d}{dx}(\arccos x) = -\frac{1}{\sqrt{1-x^2}}$$

$$\therefore \int \left( -\frac{1}{\sqrt{1-x^2}} \right) dx = \arccos x + c$$

$$\mathbf{10} \quad \mathbf{a} \quad \text{For } x > 0, \quad \frac{d}{dx}(\ln x) = \frac{1}{x}$$

$$\mathbf{c} \quad \int \frac{1}{x} dx = \begin{cases} \ln x + c & \text{if } x > 0 \\ \ln(-x) + c & \text{if } x < 0 \end{cases}$$

$$\therefore \int \frac{1}{x} dx = \ln|x| + c, \quad x \neq 0$$

$$\mathbf{b} \quad \text{For } x < 0, \quad \frac{d}{dx}(\ln(-x)) = \frac{-1}{-x} = \frac{1}{x}$$

$$\mathbf{11} \quad \mathbf{a} \quad \text{For } x > 0, \quad \frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$$

$$\mathbf{b} \quad \text{For } x < 0, \quad \frac{d}{dx}(\log_a(-x)) = \frac{-1}{-x \ln a} = \frac{1}{x \ln a}$$

$$\mathbf{c} \quad \int \frac{1}{x \ln a} dx = \begin{cases} \log_a x + c & \text{if } x > 0 \\ \log_a(-x) + c & \text{if } x < 0 \end{cases}$$

$$\therefore \int \frac{1}{x \ln a} dx = \log_a|x| + c, \quad x \neq 0$$

$$\mathbf{12} \quad \mathbf{a} \quad \frac{d}{dx}(\sin 3x) = \cos 3x \times 3$$

$$= 3 \cos 3x$$

$$\therefore \int 3 \cos 3x dx = \sin 3x + c$$

$$\therefore 3 \int \cos 3x dx = \sin 3x + c$$

$$\therefore \int \cos 3x dx = \frac{1}{3} \sin 3x + c$$

$$\mathbf{b} \quad \frac{d}{dx}(\cos(\frac{\pi}{3} - x)) = -\sin(\frac{\pi}{3} - x) \times (-1)$$

$$= \sin(\frac{\pi}{3} - x)$$

$$\therefore \int \sin(\frac{\pi}{3} - x) dx = \cos(\frac{\pi}{3} - x) + c$$

$$\mathbf{c} \quad \frac{d}{dx}(\sin(1-5x)) = \cos(1-5x) \times (-5)$$

$$= -5 \cos(1-5x)$$

$$\therefore \int (-5 \cos(1-5x)) dx = \sin(1-5x) + c$$

$$\therefore -5 \int \cos(1-5x) dx = \sin(1-5x) + c$$

$$\therefore \int \cos(1-5x) dx = -\frac{1}{5} \sin(1-5x) + c$$



$$\begin{aligned}
 \text{d} \quad & \frac{d}{dx} (e^{3x+1}) = 3e^{3x+1} \\
 \therefore & \int 3e^{3x+1} dx = e^{3x+1} + c \\
 \therefore & 3 \int e^{3x+1} dx = e^{3x+1} + c \\
 \therefore & \int e^{3x+1} dx = \frac{1}{3}e^{3x+1} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & \frac{d}{dx} (\sqrt{5x-1}) = \frac{d}{dx} ((5x-1)^{\frac{1}{2}}) \\
 & = \frac{1}{2}(5x-1)^{-\frac{1}{2}} \\
 & = \frac{5}{2\sqrt{5x-1}} \\
 \therefore & \int \frac{5}{2\sqrt{5x-1}} dx = \sqrt{5x-1} + c \\
 \therefore & \frac{5}{2} \int \frac{1}{\sqrt{5x-1}} dx = \sqrt{5x-1} + c \\
 \therefore & \int \frac{1}{\sqrt{5x-1}} dx = \frac{2}{5}\sqrt{5x-1} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & \frac{d}{dx} ((2x+1)^4) = 4(2x+1)^3 \times 2 \\
 & = 8(2x+1)^3 \\
 \therefore & \int 8(2x+1)^3 dx = (2x+1)^4 + c \\
 \therefore & 8 \int (2x+1)^3 dx = (2x+1)^4 + c \\
 \therefore & \int (2x+1)^3 dx = \frac{1}{8}(2x+1)^4 + c
 \end{aligned}$$

$$\begin{aligned}
 \text{g} \quad & \frac{d}{dx} ((x^2-x)^3) = 3(x^2-x)^2 \times (2x-1) \\
 & = 3(2x-1)(x^2-x)^2 \\
 \therefore & \int 3(2x-1)(x^2-x)^2 dx = (x^2-x)^3 + c \\
 \therefore & 3 \int (2x-1)(x^2-x)^2 dx = (x^2-x)^3 + c \\
 \therefore & \int (2x-1)(x^2-x)^2 dx = \frac{1}{3}(x^2-x)^3 + c
 \end{aligned}$$

$$\begin{aligned}
 \text{h} \quad & \frac{d}{dx} (2^{3x-5}) = 2^{3x-5} \times \ln 2 \times 3 \\
 & = 3 \ln 2 \times 2^{3x-5} \\
 \therefore & \int 3 \ln 2 \times 2^{3x-5} dx = 2^{3x-5} + c \\
 \therefore & 3 \ln 2 \int 2^{3x-5} dx = 2^{3x-5} + c \\
 \therefore & \int 2^{3x-5} dx = \frac{2^{3x-5}}{3 \ln 2} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{i} \quad & \frac{d}{dx} \left( \arctan \frac{x}{2} \right) = \frac{\frac{1}{2}}{1 + \left(\frac{x}{2}\right)^2} \\
 & = \frac{\frac{1}{2}}{1 + \frac{x^2}{4}} \times \frac{4}{4} \\
 & = \frac{2}{x^2 + 4} \\
 \therefore & \int \frac{2}{x^2 + 4} dx = \arctan \frac{x}{2} + c \\
 \therefore & -\frac{3}{2} \int \frac{2}{x^2 + 4} dx = -\frac{3}{2} \arctan \frac{x}{2} + c \\
 \therefore & \int \frac{-3}{x^2 + 4} dx = -\frac{3}{2} \arctan \frac{x}{2} + c
 \end{aligned}$$

**EXERCISE 21B**

$$\begin{aligned} \mathbf{1} \quad \mathbf{a} \quad & \int (x^2 + 3x - 2) dx \\ &= \frac{1}{3}x^3 + \frac{3}{2}x^2 - 2x + c \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad & \int (-x^3 + 4x^2 - 3) dx \\ &= -\frac{1}{4}x^4 + \frac{4}{3}x^3 - 3x + c \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad & \int (x^4 - x^2 - x + 2) dx \\ &= \frac{1}{5}x^5 - \frac{1}{3}x^3 - \frac{1}{2}x^2 + 2x + c \end{aligned}$$

$$\begin{aligned} \mathbf{g} \quad & \int \left( 2\sqrt{x} - \frac{3}{\sqrt{x}} \right) dx \\ &= \int (2x^{\frac{1}{2}} - 3x^{-\frac{1}{2}}) dx \\ &= \frac{2x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{3x^{\frac{1}{2}}}{\frac{1}{2}} + c \\ &= \frac{4}{3}x^{\frac{3}{2}} - 6x^{\frac{1}{2}} + c \\ &= \frac{4}{3}x\sqrt{x} - 6\sqrt{x} + c \end{aligned}$$

$$\begin{aligned} \mathbf{i} \quad & \int (x\sqrt{x} - 9) dx = \int (x^{\frac{3}{2}} - 9) dx \\ &= \frac{x^{\frac{5}{2}}}{\frac{5}{2}} - 9x + c \\ &= \frac{2}{5}x^{\frac{5}{2}} - 9x + c \\ &= \frac{2}{3}x^2\sqrt{x} - 9x + c \end{aligned}$$

$$\begin{aligned} \mathbf{2} \quad \mathbf{a} \quad & \int (2e^x - 3x) dx \\ &= 2e^x - \frac{3}{2}x^2 + c \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad & \int (5e^x + \frac{1}{2}x^2) dx \\ &= 5e^x + \frac{1}{6}x^3 + c \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad & \int (2x^2 - 3x + 1) dx \\ &= \frac{2}{3}x^3 - \frac{3}{2}x^2 + x + c \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad & \int (\frac{1}{2}x + x^2 + x^3) dx \\ &= \frac{1}{4}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + c \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad & \int \left( 4x^2 + \frac{1}{x} \right) dx \\ &= \frac{4}{3}x^3 + \ln|x| + c \end{aligned}$$

$$\begin{aligned} \mathbf{h} \quad & \int \left( \frac{1}{3x} - \frac{2}{x^2} \right) dx \\ &= \int \left( \frac{1}{3}x^{-1} - 2x^{-2} \right) dx \\ &= \frac{1}{3} \ln|x| - \frac{2x^{-1}}{(-1)} + c \\ &= \frac{1}{3} \ln|x| + \frac{2}{x} + c \end{aligned}$$

$$\begin{aligned} \mathbf{j} \quad & \int (3x^{-\frac{3}{2}} + x^{\frac{1}{4}}) dx \\ &= \frac{3x^{-\frac{1}{2}}}{(-\frac{1}{2})} + \frac{x^{\frac{5}{4}}}{\frac{5}{4}} + c \\ &= -6x^{-\frac{1}{2}} + \frac{4}{5}x^{\frac{5}{4}} + c \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad & \int \left( \frac{4}{x} + x^2 - e^x \right) dx \\ &= 4 \ln|x| + \frac{1}{3}x^3 - e^x + c \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad & \int (\sqrt{x} + e^x) dx = \int (x^{\frac{1}{2}} + e^x) dx \\ &= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + e^x + c \\ &= \frac{2}{3}x^{\frac{3}{2}} + e^x + c \\ &= \frac{2}{3}x\sqrt{x} + e^x + c \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & \int \left( 3e^x - \frac{1}{x\sqrt{x}} \right) dx \\
 &= \int (3e^x - x^{-\frac{3}{2}}) dx \\
 &= 3e^x - \frac{x^{-\frac{1}{2}}}{-\frac{1}{2}} + c \\
 &= 3e^x + 2\sqrt{x} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & \int \left( \frac{1}{2x} + x^2 - \frac{1}{2}e^x \right) dx \\
 &= \frac{1}{2} \ln |x| + \frac{1}{3}x^3 - \frac{1}{2}e^x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{3 a} \quad & \int (3 \sin x - 2) dx \\
 &= -3 \cos x - 2x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & \int (4x - 2 \cos x) dx \\
 &= 2x^2 - 2 \sin x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & \int (\sin x - 2 \cos x + e^x) dx \\
 &= -\cos x - 2 \sin x + e^x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad & \int (x^2 \sqrt{x} - 10 \sin x) dx \\
 &= \int (x^{\frac{5}{2}} - 10 \sin x) dx \\
 &= \frac{x^{\frac{7}{2}}}{\frac{7}{2}} + 10 \cos x + c \\
 &= \frac{2}{7} x^{\frac{7}{2}} + 10 \cos x + c \\
 &= \frac{2}{7} x^3 \sqrt{x} + 10 \cos x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & \int \left( \frac{x(x-1)}{3} + \sec^2 x \right) dx \\
 &= \int \left( \frac{1}{3}x^2 - \frac{1}{3}x + \sec^2 x \right) dx \\
 &= \frac{1}{9}x^3 - \frac{1}{6}x^2 + \tan x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & \int (-\sin x + 2\sqrt{x}) dx \\
 &= \int (-\sin x + 2x^{\frac{1}{2}}) dx \\
 &= \cos x + \frac{2x^{\frac{3}{2}}}{\frac{3}{2}} + c \\
 &= \cos x + \frac{4}{3}x^{\frac{3}{2}} + c \\
 &= \cos x + \frac{4}{3}x\sqrt{x} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{4 a} \quad & \frac{dy}{dx} = 6 \\
 \therefore y &= \int 6 dx \\
 &= 6x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & \frac{dy}{dx} = 4x^2 \\
 \therefore y &= \int 4x^2 dx \\
 &= \frac{4}{3}x^3 + c
 \end{aligned}$$

$$\begin{aligned}\text{c} \quad \frac{dy}{dx} &= \frac{1}{x^2} = x^{-2} \\ \therefore y &= \int x^{-2} dx \\ &= \frac{x^{-1}}{(-1)} + c \\ &= -\frac{1}{x} + c\end{aligned}$$

$$\begin{aligned}\text{e} \quad \frac{dy}{dx} &= 2x^3 - 4 \\ \therefore y &= \int (2x^3 - 4) dx \\ &= \frac{1}{2}x^4 - 4x + c\end{aligned}$$

$$\begin{aligned}\text{g} \quad \frac{dy}{dx} &= 2 - \frac{1}{x} \\ \therefore y &= \int \left(2 - \frac{1}{x}\right) dx \\ &= 2x - \ln|x| + c\end{aligned}$$

$$\begin{aligned}\text{i} \quad \frac{dy}{dx} &= 2e^x - 5 + x \\ \therefore y &= \int (2e^x - 5 + x) dx \\ &= 2e^x - 5x + \frac{1}{2}x^2 + c\end{aligned}$$

$$\text{5} \quad \int x^n dx = \frac{x^{n+1}}{n+1} + c, \quad n \neq -1$$

$\frac{x^{n+1}}{n+1}$  is undefined when  $n+1=0$ , as we cannot divide by zero, so we must exclude the value  $n = -1$ .

$$\text{6} \quad \text{a} \quad \int 4^x dx = \frac{4^x}{\ln 4} + c$$

$$\text{c} \quad \int (5^x - 2 \times 7^x) dx = \frac{5^x}{\ln 5} - \frac{2 \times 7^x}{\ln 7} + c$$

$$\begin{aligned}\text{e} \quad \int (\cos x + \operatorname{cosec}^2 x) dx \\ = \sin x - \cot x + c\end{aligned}$$

$$\begin{aligned}\text{d} \quad \frac{dy}{dx} &= \frac{2}{\sqrt[3]{x}} = 2x^{-\frac{1}{3}} \\ \therefore y &= \int 2x^{-\frac{1}{3}} dx \\ &= \frac{2x^{\frac{2}{3}}}{\frac{2}{3}} + c \\ &= 3x^{\frac{2}{3}} + c\end{aligned}$$

$$\begin{aligned}\text{f} \quad \frac{dy}{dx} &= 4x^3 + 3x^2 \\ \therefore y &= \int (4x^3 + 3x^2) dx \\ &= x^4 + x^3 + c\end{aligned}$$

$$\begin{aligned}\text{h} \quad \frac{dy}{dx} &= \sin x + 2 \cos x \\ \therefore y &= \int (\sin x + 2 \cos x) dx \\ &= -\cos x + 2 \sin x + c\end{aligned}$$

$$\begin{aligned}\text{b} \quad \int \left(\frac{3}{x} - \frac{1}{x \ln 2}\right) dx \\ = 3 \ln|x| - \log_2|x| + c \\ \{\text{using Exercise 21A question 11 c}\}\end{aligned}$$

$$\text{d} \quad \int \operatorname{cosec} x \cot x dx = -\operatorname{cosec} x + c$$

$$\begin{aligned}\text{f} \quad \int \left(\frac{4}{x^2 \sqrt{x}} - \frac{\sec x \tan x}{9}\right) dx \\ = \int \left(4x^{-\frac{5}{2}} - \frac{\sec x \tan x}{9}\right) dx \\ = \frac{4x^{-\frac{3}{2}}}{-\frac{3}{2}} - \frac{1}{9} \sec x + c \\ = -\frac{8}{3x\sqrt{x}} - \frac{1}{9} \sec x + c\end{aligned}$$



$$\begin{aligned} 7 \quad \mathbf{a} \quad & \int (2e^t - 4 \sin t) dt \\ &= 2e^t + 4 \cos t + c \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad & \int (5 \sin t - \sqrt{t}) dt \\ &= \int (5 \sin t - t^{\frac{1}{2}}) dt \\ &= -5 \cos t - \frac{t^{\frac{3}{2}}}{\frac{3}{2}} + c \\ &= -5 \cos t - \frac{2}{3} t \sqrt{t} + c \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad & \int (\theta - \sin \theta) d\theta \\ &= \frac{1}{2} \theta^2 + \cos \theta + c \end{aligned}$$

$$\begin{aligned} \mathbf{g} \quad & \int (2 \operatorname{cosec}^2 \theta + 3 \operatorname{cosec} \theta \cot \theta) d\theta \\ &= -2 \cot \theta - 3 \operatorname{cosec} \theta + c \end{aligned}$$

$$\begin{aligned} \mathbf{i} \quad & \int \tan^2 \theta d\theta \\ &= \int (\sec^2 \theta - 1) d\theta \\ &= \tan \theta - \theta + c \end{aligned}$$

$$\begin{aligned} 8 \quad \mathbf{a} \quad & \int \frac{4}{\sqrt{1-x^2}} dx \\ &= 4 \arcsin x + c \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad & \int \frac{7}{1+x^2} dx \\ &= 7 \arctan x + c \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad & \int \left( -\frac{1}{\sqrt{1-x^2}} + 3x \right) dx \\ &= \arccos x + \frac{3}{2} x^2 + c \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad & \int \left( 3 \cos t - \frac{1}{t} \right) dt \\ &= 3 \sin t - \ln |t| + c \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad & \int (\sec^2 x + 2 \sin x) dx \\ &= \tan x - 2 \cos x + c \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad & \int \left( \frac{2}{\theta} - \sec^2 \theta \right) d\theta \\ &= 2 \ln |\theta| - \tan \theta + c \end{aligned}$$

$$\begin{aligned} \mathbf{h} \quad & \int 4 \sec \theta \tan \theta d\theta \\ &= 4 \sec \theta + c \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad & \int -\frac{5}{\sqrt{1-x^2}} dx \\ &= 5 \arccos x + c \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad & \int \left( \frac{3}{x} + \frac{5}{\sqrt{1-x^2}} \right) dx \\ &= 3 \ln |x| + 5 \arcsin x + c \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad & \int \left( 6 - \frac{2}{1+x^2} \right) dx \\ &= 6x - 2 \arctan x + c \end{aligned}$$

9 Both are correct. Notice that:

$$\begin{aligned} \cos(-\arcsin x + \frac{\pi}{2}) &= \cos(-\arcsin x) \cos \frac{\pi}{2} - \sin(-\arcsin x) \sin \frac{\pi}{2} \\ &= \sin(\arcsin x) \\ &= x \end{aligned}$$

$$\therefore -\arcsin x + \frac{\pi}{2} = \arccos x \quad \{\text{as } 0 \leq -\arcsin x + \frac{\pi}{2} \leq \pi\}$$

So,  $\arccos x$  and  $-\arcsin x$  differ by a constant, which is accounted for by  $c$ . Thus, the answers are equivalent.

$$\begin{aligned}
 \mathbf{10} \quad \mathbf{a} \quad & \int (2x+1)^2 dx \\
 &= \int (4x^2 + 4x + 1) dx \\
 &= \frac{4x^3}{3} + \frac{4x^2}{2} + x + c \\
 &= \frac{4}{3}x^3 + 2x^2 + x + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & \int \frac{1-4x}{x\sqrt{x}} dx \\
 &= \int \left( \frac{1}{x\sqrt{x}} - \frac{4}{\sqrt{x}} \right) dx \\
 &= \int (x^{-\frac{3}{2}} - 4x^{-\frac{1}{2}}) dx \\
 &= \frac{x^{-\frac{1}{2}}}{(-\frac{1}{2})} - \frac{4x^{\frac{1}{2}}}{\frac{1}{2}} + c \\
 &= -2x^{-\frac{1}{2}} - 8x^{\frac{1}{2}} + c \\
 &= -\frac{2}{\sqrt{x}} - 8\sqrt{x} + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{e} \quad & \int \left( \sqrt{x} - \frac{1}{\sqrt{x}} \right)^2 dx \\
 &= \int \left( x - 2 + \frac{1}{x} \right) dx \\
 &= \frac{1}{2}x^2 - 2x + \ln|x| + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{g} \quad & \int \left( \frac{2}{x} + 1 \right)^2 dx \\
 &= \int \left( \frac{4}{x^2} + \frac{4}{x} + 1 \right) dx \\
 &= \int (4x^{-2} + 4x^{-1} + 1) dx \\
 &= \frac{4x^{-1}}{(-1)} + 4\ln|x| + x + c \\
 &= -\frac{4}{x} + 4\ln|x| + x + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \int \left( x + \frac{1}{x} \right)^2 dx \\
 &= \int \left( x^2 + 2 + \frac{1}{x^2} \right) dx \\
 &= \int (x^2 + 2 + x^{-2}) dx \\
 &= \frac{x^3}{3} + 2x + \frac{x^{-1}}{(-1)} + c \\
 &= \frac{1}{3}x^3 + 2x - \frac{1}{x} + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad & \int \frac{2x-1}{\sqrt{x}} dx \\
 &= \int \left( 2\sqrt{x} - \frac{1}{\sqrt{x}} \right) dx \\
 &= \int (2x^{\frac{1}{2}} - x^{-\frac{1}{2}}) dx \\
 &= \frac{2x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c \\
 &= \frac{4}{3}x^{\frac{3}{2}} - 2x^{\frac{1}{2}} + c \\
 &= \frac{4}{3}x\sqrt{x} - 2\sqrt{x} + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{f} \quad & \int \frac{1-x^2}{x} dx \\
 &= \int \left( \frac{1}{x} - x \right) dx \\
 &= \ln|x| - \frac{1}{2}x^2 + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{h} \quad & \int (x+1)^3 dx \\
 &= \int (x^3 + 3x^2 + 3x + 1) dx \\
 & \quad \text{\{binomial theorem\}} \\
 &= \frac{x^4}{4} + \frac{3x^3}{3} + \frac{3x^2}{2} + x + c \\
 &= \frac{1}{4}x^4 + x^3 + \frac{3}{2}x^2 + x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{i} \quad & \int (x-1)^4 dx \\
 &= \int (x^4 - 4x^3 + 6x^2 - 4x + 1) dx \\
 &\quad \{\text{binomial theorem}\} \\
 &= \frac{x^5}{5} - \frac{4x^4}{4} + \frac{6x^3}{3} - \frac{4x^2}{2} + x + c \\
 &= \frac{1}{5}x^5 - x^4 + 2x^3 - 2x^2 + x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{j} \quad & \int \frac{x^2 - 4}{x\sqrt{x}} dx \\
 &= \int \frac{x^2 - 4}{x^{\frac{3}{2}}} dx \\
 &= \int (x^{\frac{1}{2}} - 4x^{-\frac{3}{2}}) dx \\
 &= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{4x^{-\frac{1}{2}}}{(-\frac{1}{2})} + c \\
 &= \frac{2}{3}x^{\frac{3}{2}} + 8x^{-\frac{1}{2}} + c \\
 &= \frac{2}{3}x\sqrt{x} + \frac{8}{\sqrt{x}} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{k} \quad & \int \frac{x^2 - 4x + 10}{x} dx \\
 &= \int \left(x - 4 + \frac{10}{x}\right) dx \\
 &= \frac{1}{2}x^2 - 4x + 10 \ln|x| + c
 \end{aligned}$$

$$\begin{aligned}
 \text{l} \quad & \int \frac{3x^3 - 2x^2 + 5}{x^2} dx \\
 &= \int \left(3x - 2 + \frac{5}{x^2}\right) dx \\
 &= \int (3x - 2 + 5x^{-2}) dx \\
 &= \frac{3x^2}{2} - 2x + \frac{5x^{-1}}{(-1)} + c \\
 &= \frac{3}{2}x^2 - 2x - \frac{5}{x} + c
 \end{aligned}$$

$$\begin{aligned}
 11 \quad \text{a} \quad & f'(x) = (1 - 2x)^2 \\
 \therefore f(x) &= \int (1 - 2x)^2 dx \\
 &= \int (1 - 4x + 4x^2) dx \\
 &= x - \frac{4x^2}{2} + \frac{4x^3}{3} + c \\
 &= x - 2x^2 + \frac{4}{3}x^3 + c
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & f'(x) = \sqrt{x} - \frac{2}{\sqrt{x}} \\
 &= x^{\frac{1}{2}} - 2x^{-\frac{1}{2}} \\
 \therefore f(x) &= \int (x^{\frac{1}{2}} - 2x^{-\frac{1}{2}}) dx \\
 &= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{2x^{\frac{1}{2}}}{\frac{1}{2}} + c \\
 &= \frac{2}{3}x^{\frac{3}{2}} - 4x^{\frac{1}{2}} + c \\
 &= \frac{2}{3}x\sqrt{x} - 4\sqrt{x} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & f'(x) = \frac{x^2 - 5}{x^2} \\
 &= 1 - 5x^{-2} \\
 \therefore f(x) &= \int (1 - 5x^{-2}) dx \\
 &= x - \frac{5x^{-1}}{(-1)} + c \\
 &= x + \frac{5}{x} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad & f'(x) = \frac{2x^2 + 5}{x^2 + 1} \\
 &= \frac{2(x^2 + 1) + 3}{x^2 + 1} \\
 &= 2 + \frac{3}{x^2 + 1} \\
 \therefore f(x) &= \int \left(2 + \frac{3}{x^2 + 1}\right) dx \\
 &= 2x + 3 \arctan x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad f'(x) &= \operatorname{cosec} x \left( \frac{\cot x}{2} - \frac{\operatorname{cosec} x}{3} \right) \\
 &= \frac{\operatorname{cosec} x \cot x}{2} - \frac{\operatorname{cosec}^2 x}{3} \\
 \therefore f(x) &= \int \left( \frac{\operatorname{cosec} x \cot x}{2} - \frac{\operatorname{cosec}^2 x}{3} \right) dx \\
 &= -\frac{1}{2} \operatorname{cosec} x + \frac{1}{3} \cot x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{12 a} \quad & \int \left( x^2 - \frac{1}{x} \right)^2 dx \\
 &= \int \left( x^4 - 2x + \frac{1}{x^2} \right) dx \\
 &= \int (x^4 - 2x + x^{-2}) dx \\
 &= \frac{x^5}{5} - \frac{2x^2}{2} + \frac{x^{-1}}{(-1)} + c \\
 &= \frac{1}{5}x^5 - x^2 - \frac{1}{x} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & \int \sqrt{x}(3x-1)^2 dx \\
 &= \int x^{\frac{1}{2}} \times (9x^2 - 6x + 1) dx \\
 &= \int (9x^{\frac{5}{2}} - 6x^{\frac{3}{2}} + x^{\frac{1}{2}}) dx \\
 &= \frac{9x^{\frac{7}{2}}}{\frac{7}{2}} - \frac{6x^{\frac{5}{2}}}{\frac{5}{2}} + \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c \\
 &= \frac{18}{7}x^{\frac{7}{2}} - \frac{12}{5}x^{\frac{5}{2}} + \frac{2}{3}x^{\frac{3}{2}} + c \\
 &= \frac{18}{7}x^3\sqrt{x} - \frac{12}{5}x^2\sqrt{x} + \frac{2}{3}x\sqrt{x} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & \int \frac{x^2 - 4x + 2}{\sqrt{x}} dx \\
 &= \int \frac{x^2 - 4x + 2}{x^{\frac{1}{2}}} dx \\
 &= \int (x^{\frac{3}{2}} - 4x^{\frac{1}{2}} + 2x^{-\frac{1}{2}}) dx \\
 &= \frac{x^{\frac{5}{2}}}{\frac{5}{2}} - \frac{4x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{2x^{\frac{1}{2}}}{\frac{1}{2}} + c \\
 &= \frac{2}{5}x^{\frac{5}{2}} - \frac{8}{3}x^{\frac{3}{2}} + 4x^{\frac{1}{2}} + c \\
 &= \frac{2}{5}x^2\sqrt{x} - \frac{8}{3}x\sqrt{x} + 4\sqrt{x} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{13} \quad \frac{d}{dx}(x \cos x) &= (1) \cos x + (x)(-\sin x) \\
 &= \cos x - x \sin x \\
 \therefore \int (\cos x - x \sin x) dx &= x \cos x + c \\
 \therefore \int \cos x dx - \int x \sin x dx &= x \cos x + c \\
 \therefore \sin x - \int x \sin x dx &= x \cos x + c \\
 \therefore \int x \sin x dx &= \sin x - x \cos x + c
 \end{aligned}$$



$$\mathbf{14} \quad \frac{d}{dx}(xe^x) = e^x + xe^x$$

$$\therefore \int (e^x + xe^x) dx = xe^x + c$$

$$\therefore \int e^x dx + \int xe^x dx = xe^x + c$$

$$\therefore e^x + \int xe^x dx = xe^x + c$$

$$\therefore \int xe^x dx = xe^x - e^x + c$$

$$\mathbf{15} \quad \frac{d}{dx}(x \ln x) = (1) \ln x + x \left(\frac{1}{x}\right) \\ = \ln x + 1$$

$$\therefore \int (\ln x + 1) dx = x \ln x + c$$

$$\therefore \int \ln x dx + \int 1 dx = x \ln x + c$$

$$\therefore \int \ln x dx + x = x \ln x + c$$

$$\therefore \int \ln x dx = x \ln x - x + c$$

$$\mathbf{16} \quad \mathbf{a} \quad \frac{d}{dx}(2x \sin x) = (2) \sin x + 2x(\cos x) \\ = 2 \sin x + 2x \cos x$$

$$\frac{d}{dx}(-x^2 \cos x) = (-2x) \cos x - x^2(-\sin x) \\ = -2x \cos x + x^2 \sin x$$

$$\mathbf{b} \quad \int (-2x \cos x + x^2 \sin x) dx = -x^2 \cos x + c \quad \{\text{from } \mathbf{a}\}$$

$$\therefore - \int (2x \cos x) dx + \int x^2 \sin x dx = -x^2 \cos x + c \quad \dots (1)$$

$$\text{Now } \int (2 \sin x + 2x \cos x) dx = 2x \sin x + c \quad \{\text{from } \mathbf{a}\}$$

$$\therefore \int 2 \sin x dx + \int 2x \cos x dx = 2x \sin x + c$$

$$\therefore -2 \cos x + \int 2x \cos x dx = 2x \sin x + c$$

$$\therefore \int 2x \cos x dx = 2x \sin x + 2 \cos x + c \quad \dots (2)$$

Substituting (2) into (1):

$$-(2x \sin x + 2 \cos x) + \int x^2 \sin x dx = -x^2 \cos x + c$$

$$\therefore \int x^2 \sin x dx = -x^2 \cos x + 2x \sin x + 2 \cos x + c$$

**EXERCISE 21C**

**1 a**  $f'(x) = 2x - 1$

$$\begin{aligned}\therefore f(x) &= \int (2x - 1) dx \\ &= \frac{2x^2}{2} - x + c \\ &= x^2 - x + c\end{aligned}$$

But  $f(0) = 3$ , so  $0 - 0 + c = 3$   
 $\therefore c = 3$

$$\therefore f(x) = x^2 - x + 3$$

**c**  $f'(x) = e^x + \frac{1}{\sqrt{x}} = e^x + x^{-\frac{1}{2}}$

$$\begin{aligned}\therefore f(x) &= \int (e^x + x^{-\frac{1}{2}}) dx \\ &= e^x + \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c \\ &= e^x + 2x^{\frac{1}{2}} + c\end{aligned}$$

But  $f(1) = 1$ , so  $e + 2 + c = 1$   
 $\therefore c = -1 - e$

$$\therefore f(x) = e^x + 2\sqrt{x} - 1 - e$$

**e**  $f'(x) = \sqrt{x} - 2 = x^{\frac{1}{2}} - 2$

$$\begin{aligned}\therefore f(x) &= \int (x^{\frac{1}{2}} - 2) dx \\ &= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - 2x + c \\ &= \frac{2}{3}x^{\frac{3}{2}} - 2x + c\end{aligned}$$

But  $f(4) = 0$ , so  $\frac{2}{3}(4)^{\frac{3}{2}} - 2(4) + c = 0$   
 $\therefore \frac{16}{3} - 8 + c = 0$   
 $\therefore c = \frac{8}{3}$

$$\therefore f(x) = \frac{2}{3}x\sqrt{x} - 2x + \frac{8}{3}$$

**b**  $f'(x) = 3x^2 + 2x$

$$\begin{aligned}\therefore f(x) &= \int (3x^2 + 2x) dx \\ &= \frac{3x^3}{3} + \frac{2x^2}{2} + c \\ &= x^3 + x^2 + c\end{aligned}$$

But  $f(2) = 5$ , so  $8 + 4 + c = 5$   
 $\therefore c = -7$

$$\therefore f(x) = x^3 + x^2 - 7$$

**d**  $f'(x) = x - \frac{2}{\sqrt{x}} = x - 2x^{-\frac{1}{2}}$

$$\begin{aligned}\therefore f(x) &= \int (x - 2x^{-\frac{1}{2}}) dx \\ &= \frac{x^2}{2} - \frac{2x^{\frac{1}{2}}}{\frac{1}{2}} + c \\ &= \frac{1}{2}x^2 - 4\sqrt{x} + c\end{aligned}$$

But  $f(1) = 2$ , so  $\frac{1}{2} - 4 + c = 2$   
 $\therefore c = \frac{11}{2}$

$$\therefore f(x) = \frac{1}{2}x^2 - 4\sqrt{x} + \frac{11}{2}$$

**f**  $f'(x) = \frac{1}{x}$

$$\begin{aligned}\therefore f(x) &= \int \frac{1}{x} dx \\ &= \ln|x| + c\end{aligned}$$

But  $f(e) = 2$ , so  $\ln e + c = 2$   
 $\therefore c = 1$

$$\therefore f(x) = \ln|x| + 1$$

$$2 \quad \frac{dy}{dx} = x - 2x^2$$

$$\begin{aligned} \therefore y &= \int (x - 2x^2) dx \\ &= \frac{1}{2}x^2 - \frac{2}{3}x^3 + c \end{aligned}$$

But the curve passes through (2, 4),  
so when  $x = 2$ ,  $y = 4$ .

$$\therefore 4 = \frac{1}{2}(2)^2 - \frac{2}{3}(2)^3 + c$$

$$\therefore 4 = 2 - \frac{16}{3} + c$$

$$\therefore c = \frac{22}{3}$$

$$\therefore y = \frac{1}{2}x^2 - \frac{2}{3}x^3 + \frac{22}{3}$$

$$3 \quad \frac{dy}{dx} = 1 - e^x$$

$$\begin{aligned} \therefore y &= \int (1 - e^x) dx \\ &= x - e^x + c \end{aligned}$$

But the curve passes through (3,  $e^3$ ),  
so when  $x = 3$ ,  $y = e^3$ .

$$\therefore e^3 = 3 - e^3 + c$$

$$\therefore c = 2e^3 - 3$$

$$\therefore y = x - e^x + 2e^3 - 3$$

$$4 \quad a \quad f'(x) = x^2 - 4 \cos x$$

$$\begin{aligned} \therefore f(x) &= \int (x^2 - 4 \cos x) dx \\ &= \frac{1}{3}x^3 - 4 \sin x + c \end{aligned}$$

But  $f(0) = 3$ , so  $c = 3$

$$\therefore f(x) = \frac{1}{3}x^3 - 4 \sin x + 3$$

$$b \quad f'(x) = 2 \cos x - 3 \sin x$$

$$\begin{aligned} \therefore f(x) &= \int (2 \cos x - 3 \sin x) dx \\ &= 2 \sin x + 3 \cos x + c \end{aligned}$$

But  $f(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$ ,

$$\text{so } 2 \sin \frac{\pi}{4} + 3 \cos \frac{\pi}{4} + c = \frac{1}{\sqrt{2}}$$

$$\therefore \frac{2}{\sqrt{2}} + \frac{3}{\sqrt{2}} + c = \frac{1}{\sqrt{2}}$$

$$\therefore c = -\frac{4}{\sqrt{2}}$$

$$\therefore c = -2\sqrt{2}$$

$$\therefore f(x) = 2 \sin x + 3 \cos x - 2\sqrt{2}$$

$$c \quad f'(x) = \sqrt{x} - 2 \sec^2 x = x^{\frac{1}{2}} - 2 \sec^2 x$$

$$\begin{aligned} \therefore f(x) &= \int (x^{\frac{1}{2}} - 2 \sec^2 x) dx \\ &= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - 2 \tan x + c \\ &= \frac{2}{3}x^{\frac{3}{2}} - 2 \tan x + c \end{aligned}$$

But  $f(\pi) = 0$ ,

$$\text{so } \frac{2}{3}\pi^{\frac{3}{2}} - 2 \tan \pi + c = 0$$

$$\therefore c = -\frac{2}{3}\pi^{\frac{3}{2}}$$

$$\begin{aligned} f(x) &= \frac{2}{3}x^{\frac{3}{2}} - 2 \tan x - \frac{2}{3}\pi^{\frac{3}{2}} \\ &= \frac{2}{3}x\sqrt{x} - 2 \tan x - \frac{2}{3}\pi\sqrt{\pi} \end{aligned}$$

$$d \quad f'(x) = e^x + 3 \cos x$$

$$\begin{aligned} \therefore f(x) &= \int (e^x + 3 \cos x) dx \\ &= e^x + 3 \sin x + c \end{aligned}$$

But  $f(\pi) = 0$ , so  $e^\pi + 3 \sin \pi + c = 0$

$$\therefore e^\pi + c = 0$$

$$\therefore c = -e^\pi$$

$$\therefore f(x) = e^x + 3 \sin x - e^\pi$$

**e**  $f'(x) = 2 \sec x \tan x$   
 $\therefore f(x) = \int 2 \sec x \tan x \, dx$   
 $= 2 \sec x + c$   
 But  $f(\frac{\pi}{3}) = 7$   
 so  $2 \sec \frac{\pi}{3} + c = 7$   
 $\therefore \frac{2}{\cos \frac{\pi}{3}} + c = 7$   
 $\therefore \frac{2}{\frac{1}{2}} + c = 7$   
 $\therefore 4 + c = 7$   
 $\therefore c = 3$   
 $\therefore f(x) = 2 \sec x + 3$

**5**  $f'(x) = ax + 1$   
 $\therefore f(x) = \int (ax + 1) \, dx$   
 $= \frac{ax^2}{2} + x + c$   
 Now  $f(0) = 3$ , so  $c = 3$   
 $\therefore f(x) = \frac{1}{2}ax^2 + x + 3$   
 and  $f(3) = -3$   
 $\therefore \frac{1}{2}a(3)^2 + 3 + 3 = -3$   
 $\therefore \frac{9}{2}a + 6 = -3$   
 $\therefore \frac{9}{2}a = -9$   
 $\therefore a = -2$   
 $\therefore f(x) = \frac{1}{2}(-2)x^2 + x + 3$   
 $= -x^2 + x + 3$

**f**  $f'(x) = \frac{10}{\pi\sqrt{1-x^2}} - \frac{3}{\pi(x^2+1)}$   
 $\therefore f(x) = \int \left( \frac{10}{\pi\sqrt{1-x^2}} - \frac{3}{\pi(x^2+1)} \right) dx$   
 $= \frac{10}{\pi} \arcsin x - \frac{3}{\pi} \arctan x + c$   
 But  $f(1) = 4$   
 so  $\frac{10}{\pi} \arcsin 1 - \frac{3}{\pi} \arctan 1 + c = 4$   
 $\therefore \frac{10}{\pi} \times \frac{\pi}{2} - \frac{3}{\pi} \times \frac{\pi}{4} + c = 4$   
 $\therefore 5 - \frac{3}{4} + c = 4$   
 $\therefore c = -\frac{1}{4}$   
 $\therefore f(x) = \frac{10}{\pi} \arcsin x - \frac{3}{\pi} \arctan x - \frac{1}{4}$

**6**  $f'(x) = ax^2 + bx$   
 $\therefore f(x) = \int (ax^2 + bx) \, dx$   
 $= \frac{ax^3}{3} + \frac{bx^2}{2} + c$   
 Now  $f(0) = 1$ , so  $c = 1$   
 $\therefore f(x) = \frac{1}{3}ax^3 + \frac{1}{2}bx^2 + 1$   
 $f(-1) = -2$   
 $\therefore \frac{1}{3}a(-1)^3 + \frac{1}{2}b(-1)^2 + 1 = -2$   
 $\therefore -\frac{1}{3}a + \frac{1}{2}b + 1 = -2$   
 $\therefore -\frac{1}{3}a + \frac{1}{2}b = -3 \quad \dots (1)$   
 and  $f(1) = 4$   
 $\therefore \frac{1}{3}a + \frac{1}{2}b + 1 = 4$   
 $\therefore \frac{1}{3}a + \frac{1}{2}b = 3 \quad \dots (2)$

Adding (1) and (2) together gives:  $b = 0$

Substituting  $b = 0$  into (1) gives:

$$-\frac{1}{3}a = -3$$

$$\therefore a = 9$$

$$\therefore f(x) = \frac{1}{3}(9)x^3 + \frac{1}{2}(0)x^2 + 1$$

$$= 3x^3 + 1$$



**7 a**  $f''(x) = 2x + 1$

$$\begin{aligned}\therefore f'(x) &= \int (2x + 1) dx \\ &= x^2 + x + c\end{aligned}$$

But  $f'(1) = 3$ , so  $1 + 1 + c = 3$   
 $\therefore c = 1$

$$\therefore f'(x) = x^2 + x + 1$$

$$\begin{aligned}\therefore f(x) &= \int (x^2 + x + 1) dx \\ &= \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + d\end{aligned}$$

But  $f(2) = 7$ ,  
 so  $\frac{1}{3}(2)^3 + \frac{1}{2}(2)^2 + 2 + d = 7$   
 $\therefore \frac{8}{3} + 2 + 2 + d = 7$   
 $\therefore d = \frac{1}{3}$

$$\therefore f(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + \frac{1}{3}$$

**c**  $f''(x) = \cos x$

$$\begin{aligned}\therefore f'(x) &= \int \cos x dx \\ &= \sin x + c\end{aligned}$$

But  $f'(\frac{\pi}{2}) = 0$ , so  $\sin \frac{\pi}{2} + c = 0$   
 $\therefore 1 + c = 0$   
 $\therefore c = -1$

$$\therefore f'(x) = \sin x - 1$$

$$\begin{aligned}\therefore f(x) &= \int (\sin x - 1) dx \\ &= -\cos x - x + d\end{aligned}$$

But  $f(0) = 3$ , so  $-\cos 0 + d = 3$   
 $\therefore -1 + d = 3$   
 $\therefore d = 4$

$$\therefore f(x) = -\cos x - x + 4$$

**b**  $f''(x) = 15\sqrt{x} + \frac{3}{\sqrt{x}} = 15x^{\frac{1}{2}} + 3x^{-\frac{1}{2}}$

$$\begin{aligned}\therefore f'(x) &= \int (15x^{\frac{1}{2}} + 3x^{-\frac{1}{2}}) dx \\ &= \frac{15x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{3x^{\frac{1}{2}}}{\frac{1}{2}} + c \\ &= 10x^{\frac{3}{2}} + 6x^{\frac{1}{2}} + c\end{aligned}$$

But  $f'(1) = 12$ , so  $10 + 6 + c = 12$   
 $\therefore c = -4$

$$\therefore f'(x) = 10x^{\frac{3}{2}} + 6x^{\frac{1}{2}} - 4$$

$$\begin{aligned}\therefore f(x) &= \int (10x^{\frac{3}{2}} + 6x^{\frac{1}{2}} - 4) dx \\ &= \frac{10x^{\frac{5}{2}}}{\frac{5}{2}} + \frac{6x^{\frac{3}{2}}}{\frac{3}{2}} - 4x + d \\ &= 4x^{\frac{5}{2}} + 4x^{\frac{3}{2}} - 4x + d\end{aligned}$$

But  $f(0) = 5$ , so  $d = 5$

$$\begin{aligned}\therefore f(x) &= 4x^{\frac{5}{2}} + 4x^{\frac{3}{2}} - 4x + 5 \\ &= 4x^2\sqrt{x} + 4x\sqrt{x} - 4x + 5\end{aligned}$$

**d**  $f''(x) = 3^x$

$$\begin{aligned}\therefore f'(x) &= \int 3^x dx \\ &= \frac{3^x}{\ln 3} + c\end{aligned}$$

But  $f'(0) = 0$ , so  $\frac{3^0}{\ln 3} + c = 0$   
 $\therefore c = -\frac{1}{\ln 3}$

$$\therefore f'(x) = \frac{3^x}{\ln 3} - \frac{1}{\ln 3}$$

$$\begin{aligned}\therefore f(x) &= \int \left( \frac{3^x}{\ln 3} - \frac{1}{\ln 3} \right) dx \\ &= \frac{3^x}{(\ln 3)^2} - \frac{x}{\ln 3} + d\end{aligned}$$

But  $f(1) = -\frac{1}{\ln 3}$ ,

so  $\frac{3}{(\ln 3)^2} - \frac{1}{\ln 3} + d = -\frac{1}{\ln 3}$   
 $\therefore d = -\frac{3}{(\ln 3)^2}$

$$\therefore f(x) = \frac{3^x}{(\ln 3)^2} - \frac{x}{\ln 3} - \frac{3}{(\ln 3)^2}$$

$$\text{e} \quad f''(x) = 2x$$

$$\begin{aligned}\therefore f'(x) &= \int 2x \, dx \\ &= x^2 + c\end{aligned}$$

$$\begin{aligned}\therefore f(x) &= \int (x^2 + c) \, dx \\ &= \frac{1}{3}x^3 + cx + d\end{aligned}$$

But  $(1, 0)$  and  $(0, 5)$  lie on the curve  $y = f(x)$

$$\therefore f(1) = 0$$

$$\therefore \frac{1}{3} + c + d = 0$$

$$\therefore c + d = -\frac{1}{3} \quad \dots (*)$$

$$\text{and } f(0) = 5$$

$$\therefore d = 5$$

Substituting  $d = 5$  into  $(*)$  gives:

$$c + 5 = -\frac{1}{3}$$

$$\therefore c = -\frac{16}{3}$$

$$\therefore f(x) = \frac{1}{3}x^3 - \frac{16}{3}x + 5$$

$$\text{8} \quad f''(x) = \sin x - \cos x$$

$$\begin{aligned}\therefore f'(x) &= \int (\sin x - \cos x) \, dx \\ &= -\cos x - \sin x + c\end{aligned}$$

$$\begin{aligned}\therefore f(x) &= \int (-\cos x - \sin x + c) \, dx \\ &= -\sin x + \cos x + cx + d\end{aligned}$$

$$\text{But } f\left(\frac{\pi}{2}\right) = -1$$

$$\therefore -\sin \frac{\pi}{2} + \cos \frac{\pi}{2} + \frac{c\pi}{2} + d = -1$$

$$\therefore -1 + 0 + \frac{c\pi}{2} + d = -1$$

$$\therefore d = -\frac{c\pi}{2} \quad \dots (*)$$

$$\text{and } f(\pi) = 2$$

$$\therefore -\sin \pi + \cos \pi + c\pi + d = 2$$

$$\therefore 0 - 1 + c\pi - \frac{c\pi}{2} = 2 \quad \{\text{using } (*)\}$$

$$\therefore \frac{c\pi}{2} = 3$$

$$\therefore c = \frac{6}{\pi}$$

$$\begin{aligned}\text{Substituting into } (*): \quad d &= -\frac{6}{\pi} \times \frac{\pi}{2} \\ &= -3\end{aligned}$$

$$\therefore f(x) = -\sin x + \cos x + \frac{6}{\pi}x - 3$$

**EXERCISE 21D**

$$\begin{aligned}
 \mathbf{1} \quad \mathbf{a} \quad & \int (2x + 5)^3 dx \\
 &= \frac{1}{2} \times \frac{(2x + 5)^4}{4} + c \\
 &= \frac{1}{8}(2x + 5)^4 + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & \int 3(1 - x)^4 dx \\
 &= 3 \int (1 - x)^4 dx \\
 &= 3 \times \frac{1}{-1} \times \frac{(1 - x)^5}{5} + c \\
 &= -\frac{3}{5}(1 - x)^5 + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{e} \quad & \int \frac{4}{(2x - 1)^4} dx \\
 &= 4 \int (2x - 1)^{-4} dx \\
 &= 4 \times \frac{1}{2} \times \frac{(2x - 1)^{-3}}{-3} + c \\
 &= -\frac{2}{3(2x - 1)^3} + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{g} \quad & \int \frac{10}{\sqrt{1 - 5x}} dx \\
 &= 10 \int (1 - 5x)^{-\frac{1}{2}} dx \\
 &= 10 \times \frac{1}{-5} \times \frac{(1 - 5x)^{\frac{1}{2}}}{\frac{1}{2}} + c \\
 &= -4\sqrt{1 - 5x} + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{i} \quad & \int \frac{5}{(3x - 2)^3} dx \\
 &= 5 \int (3x - 2)^{-3} dx \\
 &= 5 \times \frac{1}{3} \times \frac{(3x - 2)^{-2}}{-2} + c \\
 &= -\frac{5}{6(3x - 2)^2} + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \int (4x - 3)^7 dx \\
 &= \frac{1}{4} \times \frac{(4x - 3)^8}{8} + c \\
 &= \frac{1}{32}(4x - 3)^8 + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad & \int \frac{1}{(3 - 2x)^2} dx \\
 &= \int (3 - 2x)^{-2} dx \\
 &= \frac{1}{-2} \times \frac{(3 - 2x)^{-1}}{-1} + c \\
 &= \frac{1}{2(3 - 2x)} + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{f} \quad & \int \sqrt{3x - 4} dx \\
 &= \int (3x - 4)^{\frac{1}{2}} dx \\
 &= \frac{1}{3} \times \frac{(3x - 4)^{\frac{3}{2}}}{\frac{3}{2}} + c \\
 &= \frac{2}{9}(3x - 4)^{\frac{3}{2}} + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{h} \quad & \int \frac{4}{\sqrt{3 - 4x}} dx \\
 &= 4 \int (3 - 4x)^{-\frac{1}{2}} dx \\
 &= 4 \times \frac{1}{-4} \times \frac{(3 - 4x)^{\frac{1}{2}}}{\frac{1}{2}} + c \\
 &= -2\sqrt{3 - 4x} + c
 \end{aligned}$$

$$2 \quad \frac{dy}{dx} = \sqrt{2x-7} = (2x-7)^{\frac{1}{2}}$$

$$\begin{aligned}\therefore y &= \int (2x-7)^{\frac{1}{2}} dx \\ &= \frac{1}{2} \times \frac{(2x-7)^{\frac{3}{2}}}{\frac{3}{2}} + c\end{aligned}$$

$$\therefore y = f(x) = \frac{1}{3}(2x-7)^{\frac{3}{2}} + c$$

But  $f(8) = 11$ , so  $\frac{1}{3}(2(8)-7)^{\frac{3}{2}} + c = 11$

$$\therefore \frac{1}{3}(9)^{\frac{3}{2}} + c = 11$$

$$\therefore \frac{1}{3}(27) + c = 11$$

$$\therefore 9 + c = 11$$

$$\therefore c = 2$$

$$\therefore y = \frac{1}{3}(2x-7)^{\frac{3}{2}} + 2$$

$$3 \quad f'(x) = \frac{4}{\sqrt{1-x}} = 4(1-x)^{-\frac{1}{2}}$$

$$\begin{aligned}\therefore f(x) &= \int 4(1-x)^{-\frac{1}{2}} dx \\ &= 4 \int (1-x)^{-\frac{1}{2}} dx \\ &= 4 \times \frac{1}{-1} \times \frac{(1-x)^{\frac{1}{2}}}{\frac{1}{2}} + c \\ &= -8\sqrt{1-x} + c\end{aligned}$$

But  $y = f(x)$  passes through  $(-3, -11)$ , so  $-8\sqrt{1-(-3)} + c = -11$

$$\therefore -8\sqrt{4} + c = -11$$

$$\therefore -16 + c = -11$$

$$\therefore c = 5$$

$$\therefore f(x) = 5 - 8\sqrt{1-x}$$

$$\begin{aligned}\text{Now } f(-8) &= 5 - 8\sqrt{1-(-8)} \\ &= 5 - 8(3) \\ &= -19\end{aligned}$$

So, the point on the graph of  $y = f(x)$  with  $x$ -coordinate  $-8$  is  $(-8, -19)$ .

$$\begin{aligned}4 \quad a \quad & \int 3(2x-1)^2 dx \\ &= 3 \int (2x-1)^2 dx \\ &= 3 \times \frac{1}{2} \times \frac{(2x-1)^3}{3} + c \\ &= \frac{1}{2}(2x-1)^3 + c\end{aligned}$$

$$\begin{aligned}b \quad & \int (4x-5)^2 dx \\ &= \frac{1}{4} \times \frac{(4x-5)^3}{3} + c \\ &= \frac{1}{12}(4x-5)^3 + c\end{aligned}$$



$$\begin{aligned}
 \text{c} \quad & \int (1-3x)^3 dx \\
 &= \frac{1}{-3} \times \frac{(1-3x)^4}{4} + c \\
 &= -\frac{1}{12}(1-3x)^4 + c
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & \int 4\sqrt{5-x} dx \\
 &= 4 \int (5-x)^{\frac{1}{2}} dx \\
 &= 4 \times \frac{1}{-\frac{1}{2}} \times \frac{(5-x)^{\frac{3}{2}}}{\frac{3}{2}} + c \\
 &= -\frac{8}{3}(5-x)^{\frac{3}{2}} + c
 \end{aligned}$$

$$\begin{aligned}
 5 \quad & \frac{dy}{dx} = x - \frac{5}{(1-x)^2} \\
 &= x - 5(1-x)^{-2} \\
 \therefore y &= \int (x - 5(1-x)^{-2}) dx \\
 &= \frac{1}{2}x^2 - 5 \times \frac{1}{-1} \times \frac{(1-x)^{-1}}{-1} + c \\
 &= \frac{1}{2}x^2 - \frac{5}{1-x} + c
 \end{aligned}$$

But when  $x = 2$ ,  $y = 0$

$$\begin{aligned}
 \therefore 0 &= \frac{1}{2}(2)^2 - \frac{5}{1-2} + c \\
 \therefore 0 &= \frac{1}{2} \times 4 - \frac{5}{-1} + c \\
 \therefore 0 &= 2 + 5 + c \\
 \therefore c &= -7
 \end{aligned}$$

$$\therefore y = \frac{1}{2}x^2 - \frac{5}{1-x} - 7$$

$$\begin{aligned}
 6 \quad \text{a} \quad & \int \sin 3x dx \\
 &= -\frac{1}{3} \cos 3x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & \int \sec^2 2x dx \\
 &= \frac{1}{2} \tan 2x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad & \int (2-5x)^2 dx \\
 &= \frac{1}{-5} \times \frac{(2-5x)^3}{3} + c \\
 &= -\frac{1}{15}(2-5x)^3 + c
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & \int (7x+1)^4 dx \\
 &= \frac{1}{7} \times \frac{(7x+1)^5}{5} + c \\
 &= \frac{1}{35}(7x+1)^5 + c
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & \int (2 \cos(-4x) + 1) dx \\
 &= 2 \times \left(\frac{1}{-4}\right) \sin(-4x) + x + c \\
 &= -\frac{1}{2} \sin(-4x) + x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad & \int 3 \cos \frac{x}{2} dx \\
 &= 6 \sin \frac{x}{2} + c
 \end{aligned}$$

$$\begin{aligned} \text{e} \quad & \int (3 \sin 2x - e^{-x}) dx \\ &= -\frac{3}{2} \cos 2x + e^{-x} + c \end{aligned}$$

$$\begin{aligned} \text{g} \quad & \int 2 \sin\left(2x + \frac{\pi}{6}\right) dx \\ &= 2 \times \left(\frac{1}{2}\right) \left(-\cos\left(2x + \frac{\pi}{6}\right)\right) + c \\ &= -\cos\left(2x + \frac{\pi}{6}\right) + c \end{aligned}$$

$$\begin{aligned} \text{i} \quad & \int 4 \sec^2\left(\frac{\pi}{3} - 2x\right) dx \\ &= 4 \times \left(\frac{1}{-2}\right) \tan\left(\frac{\pi}{3} - 2x\right) + c \\ &= -2 \tan\left(\frac{\pi}{3} - 2x\right) + c \end{aligned}$$

$$\begin{aligned} \text{k} \quad & \int (2 \sin 3x + 5 \cos 4x) dx \\ &= -\frac{2}{3} \cos 3x + \frac{5}{4} \sin 4x + c \end{aligned}$$

$$\begin{aligned} \text{f} \quad & \int \left(e^{2x} - 2 \sec^2\left(\frac{x}{2}\right)\right) dx \\ &= \frac{1}{2} e^{2x} - 2 \left(\frac{1}{\frac{1}{2}}\right) \tan \frac{x}{2} + c \\ &= \frac{1}{2} e^{2x} - 4 \tan \frac{x}{2} + c \end{aligned}$$

$$\begin{aligned} \text{h} \quad & \int -3 \cos\left(\frac{\pi}{4} - x\right) dx \\ &= -3 \times \left(\frac{1}{-1}\right) \sin\left(\frac{\pi}{4} - x\right) + c \\ &= 3 \sin\left(\frac{\pi}{4} - x\right) + c \end{aligned}$$

$$\begin{aligned} \text{j} \quad & \int (\cos 2x + \sin 2x) dx \\ &= \frac{1}{2} \sin 2x - \frac{1}{2} \cos 2x + c \end{aligned}$$

$$\begin{aligned} \text{l} \quad & \int \left(\frac{1}{2} \cos 8x - 3 \sin x\right) dx \\ &= \frac{1}{2} \times \frac{1}{8} \sin 8x + 3 \cos x + c \\ &= \frac{1}{16} \sin 8x + 3 \cos x + c \end{aligned}$$

$$\begin{aligned} 7 \quad \text{a} \quad & \int (2e^x + 5e^{2x}) dx \\ &= 2e^x + 5\left(\frac{1}{2}\right)e^{2x} + c \\ &= 2e^x + \frac{5}{2}e^{2x} + c \end{aligned}$$

$$\begin{aligned} \text{c} \quad & \int (e^{7-3x}) dx \\ &= \left(\frac{1}{-3}\right)e^{7-3x} + c \\ &= -\frac{1}{3}e^{7-3x} + c \end{aligned}$$

$$\begin{aligned} \text{e} \quad & \int (e^{-x} + 2)^2 dx \\ &= \int (e^{-2x} + 4e^{-x} + 4) dx \\ &= \left(\frac{1}{-2}\right)e^{-2x} + 4\left(\frac{1}{-1}\right)e^{-x} + 4x + c \\ &= -\frac{1}{2}e^{-2x} - 4e^{-x} + 4x + c \end{aligned}$$

$$\begin{aligned} \text{b} \quad & \int (3e^{5x-2}) dx \\ &= 3\left(\frac{1}{5}\right)e^{5x-2} + c \\ &= \frac{3}{5}e^{5x-2} + c \end{aligned}$$

$$\begin{aligned} \text{d} \quad & \int (e^x + e^{-x})^2 dx \\ &= \int (e^{2x} + 2 + e^{-2x}) dx \\ &= \frac{1}{2}e^{2x} + 2x + \left(\frac{1}{-2}\right)e^{-2x} + c \\ &= \frac{1}{2}e^{2x} + 2x - \frac{1}{2}e^{-2x} + c \end{aligned}$$

$$\begin{aligned} \text{f} \quad & \int \frac{(e^{2x} - 5)^2}{e^x} dx \\ &= \int \frac{e^{4x} - 10e^{2x} + 25}{e^x} dx \\ &= \int (e^{3x} - 10e^x + 25e^{-x}) dx \\ &= \frac{1}{3}e^{3x} - 10e^x - 25e^{-x} + c \end{aligned}$$

$$8 \quad \frac{dy}{dx} = (1 - e^x)^2 = 1 - 2e^x + e^{2x}$$

$$\begin{aligned} \therefore y &= \int (1 - 2e^x + e^{2x}) dx \\ &= x - 2e^x + \frac{1}{2}e^{2x} + c \end{aligned}$$

When  $x = 0$ ,  $y = 4$

$$\begin{aligned} \therefore 0 - 2 + \frac{1}{2} + c &= 4 \\ \therefore c &= \frac{11}{2} \end{aligned}$$

$$\therefore y = x - 2e^x + \frac{1}{2}e^{2x} + \frac{11}{2}$$

$$9 \quad \text{a} \quad \frac{d}{dx} \left( \frac{k^{ax+b}}{a \ln k} \right) = \frac{a \times k^{ax+b} \ln k}{a \ln k}, \quad k > 0, \quad k \neq 1, \quad a \neq 0$$

$$= k^{ax+b}$$

$$\therefore \int k^{ax+b} dx = \frac{k^{ax+b}}{a \ln k} + c, \quad k > 0, \quad k \neq 1, \quad a \neq 0$$

$$\text{b} \quad \text{i} \quad \int 3^{2x-1} dx = \frac{3^{2x-1}}{2 \ln 3} + c \qquad \text{ii} \quad \int 5^{-x} dx = -\frac{5^{-x}}{\ln 5} + c$$

$$\begin{aligned} \text{iii} \quad \int (2^{5x} - 7^{1-2x}) dx &= \int 2^{5x} dx - \int 7^{1-2x} dx \\ &= \frac{2^{5x}}{5 \ln 2} - \frac{7^{1-2x}}{(-2) \ln 7} + c \\ &= \frac{2^{5x}}{5 \ln 2} + \frac{7^{1-2x}}{2 \ln 7} + c \end{aligned}$$

$$10 \quad f'(x) = 2e^{-2x}$$

$$\begin{aligned} \therefore f(x) &= 2\left(\frac{1}{-2}\right)e^{-2x} + c \\ &= -e^{-2x} + c \end{aligned}$$

$$\text{But } f(0) = 3, \text{ so } -e^0 + c = 3$$

$$\therefore -1 + c = 3$$

$$\therefore c = 4$$

$$\therefore f(x) = -e^{-2x} + 4$$

**11**  $f'(x) = p \sin \frac{x}{2}$

$$\begin{aligned}\therefore f(x) &= \int p \sin \frac{x}{2} dx \\ &= p \int \sin \frac{x}{2} dx \\ &= p \times 2 \left( -\cos \frac{x}{2} \right) + c \\ &= -2p \cos \frac{x}{2} + c\end{aligned}$$

But  $f(0) = 1$ , so  $-2p + c = 1$  .... (1)

and  $f(2\pi) = 0$ , so  $2p + c = 0$  .... (2)

(2) - (1) gives:  $2p + c - (-2p + c) = 0 - 1$

$$\therefore 4p = -1$$

$$\begin{aligned}\therefore p &= -\frac{1}{4} \quad \text{and} \quad \therefore c = -2p \quad \{\text{using (2)}\} \\ &= -2\left(-\frac{1}{4}\right) \\ &= \frac{1}{2}\end{aligned}$$

$$\therefore f(x) = \frac{1}{2} \cos \frac{x}{2} + \frac{1}{2}$$

**12**  $g''(x) = -\sin 2x$

$$\begin{aligned}\therefore g'(x) &= \int -\sin 2x dx \\ &= \frac{1}{2} \cos 2x + c\end{aligned}$$

Now  $g'(\pi) = \frac{1}{2} \cos 2\pi + c$

$$= \frac{1}{2} + c$$

and  $g'(-\pi) = \frac{1}{2} \cos(-2\pi) + c$

$$= \frac{1}{2} + c$$

$$= g'(\pi)$$

$\therefore$  the gradients of the tangents to  $y = g(x)$  at  $x = \pi$  and  $x = -\pi$  are equal.

**13**  $\frac{dy}{dx} = \sqrt{x} + \frac{1}{2}e^{-4x} = x^{\frac{1}{2}} + \frac{1}{2}e^{-4x}$

$$\begin{aligned}\therefore y &= \int (x^{\frac{1}{2}} + \frac{1}{2}e^{-4x}) dx \\ &= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{1}{2} \left( \frac{1}{-4} \right) e^{-4x} + c \\ &= \frac{2}{3} x^{\frac{3}{2}} - \frac{1}{8} e^{-4x} + c\end{aligned}$$

But  $y = 0$  when  $x = 1$  so  $\frac{2}{3} - \frac{1}{8}e^{-4} + c = 0$

$$\therefore c = \frac{1}{8}e^{-4} - \frac{2}{3}$$

$$\therefore y = \frac{2}{3}x\sqrt{x} - \frac{1}{8}e^{-4x} + \frac{1}{8}e^{-4} - \frac{2}{3}$$



$$\begin{aligned}
 \mathbf{14} \quad (\sin x + \cos x)^2 &= \sin^2 x + 2 \sin x \cos x + \cos^2 x \\
 &= \sin^2 x + \cos^2 x + \sin 2x && \{\sin 2x = 2 \sin x \cos x\} \\
 &= 1 + \sin 2x && \{\sin^2 x + \cos^2 x = 1\} \\
 \therefore \int (\sin x + \cos x)^2 dx &= \int (1 + \sin 2x) dx \\
 &= x + \frac{1}{2}(-\cos 2x) + c \\
 &= x - \frac{1}{2} \cos 2x + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{15} \quad \mathbf{a} \quad \cos 2x &= 1 - 2 \sin^2 x && \cos 2x = 2 \cos^2 x - 1 \\
 \therefore 2 \sin^2 x &= 1 - \cos 2x && \therefore 2 \cos^2 x = 1 + \cos 2x \\
 \therefore \sin^2 x &= \frac{1}{2} - \frac{1}{2} \cos 2x && \therefore \cos^2 x = \frac{1}{2} + \frac{1}{2} \cos 2x
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad \mathbf{i} \quad \int \sin^2 x dx &= \int \left( \frac{1}{2} - \frac{1}{2} \cos 2x \right) dx && \{\text{from } \mathbf{a}\} \\
 &= \frac{1}{2}x - \frac{1}{2} \left( \frac{1}{2} \sin 2x \right) + c \\
 &= \frac{1}{2}x - \frac{1}{4} \sin 2x + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{ii} \quad \int \cos^2 x dx &= \int \left( \frac{1}{2} + \frac{1}{2} \cos 2x \right) dx && \{\text{from } \mathbf{a}\} \\
 &= \frac{1}{2}x + \frac{1}{2} \left( \frac{1}{2} \sin 2x \right) + c \\
 &= \frac{1}{2}x + \frac{1}{4} \sin 2x + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{16} \quad \mathbf{a} \quad \int (\cos^2 x + 2) dx \\
 &= \int \left( \frac{1}{2} + \frac{1}{2} \cos 2x + 2 \right) dx \\
 &= \int \left( \frac{5}{2} + \frac{1}{2} \cos 2x \right) dx \\
 &= \frac{5}{2}x + \frac{1}{4} \sin 2x + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad \int (\sin^2 x + 4x) dx \\
 &= \int \left( \frac{1}{2} - \frac{1}{2} \cos 2x + 4x \right) dx \\
 &= \frac{1}{2}x - \frac{1}{4} \sin 2x + 2x^2 + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad \int (1 + \cos^2 2x) dx \\
 &= \int \left( 1 + \frac{1}{2} + \frac{1}{2} \cos(2(2x)) \right) dx \\
 &= \int \left( \frac{3}{2} + \frac{1}{2} \cos 4x \right) dx \\
 &= \frac{3}{2}x + \frac{1}{8} \sin 4x + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad \int (3 - \sin^2 3x) dx \\
 &= \int \left( 3 - \left( \frac{1}{2} - \frac{1}{2} \cos(2(3x)) \right) \right) dx \\
 &= \int \left( \frac{5}{2} + \frac{1}{2} \cos 6x \right) dx \\
 &= \frac{5}{2}x + \frac{1}{12} \sin 6x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & \int \frac{1}{2} \cos^2 4x \, dx \\
 &= \int \frac{1}{2} \left( \frac{1}{2} + \frac{1}{2} \cos(2(4x)) \right) dx \\
 &= \int \left( \frac{1}{4} + \frac{1}{4} \cos 8x \right) dx \\
 &= \frac{1}{4}x + \frac{1}{32} \sin 8x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & \int (1 + \cos x)^2 \, dx \\
 &= \int (1 + 2 \cos x + \cos^2 x) \, dx \\
 &= \int \left( 1 + 2 \cos x + \frac{1}{2} + \frac{1}{2} \cos 2x \right) dx \\
 &= \int \left( \frac{3}{2} + 2 \cos x + \frac{1}{2} \cos 2x \right) dx \\
 &= \frac{3}{2}x + 2 \sin x + \frac{1}{4} \sin 2x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{g} \quad & \int \sin x (2 \sin x - 1) \, dx \\
 &= \int (2 \sin^2 x - \sin x) \, dx \\
 &= \int \left( 2 \left( \frac{1}{2} - \frac{1}{2} \cos 2x \right) - \sin x \right) dx \\
 &= \int (1 - \cos 2x - \sin x) \, dx \\
 &= x - \frac{1}{2} \sin 2x + \cos x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{h} \quad & \int (1 - 3 \sin x)^2 \, dx \\
 &= \int (1 - 6 \sin x + 9 \sin^2 x) \, dx \\
 &= \int \left( 1 - 6 \sin x + 9 \left( \frac{1}{2} - \frac{1}{2} \cos 2x \right) \right) dx \\
 &= \int \left( 1 - 6 \sin x + \frac{9}{2} - \frac{9}{2} \cos 2x \right) dx \\
 &= \int \left( \frac{11}{2} - 6 \sin x - \frac{9}{2} \cos 2x \right) dx \\
 &= \frac{11}{2}x + 6 \cos x - \frac{9}{4} \sin 2x + c
 \end{aligned}$$

$$\begin{aligned}
 17 \quad \text{a} \quad & \cos^2 \theta = \frac{1}{2} + \frac{1}{2} \cos 2\theta \\
 \therefore \cos^4 x &= \left( \frac{1}{2} + \frac{1}{2} \cos 2x \right)^2 \\
 &= \frac{1}{4} + \frac{1}{2} \cos 2x + \frac{1}{4} \cos^2 2x \\
 &= \frac{1}{4} + \frac{1}{2} \cos 2x + \frac{1}{4} \left( \frac{1}{2} + \frac{1}{2} \cos 4x \right) \\
 &= \frac{1}{4} + \frac{1}{2} \cos 2x + \frac{1}{8} + \frac{1}{8} \cos 4x \\
 &= \frac{1}{8} \cos 4x + \frac{1}{2} \cos 2x + \frac{3}{8} \quad \text{as required}
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & \int \cos^4 x \, dx = \int \left( \frac{1}{8} \cos 4x + \frac{1}{2} \cos 2x + \frac{3}{8} \right) dx \\
 &= \frac{1}{32} \sin 4x + \frac{1}{4} \sin 2x + \frac{3}{8}x + c
 \end{aligned}$$

$$18 \quad \text{a} \quad \int \frac{6}{x+4} \, dx = 6 \ln |x+4| + c$$

$$\text{b} \quad \int \frac{1}{2x-1} \, dx = \frac{1}{2} \ln |2x-1| + c$$

$$\begin{aligned}
 \text{c} \quad & \int \frac{5}{1-3x} \, dx \\
 &= 5 \int \frac{1}{1-3x} \, dx \\
 &= 5 \left( -\frac{1}{3} \right) \ln |1-3x| + c \\
 &= -\frac{5}{3} \ln |1-3x| + c
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad & \int \left( 4 + \frac{1}{5x-2} \right) dx \\
 &= 4x + \frac{1}{5} \ln |5x-2| + c
 \end{aligned}$$

$$\begin{aligned} \text{e} \quad & \int \left( 1 - 2x + \frac{4}{x-3} \right) dx \\ &= x - x^2 + 4 \ln |x-3| + c \end{aligned}$$

$$\begin{aligned} \text{g} \quad & \int \left( e^{-x} - \frac{4}{2x+1} \right) dx \\ &= \left( \frac{1}{-1} \right) e^{-x} - 4 \left( \frac{1}{2} \right) \ln |2x+1| + c \\ &= -e^{-x} - 2 \ln |2x+1| + c \end{aligned}$$

$$\begin{aligned} \text{i} \quad & \int \left( \frac{5}{x-6} - \frac{2}{3x-1} \right) dx \\ &= 5 \ln |x-6| - 2 \left( \frac{1}{3} \right) \ln |3x-1| + c \\ &= 5 \ln |x-6| - \frac{2}{3} \ln |3x-1| + c \end{aligned}$$

$$\begin{aligned} \text{f} \quad & \int \left( \sin 2x - \frac{3}{1-2x} \right) dx \\ &= -\frac{1}{2} \cos 2x - 3 \left( \frac{1}{-2} \right) \ln |1-2x| + c \\ &= -\frac{1}{2} \cos 2x + \frac{3}{2} \ln |1-2x| + c \end{aligned}$$

$$\begin{aligned} \text{h} \quad & \int \left( \frac{1}{x+2} + \frac{2}{x-3} \right) dx \\ &= \ln |x+2| + 2 \ln |x-3| + c \end{aligned}$$

**19** Differentiating Tracy's answer:  $\frac{d}{dx} \left( \frac{1}{4} \ln 4x + c \right) = \frac{1}{4} \left( \frac{4}{4x} \right) + 0, \quad x > 0$

$$= \frac{1}{4x}, \quad x > 0$$

Differentiating Nadine's answer:  $\frac{d}{dx} \left( \frac{1}{4} \ln x + c \right) = \frac{1}{4} \left( \frac{1}{x} \right) + 0, \quad x > 0$

$$= \frac{1}{4x}, \quad x > 0$$

Both answers give the correct derivative and both are correct. This result occurs because  $\ln 4x = \ln 4 + \ln x$ . So, their answers differ by a constant which is accounted for by  $c$ .

**20**  $\frac{3x-1}{x+2} = \frac{3(x+2)-6-1}{x+2}$

$$= \frac{3(x+2)-7}{x+2}$$

$$= 3 - \frac{7}{x+2}$$

$$\begin{aligned} \therefore \int \frac{3x-1}{x+2} dx &= \int \left( 3 - \frac{7}{x+2} \right) dx \\ &= 3x - 7 \ln |x+2| + c \end{aligned}$$

**21**

$$\begin{array}{r} x^2 + 0x + 4 \\ x-1 \overline{) \begin{array}{r} x^3 - x^2 + 4x - 3 \\ -(x^3 - x^2) \phantom{+ 4x - 3} \\ \hline 0x^2 + 4x \phantom{- 3} \\ -(0x^2 - 0x) \phantom{- 3} \\ \hline 4x - 3 \\ -(4x - 4) \\ \hline 1 \end{array}} \end{array}$$

$$\therefore \frac{x^3 - x^2 + 4x - 3}{x-1} = x^2 + 4 + \frac{1}{x-1}$$

$$\begin{aligned} \therefore \int \frac{x^3 - x^2 + 4x - 3}{x-1} dx &= \int \left( x^2 + 4 + \frac{1}{x-1} \right) dx \\ &= \frac{1}{3} x^3 + 4x + \ln |x-1| + c \end{aligned}$$

$$\mathbf{22} \quad f'(x) = 2x - \frac{2}{1-x}$$

$$\begin{aligned} \therefore f(x) &= \int \left( 2x - \frac{2}{1-x} \right) dx \\ &= x^2 - 2\left(\frac{1}{-1}\right) \ln|1-x| + c \\ &= x^2 + 2\ln|1-x| + c \end{aligned}$$

$$\text{But } f(-1) = 3$$

$$\therefore (-1)^2 + 2\ln|1-(-1)| + c = 3$$

$$\therefore 1 + 2\ln 2 + c = 3$$

$$\therefore c = 2 - 2\ln 2$$

$$\therefore f(x) = x^2 + 2\ln|1-x| + 2 - 2\ln 2$$

$$\begin{aligned} \mathbf{23} \quad \mathbf{a} \quad & \frac{d}{dx} \left( \frac{1}{a} \log_k |ax+b| \right) \\ &= \frac{1}{a} \times \frac{a}{(ax+b) \ln k}, \quad k > 0, \quad k \neq 1, \quad a \neq 0 \quad \{\text{using Exercise 21A question 11 c}\} \\ &= \frac{1}{(ax+b) \ln k} \end{aligned}$$

$$\therefore \int \frac{1}{(ax+b) \ln k} dx = \frac{1}{a} \log_k |ax+b| + c, \quad k > 0, \quad k \neq 1, \quad a \neq 0$$

$$\mathbf{b} \quad \mathbf{i} \quad \int \frac{1}{(2x+5) \ln 3} dx = \frac{1}{2} \log_3 |2x+5| + c$$

$$\begin{aligned} \mathbf{ii} \quad \int \frac{3}{(1-x) \ln 2} dx &= \frac{3}{-1} \log_2 |1-x| + c \\ &= -3 \log_2 |1-x| + c \end{aligned}$$

$$\begin{aligned} \mathbf{24} \quad \mathbf{a} \quad & \frac{d}{dx} \left( \frac{1}{a} \operatorname{cosec}(ax+b) \right) = \frac{1}{a} \times -\operatorname{cosec}(ax+b) \times \cot(ax+b) \times a, \quad a \neq 0 \\ &= -\operatorname{cosec}(ax+b) \cot(ax+b) \end{aligned}$$

$$\therefore \int (-\operatorname{cosec}(ax+b) \cot(ax+b)) dx = \frac{1}{a} \operatorname{cosec}(ax+b) + c, \quad a \neq 0$$

$$\begin{aligned} \mathbf{b} \quad \mathbf{i} \quad & \frac{d}{dx} \left( \frac{1}{a} \sec(ax+b) \right) = \frac{1}{a} \times \sec(ax+b) \tan(ax+b) \times a, \quad a \neq 0 \\ &= \sec(ax+b) \tan(ax+b) \end{aligned}$$

$$\therefore \int \sec(ax+b) \tan(ax+b) dx = \frac{1}{a} \sec(ax+b) + c, \quad a \neq 0$$

$$\begin{aligned} \mathbf{ii} \quad & \frac{d}{dx} \left( \frac{1}{a} \cot(ax+b) \right) = \frac{1}{a} \times -\operatorname{cosec}^2(ax+b) \times a, \quad a \neq 0 \\ &= -\operatorname{cosec}^2(ax+b) \end{aligned}$$

$$\therefore \int (-\operatorname{cosec}^2(ax+b)) dx = \frac{1}{a} \cot(ax+b) + c, \quad a \neq 0$$



$$\text{c} \quad \text{i} \quad \int \operatorname{cosec}\left(2x - \frac{\pi}{6}\right) \cot\left(2x - \frac{\pi}{6}\right) dx = -\frac{1}{2} \operatorname{cosec}\left(2x - \frac{\pi}{6}\right) + c \quad \{\text{using a}\}$$

$$\begin{aligned} \text{ii} \quad \int 3 \sec\left(\frac{\pi}{3} - \theta\right) \tan\left(\frac{\pi}{3} - \theta\right) d\theta &= 3 \int \sec\left(\frac{\pi}{3} - \theta\right) \tan\left(\frac{\pi}{3} - \theta\right) d\theta \\ &= -3 \sec\left(\frac{\pi}{3} - \theta\right) + c \quad \{\text{using b i}\} \end{aligned}$$

$$\begin{aligned} \text{iii} \quad \int \frac{\operatorname{cosec}^2(4x - \pi)}{5} dx &= \frac{1}{5} \int \operatorname{cosec}^2(4x - \pi) dx \\ &= \frac{1}{5} \times -\frac{1}{4} \cot(4x - \pi) + c \quad \{\text{using b ii}\} \\ &= -\frac{1}{20} \cot(4x - \pi) + c \end{aligned}$$

$$\begin{aligned} \text{25} \quad \text{a} \quad \frac{d}{dx} \left( \frac{1}{a} \arcsin(ax + b) \right) &= \frac{1}{a} \times \frac{a}{\sqrt{1 - (ax + b)^2}} \\ &= \frac{1}{\sqrt{1 - (ax + b)^2}} \end{aligned}$$

$$\therefore \int \frac{1}{\sqrt{1 - (ax + b)^2}} dx = \frac{1}{a} \arcsin(ax + b) + c, \quad a \neq 0$$

$$\begin{aligned} \text{b} \quad \text{i} \quad \frac{d}{dx} \left( \frac{1}{a} \arccos(ax + b) \right) &= \frac{1}{a} \times \frac{-a}{\sqrt{1 - (ax + b)^2}} \\ &= \frac{-1}{\sqrt{1 - (ax + b)^2}} \end{aligned}$$

$$\therefore \int \frac{-1}{\sqrt{1 - (ax + b)^2}} dx = \frac{1}{a} \arccos(ax + b) + c, \quad a \neq 0$$

$$\begin{aligned} \text{ii} \quad \frac{d}{dx} \left( \frac{1}{a} \arctan(ax + b) \right) &= \frac{1}{a} \times \frac{a}{1 + (ax + b)^2} \\ &= \frac{1}{1 + (ax + b)^2} \end{aligned}$$

$$\therefore \int \frac{1}{1 + (ax + b)^2} dx = \frac{1}{a} \arctan(ax + b) + c, \quad a \neq 0$$

$$\text{c} \quad \text{i} \quad \int \frac{1}{\sqrt{1 - (x + 2)^2}} dx = \arcsin(x + 2) + c \quad \{\text{using a}\}$$

$$\begin{aligned} \text{ii} \quad \int \frac{-2}{\sqrt{-x^2 + 10x - 24}} dx &= 2 \int \frac{-1}{\sqrt{1 - (x - 5)^2}} dx \\ &= 2 \arccos(x - 5) + c \quad \{\text{using b i}\} \end{aligned}$$

$$\begin{aligned} \text{iii} \quad \int \frac{1}{4x^2 + 12x + 10} dx &= \int \frac{1}{1 + (2x + 3)^2} dx \\ &= \frac{1}{2} \arctan(2x + 3) + c \quad \{\text{using b ii}\} \end{aligned}$$

$$\begin{aligned}
 \text{26 a i} \quad \frac{d}{dx} \left( \arcsin \frac{x}{a} \right) &= \frac{\frac{1}{a}}{\sqrt{1 - \left(\frac{x}{a}\right)^2}} \\
 &= \frac{\frac{1}{a}}{\sqrt{1 - \frac{x^2}{a^2}}} \times \frac{\sqrt{a^2}}{\sqrt{a^2}} \\
 &= \frac{1}{\sqrt{a^2 - x^2}}
 \end{aligned}$$

$$\therefore \int \frac{1}{\sqrt{a^2 - x^2}} dx = \arcsin \frac{x}{a} + c$$

$$\begin{aligned}
 \text{ii} \quad \frac{d}{dx} \left( \arccos \frac{x}{a} \right) &= \frac{-\frac{1}{a}}{\sqrt{1 - \left(\frac{x}{a}\right)^2}} \\
 &= \frac{-\frac{1}{a}}{\sqrt{1 - \frac{x^2}{a^2}}} \times \frac{\sqrt{a^2}}{\sqrt{a^2}} \\
 &= \frac{-1}{\sqrt{a^2 - x^2}}
 \end{aligned}$$

$$\therefore \int \frac{-1}{\sqrt{a^2 - x^2}} dx = \arccos \frac{x}{a} + c$$

$$\begin{aligned}
 \text{iii} \quad \frac{d}{dx} \left( \arctan \frac{x}{a} \right) &= \frac{\frac{1}{a}}{1 + \left(\frac{x}{a}\right)^2} \\
 &= \frac{\frac{1}{a}}{1 + \frac{x^2}{a^2}} \times \frac{a^2}{a^2} \\
 &= \frac{a}{a^2 + x^2}
 \end{aligned}$$

$$\therefore \int \frac{a}{a^2 + x^2} dx = \arctan \frac{x}{a} + c$$

$$\begin{aligned}
 \text{b i} \quad \int \frac{1}{\sqrt{25 - x^2}} dx \\
 &= \int \frac{1}{\sqrt{5^2 - x^2}} dx \\
 &= \arcsin \frac{x}{5} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{ii} \quad \int -\frac{4}{\sqrt{9 - x^2}} dx \\
 &= 4 \int -\frac{1}{\sqrt{3^2 - x^2}} dx \\
 &= 4 \arccos \frac{x}{3} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{iii} \quad \int \frac{18}{9 + x^2} dx \\
 &= 6 \int \frac{3}{3^2 + x^2} dx \\
 &= 6 \arctan \frac{x}{3} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{iv} \quad \int \frac{1}{\sqrt{\frac{1}{4} - x^2}} dx \\
 &= \int \frac{1}{\sqrt{\left(\frac{1}{2}\right)^2 - x^2}} dx \\
 &= \arcsin \left( \frac{x}{\frac{1}{2}} \right) + c \\
 &= \arcsin 2x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{v} \quad \int -\frac{5}{\sqrt{\frac{1}{9} - x^2}} dx \\
 &= 5 \int -\frac{1}{\sqrt{\left(\frac{1}{3}\right)^2 - x^2}} dx \\
 &= 5 \arccos \left( \frac{x}{\frac{1}{3}} \right) + c \\
 &= 5 \arccos 3x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{vi} \quad \int \frac{1}{\frac{1}{16} + x^2} dx \\
 &= 4 \int \frac{\frac{1}{4}}{\left(\frac{1}{4}\right)^2 + x^2} dx \\
 &= 4 \arctan \left( \frac{x}{\frac{1}{4}} \right) + c \\
 &= 4 \arctan 4x + c
 \end{aligned}$$

## EXERCISE 21E

$$\begin{aligned}
 \text{1 a} \quad \frac{3}{x+2} - \frac{1}{x-2} \\
 &= \frac{3(x-2) - (x+2)}{(x+2)(x-2)} \\
 &= \frac{3x - 6 - x - 2}{(x+2)(x-2)} \\
 &= \frac{2x - 8}{(x+2)(x-2)}
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad \int \frac{2x-8}{x^2-4} dx \\
 &= \int \frac{2x-8}{(x+2)(x-2)} dx \\
 &= \int \left( \frac{3}{x+2} - \frac{1}{x-2} \right) dx \\
 &= 3 \ln |x+2| - \ln |x-2| + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{2} \quad \mathbf{a} \quad & \frac{1}{2x-1} - \frac{1}{2x+1} \\
 &= \frac{2x+1 - (2x-1)}{(2x-1)(2x+1)} \\
 &= \frac{2x+1 - 2x+1}{(2x-1)(2x+1)} \\
 &= \frac{2}{(2x-1)(2x+1)}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \int \frac{2}{4x^2-1} dx \\
 &= \int \frac{2}{(2x-1)(2x+1)} dx \\
 &= \int \left( \frac{1}{2x-1} - \frac{1}{2x+1} \right) dx \\
 &= \frac{1}{2} \ln |2x-1| - \frac{1}{2} \ln |2x+1|
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{3} \quad \mathbf{a} \quad \text{Let } & \frac{20}{(2x-3)(x+1)} = \frac{A}{2x-3} + \frac{B}{x+1} \\
 & \therefore 20 = A(x+1) + B(2x-3)
 \end{aligned}$$

$$\begin{aligned}
 \text{Substituting } x = -1, \quad & 20 = B(-2-3) \\
 & \therefore 20 = -5B \\
 & \therefore B = -4
 \end{aligned}$$

$$\begin{aligned}
 \text{Substituting } x = \frac{3}{2}, \quad & 20 = A\left(\frac{3}{2}+1\right) \\
 & \therefore 20 = \frac{5}{2}A \\
 & \therefore A = 8
 \end{aligned}$$

$$\therefore \frac{20}{(2x-3)(x+1)} = \frac{8}{2x-3} - \frac{4}{x+1}$$

$$\begin{aligned}
 \mathbf{b} \quad & \int \frac{20}{2x^2-x-3} dx = \int \frac{20}{(2x-3)(x+1)} dx \\
 &= \int \left( \frac{8}{2x-3} - \frac{4}{x+1} \right) dx \\
 &= \frac{8}{2} \times \ln |2x-3| - 4 \ln |x+1| + c \\
 &= 4 \ln |2x-3| - 4 \ln |x+1| + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{4} \quad \mathbf{a} \quad & x^2 - 2x - 3 = (x+1)(x-3) \\
 \text{Let } & \frac{x-9}{x^2-2x-3} = \frac{A}{x+1} + \frac{B}{x-3} \\
 & \therefore x-9 = A(x-3) + B(x+1)
 \end{aligned}$$

$$\begin{aligned}
 \text{Substituting } x = 3, \quad & 3-9 = B(3+1) \\
 & \therefore -6 = 4B \\
 & \therefore B = -\frac{3}{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{Substituting } x = -1, \quad & -1-9 = A(-1-3) \\
 & \therefore -10 = -4A \\
 & \therefore A = \frac{5}{2}
 \end{aligned}$$

$$\therefore \frac{x-9}{x^2-2x-3} = \frac{5}{2(x+1)} - \frac{3}{2(x-3)}$$

$$\begin{aligned}
 \mathbf{b} \quad & \int \frac{x-9}{x^2-2x-3} dx \\
 &= \int \left( \frac{5}{2(x+1)} - \frac{3}{2(x-3)} \right) dx \\
 &= \frac{5}{2} \ln |x+1| - \frac{3}{2} \ln |x-3| + c
 \end{aligned}$$

**5 a** Let  $\frac{3}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1}$   
 $\therefore 3 = A(x+1) + B(x-2)$

Substituting  $x = -1$ ,  $3 = B(-1-2)$   
 $\therefore 3 = -3B$   
 $\therefore B = -1$

Substituting  $x = 2$ ,  $3 = A(2+1)$   
 $\therefore 3 = 3A$   
 $\therefore A = 1$

$\therefore \frac{3}{(x-2)(x+1)} = \frac{1}{x-2} - \frac{1}{x+1}$   
 $\therefore \int \frac{3}{(x-2)(x+1)} dx = \int \left( \frac{1}{x-2} - \frac{1}{x+1} \right) dx$   
 $= \ln|x-2| - \ln|x+1| + c$

**b**  $x^2 + 2x - 3 = (x+3)(x-1)$

Let  $\frac{4x}{x^2 + 2x - 3} = \frac{A}{x+3} + \frac{B}{x-1}$   
 $\therefore 4x = A(x-1) + B(x+3)$

Substituting  $x = 1$ ,  $4(1) = B(1+3)$   
 $\therefore 4 = 4B$   
 $\therefore B = 1$

Substituting  $x = -3$ ,  $4(-3) = A(-3-1)$   
 $\therefore -12 = -4A$   
 $\therefore A = 3$

$\therefore \frac{4x}{x^2 + 2x - 3} = \frac{3}{x+3} + \frac{1}{x-1}$   
 $\therefore \int \frac{4x}{x^2 + 2x - 3} dx = \int \left( \frac{3}{x+3} + \frac{1}{x-1} \right) dx$   
 $= 3 \ln|x+3| + \ln|x-1| + c$

**c**  $x^2 + 2x = x(x+2)$

Let  $\frac{1-x}{x^2 + 2x} = \frac{A}{x} + \frac{B}{x+2}$   
 $\therefore 1-x = A(x+2) + Bx$

Substituting  $x = -2$ ,  $1 - (-2) = B(-2)$   
 $\therefore 3 = -2B$   
 $\therefore B = -\frac{3}{2}$

Substituting  $x = 0$ ,  $1 = 2A$   
 $\therefore A = \frac{1}{2}$

$\therefore \frac{1-x}{x^2 + 2x} = \frac{1}{2x} - \frac{3}{2(x+2)}$   
 $\therefore \int \frac{1-x}{x^2 + 2x} dx = \int \left( \frac{1}{2x} - \frac{3}{2(x+2)} \right) dx$   
 $= \frac{1}{2} \ln|x| - \frac{3}{2} \ln|x+2| + c$



**d**  $x^2 - x - 2 = (x + 1)(x - 2)$

Let  $\frac{2x - 5}{x^2 - x - 2} = \frac{A}{x + 1} + \frac{B}{x - 2}$

$$\therefore 2x - 5 = A(x - 2) + B(x + 1)$$

Substituting  $x = 2$ ,  $2(2) - 5 = B(2 + 1)$

$$\therefore -1 = 3B$$

$$\therefore B = -\frac{1}{3}$$

Substituting  $x = -1$ ,  $2(-1) - 5 = A(-1 - 2)$

$$\therefore -7 = -3A$$

$$\therefore A = \frac{7}{3}$$

$$\therefore \frac{2x - 5}{x^2 - x - 2} = \frac{7}{3(x + 1)} - \frac{1}{3(x - 2)}$$

$$\begin{aligned} \therefore \int \frac{2x - 5}{x^2 - x - 2} dx &= \int \left( \frac{7}{3(x + 1)} - \frac{1}{3(x - 2)} \right) dx \\ &= \frac{7}{3} \ln |x + 1| - \frac{1}{3} \ln |x - 2| + c \end{aligned}$$

**e**  $2x^2 - 5x - 3 = (2x + 1)(x - 3)$

Let  $\frac{2 - x}{2x^2 - 5x - 3} = \frac{A}{2x + 1} + \frac{B}{x - 3}$

$$\therefore 2 - x = A(x - 3) + B(2x + 1)$$

Substituting  $x = 3$ ,  $2 - 3 = B(2(3) + 1)$

$$\therefore -1 = 7B$$

$$\therefore B = -\frac{1}{7}$$

Substituting  $x = -\frac{1}{2}$ ,  $2 - \left(-\frac{1}{2}\right) = A\left(-\frac{1}{2} - 3\right)$

$$\therefore \frac{5}{2} = A\left(-\frac{7}{2}\right)$$

$$\therefore A = -\frac{5}{7}$$

$$\therefore \frac{2 - x}{2x^2 - 5x - 3} = -\frac{5}{7(2x + 1)} - \frac{1}{7(x - 3)}$$

$$\begin{aligned} \therefore \int \frac{2 - x}{2x^2 - 5x - 3} dx &= \int \left( -\frac{5}{7(2x + 1)} - \frac{1}{7(x - 3)} \right) dx \\ &= -\frac{5}{7} \times \frac{1}{2} \ln |2x + 1| - \frac{1}{7} \ln |x - 3| + c \\ &= -\frac{5}{14} \ln |2x + 1| - \frac{1}{7} \ln |x - 3| + c \end{aligned}$$

$$\mathbf{f} \quad 6x^2 - x - 15 = (3x - 5)(2x + 3)$$

$$\text{Let } \frac{2x}{6x^2 - x - 15} = \frac{A}{3x - 5} + \frac{B}{2x + 3}$$

$$\therefore 2x = A(2x + 3) + B(3x - 5)$$

$$\text{Substituting } x = -\frac{3}{2}, \quad 2\left(-\frac{3}{2}\right) = B\left(3\left(-\frac{3}{2}\right) - 5\right)$$

$$\therefore -3 = -\frac{19}{2}B$$

$$\therefore B = \frac{6}{19}$$

$$\text{Substituting } x = \frac{5}{3}, \quad 2\left(\frac{5}{3}\right) = A\left(2\left(\frac{5}{3}\right) + 3\right)$$

$$\therefore \frac{10}{3} = \frac{19}{3}A$$

$$\therefore A = \frac{10}{19}$$

$$\therefore \frac{2x}{6x^2 - x - 15} = \frac{10}{19(3x - 5)} + \frac{6}{19(2x + 3)}$$

$$\begin{aligned} \therefore \int \frac{2x}{6x^2 - x - 15} dx &= \frac{10}{19} \times \frac{1}{3} \ln |3x - 5| + \frac{6}{19} \times \frac{1}{2} \ln |2x + 3| + c \\ &= \frac{10}{57} \ln |3x - 5| + \frac{3}{19} \ln |2x + 3| + c \end{aligned}$$

$$\mathbf{6} \quad \mathbf{a} \quad \frac{6x^2 + x - 19}{(x + 3)(x - 1)^2} = \frac{A}{x + 3} + \frac{B}{x - 1} + \frac{C}{(x - 1)^2}$$

$$\therefore 6x^2 + x - 19 = A(x - 1)^2 + B(x + 3)(x - 1) + C(x + 3)$$

$$\text{Substituting } x = 1, \quad 6(1)^2 + 1 - 19 = C(1 + 3)$$

$$\therefore 6 + 1 - 19 = 4C$$

$$\therefore -12 = 4C$$

$$\therefore C = -3$$

$$\text{Substituting } x = -3, \quad 6(-3)^2 - 3 - 19 = A(-3 - 1)^2$$

$$\therefore 54 - 3 - 19 = 16A$$

$$\therefore 32 = 16A$$

$$\therefore A = 2$$

$$\text{Substituting } x = 0, \quad 6(0)^2 + 0 - 19 = 2(-1)^2 + B(3)(-1) - 3(3)$$

$$\therefore -19 = 2 - 3B - 9$$

$$\therefore -12 = -3B$$

$$\therefore B = 4$$

$$\therefore \frac{6x^2 + x - 19}{(x + 3)(x - 1)^2} = \frac{2}{x + 3} + \frac{4}{x - 1} - \frac{3}{(x - 1)^2}$$

$$\begin{aligned} \mathbf{b} \quad \int \frac{6x^2 + x - 19}{(x + 3)(x - 1)^2} dx &= \int \left( \frac{2}{x + 3} + \frac{4}{x - 1} - \frac{3}{(x - 1)^2} \right) dx \\ &= \int \frac{2}{x + 3} dx + \int \frac{4}{x - 1} dx - \int 3(x - 1)^{-2} dx \\ &= 2 \ln |x + 3| + 4 \ln |x - 1| - \frac{3(x - 1)^{-1}}{-1} + c \\ &= 2 \ln |x + 3| + 4 \ln |x - 1| + \frac{3}{x - 1} + c \end{aligned}$$

**EXERCISE 21F**

$$1 \quad \frac{d}{dx} ((2x^2 - 5x)^3) = 3(4x - 5)(2x^2 - 5x)^2$$

$$\therefore \int 3(4x - 5)(2x^2 - 5x)^2 dx = (2x^2 - 5x)^3 + c$$

$$\therefore 3 \int (4x - 5)(2x^2 - 5x)^2 dx = (2x^2 - 5x)^3 + c$$

$$\therefore \int (4x - 5)(2x^2 - 5x)^2 dx = \frac{1}{3}(2x^2 - 5x)^3 + c$$

$$2 \quad \begin{aligned} \frac{d}{dx} (\sin(x^2)) &= \cos(x^2) \times 2x \\ &= 2x \cos(x^2) \end{aligned}$$

$$\therefore \int 2x \cos(x^2) dx = \sin(x^2) + c$$

$$\therefore 2 \int x \cos(x^2) dx = \sin(x^2) + c$$

$$\therefore \int x \cos(x^2) dx = \frac{1}{2} \sin(x^2) + c$$

$$3 \quad \frac{d}{dx} (\ln(5 - 3x + x^2)) = \frac{2x - 3}{5 - 3x + x^2}$$

$$\therefore \int \frac{2x - 3}{5 - 3x + x^2} dx = \ln |5 - 3x + x^2| + c$$

$$\therefore 2 \int \frac{2x - 3}{5 - 3x + x^2} dx = 2 \ln |5 - 3x + x^2| + c$$

$$\therefore \int \frac{4x - 6}{5 - 3x + x^2} dx = 2 \ln |5 - 3x + x^2| + c$$

$$4 \quad \text{a} \quad u = x^3 + 1, \quad \frac{du}{dx} = 3x^2$$

$$\begin{aligned} \therefore \int 3x^2(x^3 + 1)^4 dx &= \int u^4 \frac{du}{dx} dx \\ &= \int u^4 du \\ &= \frac{1}{5} u^5 + c \\ &= \frac{1}{5} (x^3 + 1)^5 + c \end{aligned}$$

$$\text{b} \quad u = x^3 + 1, \quad \frac{du}{dx} = 3x^2$$

$$\begin{aligned} \therefore \int x^2 e^{x^3+1} dx &= \frac{1}{3} \int (3x^2) e^{x^3+1} dx \\ &= \frac{1}{3} \int e^u \frac{du}{dx} dx \\ &= \frac{1}{3} \int e^u du \\ &= \frac{1}{3} e^u + c \\ &= \frac{1}{3} e^{x^3+1} + c \end{aligned}$$

$$\text{c } u = \sin x, \quad \frac{du}{dx} = \cos x$$

$$\begin{aligned} \therefore \int \sin^4 x \cos x \, dx &= \int u^4 \frac{du}{dx} \, dx \\ &= \int u^4 \, du \\ &= \frac{u^5}{5} + c \\ &= \frac{1}{5} \sin^5 x + c \end{aligned}$$

$$\text{d } u = x^2 - 3, \quad \frac{du}{dx} = 2x$$

$$\begin{aligned} \therefore \int 2x \cos(x^2 - 3) \, dx &= \int \cos u \frac{du}{dx} \, dx \\ &= \int \cos u \, du \\ &= \sin u + c \\ &= \sin(x^2 - 3) + c \end{aligned}$$

$$\text{e } u = \frac{x-1}{x} = 1 - \frac{1}{x}, \quad \frac{du}{dx} = \frac{1}{x^2}$$

$$\begin{aligned} \therefore \int \frac{e^{\frac{x-1}{x}}}{x^2} \, dx &= \int e^u \frac{du}{dx} \, dx \\ &= \int e^u \, du \\ &= e^u + c \\ &= e^{\frac{x-1}{x}} + c \end{aligned}$$

$$\begin{aligned} \text{5 a } \int 4x^3(2+x^4)^3 \, dx &= \int u^3 \frac{du}{dx} \, dx \quad \{u = 2 + x^4, \quad \frac{du}{dx} = 4x^3\} \\ &= \int u^3 \, du \\ &= \frac{u^4}{4} + c \\ &= \frac{1}{4}(2+x^4)^4 + c \end{aligned}$$

$$\begin{aligned} \text{b } \int \frac{2x}{\sqrt{x^2+3}} \, dx &= \int 2x(x^2+3)^{-\frac{1}{2}} \, dx \\ &= \int u^{-\frac{1}{2}} \frac{du}{dx} \, dx \\ &\quad \{u = x^2 + 3, \quad \frac{du}{dx} = 2x\} \\ &= \int u^{-\frac{1}{2}} \, du \\ &= \frac{u^{\frac{1}{2}}}{\frac{1}{2}} + c \\ &= 2\sqrt{u} + c \\ &= 2\sqrt{x^2+3} + c \end{aligned}$$

$$\begin{aligned} \text{c } \int \frac{6x^2}{(2x^3-1)^4} \, dx &= \int 6x^2(2x^3-1)^{-4} \, dx \\ &= \int u^{-4} \frac{du}{dx} \, dx \\ &\quad \{u = 2x^3 - 1, \quad \frac{du}{dx} = 6x^2\} \\ &= \int u^{-4} \, du \\ &= \frac{u^{-3}}{-3} + c \\ &= -\frac{1}{3u^3} + c \\ &= -\frac{1}{3(2x^3-1)^3} + c \end{aligned}$$



$$\begin{aligned}
 \text{d} \quad & \int \sqrt{x^3 + x} (3x^2 + 1) \, dx \\
 &= \int \sqrt{u} \frac{du}{dx} \, dx \\
 &\quad \{u = x^3 + x, \quad \frac{du}{dx} = 3x^2 + 1\} \\
 &= \int u^{\frac{1}{2}} \, du \\
 &= \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c \\
 &= \frac{2}{3} u^{\frac{3}{2}} + c \\
 &= \frac{2}{3} (x^3 + x)^{\frac{3}{2}}
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & \int \frac{x}{(1 - x^2)^5} \, dx \\
 &= -\frac{1}{2} \int (1 - x^2)^{-5} \times (-2x) \, dx \\
 &= -\frac{1}{2} \int u^{-5} \frac{du}{dx} \, dx \\
 &\quad \{u = 1 - x^2, \quad \frac{du}{dx} = -2x\} \\
 &= -\frac{1}{2} \int u^{-5} \, du \\
 &= -\frac{1}{2} \frac{u^{-4}}{-4} + c \\
 &= \frac{1}{8(1 - x^2)^4} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{h} \quad & \int \frac{x^2}{(3x^3 - 1)^4} \, dx \\
 &= \frac{1}{9} \int 9x^2 (3x^3 - 1)^{-4} \, dx \\
 &= \frac{1}{9} \int u^{-4} \frac{du}{dx} \, dx \\
 &\quad \{u = 3x^3 - 1, \quad \frac{du}{dx} = 9x^2\} \\
 &= \frac{1}{9} \int u^{-4} \, du \\
 &= \frac{1}{9} \frac{u^{-3}}{-3} + c \\
 &= -\frac{1}{27(3x^3 - 1)^3} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & \int (x^3 + 2x + 1)^4 (3x^2 + 2) \, dx \\
 &= \int u^4 \frac{du}{dx} \, dx \\
 &\quad \{u = x^3 + 2x + 1, \quad \frac{du}{dx} = 3x^2 + 2\} \\
 &= \int u^4 \, du \\
 &= \frac{u^5}{5} + c \\
 &= \frac{1}{5} (x^3 + 2x + 1)^5 + c
 \end{aligned}$$

$$\begin{aligned}
 \text{g} \quad & \int \frac{x + 2}{(x^2 + 4x - 3)^2} \, dx \\
 &= \frac{1}{2} \int (x^2 + 4x - 3)^{-2} (2x + 4) \, dx \\
 &= \frac{1}{2} \int u^{-2} \frac{du}{dx} \, dx \\
 &\quad \{u = x^2 + 4x - 3, \quad \frac{du}{dx} = 2x + 4\} \\
 &= \frac{1}{2} \int u^{-2} \, du \\
 &= \frac{1}{2} \frac{u^{-1}}{-1} + c \\
 &= -\frac{1}{2(x^2 + 4x - 3)} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{i} \quad & \int x^4 (x + 1)^4 (2x + 1) \, dx \\
 &= \int (x^2 + x)^4 (2x + 1) \, dx \\
 &= \int u^4 \frac{du}{dx} \, dx \\
 &\quad \{u = x^2 + x, \quad \frac{du}{dx} = 2x + 1\} \\
 &= \int u^4 \, du \\
 &= \frac{1}{5} u^5 + c \\
 &= \frac{1}{5} (x^2 + x)^5 + c
 \end{aligned}$$

$$\begin{aligned}
 \text{6 a} \quad & \int -2e^{1-2x} dx \\
 &= \int e^u \frac{du}{dx} dx \\
 &\quad \{u = 1 - 2x, \quad \frac{du}{dx} = -2\} \\
 &= \int e^u du \\
 &= e^u + c \\
 &= e^{1-2x} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & \int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx \\
 &= 2 \int \frac{e^{\sqrt{x}}}{2\sqrt{x}} dx \\
 &= 2 \int e^u \frac{du}{dx} dx \\
 &\quad \{u = \sqrt{x}, \quad \frac{du}{dx} = \frac{1}{2\sqrt{x}}\} \\
 &= 2 \int e^u du \\
 &= 2e^u + c \\
 &= 2e^{\sqrt{x}} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{7 a} \quad & \int \frac{2x}{x^2 + 1} dx \\
 &= \int \frac{1}{x^2 + 1} (2x) dx \\
 &= \int \frac{1}{u} \frac{du}{dx} dx \\
 &\quad \{u = x^2 + 1, \quad \frac{du}{dx} = 2x\} \\
 &= \int \frac{1}{u} du \\
 &= \ln |u| + c \\
 &= \ln(x^2 + 1) + c \quad \{x^2 + 1 > 0\}
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & \int 2xe^{x^2} dx \\
 &= \int e^u \frac{du}{dx} dx \quad \{u = x^2, \quad \frac{du}{dx} = 2x\} \\
 &= \int e^u du \\
 &= e^u + c \\
 &= e^{x^2} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad & \int (2x - 1)e^{x-x^2} dx \\
 &= - \int (1 - 2x)e^{x-x^2} dx \\
 &= - \int e^u \frac{du}{dx} dx \\
 &\quad \{u = x - x^2, \quad \frac{du}{dx} = 1 - 2x\} \\
 &= - \int e^u du \\
 &= -e^{x-x^2} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & \int \frac{x}{2-x^2} dx \\
 &= -\frac{1}{2} \int \frac{1}{2-x^2} (-2x) dx \\
 &= -\frac{1}{2} \int \frac{1}{u} \frac{du}{dx} dx \\
 &\quad \{u = 2 - x^2, \quad \frac{du}{dx} = -2x\} \\
 &= -\frac{1}{2} \int \frac{1}{u} du \\
 &= -\frac{1}{2} \ln |u| + c \\
 &= -\frac{1}{2} \ln |2 - x^2| + c
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & \int \frac{4x-10}{5x-x^2} dx \\
 &= -2 \int \frac{5-2x}{5x-x^2} dx \\
 &= -2 \int \frac{1}{u} \frac{du}{dx} dx \\
 &\quad \{u = 5x - x^2, \quad \frac{du}{dx} = 5 - 2x\} \\
 &= -2 \int \frac{1}{u} du \\
 &= -2 \ln |u| + c \\
 &= -2 \ln |5x - x^2| + c
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & \int \frac{e^x + e^{-x}}{e^x - e^{-x}} dx \\
 &= \int \frac{1}{u} \frac{du}{dx} dx \\
 &\quad \{u = e^x - e^{-x}, \quad \frac{du}{dx} = e^x + e^{-x}\} \\
 &= \int \frac{1}{u} du \\
 &= \ln |u| + c \\
 &= \ln |e^x - e^{-x}| + c
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad & \int \frac{1-x^2}{x^3-3x} dx \\
 &= -\frac{1}{3} \int \frac{3x^2-3}{x^3-3x} dx \\
 &= -\frac{1}{3} \int \frac{1}{u} \frac{du}{dx} dx \\
 &\quad \{u = x^3 - 3x, \quad \frac{du}{dx} = 3x^2 - 3\} \\
 &= -\frac{1}{3} \int \frac{1}{u} du \\
 &= -\frac{1}{3} \ln |u| + c \\
 &= -\frac{1}{3} \ln |x^3 - 3x| + c
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & \int \frac{x^3}{1+x^2} dx \\
 &= \int \frac{x(1+x^2)-x}{1+x^2} dx \\
 &= \int \left( x - \frac{x}{1+x^2} \right) dx \\
 &= \int \left( x - \frac{1}{2} \left( \frac{2x}{1+x^2} \right) \right) dx \\
 &= \int x dx - \frac{1}{2} \int \frac{2x}{1+x^2} dx \\
 &= \frac{1}{2} x^2 - \frac{1}{2} \int \frac{1}{u} \frac{du}{dx} dx \\
 &\quad \{u = 1 + x^2, \quad \frac{du}{dx} = 2x\} \\
 &= \frac{1}{2} x^2 - \frac{1}{2} \int \frac{1}{u} du \\
 &= \frac{1}{2} x^2 - \frac{1}{2} \ln |u| + c \\
 &= \frac{1}{2} x^2 - \frac{1}{2} \ln(1+x^2) + c \quad \{1+x^2 > 0\}
 \end{aligned}$$

$$\begin{aligned}
 \text{8 a} \quad & \int x^2(3-x^3)^2 dx \\
 &= -\frac{1}{3} \int (-3x^2)(3-x^3)^2 dx \\
 &= -\frac{1}{3} \int u^2 \frac{du}{dx} dx \\
 &\quad \{u = 3 - x^3, \quad \frac{du}{dx} = -3x^2\} \\
 &= -\frac{1}{3} \int u^2 du \\
 &= -\frac{1}{3} \times \frac{u^3}{3} + c \\
 &= -\frac{1}{9}(3-x^3)^3 + c
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & \int x\sqrt{1-x^2} dx \\
 &= -\frac{1}{2} \int (-2x)(1-x^2)^{\frac{1}{2}} dx \\
 &= -\frac{1}{2} \int u^{\frac{1}{2}} \frac{du}{dx} dx \\
 &\quad \{u = 1 - x^2, \quad \frac{du}{dx} = -2x\} \\
 &= -\frac{1}{2} \int u^{\frac{1}{2}} du \\
 &= -\frac{1}{2} \times \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c \\
 &= -\frac{1}{3} u^{\frac{3}{2}} + c \\
 &= -\frac{1}{3}(1-x^2)^{\frac{3}{2}} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & \int x e^{1-x^2} dx \\
 &= -\frac{1}{2} \int (-2x) e^{1-x^2} dx \\
 &= -\frac{1}{2} \int e^u \frac{du}{dx} dx \\
 &\quad \{u = 1 - x^2, \quad \frac{du}{dx} = -2x\} \\
 &= -\frac{1}{2} \int e^u du \\
 &= -\frac{1}{2} e^u + c \\
 &= -\frac{1}{2} e^{1-x^2} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad & \int \frac{(\ln x)^3}{x} dx \\
 &= \int u^3 \frac{du}{dx} dx \\
 &\quad \{u = \ln x, \quad \frac{du}{dx} = \frac{1}{x}\} \\
 &= \int u^3 du \\
 &= \frac{u^4}{4} + c \\
 &= \frac{1}{4}(\ln x)^4 + c
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & \int \frac{4}{x \ln x} dx \\
 &= 4 \int \frac{1}{\ln x} \times \frac{1}{x} dx \\
 &= 4 \int \frac{1}{u} \frac{du}{dx} dx \\
 &\quad \{u = \ln x, \quad \frac{du}{dx} = \frac{1}{x}\} \\
 &= 4 \int \frac{1}{u} du \\
 &= 4 \ln |u| + c \\
 &= 4 \ln |\ln x| + c
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & \int x^2 2^{x^3} dx \\
 &= \frac{1}{3} \int 3x^2 2^{x^3} dx \\
 &= \frac{1}{3} \int 2^u \frac{du}{dx} dx \\
 &\quad \{u = x^3, \quad \frac{du}{dx} = 3x^2\} \\
 &= \frac{1}{3} \int 2^u du \\
 &= \frac{1}{3} \times \frac{2^u}{\ln 2} + c \\
 &= \frac{2^{x^3}}{3 \ln 2} + c
 \end{aligned}$$



$$\begin{aligned}
 \mathbf{g} \quad & \int \frac{e^{2x}}{\sqrt{5-e^{2x}}} dx \\
 &= - \int \frac{1}{\sqrt{u}} \frac{du}{dx} dx \\
 &\quad \{u = 5 - e^{2x}, \quad \frac{du}{dx} = -e^{2x}\} \\
 &= - \int u^{-\frac{1}{2}} du \\
 &= -\frac{u^{\frac{1}{2}}}{\frac{1}{2}} + c \\
 &= -2\sqrt{u} + c \\
 &= -2\sqrt{5-e^{2x}} + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{h} \quad & \int \cos x \, 3^{\sin x} dx \\
 &= \int 3^u \frac{du}{dx} dx \\
 &\quad \{u = \sin x, \quad \frac{du}{dx} = \cos x\} \\
 &= \int 3^u du \\
 &= \frac{3^u}{\ln 3} + c \\
 &= \frac{3^{\sin x}}{\ln 3} + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{9} \quad \mathbf{a} \quad & \int \sin^7 x \cos x dx \\
 &= \int u^7 \frac{du}{dx} dx \\
 &\quad \{u = \sin x, \quad \frac{du}{dx} = \cos x\} \\
 &= \int u^7 du \\
 &= \frac{u^8}{8} + c \\
 &= \frac{1}{8} \sin^8 x + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \int \cos^5 x \sin x dx \\
 &= \int u^5 \left(-\frac{du}{dx}\right) dx \\
 &\quad \{u = \cos x, \quad \frac{du}{dx} = -\sin x\} \\
 &= - \int u^5 du \\
 &= -\frac{1}{6} u^6 + c \\
 &= -\frac{1}{6} \cos^6 x + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & \int \frac{\sin x}{\sqrt{\cos x}} dx \\
 &= \int u^{-\frac{1}{2}} \left(-\frac{du}{dx}\right) dx \\
 &\quad \{u = \cos x, \quad \frac{du}{dx} = -\sin x\} \\
 &= - \int u^{-\frac{1}{2}} du \\
 &= -\frac{u^{\frac{1}{2}}}{\frac{1}{2}} + c \\
 &= -2(\cos x)^{\frac{1}{2}} + c \\
 &= -2\sqrt{\cos x} + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad & \int \tan x dx \\
 &= \int \frac{\sin x}{\cos x} dx \\
 &= \int \frac{1}{u} \left(-\frac{du}{dx}\right) dx \\
 &\quad \{u = \cos x, \quad \frac{du}{dx} = -\sin x\} \\
 &= - \int \frac{1}{u} du \\
 &= -\ln |u| + c \\
 &= -\ln |\cos x| + c
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & \int \sqrt{\sin x} \cos x \, dx \\
 &= \int u^{\frac{1}{2}} \frac{du}{dx} \, dx \\
 &\quad \{u = \sin x, \quad \frac{du}{dx} = \cos x\} \\
 &= \int u^{\frac{1}{2}} \, du \\
 &= \frac{2}{3} u^{\frac{3}{2}} + c \\
 &= \frac{2}{3} (\sin x)^{\frac{3}{2}} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{g} \quad & \int \frac{\sin x}{\cos^3 x} \, dx \\
 &= \int \frac{1}{u^3} \left( -\frac{du}{dx} \right) dx \\
 &\quad \{u = \cos x, \quad \frac{du}{dx} = -\sin x\} \\
 &= - \int u^{-3} \, dx \\
 &= -\frac{u^{-2}}{-2} + c \\
 &= \frac{1}{2} \times \frac{1}{u^2} + c \\
 &= \frac{1}{2 \cos^2 x} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{i} \quad & \int \frac{\sin x + \cos x}{\sin x - \cos x} \, dx \\
 &= \int \frac{1}{u} \frac{du}{dx} \, dx \\
 &\quad \{u = \sin x - \cos x, \quad \frac{du}{dx} = \cos x + \sin x\} \\
 &= \int \frac{1}{u} \, du \\
 &= \ln |u| + c \\
 &= \ln |\sin x - \cos x| + c
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & \int \frac{\cos x}{(2 + \sin x)^2} \, dx \\
 &= \int u^{-2} \frac{du}{dx} \, dx \\
 &\quad \{u = 2 + \sin x, \quad \frac{du}{dx} = \cos x\} \\
 &= \int u^{-2} \, du \\
 &= -u^{-1} + c \\
 &= -(2 + \sin x)^{-1} + c \\
 &= -\frac{1}{2 + \sin x} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{h} \quad & \int \frac{\sin x}{1 - \cos x} \, dx \\
 &= \int \frac{1}{u} \frac{du}{dx} \, dx \\
 &\quad \{u = 1 - \cos x, \quad \frac{du}{dx} = \sin x\} \\
 &= \int \frac{1}{u} \, du \\
 &= \ln |u| + c \\
 &= \ln |1 - \cos x| + c
 \end{aligned}$$

$$\begin{aligned}
 \text{j} \quad & \int \frac{\cos 2x}{\sin 2x - 3} \, dx \\
 &= \int \frac{1}{u} \left( \frac{1}{2} \frac{du}{dx} \right) dx \\
 &\quad \{u = \sin 2x - 3, \quad \frac{du}{dx} = 2 \cos 2x\} \\
 &= \frac{1}{2} \int \frac{1}{u} \, du \\
 &= \frac{1}{2} \ln |u| + c \\
 &= \frac{1}{2} \ln |\sin 2x - 3| + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{k} \quad & \int x \sin(x^2) \, dx \\
 &= \int \sin u \left( \frac{1}{2} \frac{du}{dx} \right) dx \\
 &\quad \{u = x^2, \quad \frac{du}{dx} = 2x\} \\
 &= \frac{1}{2} \int \sin u \, du \\
 &= \frac{1}{2} (-\cos u) + c \\
 &= -\frac{1}{2} \cos(x^2) + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{m} \quad & \int \frac{\arctan x}{1+x^2} \, dx \\
 &= \int u \frac{du}{dx} \, dx \\
 &\quad \{u = \arctan x, \quad \frac{du}{dx} = \frac{1}{1+x^2}\} \\
 &= \int u \, du \\
 &= \frac{1}{2} u^2 + c \\
 &= \frac{1}{2} \arctan^2 x + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{l} \quad & \int \frac{\sin^3 x}{\cos^5 x} \, dx \\
 &= \int \tan^3 x \sec^2 x \, dx \\
 &= \int u^3 \frac{du}{dx} \, dx \\
 &\quad \{u = \tan x, \quad \frac{du}{dx} = \sec^2 x\} \\
 &= \int u^3 \, du \\
 &= \frac{1}{4} u^4 + c \\
 &= \frac{1}{4} \tan^4 x + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{n} \quad & \int \tan^3 x \, dx \\
 &= \int \frac{\sin^3 x}{\cos^3 x} \, dx \\
 &= \int \frac{\sin x (1 - \cos^2 x)}{\cos^3 x} \, dx \\
 &= \int \left( \frac{1}{\cos x} - \frac{1}{\cos^3 x} \right) (-\sin x) \, dx \\
 &= \int \left( \frac{1}{u} - \frac{1}{u^3} \right) \frac{du}{dx} \, dx \\
 &\quad \{u = \cos x, \quad \frac{du}{dx} = -\sin x\} \\
 &= \int (u^{-1} - u^{-3}) \, du \\
 &= \ln |u| - \frac{u^{-2}}{-2} + c \\
 &= \ln |u| + \frac{1}{2} \times \frac{1}{u^2} + c \\
 &= \ln |\cos x| + \frac{1}{2 \cos^2 x} + c
 \end{aligned}$$

$$\begin{aligned}
& \bullet \quad \int \cos^3 x \, dx \\
&= \int \cos^2 x \cos x \, dx \\
&= \int (1 - \sin^2 x) \cos x \, dx \\
&= \int (1 - u^2) \frac{du}{dx} \, dx \quad \left\{ u = \sin x, \quad \frac{du}{dx} = \cos x \right\} \\
&= \int (1 - u^2) \, du \\
&= u - \frac{u^3}{3} + c \\
&= \sin x - \frac{\sin^3 x}{3} + c \\
&= \sin x - \frac{1}{3} \sin^3 x + c
\end{aligned}$$

$$\begin{aligned}
& \text{p} \quad \int \sin^5 x \cos^5 x \, dx \\
&= \int \sin^5 x (1 - \sin^2 x)(1 - \sin^2 x) \cos x \, dx \\
&= \int u^5 (1 - u^2)(1 - u^2) \frac{du}{dx} \, dx \quad \left\{ u = \sin x, \quad \frac{du}{dx} = \cos x \right\} \\
&= \int u^5 (1 - 2u^2 + u^4) \, du \\
&= \int (u^5 - 2u^7 + u^9) \, du \\
&= \frac{1}{6} u^6 - \frac{1}{4} u^8 + \frac{1}{10} u^{10} + c \\
&= \frac{1}{6} \sin^6 x - \frac{1}{4} \sin^8 x + \frac{1}{10} \sin^{10} x + c
\end{aligned}$$

$$\begin{aligned}
10 \quad \text{a} \quad \int \sin^5 x \, dx &= \int \sin^4 x \sin x \, dx \\
&= \int (1 - \cos^2 x)^2 \sin x \, dx \\
&= \int (1 - 2\cos^2 x + \cos^4 x) \sin x \, dx \\
&= \int (1 - 2u^2 + u^4) \left( -\frac{du}{dx} \right) \, dx \quad \left\{ u = \cos x, \quad \frac{du}{dx} = -\sin x \right\} \\
&= - \int (1 - 2u^2 + u^4) \, du \\
&= -u + \frac{2}{3} u^3 - \frac{1}{5} u^5 + c \\
&= -\cos x + \frac{2}{3} \cos^3 x - \frac{1}{5} \cos^5 x + c
\end{aligned}$$

$$\begin{aligned}
\text{b} \quad \int \sin^4 x \cos^3 x \, dx &= \int \sin^4 x \cos^2 x \cos x \, dx \\
&= \int \sin^4 x (1 - \sin^2 x) \cos x \, dx \\
&= \int (\sin^4 x - \sin^6 x) \cos x \, dx \\
&= \int (u^4 - u^6) \frac{du}{dx} \, dx \quad \left\{ u = \sin x, \quad \frac{du}{dx} = \cos x \right\} \\
&= \int (u^4 - u^6) \, du \\
&= \frac{1}{5} u^5 - \frac{1}{7} u^7 + c \\
&= \frac{1}{5} \sin^5 x - \frac{1}{7} \sin^7 x + c
\end{aligned}$$



$$\begin{aligned}
 \text{c} \quad \int \sin^3 2x \cos 2x \, dx &= \int u^3 \left( \frac{1}{2} \frac{du}{dx} \right) dx \quad \{u = \sin 2x, \quad \frac{du}{dx} = 2 \cos 2x\} \\
 &= \frac{1}{2} \int u^3 \, du \\
 &= \frac{1}{2} \times \frac{u^4}{4} + c \\
 &= \frac{1}{8} \sin^4 2x + c
 \end{aligned}$$

$$\begin{aligned}
 11 \quad \text{a} \quad f'(x) &= \sin x e^{\cos x} \\
 \therefore f(x) &= \int \sin x e^{\cos x} \, dx \\
 &= \int e^u \left( -\frac{du}{dx} \right) dx \\
 &\quad \{u = \cos x, \quad \frac{du}{dx} = -\sin x\} \\
 &= - \int e^u \, du \\
 &= -e^u + c \\
 &= -e^{\cos x} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad f'(x) &= \frac{e^{\tan x}}{\cos^2 x} \\
 \therefore f(x) &= \int \frac{e^{\tan x}}{\cos^2 x} \, dx \\
 &= \int e^{\tan x} \sec^2 x \, dx \\
 &= \int e^u \frac{du}{dx} \, dx \\
 &\quad \{u = \tan x, \quad \frac{du}{dx} = \sec^2 x\} \\
 &= \int e^u \, du \\
 &= e^u + c \\
 &= e^{\tan x} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad f'(x) &= \frac{e^{\cot x}}{\sin^2 x} \\
 \therefore f(x) &= \int \frac{e^{\cot x}}{\sin^2 x} \, dx \\
 &= \int e^{\cot x} \operatorname{cosec}^2 x \, dx \\
 &= \int e^u \left( -\frac{du}{dx} \right) dx \quad \{u = \cot x, \quad \frac{du}{dx} = -\operatorname{cosec}^2 x\} \\
 &= - \int e^u \, du \\
 &= -e^u + c \\
 &= -e^{\cot x} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{12 a} \quad & \int \cot 3x \, dx \\
 &= \int \frac{\cos 3x}{\sin 3x} \, dx \\
 &= \int \frac{1}{u} \left( \frac{1}{3} \frac{du}{dx} \right) dx \\
 &\quad \{u = \sin 3x, \quad \frac{du}{dx} = 3 \cos 3x\} \\
 &= \frac{1}{3} \int \frac{1}{u} \, du \\
 &= \frac{1}{3} \ln |u| + c \\
 &= \frac{1}{3} \ln |\sin 3x| + c
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & \int \sec^3 x \sin x \, dx \\
 &= \int \frac{\sin x}{\cos^3 x} \, dx \\
 &= \int \frac{1}{u^3} \left( -\frac{du}{dx} \right) dx \\
 &\quad \{u = \cos x, \quad \frac{du}{dx} = -\sin x\} \\
 &= - \int u^{-3} \, du \\
 &= -\frac{u^{-2}}{-2} + c \\
 &= \frac{1}{2} \times \frac{1}{u^2} + c \\
 &= \frac{1}{2 \cos^2 x} + c \\
 &= \frac{1}{2} \sec^2 x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & \int \operatorname{cosec}^4 x \cos x \, dx \\
 &= \int \frac{\cos x}{\sin^4 x} \, dx \\
 &= \int \frac{1}{u^4} \frac{du}{dx} \, dx \\
 &\quad \{u = \sin x, \quad \frac{du}{dx} = \cos x\} \\
 &= \int u^{-4} \, du \\
 &= \frac{u^{-3}}{-3} + c \\
 &= -\frac{1}{3} \times \frac{1}{u^3} + c \\
 &= -\frac{1}{3 \sin^3 x} \\
 &= -\frac{1}{3} \operatorname{cosec}^3 x
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad & \int \sec^2 x \sqrt{\tan x} \, dx \\
 &= \int \sqrt{u} \frac{du}{dx} \, dx \\
 &\quad \{u = \tan x, \quad \frac{du}{dx} = \sec^2 x\} \\
 &= \int u^{\frac{1}{2}} \, du \\
 &= \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c \\
 &= \frac{2}{3} u^{\frac{3}{2}} + c \\
 &= \frac{2}{3} (\tan x)^{\frac{3}{2}} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & \int x \operatorname{cosec}^2(x^2) dx \\
 &= \int \operatorname{cosec}^2 u \left( \frac{1}{2} \frac{du}{dx} \right) dx \\
 &\quad \{u = x^2, \quad \frac{du}{dx} = 2x\} \\
 &= \frac{1}{2} \int \operatorname{cosec}^2 u du \\
 &= -\frac{1}{2} \cot u + c \\
 &= -\frac{1}{2} \cot(x^2) + c
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & \int \frac{\sec x \tan x}{\sqrt[3]{\sec x}} dx \\
 &= \int \frac{1}{\sqrt[3]{u}} \frac{du}{dx} dx \\
 &\quad \{u = \sec x, \quad \frac{du}{dx} = \sec x \tan x\} \\
 &= \int u^{-\frac{1}{3}} du \\
 &= \frac{u^{\frac{2}{3}}}{\frac{2}{3}} + c \\
 &= \frac{3}{2} u^{\frac{2}{3}} + c \\
 &= \frac{3}{2} (\sec x)^{\frac{2}{3}} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{g} \quad & \int \frac{\operatorname{cosec}^2 x}{\sqrt{\cot x}} dx = \int \frac{1}{\sqrt{u}} \left( -\frac{du}{dx} \right) dx \quad \{u = \cot x, \quad \frac{du}{dx} = -\operatorname{cosec}^2 x\} \\
 &= - \int u^{-\frac{1}{2}} du \\
 &= -\frac{u^{\frac{1}{2}}}{\frac{1}{2}} + c \\
 &= -2\sqrt{u} + c \\
 &= -2\sqrt{\cot x} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{h} \quad & \int \operatorname{cosec}^3(2x) \cot 2x dx = \int \operatorname{cosec}^2(2x) \operatorname{cosec} 2x \cot 2x dx \\
 &= \int u^2 \left( -\frac{1}{2} \frac{du}{dx} \right) dx \quad \{u = \operatorname{cosec} 2x, \quad \frac{du}{dx} = -2 \operatorname{cosec} 2x \cot 2x\} \\
 &= -\frac{1}{2} \int u^2 du \\
 &= -\frac{1}{2} \times \frac{1}{3} u^3 + c \\
 &= -\frac{1}{6} \operatorname{cosec}^3 2x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{i} \quad & \int \sec^5 x \tan x dx = \int \sec^4 x \sec x \tan x dx \\
 &= \int u^4 \frac{du}{dx} dx \quad \{u = \sec x, \quad \frac{du}{dx} = \sec x \tan x\} \\
 &= \int u^4 du \\
 &= \frac{u^5}{5} + c \\
 &= \frac{1}{5} \sec^5 x + c
 \end{aligned}$$

**13 a** Let  $u = \ln(x^2 + 7)$   $\therefore \frac{du}{dx} = \frac{2x}{x^2 + 7}$

$$\begin{aligned}\therefore \int \frac{x \ln(x^2 + 7)}{x^2 + 7} dx &= \frac{1}{2} \int \frac{2x}{x^2 + 7} \ln(x^2 + 7) dx \\ &= \frac{1}{2} \int u \frac{du}{dx} dx \\ &= \frac{1}{2} \int u du \\ &= \frac{1}{2} \times \frac{1}{2} u^2 + c \\ &= \frac{1}{4} [\ln(x^2 + 7)]^2 + c\end{aligned}$$

**b** Let  $x = \sin^2 \theta$   $\therefore \frac{dx}{d\theta} = 2 \sin \theta \cos \theta$

$$\begin{aligned}\therefore \int \frac{4}{\sqrt{x}\sqrt{1-x}} dx &= \int \frac{4}{\sqrt{\sin^2 \theta} \sqrt{1 - \sin^2 \theta}} 2 \sin \theta \cos \theta d\theta \\ &= \int \frac{8 \sin \theta \cos \theta}{\sin \theta \cos \theta} d\theta \\ &= \int 8 d\theta \\ &= 8\theta + c\end{aligned}$$

Now  $x = \sin^2 \theta$

$\therefore \sin \theta = \sqrt{x} \quad \{x > 0\}$

$\therefore \theta = \arcsin \sqrt{x}$

$\therefore \int \frac{4}{\sqrt{x}\sqrt{1-x}} dx = 8 \arcsin \sqrt{x} + c$

**14** Let  $u = x - 16$   $\therefore \frac{du}{dx} = 1$

$$\begin{aligned}\therefore \int x^2 \sqrt{x - 16} dx &= \int (u + 16)^2 \sqrt{u} \frac{du}{dx} dx \\ &= \int (u^2 + 32u + 256) u^{\frac{1}{2}} du \\ &= \int (u^{\frac{5}{2}} + 32u^{\frac{3}{2}} + 256u^{\frac{1}{2}}) du \\ &= \frac{u^{\frac{7}{2}}}{\frac{7}{2}} + \frac{32u^{\frac{5}{2}}}{\frac{5}{2}} + \frac{256u^{\frac{3}{2}}}{\frac{3}{2}} + c \\ &= \frac{2}{7}(x - 16)^{\frac{7}{2}} + \frac{64}{5}(x - 16)^{\frac{5}{2}} + \frac{512}{3}(x - 16)^{\frac{3}{2}} + c\end{aligned}$$



**15 a** Let  $u = x - 3$   $\therefore \frac{du}{dx} = 1$

$$\begin{aligned}\therefore \int x\sqrt{x-3} \, dx &= \int (u+3)\sqrt{u} \frac{du}{dx} \, dx \\ &= \int (u+3)u^{\frac{1}{2}} \, du \\ &= \int (u^{\frac{3}{2}} + 3u^{\frac{1}{2}}) \, du \\ &= \frac{u^{\frac{5}{2}}}{\frac{5}{2}} + \frac{3u^{\frac{3}{2}}}{\frac{3}{2}} + c \\ &= \frac{2}{5}(x-3)^{\frac{5}{2}} + 2(x-3)^{\frac{3}{2}} + c\end{aligned}$$

**b** Let  $u = 1 - x$   $\therefore \frac{du}{dx} = -1$

$$\begin{aligned}\therefore \int x\sqrt{1-x} \, dx &= \int (1-u)\sqrt{u} \left(-\frac{du}{dx}\right) \, dx \\ &= -\int (1-u)u^{\frac{1}{2}} \, du \\ &= -\int u^{\frac{1}{2}} - u^{\frac{3}{2}} \, du \\ &= -\frac{u^{\frac{3}{2}}}{\frac{3}{2}} + \frac{u^{\frac{5}{2}}}{\frac{5}{2}} + c \\ &= \frac{2}{5}(1-x)^{\frac{5}{2}} - \frac{2}{3}(1-x)^{\frac{3}{2}} + c\end{aligned}$$

**c** Let  $u = x + 1$   $\therefore \frac{du}{dx} = 1$

$$\begin{aligned}\therefore \int x^2\sqrt{x+1} \, dx &= \int (u-1)^2\sqrt{u} \frac{du}{dx} \, dx \\ &= \int (u^2 - 2u + 1)u^{\frac{1}{2}} \, du \\ &= \int (u^{\frac{5}{2}} - 2u^{\frac{3}{2}} + u^{\frac{1}{2}}) \, du \\ &= \frac{u^{\frac{7}{2}}}{\frac{7}{2}} - \frac{2u^{\frac{5}{2}}}{\frac{5}{2}} + \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c \\ &= \frac{2}{7}(x+1)^{\frac{7}{2}} - \frac{4}{5}(x+1)^{\frac{5}{2}} + \frac{2}{3}(x+1)^{\frac{3}{2}} + c\end{aligned}$$

**d** Let  $u = x - 3 \quad \therefore \frac{du}{dx} = 1$

$$\begin{aligned}
 \therefore \int \frac{x}{\sqrt{x-3}} dx &= \int \frac{u+3}{\sqrt{u}} \frac{du}{dx} dx \\
 &= \int (u+3)u^{-\frac{1}{2}} du \\
 &= \int (u^{\frac{1}{2}} + 3u^{-\frac{1}{2}}) du \\
 &= \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + \frac{3u^{\frac{1}{2}}}{\frac{1}{2}} + c \\
 &= \frac{2}{3}(x-3)^{\frac{3}{2}} + 6\sqrt{x-3} + c
 \end{aligned}$$

**e** Let  $u = 3 - x^2 \quad \therefore \frac{du}{dx} = -2x$

$$\begin{aligned}
 \therefore \int x^3 \sqrt{3-x^2} dx &= -\frac{1}{2} \int x^2(-2x)\sqrt{3-x^2} dx \\
 &= -\frac{1}{2} \int (3-u) \frac{du}{dx} \sqrt{u} dx \\
 &= -\frac{1}{2} \int (3-u)u^{\frac{1}{2}} du \\
 &= -\frac{1}{2} \int (3u^{\frac{1}{2}} - u^{\frac{3}{2}}) du \\
 &= -\frac{1}{2} \left( \frac{3u^{\frac{3}{2}}}{\frac{3}{2}} - \frac{u^{\frac{5}{2}}}{\frac{5}{2}} \right) + c \\
 &= -\frac{1}{2} \left( 2u^{\frac{3}{2}} - \frac{2}{5}u^{\frac{5}{2}} \right) + c \\
 &= -u^{\frac{3}{2}} + \frac{1}{5}u^{\frac{5}{2}} + c \\
 &= -(3-x^2)^{\frac{3}{2}} + \frac{1}{5}(3-x^2)^{\frac{5}{2}} + c
 \end{aligned}$$

**f** Let  $u = t^2 + 2 \quad \therefore \frac{du}{dt} = 2t$

$$\begin{aligned}
 \therefore \int t^3 \sqrt{t^2 + 2} \, dt &= \frac{1}{2} \int t^2 (2t) \sqrt{t^2 + 2} \, dt \\
 &= \frac{1}{2} \int (u - 2) \frac{du}{dt} \sqrt{u} \, dt \\
 &= \frac{1}{2} \int (u - 2) u^{\frac{1}{2}} \, du \\
 &= \frac{1}{2} \int (u^{\frac{3}{2}} - 2u^{\frac{1}{2}}) \, du \\
 &= \frac{1}{2} \left( \frac{u^{\frac{5}{2}}}{\frac{5}{2}} - \frac{2u^{\frac{3}{2}}}{\frac{3}{2}} \right) + c \\
 &= \frac{1}{2} \left( \frac{2}{5} u^{\frac{5}{2}} - \frac{4}{3} u^{\frac{3}{2}} \right) + c \\
 &= \frac{1}{5} u^{\frac{5}{2}} - \frac{2}{3} u^{\frac{3}{2}} + c \\
 &= \frac{1}{5} (t^2 + 2)^{\frac{5}{2}} - \frac{2}{3} (t^2 + 2)^{\frac{3}{2}} + c
 \end{aligned}$$

**16** Let  $u = \sqrt{x-1}$

$$\therefore u^2 = x - 1$$

$$\therefore x = 1 + u^2$$

and  $\frac{du}{dx} = \frac{1}{2}(x-1)^{-\frac{1}{2}}$

$$= \frac{1}{2\sqrt{x-1}}$$

$$\begin{aligned}
 \int \frac{\sqrt{x-1}}{x} \, dx &= 2 \int \frac{x-1}{x} \frac{1}{2\sqrt{x-1}} \, dx \\
 &= 2 \int u^2 \times \frac{1}{1+u^2} \times \frac{du}{dx} \, dx \\
 &= 2 \int \frac{u^2}{1+u^2} \, du \\
 &= 2 \int \left( 1 - \frac{1}{1+u^2} \right) \, du \\
 &= 2(u - \arctan u) + c \\
 &= 2\sqrt{x-1} - 2\arctan \sqrt{x-1} + c
 \end{aligned}$$

**17 a** Let  $x = 3 \tan \theta$   $\therefore \frac{dx}{d\theta} = 3 \sec^2 \theta$

$$\begin{aligned} \therefore \int \frac{1}{36 + 4x^2} dx &= \int \frac{1}{36 + 4(3 \tan \theta)^2} dx \\ &= \int \frac{1}{36 + 36 \tan^2 \theta} 3 \sec^2 \theta d\theta \\ &= \int \frac{\cancel{3 \sec^2 \theta}}{\cancel{36(1 + \tan^2 \theta)}} d\theta \\ &= \int \frac{1}{12} d\theta \\ &= \frac{1}{12} \theta + c \end{aligned}$$

Since  $\tan \theta = \frac{x}{3}$ ,  $\theta = \arctan \frac{x}{3}$

$$\therefore \int \frac{1}{36 + 4x^2} dx = \frac{1}{12} \arctan \frac{x}{3} + c$$

**b** Let  $x = \frac{1}{2} \sec \theta$   
 $\therefore x^2 = \frac{1}{4} \sec^2 \theta$   
 $\therefore x^2 = \frac{1}{4}(1 + \tan^2 \theta)$   
 $\therefore 4x^2 - 1 = \tan^2 \theta$

$$\therefore \sqrt{4x^2 - 1} = \tan \theta$$

Also,  $\frac{dx}{d\theta} = \frac{1}{2} \sec \theta \tan \theta$ ,  $2x = \sec \theta$   
 $\therefore \cos \theta = \frac{1}{2x}$   
 $\therefore \theta = \arccos \frac{1}{2x}$

$$\begin{aligned} \text{Now, } \int \frac{\sqrt{4x^2 - 1}}{5x} dx &= \frac{1}{5} \int \frac{\tan \theta}{\frac{1}{2} \sec \theta} \times \frac{dx}{d\theta} d\theta \\ &= \frac{1}{5} \int \frac{\tan \theta}{\frac{1}{2} \sec \theta} \times \frac{1}{2} \sec \theta \tan \theta d\theta \\ &= \frac{1}{5} \int \tan^2 \theta d\theta \\ &= \frac{1}{5} \int (\sec^2 \theta - 1) d\theta \\ &= \frac{1}{5} (\tan \theta - \theta) + c \\ &= \frac{1}{5} \sqrt{4x^2 - 1} - \frac{1}{5} \arccos \frac{1}{2x} + c \end{aligned}$$

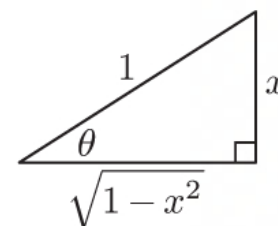


**18 a** Let  $x = 3 \tan \theta$   $\therefore \frac{dx}{d\theta} = 3 \sec^2 \theta$  and  $\theta = \arctan \frac{x}{3}$

$$\begin{aligned} \therefore \int \frac{x^2}{9+x^2} dx &= \int \frac{9 \tan^2 \theta}{9+9 \tan^2 \theta} 3 \sec^2 \theta d\theta \\ &= 3 \int \frac{\tan^2 \theta}{1+\tan^2 \theta} \cancel{\sec^2 \theta} d\theta \\ &= 3 \int \tan^2 \theta d\theta \\ &= 3 \int (\sec^2 \theta - 1) d\theta \\ &= 3 \tan \theta - 3\theta + c \\ &= x - 3 \arctan \frac{x}{3} + c \end{aligned}$$

**b** Let  $x = \sin \theta$   $\therefore \frac{dx}{d\theta} = \cos \theta$  and  $\theta = \arcsin x$

$$\begin{aligned} \therefore \int \frac{x^2}{\sqrt{1-x^2}} dx &= \int \frac{\sin^2 \theta}{\sqrt{1-\sin^2 \theta}} \cos \theta d\theta \\ &= \int \frac{\sin^2 \theta}{\cancel{\cos \theta}} \cancel{\cos \theta} d\theta \\ &= \int \sin^2 \theta d\theta \\ &= \int \left( \frac{1}{2} - \frac{1}{2} \cos 2\theta \right) d\theta \\ &= \frac{1}{2} \theta - \frac{1}{4} \sin 2\theta + c \\ &= \frac{1}{2} \arcsin x - \frac{1}{2} \sin \theta \cos \theta + c \\ &= \frac{1}{2} \arcsin x - \frac{1}{2} x \sqrt{1-x^2} + c \quad \{ \cos \theta = \sqrt{1-\sin^2 \theta} \} \end{aligned}$$



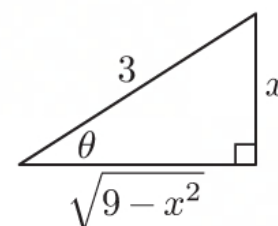
**c** Let  $x = 3 \sin \theta$   $\therefore \frac{dx}{d\theta} = 3 \cos \theta$  and  $\theta = \arcsin \frac{x}{3}$

$$\begin{aligned} \therefore \int \sqrt{9-x^2} dx &= \int \sqrt{9-9 \sin^2 \theta} \times 3 \cos \theta d\theta \\ &= 9 \int \cos^2 \theta d\theta \\ &= 9 \int \left( \frac{1}{2} + \frac{1}{2} \cos 2\theta \right) d\theta \\ &= 9 \left( \frac{1}{2} \theta + \frac{1}{4} \sin 2\theta \right) + c \\ &= \frac{9}{2} \arcsin \frac{x}{3} + \frac{9}{4} \sin 2\theta + c \end{aligned}$$

Now  $\sin 2\theta = 2 \sin \theta \cos \theta$  and  $\sin \theta = \frac{x}{3}$

$$\therefore \sin 2\theta = 2 \left( \frac{x}{3} \right) \frac{\sqrt{9-x^2}}{3} \quad \{ \cos \theta = \sqrt{1-\sin^2 \theta} \}$$

$$\begin{aligned} \therefore \int \sqrt{9-x^2} dx &= \frac{9}{2} \arcsin \frac{x}{3} + \left( \frac{9}{4} \right) \left( \frac{2x\sqrt{9-x^2}}{9} \right) + c \\ &= \frac{9}{2} \arcsin \frac{x}{3} + \frac{1}{2} x \sqrt{9-x^2} + c \end{aligned}$$



**d** Let  $u = \ln x \quad \therefore \frac{du}{dx} = \frac{1}{x}$

$$\begin{aligned}
 \therefore \int \frac{4 \ln x}{x(1 + [\ln x]^2)} dx &= \int \frac{4u}{1 + u^2} \frac{du}{dx} dx \\
 &= \int \frac{4u}{1 + u^2} du \\
 &= \int \frac{2}{v} \frac{dv}{du} du \quad \{v = 1 + u^2, \quad \frac{dv}{du} = 2u\} \\
 &= 2 \ln |v| + c \\
 &= 2 \ln |1 + u^2| + c \\
 &= 2 \ln(1 + u^2) + c \quad \{1 + u^2 > 0\} \\
 &= 2 \ln(1 + [\ln x]^2) + c
 \end{aligned}$$

**e** Let  $x = \sin \theta \quad \therefore \frac{dx}{d\theta} = \cos \theta$

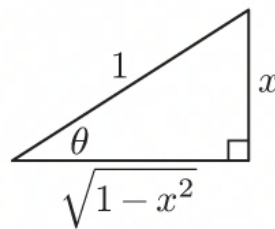
$$\begin{aligned}
 \therefore \int x^2 \sqrt{1 - x^2} dx &= \int \sin^2 \theta \sqrt{1 - \sin^2 \theta} \cos \theta d\theta \\
 &= \frac{1}{4} \int 4 \sin^2 \theta \cos^2 \theta d\theta \\
 &= \frac{1}{4} \int \sin^2 2\theta d\theta \\
 &= \frac{1}{4} \int \left( \frac{1}{2} - \frac{1}{2} \cos 4\theta \right) d\theta \\
 &= \frac{1}{4} \left( \frac{1}{2} \theta - \frac{1}{8} \sin 4\theta \right) + c \\
 &= \frac{1}{8} \theta - \frac{1}{32} \sin 4\theta + c
 \end{aligned}$$

Now  $\sin \theta = x$ , so  $\cos \theta = \sqrt{1 - x^2}$

$$\therefore \sin 2\theta = 2x\sqrt{1 - x^2}$$

and  $\cos 2\theta = 1 - 2\sin^2 \theta = 1 - 2x^2$

$$\begin{aligned}
 \therefore \sin 4\theta &= 2 \left( 2x\sqrt{1 - x^2} \right) (1 - 2x^2) \\
 &= 4x\sqrt{1 - x^2} (1 - 2x^2)
 \end{aligned}$$



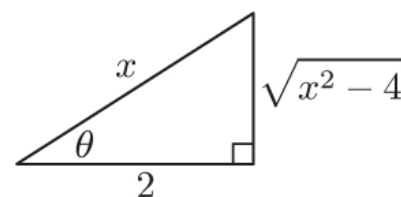
$$\begin{aligned}
 \therefore \int x^2 \sqrt{1 - x^2} dx &= \frac{1}{8} \arcsin x - \frac{1}{32} (4x\sqrt{1 - x^2} (1 - 2x^2)) + c \\
 &= \frac{1}{8} \arcsin x - \frac{1}{8} x\sqrt{1 - x^2} (1 - 2x^2) + c \\
 &= \frac{1}{8} \arcsin x - \frac{1}{8} x\sqrt{1 - x^2} + \frac{1}{4} x^3 \sqrt{1 - x^2} + c
 \end{aligned}$$

**f** Let  $x = 2 \sec \theta$   $\therefore \frac{dx}{d\theta} = 2 \sec \theta \tan \theta$

$$\begin{aligned}
 \therefore \int \frac{\sqrt{x^2 - 4}}{x} dx &= \int \frac{\sqrt{4 \sec^2 \theta - 4}}{2 \sec \theta} 2 \sec \theta \tan \theta d\theta \\
 &= \frac{2}{2} \int \frac{\sqrt{\sec^2 \theta - 1}}{\sec \theta} \times 2 \sec \theta \tan \theta d\theta \\
 &= 2 \int \sqrt{\sec^2 \theta - 1} \tan \theta d\theta \\
 &= 2 \int \tan \theta \tan \theta d\theta \quad \{\sec^2 \theta - 1 = \tan^2 \theta\} \\
 &= 2 \int \tan^2 \theta d\theta \\
 &= 2 \int (\sec^2 \theta - 1) d\theta \\
 &= 2 \tan \theta - 2\theta + c
 \end{aligned}$$

Now,  $\cos \theta = \frac{2}{x}$ , so  $\tan \theta = \frac{\sqrt{x^2 - 4}}{2}$

$$\begin{aligned}
 \therefore \int \frac{\sqrt{x^2 - 4}}{x} dx &= 2 \frac{\sqrt{x^2 - 4}}{2} - 2 \arccos \frac{2}{x} + c \\
 &= \sqrt{x^2 - 4} - 2 \arccos \frac{2}{x} + c
 \end{aligned}$$



**g**  $\int \frac{1}{\sqrt{9 - 4x^2}} dx = \frac{1}{2} \int \frac{1}{\sqrt{\frac{9}{4} - x^2}} dx$

Let  $x = \frac{3}{2} \sin \theta$   $\therefore \frac{dx}{d\theta} = \frac{3}{2} \cos \theta$

$$\begin{aligned}
 \therefore \int \frac{1}{\sqrt{9 - 4x^2}} dx &= \frac{1}{2} \int \frac{1}{\sqrt{\frac{9}{4} - \frac{9}{4} \sin^2 \theta}} \frac{3}{2} \cos \theta d\theta \\
 &= \frac{1}{2} \int \frac{1}{\frac{3}{2} \sqrt{1 - \sin^2 \theta}} \times \frac{3}{2} \cos \theta d\theta \\
 &= \frac{1}{2} \int \frac{\cos \theta}{\cos \theta} d\theta \\
 &= \frac{1}{2} \int 1 d\theta \\
 &= \frac{1}{2} \theta + c \\
 &= \frac{1}{2} \arcsin \frac{2x}{3} + c \quad \{\text{since } \sin \theta = \frac{2x}{3}\}
 \end{aligned}$$

**h** Let  $u = \ln x \quad \therefore \quad \frac{du}{dx} = \frac{1}{x}$

$$\begin{aligned}
 \therefore \int \frac{1}{x(9 + 4[\ln x]^2)} dx &= \int \frac{1}{9 + 4u^2} \frac{du}{dx} dx \\
 &= \int \frac{1}{9 + 4u^2} du \\
 &= \frac{1}{4} \int \frac{1}{u^2 + \frac{9}{4}} du \\
 &= \frac{1}{4} \left( \frac{1}{\frac{3}{2}} \right) \arctan \left( \frac{u}{\frac{3}{2}} \right) + c \\
 &= \frac{1}{6} \arctan \left( \frac{2u}{3} \right) + c \\
 &= \frac{1}{6} \arctan \left( \frac{2 \ln x}{3} \right) + c
 \end{aligned}$$

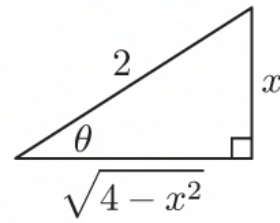
**i** Let  $x = 2 \sin \theta \quad \therefore \quad \frac{dx}{d\theta} = 2 \cos \theta$

$$\begin{aligned}
 \therefore \int \frac{1 - 2x}{\sqrt{4 - x^2}} dx &= \int \frac{1 - 4 \sin \theta}{\sqrt{4 - 4 \sin^2 \theta}} (2 \cos \theta) d\theta \\
 &= \int \frac{1 - 4 \sin \theta}{2 \sqrt{1 - \sin^2 \theta}} (2 \cos \theta) d\theta \\
 &= \int \frac{1 - 4 \sin \theta}{2 \cos \theta} (2 \cos \theta) d\theta \\
 &= \int (1 - 4 \sin \theta) d\theta \\
 &= \theta + 4 \cos \theta + c
 \end{aligned}$$

Now  $\sin \theta = \frac{x}{2}$ , so  $\cos \theta = \frac{\sqrt{4 - x^2}}{2}$

$\therefore \theta = \arcsin \frac{x}{2}$

$$\begin{aligned}
 \therefore \int \frac{1 - 2x}{\sqrt{4 - x^2}} dx &= \arcsin \frac{x}{2} + 4 \left( \frac{\sqrt{4 - x^2}}{2} \right) + c \\
 &= \arcsin \frac{x}{2} + 2\sqrt{4 - x^2} + c
 \end{aligned}$$





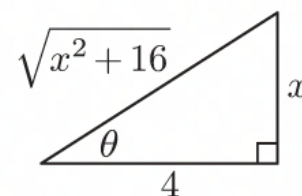
$$\begin{aligned}
 \text{j} \quad \int \frac{x+4}{x^2+4} dx &= \frac{1}{2} \int \frac{2x}{x^2+4} dx + \int \frac{4}{x^2+4} dx \\
 &= \frac{1}{2} \ln|x^2+4| + \int \frac{4}{x^2+2^2} dx \quad \text{where } x^2+4 > 0
 \end{aligned}$$

$$\text{Let } x = 2 \tan \theta \quad \therefore \frac{dx}{d\theta} = 2 \sec^2 \theta$$

$$\begin{aligned}
 \therefore \int \frac{x+4}{x^2+4} dx &= \frac{1}{2} \ln(x^2+4) + \int \frac{4}{4 \tan^2 \theta + 4} \times 2 \sec^2 \theta d\theta \\
 &= \frac{1}{2} \ln(x^2+4) + \int \frac{8 \sec^2 \theta}{4(\tan^2 \theta + 1)} d\theta \\
 &= \frac{1}{2} \ln(x^2+4) + \int 2 d\theta \\
 &= \frac{1}{2} \ln(x^2+4) + 2\theta + c \\
 &= \frac{1}{2} \ln(x^2+4) + 2 \arctan \frac{x}{2} + c
 \end{aligned}$$

$$\text{k} \quad \text{Let } x = 4 \tan \theta \quad \therefore \frac{dx}{d\theta} = 4 \sec^2 \theta$$

$$\begin{aligned}
 \therefore \int \frac{1}{x(x^2+16)} dx &= \int \frac{1}{4 \tan \theta (16 \tan^2 \theta + 16)} \times 4 \sec^2 \theta d\theta \\
 &= \int \frac{1}{4 \tan \theta \times 16 \sec^2 \theta} \times 4 \sec^2 \theta d\theta \\
 &= \frac{1}{16} \int \frac{1}{\tan \theta} d\theta \\
 &= \frac{1}{16} \int \frac{\cos \theta}{\sin \theta} d\theta \\
 &= \frac{1}{16} \ln |\sin \theta| + c \\
 &= \frac{1}{16} \ln \left| \frac{x}{\sqrt{x^2+16}} \right| + c
 \end{aligned}$$



$$\text{l} \quad \text{Let } x = 2 \sec \theta \quad \therefore \frac{dx}{d\theta} = 2 \sec \theta \tan \theta$$

$$\begin{aligned}
 \therefore \int \frac{3}{x\sqrt{x^2-4}} dx &= \int \frac{3}{\cancel{2 \sec \theta} \sqrt{4 \sec^2 \theta - 4}} \cancel{2 \sec \theta} \tan \theta d\theta \\
 &= \int \frac{3 \tan \theta}{2 \sqrt{\sec^2 \theta - 1}} d\theta \\
 &= \int \frac{3}{2} d\theta \quad \{\sqrt{\sec^2 \theta - 1} = \tan \theta\} \\
 &= \frac{3}{2} \theta + c
 \end{aligned}$$

$$\text{Now } x = \frac{2}{\cos \theta} \text{ so } \cos \theta = \frac{2}{x}, \quad \theta = \arccos \frac{2}{x}$$

$$\therefore \int \frac{3}{x\sqrt{x^2-4}} dx = \frac{3}{2} \arccos \frac{2}{x} + c$$

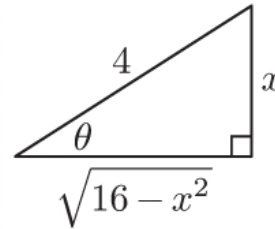
**m** Let  $x = 4 \sin \theta$   $\therefore \frac{dx}{d\theta} = 4 \cos \theta$

$$\begin{aligned} \therefore \int \frac{1}{x^2 \sqrt{16 - x^2}} dx &= \int \frac{1}{16 \sin^2 \theta \sqrt{16 - 16 \sin^2 \theta}} 4 \cos \theta d\theta \\ &= \int \frac{1}{16 \sin^2 \theta \times 4 \cos \theta} 4 \cos \theta d\theta \\ &= \frac{1}{16} \int \frac{1}{\sin^2 \theta} d\theta \\ &= \frac{1}{16} \int \operatorname{cosec}^2 \theta d\theta \\ &= \frac{1}{16} (-\cot \theta) + c \\ &= -\frac{1}{16} \cot \theta + c \end{aligned}$$

Now  $\sin \theta = \frac{x}{4}$ , so  $\tan \theta = \frac{x}{\sqrt{16 - x^2}}$

$$\therefore \cot \theta = \frac{\sqrt{16 - x^2}}{x}$$

$$\begin{aligned} \therefore \int \frac{1}{x^2 \sqrt{16 - x^2}} dx &= -\frac{1}{16} \frac{\sqrt{16 - x^2}}{x} + c \\ &= -\frac{\sqrt{16 - x^2}}{16x} + c \end{aligned}$$



**n**  $\frac{1}{x^2 + 2x + 3} = \frac{1}{(x + 1)^2 + 2}$

$$= \frac{1}{(x + 1)^2 + (\sqrt{2})^2}$$

Let  $x + 1 = \sqrt{2} \tan \theta$   $\therefore \frac{dx}{d\theta} = \sqrt{2} \sec^2 \theta$

$$\begin{aligned} \therefore \int \frac{1}{x^2 + 2x + 3} dx &= \int \frac{1}{(x + 1)^2 + (\sqrt{2})^2} dx \\ &= \int \frac{1}{2 \tan^2 \theta + 2} \times \sqrt{2} \sec^2 \theta d\theta \\ &= \int \frac{\sqrt{2} \sec^2 \theta}{2 \sec^2 \theta} d\theta \\ &= \int \frac{1}{\sqrt{2}} d\theta \\ &= \frac{1}{\sqrt{2}} \theta + c \end{aligned}$$

Now  $\tan \theta = \frac{x + 1}{\sqrt{2}}$ , so  $\theta = \arctan\left(\frac{x + 1}{\sqrt{2}}\right)$

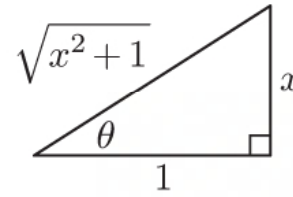
$$\therefore \int \frac{1}{x^2 + 2x + 3} dx = \frac{1}{\sqrt{2}} \arctan\left(\frac{x + 1}{\sqrt{2}}\right) + c$$

• Let  $x = \tan \theta \quad \therefore \frac{dx}{d\theta} = \sec^2 \theta$

$$\begin{aligned} \therefore \int \frac{1}{x(1+x^2)} dx &= \int \frac{1}{\tan \theta (1 + \tan^2 \theta)} \times \sec^2 \theta d\theta \\ &= \int \frac{1}{\tan \theta} d\theta \quad \{1 + \tan^2 \theta = \sec^2 \theta\} \\ &= \int \frac{\cos \theta}{\sin \theta} d\theta \\ &= \ln |\sin \theta| + c \end{aligned}$$

Since  $\tan \theta = x$ ,  $\sin \theta = \frac{x}{\sqrt{x^2 + 1}}$

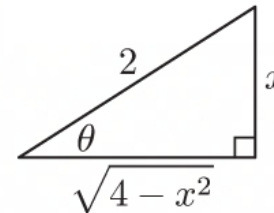
$$\therefore \int \frac{1}{x(1+x^2)} dx = \ln \left| \frac{x}{\sqrt{x^2 + 1}} \right| + c$$



• Let  $x = 2 \sin \theta \quad \therefore \frac{dx}{d\theta} = 2 \cos \theta$

$$\begin{aligned} \therefore \int x^2 \sqrt{4 - x^2} dx &= \int 4 \sin^2 \theta \sqrt{4 - 4 \sin^2 \theta} 2 \cos \theta d\theta \\ &= \int 4 \sin^2 \theta 2 \cos \theta 2 \cos \theta d\theta \\ &= 4 \int 4 \sin^2 \theta \cos^2 \theta d\theta \\ &= 4 \int \sin^2 2\theta d\theta \\ &= 4 \int \left( \frac{1}{2} - \frac{1}{2} \cos 4\theta \right) d\theta \\ &= 2\theta - 2 \left( \frac{1}{4} \right) \sin 4\theta + c \\ &= 2\theta - \frac{1}{2} \sin 4\theta + c \end{aligned}$$

Now  $\sin \theta = \frac{x}{2}$ , so  $\cos \theta = \frac{\sqrt{4 - x^2}}{2}$



$$\begin{aligned} \therefore \sin 2\theta &= 2 \left( \frac{x}{2} \right) \frac{\sqrt{4 - x^2}}{2} = \frac{x\sqrt{4 - x^2}}{2} \quad \text{and} \quad \cos 2\theta = \cos^2 \theta - \sin^2 \theta \\ &= \frac{4 - x^2}{4} - \frac{x^2}{4} \\ &= \frac{4 - 2x^2}{4} \end{aligned}$$

Now,  $\sin 4\theta = 2 \sin 2\theta \cos 2\theta$

$$\begin{aligned} &= 2 \left( \frac{x\sqrt{4 - x^2}}{2} \right) \left( \frac{4 - 2x^2}{4} \right) \\ &= \frac{x\sqrt{4 - x^2}(2 - x^2)}{2} \end{aligned}$$

$$\therefore \int x^2 \sqrt{4 - x^2} dx = 2 \arcsin \frac{x}{2} - \frac{1}{4} x \sqrt{4 - x^2} (2 - x^2) + c$$

**19 a** Let  $u = \sqrt{\frac{x-\xi}{x+\xi}}$

$$\begin{aligned}\therefore u^2 &= \frac{x-\xi}{x+\xi} \\ &= \frac{x+\xi-2\xi}{x+\xi} \\ &= 1 - \frac{2\xi}{x+\xi} \\ &= 1 - 2\xi(x+\xi)^{-1}\end{aligned}$$

$$\therefore 2u \frac{du}{dx} = \frac{2\xi}{(x+\xi)^2}$$

$$\therefore \frac{u}{\xi} du = \frac{1}{(x+\xi)^2} dx$$

$$\begin{aligned}\therefore \int \frac{x-\xi}{(x+\xi)^3} dx &= \int \frac{x-\xi}{x+\xi} \times \frac{1}{(x+\xi)^2} dx \\ &= \int u^2 \times \frac{u}{\xi} du \\ &= \frac{1}{\xi} \int u^3 du \\ &= \frac{1}{\xi} \left( \frac{u^4}{4} + c \right) \\ &= \frac{1}{4\xi} \left( \frac{x-\xi}{x+\xi} \right)^2 + c \\ &= \frac{x^2 - 2\xi x + \xi^2}{4\xi(x+\xi)^2} + c \\ &= \frac{x^2 - 4\xi x + 2\xi x + \xi^2}{4\xi(x+\xi)^2} + c \\ &= \frac{(x+\xi)^2 - 4\xi x}{4\xi(x+\xi)^2} + c \\ &= \frac{1}{4\xi} - \frac{x}{(x+\xi)^2} + c\end{aligned}$$

**b** 
$$\begin{aligned}\int (x-\xi)^{\frac{1}{2}} (x+\xi)^{-\frac{5}{2}} dx &= \int \sqrt{\frac{x-\xi}{x+\xi}} \times \frac{1}{(x+\xi)^2} dx \\ &= \int u \times \frac{u}{\xi} du \quad \{\text{from part a, } \frac{u}{\xi} du = \frac{1}{(x+\xi)^2} dx\} \\ &= \frac{1}{\xi} \int u^2 du \\ &= \frac{1}{\xi} \left( \frac{u^3}{3} + c \right) \\ &= \frac{1}{3\xi} \left( \frac{x-\xi}{x+\xi} \right)^{\frac{3}{2}} + c\end{aligned}$$



**20** Let  $u = x^{\frac{1}{6}}$

$$\therefore x = u^6$$

$$\therefore \frac{dx}{du} = 6u^5$$

$$\begin{aligned} \therefore \int \frac{1}{\sqrt{x} + \sqrt[3]{x}} dx &= \int \frac{1}{\sqrt{u^6} + \sqrt[3]{u^6}} \frac{dx}{du} du \\ &= \int \frac{1}{u^3 + u^2} \times 6u^5 du \\ &= 6 \int \frac{u^5}{u^3 + u^2} du \\ &= 6 \int \frac{u^3}{u + 1} du \end{aligned}$$

$$\text{Let } v = u + 1 \quad \therefore u = v - 1 \quad \text{and} \quad \frac{dv}{du} = 1$$

$$\begin{aligned} \therefore \int \frac{1}{\sqrt{x} + \sqrt[3]{x}} dx &= 6 \int \frac{(v-1)^3}{v-1+1} \frac{dv}{du} du \\ &= 6 \int \frac{v^3 - 3v^2 + 3v - 1}{v} dv \\ &= 6 \int \left( v^2 - 3v + 3 - \frac{1}{v} \right) dv \\ &= 6 \left( \frac{v^3}{3} - \frac{3v^2}{2} + 3v - \ln|v| + c \right) \\ &= 2v^3 - 9v^2 + 18v - 6\ln|v| + c \\ &= 2(u+1)^3 - 9(u+1)^2 + 18(u+1) - 6\ln|u+1| + c \quad \{v = u+1\} \\ &= 2(u^3 + 3u^2 + 3u + 1) - 9(u^2 + 2u + 1) + 18u + 18 - 6\ln|u+1| + c \\ &= 2u^3 + 6u^2 + 6u + 2 - 9u^2 - 18u - 9 + 18u + 18 - 6\ln|u+1| + c \\ &= 2u^3 - 3u^2 + 6u - 6\ln|u+1| + c \\ &= 2(x^{\frac{1}{6}})^3 - 3(x^{\frac{1}{6}})^2 + 6(x^{\frac{1}{6}}) - 6\ln(x^{\frac{1}{6}} + 1) + c \quad \{u = x^{\frac{1}{6}}\} \\ &= 2x^{\frac{1}{2}} - 3x^{\frac{1}{3}} + 6x^{\frac{1}{6}} - 6\ln(x^{\frac{1}{6}} + 1) + c \end{aligned}$$

**21** 
$$\int \frac{1}{ax^2 + bx + c} dx = \frac{1}{a} \int \frac{1}{x^2 + \frac{b}{a}x + \frac{c}{a}} dx, \quad a \neq 0$$

$$= \frac{1}{a} \int \frac{1}{\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{4ac}{4a^2}} dx$$

$$= \frac{1}{a} \int \frac{1}{\left(x + \frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a^2}} dx, \quad a \neq 0$$

We need to consider the cases  $b^2 < 4ac$ ,  $b^2 > 4ac$ , and  $b^2 = 4ac$  separately.

**Case 1:  $b^2 < 4ac$**

$$\begin{aligned}\int \frac{1}{ax^2 + bx + c} dx &= \frac{1}{a} \int \frac{1}{\left(x + \frac{b}{2a}\right)^2 + \left(\frac{\sqrt{4ac-b^2}}{2a}\right)^2} dx \\&= \frac{1}{a} \times \frac{2a}{\sqrt{4ac-b^2}} \int \frac{\frac{\sqrt{4ac-b^2}}{2a}}{\left(x + \frac{b}{2a}\right)^2 + \left(\frac{\sqrt{4ac-b^2}}{2a}\right)^2} dx \\&= \frac{2}{\sqrt{4ac-b^2}} \arctan \left( \frac{x + \frac{b}{2a}}{\frac{\sqrt{4ac-b^2}}{2a}} \right) + d \\&\quad \text{(compare with Exercise 21D question 26 a iii)} \\&= \frac{2}{\sqrt{4ac-b^2}} \arctan \left( \frac{2ax + b}{\sqrt{4ac-b^2}} \right) + d\end{aligned}$$

**Case 2:  $b^2 > 4ac$**

$$\begin{aligned}\int \frac{1}{ax^2 + bx + c} dx &= \frac{1}{a} \int \frac{1}{\left(x + \frac{b}{2a}\right)^2 - \left(\frac{\sqrt{b^2-4ac}}{2a}\right)^2} dx \\&= \frac{1}{a} \int \frac{1}{\left[\left(x + \frac{b}{2a}\right) + \frac{\sqrt{b^2-4ac}}{2a}\right] \left[\left(x + \frac{b}{2a}\right) - \frac{\sqrt{b^2-4ac}}{2a}\right]} dx\end{aligned}$$

Writing  $\frac{1}{\left[\left(x + \frac{b}{2a}\right) + \frac{\sqrt{b^2-4ac}}{2a}\right] \left[\left(x + \frac{b}{2a}\right) - \frac{\sqrt{b^2-4ac}}{2a}\right]}$  as the sum of partial fractions:

$$\begin{aligned}\frac{1}{\left[\left(x + \frac{b}{2a}\right) + \frac{\sqrt{b^2-4ac}}{2a}\right] \left[\left(x + \frac{b}{2a}\right) - \frac{\sqrt{b^2-4ac}}{2a}\right]} &= \frac{M}{x + \left(\frac{b + \sqrt{b^2-4ac}}{2a}\right)} + \frac{N}{x + \left(\frac{b - \sqrt{b^2-4ac}}{2a}\right)} \\ \therefore 1 &= M \left( x + \left( \frac{b - \sqrt{b^2-4ac}}{2a} \right) \right) + N \left( x + \left( \frac{b + \sqrt{b^2-4ac}}{2a} \right) \right)\end{aligned}$$

$$\begin{aligned}\text{Substituting } x &= -\left(\frac{b - \sqrt{b^2-4ac}}{2a}\right): \quad 1 = N \left( \frac{-b + \sqrt{b^2-4ac}}{2a} + \frac{b + \sqrt{b^2-4ac}}{2a} \right) \\ \therefore 1 &= N \left( \frac{\sqrt{b^2-4ac}}{a} \right) \\ \therefore N &= \frac{a}{\sqrt{b^2-4ac}}\end{aligned}$$

$$\begin{aligned}\text{Substituting } x &= -\left(\frac{b + \sqrt{b^2-4ac}}{2a}\right): \quad 1 = M \left( \frac{-b - \sqrt{b^2-4ac}}{2a} + \frac{b - \sqrt{b^2-4ac}}{2a} \right) \\ \therefore 1 &= M \left( \frac{-\sqrt{b^2-4ac}}{a} \right) \\ \therefore M &= -\frac{a}{\sqrt{b^2-4ac}}\end{aligned}$$

$$\begin{aligned}
& \therefore \int \frac{1}{ax^2 + bx + c} dx \\
&= \frac{1}{a} \int \left( \frac{-\frac{a}{\sqrt{b^2 - 4ac}}}{\left(x + \frac{b}{2a}\right) + \frac{\sqrt{b^2 - 4ac}}{2a}} + \frac{\frac{a}{\sqrt{b^2 - 4ac}}}{\left(x + \frac{b}{2a}\right) - \frac{\sqrt{b^2 - 4ac}}{2a}} \right) dx \\
&= \frac{1}{a} \int -\frac{a}{\sqrt{b^2 - 4ac}} \left( \frac{1}{\left(x + \frac{b}{2a}\right) + \frac{\sqrt{b^2 - 4ac}}{2a}} - \frac{1}{\left(x + \frac{b}{2a}\right) - \frac{\sqrt{b^2 - 4ac}}{2a}} \right) dx \\
&= -\frac{1}{\sqrt{b^2 - 4ac}} \left( \ln \left| x + \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} \right| - \ln \left| x + \frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a} \right| \right) + d \\
&= \frac{1}{\sqrt{b^2 - 4ac}} \ln \left| \frac{x + \frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}}{x + \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}} \right| + d \\
&= \frac{1}{\sqrt{b^2 - 4ac}} \ln \left| \frac{2ax + b - \sqrt{b^2 - 4ac}}{2ax + b + \sqrt{b^2 - 4ac}} \right| + d
\end{aligned}$$

**Case 3:  $b^2 = 4ac$**

$$\begin{aligned}
\int \frac{1}{ax^2 + bx + c} dx &= \frac{1}{a} \int \frac{1}{\left(x + \frac{b}{2a}\right)^2} dx \\
&= \frac{1}{a} \left( -\left(x + \frac{b}{2a}\right)^{-1} \right) + d \\
&= -\frac{1}{a} \left( \frac{1}{x + \frac{b}{2a}} \right) + d \\
&= -\frac{1}{a} \left( \frac{2a}{2ax + b} \right) + d \\
&= -\frac{2}{2ax + b} + d
\end{aligned}$$

## EXERCISE 21G

$$\begin{aligned} \mathbf{1} \quad \mathbf{a} \quad \int x e^x dx &= x e^x - \int 1 e^x dx & \begin{cases} u = x & v' = e^x \\ u' = 1 & v = e^x \end{cases} \\ &= x e^x - e^x + c \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \int x \sin x dx &= x(-\cos x) - \int 1(-\cos x) dx & \begin{cases} u = x & v' = \sin x \\ u' = 1 & v = -\cos x \end{cases} \\ &= -x \cos x + \int \cos x dx \\ &= -x \cos x + \sin x + c \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad \int x^2 \ln x dx &= \ln x \left( \frac{x^3}{3} \right) - \int \frac{1}{x} \frac{x^3}{3} dx & \begin{cases} u = \ln x & v' = x^2 \\ u' = \frac{1}{x} & v = \frac{x^3}{3} \end{cases} \\ &= \frac{x^3 \ln x}{3} - \frac{1}{3} \int x^2 dx \\ &= \frac{x^3 \ln x}{3} - \frac{1}{3} \frac{x^3}{3} + c \\ &= \frac{1}{3} x^3 \ln x - \frac{1}{9} x^3 + c \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad \int \frac{\ln x}{x} dx &= (\ln x)(\ln x) - \int \left( \frac{1}{x} \right) \ln x dx & \begin{cases} u = \ln x & v' = \frac{1}{x} \\ u' = \frac{1}{x} & v = \ln x \end{cases} \\ \therefore \int \frac{\ln x}{x} dx &= (\ln x)^2 - \int \frac{\ln x}{x} dx \\ \therefore 2 \int \frac{\ln x}{x} dx &= (\ln x)^2 + c \\ \therefore \int \frac{\ln x}{x} dx &= \frac{(\ln x)^2}{2} + c \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad \int x 3^x dx &= x \left( \frac{3^x}{\ln 3} \right) - \int \frac{3^x}{\ln 3} dx & \begin{cases} u = x & v' = 3^x \\ u' = 1 & v = \frac{3^x}{\ln 3} \end{cases} \\ &= \frac{x 3^x}{\ln 3} - \frac{3^x}{(\ln 3)^2} + c \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad \int x \sin 3x dx &= x \left( -\frac{1}{3} \cos 3x \right) - \int \left( -\frac{1}{3} \cos 3x \right) dx & \begin{cases} u = x & v' = \sin 3x \\ u' = 1 & v = -\frac{1}{3} \cos 3x \end{cases} \\ &= -\frac{1}{3} x \cos 3x + \frac{1}{3} \left( \frac{1}{3} \right) \sin 3x + c \\ &= -\frac{1}{3} x \cos 3x + \frac{1}{9} \sin 3x + c \end{aligned}$$



$$\begin{aligned}
 \text{g} \quad \int x \cos 2x \, dx &= x \left( \frac{1}{2} \sin 2x \right) - \int \frac{1}{2} \sin 2x \, dx & \begin{cases} u = x & v' = \cos 2x \\ u' = 1 & v = \frac{1}{2} \sin 2x \end{cases} \\
 &= \frac{1}{2} x \sin 2x - \frac{1}{2} \left( -\frac{1}{2} \right) \cos 2x + c \\
 &= \frac{1}{2} x \sin 2x + \frac{1}{4} \cos 2x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{h} \quad \int x \sec^2 x \, dx &= x \tan x - \int \tan x \, dx & \begin{cases} u = x & v' = \sec^2 x \\ u' = 1 & v = \tan x \end{cases} \\
 &= x \tan x + \int \frac{-\sin x}{\cos x} \, dx \\
 &= x \tan x + \ln |\cos x| + c
 \end{aligned}$$

$$\begin{aligned}
 \text{2} \quad \text{a} \quad \int \ln x \, dx &= \int 1 \times \ln x \, dx \\
 &= x \ln x - \int \left( \frac{1}{x} \right) x \, dx & \begin{cases} u = \ln x & v' = 1 \\ u' = \frac{1}{x} & v = x \end{cases} \\
 &= x \ln x - \int 1 \, dx \\
 &= x \ln x - x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad &\int (\ln x)^2 \, dx \\
 &= \int (\ln x)(\ln x) \, dx \\
 &= \ln x (x \ln x - x) - \int \frac{1}{x} (x \ln x - x) \, dx & \begin{cases} u = \ln x & v' = \ln x \\ u' = \frac{1}{x} & v = x \ln x - x \end{cases} \quad \{\text{using a}\} \\
 &= x(\ln x)^2 - x \ln x - \int (\ln x - 1) \, dx \\
 &= x(\ln x)^2 - x \ln x - [x \ln x - x] + x + c \\
 &= x(\ln x)^2 - 2x \ln x + 2x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{3} \quad \int \arctan x \, dx &= \int 1 \arctan x \, dx \\
 &= x \arctan x - \int \frac{x}{1+x^2} \, dx & \begin{cases} u = \arctan x & v' = 1 \\ u' = \frac{1}{1+x^2} & v = x \end{cases} \\
 &= x \arctan x - \frac{1}{2} \int \frac{2x}{1+x^2} \, dx \\
 &= x \arctan x - \frac{1}{2} \ln |1+x^2| + c \\
 &= x \arctan x - \frac{1}{2} \ln(1+x^2) + c \quad \{\text{as } 1+x^2 > 0\}
 \end{aligned}$$

$$\begin{aligned}
4 \quad \mathbf{a} \quad \int x^2 e^{-x} dx &= -x^2 e^{-x} - \int 2x(-e^{-x}) dx \quad \leftarrow \begin{cases} u = x^2 & v' = e^{-x} \\ u' = 2x & v = -e^{-x} \end{cases} \\
&= -x^2 e^{-x} + 2 \int x e^{-x} dx \\
&= -x^2 e^{-x} + 2 \left[ x(-e^{-x}) - \int -e^{-x} dx \right] \quad \leftarrow \begin{cases} u = x & v' = e^{-x} \\ u' = 1 & v = -e^{-x} \end{cases} \\
&= -x^2 e^{-x} - 2x e^{-x} + 2 \int e^{-x} dx \\
&= -x^2 e^{-x} - 2x e^{-x} - 2e^{-x} + c
\end{aligned}$$

$$\begin{aligned}
\mathbf{b} \quad \int e^x \cos x dx &= e^x \sin x - \int e^x \sin x dx \quad \leftarrow \begin{cases} u = e^x & v' = \cos x \\ u' = e^x & v = \sin x \end{cases} \\
&= e^x \sin x - \left[ -e^x \cos x - \int e^x (-\cos x) dx \right] \quad \leftarrow \begin{cases} u = e^x & v' = \sin x \\ u' = e^x & v = -\cos x \end{cases} \\
&= e^x \sin x + e^x \cos x - \int e^x \cos x dx + c \\
\therefore 2 \int e^x \cos x dx &= e^x (\sin x + \cos x) + c \\
\therefore \int e^x \cos x dx &= \frac{1}{2} e^x (\sin x + \cos x) + c
\end{aligned}$$

$$\begin{aligned}
\mathbf{c} \quad \int e^{-x} \sin x dx &= -e^{-x} \cos x - \int -e^{-x} (-\cos x) dx \quad \leftarrow \begin{cases} u = e^{-x} & v' = \sin x \\ u' = -e^{-x} & v = -\cos x \end{cases} \\
&= -e^{-x} \cos x - \int e^{-x} \cos x dx \\
&= -e^{-x} \cos x - \left[ e^{-x} \sin x - \int -e^{-x} \sin x dx \right] \quad \leftarrow \begin{cases} u = e^{-x} & v' = \cos x \\ u' = -e^{-x} & v = \sin x \end{cases} \\
&= -e^{-x} \cos x - e^{-x} \sin x - \int e^{-x} \sin x dx + c \\
\therefore 2 \int e^{-x} \sin x dx &= -e^{-x} (\sin x + \cos x) + c \\
\therefore \int e^{-x} \sin x dx &= -\frac{1}{2} e^{-x} (\sin x + \cos x) + c
\end{aligned}$$

$$\begin{aligned}
 \text{d} \quad \int x^2 \sin x \, dx &= -x^2 \cos x - \int -2x \cos x \, dx \quad \leftarrow \begin{cases} u = x^2 & v' = \sin x \\ u' = 2x & v = -\cos x \end{cases} \\
 &= -x^2 \cos x + \int 2x \cos x \, dx \\
 &= -x^2 \cos x + \left[ 2x \sin x - \int 2 \sin x \, dx \right] \quad \leftarrow \begin{cases} u = 2x & v' = \cos x \\ u' = 2 & v = \sin x \end{cases} \\
 &= -x^2 \cos x + 2x \sin x - 2 \int \sin x \, dx \\
 &= -x^2 \cos x + 2x \sin x - 2(-\cos x) + c \\
 &= -x^2 \cos x + 2x \sin x + 2 \cos x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad \int \frac{(\ln x)^2}{x^2} \, dx &= -\frac{(\ln x)^2}{x} - \int \left( \frac{2 \ln x}{x} \right) \left( -\frac{1}{x} \right) dx \quad \leftarrow \begin{cases} u = (\ln x)^2 & v' = x^{-2} \\ u' = \frac{2 \ln x}{x} & v = -\frac{1}{x} \end{cases} \\
 &= -\frac{(\ln x)^2}{x} + 2 \int x^{-2} \ln x \, dx \\
 &= -\frac{(\ln x)^2}{x} + 2 \left[ -\frac{\ln x}{x} - \int -x^{-2} \, dx \right] \quad \leftarrow \begin{cases} u = \ln x & v' = x^{-2} \\ u' = \frac{1}{x} & v = -\frac{1}{x} \end{cases} \\
 &= -\frac{(\ln x)^2}{x} - \frac{2 \ln x}{x} - 2 \left( \frac{1}{x} \right) + c \\
 &= -\frac{(\ln x)^2 + 2 \ln x + 2}{x} + c
 \end{aligned}$$

$$\begin{aligned}
 5 \quad \text{a} \quad \int u^2 e^u \, du &= u^2 e^u - \int 2u e^u \, du \quad \leftarrow \begin{cases} a = u^2 & b' = e^u \\ a' = 2u & b = e^u \end{cases} \\
 &= u^2 e^u - 2 \int u e^u \, du \\
 &= u^2 e^u - 2 \left[ u e^u - \int e^u \, du \right] \quad \leftarrow \begin{cases} a = u & b' = e^u \\ a' = 1 & b = e^u \end{cases} \\
 &= u^2 e^u - 2u e^u + 2e^u + c
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad \text{Let } u = \ln x \quad \therefore \quad \frac{du}{dx} &= \frac{1}{x} = \frac{1}{e^u} \\
 \therefore \int (\ln x)^2 \, dx &= \int u^2 e^u \, du \\
 &= u^2 e^u - 2u e^u + 2e^u + c \quad \{\text{using a}\} \\
 &= (\ln x)^2 e^{\ln x} - 2 \ln x e^{\ln x} + 2e^{\ln x} + c \\
 &= x(\ln x)^2 - 2x \ln x + 2x + c
 \end{aligned}$$

$$\begin{aligned}
 6 \quad \text{a} \quad \int \sin x \cos x \, dx &= \frac{1}{2} \int 2 \sin x \cos x \, dx \\
 &= \frac{1}{2} \int \sin 2x \, dx \\
 &= -\frac{1}{4} \cos 2x + c
 \end{aligned}$$

**b** Let  $u = \sin x \quad \therefore \frac{du}{dx} = \cos x$

$$\begin{aligned} \therefore \int \sin x \cos x \, dx &= \int u \frac{du}{dx} \, dx \\ &= \int u \, du \\ &= \frac{1}{2}u^2 + c = \frac{1}{2}\sin^2 x + c \end{aligned}$$

**c**  $\int \sin x \cos x \, dx = \sin x \sin x - \int \cos x \sin x \, dx \quad \begin{cases} u = \sin x & v' = \cos x \\ u' = \cos x & v = \sin x \end{cases}$

$$\therefore 2 \int \sin x \cos x \, dx = \sin^2 x$$

$$\therefore \int \sin x \cos x \, dx = \frac{1}{2}\sin^2 x + c$$

**7**  $\int \sin 4x \cos x \, dx$

$$= \sin 4x \sin x - \int \sin x (4 \cos 4x) \, dx \quad \leftarrow \begin{cases} u = \sin 4x & v' = \cos x \\ u' = 4 \cos 4x & v = \sin x \end{cases}$$

$$= \sin 4x \sin x - 4 \int \sin x \cos 4x \, dx$$

$$= \sin 4x \sin x - 4 \left[ (\cos 4x)(-\cos x) - \int (-\cos x)(-4 \sin 4x) \, dx \right] \quad \begin{cases} u = \cos 4x & v' = \sin x \\ u' = -4 \sin 4x & v = -\cos x \end{cases}$$

$$= \sin 4x \sin x + 4 \cos 4x \cos x + 16 \int \sin 4x \cos x \, dx$$

$$\therefore -15 \int \sin 4x \cos x \, dx = \sin 4x \sin x + 4 \cos 4x \cos x + c$$

$$\therefore \int \sin 4x \cos x \, dx = -\frac{1}{15}(\sin 4x \sin x + 4 \cos 4x \cos x) + c$$

**8 a**  $\int u \sin u \, du = u(-\cos u) - \int (-\cos u) \, du \quad \begin{cases} a = u & b' = \sin u \\ a' = 1 & b = -\cos u \end{cases}$

$$= -u \cos u + \sin u + c$$



**b** Let  $u^2 = 2x$ ,  $2u \frac{du}{dx} = 2$

$$\therefore \frac{du}{dx} = \frac{1}{u}$$

$$\begin{aligned} \therefore \int \sin \sqrt{2x} \, dx &= \int \sin u \times u \, du \\ &= \int u \sin u \, du \\ &= -u \cos u + \sin u + c \quad \{\text{using a}\} \\ &= -\sqrt{2x} \cos \sqrt{2x} + \sin \sqrt{2x} + c \end{aligned}$$

**9** Let  $u^2 = 3x$ ,  $2u \frac{du}{dx} = 3$

$$\therefore \frac{du}{dx} = \frac{3}{2u}$$

$$\begin{aligned} \therefore \int \cos \sqrt{3x} \, dx &= \int \cos u \left( \frac{2u}{3} \right) du \\ &= \frac{2}{3} \int u \cos u \, du \\ &= \frac{2}{3} \left[ u \sin u - \int \sin u \, du \right] \quad \begin{cases} a = u & b' = \cos u \\ a' = 1 & b = \sin u \end{cases} \\ &= \frac{2}{3} u \sin u - \frac{2}{3} (-\cos u) + c \\ &= \frac{2}{3} \sqrt{3x} \sin \sqrt{3x} + \frac{2}{3} \cos \sqrt{3x} + c \end{aligned}$$

## REVIEW SET 21A

**1**  $\frac{d}{dx}(x^4 - x^2) = 4x^3 - 2x$

$$\therefore \int (4x^3 - 2x) \, dx = x^4 - x^2 + c$$

$$\therefore 2 \int (2x^3 - x) \, dx = x^4 - x^2 + c$$

$$\therefore \int (2x^3 - x) \, dx = \frac{1}{2}x^4 - \frac{1}{2}x^2 + c$$

**2**  $\frac{d}{dx}(\sin(\frac{\pi}{3} - 2x)) = \cos(\frac{\pi}{3} - 2x) \times (-2)$   
 $= -2 \cos(\frac{\pi}{3} - 2x)$

$$\therefore \int -2 \cos(\frac{\pi}{3} - 2x) \, dx = \sin(\frac{\pi}{3} - 2x) + c$$

$$\therefore -2 \int \cos(\frac{\pi}{3} - 2x) \, dx = \sin(\frac{\pi}{3} - 2x) + c$$

$$\therefore \int \cos(\frac{\pi}{3} - 2x) \, dx = -\frac{1}{2} \sin(\frac{\pi}{3} - 2x) + c$$

$$\begin{aligned}
 \text{3 a } \int \left( \sqrt{x} - \frac{2}{x^2} \right) dx &= \int (x^{\frac{1}{2}} - 2x^{-2}) dx \\
 &= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - 2 \times \frac{x^{-1}}{-1} + c \\
 &= \frac{2}{3}x^{\frac{3}{2}} + 2x^{-1} + c \\
 &= \frac{2}{3}x\sqrt{x} + \frac{2}{x} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{b } \int \left( 2x - \frac{3}{\sqrt[3]{x}} \right) dx &= \int (2x - 3x^{-\frac{1}{3}}) dx \\
 &= x^2 - \frac{3x^{\frac{2}{3}}}{\frac{2}{3}} + c \\
 &= x^2 - \frac{9}{2}x^{\frac{2}{3}} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{c } \int \frac{6x+5}{\sqrt{x}} dx &= \int (6x^{\frac{1}{2}} + 5x^{-\frac{1}{2}}) dx \\
 &= 6 \times \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + 5 \times \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c \\
 &= 4x^{\frac{3}{2}} + 10x^{\frac{1}{2}} + c \\
 &= 4x\sqrt{x} + 10\sqrt{x} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{4 a } \int \frac{4}{\sqrt{x}} dx &= 4 \int x^{-\frac{1}{2}} dx \\
 &= 4 \times \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c \\
 &= 8\sqrt{x} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{b } \int \frac{3}{1-2x} dx &= -\frac{3}{2} \int \frac{-2}{1-2x} dx \\
 &= -\frac{3}{2} \ln |1-2x| + c
 \end{aligned}$$

$$\begin{aligned}
 \text{c } \int \left( \frac{1}{3}x^3 + 2x \right) dx &= \frac{1}{3} \times \frac{x^4}{4} + x^2 + c \\
 &= \frac{1}{12}x^4 + x^2 + c
 \end{aligned}$$

$$\begin{aligned}
 \text{d } \int \frac{1-2x}{x^3} dx &= \int \left( \frac{1}{x^3} - \frac{2}{x^2} \right) dx \\
 &= \int (x^{-3} - 2x^{-2}) dx \\
 &= \frac{x^{-2}}{-2} - \frac{2x^{-1}}{-1} + c \\
 &= -\frac{1}{2}x^{-2} + 2x^{-1} + c \\
 &= -\frac{1}{2x^2} + \frac{2}{x} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{5 a } \int (-3x^4 + 6x^2) dx &= -\frac{3x^5}{5} + \frac{6x^3}{3} + c \\
 &= -\frac{3}{5}x^5 + 2x^3 + c
 \end{aligned}$$

$$\begin{aligned}
 \text{b } \int \frac{3x^3 - x^2 - 1}{x^2} dx &= \int \left( 3x - 1 - \frac{1}{x^2} \right) dx \\
 &= \int (3x - 1 - x^{-2}) dx \\
 &= \frac{3x^2}{2} - x - \frac{x^{-1}}{-1} + c \\
 &= \frac{3}{2}x^2 - x + \frac{1}{x} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{c } \int (2x - \sqrt{x})^2 dx &= \int (4x^2 - 4x\sqrt{x} + x) dx \\
 &= \int (4x^2 - 4x^{\frac{3}{2}} + x) dx \\
 &= \frac{4x^3}{3} - \frac{4x^{\frac{5}{2}}}{\frac{5}{2}} + \frac{x^2}{2} + c \\
 &= \frac{4}{3}x^3 - \frac{8}{5}x^{\frac{5}{2}} + \frac{1}{2}x^2 + c
 \end{aligned}$$

$$\begin{aligned}
 \text{d } \int \left( 4e^x - \frac{3}{x} \right) dx &= 4e^x - 3 \ln |x| + c
 \end{aligned}$$

$$\begin{aligned}
 \text{e } \int \sin(4x - 5) dx &= -\frac{1}{4} \cos(4x - 5) + c
 \end{aligned}$$

$$\begin{aligned}
 \text{f } \int e^{4-3x} dx &= \left( \frac{1}{-3} \right) e^{4-3x} + c \\
 &= -\frac{1}{3}e^{4-3x} + c
 \end{aligned}$$

$$\begin{aligned} \mathbf{g} \quad & \int 5^{2-x} dx \\ &= \left(\frac{1}{-1}\right) \frac{5^{2-x}}{\ln 5} + c \\ &= -\frac{5^{2-x}}{\ln 5} + c \end{aligned}$$

$$\begin{aligned} \mathbf{h} \quad & \int \sec^2\left(\frac{x}{2}\right) dx \\ &= \left(\frac{1}{\frac{1}{2}}\right) \tan \frac{x}{2} + c \\ &= 2 \tan \frac{x}{2} + c \end{aligned}$$

$$\begin{aligned} \mathbf{i} \quad & \int \operatorname{cosec}^2\left(x - \frac{\pi}{4}\right) dx \\ &= -\cot\left(x - \frac{\pi}{4}\right) + c \end{aligned}$$

$$\begin{aligned} \mathbf{6} \quad \mathbf{a} \quad & \frac{dy}{dx} = 3e^{-x} - 2 \sin\left(\frac{\pi}{2} - x\right) \\ \therefore y &= \int (3e^{-x} - 2 \sin(\frac{\pi}{2} - x)) dx \\ &= 3\left(\frac{1}{-1}\right) e^{-x} - 2\left(\frac{1}{-1}\right) (-\cos(\frac{\pi}{2} - x)) + c \\ &= -3e^{-x} - 2 \cos\left(\frac{\pi}{2} - x\right) + c \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad & \frac{dy}{dx} = \cos 4x - \frac{1}{2}x^2 \\ \therefore y &= \int (\cos 4x - \frac{1}{2}x^2) dx \\ &= \frac{1}{4} \sin 4x - \frac{1}{2} \times \frac{x^3}{3} + c \\ &= \frac{1}{4} \sin 4x - \frac{1}{6}x^3 + c \end{aligned}$$

$$\begin{aligned} \mathbf{7} \quad & f'(x) = 3x^2 - 4x + 1 \\ \therefore f(x) &= \int (3x^2 - 4x + 1) dx \\ &= \frac{3x^3}{3} - \frac{4x^2}{2} + x + c \\ &= x^3 - 2x^2 + x + c \end{aligned}$$

But  $f(0) = 2$ , so  $c = 2$

$$\therefore f(x) = x^3 - 2x^2 + x + 2$$

$$\begin{aligned} \mathbf{8} \quad & f'(x) = ax + 3 \\ \therefore f(x) &= \int (ax + 3) dx \\ &= \frac{ax^2}{2} + 3x + c \end{aligned}$$

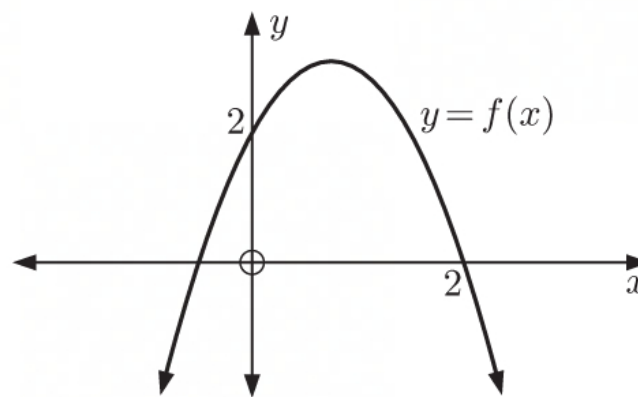
But the  $y$ -intercept is 2, so  $f(0) = 2$   
 $\therefore c = 2$

$$\therefore f(x) = \frac{ax^2}{2} + 3x + 2$$

and the  $x$ -intercept is 2, so  $f(2) = 0$

$$\begin{aligned} \therefore \frac{a(2)^2}{2} + 3(2) + 2 &= 0 \\ \therefore 2a + 6 + 2 &= 0 \\ \therefore 2a &= -8 \\ \therefore a &= -4 \end{aligned}$$

$$\begin{aligned} \therefore \text{the equation of the curve is } y = f(x) &= \frac{(-4)x^2}{2} + 3x + 2 \\ &= -2x^2 + 3x + 2 \end{aligned}$$



$$9 \quad f'(x) = 3e^{2x}$$

$$\begin{aligned}\therefore f(x) &= \int 3e^{2x} dx \\ &= \frac{3}{2}e^{2x} + c\end{aligned}$$

But  $f(0) = 2$ , so  $\frac{3}{2} + c = 2$

$$\therefore c = \frac{1}{2}$$

$$\therefore f(x) = \frac{3}{2}e^{2x} + \frac{1}{2}$$

$$10 \quad f'(x) = a \cos 3x$$

$$\begin{aligned}\therefore f(x) &= \int a \cos 3x dx \\ &= \frac{a}{3} \sin 3x + c\end{aligned}$$

But  $f(0) = -1$ , so  $c = -1$

$$\therefore f(x) = \frac{a}{3} \sin 3x - 1$$

and  $f(\frac{\pi}{4}) = 1$

$$\therefore \frac{a}{3} \sin \frac{3\pi}{4} - 1 = 1$$

$$\therefore \frac{a}{3} \times \frac{1}{\sqrt{2}} = 2$$

$$\therefore \frac{a}{3\sqrt{2}} = 2$$

$$\therefore a = 6\sqrt{2}$$

$$\therefore f(x) = \frac{6\sqrt{2}}{3} \sin 3x - 1$$

$$= 2\sqrt{2} \sin 3x - 1$$

$$11 \quad a \quad \int \frac{x^2 - 7}{x} dx$$

$$= \int \left(x - \frac{7}{x}\right) dx$$

$$= \frac{1}{2}x^2 - 7 \ln |x| + c$$

$$\begin{aligned}c \quad & \int ((4 - 3x)^3 + \sin(-2x)) dx \\ &= \left(\frac{1}{-3}\right) \frac{(4 - 3x)^4}{4} + \left(\frac{1}{-2}\right)(-\cos(-2x)) + c \\ &= -\frac{1}{12}(4 - 3x)^4 + \frac{1}{2} \cos(-2x) + c\end{aligned}$$

$$\begin{aligned}b \quad & \int \left(e^{2x-3} - \frac{2}{3x-1}\right) dx \\ &= \int e^{2x-3} dx - \frac{2}{3} \int \frac{3}{3x-1} dx \\ &= \frac{1}{2}e^{2x-3} - \frac{2}{3} \ln |3x-1| + c\end{aligned}$$

$$\begin{aligned}d \quad & \int \left(\frac{3}{1+x^2} - \frac{2}{5-x}\right) dx \\ &= 3 \int \frac{1}{1+x^2} dx + 2 \int \frac{-1}{5-x} dx \\ &= 3 \arctan x + 2 \ln |5-x| + c\end{aligned}$$

$$\begin{aligned}12 \quad a \quad & \int (1 - \sin x)^2 dx \\ &= \int (1 - 2 \sin x + \sin^2 x) dx \\ &= \int \left(1 - 2 \sin x + \frac{1}{2} - \frac{1}{2} \cos 2x\right) dx \\ & \quad \{ \sin^2 x = \frac{1}{2} - \frac{1}{2} \cos 2x \} \\ &= \int \left(\frac{3}{2} - 2 \sin x - \frac{1}{2} \cos 2x\right) dx \\ &= \frac{3}{2}x + 2 \cos x - \frac{1}{4} \sin 2x + c\end{aligned}$$

$$\begin{aligned}b \quad & \sqrt{x^2 - 4} = (x^2 - 4)^{\frac{1}{2}} \\ \therefore \frac{d}{dx} \left( \sqrt{x^2 - 4} \right) &= \frac{1}{2}(x^2 - 4)^{-\frac{1}{2}} \times 2x \\ &= \frac{x}{\sqrt{x^2 - 4}} \\ \therefore \int \frac{x}{\sqrt{x^2 - 4}} dx &= \sqrt{x^2 - 4} + c\end{aligned}$$



$$\text{c } u = x^2 + \frac{\pi}{3} \quad \therefore \quad \frac{du}{dx} = 2x$$

$$\begin{aligned} \therefore \int x \sin\left(x^2 + \frac{\pi}{3}\right) dx &= \frac{1}{2} \int 2x \sin\left(x^2 + \frac{\pi}{3}\right) dx \\ &= \frac{1}{2} \int \sin u \frac{du}{dx} dx \\ &= \frac{1}{2} \int \sin u du \\ &= \frac{1}{2}(-\cos u) + c \\ &= -\frac{1}{2} \cos\left(x^2 + \frac{\pi}{3}\right) + c \end{aligned}$$

13 a

$$\begin{aligned} &\frac{4}{x-2} + \frac{3}{x+1} \\ &= \frac{4(x+1)}{(x-2)(x+1)} + \frac{3(x-2)}{(x-2)(x+1)} \\ &= \frac{4x+4+3x-6}{(x-2)(x+1)} \\ &= \frac{7x-2}{(x-2)(x+1)} \end{aligned}$$

b

$$\begin{aligned} &\int \frac{7x-2}{x^2-x-2} dx \\ &= \int \frac{7x-2}{(x-2)(x+1)} dx \\ &= \int \left( \frac{4}{x-2} + \frac{3}{x+1} \right) dx \\ &= 4 \ln|x-2| + 3 \ln|x+1| + c \end{aligned}$$

14 a  $x^2 - 2x - 8 = (x+2)(x-4)$ 

$$\begin{aligned} \text{Let } \frac{x}{x^2-2x-8} &= \frac{A}{x+2} + \frac{B}{x-4} \\ \therefore x &= A(x-4) + B(x+2) \end{aligned}$$

Substituting  $x = 4$ ,  $4 = B(4+2)$ 

$$\therefore 4 = 6B$$

$$\therefore B = \frac{2}{3}$$

Substituting  $x = -2$ ,  $-2 = A(-2-4)$ 

$$\therefore -2 = -6A$$

$$\therefore A = \frac{1}{3}$$

$$\therefore \frac{x}{x^2-2x-8} = \frac{1}{3(x+2)} + \frac{2}{3(x-4)}$$

$$\begin{aligned} \therefore \int \frac{x}{x^2-2x-8} dx &= \int \left( \frac{1}{3(x+2)} + \frac{2}{3(x-4)} \right) dx \\ &= \frac{1}{3} \ln|x+2| + \frac{2}{3} \ln|x-4| + c \end{aligned}$$

**b**  $2x^2 - 3x - 5 = (2x - 5)(x + 1)$

Let  $\frac{3x - 4}{2x^2 - 3x - 5} = \frac{A}{2x - 5} + \frac{B}{x + 1}$

$$\therefore 3x - 4 = A(x + 1) + B(2x - 5)$$

Substituting  $x = -1$ ,  $3(-1) - 4 = B(2(-1) - 5)$

$$\therefore -7 = -7B$$

$$\therefore B = 1$$

Substituting  $x = \frac{5}{2}$ ,  $3\left(\frac{5}{2}\right) - 4 = A\left(\frac{5}{2} + 1\right)$

$$\therefore \frac{7}{2} = \frac{7}{2}A$$

$$\therefore A = 1$$

$$\therefore \frac{3x - 4}{2x^2 - 3x - 5} = \frac{1}{2x - 5} + \frac{1}{x + 1}$$

$$\begin{aligned} \therefore \int \frac{3x - 4}{2x^2 - 3x - 5} dx &= \int \left( \frac{1}{2x - 5} + \frac{1}{x + 1} \right) dx \\ &= \frac{1}{2} \ln |2x - 5| + \ln |x + 1| + c \end{aligned}$$

**15**  $\frac{d}{dx} (\tan^3 x) = 3 \tan^2 x (\sec^2 x)$   
 $= 3(\sec^2 x - 1) \sec^2 x$   
 $= 3 \sec^4 x - \sec^2 x$

$$\therefore \int (3 \sec^4 x - \sec^2 x) dx = \tan^3 x + c$$

$$\therefore 3 \int \sec^4 x dx - \int \sec^2 x dx = \tan^3 x + c$$

$$\therefore 3 \int \sec^4 x dx - \tan x = \tan^3 x + c$$

$$\therefore 3 \int \sec^4 x dx = \tan^3 x + \tan x + c$$

$$\therefore \int \sec^4 x dx = \frac{1}{3} \tan^3 x + \frac{1}{3} \tan x + c$$

**16 a**  $\int \frac{x + 2}{x^2 + 4x} dx$   
 $= \int \frac{1}{u} \left( \frac{1}{2} \frac{du}{dx} \right) dx$   
 $\{u = x^2 + 4x, \quad \frac{du}{dx} = 2x + 4\}$   
 $= \frac{1}{2} \int \frac{1}{u} du$   
 $= \frac{1}{2} \ln |u| + c$   
 $= \frac{1}{2} \ln |x^2 + 4x| + c$

**b**  $\int 2xe^{x^2-1} dx$   
 $= \int e^u \frac{du}{dx} dx$   
 $\{u = x^2 - 1, \quad \frac{du}{dx} = 2x\}$   
 $= \int e^u du$   
 $= e^u + c$   
 $= e^{x^2-1} + c$

$$\text{c} \quad \int \sin^9 x \cos x \, dx$$

$$= \int u^9 \frac{du}{dx} \, dx$$

$$\{u = \sin x, \quad \frac{du}{dx} = \cos x\}$$

$$= \int u^9 \, du$$

$$= \frac{1}{10} u^{10} + c$$

$$= \frac{1}{10} \sin^{10} x + c$$

$$\text{d} \quad \int \tan 2x \, dx$$

$$= \int \frac{\sin 2x}{\cos 2x} \, dx$$

$$= \int \frac{1}{u} \left(-\frac{1}{2} \frac{du}{dx}\right) \, dx$$

$$\{u = \cos 2x, \quad \frac{du}{dx} = -2 \sin 2x\}$$

$$= -\frac{1}{2} \int \frac{1}{u} \, du$$

$$= -\frac{1}{2} \ln |u| + c$$

$$= -\frac{1}{2} \ln |\cos 2x| + c$$

$$\text{e} \quad \int e^{\sin x} \cos x \, dx$$

$$= \int e^u \frac{du}{dx} \, dx$$

$$\{u = \sin x, \quad \frac{du}{dx} = \cos x\}$$

$$= \int e^u \, du$$

$$= e^u + c$$

$$= e^{\sin x} + c$$

$$\text{f} \quad \int \frac{\arcsin x}{\sqrt{1-x^2}} \, dx$$

$$= \int u \frac{du}{dx} \, dx$$

$$\{u = \arcsin x, \quad \frac{du}{dx} = \frac{1}{\sqrt{1-x^2}}\}$$

$$= \int u \, du$$

$$= \frac{1}{2} u^2 + c$$

$$= \frac{1}{2} (\arcsin x)^2 + c$$

$$\text{17 a} \quad \text{Let } u = 4 - x \quad \therefore \frac{du}{dx} = -1$$

$$\therefore \int x^2 \sqrt{4-x} \, dx = \int (4-u)^2 \sqrt{u} \left(-\frac{du}{dx}\right) \, dx$$

$$= - \int (4-u)^2 \sqrt{u} \, du$$

$$= - \int (16 - 8u + u^2) u^{\frac{1}{2}} \, du$$

$$= - \int (16u^{\frac{1}{2}} - 8u^{\frac{3}{2}} + u^{\frac{5}{2}}) \, du$$

$$= - \left( \frac{16u^{\frac{3}{2}}}{\frac{3}{2}} - \frac{8u^{\frac{5}{2}}}{\frac{5}{2}} + \frac{u^{\frac{7}{2}}}{\frac{7}{2}} \right) + c$$

$$= -\frac{32}{3} u^{\frac{3}{2}} + \frac{16}{5} u^{\frac{5}{2}} - \frac{2}{7} u^{\frac{7}{2}} + c$$

$$= -\frac{32}{3} (4-x)^{\frac{3}{2}} + \frac{16}{5} (4-x)^{\frac{5}{2}} - \frac{2}{7} (4-x)^{\frac{7}{2}} + c$$

**b** Let  $u = 2 - x \quad \therefore \frac{du}{dx} = -1$

$$\begin{aligned}
 \therefore \int \frac{x^3}{(2-x)^3} dx &= \int \frac{(2-u)^3}{u^3} \left(-\frac{du}{dx}\right) dx \\
 &= - \int \frac{(2-u)^3}{u^3} du \\
 &= - \int \frac{2^3 + 3(2)^2(-u) + 3(2)(-u)^2 + (-u)^3}{u^3} du \quad \{\text{binomial theorem}\} \\
 &= - \int \frac{8 - 12u + 6u^2 - u^3}{u^3} du \\
 &= - \int \left( \frac{8}{u^3} - \frac{12}{u^2} + \frac{6}{u} - 1 \right) du \\
 &= \int \left( 1 - \frac{6}{u} + \frac{12}{u^2} - \frac{8}{u^3} \right) du \\
 &= \int \left( 1 - \frac{6}{u} + 12u^{-2} - 8u^{-3} \right) du \\
 &= u - 6 \ln |u| + \frac{12u^{-1}}{-1} - \frac{8u^{-2}}{-2} + c \\
 &= u - 6 \ln |u| - \frac{12}{u} + \frac{4}{u^2} + c \\
 &= 2 - x - 6 \ln |2 - x| - \frac{12}{2 - x} + \frac{4}{(2 - x)^2} + c \\
 &= -x - 6 \ln |2 - x| - \frac{12}{2 - x} + \frac{4}{(2 - x)^2} + c
 \end{aligned}$$

**c** Let  $u = \sqrt{x+2}$ , so  $u^2 = x+2$

$$\therefore 2u \frac{du}{dx} = 1$$

$$\begin{aligned}
 \therefore \int \frac{x}{1 + \sqrt{x+2}} dx &= \int \left( \frac{u^2 - 2}{1 + u} \right) 2u \frac{du}{dx} dx \\
 &= \int \frac{2u^3 - 4u}{u + 1} du \\
 &= \int \left( 2u^2 - 2u - 2 + \frac{2}{u + 1} \right) du \\
 &= \frac{2u^3}{3} - \frac{2u^2}{2} - 2u + 2 \ln |u + 1| + c \\
 &= \frac{2}{3}(x+2)^{\frac{3}{2}} - (x+2) - 2\sqrt{x+2} + 2 \ln(\sqrt{x+2} + 1) + c \\
 &= \frac{2}{3}(x+2)^{\frac{3}{2}} - x - 2\sqrt{x+2} + 2 \ln(\sqrt{x+2} + 1) + c
 \end{aligned}$$

$$\begin{array}{r}
 \phantom{u+1} \overline{2u^2 - 2u - 2} \\
 u+1 \overline{2u^3 + 0u^2 - 4u + 0} \\
 \underline{-(2u^3 + 2u^2)} \phantom{+ 0} \downarrow \\
 \phantom{u+1} -2u^2 - 4u \phantom{+ 0} \downarrow \\
 \underline{-(-2u^2 - 2u)} \phantom{+ 0} \downarrow \\
 \phantom{u+1} \phantom{-2u^2 - 4u} -2u + 0 \phantom{+ 0} \downarrow \\
 \underline{-(-2u - 2)} \\
 \phantom{u+1} \phantom{-2u^2 - 4u} \phantom{-2u + 0} 2
 \end{array}$$



**18 a**  $\int x^2 \ln(x^3) dx = \frac{1}{3}x^3 \ln(x^3) - \int \frac{3}{x} \times \frac{1}{3}x^3 dx$   $\left\{ \begin{array}{ll} u = \ln(x^3) & v' = x^2 \\ u' = \frac{1}{x^3} \times 3x^2 & v = \frac{1}{3}x^3 \\ & = \frac{3}{x} \end{array} \right.$

$$= \frac{1}{3}x^3 \ln(x^3) - \int x^2 dx$$

$$= \frac{1}{3}x^3 \ln(x^3) - \frac{1}{3}x^3 + c$$

$$= \frac{1}{3}x^3 (\ln(x^3) - 1) + c$$

**b**  $\int e^{2x} \sin x dx$

$$= \frac{1}{2}e^{2x} \sin x - \int \cos x \times \frac{1}{2}e^{2x} dx$$

$$= \frac{1}{2}e^{2x} \sin x - \frac{1}{2} \int e^{2x} \cos x dx$$

$$= \frac{1}{2}e^{2x} \sin x - \frac{1}{2} \left[ \frac{1}{2}e^{2x} \cos x - \int (-\sin x) \times \frac{1}{2}e^{2x} dx \right]$$

$$\therefore \int e^{2x} \sin x dx = \frac{1}{2}e^{2x} \sin x - \frac{1}{4}e^{2x} \cos x - \frac{1}{4} \int e^{2x} \sin x dx$$

$$\therefore \frac{5}{4} \int e^{2x} \sin x dx = \frac{1}{2}e^{2x} \sin x - \frac{1}{4}e^{2x} \cos x + c$$

$$\therefore \int e^{2x} \sin x dx = \frac{2}{5}e^{2x} \sin x - \frac{1}{5}e^{2x} \cos x + c$$

$$= \frac{1}{5}e^{2x} (2 \sin x - \cos x) + c$$

**19 a**  $\int \frac{x}{x^2 - 9} dx = \int \frac{1}{u} \left( \frac{1}{2} \frac{du}{dx} \right) dx \quad \{u = x^2 - 9, \quad \frac{du}{dx} = 2x\}$

$$= \frac{1}{2} \int \frac{1}{u} du$$

$$= \frac{1}{2} \ln |u| + c$$

$$= \frac{1}{2} \ln |x^2 - 9| + c$$

**b**  $\frac{x}{x^2 - 9} = \frac{x}{(x+3)(x-3)}$

Let  $\frac{x}{x^2 - 9} = \frac{A}{x+3} + \frac{B}{x-3}$

$$\therefore x = A(x-3) + B(x+3)$$

Substituting  $x = 3$ ,  $3 = 6B$

$$\therefore B = \frac{1}{2}$$

Substituting  $x = -3$ ,  $-3 = -6A$

$$\therefore A = \frac{1}{2}$$

$$\therefore \frac{x}{x^2 - 9} = \frac{1}{2(x+3)} + \frac{1}{2(x-3)}$$

$$\therefore \int \frac{x}{x^2 - 9} dx = \frac{1}{2} \ln |x+3| + \frac{1}{2} \ln |x-3| + c$$

**c** Let  $x = 3 \sin t$ ,  $\frac{dx}{dt} = 3 \cos t$

$$\begin{aligned} \therefore \int \frac{x}{x^2 - 9} dx &= \int \frac{3 \sin t}{9 \sin^2 t - 9} \times 3 \cos t dt \\ &= \int \frac{3 \sin t}{-9(1 - \sin^2 t)} \times 3 \cos t dt \\ &= \int \frac{3 \sin t}{-9 \cos^2 t} \times 3 \cos t dt \\ &= \int -\tan t dt \\ &= \ln |\cos t| + c \end{aligned}$$

$x = 3 \sin t$ , so  $t = \arcsin \frac{x}{3}$

$$\therefore \int \frac{x}{x^2 - 9} dx = \ln \left| \cos \left( \arcsin \frac{x}{3} \right) \right| + c$$

**20 a**

$$\begin{aligned} &\int e^u \sin u du \\ &= e^u(-\cos u) - \int e^u(-\cos u) du \quad \leftarrow \begin{cases} a = e^u & b' = \sin u \\ a' = e^u & b = -\cos u \end{cases} \\ &= -e^u \cos u + \int e^u \cos u du \\ &= -e^u \cos u + \left[ e^u \sin u - \int e^u \sin u du \right] \quad \leftarrow \begin{cases} a = e^u & b' = \cos u \\ a' = e^u & b = \sin u \end{cases} \\ \therefore \int e^u \sin u du &= -e^u \cos u + e^u \sin u - \int e^u \sin u du \\ \therefore 2 \int e^u \sin u du &= -e^u \cos u + e^u \sin u + c \\ \therefore \int e^u \sin u du &= \frac{1}{2} e^u (\sin u - \cos u) + c \end{aligned}$$

**b**  $u = \ln x \quad \therefore \frac{du}{dx} = \frac{1}{x}$

Now,  $\int e^u \sin u du = \frac{1}{2} e^u (\sin u - \cos u) + c \quad \{\text{from a}\}$

$$\therefore \int e^{\ln x} \sin(\ln x) \times \frac{1}{x} dx = \frac{1}{2} e^{\ln x} (\sin(\ln x) - \cos(\ln x)) + c$$

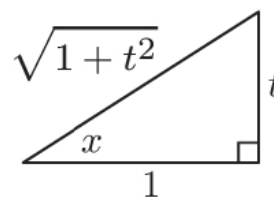
$$\therefore \int x \sin(\ln x) \times \frac{1}{x} dx = \frac{1}{2} x (\sin(\ln x) - \cos(\ln x)) + c$$

$$\therefore \int \sin(\ln x) dx = \frac{1}{2} x (\sin(\ln x) - \cos(\ln x)) + c$$

**21** Let  $t = \tan x$

$$\therefore \cos x = \frac{1}{\sqrt{1+t^2}}$$

Now,  $x = \arctan t \quad \therefore \frac{dx}{dt} = \frac{1}{1+t^2}$



$$\begin{aligned} \therefore \int \frac{1}{1+\cos^2 x} dx &= \int \frac{1}{1+\frac{1}{1+t^2}} \times \frac{1}{1+t^2} dt \\ &= \int \frac{1+t^2}{1+t^2+1} \times \frac{1}{1+t^2} dt \\ &= \int \frac{1}{2+t^2} dt \\ &= \frac{1}{\sqrt{2}} \int \frac{\sqrt{2}}{(\sqrt{2})^2+t^2} dt \\ &= \frac{1}{\sqrt{2}} \arctan\left(\frac{t}{\sqrt{2}}\right) + c \\ &= \frac{1}{\sqrt{2}} \arctan\left(\frac{1}{\sqrt{2}} \tan x\right) + c \end{aligned}$$

## REVIEW SET 21B

**1**  $\frac{d}{dx}(e^{-2x}) = e^{-2x} \times (-2)$   
 $= -2e^{-2x}$   
 $\therefore \int -2e^{-2x} dx = e^{-2x} + c$   
 $\therefore -2 \int e^{-2x} dx = e^{-2x} + c$   
 $\therefore \int e^{-2x} dx = -\frac{1}{2}e^{-2x} + c$

**2**  $\frac{d}{dx}(\ln(2x+1)) = \frac{2}{2x+1}$   
 $\therefore \int \frac{2}{2x+1} dx = \ln|2x+1| + c$   
 $\therefore 2 \int \frac{1}{2x+1} dx = \ln|2x+1| + c$   
 $\therefore \int \frac{1}{2x+1} dx = \frac{1}{2} \ln|2x+1| + c$

**3 a**  $\int \frac{x^2-2}{x^2} dx$   
 $= \int \left(1 - \frac{2}{x^2}\right) dx$   
 $= \int (1 - 2x^{-2}) dx$   
 $= x - \frac{2x^{-1}}{-1} + c$   
 $= x + \frac{2}{x} + c$

**b**  $\int (3x-4)^2 dx$   
 $= \int (9x^2 - 24x + 16) dx$   
 $= \frac{9x^3}{3} - \frac{24x^2}{2} + 16x + c$   
 $= 3x^3 - 12x^2 + 16x + c$

**c**  $\int (4-2x^2) dx$   
 $= 4x - \frac{2}{3}x^3 + c$

$$\begin{aligned}
 \text{4 a } \int (x^{\frac{1}{3}} + 3) dx \\
 &= \frac{x^{\frac{4}{3}}}{\frac{4}{3}} + 3x + c \\
 &= \frac{3}{4}x^{\frac{4}{3}} + 3x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{b } \int (3x^2 - 2) dx \\
 &= \frac{3x^3}{3} - 2x + c \\
 &= x^3 - 2x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{c } \int (3 + 2x)^2 dx \\
 &= \int (9 + 12x + 4x^2) dx \\
 &= 9x + \frac{12x^2}{2} + \frac{4x^3}{3} + c \\
 &= 9x + 6x^2 + \frac{4}{3}x^3 + c
 \end{aligned}$$

$$\begin{aligned}
 \text{d } \int \left( \sqrt{x} - \frac{1}{\sqrt{x}} \right)^2 dx \\
 &= \int \left( x - 2 + \frac{1}{x} \right) dx \\
 &= \frac{1}{2}x^2 - 2x + \ln|x| + c
 \end{aligned}$$

$$\begin{aligned}
 \text{e } \int \frac{5}{x \ln 2} dx \\
 &= 5 \log_2 |x| + c
 \end{aligned}$$

$$\begin{aligned}
 \text{f } \int \frac{-3}{\sqrt{1-x^2}} dx \\
 &= 3 \int \frac{-1}{\sqrt{1-x^2}} dx \\
 &= 3 \arccos x + c
 \end{aligned}$$

$$\text{5 } f'(x) = x^2 - 3x + 2$$

$$\begin{aligned}
 \therefore f(x) &= \int (x^2 - 3x + 2) dx \\
 &= \frac{x^3}{3} - \frac{3x^2}{2} + 2x + c \\
 &= \frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{But } f(1) &= 3, \text{ so } \frac{1}{3} - \frac{3}{2} + 2 + c = 3 \\
 \therefore c &= \frac{13}{6}
 \end{aligned}$$

$$\therefore f(x) = \frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x + \frac{13}{6}$$

$$\begin{aligned}
 \text{6 a } \frac{dy}{dx} &= (x^2 - 1)^2 = x^4 - 2x^2 + 1 \\
 \therefore y &= \int (x^4 - 2x^2 + 1) dx \\
 &= \frac{1}{5}x^5 - \frac{2}{3}x^3 + x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{b } \frac{dy}{dx} &= 400 - 20x^{-\frac{1}{2}} \\
 \therefore y &= \int (400 - 20x^{-\frac{1}{2}}) dx \\
 &= 400x - \frac{20x^{\frac{1}{2}}}{\frac{1}{2}} + c \\
 &= 400x - 40x^{\frac{1}{2}} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{c } \frac{dy}{dx} &= \frac{5}{7-3x} + 7^x \\
 &= \left(-\frac{5}{3}\right) \times \frac{-3}{7-3x} + 7^x \\
 \therefore y &= \int \left( \left(-\frac{5}{3}\right) \times \frac{-3}{7-3x} + 7^x \right) dx \\
 &= -\frac{5}{3} \ln|7-3x| + \frac{7^x}{\ln 7} + c
 \end{aligned}$$



$$\begin{aligned}
 \text{7 a} \quad & \int (2x^3 - 5x + 7) dx \\
 &= \frac{2x^4}{4} - \frac{5x^2}{2} + 7x + c \\
 &= \frac{1}{2}x^4 - \frac{5}{2}x^2 + 7x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & \int \left(3x - \frac{1}{x}\right) dx \\
 &= \frac{3}{2}x^2 - \ln|x| + c
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & \int (1 - x^2)^3 dx = \int (1^3 + 3(1)^2(-x^2) + 3(1)(-x^2)^2 + (-x^2)^3) dx \quad \{\text{binomial theorem}\} \\
 &= \int (1 - 3x^2 + 3x^4 - x^6) dx \\
 &= x - \frac{3x^3}{3} + \frac{3x^5}{5} - \frac{x^7}{7} + c \\
 &= x - x^3 + \frac{3}{5}x^5 - \frac{1}{7}x^7 + c
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad & \int (2e^{-x} + 3) dx \\
 &= 2\left(\frac{1}{-1}\right)e^{-x} + 3x + c \\
 &= -2e^{-x} + 3x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & \int 4 \cos 2x dx \\
 &= 4 \times \frac{1}{2} \sin 2x + c \\
 &= 2 \sin 2x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & \int (3 + e^{2x-1})^2 dx = \int (9 + 6e^{2x-1} + (e^{2x-1})^2) dx \\
 &= \int (9 + 6e^{2x-1} + e^{4x-2}) dx \\
 &= 9x + 6\left(\frac{1}{2}\right)e^{2x-1} + \frac{1}{4}e^{4x-2} + c \\
 &= 9x + 3e^{2x-1} + \frac{1}{4}e^{4x-2} + c
 \end{aligned}$$

$$\text{8} \quad f'(x) = \frac{2}{x} - 1$$

$$\begin{aligned}
 \therefore f(x) &= \int \left(\frac{2}{x} - 1\right) dx \\
 &= 2 \ln|x| - x + c
 \end{aligned}$$

$$\text{But } f(2) = e, \text{ so } 2 \ln 2 - 2 + c = e$$

$$\therefore c = e + 2 - 2 \ln 2$$

$$\therefore f(x) = 2 \ln|x| - x + e + 2 - 2 \ln 2$$

$$9 \quad \frac{dy}{dx} = ax^2 + b\sqrt{x-1} = ax^2 + b(x-1)^{\frac{1}{2}}$$

$$\therefore y = \int (ax^2 + b(x-1)^{\frac{1}{2}}) dx$$

$$= \frac{ax^3}{3} + \frac{b(x-1)^{\frac{3}{2}}}{\frac{3}{2}} + c$$

$$= \frac{1}{3}ax^3 + \frac{2}{3}b(x-1)^{\frac{3}{2}} + c$$

But the curve passes through (1, 4), (2, 4), and (5, 1)

$$\therefore \frac{1}{3}a(1)^3 + \frac{2}{3}b(1-1)^{\frac{3}{2}} + c = 4$$

$$\therefore \frac{1}{3}a + c = 4 \quad \dots (1)$$

$$\text{and } \frac{1}{3}a(2)^3 + \frac{2}{3}b(2-1)^{\frac{3}{2}} + c = 4$$

$$\therefore \frac{8}{3}a + \frac{2}{3}b + c = 4 \quad \dots (2)$$

$$\text{and } \frac{1}{3}a(5)^3 + \frac{2}{3}b(5-1)^{\frac{3}{2}} + c = 1$$

$$\therefore \frac{125}{3}a + \frac{2}{3}b(4)^{\frac{3}{2}} + c = 1$$

$$\therefore \frac{125}{3}a + \frac{16}{3}b + c = 1 \quad \dots (3)$$

We solve (1), (2), and (3) simultaneously using technology.

$$\therefore a = -\frac{9}{68}, \quad b = \frac{63}{136}, \quad c = \frac{275}{68}$$

$$\therefore y = \frac{1}{3}\left(-\frac{9}{68}\right)x^3 + \frac{2}{3}\left(\frac{63}{136}\right)(x-1)^{\frac{3}{2}} + \frac{275}{68}$$

$$\therefore y = -\frac{3}{68}x^3 + \frac{21}{68}(x-1)^{\frac{3}{2}} + \frac{275}{68}$$

Math Rad Norm1 d/c Real  
 $a_n X + b_n Y + c_n Z = d_n$   
 X  $-0.132$   
 Y  $0.4632$   
 Z  $4.0441$   
 $-\frac{9}{68}$   
 REPEAT

$$10 \quad f'(x) = \frac{3}{\sqrt{4-3x}} = 3(4-3x)^{-\frac{1}{2}}$$

$$\therefore f(x) = \int 3(4-3x)^{-\frac{1}{2}} dx$$

$$= 3\left(\frac{1}{-3}\right) \frac{(4-3x)^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$= -2\sqrt{4-3x} + c$$

$$\text{But } f(-4) = 0, \text{ so } -2\sqrt{4-3(-4)} + c = 0$$

$$\therefore -2\sqrt{16} + c = 0$$

$$\therefore -8 + c = 0$$

$$\therefore c = 8$$

$$\therefore f(x) = -2\sqrt{4-3x} + 8$$

$$\begin{aligned}
 \mathbf{11} \quad \mathbf{a} \quad \int \frac{1}{3-2x} dx &= -\frac{1}{2} \int \frac{-2}{3-2x} dx & \mathbf{b} \quad \int \frac{4}{5x+1} dx &= 4 \int \frac{1}{5x+1} dx \\
 &= -\frac{1}{2} \ln |3-2x| + c & &= \frac{4}{5} \ln |5x+1| + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad (\sin x - \cos x)^2 &= \sin^2 x - 2 \sin x \cos x + \cos^2 x \\
 &= 1 - \sin 2x \quad \{ \sin^2 x + \cos^2 x = 1, \quad \sin 2x = 2 \sin x \cos x \}
 \end{aligned}$$

$$\begin{aligned}
 \therefore \int (\sin x - \cos x)^2 dx &= \int (1 - \sin 2x) dx \\
 &= x + \frac{1}{2} \cos 2x + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{12} \quad \mathbf{a} \quad \text{Let } \frac{x+20}{(x-6)(2x+1)} &= \frac{A}{x-6} + \frac{B}{2x+1} \\
 \therefore x+20 &= A(2x+1) + B(x-6)
 \end{aligned}$$

$$\begin{aligned}
 \text{Substituting } x = -\frac{1}{2}, \quad -\frac{1}{2} + 20 &= B\left(-\frac{1}{2} - 6\right) \\
 \therefore \frac{39}{2} &= -\frac{13}{2}B \\
 \therefore B &= -3
 \end{aligned}$$

$$\begin{aligned}
 \text{Substituting } x = 6, \quad 6 + 20 &= A(2(6) + 1) \\
 \therefore 26 &= 13A \\
 \therefore A &= 2
 \end{aligned}$$

$$\therefore \frac{x+20}{(x-6)(2x+1)} = \frac{2}{x-6} - \frac{3}{2x+1}$$

$$\begin{aligned}
 \mathbf{b} \quad \int \frac{x+20}{2x^2-11x-6} dx &= \int \frac{x+20}{(x-6)(2x+1)} dx \\
 &= \int \left( \frac{2}{x-6} - \frac{3}{2x+1} \right) dx \quad \{\text{using } \mathbf{a}\} \\
 &= 2 \ln |x-6| - \frac{3}{2} \ln |2x+1| + c
 \end{aligned}$$

$$\mathbf{13} \quad x^2 + x - 2 = (x+2)(x-1)$$

$$\begin{aligned}
 \text{Let } \frac{x+5}{x^2+x-2} &= \frac{A}{x+2} + \frac{B}{x-1} \\
 \therefore x+5 &= A(x-1) + B(x+2)
 \end{aligned}$$

$$\begin{aligned}
 \text{Substituting } x = 1, \quad 1 + 5 &= B(1+2) \\
 \therefore 6 &= 3B \\
 \therefore B &= 2
 \end{aligned}$$

$$\begin{aligned}
 \text{Substituting } x = -2, \quad -2 + 5 &= A(-2-1) \\
 \therefore 3 &= -3A \\
 \therefore A &= -1
 \end{aligned}$$

$$\begin{aligned}
 \therefore \frac{x+5}{x^2+x-2} &= \frac{-1}{x+2} + \frac{2}{x-1} \\
 \therefore \int \frac{x+5}{x^2+x-2} dx &= \int \left( -\frac{1}{x+2} + \frac{2}{x-1} \right) dx \\
 &= -\ln |x+2| + 2 \ln |x-1| + c
 \end{aligned}$$

$$14 \quad \frac{1}{a^2 - x^2} = \frac{1}{(a+x)(a-x)}$$

$$\text{Let } \frac{1}{a^2 - x^2} = \frac{A}{a+x} + \frac{B}{a-x}$$

$$\therefore 1 = A(a-x) + B(a+x)$$

$$\text{Substituting } x = a, \quad 1 = B(a+a)$$

$$\therefore 1 = 2aB$$

$$\therefore B = \frac{1}{2a}$$

$$\text{Substituting } x = -a, \quad 1 = A(a - (-a))$$

$$\therefore 1 = 2aA$$

$$\therefore A = \frac{1}{2a}$$

$$\therefore \frac{1}{a^2 - x^2} = \frac{\frac{1}{2a}}{a+x} + \frac{\frac{1}{2a}}{a-x}$$

$$\begin{aligned} \therefore \int \frac{1}{a^2 - x^2} dx &= \int \left( \frac{\frac{1}{2a}}{a+x} + \frac{\frac{1}{2a}}{a-x} \right) dx \\ &= \frac{1}{2a} \int \left( \frac{1}{a+x} + \frac{1}{a-x} \right) dx \\ &= \frac{1}{2a} (\ln |a+x| - \ln |a-x|) + c \\ &= \frac{1}{2a} \ln \left| \frac{a+x}{a-x} \right| + c \end{aligned}$$

$$15 \quad \text{a} \quad \frac{x^3 - 3x^2 + 5x + 4}{x-3} = x^2 + 5 + \frac{19}{x-3}$$

$$\begin{array}{r} x^2 + 0x + 5 \\ x-3 \overline{) \begin{array}{r} x^3 - 3x^2 + 5x + 4 \\ -(x^3 - 3x^2) \phantom{+ 5x + 4} \\ \hline 0x^2 + 5x \phantom{+ 4} \\ -(0x^2 + 0x) \phantom{+ 4} \\ \hline 5x + 4 \\ -(5x - 15) \\ \hline 19 \end{array}} \end{array}$$

$$\begin{aligned} \text{b} \quad \int \frac{x^3 - 3x^2 + 5x + 4}{x-3} dx &= \int \left( x^2 + 5 + \frac{19}{x-3} \right) dx \\ &= \frac{1}{3}x^3 + 5x + 19 \ln |x-3| + c \end{aligned}$$



$$\begin{aligned} \mathbf{16} \quad \mathbf{a} \quad & \int (3x^2 + x)^2 (6x + 1) \, dx \\ &= \int u^2 \frac{du}{dx} \, dx \\ & \quad \{u = 3x^2 + x, \quad \frac{du}{dx} = 6x + 1\} \\ &= \int u^2 \, du \\ &= \frac{1}{3} u^3 + c \\ &= \frac{1}{3} (3x^2 + x)^3 + c \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad & \int \frac{2x}{\sqrt{x^2 - 5}} \, dx \\ &= \int 2x(x^2 - 5)^{-\frac{1}{2}} \, dx \\ &= \int u^{-\frac{1}{2}} \frac{du}{dx} \, dx \\ & \quad \{u = x^2 - 5, \quad \frac{du}{dx} = 2x\} \\ &= \int u^{-\frac{1}{2}} \, du \\ &= \frac{u^{\frac{1}{2}}}{\frac{1}{2}} + c \\ &= 2\sqrt{u} + c \\ &= 2\sqrt{x^2 - 5} + c \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad & \int \frac{\sin x}{\cos^4 x} \, dx \\ &= \int u^{-4} \left(-\frac{du}{dx}\right) \, dx \\ & \quad \{u = \cos x, \quad \frac{du}{dx} = -\sin x\} \\ &= -\int u^{-4} \, du \\ &= -\frac{u^{-3}}{-3} + c \\ &= \frac{1}{3 \cos^3 x} + c \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad & \int 4xe^{-x^2} \, dx \\ &= \int e^u \left(-2 \frac{du}{dx}\right) \, dx \\ & \quad \{u = -x^2, \quad \frac{du}{dx} = -2x\} \\ &= -2 \int e^u \, du \\ &= -2e^u + c \\ &= -2e^{-x^2} + c \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & \int \sin^3 x \, dx \\
 &= \int \sin^2 x \sin x \, dx \\
 &= \int (1 - \cos^2 x) \sin x \, dx \\
 &= \int (1 - u^2) \left(-\frac{du}{dx}\right) dx \\
 &\quad \{u = \cos x, \quad \frac{du}{dx} = -\sin x\} \\
 &= -\int (1 - u^2) \, du \\
 &= -(u - \frac{1}{3}u^3 + c) \\
 &= -u + \frac{1}{3}u^3 + c \\
 &= -\cos x + \frac{1}{3}\cos^3 x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & \int \sin x \cos 2x \, dx \\
 &= \int \sin x (2\cos^2 x - 1) \, dx \\
 &= \int (1 - 2\cos^2 x) \times (-\sin x) \, dx \\
 &= \int (1 - 2u^2) \frac{du}{dx} \, dx \\
 &\quad \{u = \cos x, \quad \frac{du}{dx} = -\sin x\} \\
 &= \int (1 - 2u^2) \, du \\
 &= u - \frac{2}{3}u^3 + c \\
 &= \cos x - \frac{2}{3}\cos^3 x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{g} \quad & \text{Let } x = 4 \sec \theta, \quad \frac{dx}{d\theta} = 4 \sec \theta \tan \theta \\
 \therefore & \int \frac{1}{x\sqrt{x^2 - 16}} \, dx \\
 &= \int \frac{1}{\cancel{4 \sec \theta} \sqrt{16 \sec^2 \theta - 16}} \cancel{4 \sec \theta} \tan \theta \, d\theta \\
 &= \int \frac{\tan \theta}{4\sqrt{\sec^2 \theta - 1}} \, d\theta \\
 &= \int \frac{1}{4} \, d\theta \quad \{\sqrt{\sec^2 \theta - 1} = \tan \theta\} \\
 &= \frac{1}{4}\theta + c
 \end{aligned}$$

$$\begin{aligned}
 \text{Now } x &= \frac{4}{\cos \theta}, \text{ so } \cos \theta = \frac{4}{x} \\
 \therefore \theta &= \arccos \frac{4}{x}
 \end{aligned}$$

$$\therefore \int \frac{1}{x\sqrt{x^2 - 16}} \, dx = \frac{1}{4} \arccos \frac{4}{x} + c$$

$$\begin{aligned}
 \text{h} \quad & \text{Let } u = 9 - x^2, \quad \frac{du}{dx} = -2x \\
 \therefore & \int \frac{x^3}{\sqrt{9 - x^2}} \, dx \\
 &= \int \frac{9 - u}{\sqrt{u}} \left(-\frac{1}{2} \frac{du}{dx}\right) \, dx \\
 &= -\frac{1}{2} \int \frac{9 - u}{\sqrt{u}} \, du \\
 &= -\frac{1}{2} \int (9u^{-\frac{1}{2}} - u^{\frac{1}{2}}) \, du \\
 &= -\frac{1}{2} \left[ \frac{9u^{\frac{1}{2}}}{\frac{1}{2}} - \frac{u^{\frac{3}{2}}}{\frac{3}{2}} \right] + c \\
 &= -9u^{\frac{1}{2}} + \frac{1}{3}u^{\frac{3}{2}} + c \\
 &= -9\sqrt{9 - x^2} + \frac{1}{3}(9 - x^2)^{\frac{3}{2}} + c
 \end{aligned}$$

**17** Let  $u = \cos x \quad \therefore \frac{du}{dx} = -\sin x$

$$\begin{aligned}\therefore \int \frac{\sin x}{\sqrt{\cos^n x}} dx &= \int \frac{1}{\sqrt{u^n}} \left(-\frac{du}{dx}\right) dx \\ &= -\int u^{-\frac{n}{2}} du \\ &= -\frac{u^{1-\frac{n}{2}}}{1-\frac{n}{2}} + c \quad \text{for } n \neq 2 \\ &= \frac{(\cos x)^{1-\frac{n}{2}}}{\frac{n}{2}-1} + c, \quad \text{for } n \neq 2 \\ &= \frac{2}{n-2}(\cos x)^{\frac{2-n}{2}} + c \quad \text{for } n \neq 2\end{aligned}$$

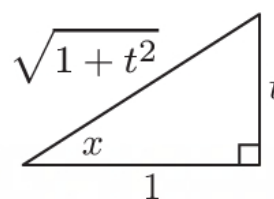
If  $n = 2$ ,  $\int \frac{\sin x}{\sqrt{\cos^2 x}} dx = -\int u^{-1} du$

$$\begin{aligned}&= -\ln|u| + c \\ &= -\ln|\cos x| + c\end{aligned}$$

So, the integral is defined for all  $n$ .

**18** Let  $t = \tan x$

$$\begin{aligned}\therefore \cos 2x &= \cos^2 x - \sin^2 x \\ &= \frac{1}{1+t^2} - \frac{t^2}{1+t^2} \\ &= \frac{1-t^2}{1+t^2}\end{aligned}$$



Now,  $x = \arctan t$

$$\therefore \frac{dx}{dt} = \frac{1}{1+t^2}$$

$$\begin{aligned}\therefore \int \frac{6 \tan x}{\cos 2x} dx &= \int \frac{6t}{\frac{1-t^2}{1+t^2}} \times \frac{1}{1+t^2} dt \\ &= \int \frac{6t}{1-t^2} dt \\ &= -3 \int \frac{-2t}{1-t^2} dt \\ &= -3 \ln|1-t^2| + c \\ &= -3 \ln|1-\tan^2 x| + c\end{aligned}$$

**19 a**

$$\begin{aligned}
 & \int e^{-x} \cos x \, dx \\
 &= e^{-x} \sin x - \int -e^{-x} \sin x \, dx \quad \leftarrow \begin{cases} u = e^{-x} & v' = \cos x \\ u' = -e^{-x} & v = \sin x \end{cases} \\
 &= e^{-x} \sin x + \int e^{-x} \sin x \, dx \\
 &= e^{-x} \sin x + \left[ e^{-x}(-\cos x) - \int (-e^{-x})(-\cos x) \, dx \right] \quad \leftarrow \begin{cases} u = e^{-x} & v' = \sin x \\ u' = -e^{-x} & v = -\cos x \end{cases} \\
 &= e^{-x} \sin x - e^{-x} \cos x - \int e^{-x} \cos x \, dx + c \\
 \therefore 2 \int e^{-x} \cos x \, dx &= e^{-x}(\sin x - \cos x) + c \\
 \therefore \int e^{-x} \cos x \, dx &= \frac{1}{2}e^{-x}(\sin x - \cos x) + c
 \end{aligned}$$

**b**

$$\begin{aligned}
 & \int 2^x \sin x \, dx \\
 &= \frac{2^x \sin x}{\ln 2} - \int \cos x \left( \frac{2^x}{\ln 2} \right) dx \quad \leftarrow \begin{cases} u = \sin x & v' = 2^x \\ u' = \cos x & v = \frac{2^x}{\ln 2} \end{cases} \\
 &= \frac{2^x \sin x}{\ln 2} - \frac{1}{\ln 2} \int 2^x \cos x \, dx \\
 &= \frac{2^x \sin x}{\ln 2} - \frac{1}{\ln 2} \left[ \frac{2^x \cos x}{\ln 2} - \int (-\sin x) \left( \frac{2^x}{\ln 2} \right) dx \right] \quad \leftarrow \begin{cases} u = \cos x & v' = 2^x \\ u' = -\sin x & v = \frac{2^x}{\ln 2} \end{cases} \\
 &= \frac{2^x \sin x}{\ln 2} - \frac{2^x \cos x}{(\ln 2)^2} - \frac{1}{(\ln 2)^2} \int 2^x \sin x \, dx + c \\
 \therefore \frac{1 + (\ln 2)^2}{(\ln 2)^2} \int 2^x \sin x \, dx &= \frac{2^x \sin x}{\ln 2} - \frac{2^x \cos x}{(\ln 2)^2} \\
 \therefore 1 + (\ln 2)^2 \int 2^x \sin x \, dx &= 2^x \times \ln 2 \times \sin x - 2^x \cos x + c \\
 \therefore \int 2^x \sin x \, dx &= \frac{2^x(\ln 2 \times \sin x - \cos x)}{1 + (\ln 2)^2} + c
 \end{aligned}$$

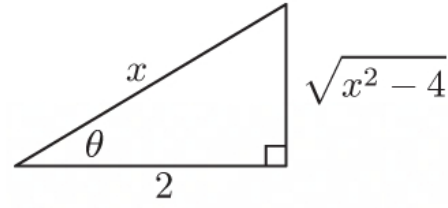


**c** Let  $x = 2 \sec \theta \quad \therefore \frac{dx}{d\theta} = 2 \sec \theta \tan \theta$

$$\begin{aligned} \therefore \int \frac{\sqrt{x^2 - 4}}{x} dx &= \int \frac{\sqrt{4 \sec^2 \theta - 4}}{2 \sec \theta} \times 2 \sec \theta \tan \theta d\theta \\ &= \int \sqrt{4(\sec^2 \theta - 1)} \tan \theta d\theta \\ &= \int 2 \tan \theta \tan \theta d\theta \\ &= 2 \int \tan^2 \theta d\theta \\ &= 2 \int (\sec^2 \theta - 1) d\theta \\ &= 2[\tan \theta - \theta] + c \\ &= 2 \tan \theta - 2\theta + c \end{aligned}$$

Now  $x = \frac{2}{\cos \theta}$

$\therefore \theta = \arccos \frac{2}{x} \text{ and } \tan \theta = \frac{\sqrt{x^2 - 4}}{2}$



$$\begin{aligned} \therefore \int \frac{\sqrt{x^2 - 4}}{x} dx &= 2 \left( \frac{\sqrt{x^2 - 4}}{2} \right) - 2 \arccos \frac{2}{x} + c \\ &= \sqrt{x^2 - 4} - 2 \arccos \frac{2}{x} + c \end{aligned}$$

**20 a**

$$\begin{aligned} &\int \frac{5}{\sqrt{9 - x^2}} dx \\ &= 5 \int \frac{1}{\sqrt{3^2 - x^2}} dx \\ &= 5 \arcsin \frac{x}{3} + c \end{aligned}$$

{using  $\int \frac{1}{\sqrt{a^2 - x^2}} dx = \arcsin \frac{x}{a} + c, \quad a \neq 0$  from **Exercise 21D**, question **26 a i**}

**b**

$$\begin{aligned} &\int \frac{1}{9 + 4x^2} dx \\ &= \frac{1}{4} \int \frac{1}{\frac{9}{4} + x^2} dx \\ &= \frac{1}{4} \left( \frac{1}{\frac{3}{2}} \right) \arctan \left( \frac{x}{\frac{3}{2}} \right) + c \end{aligned}$$

{using  $\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \arctan \frac{x}{a} + c, \quad a \neq 0$  from **Exercise 21D**, question **26 a iii**}

$$= \frac{1}{6} \arctan \frac{2x}{3} + c$$

$$\begin{aligned}
\bullet \quad \int \frac{5+3x}{1+x^2} dx &= \int \left( \frac{5}{1+x^2} + \frac{3x}{1+x^2} \right) dx \\
&= 5 \int \frac{1}{1+x^2} dx + \frac{3}{2} \int \frac{2x}{1+x^2} dx \\
&= 5 \arctan x + \frac{3}{2} \ln(1+x^2) + c \quad \{x^2 + 1 > 0\}
\end{aligned}$$

- 21** The argument has not accounted for the constant of integration  $c$ .  
The argument should read:

$$\begin{aligned}
\text{Let } I &= \int \frac{1}{x} dx \\
\therefore I &= \int 1 \times \frac{1}{x} dx \\
\therefore I &= x \left( \frac{1}{x} \right) - \int x \left( -\frac{1}{x^2} \right) dx & \begin{cases} u = \frac{1}{x} & v' = 1 \\ u' = -\frac{1}{x^2} & v = x \end{cases} \\
\therefore I &= 1 + \int \frac{1}{x} dx \\
\therefore I &= 1 + I + c \\
\therefore 0 &= 1 + c \quad \text{which is valid if } c = -1.
\end{aligned}$$

# Chapter 22

## DEFINITE INTEGRALS

### EXERCISE 22A

$$\begin{aligned} \mathbf{1} \quad \mathbf{a} \quad \int_1^4 \sqrt{x} \, dx &= \int_1^4 x^{\frac{1}{2}} \, dx & \int_1^4 (-\sqrt{x}) \, dx &= \int_1^4 (-x^{\frac{1}{2}}) \, dx \\ &= \left[ \frac{x^{\frac{3}{2}}}{\frac{3}{2}} \right]_1^4 & &= \left[ -\frac{x^{\frac{3}{2}}}{\frac{3}{2}} \right]_1^4 \\ &= \left[ \frac{2}{3} x^{\frac{3}{2}} \right]_1^4 & &= \left[ -\frac{2}{3} x^{\frac{3}{2}} \right]_1^4 \\ &= \frac{2}{3}(8) - \frac{2}{3}(1) & &= -\frac{2}{3}(8) - \left(-\frac{2}{3}(1)\right) \\ &= \frac{14}{3} & &= -\frac{14}{3} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \int_0^1 x^7 \, dx &= \left[ \frac{x^8}{8} \right]_0^1 & \int_0^1 (-x^7) \, dx &= \left[ -\frac{x^8}{8} \right]_0^1 \\ &= \frac{1}{8} - 0 & &= -\frac{1}{8} - 0 \\ &= \frac{1}{8} & &= -\frac{1}{8} \end{aligned}$$

Property:  $\int_a^b [-f(x)] \, dx = -\int_a^b f(x) \, dx$

$$\begin{aligned} \mathbf{2} \quad \mathbf{a} \quad \int_0^1 x^2 \, dx &= \left[ \frac{x^3}{3} \right]_0^1 = \frac{1}{3} - 0 = \frac{1}{3} & \mathbf{b} \quad \int_1^2 x^2 \, dx &= \left[ \frac{x^3}{3} \right]_1^2 = \frac{8}{3} - \frac{1}{3} = \frac{7}{3} \\ \mathbf{c} \quad \int_0^2 x^2 \, dx &= \left[ \frac{x^3}{3} \right]_0^2 = \frac{8}{3} - 0 = \frac{8}{3} & \mathbf{d} \quad \int_0^1 3x^2 \, dx &= [x^3]_0^1 = 1 - 0 = 1 \end{aligned}$$

Properties:  $\int_a^b f(x) \, dx + \int_b^c f(x) \, dx = \int_a^c f(x) \, dx$   
 $\int_a^b c f(x) \, dx = c \int_a^b f(x) \, dx$ , where  $c$  is a constant

$$\begin{aligned}
 \text{3 a } \int_0^2 (x^3 - 4x) \, dx &= \left[ \frac{x^4}{4} - 2x^2 \right]_0^2 \\
 &= \left( \frac{16}{4} - 2(4) \right) - (0 - 0) \\
 &= -4
 \end{aligned}$$

$$\begin{aligned}
 \text{c } \int_0^3 (x^3 - 4x) \, dx &= \left[ \frac{x^4}{4} - 2x^2 \right]_0^3 \\
 &= \left( \frac{81}{4} - 2(9) \right) - (0 - 0) \\
 &= \frac{9}{4} \\
 &= 2\frac{1}{4}
 \end{aligned}$$

$$\text{Property: } \int_a^b f(x) \, dx + \int_b^c f(x) \, dx = \int_a^c f(x) \, dx$$

$$\begin{aligned}
 \text{4 a } \int_0^1 x^2 \, dx &= \left[ \frac{x^3}{3} \right]_0^1 \\
 &= \frac{1}{3}(1) - 0 \\
 &= \frac{1}{3}
 \end{aligned}$$

$$\begin{aligned}
 \text{c } \int_0^1 (x^2 + \sqrt{x}) \, dx &= \int_0^1 (x^2 + x^{\frac{1}{2}}) \, dx \\
 &= \left[ \frac{x^3}{3} + \frac{x^{\frac{3}{2}}}{\frac{3}{2}} \right]_0^1 \\
 &= \left[ \frac{x^3}{3} + \frac{2}{3}x^{\frac{3}{2}} \right]_0^1 \\
 &= \left( \frac{1}{3} + \frac{2}{3}(1) \right) - (0 + 0) \\
 &= 1
 \end{aligned}$$

$$\text{Property: } \int_a^b f(x) \, dx + \int_a^b g(x) \, dx = \int_a^b [f(x) + g(x)] \, dx$$

$$\begin{aligned}
 \text{5 a } \int_0^1 x^3 \, dx &= \left[ \frac{x^4}{4} \right]_0^1 \\
 &= \frac{1}{4} - 0 \\
 &= \frac{1}{4}
 \end{aligned}$$

$$\begin{aligned}
 \text{b } \int_2^3 (x^3 - 4x) \, dx &= \left[ \frac{x^4}{4} - 2x^2 \right]_2^3 \\
 &= \left( \frac{81}{4} - 2(9) \right) - \left( \frac{16}{4} - 2(4) \right) \\
 &= \frac{25}{4} \\
 &= 6\frac{1}{4}
 \end{aligned}$$

$$\begin{aligned}
 \text{b } \int_0^1 \sqrt{x} \, dx &= \int_0^1 x^{\frac{1}{2}} \, dx \\
 &= \left[ \frac{x^{\frac{3}{2}}}{\frac{3}{2}} \right]_0^1 \\
 &= \left[ \frac{2}{3}x^{\frac{3}{2}} \right]_0^1 \\
 &= \frac{2}{3}(1) - 0 \\
 &= \frac{2}{3}
 \end{aligned}$$

$$\begin{aligned}
 \text{b } \int_0^2 (x^2 - x) \, dx &= \left[ \frac{x^3}{3} - \frac{x^2}{2} \right]_0^2 \\
 &= \left( \frac{8}{3} - 2 \right) - (0 - 0) \\
 &= \frac{2}{3}
 \end{aligned}$$



$$\begin{aligned}
 \text{c} \quad & \int_0^2 (3x^2 - x + 6) dx \\
 &= \left[ \frac{3x^3}{3} - \frac{x^2}{2} + 6x \right]_0^2 \\
 &= \left[ x^3 - \frac{x^2}{2} + 6x \right]_0^2 \\
 &= (8 - 2 + 12) - 0 \\
 &= 18
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & \int_1^4 (x + 2\sqrt{x}) dx \\
 &= \int_1^4 (x + 2x^{\frac{1}{2}}) dx \\
 &= \left[ \frac{x^2}{2} + \frac{2x^{\frac{3}{2}}}{\frac{3}{2}} \right]_1^4 \\
 &= \left[ \frac{x^2}{2} + \frac{4}{3}x^{\frac{3}{2}} \right]_1^4 \\
 &= \left( \frac{16}{2} + \frac{4}{3}(8) \right) - \left( \frac{1}{2} + \frac{4}{3} \right) \\
 &= \frac{101}{6} = 16\frac{5}{6}
 \end{aligned}$$

$$\begin{aligned}
 \text{g} \quad & \int_1^3 \frac{1}{x^2} dx = \int_1^3 x^{-2} dx \\
 &= \left[ \frac{x^{-1}}{-1} \right]_1^3 \\
 &= \left[ -\frac{1}{x} \right]_1^3 \\
 &= -\frac{1}{3} - (-1) \\
 &= \frac{2}{3}
 \end{aligned}$$

$$\begin{aligned}
 \text{i} \quad & \int_1^4 \left( x^2 + \frac{1}{x} \right) dx = \left[ \frac{x^3}{3} + \ln|x| \right]_1^4 \\
 &= \left( \frac{64}{3} + \ln 4 \right) - \left( \frac{1}{3} + \ln 1 \right) \\
 &= 21 + \ln 4
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad & \int_1^4 \left( x - \frac{3}{\sqrt{x}} \right) dx \\
 &= \int_1^4 (x - 3x^{-\frac{1}{2}}) dx \\
 &= \left[ \frac{x^2}{2} - \frac{3x^{\frac{1}{2}}}{\frac{1}{2}} \right]_1^4 \\
 &= \left[ \frac{x^2}{2} - 6\sqrt{x} \right]_1^4 \\
 &= \left( \frac{16}{2} - 12 \right) - \left( \frac{1}{2} - 6 \right) \\
 &= \frac{3}{2} = 1\frac{1}{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & \int_4^9 \frac{x-3}{\sqrt{x}} dx \\
 &= \int_4^9 (x^{\frac{1}{2}} - 3x^{-\frac{1}{2}}) dx \\
 &= \left[ \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{3x^{\frac{1}{2}}}{\frac{1}{2}} \right]_4^9 \\
 &= \left[ \frac{2}{3}x^{\frac{3}{2}} - 6\sqrt{x} \right]_4^9 \\
 &= \left( \frac{2}{3}(27) - 6(3) \right) - \left( \frac{2}{3}(8) - 6(2) \right) \\
 &= (18 - 18) - \left( \frac{16}{3} - 12 \right) \\
 &= \frac{20}{3} = 6\frac{2}{3}
 \end{aligned}$$

$$\begin{aligned}
 \text{h} \quad & \int_1^2 (x+3)^2 dx \\
 &= \int_1^2 (x^2 + 6x + 9) dx \\
 &= \left[ \frac{x^3}{3} + \frac{6x^2}{2} + 9x \right]_1^2 \\
 &= \left[ \frac{x^3}{3} + 3x^2 + 9x \right]_1^2 \\
 &= \left( \frac{8}{3} + 12 + 18 \right) - \left( \frac{1}{3} + 3 + 9 \right) \\
 &= \frac{61}{3} = 20\frac{1}{3}
 \end{aligned}$$

$$\begin{aligned}
 \text{6 a } \int_0^1 (3x+1)^4 dx &= \left[ \frac{1}{3} \frac{(3x+1)^5}{5} \right]_0^1 \\
 &= \left[ \frac{(3x+1)^5}{15} \right]_0^1 \\
 &= \frac{4^5}{15} - \frac{1^5}{15} \\
 &= \frac{1024}{15} - \frac{1}{15} \\
 &= \frac{341}{5} = 68\frac{1}{5}
 \end{aligned}$$

$$\begin{aligned}
 \text{b } \int_2^6 \frac{1}{\sqrt{2x-3}} dx &= \int_2^6 (2x-3)^{-\frac{1}{2}} dx \\
 &= \left[ \frac{1}{\frac{1}{2}} \frac{(2x-3)^{\frac{1}{2}}}{\frac{1}{2}} \right]_2^6 \\
 &= [\sqrt{2x-3}]_2^6 \\
 &= \sqrt{9} - \sqrt{1} \\
 &= 2
 \end{aligned}$$

$$\begin{aligned}
 \text{c } \int_{-3}^0 \sqrt{1-x} dx &= \int_{-3}^0 (1-x)^{\frac{1}{2}} dx \\
 &= \left[ \left( \frac{1}{-1} \right) \frac{(1-x)^{\frac{3}{2}}}{\frac{3}{2}} \right]_{-3}^0 \\
 &= \left[ -\frac{2}{3} (1-x)^{\frac{3}{2}} \right]_{-3}^0 \\
 &= \left( -\frac{2}{3} (1)^{\frac{3}{2}} \right) - \left( -\frac{2}{3} (4)^{\frac{3}{2}} \right) \\
 &= -\frac{2}{3} - \left( -\frac{16}{3} \right) \\
 &= \frac{14}{3} = 4\frac{2}{3}
 \end{aligned}$$

$$\begin{aligned}
 \text{7 a } \int_0^1 e^x dx &= [e^x]_0^1 \\
 &= e^1 - e^0 \\
 &= e - 1
 \end{aligned}$$

$$\begin{aligned}
 \text{b } \int_0^3 (2e^x - 3) dx &= [2e^x - 3x]_0^3 \\
 &= (2e^3 - 9) - (2 - 0) \\
 &= 2e^3 - 11
 \end{aligned}$$

$$\begin{aligned}
 \text{c } \int_0^2 e^{3x} dx &= \left[ \frac{1}{3} e^{3x} \right]_0^2 \\
 &= \frac{1}{3} e^6 - \frac{1}{3} e^0 \\
 &= \frac{1}{3} e^6 - \frac{1}{3} \\
 &= \frac{1}{3} (e^6 - 1)
 \end{aligned}$$

$$\begin{aligned}
 \text{d } \int_0^1 e^{1-x} dx &= \left[ \left( \frac{1}{-1} \right) e^{1-x} \right]_0^1 \\
 &= [-e^{1-x}]_0^1 \\
 &= -e^0 - (-e^1) \\
 &= e - 1
 \end{aligned}$$

$$\begin{aligned}
 \text{e } \int_0^{\ln 4} e^x (e^x - 2) dx &= \int_0^{\ln 4} (e^{2x} - 2e^x) dx \\
 &= \left[ \frac{1}{2} e^{2x} - 2e^x \right]_0^{\ln 4} \\
 &= \left( \frac{1}{2} e^{2 \ln 4} - 2e^{\ln 4} \right) - \left( \frac{1}{2} e^0 - 2e^0 \right) \\
 &= \left( \frac{1}{2} e^{\ln 16} - 2e^{\ln 4} \right) - \left( \frac{1}{2} - 2 \right) \\
 &= \left( \frac{1}{2} (16) - 2(4) \right) - \left( -\frac{3}{2} \right) \\
 &= \frac{3}{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad \int_1^2 (e^{-x} + 1)^2 dx &= \int_1^2 (e^{-2x} + 2e^{-x} + 1) dx \\
 &= \left[ \left(\frac{1}{-2}\right)e^{-2x} + 2\left(\frac{1}{-1}\right)e^{-x} + x \right]_1^2 \\
 &= \left[ -\frac{e^{-2x}}{2} - 2e^{-x} + x \right]_1^2 \\
 &= \left( -\frac{e^{-4}}{2} - 2e^{-2} + 2 \right) - \left( -\frac{e^{-2}}{2} - 2e^{-1} + 1 \right) \\
 &= -\frac{1}{2e^4} - \frac{3}{2e^2} + \frac{2}{e} + 1
 \end{aligned}$$

$$\begin{aligned}
 8 \quad \text{a} \quad \int_0^{\frac{\pi}{6}} \cos x dx &= \left[ \sin x \right]_0^{\frac{\pi}{6}} \\
 &= \sin \frac{\pi}{6} - \sin 0 \\
 &= \frac{1}{2} - 0 \\
 &= \frac{1}{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin x dx &= \left[ -\cos x \right]_{\frac{\pi}{3}}^{\frac{\pi}{2}} \\
 &= \left( -\cos \frac{\pi}{2} \right) - \left( -\cos \frac{\pi}{3} \right) \\
 &= 0 - \left( -\frac{1}{2} \right) \\
 &= \frac{1}{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sec^2 x dx &= \left[ \tan x \right]_{\frac{\pi}{4}}^{\frac{\pi}{3}} \\
 &= \tan \frac{\pi}{3} - \tan \frac{\pi}{4} \\
 &= \sqrt{3} - 1
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad \int_0^{\frac{\pi}{6}} \sin 3x dx &= \left[ -\frac{1}{3} \cos 3x \right]_0^{\frac{\pi}{6}} \\
 &= \left( -\frac{1}{3} \cos \frac{\pi}{2} \right) - \left( -\frac{1}{3} \cos 0 \right) \\
 &= 0 - \left( -\frac{1}{3} \right) \\
 &= \frac{1}{3}
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos \left( x - \frac{\pi}{3} \right) dx &= \left[ \sin \left( x - \frac{\pi}{3} \right) \right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} \\
 &= \sin \left( \frac{\pi}{2} - \frac{\pi}{3} \right) - \sin \left( \frac{\pi}{6} - \frac{\pi}{3} \right) \\
 &= \sin \frac{\pi}{6} - \sin \left( -\frac{\pi}{6} \right) \\
 &= \frac{1}{2} - \left( -\frac{1}{2} \right) \\
 &= 1
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin \left( 2x - \frac{\pi}{4} \right) dx &= \left[ -\frac{1}{2} \cos \left( 2x - \frac{\pi}{4} \right) \right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} \\
 &= \left( -\frac{1}{2} \cos \frac{3\pi}{4} \right) - \left( -\frac{1}{2} \cos \frac{\pi}{4} \right) \\
 &= \frac{1}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} \\
 &= \frac{1}{\sqrt{2}}
 \end{aligned}$$

$$\begin{aligned}
 \text{g} \quad \cos^2 x &= \frac{1}{2} + \frac{1}{2} \cos 2x \\
 \therefore \int_0^{\frac{\pi}{4}} \cos^2 x dx &= \int_0^{\frac{\pi}{4}} \left( \frac{1}{2} + \frac{1}{2} \cos 2x \right) dx \\
 &= \left[ \frac{1}{2}x + \frac{1}{4} \sin 2x \right]_0^{\frac{\pi}{4}} \\
 &= \left( \frac{\pi}{8} + \frac{1}{4}(1) \right) - \left( 0 + \frac{1}{4}(0) \right) \\
 &= \frac{\pi}{8} + \frac{1}{4}
 \end{aligned}$$

$$\begin{aligned}
 \text{h} \quad \sin^2 x &= \frac{1}{2} - \frac{1}{2} \cos 2x \\
 \therefore \int_0^{\frac{\pi}{2}} \sin^2 x dx &= \int_0^{\frac{\pi}{2}} \left( \frac{1}{2} - \frac{1}{2} \cos 2x \right) dx \\
 &= \left[ \frac{1}{2}x - \frac{1}{4} \sin 2x \right]_0^{\frac{\pi}{2}} \\
 &= \left( \frac{\pi}{4} - \frac{1}{4}(0) \right) - \left( 0 - \frac{1}{4}(0) \right) \\
 &= \frac{\pi}{4}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{i} \quad \int_0^{\frac{\pi}{6}} (\sin 3x - \cos x) dx &= \left[ -\frac{1}{3} \cos 3x - \sin x \right]_0^{\frac{\pi}{6}} \\
 &= \left( -\frac{1}{3}(0) - \frac{1}{2} \right) - \left( -\frac{1}{3}(1) - 0 \right) \\
 &= -\frac{1}{6}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{9} \quad \mathbf{a} \quad \int_{-6}^{-2} \frac{1}{x} dx &= [\ln |x|]_{-6}^{-2} \\
 &= \ln 2 - \ln 6 \\
 &= \ln\left(\frac{2}{6}\right) \quad \left\{ \ln a - \ln b = \ln\left(\frac{a}{b}\right) \right\} \\
 &= \ln\left(\frac{1}{3}\right) \\
 &= \ln(3^{-1}) \\
 &= -\ln 3
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad \int_1^8 \frac{2}{3x+4} dx &= \left[ 2\left(\frac{1}{3}\right) \ln |3x+4| \right]_1^8 \\
 &= \left[ \frac{2}{3} \ln |3x+4| \right]_1^8 \\
 &= \frac{2}{3} \ln 28 - \frac{2}{3} \ln 7 \\
 &= \frac{2}{3} (\ln 28 - \ln 7) \\
 &= \frac{2}{3} \ln\left(\frac{28}{7}\right) \quad \left\{ \ln a - \ln b = \ln\left(\frac{a}{b}\right) \right\} \\
 &= \frac{2}{3} \ln 4 \\
 &= \frac{4}{3} \ln 2
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad \int_{-1}^5 \frac{1}{x+4} dx &= [\ln |x+4|]_{-1}^5 \\
 &= \ln 9 - \ln 3 \\
 &= \ln\left(\frac{9}{3}\right) \quad \left\{ \ln a - \ln b = \ln\left(\frac{a}{b}\right) \right\} \\
 &= \ln 3
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad \int_{-4}^0 \frac{4}{5-2x} dx &= \left[ 4\left(\frac{1}{-2}\right) \ln |5-2x| \right]_{-4}^0 \\
 &= [-2 \ln |5-2x|]_{-4}^0 \\
 &= -2 \ln 5 - (-2 \ln 13) \\
 &= -2 \ln 5 + 2 \ln 13 \\
 &= 2(\ln 13 - \ln 5) \\
 &= 2 \ln\left(\frac{13}{5}\right) \quad \left\{ \ln a - \ln b = \ln\left(\frac{a}{b}\right) \right\}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{10} \quad \frac{4x+1}{x-1} &= \frac{4x-4+1+4}{x-1} \\
 &= \frac{4(x-1)+5}{x-1} \\
 &= \frac{4(x-1)}{x-1} + \frac{5}{x-1} \\
 &= 4 + \frac{5}{x-1} \quad \text{as required}
 \end{aligned}$$

$$\begin{aligned}
 \therefore \int_3^5 \frac{4x+1}{x-1} dx &= \int_3^5 \left( 4 + \frac{5}{x-1} \right) dx \\
 &= [4x + 5 \ln |x-1|]_3^5 \\
 &= (4(5) + 5 \ln |5-1|) - (4(3) + 5 \ln |3-1|) \\
 &= 20 + 5 \ln 4 - 12 - 5 \ln 2 \\
 &= 20 + 5 \ln(2^2) - 12 - 5 \ln 2 \\
 &= 8 + 10 \ln 2 - 5 \ln 2 \\
 &= 8 + 5 \ln 2
 \end{aligned}$$



**11 a**

$$\int_m^{2m} (2x - 1) dx = 4$$

$$\therefore [x^2 - x]_m^{2m} = 4$$

$$\therefore (4m^2 - 2m) - (m^2 - m) = 4$$

$$\therefore 3m^2 - m - 4 = 0$$

$$\therefore (3m - 4)(m + 1) = 0$$

$$\therefore m = -1 \text{ or } \frac{4}{3}$$

**b**

$$\int_m^{-2} \frac{1}{4-x} dx = \ln\left(\frac{3}{2}\right)$$

$$\therefore [-\ln|4-x|]_m^{-2} = \ln\left(\frac{3}{2}\right)$$

$$\therefore -\ln|4-(-2)| + \ln|4-m| = \ln\left(\frac{3}{2}\right)$$

$$\therefore \ln|4-m| - \ln 6 = \ln\left(\frac{3}{2}\right)$$

$$\therefore \ln\left|\frac{4-m}{6}\right| = \ln\left(\frac{3}{2}\right)$$

$$\therefore \left|\frac{4-m}{6}\right| = \frac{3}{2}$$

$$\therefore \frac{4-m}{6} = \pm \frac{3}{2}$$

$$\therefore 4-m = \pm 9$$

$$\therefore m = 4 \pm 9$$

$$\therefore m = -5 \text{ or } 13$$

However, the solution  $m = 13$  is invalid, since the vertical asymptote  $x = 4$  lies between  $-2$  and  $13$ .

$\therefore m = -5$  is the only valid answer.

**12 a**

$$\begin{aligned} & \int_2^4 2^x dx \\ &= \left[ \frac{2^x}{\ln 2} \right]_2^4 \\ &= \frac{2^4}{\ln 2} - \frac{2^2}{\ln 2} \\ &= \frac{16}{\ln 2} - \frac{4}{\ln 2} \\ &= \frac{12}{\ln 2} \end{aligned}$$

**b**

$$\begin{aligned} & \int_{-1}^1 \frac{3}{x^2+1} dx \\ &= 3 \int_{-1}^1 \frac{1}{x^2+1} dx \\ &= 3 [\arctan x]_{-1}^1 \\ &= 3(\arctan 1 - \arctan(-1)) \\ &= 3\left(\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right) \\ &= \frac{3\pi}{2} \end{aligned}$$

**c**

$$\begin{aligned} & \int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}} dx \\ &= \left[ \arcsin x \right]_0^{\frac{1}{2}} \\ &= \arcsin \frac{1}{2} - \arcsin 0 \\ &= \frac{\pi}{6} - 0 \\ &= \frac{\pi}{6} \end{aligned}$$

$$\text{d} \quad \frac{1}{x^2 - 1} = \frac{1}{(x+1)(x-1)}$$

$$\text{Let } \frac{1}{x^2 - 1} = \frac{A}{x+1} + \frac{B}{x-1}$$

$$\therefore 1 = A(x-1) + B(x+1)$$

$$\text{Substituting } x = 1: \quad 1 = 2B$$

$$\therefore B = \frac{1}{2}$$

$$\text{Substituting } x = -1: \quad 1 = -2A$$

$$\therefore A = -\frac{1}{2}$$

$$\begin{aligned} \therefore \frac{1}{x^2 - 1} &= -\frac{1}{2(x+1)} + \frac{1}{2(x-1)} \\ &= -\frac{1}{2} \left( \frac{1}{x+1} - \frac{1}{x-1} \right) \end{aligned}$$

$$\begin{aligned} \therefore \int_2^3 \frac{1}{x^2 - 1} dx &= -\frac{1}{2} \int_2^3 \left( \frac{1}{x+1} - \frac{1}{x-1} \right) dx \\ &= -\frac{1}{2} [\ln|x+1| - \ln|x-1|]_2^3 \\ &= -\frac{1}{2} ((\ln 4 - \ln 2) - (\ln 3 - \ln 1)) \\ &= -\frac{1}{2} (\ln 2 - \ln 3) \\ &= -\frac{1}{2} \ln\left(\frac{2}{3}\right) \\ &= \frac{1}{2} \ln\left(\frac{3}{2}\right) \end{aligned}$$

$$\begin{aligned} \text{e} \quad & \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} (\cos 2x - \sin x)^2 dx \\ &= \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} (\cos^2 2x - 2 \sin x \cos 2x + \sin^2 x) dx \\ &= \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \left( \frac{1}{2} + \frac{1}{2} \cos 4x - (\sin(x+2x) + \sin(x-2x)) + \frac{1}{2} - \frac{1}{2} \cos 2x \right) dx \\ &= \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \left( 1 + \frac{1}{2} \cos 4x - \sin 3x + \sin x - \frac{1}{2} \cos 2x \right) dx \\ &= \left[ x + \frac{1}{8} \sin 4x + \frac{1}{3} \cos 3x - \cos x - \frac{1}{4} \sin 2x \right]_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \\ &= \left( \frac{2\pi}{3} + \frac{1}{8} \sin \frac{8\pi}{3} + \frac{1}{3} \cos 2\pi - \cos \frac{2\pi}{3} - \frac{1}{4} \sin \frac{4\pi}{3} \right) \\ &\quad - \left( \frac{\pi}{3} + \frac{1}{8} \sin \frac{4\pi}{3} + \frac{1}{3} \cos \pi - \cos \frac{\pi}{3} - \frac{1}{4} \sin \frac{2\pi}{3} \right) \\ &= \left( \frac{2\pi}{3} + \frac{\sqrt{3}}{16} + \frac{1}{3} + \frac{1}{2} + \frac{\sqrt{3}}{8} \right) - \left( \frac{\pi}{3} - \frac{\sqrt{3}}{16} - \frac{1}{3} - \frac{1}{2} - \frac{\sqrt{3}}{8} \right) \\ &= \frac{\pi}{3} + \frac{3\sqrt{3}}{8} + \frac{5}{3} \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad \int_0^2 \frac{-6}{\sqrt{4-x^2}} dx &= -6 \int_0^2 \frac{1}{\sqrt{2^2-x^2}} dx \\
 &= -6 \left[ \arcsin \frac{x}{2} \right]_0^2 \\
 &= -6(\arcsin 1 - \arcsin 0) \\
 &= -6\left(\frac{\pi}{2} - 0\right) \\
 &= -3\pi
 \end{aligned}$$

$$\begin{aligned}
 \text{13 a} \quad \frac{x+4}{x^2-5x+6} &= \frac{x+4}{(x-2)(x-3)} \\
 \text{Let } \frac{x+4}{x^2-5x+6} &= \frac{A}{x-2} + \frac{B}{x-3} \\
 \therefore x+4 &= A(x-3) + B(x-2)
 \end{aligned}$$

$$\begin{aligned}
 \text{Substituting } x=3: \quad 3+4 &= B(3-2) \\
 \therefore B &= 7
 \end{aligned}$$

$$\begin{aligned}
 \text{Substituting } x=2: \quad 2+4 &= A(2-3) \\
 \therefore 6 &= -A \\
 \therefore A &= -6
 \end{aligned}$$

$$\begin{aligned}
 \therefore \frac{x+4}{x^2-5x+6} &= \frac{-6}{x-2} + \frac{7}{x-3} \\
 \therefore \int_0^1 \frac{x+4}{x^2-5x+6} dx &= \int_0^1 \left( -\frac{6}{x-2} + \frac{7}{x-3} \right) dx \\
 &= \left[ -6 \ln |x-2| + 7 \ln |x-3| \right]_0^1 \\
 &= (-6 \ln 1 + 7 \ln 2) - (-6 \ln 2 + 7 \ln 3) \\
 &= 13 \ln 2 - 7 \ln 3
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad \frac{3x-8}{x^2-x-42} &= \frac{3x-8}{(x+6)(x-7)} \\
 \text{Let } \frac{3x-8}{x^2-x-42} &= \frac{A}{x+6} + \frac{B}{x-7} \\
 \therefore 3x-8 &= A(x-7) + B(x+6)
 \end{aligned}$$

$$\begin{aligned}
 \text{Substituting } x=7: \quad 3(7)-8 &= B(7+6) \\
 \therefore 13 &= 13B \\
 \therefore B &= 1
 \end{aligned}$$

$$\begin{aligned}
 \text{Substituting } x=-6: \quad 3(-6)-8 &= A(-6-7) \\
 \therefore -26 &= -13A \\
 \therefore A &= 2
 \end{aligned}$$

$$\begin{aligned}
 \therefore \frac{3x-8}{x^2-x-42} &= \frac{2}{x+6} + \frac{1}{x-7} \\
 \therefore \int_{-2}^3 \frac{3x-8}{x^2-x-42} dx &= \int_{-2}^3 \left( \frac{2}{x+6} + \frac{1}{x-7} \right) dx \\
 &= \left[ 2 \ln |x+6| + \ln |x-7| \right]_{-2}^3 \\
 &= (2 \ln 9 + \ln 4) - (2 \ln 4 + \ln 9) \\
 &= \ln 9 - \ln 4 \\
 &= 2 \ln 3 - 2 \ln 2
 \end{aligned}$$

$$\text{c} \quad \frac{2x+12}{3x^2+4x-4} = \frac{2x+12}{(3x-2)(x+2)}$$

$$\text{Let } \frac{2x+12}{3x^2+4x-4} = \frac{A}{3x-2} + \frac{B}{x+2}$$

$$\therefore 2x+12 = A(x+2) + B(3x-2)$$

$$\text{Substituting } x = -2: \quad 2(-2) + 12 = B(3(-2) - 2)$$

$$\therefore 8 = -8B$$

$$\therefore B = -1$$

$$\text{Substituting } x = \frac{2}{3}: \quad 2\left(\frac{2}{3}\right) + 12 = A\left(\frac{2}{3} + 2\right)$$

$$\therefore \frac{40}{3} = \frac{8}{3}A$$

$$\therefore A = 5$$

$$\therefore \frac{2x+12}{3x^2+4x-4} = \frac{5}{3x-2} - \frac{1}{x+2}$$

$$\begin{aligned} \therefore \int_1^4 \frac{2x+12}{3x^2+4x-4} dx &= \int_1^4 \left( \frac{5}{3x-2} - \frac{1}{x+2} \right) dx \\ &= \left[ \frac{5}{3} \ln |3x-2| - \ln |x+2| \right]_1^4 \\ &= \left( \frac{5}{3} \ln 10 - \ln 6 \right) - \left( \frac{5}{3} \ln 1 - \ln 3 \right) \\ &= \frac{5}{3} \ln 10 - (\ln 6 - \ln 3) \\ &= \frac{5}{3} \ln 10 - \ln 2 \end{aligned}$$

$$\begin{aligned} \text{14 a} \quad \int (-xe^{-x}) dx &= xe^{-x} - \int (-1)(-e^{-x}) dx \quad \leftarrow \begin{cases} u = -x & v' = e^{-x} \\ u' = -1 & v = -e^{-x} \end{cases} \\ &= xe^{-x} - \int e^{-x} dx \\ &= xe^{-x} + e^{-x} + c \\ &= (x+1)e^{-x} + c \\ \therefore \int_0^1 (-xe^{-x}) dx &= [(x+1)e^{-x}]_0^1 \\ &= 2e^{-1} - e^0 \\ &= \frac{2}{e} - 1 \end{aligned}$$

$$\begin{aligned} \text{b} \quad \int x \sin x dx &= -x \cos x - \int -\cos x dx \quad \leftarrow \begin{cases} u = x & v' = \sin x \\ u' = 1 & v = -\cos x \end{cases} \\ &= -x \cos x + \int \cos x dx \\ &= -x \cos x + \sin x + c \\ \therefore \int_0^{\frac{\pi}{2}} x \sin x dx &= \left[ -x \cos x + \sin x \right]_0^{\frac{\pi}{2}} \\ &= \left( -\frac{\pi}{2} \cos \frac{\pi}{2} + \sin \frac{\pi}{2} \right) - (-0 \cos 0 + \sin 0) \\ &= 1 \end{aligned}$$



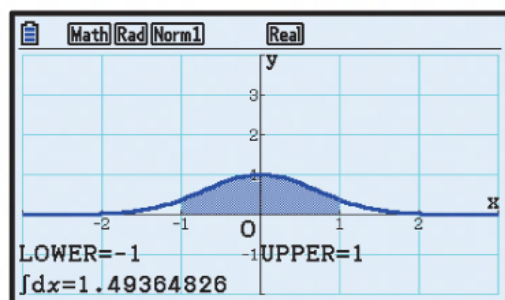
$$\begin{aligned} \text{c} \quad \int \ln x \, dx &= x \ln x - \int \left(\frac{1}{x}\right) x \, dx \leftarrow \begin{cases} u = \ln x & v' = 1 \\ u' = \frac{1}{x} & v = x \end{cases} \\ &= x \ln x - \int 1 \, dx \\ &= x \ln x - x + c \end{aligned}$$

$$\begin{aligned} \therefore \int_1^3 \ln x \, dx &= [x \ln x - x]_1^3 \\ &= (3 \ln 3 - 3) - (\ln 1 - 1) \\ &= 3 \ln 3 - 2 \end{aligned}$$

$$\begin{aligned} \text{d} \quad \int x \ln x^2 \, dx &= \frac{1}{2} x^2 \ln x^2 - \int \left(\frac{2}{x}\right) \left(\frac{1}{2} x^2\right) dx \leftarrow \begin{cases} u = \ln x^2 & v' = x \\ u' = \frac{2}{x} & v = \frac{1}{2} x^2 \end{cases} \\ &= \frac{1}{2} x^2 \ln x^2 - \int x \, dx \\ &= \frac{1}{2} x^2 \ln x^2 - \frac{1}{2} x^2 + c \\ &= \frac{1}{2} x^2 (\ln x^2 - 1) + c \end{aligned}$$

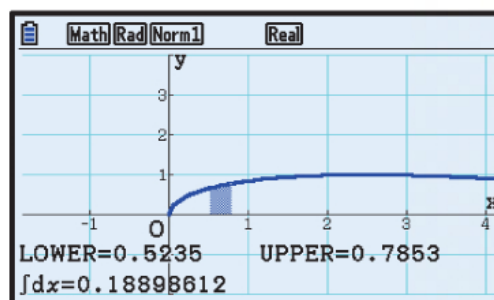
$$\begin{aligned} \therefore \int_1^e x \ln x^2 \, dx &= \left[ \frac{1}{2} x^2 (\ln x^2 - 1) \right]_1^e \\ &= \left( \frac{1}{2} e^2 (\ln e^2 - 1) \right) - \left( \frac{1}{2} (\ln 1 - 1) \right) \\ &= \frac{1}{2} e^2 (2 - 1) - \frac{1}{2} (-1) \\ &= \frac{1}{2} e^2 + \frac{1}{2} \\ &= \frac{e^2 + 1}{2} \end{aligned}$$

15 a



$$\therefore \int_{-1}^1 e^{-x^2} \, dx \approx 1.49$$

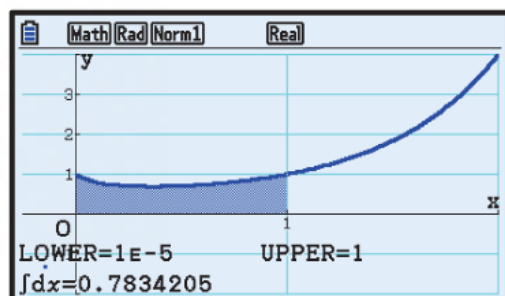
b



$$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \sin(\sqrt{x}) \, dx \approx 0.189$$

$$\therefore \int_{\frac{\pi}{4}}^{\frac{\pi}{6}} \sin(\sqrt{x}) \, dx \approx -0.189$$

c



$$\therefore \int_0^1 x^x \, dx \approx 0.783$$

**Note:** We have used a value very close to 0 since the fraction is undefined at  $x = 0$ .

$$\begin{aligned}
 \text{16 a} \quad \int x \sec^2 x \, dx &= x \tan x - \int \tan x \, dx \quad \leftarrow \begin{cases} u = x & v' = \sec^2 x \\ u' = 1 & v = \tan x \end{cases} \\
 &= x \tan x + \ln |\cos x| + c
 \end{aligned}$$

$$\begin{aligned}
 \therefore \int_0^{\frac{\pi}{4}} x \sec^2 x \, dx &= \left[ x \tan x + \ln |\cos x| \right]_0^{\frac{\pi}{4}} \\
 &= \left( \frac{\pi}{4} \tan \frac{\pi}{4} + \ln \left| \cos \frac{\pi}{4} \right| \right) - (0 \tan 0 + \ln |\cos 0|) \\
 &= \frac{\pi}{4} + \ln \left( \frac{1}{\sqrt{2}} \right) \\
 &= \frac{\pi}{4} + \ln(2^{-\frac{1}{2}}) \\
 &= \frac{\pi}{4} - \frac{\ln 2}{2}
 \end{aligned}$$

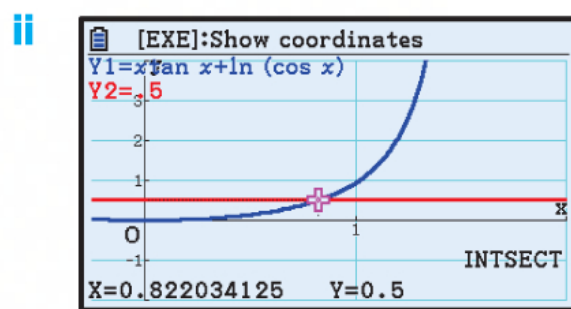
$$\text{b i} \quad \int x \sec^2 x \, dx = x \tan x + \ln |\cos x| + c \quad \{\text{from a}\}$$

$$\text{If } \int_0^a x \sec^2 x \, dx = \frac{1}{2},$$

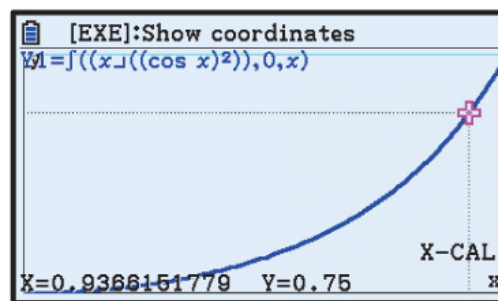
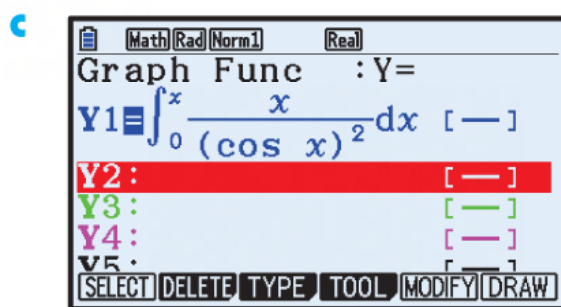
$$\text{then } \left[ x \tan x + \ln |\cos x| \right]_0^a = \frac{1}{2}$$

$$\therefore (a \tan a + \ln |\cos a|) - (0 \tan 0 + \ln |\cos 0|) = \frac{1}{2}$$

$$\therefore a \tan a + \ln(\cos a) = \frac{1}{2}, \quad 0 < a < \frac{\pi}{2}$$



Using technology,  $a \approx 0.822$



$$\text{If } \int_0^a x \sec^2 x \, dx = \frac{3}{4}, \quad 0 < a < \frac{\pi}{2}, \quad \text{then } a \approx 0.937 \quad \{\text{using technology}\}$$

17 Suppose  $f(x)$  and  $g(x)$  are continuous functions with antiderivatives  $F(x)$  and  $G(x)$  respectively.

$$\begin{aligned}
 \text{a} \quad \int_a^b f(x) \, dx &= F(b) - F(a) \\
 &= -(F(a) - F(b)) \\
 &= - \int_b^a f(x) \, dx
 \end{aligned}$$

**b**  $\frac{d}{dx}(kF(x)) = kf(x)$ , so  $kF(x)$  is the antiderivative of  $kf(x)$ .

$$\begin{aligned}\int_a^b kf(x) dx &= kF(b) - kF(a) \quad \{\text{since } k \text{ is a constant}\} \\ &= k(F(b) - F(a)) \\ &= k \int_a^b f(x) dx\end{aligned}$$

**c** 
$$\begin{aligned}\int_a^b f(x) dx + \int_b^c f(x) dx &= \cancel{F(b)} - F(a) + F(c) - \cancel{F(b)} \\ &= F(c) - F(a) \\ &= \int_a^c f(x) dx\end{aligned}$$

**d**  $\frac{d}{dx}(F(x) + G(x)) = f(x) + g(x)$ , so  $F(x) + G(x)$  is the antiderivative of  $f(x) + g(x)$ .

$$\begin{aligned}\int_a^b [f(x) + g(x)] dx &= F(b) + G(b) - (F(a) + G(a)) \\ &= F(b) + G(b) - F(a) - G(a) \\ &= F(b) - F(a) + G(b) - G(a) \\ &= \int_a^b f(x) dx + \int_a^b g(x) dx\end{aligned}$$

**18 a** 
$$\int_2^4 f(x) dx + \int_4^7 f(x) dx = \int_2^7 f(x) dx$$

**b** 
$$\begin{aligned}\int_4^5 f(x) dx - \int_6^5 f(x) dx &= \int_4^5 f(x) dx + \int_5^6 f(x) dx \\ &= \int_4^6 f(x) dx\end{aligned}$$

**c** 
$$\int_1^3 g(x) dx + \int_3^8 g(x) dx + \int_8^9 g(x) dx = \int_1^9 g(x) dx$$

**19 a** 
$$\begin{aligned}\int_1^3 f(x) dx + \int_3^6 f(x) dx &= \int_1^6 f(x) dx \\ \therefore \int_3^6 f(x) dx &= \int_1^6 f(x) dx - \int_1^3 f(x) dx \\ &= -3 - 2 \\ &= -5\end{aligned}$$

$$\begin{aligned}
 \text{b} \quad \int_0^2 f(x) \, dx + \int_2^4 f(x) \, dx + \int_4^6 f(x) \, dx &= \int_0^6 f(x) \, dx \\
 \therefore \int_2^4 f(x) \, dx &= \int_0^6 f(x) \, dx - \int_4^6 f(x) \, dx - \int_0^2 f(x) \, dx \\
 &= 7 - (-2) - 5 \\
 &= 4
 \end{aligned}$$

$$\begin{aligned}
 \text{20 a} \quad \int_{-a}^a f(x) \, dx &= \int_{-a}^0 f(x) \, dx + \int_0^a f(x) \, dx \\
 &= \int_0^a f(-x) \, dx + \int_0^a f(x) \, dx \\
 &= - \int_0^a f(x) \, dx + \int_0^a f(x) \, dx \quad \{f(-x) = -f(x)\} \\
 &= 0
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad \int_{-a}^a f(x) \, dx &= \int_{-a}^0 f(x) \, dx + \int_0^a f(x) \, dx \\
 &= \int_0^a f(-x) \, dx + \int_0^a f(x) \, dx \quad \text{or} \quad \int_{-a}^0 f(x) \, dx + \int_{-a}^0 f(-x) \, dx \\
 &= \int_0^a f(x) \, dx + \int_0^a f(x) \, dx \quad \text{or} \quad \int_{-a}^0 f(x) \, dx + \int_{-a}^0 f(x) \, dx \\
 &\quad \{f(-x) = f(x)\} \\
 &= 2 \int_0^a f(x) \, dx \quad \text{or} \quad 2 \int_{-a}^0 f(x) \, dx \\
 \therefore \int_{-a}^a f(x) \, dx &= 2 \int_0^a f(x) \, dx = 2 \int_{-a}^0 f(x) \, dx
 \end{aligned}$$

$$\begin{aligned}
 \text{21 a} \quad \int_1^{-1} f(x) \, dx &= - \int_{-1}^1 f(x) \, dx & \text{b} \quad \int_{-1}^1 (2 + f(x)) \, dx &= \int_{-1}^1 2 \, dx + \int_{-1}^1 f(x) \, dx \\
 &= -(-4) & &= [2x]_{-1}^1 + (-4) \\
 &= 4 & &= (2 - (-2)) - 4 \\
 & & &= 0
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad \int_{-1}^1 2f(x) \, dx &= 2 \int_{-1}^1 f(x) \, dx & \text{d} \quad \int_{-1}^1 k f(x) \, dx &= 7 \\
 &= 2(-4) & \therefore k \int_{-1}^1 f(x) \, dx &= 7 \\
 &= -8 & \therefore k(-4) &= 7 \\
 & & \therefore k &= -\frac{7}{4}
 \end{aligned}$$



**22** Since  $\frac{d}{dx}(g(x)) = g'(x)$ ,  $g(x)$  is the antiderivative of  $g'(x)$ .

$$\begin{aligned}\int_2^3 (g'(x) - 1) dx &= \int_2^3 g'(x) dx + \int_2^3 -1 dx \\ &= [g(x)]_2^3 + [-x]_2^3 \\ &= (g(3) - g(2)) + (-3 - (-2)) \\ &= 5 - 4 - 1 \\ &= 0\end{aligned}$$

## EXERCISE 22B

**1 a** Let  $u = x^2 - 1 \quad \therefore \frac{du}{dx} = 2x$

When  $x = 1$ ,  $u = 0$

When  $x = 2$ ,  $u = 3$

$$\begin{aligned}\therefore \int_1^2 2x(x^2 - 1)^3 dx &= \int_0^3 u^3 \frac{du}{dx} dx \\ &= \int_0^3 u^3 du \\ &= \left[ \frac{u^4}{4} \right]_0^3 \\ &= \left( \frac{81}{4} - 0 \right) \\ &= \frac{81}{4} = 20\frac{1}{4}\end{aligned}$$

Math Rad Norm1 ab/c Real

$\int_1^2 2x(x^2-1)^3 dx$

$20\frac{1}{4}$

$\int dx$   $\Sigma$   $\triangleright$

**b** Let  $u = x^2 + 2 \quad \therefore \frac{du}{dx} = 2x$

When  $x = 1$ ,  $u = 3$

When  $x = 2$ ,  $u = 6$

$$\begin{aligned}\therefore \int_1^2 \frac{x}{(x^2 + 2)^2} dx &= \int_3^6 u^{-2} \left( \frac{1}{2} \frac{du}{dx} \right) dx \\ &= \frac{1}{2} \int_3^6 u^{-2} du \\ &= \frac{1}{2} \left[ \frac{u^{-1}}{-1} \right]_3^6 \\ &= \frac{1}{2} \left[ -\frac{1}{6} - \left( -\frac{1}{3} \right) \right] \\ &= \frac{1}{12}\end{aligned}$$

Math Rad Norm1 ab/c Real

$\int_1^2 \frac{x}{(x^2+2)^2} dx$

$\frac{1}{12}$

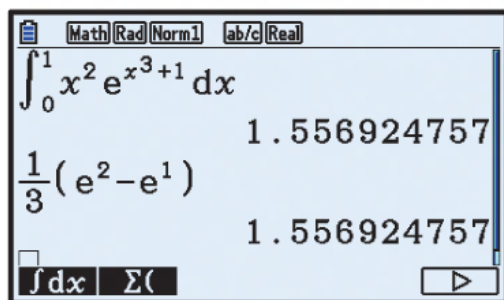
$\int dx$   $\Sigma$   $\triangleright$

**c** Let  $u = x^3 + 1 \quad \therefore \frac{du}{dx} = 3x^2$

When  $x = 0$ ,  $u = 1$

When  $x = 1$ ,  $u = 2$

$$\begin{aligned} \therefore \int_0^1 x^2 e^{x^3+1} dx &= \int_1^2 e^u \left( \frac{1}{3} \frac{du}{dx} \right) dx \\ &= \frac{1}{3} \int_1^2 e^u du \\ &= \frac{1}{3} [e^u]_1^2 \\ &= \frac{1}{3} (e^2 - e) \\ &\approx 1.56 \end{aligned}$$

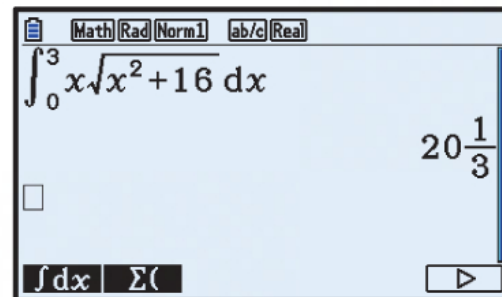


**d** Let  $u = x^2 + 16 \quad \therefore \frac{du}{dx} = 2x$

When  $x = 0$ ,  $u = 16$

When  $x = 3$ ,  $u = 25$

$$\begin{aligned} \therefore \int_0^3 x \sqrt{x^2 + 16} dx &= \int_{16}^{25} u^{\frac{1}{2}} \left( \frac{1}{2} \frac{du}{dx} \right) dx \\ &= \frac{1}{2} \int_{16}^{25} u^{\frac{1}{2}} du \\ &= \frac{1}{2} \left[ \frac{u^{\frac{3}{2}}}{\frac{3}{2}} \right]_{16}^{25} \\ &= \frac{1}{2} \times \frac{2}{3} \left[ u^{\frac{3}{2}} \right]_{16}^{25} \\ &= \frac{1}{3} (125 - 64) = \frac{61}{3} = 20\frac{1}{3} \end{aligned}$$

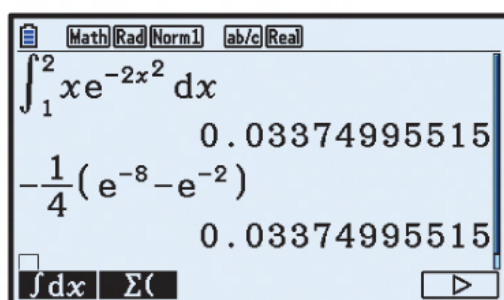


**e** Let  $u = -2x^2 \quad \therefore \frac{du}{dx} = -4x$

When  $x = 1$ ,  $u = -2$

When  $x = 2$ ,  $u = -8$

$$\begin{aligned} \therefore \int_1^2 x e^{-2x^2} dx &= \int_{-2}^{-8} e^u \left( -\frac{1}{4} \frac{du}{dx} \right) dx \\ &= -\frac{1}{4} \int_{-2}^{-8} e^u du \\ &= -\frac{1}{4} [e^u]_{-2}^{-8} \\ &= -\frac{1}{4} (e^{-8} - e^{-2}) \\ &\approx 0.0337 \end{aligned}$$

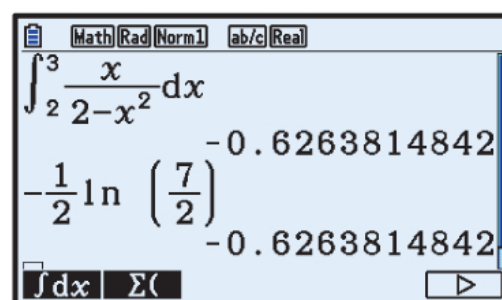


**f** Let  $u = 2 - x^2 \quad \therefore \frac{du}{dx} = -2x$

When  $x = 2$ ,  $u = -2$

When  $x = 3$ ,  $u = -7$

$$\begin{aligned} \therefore \int_2^3 \frac{x}{2-x^2} dx &= \int_{-2}^{-7} \frac{1}{u} \left( -\frac{1}{2} \frac{du}{dx} \right) dx \\ &= -\frac{1}{2} \int_{-2}^{-7} \frac{1}{u} du \\ &= -\frac{1}{2} [\ln |u|]_{-2}^{-7} \\ &= -\frac{1}{2} (\ln 7 - \ln 2) \\ &= -\frac{1}{2} \ln \left( \frac{7}{2} \right) \\ &\approx -0.626 \end{aligned}$$



**g** Let  $u = \ln x \quad \therefore \frac{du}{dx} = \frac{1}{x}$

When  $x = 1$ ,  $u = 0$

When  $x = 2$ ,  $u = \ln 2$

$$\begin{aligned} \therefore \int_1^2 \frac{\ln x}{x} dx &= \int_1^2 u \frac{du}{dx} dx \\ &= \int_0^{\ln 2} u du \\ &= \left[ \frac{u^2}{2} \right]_0^{\ln 2} \\ &= \frac{(\ln 2)^2}{2} - 0 \\ &\approx 0.240 \end{aligned}$$

Calculator screen showing the definite integral  $\int_1^2 \frac{\ln x}{x} dx$  and the result  $0.240226507$ . The manual calculation  $\frac{(\ln 2)^2}{2}$  is also shown, resulting in the same value.

**h** Let  $u = 1 - x^3 + x$

$\therefore \frac{du}{dx} = -3x^2 + 1$

When  $x = 0$ ,  $u = 1$

When  $x = 1$ ,  $u = 1$

$$\begin{aligned} \therefore \int_0^1 \frac{1 - 3x^2}{1 - x^3 + x} dx &= \int_0^1 \frac{1}{u} \frac{du}{dx} dx \\ &= \int_1^1 \frac{1}{u} du \\ &= 0 \end{aligned}$$

Calculator screen showing the definite integral  $\int_0^1 \frac{1 - 3x^2}{1 - x^3 + x} dx$  and the result  $0$ .

**i** Let  $u = x^3 - x^2 + 2x \quad \therefore \frac{du}{dx} = 3x^2 - 2x + 2$

When  $x = 2$ ,  $u = 8$

When  $x = 4$ ,  $u = 56$

$$\begin{aligned} \therefore \int_2^4 \frac{6x^2 - 4x + 4}{x^3 - x^2 + 2x} dx &= \int_2^4 \frac{1}{u} \left( 2 \frac{du}{dx} \right) dx \\ &= 2 \int_8^{56} \frac{1}{u} du \\ &= 2 [\ln |u|]_8^{56} \\ &= 2(\ln 56 - \ln 8) \\ &= 2 \ln 7 \\ &\approx 3.89 \end{aligned}$$

Calculator screen showing the definite integral  $\int_2^4 \frac{6x^2 - 4x + 4}{x^3 - x^2 + 2x} dx$  and the result  $3.891820298$ . The manual calculation  $2 \ln 7$  is also shown, resulting in the same value.

**2 a** Let  $u = \cos x \quad \therefore \frac{du}{dx} = -\sin x$

When  $x = 0$ ,  $u = \cos 0 = 1$

When  $x = \frac{\pi}{3}$ ,  $u = \cos \frac{\pi}{3} = \frac{1}{2}$

$$\begin{aligned} \therefore \int_0^{\frac{\pi}{3}} \frac{\sin x}{\sqrt{\cos x}} dx &= \int_0^{\frac{\pi}{3}} u^{-\frac{1}{2}} \left(-\frac{du}{dx}\right) dx \\ &= -\int_1^{\frac{1}{2}} u^{-\frac{1}{2}} du \\ &= -\left[2u^{\frac{1}{2}}\right]_1^{\frac{1}{2}} \\ &= -\left(2\sqrt{\frac{1}{2}} - 2\sqrt{1}\right) \\ &= 2 - \sqrt{2} \end{aligned}$$

**c** Let  $u = \cos x \quad \therefore \frac{du}{dx} = -\sin x$

When  $x = 0$ ,  $u = \cos 0 = 1$

When  $x = \frac{\pi}{4}$ ,  $u = \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$$\begin{aligned} \therefore \int_0^{\frac{\pi}{4}} \tan x dx &= \int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos x} dx \\ &= \int_0^{\frac{\pi}{4}} \frac{1}{u} \left(-\frac{du}{dx}\right) dx \\ &= -\int_1^{\frac{1}{\sqrt{2}}} \frac{1}{u} du \\ &= \int_{\frac{1}{\sqrt{2}}}^1 \frac{1}{u} du \\ &= \left[\ln |u|\right]_{\frac{1}{\sqrt{2}}}^1 \\ &= \ln 1 - \ln \left(\frac{1}{\sqrt{2}}\right) \\ &= \ln \sqrt{2} \\ &= \frac{1}{2} \ln 2 \end{aligned}$$

**b** Let  $u = \sin x \quad \therefore \frac{du}{dx} = \cos x$

When  $x = 0$ ,  $u = \sin 0 = 0$

When  $x = \frac{\pi}{6}$ ,  $u = \sin \frac{\pi}{6} = \frac{1}{2}$

$$\begin{aligned} \therefore \int_0^{\frac{\pi}{6}} \sin^2 x \cos x dx &= \int_0^{\frac{\pi}{6}} u^2 \frac{du}{dx} dx \\ &= \int_0^{\frac{1}{2}} u^2 du \\ &= \left[\frac{u^3}{3}\right]_0^{\frac{1}{2}} \\ &= \frac{1}{3} \left(\frac{1}{2}\right)^3 \\ &= \frac{1}{24} \end{aligned}$$

**d** Let  $u = \sin x \quad \therefore \frac{du}{dx} = \cos x$

When  $x = \frac{\pi}{6}$ ,  $u = \sin \frac{\pi}{6} = \frac{1}{2}$

When  $x = \frac{\pi}{2}$ ,  $u = \sin \frac{\pi}{2} = 1$

$$\begin{aligned} \therefore \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cot x dx &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos x}{\sin x} dx \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1}{u} \frac{du}{dx} dx \\ &= \int_{\frac{1}{2}}^1 \frac{1}{u} du \\ &= \left[\ln |u|\right]_{\frac{1}{2}}^1 \\ &= \ln 1 - \ln \frac{1}{2} \\ &= \ln 2 \end{aligned}$$



**e** Let  $u = 1 - \sin x \quad \therefore \frac{du}{dx} = -\cos x$

When  $x = 0, \quad u = 1 - \sin 0 = 1$

When  $x = \frac{\pi}{6}, \quad u = 1 - \sin \frac{\pi}{6} = \frac{1}{2}$

$$\begin{aligned} \therefore \int_0^{\frac{\pi}{6}} \frac{\cos x}{1 - \sin x} dx &= \int_0^{\frac{\pi}{6}} \frac{1}{u} \left( -\frac{du}{dx} \right) dx \\ &= - \int_1^{\frac{1}{2}} \frac{1}{u} du \\ &= \int_{\frac{1}{2}}^1 \frac{1}{u} du \\ &= \left[ \ln |u| \right]_{\frac{1}{2}}^1 \\ &= \ln 1 - \ln \frac{1}{2} \\ &= \ln 2 \end{aligned}$$

**f** Let  $u = \tan x \quad \therefore \frac{du}{dx} = \sec^2 x$

When  $x = 0, \quad u = \tan 0 = 0$

When  $x = \frac{\pi}{4}, \quad u = \tan \frac{\pi}{4} = 1$

$$\begin{aligned} \therefore \int_0^{\frac{\pi}{4}} \sec^2 x \tan^3 x dx &= \int_0^{\frac{\pi}{4}} u^3 \frac{du}{dx} dx \\ &= \int_0^1 u^3 du \\ &= \left[ \frac{u^4}{4} \right]_0^1 \\ &= \frac{1}{4} \end{aligned}$$

**3** Let  $u = x^2 + 2x \quad \therefore \frac{du}{dx} = 2x + 2$

When  $x = 0, \quad u = 0$

When  $x = 1, \quad u = 3$

$$\begin{aligned} \therefore \int_0^1 (x^2 + 2x)^n (x + 1) dx &= \frac{1}{2} \int_0^1 (x^2 + 2x)^n (2x + 2) dx \\ &= \frac{1}{2} \int_0^1 u^n \frac{du}{dx} dx \\ &= \frac{1}{2} \int_0^3 u^n du \end{aligned}$$

If  $n \neq -1$ , the integral  $= \frac{1}{2} \left[ \frac{u^{n+1}}{n+1} \right]_0^3 = \frac{1}{2} \left( \frac{3^{n+1}}{n+1} \right) = \frac{3^{n+1}}{2n+2}$

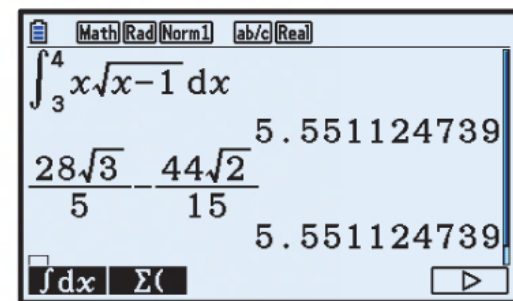
If  $n = -1$ , the integral  $= \frac{1}{2} \int_0^3 \frac{1}{u} du = \frac{1}{2} [\ln |u|]_0^3$  which is undefined as  $\ln 0$  is not defined.

**4 a** Let  $u = x - 1 \quad \therefore \frac{du}{dx} = 1$

When  $x = 3$ ,  $u = 2$

When  $x = 4$ ,  $u = 3$

$$\begin{aligned} \therefore \int_3^4 x\sqrt{x-1} \, dx &= \int_2^3 (u+1)\sqrt{u} \, du \\ &= \int_2^3 (u^{\frac{3}{2}} + u^{\frac{1}{2}}) \, du \\ &= \left[ \frac{u^{\frac{5}{2}}}{\frac{5}{2}} + \frac{u^{\frac{3}{2}}}{\frac{3}{2}} \right]_2^3 \\ &= \left( \frac{2}{5}(3)^{\frac{5}{2}} + \frac{2}{3}(3)^{\frac{3}{2}} \right) - \left( \frac{2}{5}(2)^{\frac{5}{2}} + \frac{2}{3}(2)^{\frac{3}{2}} \right) \\ &= \frac{2}{5}(9\sqrt{3}) + \frac{2}{3}(3\sqrt{3}) - \frac{2}{5}(4\sqrt{2}) - \frac{2}{3}(2\sqrt{2}) \\ &= \left( \frac{18}{5} + 2 \right) \sqrt{3} - \left( \frac{8}{5} + \frac{4}{3} \right) \sqrt{2} \\ &= \frac{28\sqrt{3}}{5} - \frac{44\sqrt{2}}{15} \end{aligned}$$



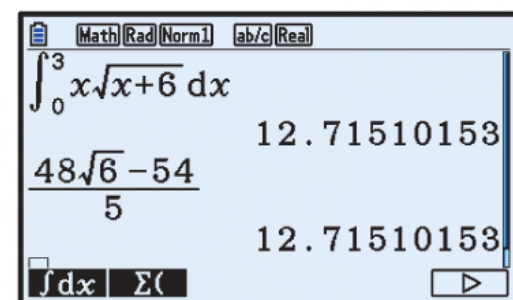
Calculator screen showing the integral  $\int_3^4 x\sqrt{x-1} \, dx$  evaluated to 5.551124739. The expression  $\frac{28\sqrt{3}}{5} - \frac{44\sqrt{2}}{15}$  is also shown, matching the manual calculation.

**b** Let  $u = x + 6 \quad \therefore \frac{du}{dx} = 1$

When  $x = 0$ ,  $u = 6$

When  $x = 3$ ,  $u = 9$

$$\begin{aligned} \therefore \int_0^3 x\sqrt{x+6} \, dx &= \int_6^9 (u-6)\sqrt{u} \, du \\ &= \int_6^9 (u^{\frac{3}{2}} - 6u^{\frac{1}{2}}) \, du \\ &= \left[ \frac{u^{\frac{5}{2}}}{\frac{5}{2}} - \frac{6u^{\frac{3}{2}}}{\frac{3}{2}} \right]_6^9 \\ &= \left( \frac{2}{5}(9)^{\frac{5}{2}} - 4(9)^{\frac{3}{2}} \right) - \left( \frac{2}{5}(6)^{\frac{5}{2}} - 4(6)^{\frac{3}{2}} \right) \\ &= \frac{2}{5}(3^5) - 4(3^3) - \frac{2}{5} \times 36\sqrt{6} + 4 \times 6\sqrt{6} \\ &= \frac{486}{5} - 108 - \frac{72}{5}\sqrt{6} + 24\sqrt{6} \\ &= -\frac{54}{5} + \frac{48}{5}\sqrt{6} \\ &= \frac{48\sqrt{6} - 54}{5} \end{aligned}$$



Calculator screen showing the integral  $\int_0^3 x\sqrt{x+6} \, dx$  evaluated to 12.71510153. The expression  $\frac{48\sqrt{6} - 54}{5}$  is also shown, matching the manual calculation.

c Let  $u = x - 2 \quad \therefore \frac{du}{dx} = 1$

When  $x = 2$ ,  $u = 0$

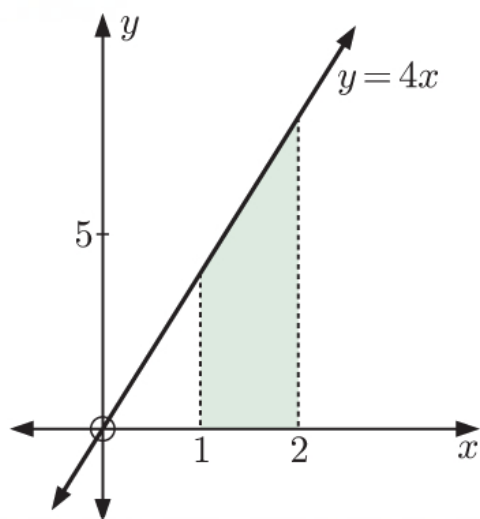
When  $x = 5$ ,  $u = 3$

$$\begin{aligned} \therefore \int_2^5 x^2 \sqrt{x-2} \, dx &= \int_0^3 (u+2)^2 \sqrt{u} \, du \\ &= \int_0^3 (u^2 + 4u + 4) \sqrt{u} \, du \\ &= \int_0^3 (u^{\frac{5}{2}} + 4u^{\frac{3}{2}} + 4u^{\frac{1}{2}}) \, du \\ &= \left[ \frac{u^{\frac{7}{2}}}{\frac{7}{2}} + \frac{4u^{\frac{5}{2}}}{\frac{5}{2}} + \frac{4u^{\frac{3}{2}}}{\frac{3}{2}} \right]_0^3 \\ &= \frac{2}{7}(3)^{\frac{7}{2}} + \frac{8}{5}(3)^{\frac{5}{2}} + \frac{8}{3}(3)^{\frac{3}{2}} - 0 \\ &= \frac{2}{7}(27\sqrt{3}) + \frac{8}{5}(9\sqrt{3}) + \frac{8}{3}(3\sqrt{3}) \\ &= \frac{1054\sqrt{3}}{35} \end{aligned}$$

Calculator screen showing the integral result:  $\int_2^5 x^2 \sqrt{x-2} \, dx = \frac{1054\sqrt{3}}{35} \approx 52.15947289$

## EXERCISE 22C

1 a



When  $x = 1$ ,  $y = 4(1) = 4$

When  $x = 2$ ,  $y = 4(2) = 8$

Area = area of trapezium

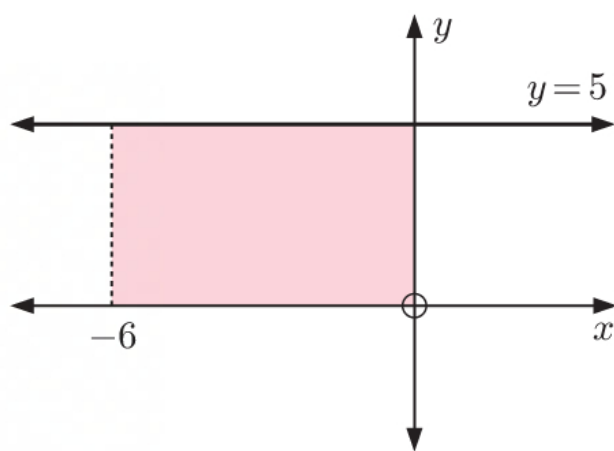
$$= \left( \frac{4+8}{2} \right) \times 1$$

$$= 6 \text{ units}^2$$

b Area =  $\int_1^2 4x \, dx$

$$\begin{aligned} &= [2x^2]_1^2 \\ &= 2(2)^2 - 2(1)^2 \\ &= 6 \text{ units}^2 \end{aligned}$$

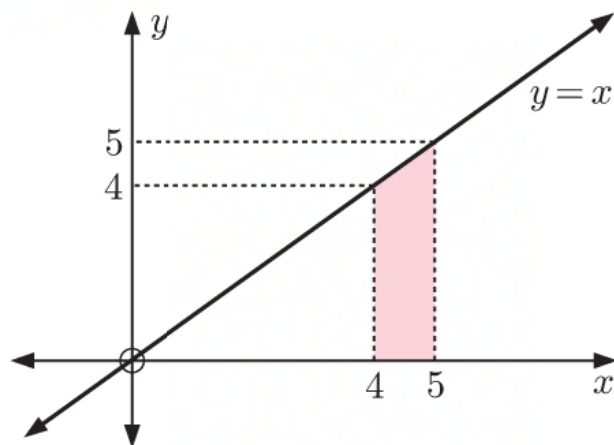
2 a



$$\begin{aligned} \text{i Area} &= 6 \times 5 \\ &= 30 \text{ units}^2 \end{aligned}$$

$$\begin{aligned} \text{ii Area} &= \int_{-6}^0 5 \, dx \\ &= [5x]_{-6}^0 \\ &= 5(0) - 5(-6) \\ &= 30 \text{ units}^2 \end{aligned}$$

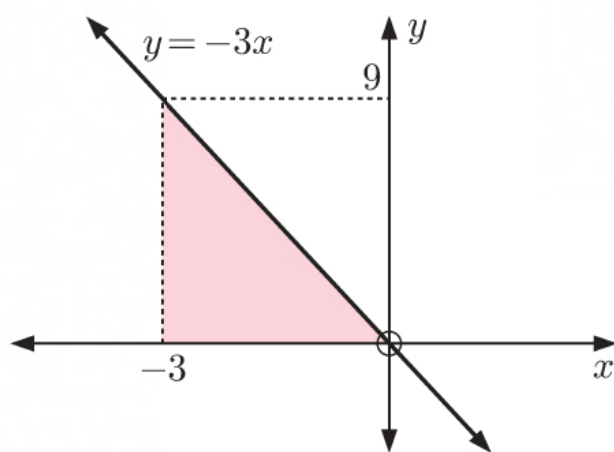
b



$$\begin{aligned} \text{i Area} &= \text{area of trapezium} \\ &= \left( \frac{4+5}{2} \right) \times 1 \\ &= \frac{9}{2} \\ &= 4\frac{1}{2} \text{ units}^2 \end{aligned}$$

$$\begin{aligned} \text{ii Area} &= \int_4^5 x \, dx \\ &= \left[ \frac{1}{2}x^2 \right]_4^5 \\ &= \frac{1}{2}(5)^2 - \frac{1}{2}(4)^2 \\ &= \frac{25}{2} - \frac{16}{2} \\ &= 4\frac{1}{2} \text{ units}^2 \end{aligned}$$

c

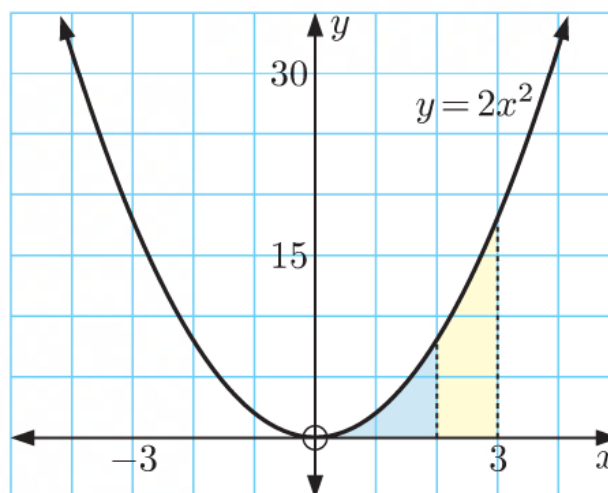


$$\begin{aligned} \text{i Area} &= \frac{1}{2} \times 3 \times 9 \\ &= 13\frac{1}{2} \text{ units}^2 \end{aligned}$$

$$\begin{aligned} \text{ii Area} &= \int_{-3}^0 -3x \, dx \\ &= \left[ -\frac{3}{2}x^2 \right]_{-3}^0 \\ &= 0 - \left( -\frac{3}{2}(9) \right) \\ &= 13\frac{1}{2} \text{ units}^2 \end{aligned}$$

3 a Area of blue shaded region

$$\begin{aligned} &= \int_0^2 2x^2 \, dx \\ &= \left[ \frac{2}{3}x^3 \right]_0^2 \\ &= \frac{16}{3} - 0 \\ &= 5\frac{1}{3} \text{ units}^2 \end{aligned}$$

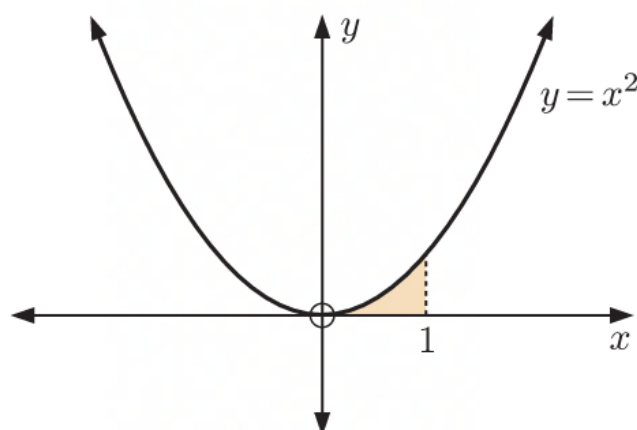




**b** Area of yellow shaded region

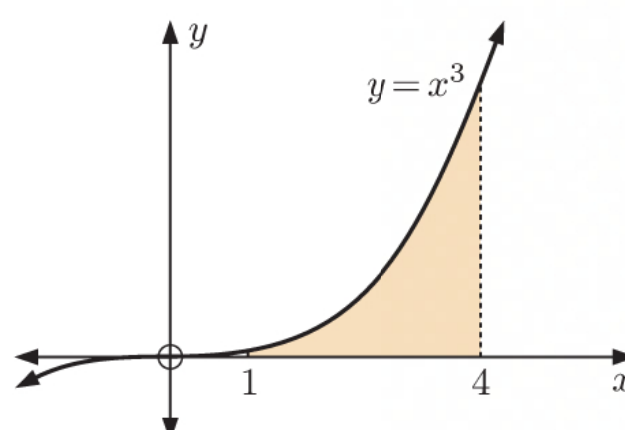
$$\begin{aligned}
 &= \int_2^3 2x^2 dx \\
 &= \left[ \frac{2}{3}x^3 \right]_2^3 \\
 &= \frac{54}{3} - \frac{16}{3} \\
 &= 12\frac{2}{3} \text{ units}^2
 \end{aligned}$$

**4 a**



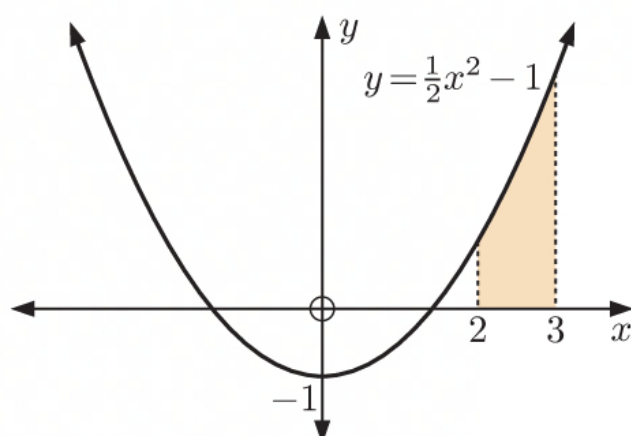
$$\begin{aligned}
 \text{Area} &= \int_0^1 x^2 dx \\
 &= \left[ \frac{1}{3}x^3 \right]_0^1 \\
 &= \frac{1}{3} - 0 \\
 &= \frac{1}{3} \text{ units}^2
 \end{aligned}$$

**b**



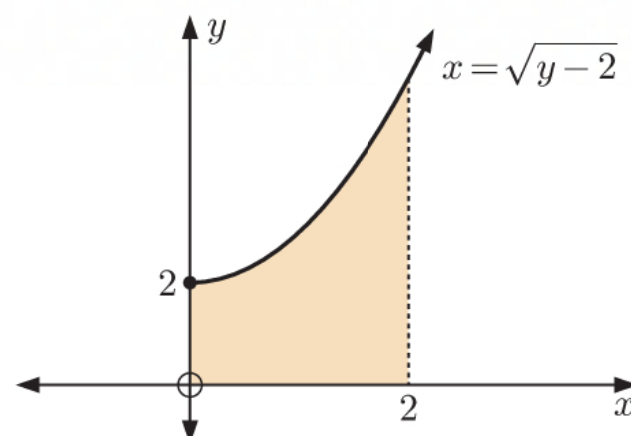
$$\begin{aligned}
 \text{Area} &= \int_1^4 x^3 dx \\
 &= \left[ \frac{1}{4}x^4 \right]_1^4 \\
 &= 64 - \frac{1}{4} \\
 &= 63\frac{3}{4} \text{ units}^2
 \end{aligned}$$

**c**



$$\begin{aligned}
 \text{Area} &= \int_2^3 \left( \frac{1}{2}x^2 - 1 \right) dx \\
 &= \left[ \frac{1}{6}x^3 - x \right]_2^3 \\
 &= \left( \frac{27}{6} - 3 \right) - \left( \frac{8}{6} - 2 \right) \\
 &= 2\frac{1}{6} \text{ units}^2
 \end{aligned}$$

**d**



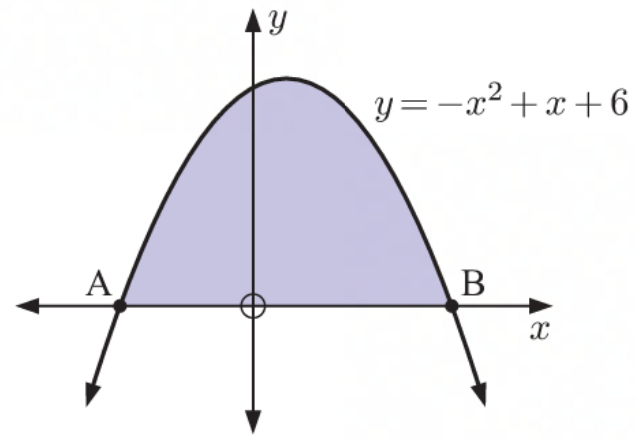
The curve has equation  $x = \sqrt{y - 2}$   
or  $y = x^2 + 2, x \geq 0$

$$\begin{aligned}
 \therefore \text{area} &= \int_0^2 (x^2 + 2) dx \\
 &= \left[ \frac{1}{3}x^3 + 2x \right]_0^2 \\
 &= \left( \frac{8}{3} + 4 \right) - 0 \\
 &= 6\frac{2}{3} \text{ units}^2
 \end{aligned}$$

- 5 a** A and B are the  $x$ -intercepts of  $y = -x^2 + x + 6$ .

$$\begin{aligned}\text{When } y = 0, \quad -x^2 + x + 6 &= 0 \\ \therefore x^2 - x - 6 &= 0 \\ \therefore (x + 2)(x - 3) &= 0 \\ \therefore x &= -2 \text{ or } 3\end{aligned}$$

$\therefore$  A is  $(-2, 0)$  and B is  $(3, 0)$ .



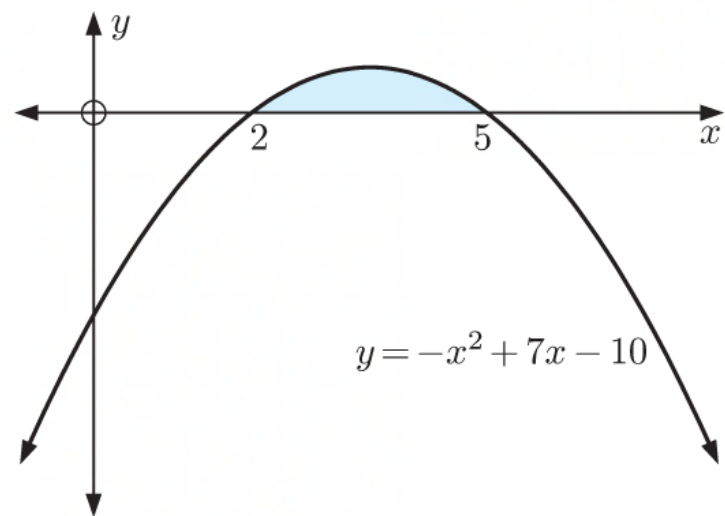
$$\begin{aligned}\text{b Area} &= \int_{-2}^3 (-x^2 + x + 6) dx \\ &= \left[ -\frac{1}{3}x^3 + \frac{1}{2}x^2 + 6x \right]_{-2}^3 \\ &= \left( -9 + \frac{9}{2} + 18 \right) - \left( \frac{8}{3} + 2 - 12 \right) \\ &= 13\frac{1}{2} - \left( -7\frac{1}{3} \right) \\ &= 20\frac{5}{6} \text{ units}^2\end{aligned}$$

- 6 a**  $y = -x^2 + 7x - 10$

$$\begin{aligned}\text{When } y = 0, \quad -x^2 + 7x - 10 &= 0 \\ \therefore x^2 - 7x + 10 &= 0 \\ \therefore (x - 2)(x - 5) &= 0 \\ \therefore x &= 2 \text{ or } 5\end{aligned}$$

$\therefore$  the  $x$ -intercepts are 2 and 5.

$$\begin{aligned}\therefore \text{enclosed area} &= \int_2^5 (-x^2 + 7x - 10) dx \\ &= \left[ -\frac{1}{3}x^3 + \frac{7}{2}x^2 - 10x \right]_2^5 \\ &= \left( -\frac{125}{3} + \frac{175}{2} - 50 \right) - \left( -\frac{8}{3} + 14 - 20 \right) \\ &= 4\frac{1}{2} \text{ units}^2\end{aligned}$$

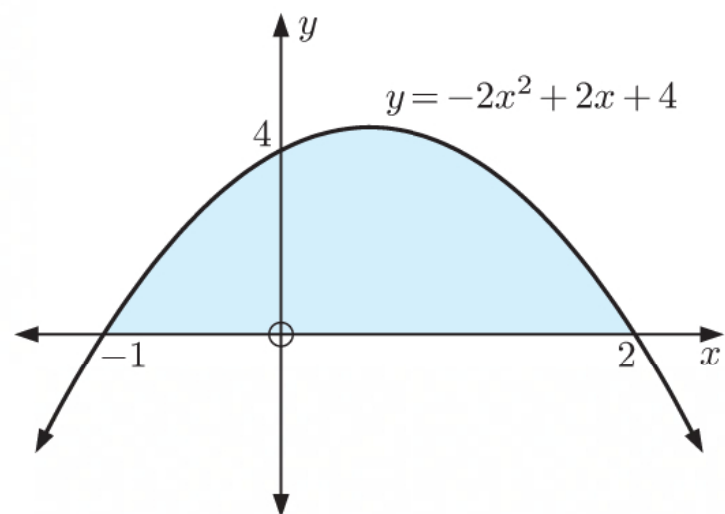


- b**  $y = -2x^2 + 2x + 4$

$$\begin{aligned}\text{When } y = 0, \quad -2x^2 + 2x + 4 &= 0 \\ \therefore -2(x^2 - x - 2) &= 0 \\ \therefore -2(x + 1)(x - 2) &= 0 \\ \therefore x &= -1 \text{ or } 2\end{aligned}$$

$\therefore$  the  $x$ -intercepts are  $-1$  and  $2$ .

$$\begin{aligned}\therefore \text{enclosed area} &= \int_{-1}^2 (-2x^2 + 2x + 4) dx \\ &= \left[ -\frac{2}{3}x^3 + x^2 + 4x \right]_{-1}^2 \\ &= \left( -\frac{16}{3} + 4 + 8 \right) - \left( \frac{2}{3} + 1 - 4 \right) \\ &= 9 \text{ units}^2\end{aligned}$$



•  $y = 3 - x^2$

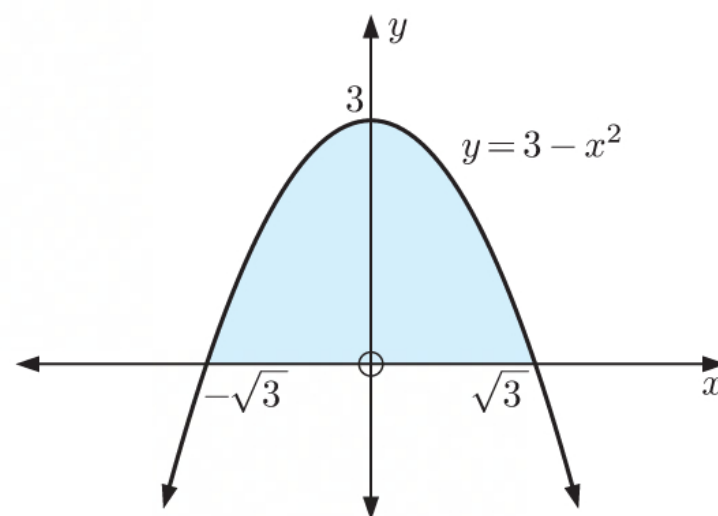
When  $y = 0$ ,  $3 - x^2 = 0$

$$\therefore x^2 = 3$$

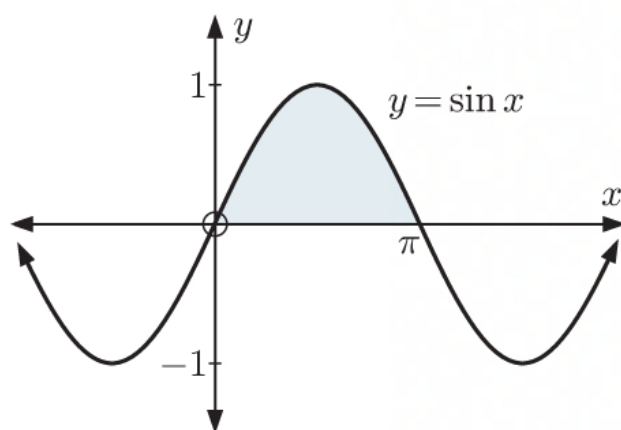
$$\therefore x = \pm\sqrt{3}$$

$\therefore$  the  $x$ -intercepts are  $\sqrt{3}$  and  $-\sqrt{3}$ .

$$\begin{aligned}\therefore \text{enclosed area} &= \int_{-\sqrt{3}}^{\sqrt{3}} (3 - x^2) dx \\ &= \left[ 3x - \frac{1}{3}x^3 \right]_{-\sqrt{3}}^{\sqrt{3}} \\ &= (3\sqrt{3} - \sqrt{3}) - (-3\sqrt{3} + \sqrt{3}) \\ &= 4\sqrt{3} \text{ units}^2\end{aligned}$$

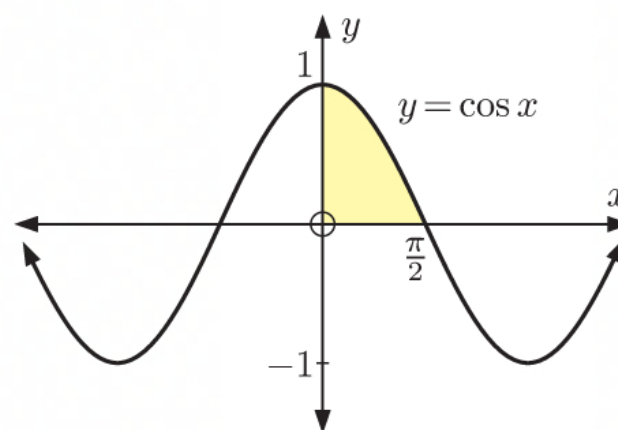


7



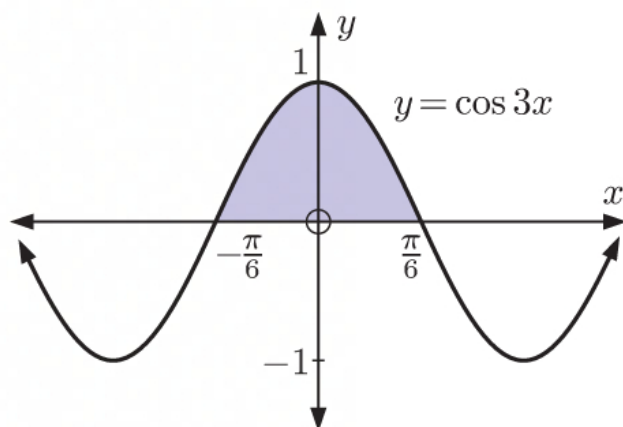
$$\begin{aligned}\text{Area} &= \int_0^{\pi} \sin x \, dx \\ &= [-\cos x]_0^{\pi} \\ &= [-\cos \pi + \cos 0] \\ &= -(-1) + 1 \\ &= 2 \text{ units}^2\end{aligned}$$

8



$$\begin{aligned}\text{Area} &= \int_0^{\frac{\pi}{2}} \cos x \, dx \\ &= [\sin x]_0^{\frac{\pi}{2}} \\ &= \sin \frac{\pi}{2} - \sin 0 \\ &= 1 \text{ unit}^2\end{aligned}$$

9



The period of  $y = \cos 3x$  is  $\frac{2\pi}{3}$ , so the  $x$ -intercepts under one arch are  $-\frac{\pi}{6}$  and  $\frac{\pi}{6}$ .

$$\begin{aligned}\text{The required area} &= \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \cos 3x \, dx \\ &= \left[ \frac{1}{3} \sin 3x \right]_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \\ &= \frac{1}{3} \left( \sin \frac{\pi}{2} - \sin \left( -\frac{\pi}{2} \right) \right) \\ &= \frac{1}{3} (1 - (-1)) \\ &= \frac{2}{3} \text{ units}^2\end{aligned}$$

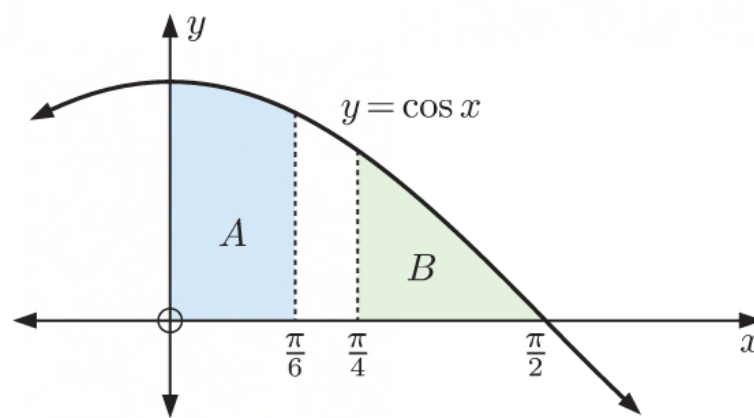
**10 a** Region  $A$  appears to be larger.

$$\begin{aligned}
 \text{b Area of region } A &= \int_0^{\frac{\pi}{6}} \cos x \, dx \\
 &= \left[ \sin x \right]_0^{\frac{\pi}{6}} \\
 &= \frac{1}{2} - 0 \\
 &= \frac{1}{2} \text{ units}^2
 \end{aligned}$$

$$\begin{aligned}
 y = \cos x = 0 \text{ when } x &= \frac{\pi}{2} \\
 \therefore \text{ the } x\text{-intercept is } &\frac{\pi}{2}.
 \end{aligned}$$

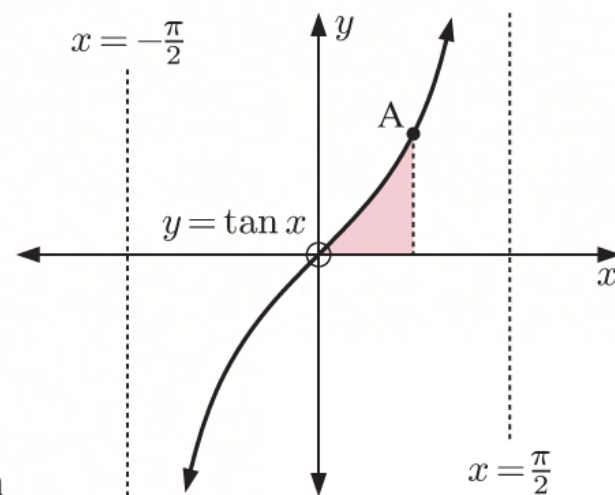
$$\begin{aligned}
 \therefore \text{ area of region } B &= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos x \, dx \\
 &= \left[ \sin x \right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} \\
 &= \left( 1 - \frac{1}{\sqrt{2}} \right) \text{ units}^2 \\
 &\approx 0.293 \text{ units}^2
 \end{aligned}$$

$\therefore$  region  $A$  is larger than region  $B$ .



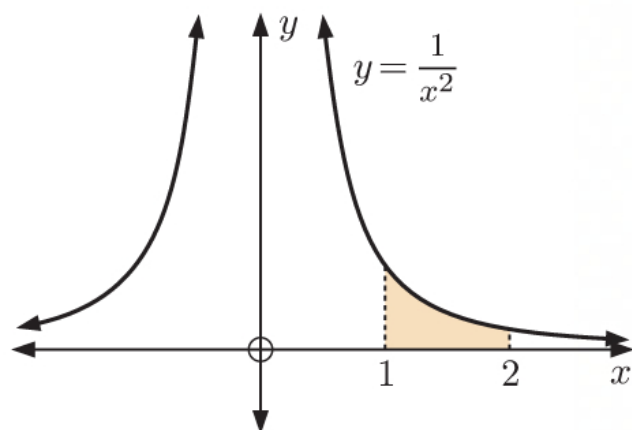
**11 a**  $y = \tan x$ ,  $-\frac{\pi}{2} < x < \frac{\pi}{2}$   
 $y = 1$  when  $x = \frac{\pi}{4}$   
 $\therefore$  A has  $x$ -coordinate  $\frac{\pi}{4}$ .

$$\begin{aligned}
 \text{b} \quad &\int \tan x \, dx \\
 &= \int \frac{\sin x}{\cos x} \, dx \\
 &= \int \frac{1}{u} \left( -\frac{du}{dx} \right) dx \quad \{u = \cos x, \frac{du}{dx} = -\sin x\} \\
 &= - \int \frac{1}{u} \, du \\
 &= -\ln |u| + c \\
 &= -\ln |\cos x| + c
 \end{aligned}$$

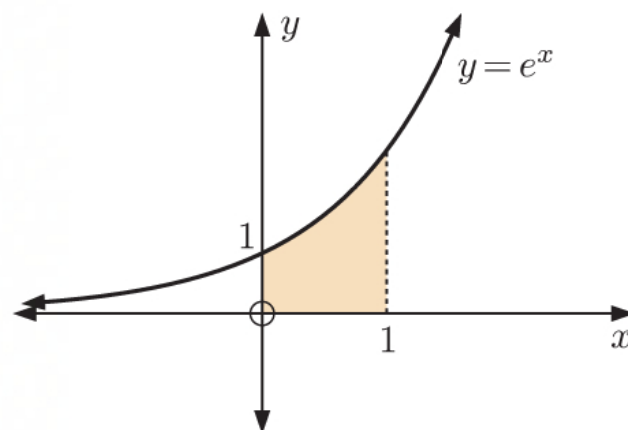


$$\begin{aligned}
 \text{c Shaded area} &= \int_0^{\frac{\pi}{4}} \tan x \, dx \\
 &= \left[ -\ln |\cos x| \right]_0^{\frac{\pi}{4}} \\
 &= -\ln \left( \frac{1}{\sqrt{2}} \right) - (-\ln 1) \\
 &= -\ln(2^{-\frac{1}{2}}) \\
 &= \ln(2^{\frac{1}{2}}) \\
 &= \ln \sqrt{2} \text{ units}^2
 \end{aligned}$$

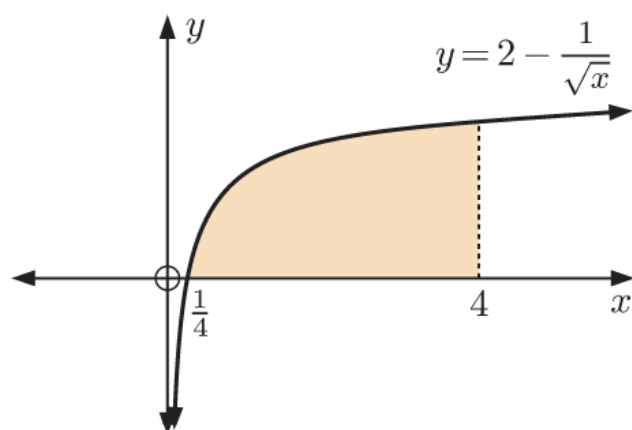


**12 a**

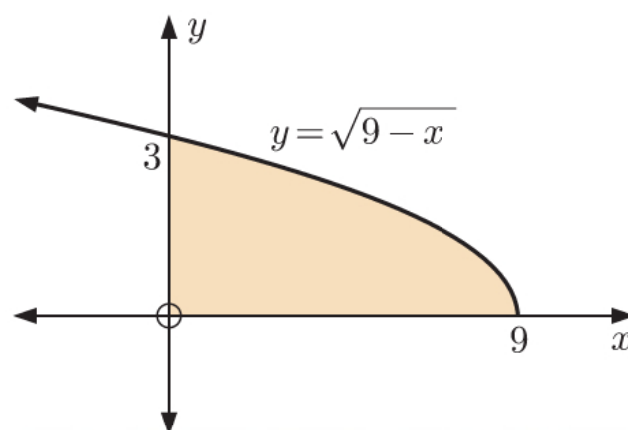
$$\begin{aligned}
 \text{Area} &= \int_1^2 \frac{1}{x^2} dx \\
 &= \int_1^2 x^{-2} dx \\
 &= \left[ -\frac{1}{x} \right]_1^2 \\
 &= -\frac{1}{2} - (-1) \\
 &= \frac{1}{2} \text{ units}^2
 \end{aligned}$$

**b**

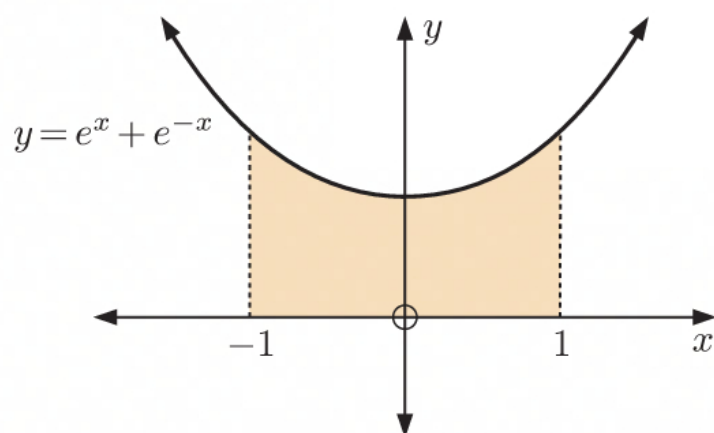
$$\begin{aligned}
 \text{Area} &= \int_0^1 e^x dx \\
 &= [e^x]_0^1 \\
 &= e^1 - e^0 \\
 &= (e - 1) \text{ units}^2
 \end{aligned}$$

**c**

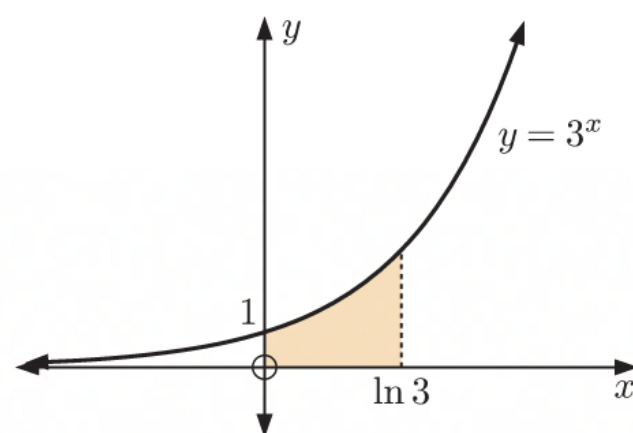
$$\begin{aligned}
 \text{When } y = 0, \quad 2 - \frac{1}{\sqrt{x}} &= 0 \\
 \therefore 2 &= \frac{1}{\sqrt{x}} \\
 \therefore \sqrt{x} &= \frac{1}{2} \\
 \therefore x &= \frac{1}{4} \quad \{x > 0\} \\
 \therefore \text{the } x\text{-intercept is } \frac{1}{4}. \\
 \therefore \text{area} &= \int_{\frac{1}{4}}^4 \left( 2 - \frac{1}{\sqrt{x}} \right) dx \\
 &= \int_{\frac{1}{4}}^4 \left( 2 - x^{-\frac{1}{2}} \right) dx \\
 &= \left[ 2x - 2x^{\frac{1}{2}} \right]_{\frac{1}{4}}^4 \\
 &= (8 - 4) - \left( \frac{1}{2} - 1 \right) \\
 &= 4\frac{1}{2} \text{ units}^2
 \end{aligned}$$

**d**

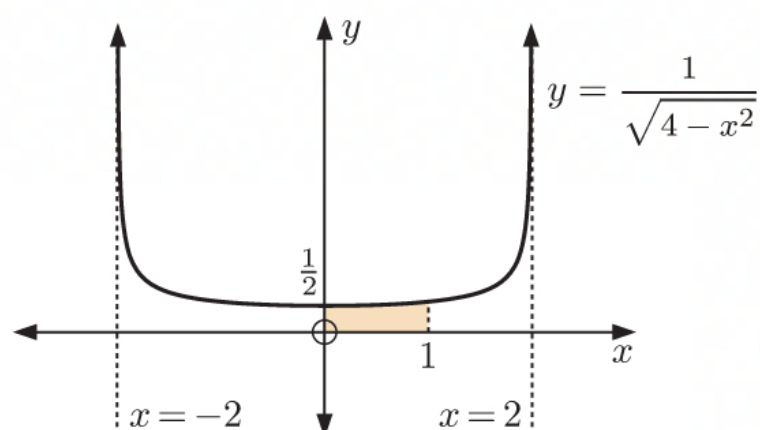
$$\begin{aligned}
 \text{When } y = 0, \quad \sqrt{9 - x} &= 0 \\
 \therefore 9 - x &= 0 \\
 \therefore x &= 9 \\
 \therefore \text{the } x\text{-intercept is } 9. \\
 \therefore \text{area} &= \int_0^9 \sqrt{9 - x} dx \\
 &= \int_0^9 (9 - x)^{\frac{1}{2}} dx \\
 &= \left[ -\frac{2}{3}(9 - x)^{\frac{3}{2}} \right]_0^9 \\
 &= 0 - \left( -\frac{2}{3}(27) \right) \\
 &= 18 \text{ units}^2
 \end{aligned}$$

**e**

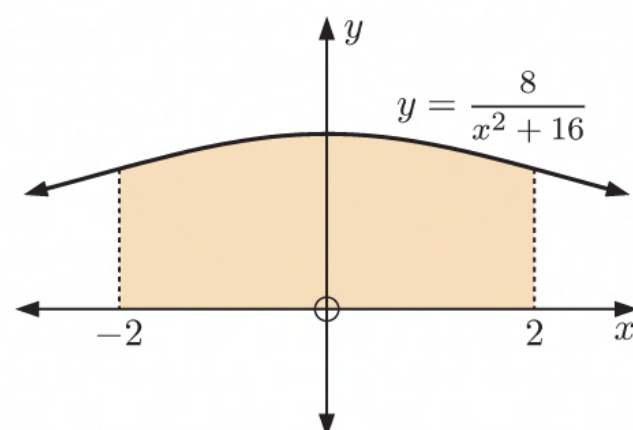
$$\begin{aligned}
 \text{Area} &= \int_{-1}^1 (e^x + e^{-x}) dx \\
 &= [e^x - e^{-x}]_{-1}^1 \\
 &= \left(e - \frac{1}{e}\right) - \left(\frac{1}{e} - e\right) \\
 &= \left(2e - \frac{2}{e}\right) \text{ units}^2
 \end{aligned}$$

**f**

$$\begin{aligned}
 \text{Area} &= \int_0^{\ln 3} 3^x dx \\
 &= \left[\frac{3^x}{\ln 3}\right]_0^{\ln 3} \\
 &= \frac{3^{\ln 3}}{\ln 3} - \frac{1}{\ln 3} \\
 &= \frac{3^{\ln 3} - 1}{\ln 3} \text{ units}^2
 \end{aligned}$$

**g**

$$\begin{aligned}
 \text{Area} &= \int_0^1 \frac{1}{\sqrt{4 - x^2}} dx \\
 &= \int_0^1 \frac{1}{\sqrt{2^2 - x^2}} dx \\
 &= [\arcsin \frac{x}{2}]_0^1 \\
 &= \arcsin \frac{1}{2} - \arcsin 0 \\
 &= \frac{\pi}{6} - 0 \\
 &= \frac{\pi}{6} \text{ units}^2
 \end{aligned}$$

**h**

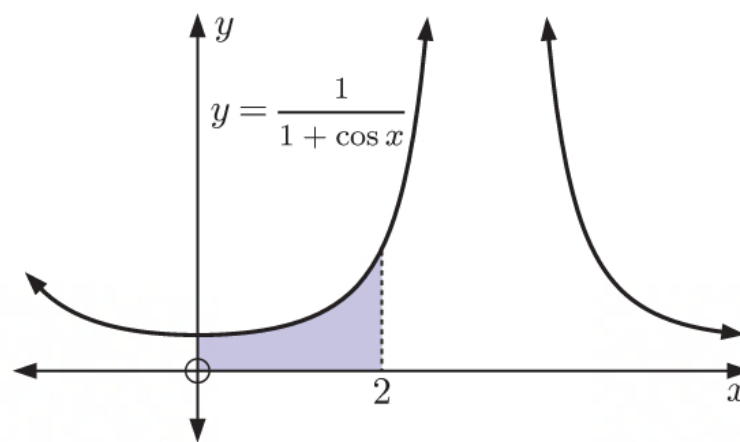
$$\begin{aligned}
 \text{Area} &= \int_{-2}^2 \frac{8}{x^2 + 16} dx \\
 &= 2 \int_{-2}^2 \frac{4}{x^2 + 4^2} dx \\
 &= [2 \arctan \frac{x}{4}]_{-2}^2 \\
 &= 2 \arctan \frac{1}{2} - 2 \arctan \left(-\frac{1}{2}\right) \\
 &= 2 \arctan \frac{1}{2} + 2 \arctan \frac{1}{2} \\
 &= 4 \arctan \frac{1}{2} \text{ units}^2
 \end{aligned}$$

**13**

$$\begin{aligned}
 \text{a } \frac{d}{dx} \left( \frac{\sin x}{1 + \cos x} \right) &= \frac{\cos x(1 + \cos x) - \sin x(-\sin x)}{(1 + \cos x)^2} \\
 &= \frac{\cos x + \cos^2 x + \sin^2 x}{(1 + \cos x)^2} \\
 &= \frac{1 + \cos x}{(1 + \cos x)^2} \\
 &= \frac{1}{1 + \cos x}
 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \frac{d}{dx} \left( \frac{\sin x}{1 + \cos x} \right) &= \frac{1}{1 + \cos x} \\ \therefore \int \frac{1}{1 + \cos x} dx &= \frac{\sin x}{1 + \cos x} + c \end{aligned}$$

$$\begin{aligned} \therefore \text{shaded area} &= \int_0^2 \frac{1}{1 + \cos x} dx \\ &= \left[ \frac{\sin x}{1 + \cos x} \right]_0^2 \\ &= \frac{\sin 2}{1 + \cos 2} - \frac{\sin 0}{1 + \cos 0} \text{ units}^2 \\ &= \frac{\sin 2}{1 + \cos 2} \text{ units}^2 \quad (\approx 1.56 \text{ units}^2) \end{aligned}$$

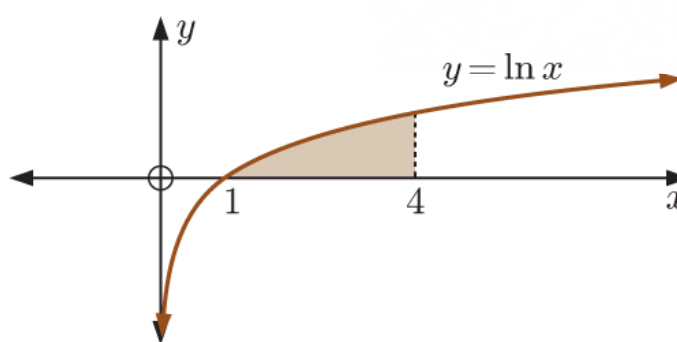


$$\begin{aligned} \mathbf{14} \quad \mathbf{a} \quad \text{We integrate by parts with } u &= \ln x \quad v' = 1 \\ u' &= \frac{1}{x} \quad v = x \end{aligned}$$

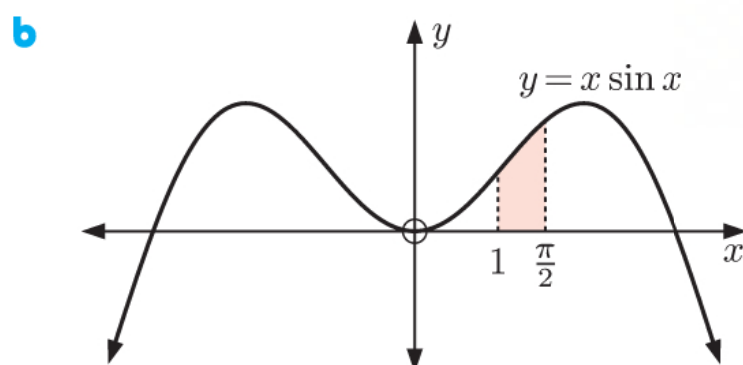
$$\begin{aligned} \therefore \int \ln x dx &= x \ln x - \int \left( \frac{1}{x} \right) x dx \\ &= x \ln x - \int 1 dx \\ &= x \ln x - x + c \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \text{When } y &= 0, \quad \ln x = 0 \\ \therefore x &= 1 \\ \therefore \text{the } x\text{-intercept is } 1. \end{aligned}$$

$$\begin{aligned} \therefore \text{area} &= \int_1^4 \ln x dx \\ &= [x \ln x - x]_1^4 \\ &= (4 \ln 4 - 4) - (\ln 1 - 1) \\ &= 8 \ln 2 - 4 + 1 \\ &= (8 \ln 2 - 3) \text{ units}^2 \quad (\approx 2.55 \text{ units}^2) \end{aligned}$$



$$\begin{aligned} \mathbf{15} \quad \mathbf{a} \quad \int x \sin x dx &= -x \cos x - \int -\cos x dx \quad \leftarrow \begin{cases} u = x & v' = \sin x \\ u' = 1 & v = -\cos x \end{cases} \\ &= -x \cos x - (-\sin x) + c \\ &= -x \cos x + \sin x + c \end{aligned}$$



$$\begin{aligned} \text{Area} &= \int_1^{\pi/2} x \sin x dx \\ &= [-x \cos x + \sin x]_1^{\pi/2} \\ &= \left( -\frac{\pi}{2} \cos \frac{\pi}{2} + \sin \frac{\pi}{2} \right) - (-\cos 1 + \sin 1) \\ &= (1 + \cos 1 - \sin 1) \text{ units}^2 \end{aligned}$$

**16 a** Area =  $\int_0^b \sqrt{x} \, dx$

$$\therefore 1 = \int_0^b x^{\frac{1}{2}} \, dx$$

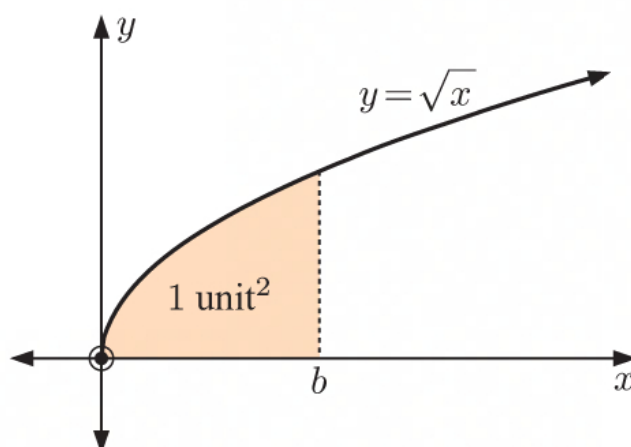
$$\therefore 1 = \left[ \frac{2}{3} x^{\frac{3}{2}} \right]_0^b$$

$$\therefore 1 = \frac{2}{3} b^{\frac{3}{2}} - 0$$

$$\therefore \frac{3}{2} = b^{\frac{3}{2}}$$

$$\therefore b = \left( \frac{3}{2} \right)^{\frac{2}{3}}$$

$$\therefore b \approx 1.3104$$



**b**

$$\text{Area} = \int_{-a}^a (x^2 + 2) \, dx$$

$$\begin{aligned} \therefore 6a &= \left[ \frac{1}{3} x^3 + 2x \right]_{-a}^a \\ &= \left( \frac{1}{3} a^3 + 2a \right) - \left( -\frac{1}{3} a^3 - 2a \right) \\ &= \frac{2}{3} a^3 + 4a \end{aligned}$$

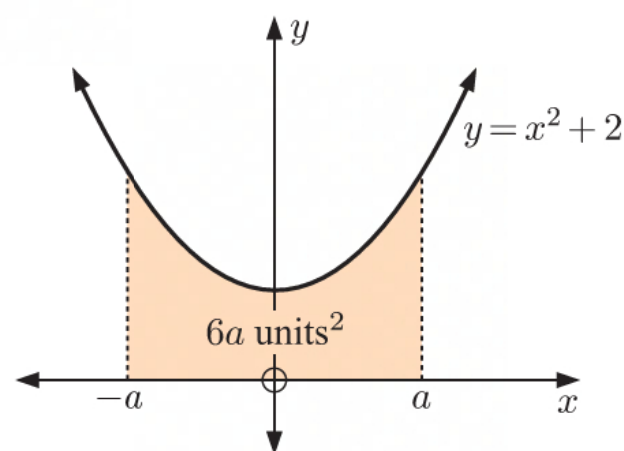
$$\therefore \frac{2}{3} a^3 - 2a = 0$$

$$\therefore a^3 - 3a = 0$$

$$\therefore a(a^2 - 3) = 0$$

$$\therefore a = 0, \pm\sqrt{3}$$

but  $a > 0$ ,  $\therefore a = \sqrt{3}$



**c**

$$\text{Area} = \int_1^k \frac{1}{1+2x} \, dx$$

$$\begin{aligned} \therefore 0.2 &= \left[ \frac{1}{2} \ln |1+2x| \right]_1^k \\ &= \frac{1}{2} \ln(1+2k) - \frac{1}{2} \ln 3 \quad \{k > 1\} \\ &= \frac{1}{2} [\ln(1+2k) - \ln 3] \end{aligned}$$

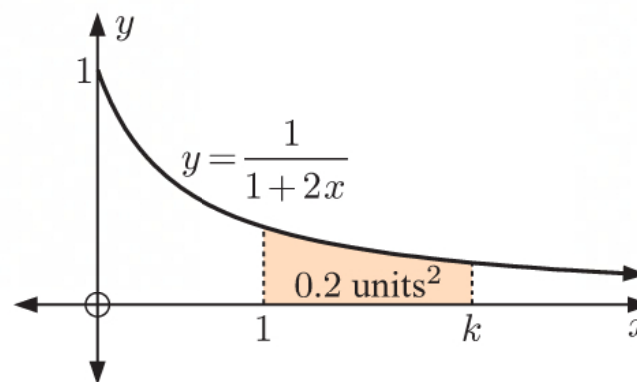
$$\therefore 0.4 = \ln \left( \frac{1+2k}{3} \right)$$

$$\therefore \frac{1+2k}{3} = e^{0.4}$$

$$\therefore 1+2k = 3e^{0.4}$$

$$\therefore 2k = 3e^{0.4} - 1$$

$$\therefore k = \frac{3e^{0.4} - 1}{2} \approx 1.7377$$





**d** Area of  $A$  = Area of  $B$

$$\therefore \int_2^k \frac{1}{x} dx = \int_k^{10} \frac{1}{x} dx$$

$$\therefore [\ln |x|]_2^k = [\ln |x|]_k^{10}$$

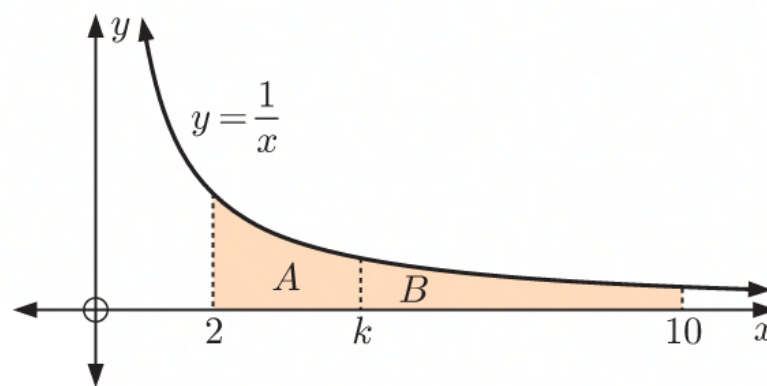
$$\therefore \ln k - \ln 2 = \ln 10 - \ln k \quad \{2 < k < 10\}$$

$$\therefore 2 \ln k = \ln 10 + \ln 2$$

$$\therefore \ln k^2 = \ln 20$$

$$\therefore k^2 = 20$$

$$\therefore k = 2\sqrt{5} \quad \{2 < k < 10\}$$



## INVESTIGATION 1

## $\int_a^b f(x) dx$ AND AREAS

**1 a**  $\int_0^1 x^3 dx = \left[ \frac{1}{4}x^4 \right]_0^1$  and  $\int_{-1}^1 x^3 dx = \left[ \frac{1}{4}x^4 \right]_{-1}^1$

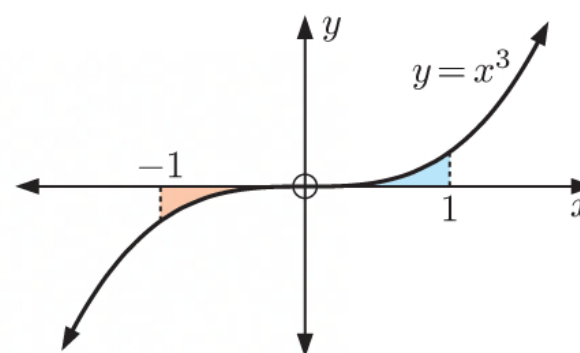
$$= \frac{1}{4} - 0 \quad = \frac{1}{4} - \frac{1}{4}(-1)^4$$

$$= \frac{1}{4} \quad = \frac{1}{4} - \frac{1}{4}$$

$$= 0$$

**b** Since the curve lies on or above the  $x$ -axis for  $0 \leq x \leq 1$ , the first integral in **a** is the area bounded by  $y = x^3$ , the  $x$ -axis, and the vertical lines  $x = 0$  and  $x = 1$ .

The second integral in **a** does *not* give an area as the curve lies on or below the  $x$ -axis for  $-1 \leq x \leq 1$ .



**c**  $\int_{-1}^0 x^3 dx = \left[ \frac{1}{4}x^4 \right]_{-1}^0$

$$= 0 - \frac{1}{4}(-1)^4$$

$$= -\frac{1}{4}$$

The answer is negative since the curve lies below the  $x$ -axis for  $-1 \leq x < 0$ .

**d**  $\int_{-1}^0 x^3 dx + \int_0^1 x^3 dx = -\frac{1}{4} + \frac{1}{4}$

$$= 0$$

$$= \int_{-1}^1 x^3 dx$$

**e**  $\int_0^{-1} x^3 dx = -\int_{-1}^0 x^3 dx$

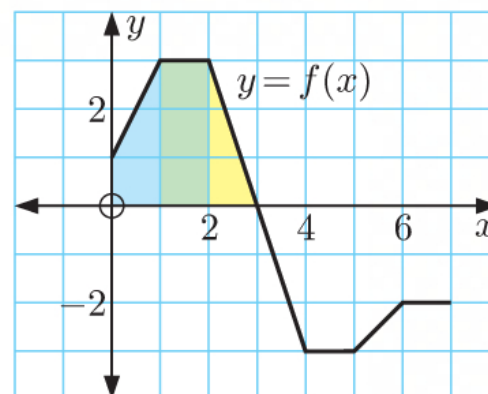
$$= -(-\frac{1}{4}) \quad \{\text{from c}\}$$

$$= \frac{1}{4}$$

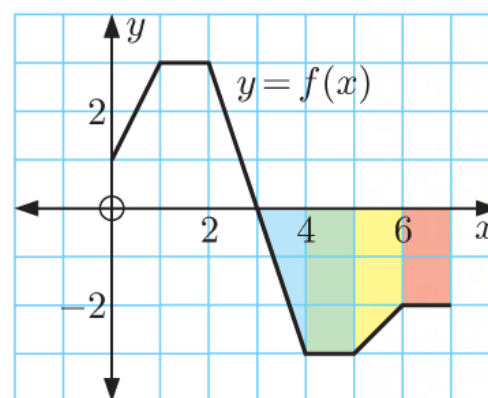
The area between the curve and the  $x$ -axis between  $x = -1$  and  $x = 0$  is  $\frac{1}{4}$  units<sup>2</sup>.

**2**  $\text{Area} = - \int_a^b f(x) \, dx$

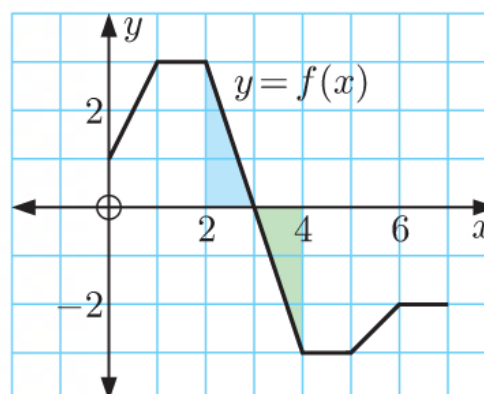
**3 a**  $\int_0^3 f(x) \, dx = \text{area of blue trapezium}$   
 $+ \text{area of green rectangle}$   
 $+ \text{area of yellow triangle}$   
 $= \left( \frac{1+3}{2} \right) \times 1 + (1 \times 3) + \left( \frac{1}{2} \times 1 \times 3 \right)$   
 $= 2 + 3 + \frac{3}{2}$   
 $= \frac{13}{2}$



**b**  $\int_3^7 f(x) \, dx$   
 $= -(\text{area of blue triangle}$   
 $+ \text{area of green rectangle}$   
 $+ \text{area of yellow trapezium}$   
 $+ \text{area of red rectangle})$   
 $= -\left[ \left( \frac{1}{2} \times 1 \times 3 \right) + (1 \times 3) + \left( \frac{2+3}{2} \right) \times 1 + (1 \times 2) \right]$   
 $= -\left( \frac{3}{2} + 3 + \frac{5}{2} + 2 \right)$   
 $= -9$



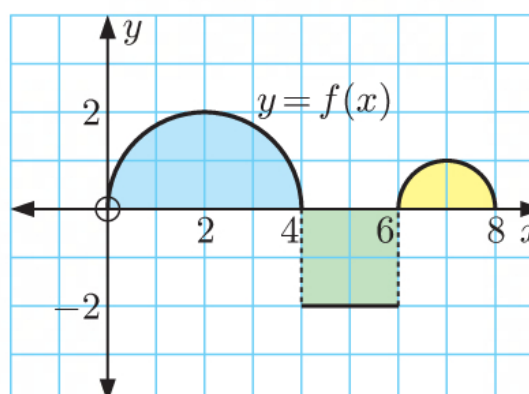
**c**  $\int_2^4 f(x) \, dx = \text{area of blue triangle}$   
 $- \text{area of green triangle}$   
 $= \left( \frac{1}{2} \times 1 \times 3 \right) - \left( \frac{1}{2} \times 1 \times 3 \right)$   
 $= 0$



**d**  $\int_0^7 f(x) \, dx = \int_0^3 f(x) \, dx + \int_3^7 f(x) \, dx$   
 $= \frac{13}{2} + (-9)$   
 $= -\frac{5}{2}$

**4 a**  $\int_0^4 f(x) \, dx = \text{area of blue semi-circle}$   
 $= \frac{1}{2} \times \pi \times 2^2$   
 $= 2\pi$

**b**  $\int_4^6 f(x) \, dx = -\text{area of green square}$   
 $= -(2 \times 2)$   
 $= -4$



$$\begin{aligned} \text{c } \int_6^8 f(x) dx &= \text{area of yellow semi-circle} \\ &= \frac{1}{2} \times \pi \times 1^2 \\ &= \frac{\pi}{2} \end{aligned}$$

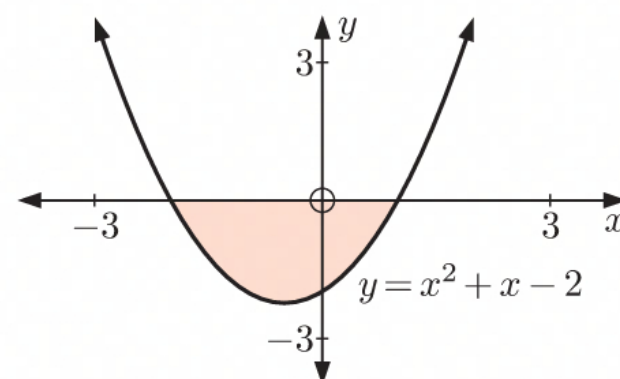
$$\begin{aligned} \text{d } \int_0^8 f(x) dx &= \int_0^4 f(x) dx + \int_4^6 f(x) dx + \int_6^8 f(x) dx \\ &= 2\pi + (-4) + \frac{\pi}{2} \\ &= \frac{5\pi}{2} - 4 \end{aligned}$$

## EXERCISE 22D

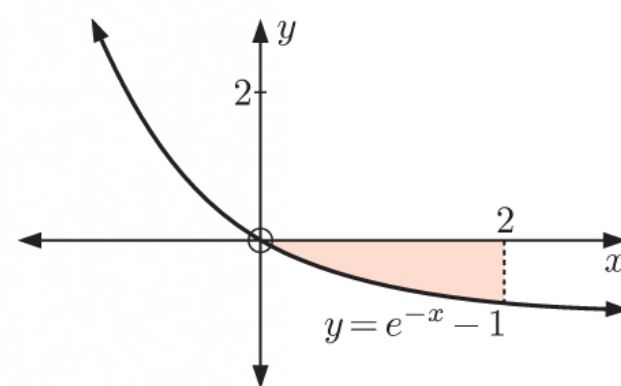
- 1 a The curve cuts the  $x$ -axis when  $y = 0$   
 $\therefore x^2 + x - 2 = 0$   
 $\therefore (x+2)(x-1) = 0$   
 $\therefore x = -2 \text{ or } 1$

$\therefore$  the  $x$ -intercepts are  $-2$  and  $1$ .

$$\begin{aligned} \text{Area} &= - \int_{-2}^1 (x^2 + x - 2) dx \\ &= - \left[ \frac{1}{3}x^3 + \frac{1}{2}x^2 - 2x \right]_{-2}^1 \\ &= - \left[ \left( \frac{1}{3} + \frac{1}{2} - 2 \right) - \left( -\frac{8}{3} + 2 + 4 \right) \right] \\ &= - \left[ -\frac{7}{6} - \frac{10}{3} \right] \\ &= 4\frac{1}{2} \text{ units}^2 \end{aligned}$$

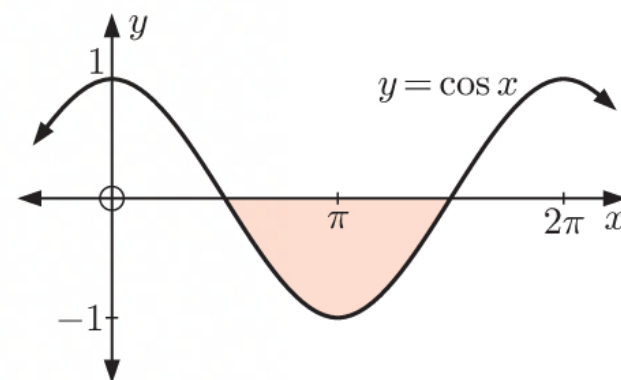


$$\begin{aligned} \text{b } \text{Area} &= - \int_0^2 (e^{-x} - 1) dx \\ &= - \left[ -e^{-x} - x \right]_0^2 \\ &= - \left[ (-e^{-2} - 2) - (-e^0 - 0) \right] \\ &= -(-e^{-2} - 1) \\ &= (1 + e^{-2}) \text{ units}^2 \end{aligned}$$



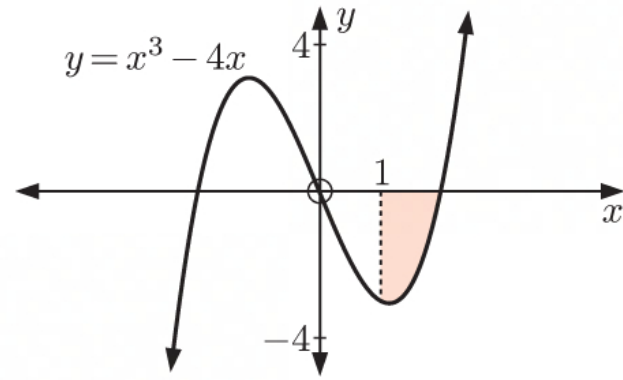
- c The curve cuts the  $x$ -axis when  $y = 0$   
 $\therefore \cos x = 0$   
 $\therefore x = \frac{\pi}{2} \text{ or } \frac{3\pi}{2}$

$$\begin{aligned} \text{Area} &= - \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos x dx \\ &= - \left[ \sin x \right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \\ &= -[(-1) - 1] \\ &= 2 \text{ units}^2 \end{aligned}$$



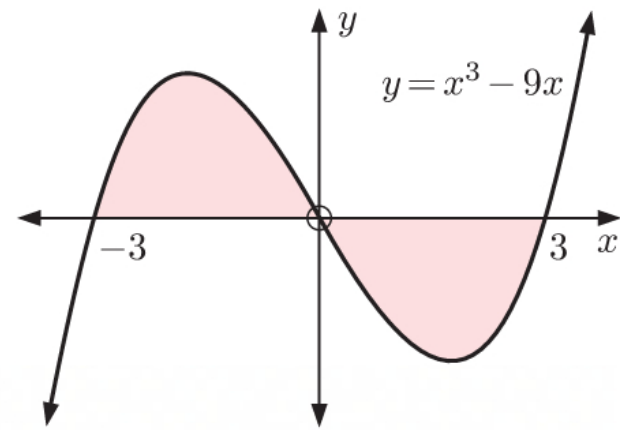
- d** The curve cuts the  $x$ -axis when  $y = 0$   
 $\therefore x^3 - 4x = 0$   
 $\therefore x(x^2 - 4) = 0$   
 $\therefore x(x+2)(x-2) = 0$   
 $\therefore$  the  $x$ -intercepts are  $-2, 0$ , and  $2$ .

$$\begin{aligned}\text{Area} &= - \int_1^2 (x^3 - 4x) dx \\ &= - \left[ \frac{1}{4}x^4 - 2x^2 \right]_1^2 \\ &= - \left[ (4 - 8) - \left( \frac{1}{4} - 2 \right) \right] \\ &= - \left[ -4 - \left( -\frac{7}{4} \right) \right] \\ &= 2\frac{1}{4} \text{ units}^2\end{aligned}$$

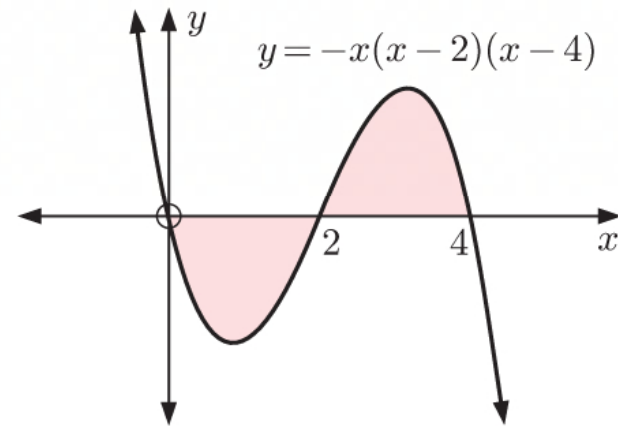


- 2 a**  $f(x) = x^3 - 9x$   
 $= x(x^2 - 9)$   
 $= x(x+3)(x-3)$   
 $\therefore y = f(x)$  cuts the  $x$ -axis at  $-3, 0$ , and  $3$ .

$$\begin{aligned}\text{Total area} &= \int_{-3}^0 (x^3 - 9x) dx - \int_0^3 (x^3 - 9x) dx \\ &= \left[ \frac{1}{4}x^4 - \frac{9}{2}x^2 \right]_{-3}^0 - \left[ \frac{1}{4}x^4 - \frac{9}{2}x^2 \right]_0^3 \\ &= \left( 0 - \left( \frac{81}{4} - \frac{81}{2} \right) \right) - \left( \left( \frac{81}{4} - \frac{81}{2} \right) - 0 \right) \\ &= 40\frac{1}{2} \text{ units}^2\end{aligned}$$



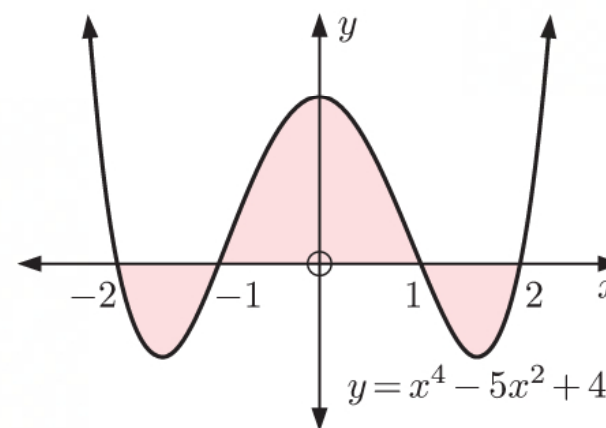
- b**  $f(x) = -x(x-2)(x-4)$   
 $\therefore y = f(x)$  cuts the  $x$ -axis at  $0, 2$ , and  $4$ .  
 $f(x) = -x(x-2)(x-4)$   
 $= -x(x^2 - 6x + 8)$   
 $= -x^3 + 6x^2 - 8x$



$$\begin{aligned}\text{Total area} &= - \int_0^2 (-x^3 + 6x^2 - 8x) dx + \int_2^4 (-x^3 + 6x^2 - 8x) dx \\ &= - \left[ -\frac{1}{4}x^4 + 2x^3 - 4x^2 \right]_0^2 + \left[ -\frac{1}{4}x^4 + 2x^3 - 4x^2 \right]_2^4 \\ &= -((-4 + 16 - 16) - 0) + ((-64 + 128 - 64) - (-4 + 16 - 16)) \\ &= 8 \text{ units}^2\end{aligned}$$

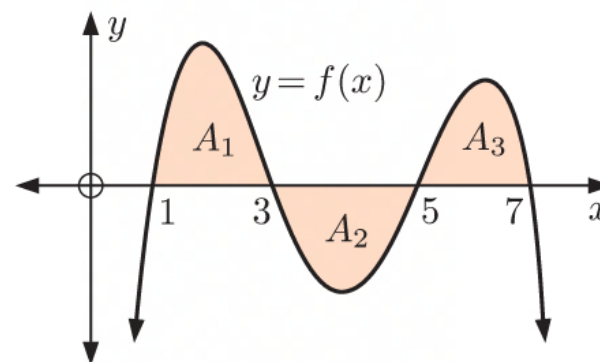


$$\begin{aligned}
 \bullet \quad f(x) &= x^4 - 5x^2 + 4 \\
 &= (x^2 - 1)(x^2 - 4) \\
 &= (x + 1)(x - 1)(x + 2)(x - 2) \\
 \therefore y = f(x) &\text{ cuts the } x\text{-axis at } -2, -1, 1, \text{ and } 2.
 \end{aligned}$$



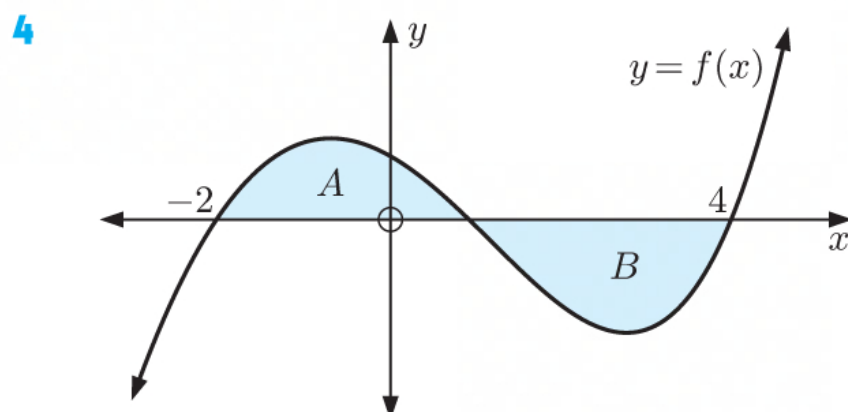
$$\begin{aligned}
 \text{Total area} &= -\int_{-2}^{-1} (x^4 - 5x^2 + 4) dx + \int_{-1}^1 (x^4 - 5x^2 + 4) dx - \int_1^2 (x^4 - 5x^2 + 4) dx \\
 &= -\left[\frac{1}{5}x^5 - \frac{5}{3}x^3 + 4x\right]_{-2}^{-1} + \left[\frac{1}{5}x^5 - \frac{5}{3}x^3 + 4x\right]_{-1}^1 - \left[\frac{1}{5}x^5 - \frac{5}{3}x^3 + 4x\right]_1^2 \\
 &= -\left[\left(-\frac{1}{5} + \frac{5}{3} - 4\right) - \left(-\frac{32}{5} + \frac{40}{3} - 8\right)\right] + \left[\left(\frac{1}{5} - \frac{5}{3} + 4\right) - \left(-\frac{1}{5} + \frac{5}{3} - 4\right)\right] \\
 &\quad - \left[\left(\frac{32}{5} - \frac{40}{3} + 8\right) - \left(\frac{1}{5} - \frac{5}{3} + 4\right)\right] \\
 &= -\left[-\frac{38}{15} + \frac{16}{15}\right] + \left[\frac{38}{15} + \frac{38}{15}\right] - \left[\frac{16}{15} - \frac{38}{15}\right] \\
 &= \frac{22}{15} + \frac{76}{15} + \frac{22}{15} \\
 &= 8 \text{ units}^2
 \end{aligned}$$

- 3 a  $\int_1^7 f(x) dx$  only gives us the correct area provided that  $f(x)$  is positive on the interval  $1 \leq x \leq 7$ . But  $f(x)$  is not positive for  $3 \leq x \leq 5$ , so
- $$\int_1^7 f(x) dx = A_1 - A_2 + A_3 \quad \text{which is not the shaded area.}$$



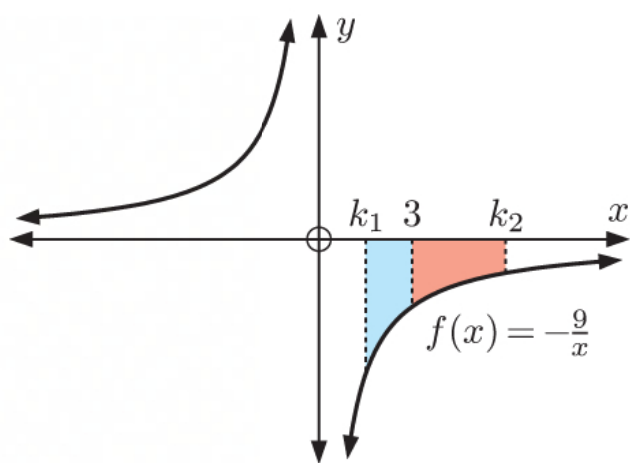
b Total shaded area =  $\int_1^3 f(x) dx + \int_3^5 (-f(x)) dx + \int_5^7 f(x) dx$

$$= \int_1^3 f(x) dx - \int_3^5 f(x) dx + \int_5^7 f(x) dx$$



$$\begin{aligned}
 \int_{-2}^4 f(x) dx &= \text{area of region } A + (-\text{area of region } B) \quad \{\text{since region } B \text{ is below the } x\text{-axis}\} \\
 &= \text{area of region } A - \text{area of region } B \\
 &= -6
 \end{aligned}$$

$\therefore$  area of region  $B >$  area of region  $A$

**5**Blue area = red area =  $9 \ln 2$  units<sup>2</sup>

$$\therefore - \int_{k_1}^3 -\frac{9}{x} dx = 9 \ln 2$$

$$\therefore \int_{k_1}^3 \frac{9}{x} dx = 9 \ln 2$$

$$\therefore [9 \ln |x|]_{k_1}^3 = 9 \ln 2$$

$$\therefore 9 \ln 3 - 9 \ln k_1 = 9 \ln 2 \quad \{k_1 > 0\}$$

$$\therefore \ln 3 - \ln k_1 = \ln 2$$

$$\therefore \ln k_1 = \ln 3 - \ln 2$$

$$\therefore \ln k_1 = \ln\left(\frac{3}{2}\right)$$

$$\therefore k_1 = \frac{3}{2}$$

$$\text{or} \quad - \int_3^{k_2} -\frac{9}{x} dx = 9 \ln 2$$

$$\therefore \int_3^{k_2} \frac{9}{x} dx = 9 \ln 2$$

$$\therefore [9 \ln |x|]_3^{k_2} = 9 \ln 2$$

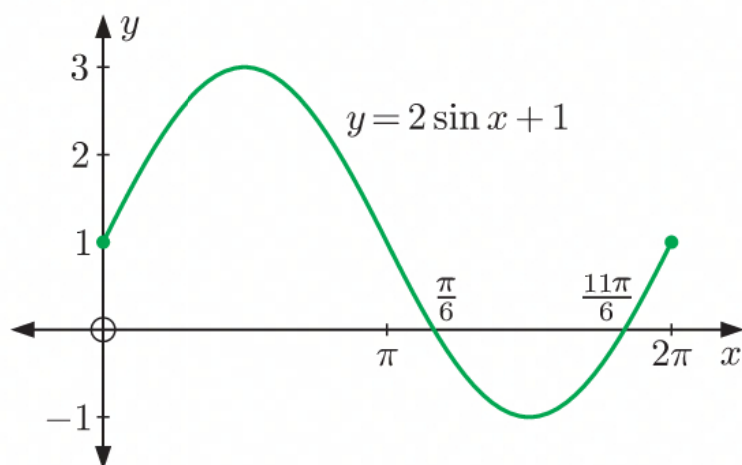
$$\therefore 9 \ln k_2 - 9 \ln 3 = 9 \ln 2 \quad \{k_2 > 0\}$$

$$\therefore \ln k_2 - \ln 3 = \ln 2$$

$$\therefore \ln k_2 = \ln 2 + \ln 3$$

$$\therefore \ln k_2 = \ln 6$$

$$\therefore k_2 = 6$$

So,  $k = \frac{3}{2}$  or 6**6 a****b** The curve cuts the  $x$ -axis when  $y = 0$ 

$$\therefore 2 \sin x + 1 = 0$$

$$\therefore 2 \sin x = -1$$

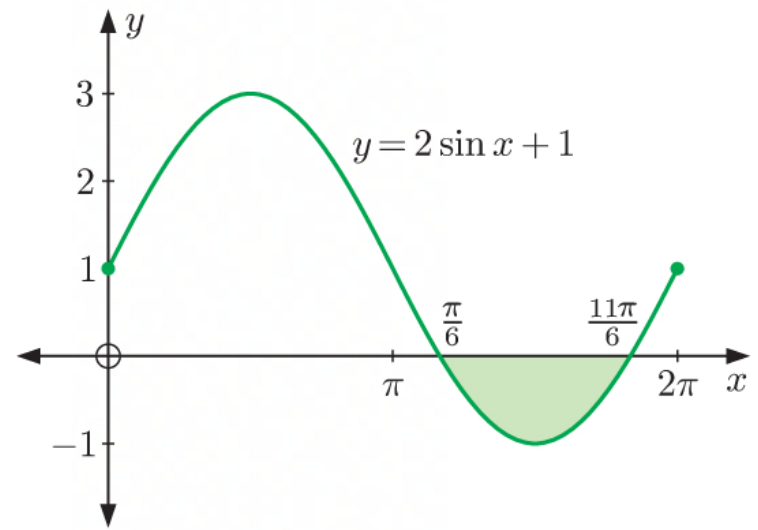
$$\therefore \sin x = -\frac{1}{2}$$

$$\therefore x = -\frac{\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}, \dots$$

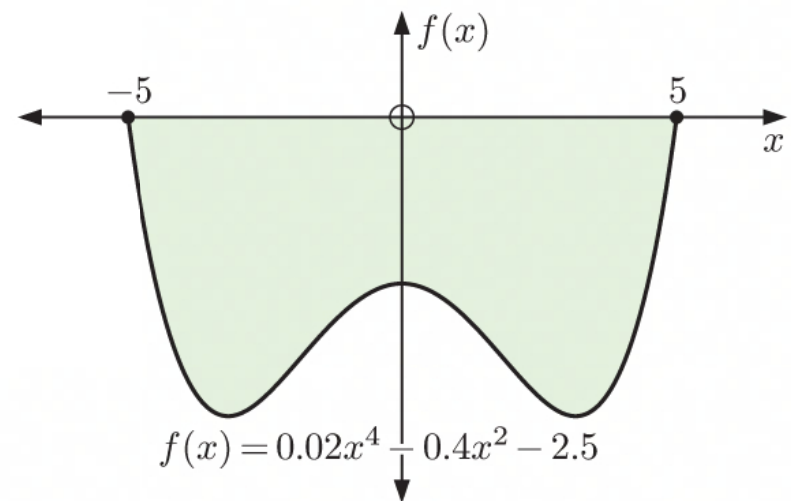
 $\therefore$  the  $x$ -intercepts are  $\frac{7\pi}{6}$  and  $\frac{11\pi}{6}$ .

Area

$$\begin{aligned}
 &= - \int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} (2 \sin x + 1) dx \\
 &= - \left[ -2 \cos x + x \right]_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} \\
 &= - \left[ \left( -2 \cos \frac{11\pi}{6} + \frac{11\pi}{6} \right) - \left( -2 \cos \frac{7\pi}{6} + \frac{7\pi}{6} \right) \right] \\
 &= - \left[ \left( -\sqrt{3} + \frac{11\pi}{6} \right) - \left( \sqrt{3} + \frac{7\pi}{6} \right) \right] \\
 &= \left( 2\sqrt{3} - \frac{2\pi}{3} \right) \text{ units}^2 \\
 &\approx 1.37 \text{ units}^2
 \end{aligned}$$

**7 a** Cross-sectional area of gutter

$$\begin{aligned}
 &= - \int_{-5}^5 (0.02x^4 - 0.4x^2 - 2.5) dx \\
 &= - \left[ 0.004x^5 - \frac{0.4}{3}x^3 - 2.5x \right]_{-5}^5 \\
 &= - \left[ -16\frac{2}{3} - 16\frac{2}{3} \right] \\
 &= 33\frac{1}{3} \text{ cm}^2
 \end{aligned}$$

**b** Volume of gutter= area of cross-section  $\times$  length of gutter

$$= 33\frac{1}{3} \text{ cm}^2 \times 20 \text{ m}$$

$$= 33\frac{1}{3} \text{ cm}^2 \times 2000 \text{ cm}$$

$$= 66\,666\frac{2}{3} \text{ cm}^3$$

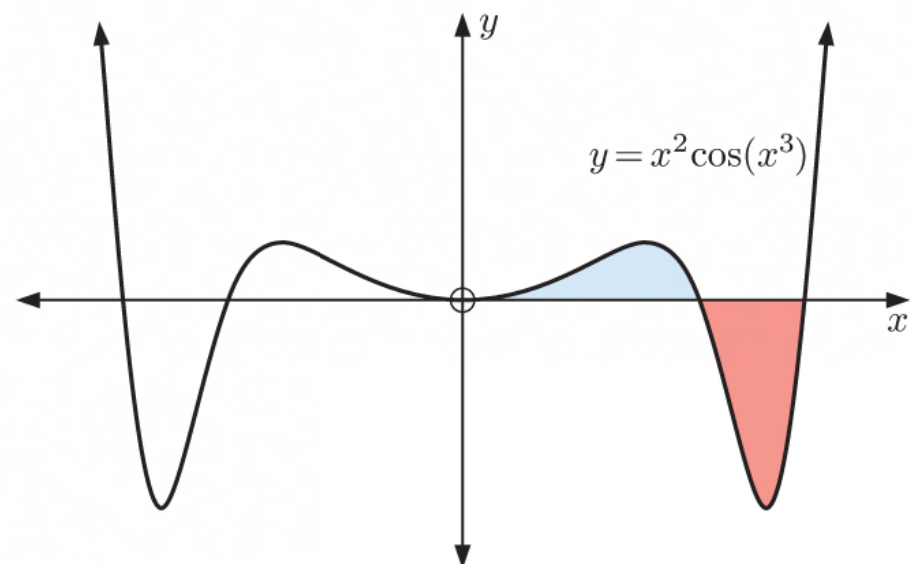
$\therefore$  the gutter can hold  $66\,666\frac{2}{3} \text{ mL}$   $\{1 \text{ cm}^3 \equiv 1 \text{ mL}\}$   
 $\approx 66.7 \text{ L}$  of water in total.

$$\begin{aligned}
 &\int x^2 \cos(x^3) dx \\
 &= \frac{1}{3} \int 3x^2 \cos(x^3) dx \\
 &= \frac{1}{3} \int \cos u \frac{du}{dx} dx \\
 &\quad \{u = x^3, \quad \frac{du}{dx} = 3x^2\} \\
 &= \frac{1}{3} \int \cos u du \\
 &= \frac{1}{3} \sin u + c \\
 &= \frac{1}{3} \sin(x^3) + c
 \end{aligned}$$

 $y = x^2 \cos(x^3)$  has  $x$ -intercepts when  $y = 0$ 

$$\therefore x^2 \cos(x^3) = 0$$

$$\therefore x^2 = 0 \text{ or } \cos(x^3) = 0$$



$\therefore x = 0$ , and the two smallest positive  $x$ -intercepts occur when  $x^3 = \frac{\pi}{2}$  or  $\frac{3\pi}{2}$

$$\therefore x = \sqrt[3]{\frac{\pi}{2}} \text{ or } \sqrt[3]{\frac{3\pi}{2}}$$

$$\begin{aligned} \therefore \text{blue area} &= \int_0^{\sqrt[3]{\frac{\pi}{2}}} x^2 \cos(x^3) dx \\ &= \left[ \frac{1}{3} \sin(x^3) \right]_0^{\sqrt[3]{\frac{\pi}{2}}} \\ &= \left( \frac{1}{3} \sin \frac{\pi}{2} \right) - \left( \frac{1}{3} \sin 0 \right) \\ &= \frac{1}{3} \text{ units}^2 \end{aligned}$$

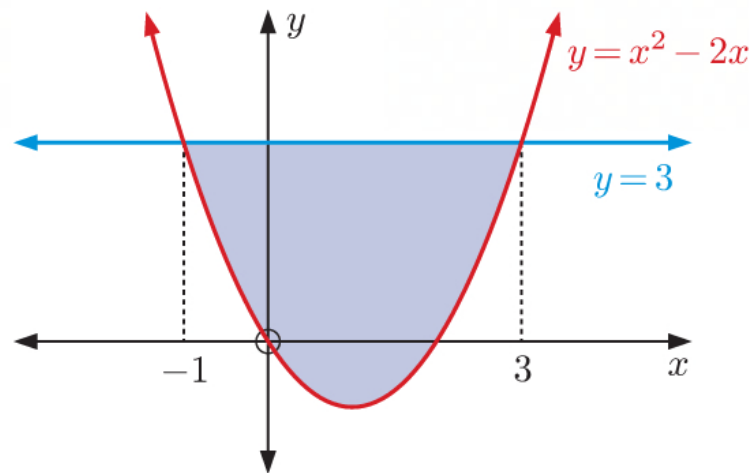
$$\begin{aligned} \therefore \text{red area} &= - \int_{\sqrt[3]{\frac{\pi}{2}}}^{\sqrt[3]{\frac{3\pi}{2}}} x^2 \cos(x^3) dx \\ &= - \left[ \frac{1}{3} \sin(x^3) \right]_{\sqrt[3]{\frac{\pi}{2}}}^{\sqrt[3]{\frac{3\pi}{2}}} \\ &= - \left[ \left( \frac{1}{3} \sin \frac{3\pi}{2} \right) - \left( \frac{1}{3} \sin \frac{\pi}{2} \right) \right] \\ &= - \left[ \left( -\frac{1}{3} \right) - \frac{1}{3} \right] \\ &= \frac{2}{3} \text{ units}^2 \end{aligned}$$

$\therefore$  the red shaded region is twice as large as the blue shaded region.

## EXERCISE 22E

- 1 a**  $y = x^2 - 2x$  meets  $y = 3$   
 where  $x^2 - 2x = 3$   
 $\therefore x^2 - 2x - 3 = 0$   
 $\therefore (x+1)(x-3) = 0$   
 $\therefore x = -1 \text{ or } 3$

$$\begin{aligned} \text{Area} &= \int_{-1}^3 [3 - (x^2 - 2x)] dx \\ &= \int_{-1}^3 (-x^2 + 2x + 3) dx \\ &= \left[ -\frac{1}{3}x^3 + x^2 + 3x \right]_{-1}^3 \\ &= (-9 + 9 + 9) - \left( \frac{1}{3} + 1 - 3 \right) \\ &= 9 + 1\frac{2}{3} \\ &= 10\frac{2}{3} \text{ units}^2 \end{aligned}$$





**b**  $y = x - 3$  meets  $y = x^2 - 3x$

where  $x^2 - 3x = x - 3$

$\therefore x^2 - 4x + 3 = 0$

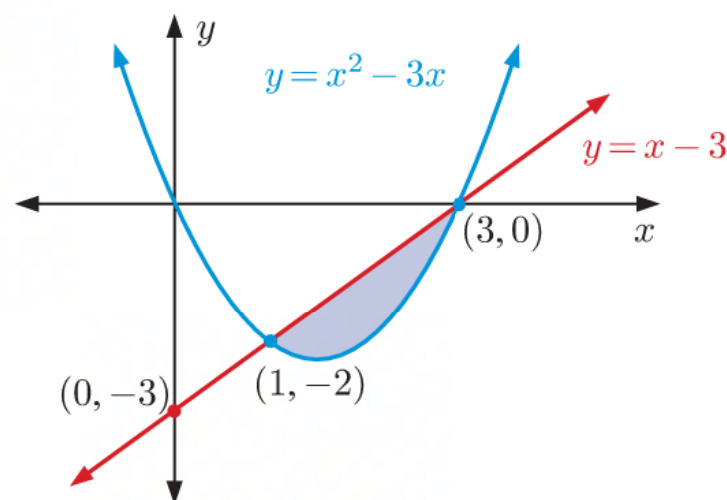
$\therefore (x - 1)(x - 3) = 0$

$\therefore x = 1$  or  $3$

When  $x = 1$ ,  $y = 1 - 3 = -2$

When  $x = 3$ ,  $y = 3 - 3 = 0$

$\therefore$  the graphs meet at the points  
(1, -2) and (3, 0).



$$\begin{aligned} \text{Area} &= \int_1^3 [(x - 3) - (x^2 - 3x)] dx \\ &= \int_1^3 (-x^2 + 4x - 3) dx \\ &= \left[ -\frac{1}{3}x^3 + 2x^2 - 3x \right]_1^3 \\ &= (-9 + 18 - 9) - \left( -\frac{1}{3} + 2 - 3 \right) \\ &= 0 - \left( -1\frac{1}{3} \right) \\ &= 1\frac{1}{3} \text{ units}^2 \end{aligned}$$

**c**  $y = 2x$  meets  $y = 4x^2$

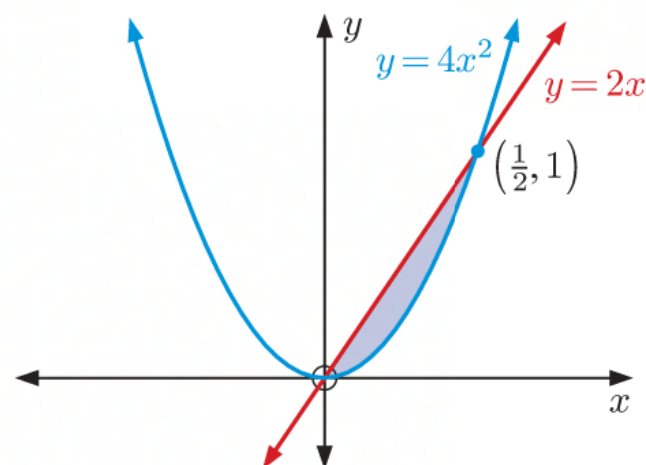
where  $4x^2 = 2x$

$\therefore 4x^2 - 2x = 0$

$\therefore 2x(2x - 1) = 0$

$\therefore x = 0$  or  $\frac{1}{2}$

$$\begin{aligned} \text{Area} &= \int_0^{\frac{1}{2}} [2x - 4x^2] dx \\ &= \left[ x^2 - \frac{4}{3}x^3 \right]_0^{\frac{1}{2}} \\ &= \left( \frac{1}{4} - \frac{4}{3} \left( \frac{1}{8} \right) \right) - 0 \\ &= \frac{1}{12} \text{ units}^2 \end{aligned}$$



**d**  $y = \sqrt{x}$  meets  $y = x^2$

where  $\sqrt{x} = x^2$

$$\therefore x = x^4$$

$$\therefore x^4 - x = 0$$

$$\therefore x(x^3 - 1) = 0$$

$$\therefore x = 0 \text{ or } 1$$

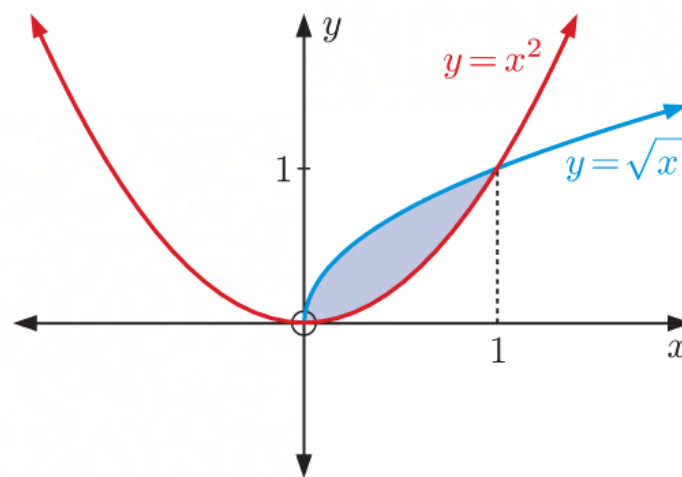
$$\text{Area} = \int_0^1 [\sqrt{x} - x^2] dx$$

$$= \int_0^1 (x^{\frac{1}{2}} - x^2) dx$$

$$= \left[ \frac{2}{3} x^{\frac{3}{2}} - \frac{1}{3} x^3 \right]_0^1$$

$$= \left( \frac{2}{3} - \frac{1}{3} \right) - 0$$

$$= \frac{1}{3} \text{ units}^2$$



**2**  $y = 2e^x$  meets  $y = e^{2x}$

where  $2e^x = e^{2x}$

$$\therefore e^{2x} - 2e^x = 0$$

$$\therefore e^x(e^x - 2) = 0$$

$$\therefore e^x = 0 \text{ or } 2$$

$$\therefore x = \ln 2 \quad \{\text{since } e^x > 0 \text{ for all } x \in \mathbb{R}\}$$

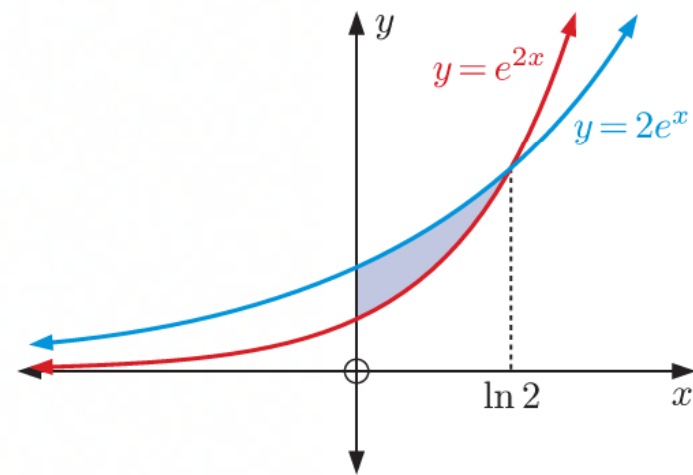
$$\text{Area} = \int_0^{\ln 2} [2e^x - e^{2x}] dx$$

$$= \left[ 2e^x - \frac{1}{2} e^{2x} \right]_0^{\ln 2}$$

$$= [2(2) - \frac{1}{2}(e^{\ln(2^2)})] - [2(1) - \frac{1}{2}(1)]$$

$$= (4 - 2) - 1\frac{1}{2}$$

$$= \frac{1}{2} \text{ units}^2$$



**3 a**  $y = x^2 - 3x + 4$  meets  $y = 2x^2 + 3x + 4$

where  $x^2 - 3x + 4 = 2x^2 + 3x + 4$

$$\therefore x^2 + 6x = 0$$

$$\therefore x(x + 6) = 0$$

$$\therefore x = 0 \text{ or } -6$$

$$\text{Area} = \int_{-6}^0 [x^2 - 3x + 4 - (2x^2 + 3x + 4)] dx$$

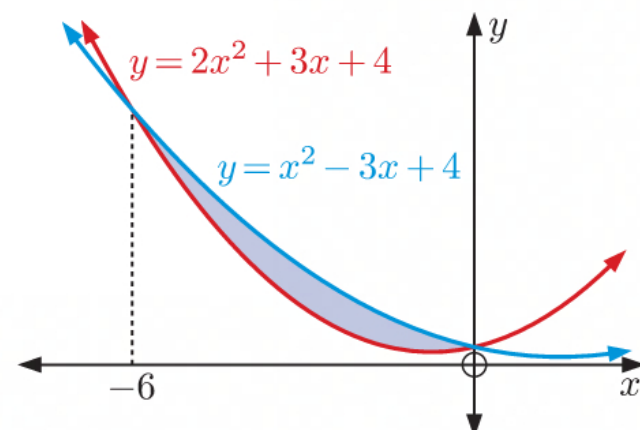
$$= \int_{-6}^0 (-x^2 - 6x) dx$$

$$= \left[ -\frac{1}{3} x^3 - 3x^2 \right]_{-6}^0$$

$$= 0 - \left( -\frac{1}{3}(-6)^3 - 3(-6)^2 \right)$$

$$= -(72 - 108)$$

$$= 36 \text{ units}^2$$



**b**  $y = -\frac{1}{2}x^2 - 2x + 1$  meets  $y = 2x^2 - 3x - 3$

where  $-\frac{1}{2}x^2 - 2x + 1 = 2x^2 - 3x - 3$

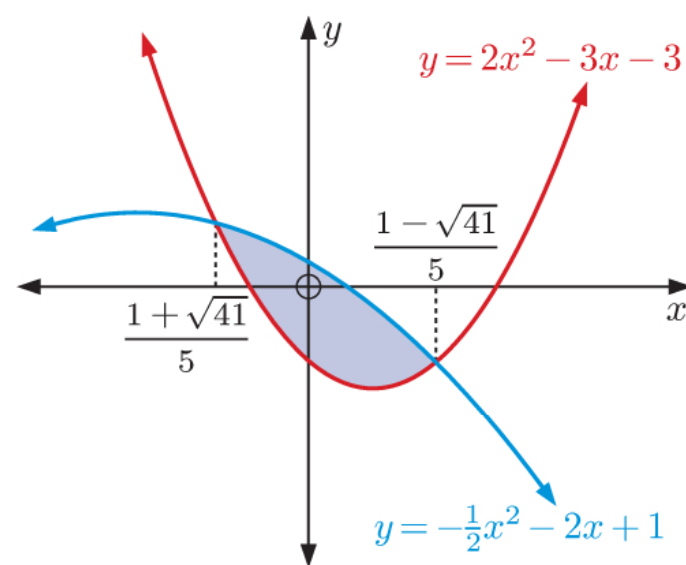
$$\therefore \frac{5}{2}x^2 - x - 4 = 0$$

$$\therefore 5x^2 - 2x - 8 = 0$$

$$\therefore x = \frac{2 \pm \sqrt{(-2)^2 - 4(5)(-8)}}{2(5)}$$

$$\therefore x = \frac{2 \pm 2\sqrt{41}}{10}$$

$$\therefore x = \frac{1 \pm \sqrt{41}}{5}$$



$$\begin{aligned} \text{Area} &= \int_{\frac{1-\sqrt{41}}{5}}^{\frac{1+\sqrt{41}}{5}} \left[ \left( -\frac{1}{2}x^2 - 2x + 1 \right) - (2x^2 - 3x - 3) \right] dx \\ &= \int_{\frac{1-\sqrt{41}}{5}}^{\frac{1+\sqrt{41}}{5}} \left( -\frac{5}{2}x^2 + x + 4 \right) dx \\ &= \left[ -\frac{5}{6}x^3 + \frac{1}{2}x^2 + 4x \right]_{\frac{1-\sqrt{41}}{5}}^{\frac{1+\sqrt{41}}{5}} \\ &= \left( -\frac{5}{6} \left( \frac{1+\sqrt{41}}{5} \right)^3 + \frac{1}{2} \left( \frac{1+\sqrt{41}}{5} \right)^2 + 4 \left( \frac{1+\sqrt{41}}{5} \right) \right) \\ &\quad - \left( -\frac{5}{6} \left( \frac{1-\sqrt{41}}{5} \right)^3 + \frac{1}{2} \left( \frac{1-\sqrt{41}}{5} \right)^2 + 4 \left( \frac{1-\sqrt{41}}{5} \right) \right) \\ &= \left( -\frac{5}{6} \left( \frac{124 + 44\sqrt{41}}{125} \right) + \frac{1}{2} \left( \frac{42 + 2\sqrt{41}}{25} \right) + \frac{4 + 4\sqrt{41}}{5} \right) \\ &\quad - \left( -\frac{5}{6} \left( \frac{124 - 44\sqrt{41}}{125} \right) + \frac{1}{2} \left( \frac{42 - 2\sqrt{41}}{25} \right) + \frac{4 - 4\sqrt{41}}{5} \right) \\ &= \left( -\frac{62 - 22\sqrt{41}}{75} + \frac{21 + \sqrt{41}}{25} + \frac{4 + 4\sqrt{41}}{5} \right) \\ &\quad - \left( -\frac{62 + 22\sqrt{41}}{75} + \frac{21 - \sqrt{41}}{25} + \frac{4 - 4\sqrt{41}}{5} \right) \\ &= -\frac{44\sqrt{41}}{75} + \frac{2\sqrt{41}}{25} + \frac{8\sqrt{41}}{5} \\ &= -\frac{44\sqrt{41}}{75} + \frac{6\sqrt{41}}{75} + \frac{120\sqrt{41}}{75} \\ &= \frac{82\sqrt{41}}{75} \text{ units}^2 \quad (\approx 7.00 \text{ units}^2) \end{aligned}$$

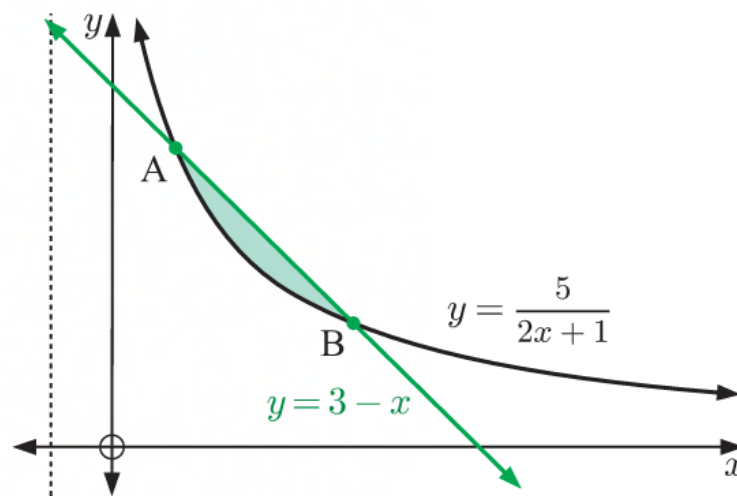
**4 a**  $y = \frac{5}{2x+1}$  and  $y = 3 - x$  meet where

$$\begin{aligned}\frac{5}{2x+1} &= 3 - x \\ \therefore 5 &= (3 - x)(2x + 1) \\ \therefore 5 &= 6x + 3 - 2x^2 - x \\ \therefore 2x^2 - 5x + 2 &= 0 \\ \therefore (2x - 1)(x - 2) &= 0 \\ \therefore x &= \frac{1}{2} \text{ or } 2\end{aligned}$$

$$\text{When } x = \frac{1}{2}, \quad y = 3 - \frac{1}{2} = \frac{5}{2}$$

$$\text{When } x = 2, \quad y = 3 - 2 = 1$$

$\therefore$  A has coordinates  $(\frac{1}{2}, \frac{5}{2})$  and B has coordinates  $(2, 1)$ .



$$\begin{aligned}\text{b Area} &= \int_{\frac{1}{2}}^2 \left( (3 - x) - \frac{5}{2x+1} \right) dx \\ &= \left[ 3x - \frac{1}{2}x^2 - \frac{5}{2} \ln |2x+1| \right]_{\frac{1}{2}}^2 \\ &= \left( 6 - 2 - \frac{5}{2} \ln 5 \right) - \left( \frac{3}{2} - \frac{1}{8} - \frac{5}{2} \ln 2 \right) \\ &= 4 - \frac{5}{2} \ln 5 - \frac{11}{8} + \frac{5}{2} \ln 2 \\ &= \frac{21}{8} - \frac{5}{2} (\ln 5 - \ln 2) \\ &= \left( \frac{21}{8} - \frac{5}{2} \ln \left( \frac{5}{2} \right) \right) \text{ units}^2\end{aligned}$$

**5**  $y = x^2$  meets  $y = k$  where  $x^2 = k$   
 $\therefore x = \pm \sqrt{k}$

$$\text{Now, the area} = \int_0^{\sqrt{k}} (k - x^2) dx$$

$$\therefore \int_0^{\sqrt{k}} (k - x^2) dx = 2.4$$

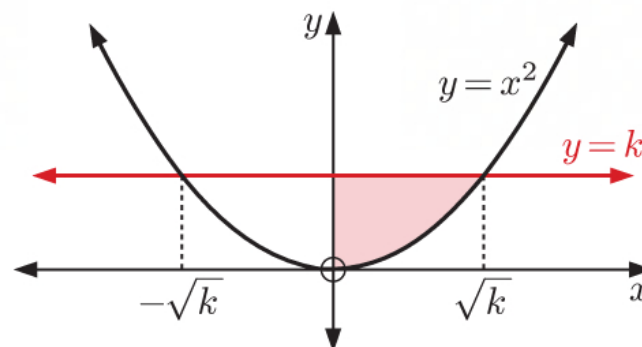
$$\therefore \left[ kx - \frac{x^3}{3} \right]_0^{\sqrt{k}} = 2.4$$

$$\therefore \left( k\sqrt{k} - \frac{k\sqrt{k}}{3} \right) - 0 = 2.4$$

$$\therefore \frac{2k\sqrt{k}}{3} = 2.4$$

$$\therefore k^{\frac{3}{2}} = 3.6$$

$$\begin{aligned}\therefore k &= (3.6)^{\frac{2}{3}} \\ &\approx 2.3489\end{aligned}$$





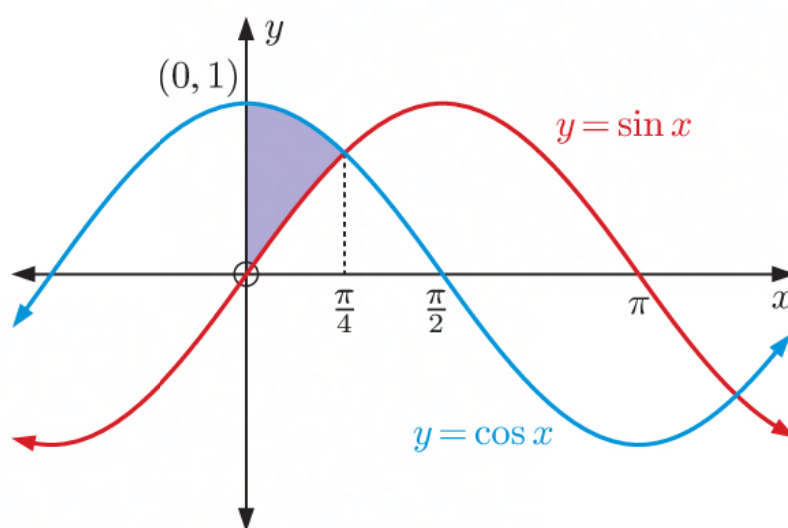
- 6**  $y = \sin x$  meets  $y = \cos x$

where  $\sin x = \cos x$

$$\therefore \tan x = 1$$

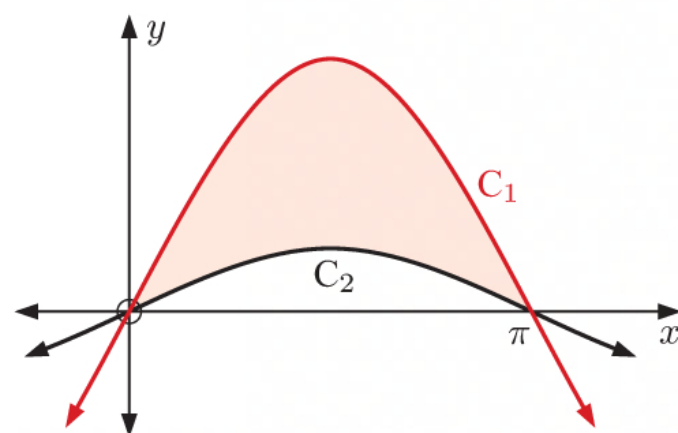
$$\therefore x = \frac{\pi}{4}$$

$$\begin{aligned} \text{Area} &= \int_0^{\frac{\pi}{4}} [\cos x - \sin x] dx \\ &= \left[ \sin x + \cos x \right]_0^{\frac{\pi}{4}} \\ &= \left( \sin \frac{\pi}{4} + \cos \frac{\pi}{4} \right) - (\sin 0 + \cos 0) \\ &= \left( \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right) - (0 + 1) \\ &= (\sqrt{2} - 1) \text{ units}^2 \end{aligned}$$

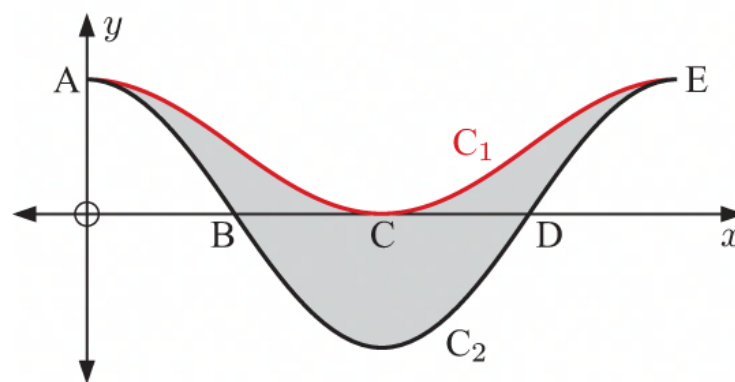


- 7 a**  $y = 4 \sin x$  has amplitude 4, which is larger than the amplitude of  $y = \sin x$  which has amplitude 1.  
 $\therefore y = 4 \sin x$  is the curve  $C_1$  and  
 $y = \sin x$  is the curve  $C_2$ .

**b** 
$$\begin{aligned} \text{Area} &= \int_0^{\pi} [4 \sin x - \sin x] dx \\ &= \int_0^{\pi} 3 \sin x dx \\ &= \left[ -3 \cos x \right]_0^{\pi} \\ &= (-3 \cos \pi) - (-3 \cos 0) \\ &= 3 - (-3) \\ &= 6 \text{ units}^2 \end{aligned}$$



- 8 a**  $\cos^2 x \geq 0$  for all  $x$ , so  $y = \cos^2 x$  must lie above the  $x$ -axis.  
 $\therefore y = \cos^2 x$  is the curve  $C_1$  and  
 $y = \cos 2x$  is the curve  $C_2$ .



- b**  $y = \cos 2x$  meets  $y = \cos^2 x$  where  $\cos 2x = \cos^2 x$   
 $\therefore 2 \cos^2 x - 1 = \cos^2 x$   
 $\therefore \cos^2 x = 1$   
 $\therefore \cos x = \pm 1$   
 $\therefore x = 0, \pi$

So, points A, B, C, D, and E have  $x$ -coordinates between 0 and  $\pi$ .

Now,  $y = \cos 2x$  cuts the  $x$ -axis where  $\cos 2x = 0$

$$\therefore 2x = \frac{\pi}{2}, \frac{3\pi}{2}$$

$$\therefore x = \frac{\pi}{4}, \frac{3\pi}{4}$$

and  $y = \cos^2 x$  cuts the  $x$ -axis where  $\cos^2 x = 0$

$$\therefore \cos x = 0$$

$$\therefore x = \frac{\pi}{2}$$

Point A lies where the curves first meet, so has  $x$ -coordinate 0.

When  $x = 0$ ,  $y = \cos 0 = 1$ , so A is  $(0, 1)$ .

Point B is the first  $x$ -intercept of  $y = \cos 2x$ , so B is  $(\frac{\pi}{4}, 0)$ .

Point C is the  $x$ -intercept of  $y = \cos^2 x$ , so C is  $(\frac{\pi}{2}, 0)$ .

Point D is the second  $x$ -intercept of  $y = \cos 2x$ , so D is  $(\frac{3\pi}{4}, 0)$ .

Point E lies where the curves meet for the second time, so has  $x$ -coordinate  $\pi$ .

When  $x = \pi$ ,  $y = \cos 2\pi = 1$ , so E is  $(\pi, 1)$ .

$$\begin{aligned} \text{c Area} &= \int_0^\pi (\cos^2 x - \cos 2x) dx \\ &= \int_0^\pi \left( \frac{1}{2} + \frac{1}{2} \cos 2x - \cos 2x \right) dx \\ &= \int_0^\pi \left( \frac{1}{2} - \frac{1}{2} \cos 2x \right) dx \\ &= \left[ \frac{x}{2} - \frac{1}{4} \sin 2x \right]_0^\pi \\ &= \left( \frac{\pi}{2} - 0 \right) - (0 - 0) \\ &= \frac{\pi}{2} \text{ units}^2 \end{aligned}$$

9 a  $y = e^x - 1$  has no vertical asymptotes.

As  $x \rightarrow \infty$ ,  $e^x - 1 \rightarrow \infty$

As  $x \rightarrow -\infty$ ,  $e^x \rightarrow 0^+$   
so  $e^x - 1 \rightarrow -1^+$

$\therefore y = -1$  is a horizontal asymptote.

$y = 0$  when  $e^x - 1 = 0$

$$\therefore e^x = 1$$

$$\therefore x = 0$$

$\therefore$  the  $x$ -intercept is 0.

This is also the  $y$ -intercept.

$y = 2 - 2e^{-x}$  has no vertical asymptotes.

As  $x \rightarrow \infty$ ,  $e^{-x} \rightarrow 0^+$

so  $2 - 2e^{-x} \rightarrow 2^-$

$\therefore y = 2$  is a horizontal asymptote.

As  $x \rightarrow -\infty$ ,  $2 - 2e^{-x} \rightarrow -\infty$

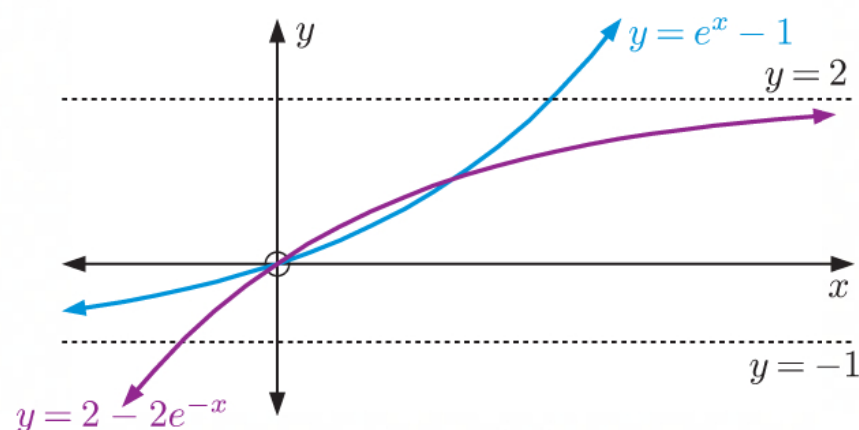
$y = 0$  when  $2 - 2e^{-x} = 0$

$$\therefore e^{-x} = 1$$

$$\therefore x = 0$$

$\therefore$  the  $x$ -intercept is 0.

This is also the  $y$ -intercept.



**b**  $y = e^x - 1$  meets  $y = 2 - 2e^{-x}$  where  $e^x - 1 = 2 - 2e^{-x}$   
 $\therefore e^{2x} - e^x = 2e^x - 2 \quad \{ \times e^x \}$   
 $\therefore e^{2x} - 3e^x + 2 = 0$   
 $\therefore (e^x - 1)(e^x - 2) = 0$   
 $\therefore e^x = 1 \text{ or } 2$   
 $\therefore x = 0 \text{ or } \ln 2$

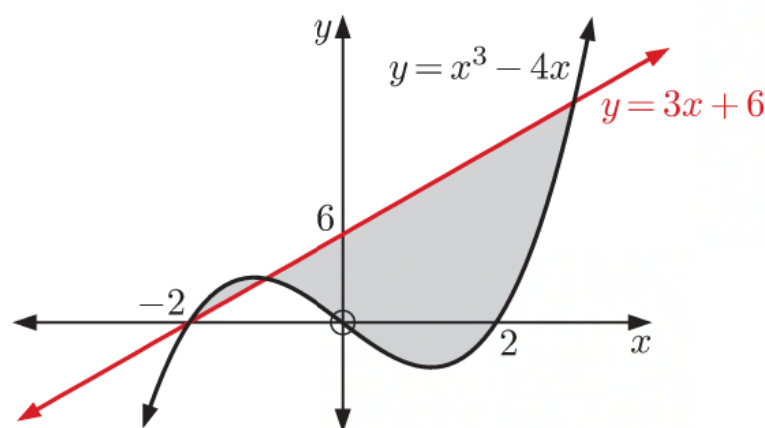
When  $x = 0$ ,  $y = e^0 - 1 = 0$

When  $x = \ln 2$ ,  $y = e^{\ln 2} - 1 = 1$

$\therefore$  the graphs meet at  $(0, 0)$  and  $(\ln 2, 1)$ .

**c** Area  $= \int_0^{\ln 2} [(2 - 2e^{-x}) - (e^x - 1)] dx$   
 $= \int_0^{\ln 2} (3 - e^x - 2e^{-x}) dx$   
 $= [3x - e^x + 2e^{-x}]_0^{\ln 2}$   
 $= (3 \ln 2 - 2 + 1) - (0 - 1 + 2)$   
 $= (3 \ln 2 - 2) \text{ units}^2$

**10 a**



The graphs meet where  $x^3 - 4x = 3x + 6$

$$\therefore x^3 - 7x - 6 = 0$$

$$\therefore (x + 2)(x^2 - 2x - 3) = 0 \quad \{\text{diagram shows intersection at } -2\}$$

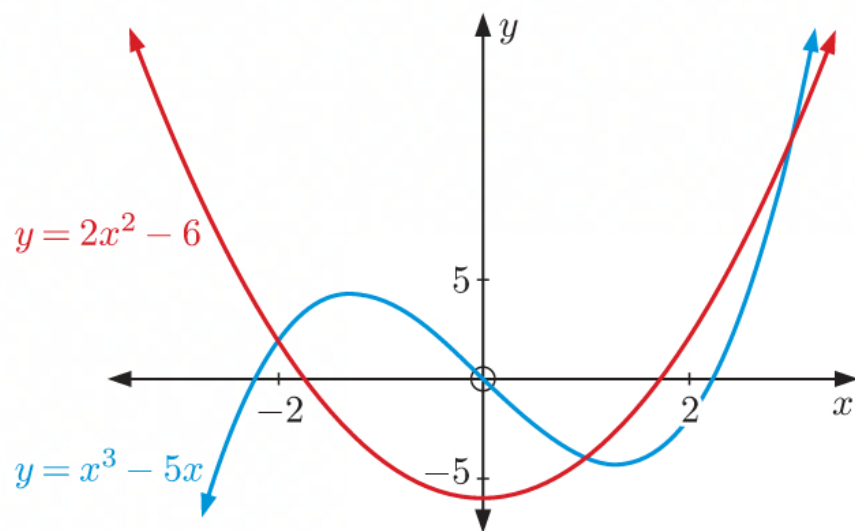
$$\therefore (x + 2)(x + 1)(x - 3) = 0$$

$$\therefore x = -2, -1, \text{ or } 3$$

$$\begin{aligned} \text{Total area} &= \int_{-2}^{-1} [(x^3 - 4x) - (3x + 6)] dx + \int_{-1}^3 [(3x + 6) - (x^3 - 4x)] dx \\ &= \int_{-2}^{-1} (x^3 - 7x - 6) dx + \int_{-1}^3 (-x^3 + 7x + 6) dx \end{aligned}$$

**b** Total area  $= \int_{-2}^{-1} (x^3 - 7x - 6) dx + \int_{-1}^3 (-x^3 + 7x + 6) dx \quad \{\text{from a}\}$

$$\begin{aligned} &= \left[ \frac{1}{4}x^4 - \frac{7}{2}x^2 - 6x \right]_{-2}^{-1} + \left[ -\frac{1}{4}x^4 + \frac{7}{2}x^2 + 6x \right]_{-1}^3 \\ &= \left[ \left( \frac{1}{4} - \frac{7}{2} - 6 \right) - (4 - 14 + 12) \right] + \left[ \left( -\frac{81}{4} + \frac{63}{2} + 18 \right) - \left( -\frac{1}{4} + \frac{7}{2} - 6 \right) \right] \\ &= \left( \frac{11}{4} - 2 \right) + \left( \frac{117}{4} + \frac{11}{4} \right) \\ &= 32\frac{3}{4} \text{ units}^2 \end{aligned}$$

**11 a****b**  $y = x^3 - 5x$  meets  $y = 2x^2 - 6$  where  $x^3 - 5x = 2x^2 - 6$ 

$$\therefore x^3 - 2x^2 - 5x + 6 = 0$$

$$\therefore (x + 2)(x - 1)(x - 3) = 0$$

$$\therefore x = -2, 1, \text{ or } 3$$

$\therefore$  the intersection points have  $x$ -coordinates  $-2, 1$ , and  $3$  respectively.

**c** Area

$$\begin{aligned} &= \int_{-2}^1 [(x^3 - 5x) - (2x^2 - 6)] dx + \int_1^3 [(2x^2 - 6) - (x^3 - 5x)] dx \\ &= \int_{-2}^1 (x^3 - 2x^2 - 5x + 6) dx + \int_1^3 (-x^3 + 2x^2 + 5x - 6) dx \\ &= \left[ \frac{1}{4}x^4 - \frac{2}{3}x^3 - \frac{5}{2}x^2 + 6x \right]_{-2}^1 + \left[ -\frac{1}{4}x^4 + \frac{2}{3}x^3 + \frac{5}{2}x^2 - 6x \right]_1^3 \\ &= \left[ \left( \frac{1}{4} - \frac{2}{3} - \frac{5}{2} + 6 \right) - \left( 4 + \frac{16}{3} - 10 - 12 \right) \right] + \left[ \left( -\frac{81}{4} + 18 + \frac{45}{2} - 18 \right) - \left( -\frac{1}{4} + \frac{2}{3} + \frac{5}{2} - 6 \right) \right] \\ &= \left( \frac{37}{12} + \frac{38}{3} \right) + \left( \frac{9}{4} + \frac{37}{12} \right) \\ &= 21\frac{1}{12} \text{ units}^2 \end{aligned}$$

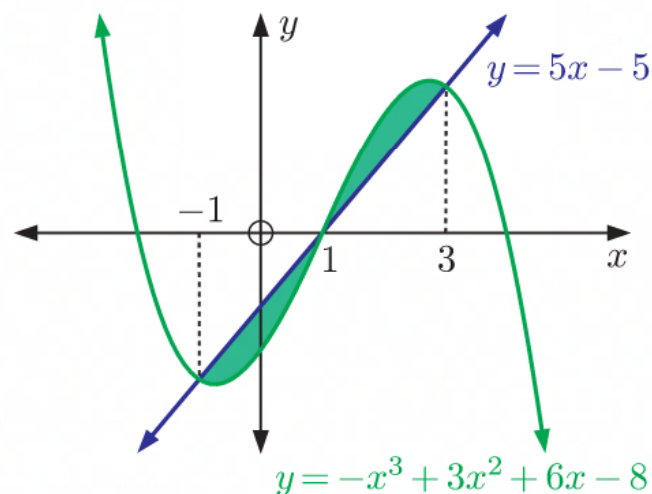
**12 a**  $y = -x^3 + 3x^2 + 6x - 8$  meets  $y = 5x - 5$ where  $-x^3 + 3x^2 + 6x - 8 = 5x - 5$ 

$$\therefore x^3 - 3x^2 - x + 3 = 0$$

$$\therefore (x - 1)(x^2 - 2x - 3) = 0$$

$$\therefore (x - 1)(x - 3)(x + 1) = 0$$

$$\therefore x = -1, 1, \text{ or } 3$$



Total area

$$\begin{aligned} &= \int_{-1}^1 [(5x - 5) - (-x^3 + 3x^2 + 6x - 8)] dx + \int_1^3 [(-x^3 + 3x^2 + 6x - 8) - (5x - 5)] dx \\ &= \int_{-1}^1 (x^3 - 3x^2 - x + 3) dx + \int_1^3 (-x^3 + 3x^2 + x - 3) dx \\ &= \left[ \frac{1}{4}x^4 - x^3 - \frac{1}{2}x^2 + 3x \right]_{-1}^1 + \left[ -\frac{1}{4}x^4 + x^3 + \frac{1}{2}x^2 - 3x \right]_1^3 \\ &= \left[ \left( \frac{1}{4} - 1 - \frac{1}{2} + 3 \right) - \left( \frac{1}{4} + 1 - \frac{1}{2} - 3 \right) \right] + \left[ \left( -\frac{81}{4} + 27 + \frac{9}{2} - 9 \right) - \left( -\frac{1}{4} + 1 + \frac{1}{2} - 3 \right) \right] \\ &= \left( \frac{7}{4} + \frac{9}{4} \right) + \left( \frac{9}{4} + \frac{7}{4} \right) \\ &= 8 \text{ units}^2 \end{aligned}$$



**b**  $y = 2x^3 - 3x^2 + 18$  meets  $y = x^3 + 10x - 6$

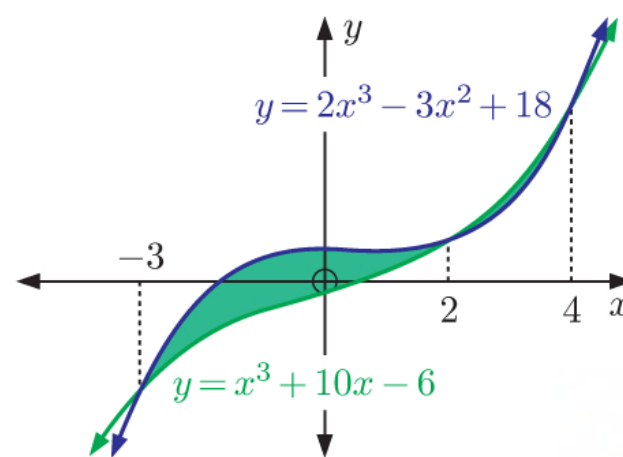
where  $2x^3 - 3x^2 + 18 = x^3 + 10x - 6$

$$\therefore x^3 - 3x^2 - 10x + 24 = 0$$

$$\therefore (x-2)(x^2 - x - 12) = 0$$

$$\therefore (x-2)(x-4)(x+3) = 0$$

$$\therefore x = -3, 2, \text{ or } 4$$

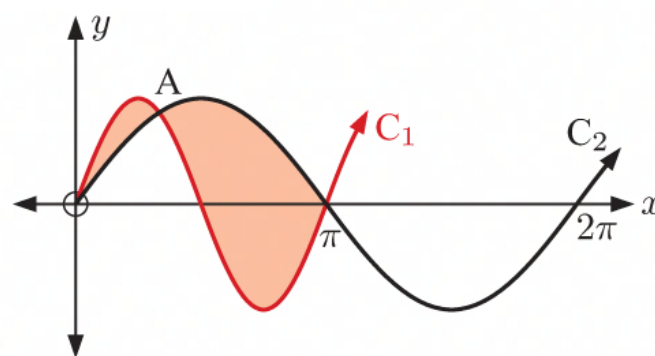


$$\begin{aligned} \text{Total area} &= \int_{-3}^2 [(2x^3 - 3x^2 + 18) - (x^3 + 10x - 6)] dx \\ &\quad + \int_2^4 [(x^3 + 10x - 6) - (2x^3 - 3x^2 + 18)] dx \\ &= \int_{-3}^2 (x^3 - 3x^2 - 10x + 24) dx + \int_2^4 (-x^3 + 3x^2 + 10x - 24) dx \\ &= \left[ \frac{1}{4}x^4 - x^3 - 5x^2 + 24x \right]_{-3}^2 + \left[ -\frac{1}{4}x^4 + x^3 + 5x^2 - 24x \right]_2^4 \\ &= [(4 - 8 - 20 + 48) - (\frac{81}{4} + 27 - 45 - 72)] \\ &\quad + [(-64 + 64 + 80 - 96) - (-4 + 8 + 20 - 48)] \\ &= (24 + \frac{279}{4}) + (-16 + 24) \\ &= 101\frac{3}{4} \text{ units}^2 \end{aligned}$$

**13 a**  $C_1$  has period  $\pi$ , and  $C_2$  has period  $2\pi$ .

$\therefore y = \sin 2x$  is the curve  $C_1$  and

$y = \sin x$  is the curve  $C_2$ .



**b** The curves meet where  $\sin 2x = \sin x$

$$\therefore 2 \sin x \cos x - \sin x = 0$$

$$\therefore \sin x(2 \cos x - 1) = 0$$

$$\therefore \sin x = 0 \text{ or } \cos x = \frac{1}{2}$$

$$\therefore x = 0 + k\pi \text{ or } x = \begin{cases} \frac{\pi}{3} \\ \frac{5\pi}{3} \end{cases} + 2k\pi, \quad k \in \mathbb{Z}$$

$\therefore$  the  $x$ -coordinate of A =  $\frac{\pi}{3}$  {smallest positive solution}

$$\text{and when } x = \frac{\pi}{3}, \quad y = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

$\therefore$  A is at  $(\frac{\pi}{3}, \frac{\sqrt{3}}{2})$

$$\begin{aligned}
 \text{Area} &= \int_0^{\frac{\pi}{3}} (\sin 2x - \sin x) dx + \int_{\frac{\pi}{3}}^{\pi} (\sin x - \sin 2x) dx \\
 &= \left[ -\frac{1}{2} \cos 2x + \cos x \right]_0^{\frac{\pi}{3}} + \left[ -\cos x + \frac{1}{2} \cos 2x \right]_{\frac{\pi}{3}}^{\pi} \\
 &= \left[ \left( -\frac{1}{2} \cos \frac{2\pi}{3} + \cos \frac{\pi}{3} \right) - \left( -\frac{1}{2} \cos 0 + \cos 0 \right) \right] \\
 &\quad + \left[ \left( -\cos \pi + \frac{1}{2} \cos 2\pi \right) - \left( -\cos \frac{\pi}{3} + \frac{1}{2} \cos \frac{2\pi}{3} \right) \right] \\
 &= \left[ \left( \frac{1}{4} + \frac{1}{2} \right) - \left( -\frac{1}{2} + 1 \right) \right] + \left[ \left( 1 + \frac{1}{2} \right) - \left( -\frac{1}{2} - \frac{1}{4} \right) \right] \\
 &= 2\frac{1}{2} \text{ units}^2
 \end{aligned}$$

**14**  $y = 2x$  meets  $y^2 = 4x$

where  $(2x)^2 = 4x$

$$\therefore 4x^2 = 4x$$

$$\therefore 4x^2 - 4x = 0$$

$$\therefore 4x(x - 1) = 0$$

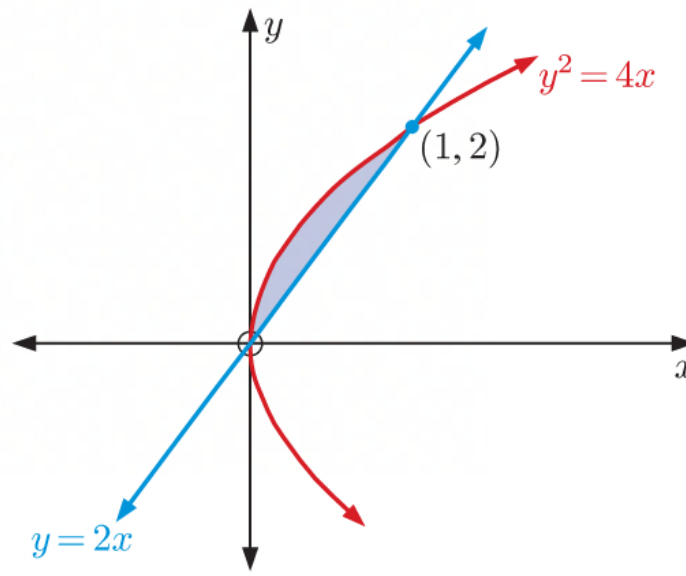
$$\therefore x = 0 \text{ or } 1$$

The upper part of  $y^2 = 4x$

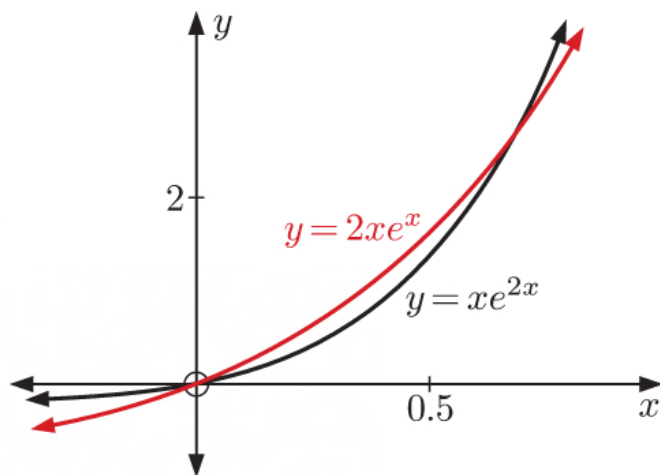
is  $y = \sqrt{4x}$

or  $y = 2\sqrt{x}$

$$\begin{aligned}
 \text{Area} &= \int_0^1 (2\sqrt{x} - 2x) dx \\
 &= \int_0^1 (2x^{\frac{1}{2}} - 2x) dx \\
 &= \left[ \frac{4}{3} x^{\frac{3}{2}} - x^2 \right]_0^1 \\
 &= \frac{4}{3} - 1 \\
 &= \frac{1}{3} \text{ unit}^2
 \end{aligned}$$



**15 a**



**b**  $y = 2xe^x$  and  $y = xe^{2x}$  meet

where  $2xe^x = xe^{2x}$

$$\therefore xe^{2x} - 2xe^x = 0$$

$$\therefore xe^x(e^x - 2) = 0$$

$$\therefore x = 0 \text{ or } e^x = 2 \quad \{e^x \neq 0\}$$

$$\therefore x = 0 \text{ or } x = \ln 2$$

$\therefore$  the  $x$ -coordinates of their points of intersection are 0 and  $\ln 2$ .

$$\begin{aligned}
 \text{Area} &= \int_0^{\ln 2} (2xe^x - xe^{2x}) dx \\
 &= 2 \int_0^{\ln 2} xe^x dx - \int_0^{\ln 2} xe^{2x} dx
 \end{aligned}$$

For  $\int xe^x dx$ , we integrate by parts with  $u = x, v' = e^x$   
 $u' = 1, v = e^x$

$$\begin{aligned}
 \therefore \int xe^x dx &= xe^x - \int e^x dx \\
 &= xe^x - e^x + c \\
 &= e^x(x - 1) + c
 \end{aligned}$$

For  $\int xe^{2x} dx$ , we integrate by parts with  $u = x, v' = e^{2x}$   
 $u' = 1, v = \frac{1}{2}e^{2x}$

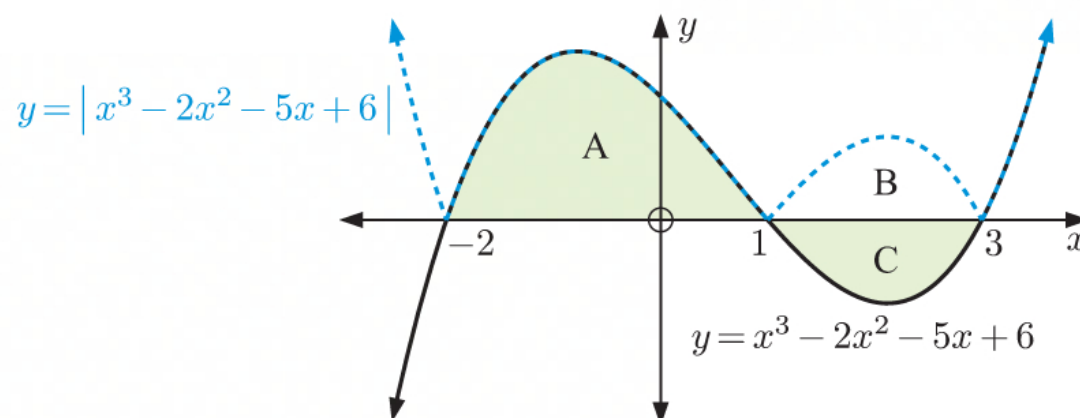
$$\begin{aligned}
 \therefore \int xe^{2x} dx &= \frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x} dx \\
 &= \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + c \\
 &= \frac{1}{4}e^{2x}(2x - 1) + c
 \end{aligned}$$

$$\begin{aligned}
 \therefore \text{area} &= 2 \int_0^{\ln 2} xe^x dx - \int_0^{\ln 2} xe^{2x} dx \\
 &= 2[e^x(x - 1)]_0^{\ln 2} - \left[\frac{1}{4}e^{2x}(2x - 1)\right]_0^{\ln 2} \\
 &= 2(2(\ln 2 - 1) - (-1)) - \left((2\ln 2 - 1) + \frac{1}{4}\right) \\
 &= 4\ln 2 - 2 - 2\ln 2 + \frac{3}{4} \\
 &= \left(2\ln 2 - \frac{5}{4}\right) \text{ units}^2
 \end{aligned}$$

## ACTIVITY 1

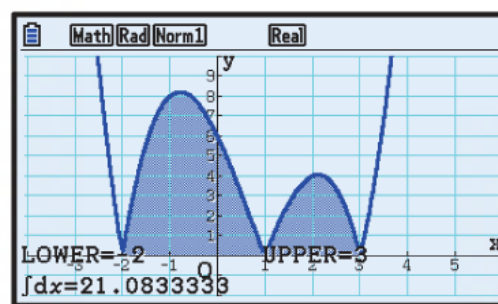
## CALCULATING AREAS USING TECHNOLOGY

1 a



$$\begin{aligned}
 \text{b Enclosed area} &= A + C \\
 &= A + B \quad \{\text{area B} = \text{area C since it is a reflection in the } x\text{-axis}\} \\
 &= \int_{-2}^3 |x^3 - 2x^2 - 5x + 6| dx
 \end{aligned}$$

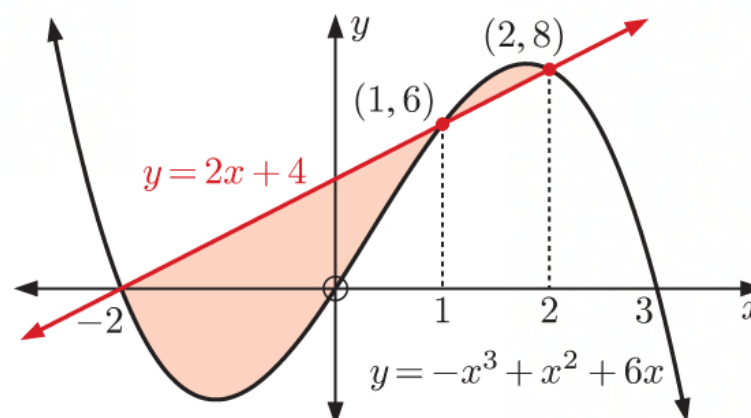
c  $\int_{-2}^3 |x^3 - 2x^2 - 5x + 6| dx \approx 21.1$



$\therefore$  the area enclosed between  $y = x^3 - 2x^2 - 5x + 6$  and the  $x$ -axis is about 21.1 units<sup>2</sup>.

2 a The graphs meet where

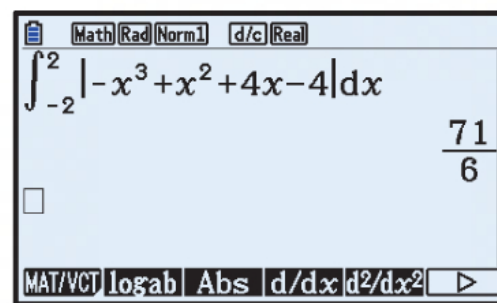
$$\begin{aligned} -x^3 + x^2 + 6x &= 2x + 4 \\ \therefore -x^3 + x^2 + 4x - 4 &= 0 \\ \therefore x^3 - x^2 - 4x + 4 &= 0 \\ \therefore (x+2)(x-2)(x-1) &= 0 \\ \therefore x &= -2, 1, \text{ or } 2 \end{aligned}$$



So, total area

$$\begin{aligned} &= \int_{-2}^1 [(2x+4) - (-x^3 + x^2 + 6x)] dx + \int_1^2 [(-x^3 + x^2 + 6x) - (2x+4)] dx \\ &= \int_{-2}^1 |(-x^3 + x^2 + 6x) - (2x+4)| dx + \int_1^2 |(-x^3 + x^2 + 6x) - (2x+4)| dx \\ &= \int_{-2}^2 |(-x^3 + x^2 + 6x) - (2x+4)| dx \\ &= \int_{-2}^2 |-x^3 + x^2 + 4x - 4| dx \end{aligned}$$

b  $\int_{-2}^2 |-x^3 + x^2 + 4x - 4| dx = \frac{71}{6}$



$\therefore$  the area enclosed between  $y = -x^3 + x^2 + 6x$  and  $y = 2x + 4$  is  $\frac{71}{6}$  units<sup>2</sup>.

3 Consider the interval  $a \leq x \leq b$ .

For intervals  $c \leq x \leq d$  within  $a \leq x \leq b$  where  $y = f(x)$  lies above  $y = g(x)$ , the contribution to the total enclosed area is  $\int_c^d [f(x) - g(x)] dx = \int_c^d |f(x) - g(x)| dx$ .

For intervals  $c \leq x \leq d$  within  $a \leq x \leq b$  where  $y = f(x)$  lies below  $y = g(x)$ , the contribution to the total enclosed area is  $\int_c^d [g(x) - f(x)] dx = \int_c^d |f(x) - g(x)| dx$ .

In each case, the integrand is  $|f(x) - g(x)|$ .

$\therefore$  total enclosed area  $= \int_a^b |f(x) - g(x)| dx$



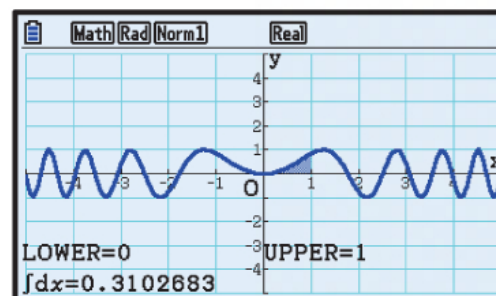
## ACTIVITY 2

## NUMERICAL INTEGRATION

1 a  $\int_0^1 \sin(x^2) dx \approx [f(0.05) + f(0.15) + f(0.25) + \dots + f(0.95)] \delta x$   
 $\approx [\sin(0.05^2) + \sin(0.15^2) + \sin(0.25^2) + \dots + \sin(0.95^2)] \times 0.1$   
 $\approx 0.310$

b

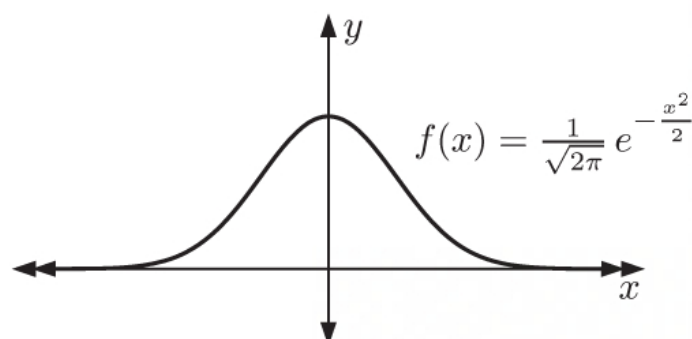
$n$	$\int_0^1 \sin(x^2) dx$
10	0.309 82
100	0.310 26
1000	0.310 27
10 000	0.310 27



Using technology,  $\int_0^1 \sin(x^2) dx \approx 0.310 27$

- c i Using the software (pressing Find integral) with  $n = 10\,000$ ,  $\int_0^2 \sin(x^2) dx \approx 0.804 78$   
 ii Using the software (pressing Find area) with  $n = 10\,000$ , the area enclosed between  $y = \sin(x^2)$ , the  $x$ -axis, and the vertical lines  $x = 0$  and  $x = 2$ , is approximately  $0.984 89$  units<sup>2</sup>.

2 a



b Using the software with  $n = 1000$ :

i  $\int_{-1}^1 f(x) dx \approx 0.682 69$     ii  $\int_{-2}^2 f(x) dx \approx 0.954 50$     iii  $\int_{-3}^3 f(x) dx \approx 0.997 30$

c Using the software with  $n = 10\,000$ :

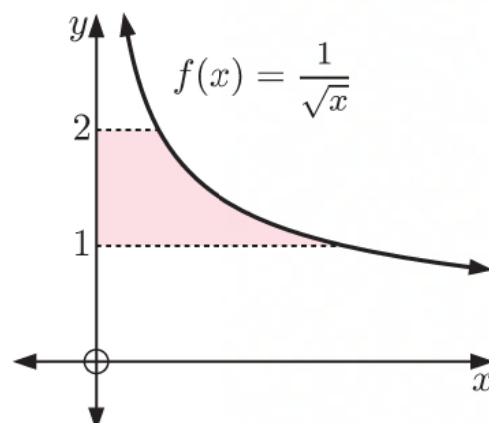
i  $\int_{-5}^5 f(x) dx \approx 1.000 00$     ii  $\int_{-10}^{10} f(x) dx \approx 1.000 00$

d  $\int_{-\infty}^{\infty} f(x) dx = 1$ , the area under the standard normal density curve is 1.

## EXERCISE 22F

1 a

$$\begin{aligned}
 y &= \frac{1}{\sqrt{x}} \\
 \therefore y^2 &= \frac{1}{x} \\
 \therefore x &= \frac{1}{y^2} \\
 \therefore f^{-1}(y) &= \frac{1}{y^2}
 \end{aligned}$$

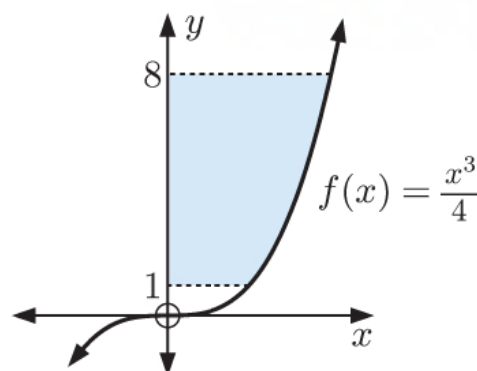


$$\begin{aligned}
 \text{b Area} &= \int_1^2 \frac{1}{y^2} dy \\
 &= [-y^{-1}]_1^2 \\
 &= -\frac{1}{2} - (-1) \\
 &= \frac{1}{2} \text{ units}^2
 \end{aligned}$$

2 a

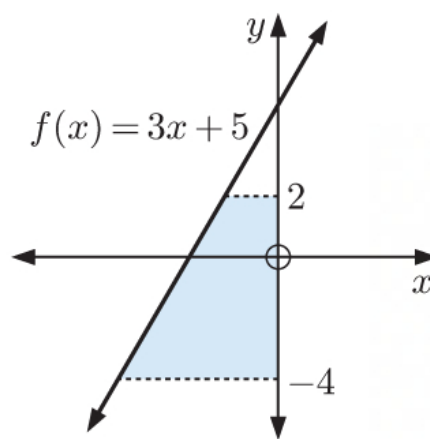
$$\begin{aligned}
 y &= \frac{x^3}{4} \\
 \therefore x^3 &= 4y \\
 \therefore x &= \sqrt[3]{4y} \\
 \therefore f^{-1}(y) &= \sqrt[3]{4y}
 \end{aligned}$$

$$\begin{aligned}
 \therefore \text{area} &= \int_1^8 (4y)^{\frac{1}{3}} dy \\
 &= 4^{\frac{1}{3}} \int_1^8 y^{\frac{1}{3}} dy \\
 &= 4^{\frac{1}{3}} \left[ \frac{3}{4} y^{\frac{4}{3}} \right]_1^8 \\
 &= 4^{\frac{1}{3}} \left( \frac{3}{4} (8)^{\frac{4}{3}} - \frac{3}{4} \right) \\
 &= 2^{\frac{2}{3}} \times \frac{45}{4} \\
 &= 2^{\frac{2}{3}} \times \frac{45}{2^2} \\
 &= \frac{45}{2^{\frac{4}{3}}} \\
 &= \frac{45}{2 \times \sqrt[3]{2}} \text{ units}^2
 \end{aligned}$$

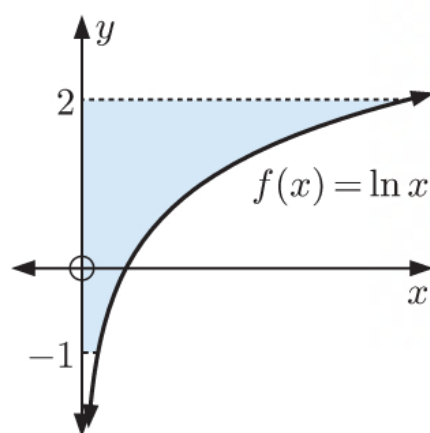


**b**

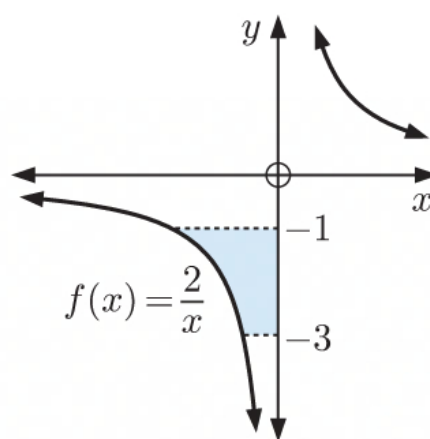
$$\begin{aligned}
 y &= 3x + 5 \\
 \therefore 3x &= y - 5 \\
 \therefore x &= \frac{y - 5}{3} \\
 \therefore f^{-1}(y) &= \frac{1}{3}y - \frac{5}{3} \\
 \therefore \text{area} &= - \int_{-4}^2 \left( \frac{1}{3}y - \frac{5}{3} \right) dy \\
 &= - \left[ \frac{1}{6}y^2 - \frac{5}{3}y \right]_{-4}^2 \\
 &= - \left[ \left( \frac{1}{6}(2)^2 - \frac{5}{3}(2) \right) - \left( \frac{1}{6}(-4)^2 - \frac{5}{3}(-4) \right) \right] \\
 &= - \left[ -\frac{8}{3} - \left( \frac{28}{3} \right) \right] \\
 &= 12 \text{ units}^2
 \end{aligned}$$

**c**

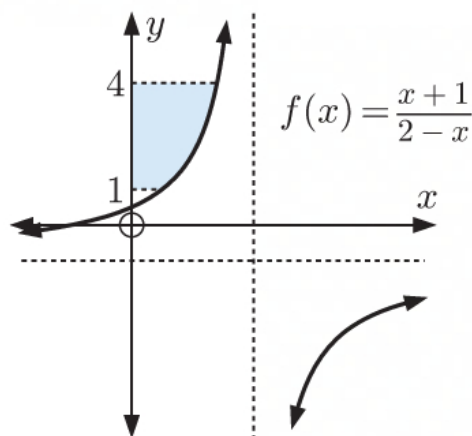
$$\begin{aligned}
 y &= \ln x \\
 \therefore x &= e^y \\
 \therefore f^{-1}(y) &= e^y \\
 \therefore \text{area} &= \int_{-1}^2 e^y dy \\
 &= [e^y]_{-1}^2 \\
 &= e^2 - e^{-1} \\
 &= e^2 - \frac{1}{e} \\
 &= \frac{e^3 - 1}{e} \text{ units}^2
 \end{aligned}$$

**d**

$$\begin{aligned}
 y &= \frac{2}{x} \\
 \therefore x &= \frac{2}{y} \\
 \therefore f^{-1}(y) &= \frac{2}{y} \\
 \therefore \text{area} &= - \int_{-3}^{-1} \frac{2}{y} dy \\
 &= - [2 \ln |y|]_{-3}^{-1} \\
 &= -(\cancel{2 \ln 1} - 2 \ln 3) \\
 &= 2 \ln 3 \text{ units}^2
 \end{aligned}$$

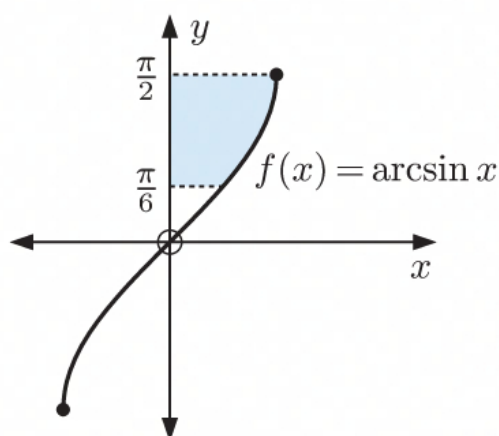


$$\begin{aligned}
 \text{e} \quad & y = \frac{x+1}{2-x} \\
 \therefore & y(2-x) = x+1 \\
 \therefore & 2y - xy = x+1 \\
 \therefore & xy + x = 2y - 1 \\
 \therefore & x(y+1) = 2y - 1 \\
 \therefore & x = \frac{2y-1}{y+1} \\
 \therefore & f^{-1}(y) = \frac{2y-1}{y+1}
 \end{aligned}$$

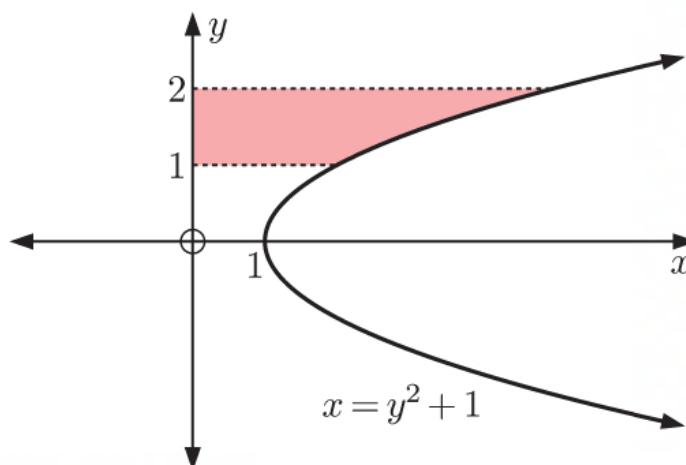


$$\begin{aligned}
 \therefore \text{ area} &= \int_1^4 \frac{2y-1}{y+1} dy \\
 &= \int_1^4 \frac{2(y+1)-3}{y+1} dy \\
 &= \int_1^4 \left( 2 - \frac{3}{y+1} \right) dy \\
 &= [2y - 3 \ln |y+1|]_1^4 \\
 &= (8 - 3 \ln 5) - (2 - 3 \ln 2) \\
 &= 6 - (3 \ln 5 - 3 \ln 2) \\
 &= 6 - 3 \ln \left( \frac{5}{2} \right) \text{ units}^2
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & y = \arcsin x \\
 \therefore & x = \sin y \\
 \therefore & f^{-1}(y) = \sin y \\
 \therefore \text{ area} &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin y dy \\
 &= [-\cos y]_{\frac{\pi}{6}}^{\frac{\pi}{2}} \\
 &= -\cancel{\cos} \frac{\pi}{2} - \left( -\cos \frac{\pi}{6} \right) \\
 &= \cos \frac{\pi}{6} \\
 &= \frac{\sqrt{3}}{2} \text{ units}^2
 \end{aligned}$$

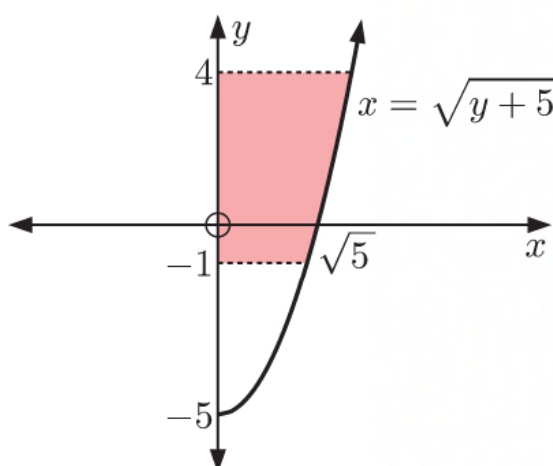


$$\begin{aligned}
 \text{3 a} \quad & x = y^2 + 1 \\
 \therefore & f^{-1}(y) = y^2 + 1 \\
 \therefore \text{ area} &= \int_1^2 (y^2 + 1) dy \\
 &= \left[ \frac{1}{3} y^3 + y \right]_1^2 \\
 &= \left( \frac{8}{3} + 2 \right) - \left( \frac{1}{3} + 1 \right) \\
 &= \frac{7}{3} + 1 \\
 &= 3\frac{1}{3} \text{ units}^2
 \end{aligned}$$

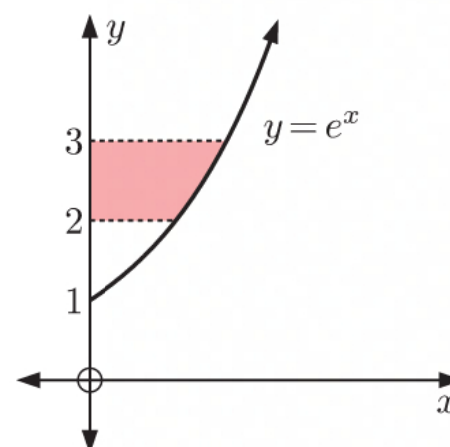




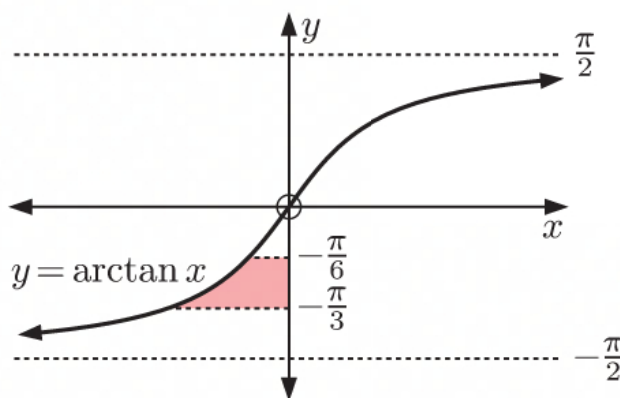
$$\begin{aligned}
 \text{b} \quad & x = \sqrt{y+5} \\
 \therefore & f^{-1}(y) = (y+5)^{\frac{1}{2}} \\
 \therefore \text{ area} &= \int_{-1}^4 (y+5)^{\frac{1}{2}} dy \\
 &= \left[ \frac{2}{3} (y+5)^{\frac{3}{2}} \right]_{-1}^4 \\
 &= \frac{2}{3} (9)^{\frac{3}{2}} - \frac{2}{3} (4)^{\frac{3}{2}} \\
 &= \frac{38}{3} \\
 &= 12\frac{2}{3} \text{ units}^2
 \end{aligned}$$



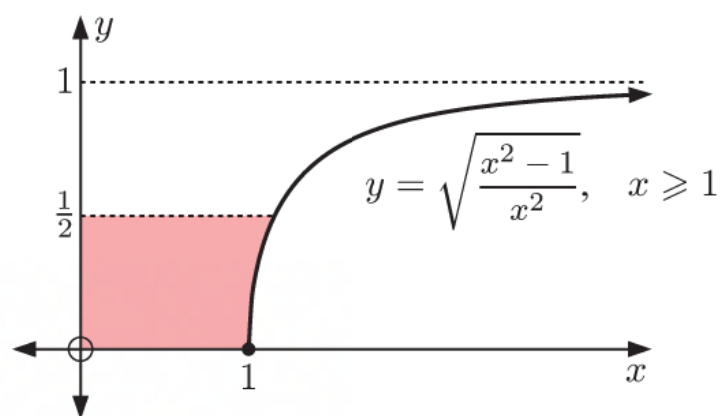
$$\begin{aligned}
 \text{4 a} \quad & y = e^x \\
 \therefore & x = \ln y \\
 \therefore & f^{-1}(y) = \ln y \\
 \therefore \text{ area} &= \int_2^3 \ln y dy \\
 &= [y(\ln y - 1)]_2^3 \\
 &\quad \{\text{from Exercise 21G question 2 a}\} \\
 &= 3(\ln 3 - 1) - 2(\ln 2 - 1) \\
 &= 3 \ln 3 - 3 - 2 \ln 2 + 2 \\
 &= \ln 27 - \ln 4 - 1 \\
 &= \left[ \ln \left( \frac{27}{4} \right) - 1 \right] \text{ units}^2
 \end{aligned}$$



$$\begin{aligned}
 \text{b} \quad & y = \arctan x \\
 \therefore & x = \tan y \\
 \therefore & f^{-1}(y) = \tan y \\
 \therefore \text{ area} &= - \int_{-\frac{\pi}{3}}^{-\frac{\pi}{6}} \tan y dy \\
 &= - \left[ -\ln |\cos y| \right]_{-\frac{\pi}{3}}^{-\frac{\pi}{6}} \\
 &= - \left[ -\ln \left| \cos \left( -\frac{\pi}{6} \right) \right| - \left( -\ln \left| \cos \left( -\frac{\pi}{3} \right) \right| \right) \right] \\
 &= - \left[ -\ln \left( \frac{\sqrt{3}}{2} \right) + \ln \left( \frac{1}{2} \right) \right] \\
 &= \ln \left( \frac{\sqrt{3}}{2} \right) - \ln \left( \frac{1}{2} \right) \\
 &= \ln \sqrt{3} - \ln 2 - \ln(2^{-1}) \\
 &= \ln(3^{\frac{1}{2}}) - \cancel{\ln 2} + \cancel{\ln 2} \\
 &= \frac{1}{2} \ln 3 \text{ units}^2
 \end{aligned}$$

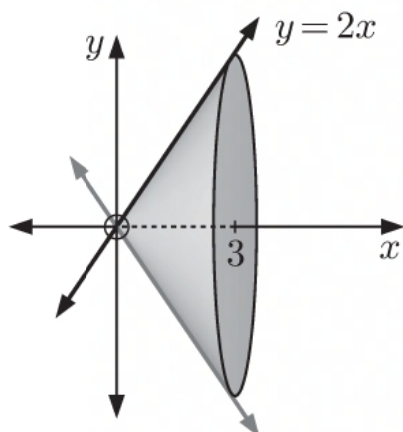


$$\begin{aligned}
 \text{c} \quad & y = \sqrt{\frac{x^2 - 1}{x^2}}, \quad x \geq 1 \\
 & \therefore y^2 = \frac{x^2 - 1}{x^2} \\
 & \therefore x^2 y^2 = x^2 - 1 \\
 & \therefore x^2 y^2 - x^2 = -1 \\
 & \therefore x^2 (y^2 - 1) = -1 \\
 & \therefore x^2 = -\frac{1}{y^2 - 1} \\
 & \therefore x = \sqrt{\frac{-1}{y^2 - 1}} \\
 & \therefore x = \sqrt{\frac{1}{1 - y^2}}, \quad 0 \leq y < 1 \\
 & \therefore f^{-1}(y) = \sqrt{\frac{1}{1 - y^2}} \\
 & \therefore \text{area} = \int_0^{\frac{1}{2}} \sqrt{\frac{1}{1 - y^2}} \, dy \\
 & = \int_0^{\frac{1}{2}} \frac{1}{\sqrt{1 - y^2}} \, dy \\
 & = \left[ \arcsin y \right]_0^{\frac{1}{2}} \\
 & = \arcsin \frac{1}{2} - \arcsin 0 \\
 & = \frac{\pi}{6} \text{ units}^2
 \end{aligned}$$



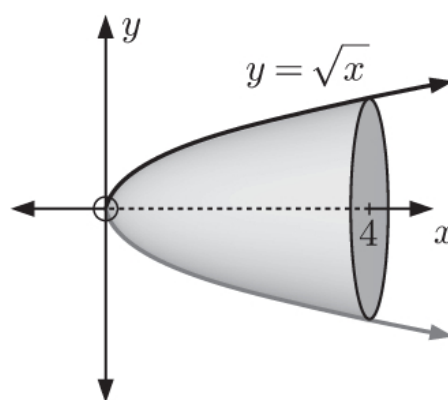
## EXERCISE 22G.1

1 a



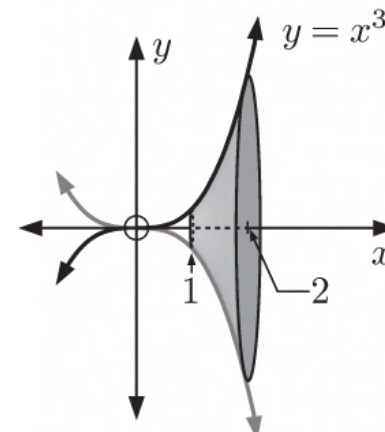
$$\begin{aligned}
 \text{Volume} &= \pi \int_0^3 y^2 dx \\
 &= \pi \int_0^3 (2x)^2 dx \\
 &= 4\pi \int_0^3 x^2 dx \\
 &= 4\pi \left[ \frac{1}{3}x^3 \right]_0^3 \\
 &= 4\pi(9 - 0) \\
 &= 36\pi \text{ units}^3
 \end{aligned}$$

b



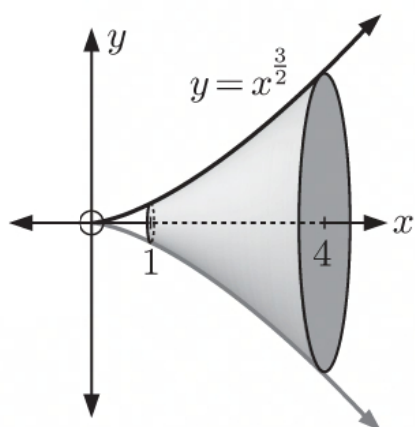
$$\begin{aligned}
 \text{Volume} &= \pi \int_0^4 y^2 dx \\
 &= \pi \int_0^4 (\sqrt{x})^2 dx \\
 &= \pi \int_0^4 x dx \\
 &= \pi \left[ \frac{1}{2}x^2 \right]_0^4 \\
 &= \pi(8 - 0) \\
 &= 8\pi \text{ units}^3
 \end{aligned}$$

c



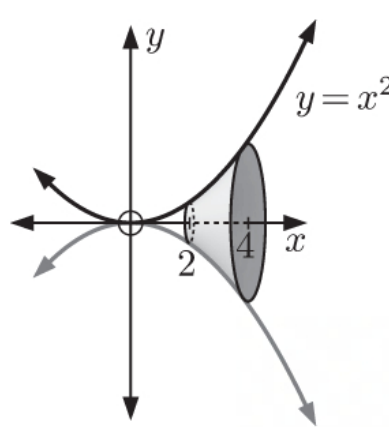
$$\begin{aligned}
 \text{Volume} &= \pi \int_1^2 y^2 dx \\
 &= \pi \int_1^2 (x^3)^2 dx \\
 &= \pi \int_1^2 x^6 dx \\
 &= \pi \left[ \frac{1}{7}x^7 \right]_1^2 \\
 &= \pi \left( \frac{128}{7} - \frac{1}{7} \right) \\
 &= \frac{127\pi}{7} \text{ units}^3
 \end{aligned}$$

d



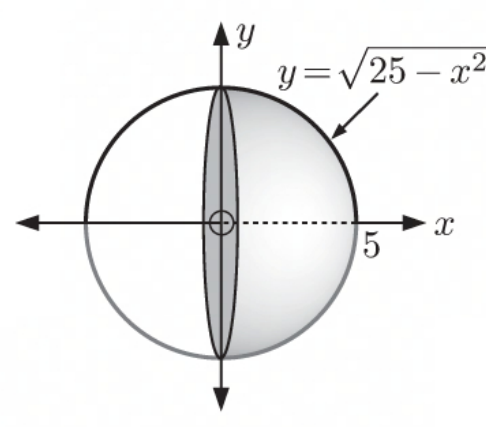
$$\begin{aligned}
 \text{Volume} &= \pi \int_1^4 y^2 dx \\
 &= \pi \int_1^4 (x^{\frac{3}{2}})^2 dx \\
 &= \pi \int_1^4 x^3 dx \\
 &= \pi \left[ \frac{1}{4}x^4 \right]_1^4 \\
 &= \pi \left( \frac{256}{4} - \frac{1}{4} \right) \\
 &= \frac{255\pi}{4} \text{ units}^3
 \end{aligned}$$

e

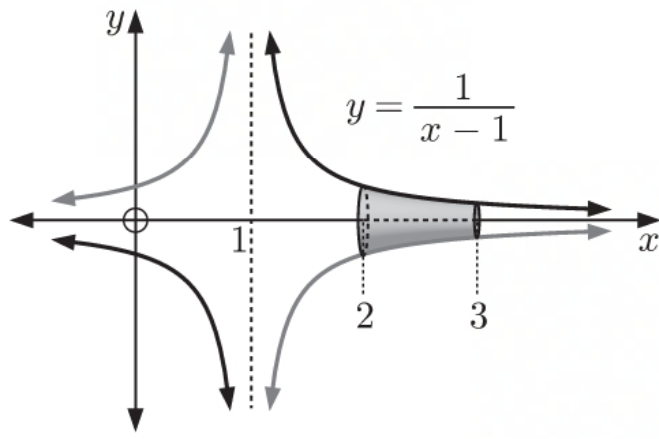


$$\begin{aligned}
 \text{Volume} &= \pi \int_2^4 y^2 dx \\
 &= \pi \int_2^4 (x^2)^2 dx \\
 &= \pi \int_2^4 x^4 dx \\
 &= \pi \left[ \frac{1}{5}x^5 \right]_2^4 \\
 &= \pi \left( \frac{1024}{5} - \frac{32}{5} \right) \\
 &= \frac{992\pi}{5} \text{ units}^3
 \end{aligned}$$

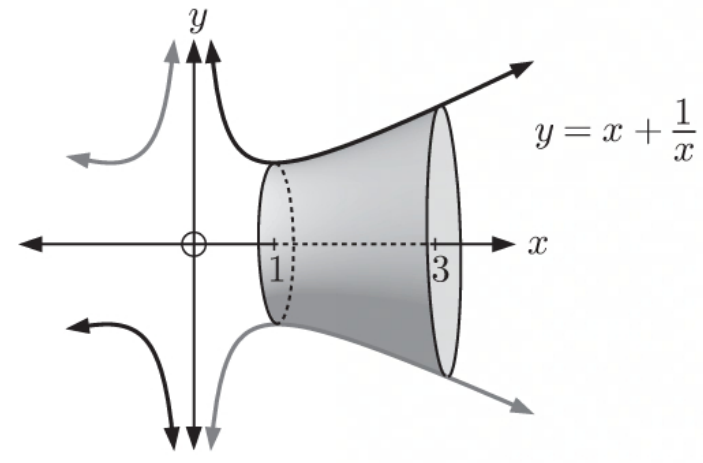
f



$$\begin{aligned}
 \text{Volume} &= \pi \int_0^5 y^2 dx \\
 &= \pi \int_0^5 (25 - x^2) dx \\
 &= \pi \left[ 25x - \frac{x^3}{3} \right]_0^5 \\
 &= \pi \left( 125 - \frac{125}{3} \right) \\
 &= \pi \left( \frac{2}{3} \right) 125 \\
 &= \frac{250\pi}{3} \text{ units}^3
 \end{aligned}$$

**g**

$$\begin{aligned}
 \text{Volume} &= \pi \int_2^3 y^2 dx \\
 &= \pi \int_2^3 \left(\frac{1}{x-1}\right)^2 dx \\
 &= \pi \int_2^3 (x-1)^{-2} dx \\
 &= \pi \left[-\frac{1}{x-1}\right]_2^3 \\
 &= \pi\left(-\frac{1}{2} + 1\right) \\
 &= \frac{\pi}{2} \text{ units}^3
 \end{aligned}$$

**h**

$$\begin{aligned}
 \text{Volume} &= \pi \int_1^3 y^2 dx \\
 &= \pi \int_1^3 \left(x + \frac{1}{x}\right)^2 dx \\
 &= \pi \int_1^3 (x^2 + 2 + x^{-2}) dx \\
 &= \pi \left[\frac{x^3}{3} + 2x - \frac{1}{x}\right]_1^3 \\
 &= \pi\left[\left(9 + 6 - \frac{1}{3}\right) - \left(\frac{1}{3} + 2 - 1\right)\right] \\
 &= \frac{40\pi}{3} \text{ units}^3
 \end{aligned}$$

**2**

**a** Volume =  $\pi \int_0^6 y^2 dx$

$$\begin{aligned}
 &= \pi \int_0^6 \left(\frac{x}{2} + 4\right)^2 dx \\
 &= \pi \int_0^6 \left(\frac{1}{4}x^2 + 4x + 16\right) dx \\
 &= \pi \left[\frac{x^3}{12} + \frac{4x^2}{2} + 16x\right]_0^6 \\
 &= \pi[(18 + 72 + 96) - 0] \\
 &= 186\pi \text{ units}^3
 \end{aligned}$$

**c**

Volume =  $\pi \int_0^4 y^2 dx$

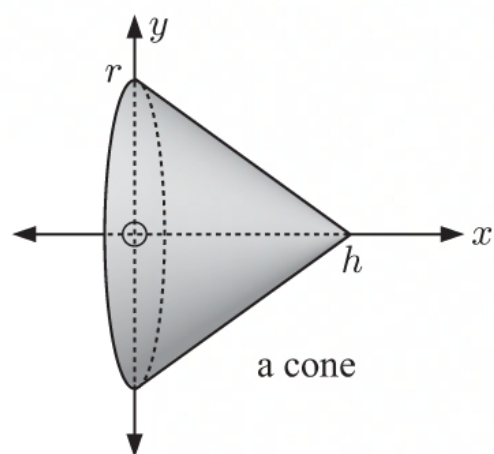
$$\begin{aligned}
 &= \pi \int_0^4 (e^x)^2 dx \\
 &= \pi \int_0^4 e^{2x} dx \\
 &= \pi \left[\frac{1}{2}e^{2x}\right]_0^4 \\
 &= \pi\left(\frac{1}{2}e^8 - \frac{1}{2}\right) \\
 &= \frac{\pi}{2}(e^8 - 1) \text{ units}^3
 \end{aligned}$$

**b** Volume

$$\begin{aligned}
 &= \pi \int_1^2 y^2 dx \\
 &= \pi \int_1^2 (x^2 + 3)^2 dx \\
 &= \pi \int_1^2 (x^4 + 6x^2 + 9) dx \\
 &= \pi \left[\frac{x^5}{5} + \frac{6x^3}{3} + 9x\right]_1^2 \\
 &= \pi\left[\left(\frac{32}{5} + 16 + 18\right) - \left(\frac{1}{5} + 2 + 9\right)\right] \\
 &= \pi\left(\frac{146}{5}\right) \\
 &= \frac{146\pi}{5} \text{ units}^3
 \end{aligned}$$



- 3 a** a cone of base radius  $r$  and height  $h$

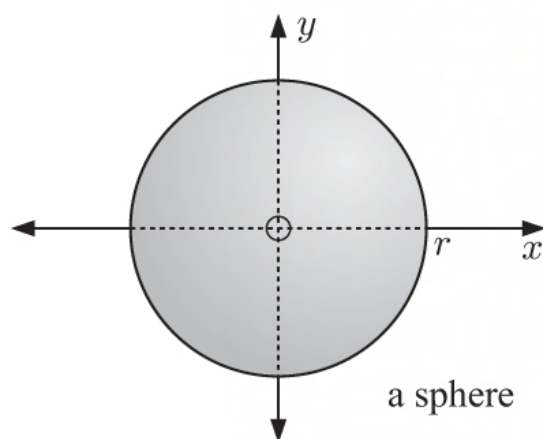


- b** (AB) has gradient  $= \frac{r-0}{0-h} = -\frac{r}{h}$

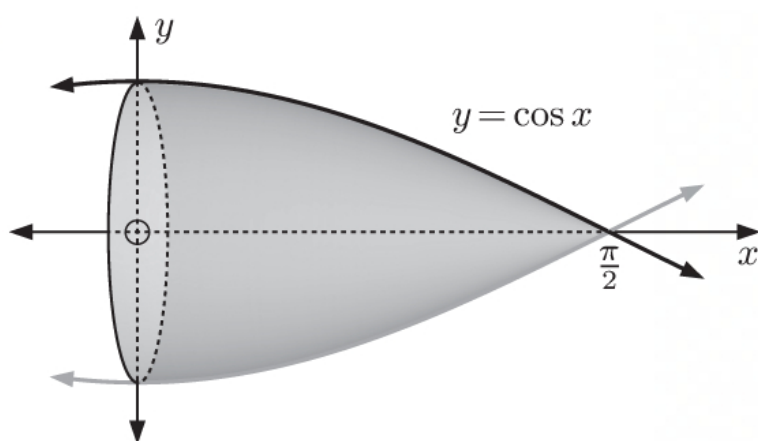
$$\therefore \text{ its equation is } y = -\left(\frac{r}{h}\right)x + r$$

$$\begin{aligned}
 \text{c } V &= \pi \int_0^h y^2 dx \\
 &= \pi \int_0^h \left(-\frac{r}{h}x + r\right)^2 dx \\
 &= \pi r^2 \int_0^h \left(-\frac{x}{h} + 1\right)^2 dx \\
 &= \pi r^2 \int_0^h \left(\frac{x^2}{h^2} - \frac{2x}{h} + 1\right) dx \\
 &= \pi r^2 \left[ \frac{x^3}{3h^2} - \frac{x^2}{h} + x \right]_0^h \\
 &= \pi r^2 \left[ \left(\frac{h}{3} - h + h\right) - 0 \right] \\
 &= \frac{1}{3}\pi r^2 h \text{ units}^3
 \end{aligned}$$

- 4 a** a sphere of radius  $r$

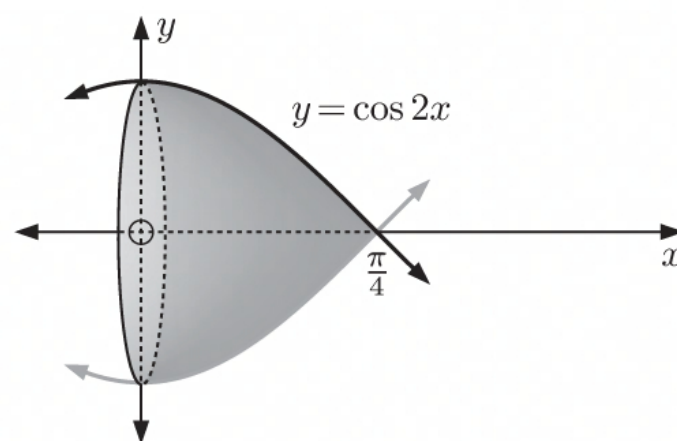


$$\begin{aligned}
 \text{b } V &= \pi \int_{-r}^r y^2 dx = 2\pi \int_0^r (r^2 - x^2) dx \\
 &= 2\pi \left[ r^2x - \frac{x^3}{3} \right]_0^r \\
 &= 2\pi \left[ \left( r^3 - \frac{r^3}{3} \right) - 0 \right] \\
 &= 2\pi \times \frac{2}{3}r^3 \\
 &= \frac{4}{3}\pi r^3 \text{ units}^3
 \end{aligned}$$

**5 a**

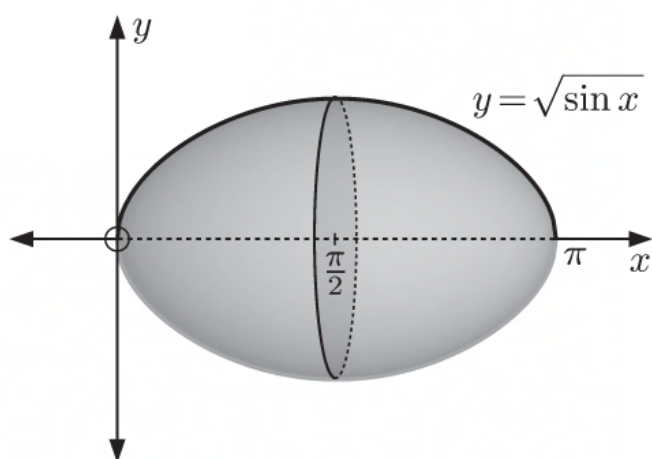
Volume

$$\begin{aligned}
 &= \pi \int_0^{\pi/2} y^2 dx \\
 &= \pi \int_0^{\pi/2} \cos^2 x dx \\
 &= \pi \int_0^{\pi/2} \left( \frac{1}{2} + \frac{1}{2} \cos 2x \right) dx \\
 &= \pi \left[ \frac{x}{2} + \frac{1}{2} \left( \frac{1}{2} \right) \sin 2x \right]_0^{\pi/2} \\
 &= \pi \left[ \left( \frac{\pi}{4} + \frac{1}{4} \sin \pi \right) - \left( 0 + \frac{1}{4} \sin 0 \right) \right] \\
 &= \pi \times \frac{\pi}{4} \\
 &= \frac{\pi^2}{4} \text{ units}^3
 \end{aligned}$$

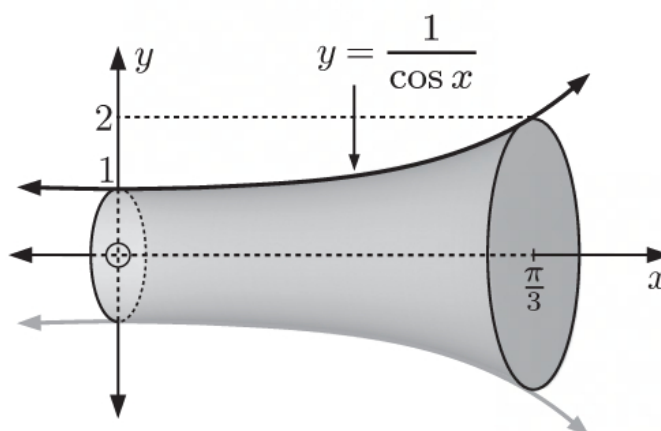
**b**

Volume

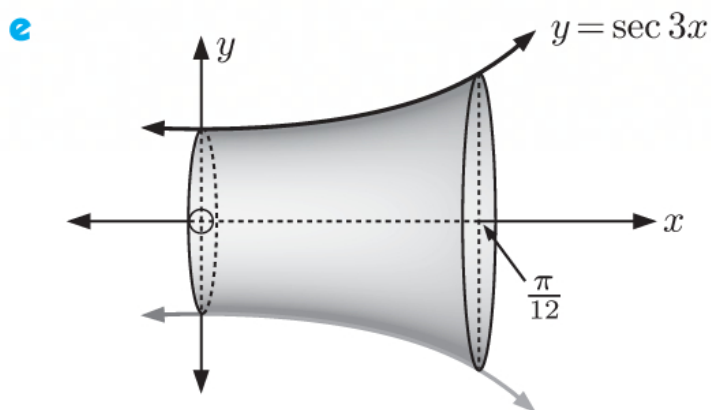
$$\begin{aligned}
 &= \pi \int_0^{\pi/4} y^2 dx \\
 &= \pi \int_0^{\pi/4} \cos^2 2x dx \\
 &= \pi \int_0^{\pi/4} \left( \frac{1}{2} + \frac{1}{2} \cos 4x \right) dx \\
 &= \pi \left[ \frac{1}{2}x + \frac{1}{2} \left( \frac{1}{4} \right) \sin 4x \right]_0^{\pi/4} \\
 &= \pi \left[ \left( \frac{\pi}{8} + \frac{1}{8} \sin \pi \right) - \left( 0 + \frac{1}{8} \sin 0 \right) \right] \\
 &= \frac{\pi^2}{8} \text{ units}^3
 \end{aligned}$$

**c**

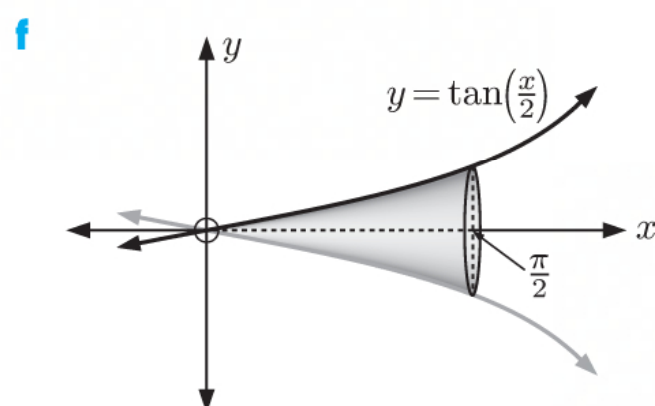
$$\begin{aligned}
 \text{Volume} &= \pi \int_0^{\pi} y^2 dx \\
 &= \pi \int_0^{\pi} \sin x dx \\
 &= \pi [-\cos x]_0^{\pi} \\
 &= \pi [-\cos \pi - (-\cos 0)] \\
 &= \pi(2) \\
 &= 2\pi \text{ units}^3
 \end{aligned}$$

**d**

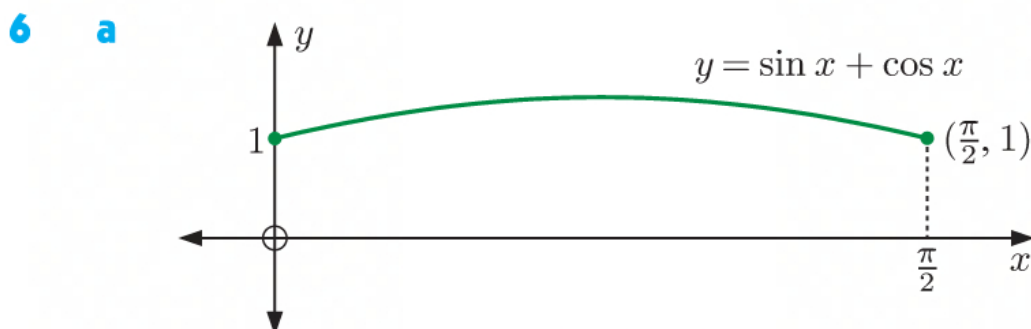
$$\begin{aligned}
 \text{Volume} &= \pi \int_0^{\pi/3} y^2 dx \\
 &= \pi \int_0^{\pi/3} \frac{1}{\cos^2 x} dx \\
 &= \pi [\tan x]_0^{\pi/3} \\
 &= \pi \left( \tan \frac{\pi}{3} - \tan 0 \right) \\
 &= \pi(\sqrt{3} - 0) \\
 &= \pi\sqrt{3} \text{ units}^3
 \end{aligned}$$



$$\begin{aligned}
 \text{Volume} &= \pi \int_0^{\frac{\pi}{12}} y^2 dx \\
 &= \pi \int_0^{\frac{\pi}{12}} \sec^2 3x dx \\
 &= \pi \left[ \frac{1}{3} \tan 3x \right]_0^{\frac{\pi}{12}} \\
 &= \frac{\pi}{3} \left( \tan \frac{\pi}{4} - \frac{1}{3} \tan 0 \right) \\
 &= \frac{\pi}{3} \text{ units}^3
 \end{aligned}$$

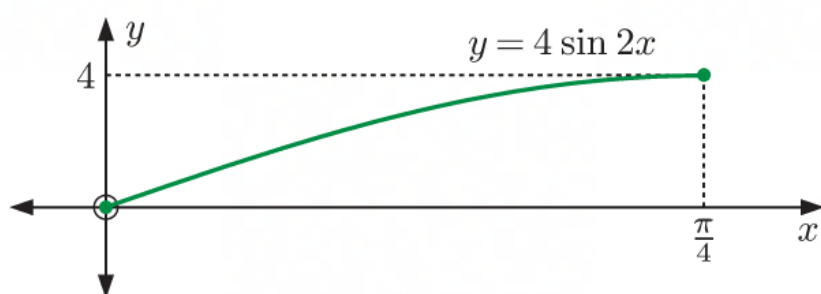


$$\begin{aligned}
 \text{Volume} &= \pi \int_0^{\frac{\pi}{2}} y^2 dx \\
 &= \pi \int_0^{\frac{\pi}{2}} \tan^2 \left( \frac{x}{2} \right) dx \\
 &= \pi \int_0^{\frac{\pi}{2}} \left( \sec^2 \left( \frac{x}{2} \right) - 1 \right) dx \\
 &= \pi \left[ 2 \tan \frac{x}{2} - x \right]_0^{\frac{\pi}{2}} \\
 &= \pi \left[ \left( 2 \tan \frac{\pi}{4} - \frac{\pi}{2} \right) - (2 \tan 0 - 0) \right] \\
 &= \pi \left( 2 - \frac{\pi}{2} \right) \text{ units}^3
 \end{aligned}$$



**b**

$$\begin{aligned}
 \text{Volume} &= \pi \int_0^{\frac{\pi}{4}} y^2 dx \\
 &= \pi \int_0^{\frac{\pi}{4}} (\sin x + \cos x)^2 dx \\
 &= \pi \int_0^{\frac{\pi}{4}} (\sin^2 x + 2 \sin x \cos x + \cos^2 x) dx \\
 &= \pi \int_0^{\frac{\pi}{4}} (1 + \sin 2x) dx \\
 &= \pi \left[ x - \frac{1}{2} \cos 2x \right]_0^{\frac{\pi}{4}} \\
 &= \pi \left[ \left( \frac{\pi}{4} - \frac{1}{2} \cos \frac{\pi}{2} \right) - \left( 0 - \frac{1}{2} \cos 0 \right) \right] \\
 &= \pi \left( \frac{\pi}{4} + \frac{1}{2} \right) \text{ units}^3
 \end{aligned}$$

**7 a****b** Volume

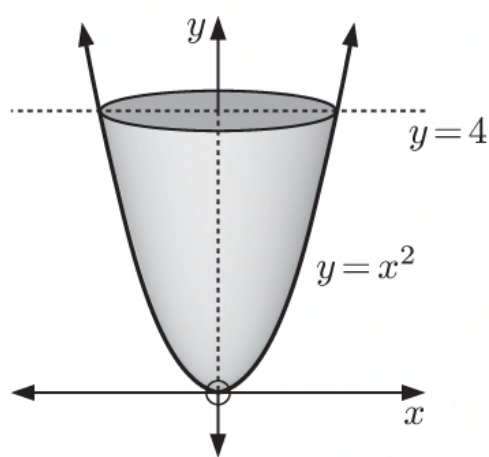
$$\begin{aligned}
 &= \pi \int_0^{\pi/4} y^2 dx \\
 &= \pi \int_0^{\pi/4} 16 \sin^2 2x dx \\
 &= 16\pi \int_0^{\pi/4} \left(\frac{1}{2} - \frac{1}{2} \cos 4x\right) dx \\
 &= 16\pi \left[ \frac{x}{2} - \frac{1}{2} \left(\frac{1}{4}\right) \sin 4x \right]_0^{\pi/4} \\
 &= 16\pi \left[ \left(\frac{\pi}{8} - \frac{1}{8} \sin \pi\right) - \left(0 - \frac{1}{8} \sin 0\right) \right] \\
 &= 2\pi^2 \text{ units}^3
 \end{aligned}$$

**8 a**

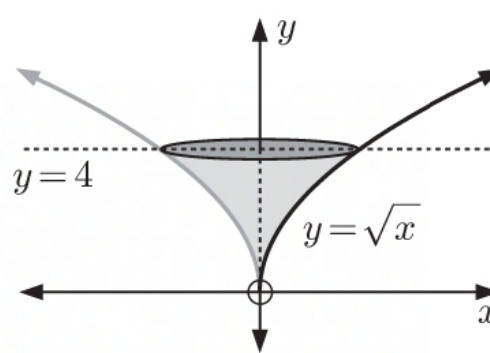
$$\begin{aligned}
 \text{Volume} &= \pi \int_1^3 y^2 dx \\
 &= \pi \int_1^3 \left( \frac{x^3}{x^2 + 1} \right)^2 dx \\
 &\approx 5.926\pi \quad \{\text{using technology}\} \\
 &\approx 18.6 \text{ units}^3
 \end{aligned}$$

**b**

$$\begin{aligned}
 \text{Volume} &= \pi \int_0^2 y^2 dx \\
 &= \pi \int_0^2 (e^{\sin x})^2 dx \\
 &\approx 9.613\pi \quad \{\text{using technology}\} \\
 &\approx 30.2 \text{ units}^3
 \end{aligned}$$

**EXERCISE 22G.2****1 a**

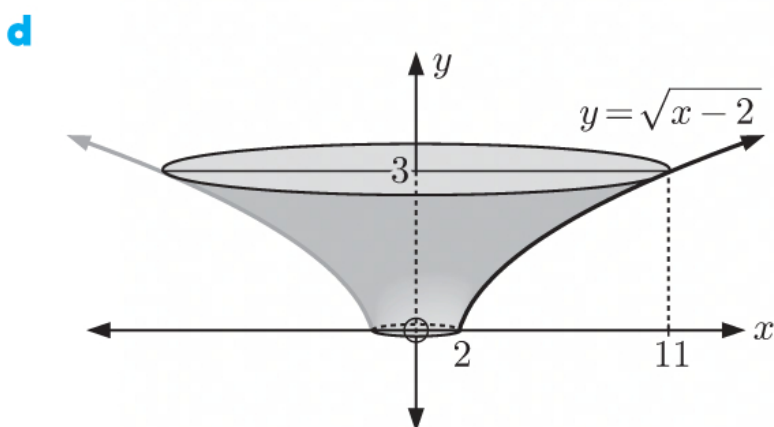
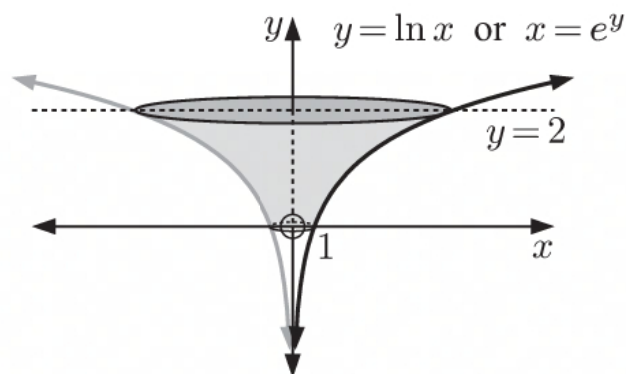
$$\begin{aligned}
 \text{Volume} &= \pi \int_0^4 x^2 dy \\
 &= \pi \int_0^4 y dy \\
 &= \pi \left[ \frac{y^2}{2} \right]_0^4 \\
 &= \pi(8 - 0) \\
 &= 8\pi \text{ units}^3
 \end{aligned}$$

**b**

$$\begin{aligned}
 \text{Volume} &= \pi \int_1^4 x^2 dy \\
 &= \pi \int_1^4 y^4 dy \\
 &= \pi \left[ \frac{y^5}{5} \right]_1^4 \\
 &= \pi \left( \frac{4^5}{5} - \frac{1}{5} \right) \\
 &= \frac{1023\pi}{5} \text{ units}^3
 \end{aligned}$$



$$\begin{aligned}
 \text{c Volume} &= \pi \int_0^2 x^2 dy \\
 &= \pi \int_0^2 (e^y)^2 dy \\
 &= \pi \int_0^2 e^{2y} dy \\
 &= \pi \left[ \frac{1}{2} e^{2y} \right]_0^2 \\
 &= \pi \left( \frac{1}{2} e^4 - \frac{1}{2} \right) \\
 &= \frac{\pi}{2} (e^4 - 1) \text{ units}^3
 \end{aligned}$$



When  $x = 2$ ,  $y = 0$

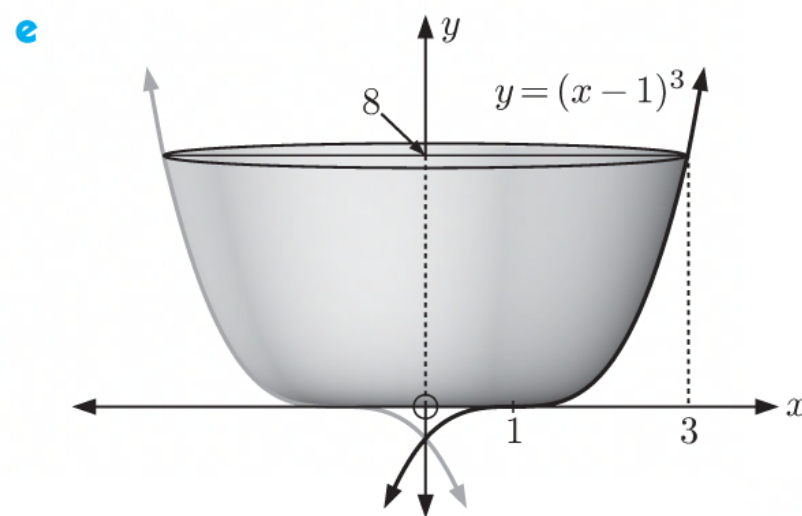
When  $x = 11$ ,  $y = 3$

Now  $y = \sqrt{x-2}$

$$\therefore y^2 = x - 2$$

$$\therefore x = y^2 + 2$$

$$\begin{aligned}
 \therefore \text{volume} &= \pi \int_0^3 x^2 dy \\
 &= \pi \int_0^3 (y^2 + 2)^2 dy \\
 &= \pi \int_0^3 (y^4 + 4y^2 + 4) dy \\
 &= \pi \left[ \frac{1}{5} y^5 + \frac{4}{3} y^3 + 4y \right]_0^3 \\
 &= \pi \left( \frac{243}{5} + 36 + 12 - 0 \right) \\
 &= \frac{483\pi}{5} \text{ units}^3
 \end{aligned}$$



When  $x = 1$ ,  $y = 0$

When  $x = 3$ ,  $y = 8$

Now  $y = (x-1)^3$

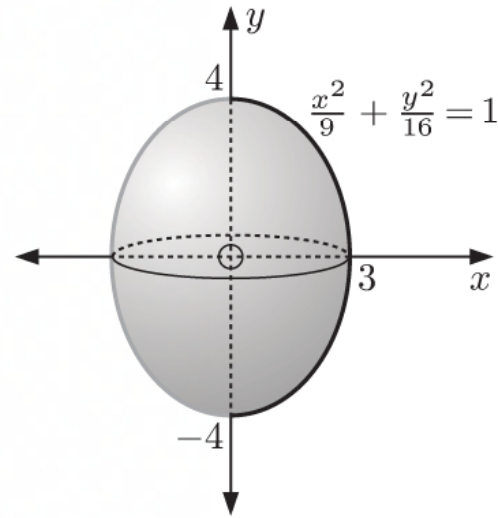
$$\therefore x - 1 = y^{\frac{1}{3}}$$

$$\therefore x = y^{\frac{1}{3}} + 1$$

$$\begin{aligned}
 \therefore \text{volume} &= \pi \int_0^8 x^2 dy \\
 &= \pi \int_0^8 (y^{\frac{1}{3}} + 1)^2 dy \\
 &= \pi \int_0^8 \left( y^{\frac{2}{3}} + 2y^{\frac{1}{3}} + 1 \right) dy \\
 &= \pi \left[ \frac{3}{5} y^{\frac{5}{3}} + \frac{3}{2} y^{\frac{4}{3}} + y \right]_0^8 \\
 &= \pi \left( \frac{3}{5} \times 32 + \frac{3}{2} \times 16 + 8 - 0 \right) \\
 &= \frac{256\pi}{5} \text{ units}^3
 \end{aligned}$$

$$2 \quad \frac{x^2}{9} + \frac{y^2}{16} = 1, \quad x \geq 0 \quad \therefore x^2 = 9 \left( 1 - \frac{y^2}{16} \right)$$

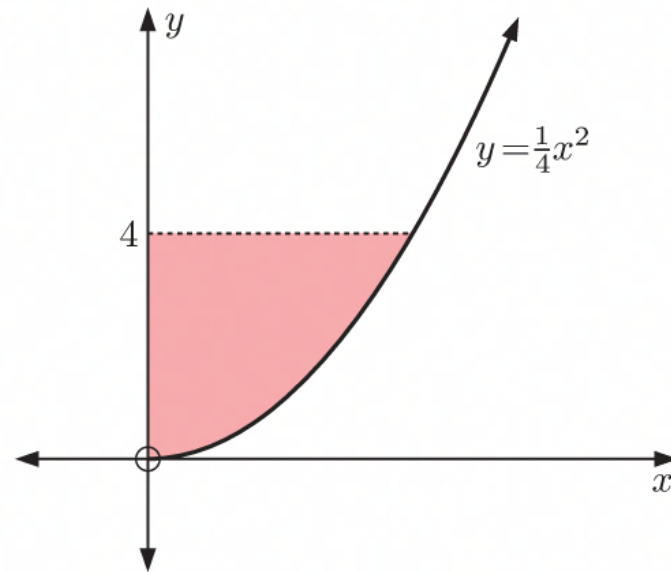
$$\begin{aligned} \therefore \text{volume} &= \pi \int_{-4}^4 x^2 dy \\ &= \pi \int_{-4}^4 \left( 9 - \frac{9}{16} y^2 \right) dy \\ &= \pi \left[ 9y - \frac{3}{16} y^3 \right]_{-4}^4 \\ &= \pi [(36 - 12) - (-36 + 12)] \\ &= 48\pi \text{ units}^3 \end{aligned}$$



$$3 \quad y = \frac{1}{4}x^2$$

$$\therefore x^2 = 4y$$

$$\begin{aligned} \therefore \text{volume} &= \pi \int_0^4 x^2 dy \\ &= \pi \int_0^4 4y dy \\ &= \pi [2y^2]_0^4 \\ &= \pi (2(4^2) - 0) \\ &= 32\pi \text{ units}^3 \end{aligned}$$

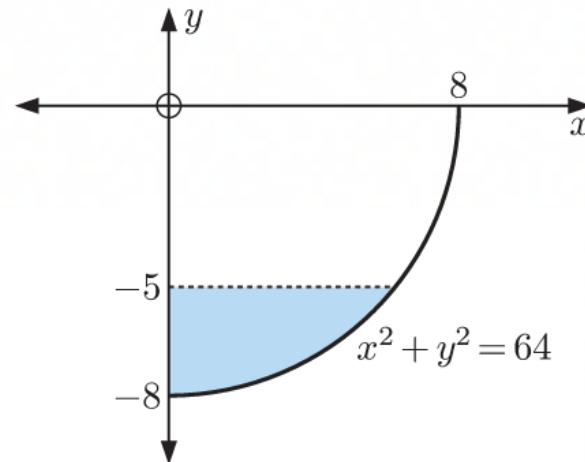


$\therefore$  the capacity of the bowl is  $32\pi$  units<sup>3</sup>.

$$4 \quad a \quad x^2 + y^2 = 64$$

$$\therefore x^2 = 64 - y^2$$

$$\begin{aligned} \therefore \text{volume} &= \pi \int_{-8}^{-5} x^2 dy \\ &= \pi \int_{-8}^{-5} (64 - y^2) dy \\ &= \pi \left[ 64y - \frac{1}{3} y^3 \right]_{-8}^{-5} \\ &= \pi \left[ (64(-5) - \frac{1}{3}(-5)^3) - (64(-8) - \frac{1}{3}(-8)^3) \right] \\ &= \pi \left[ -\frac{835}{3} + \frac{1024}{3} \right] \\ &= 63\pi \text{ units}^3 \end{aligned}$$

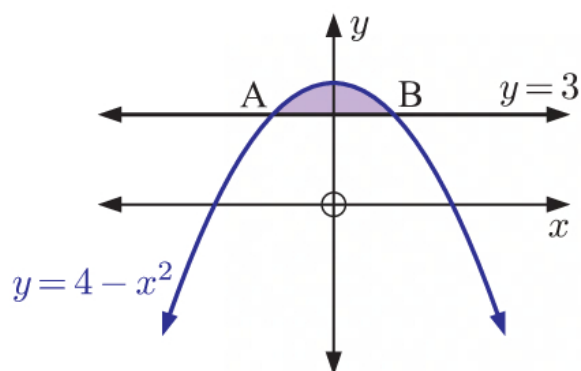


**b** The volume of a hemispherical bowl of radius 8 cm which contains water to a depth of 3 cm is described by the integral in part **a**. The units are cm.

$\therefore$  the volume of water in the bowl is  $63\pi \approx 198 \text{ cm}^3$ .

## EXERCISE 22G.3

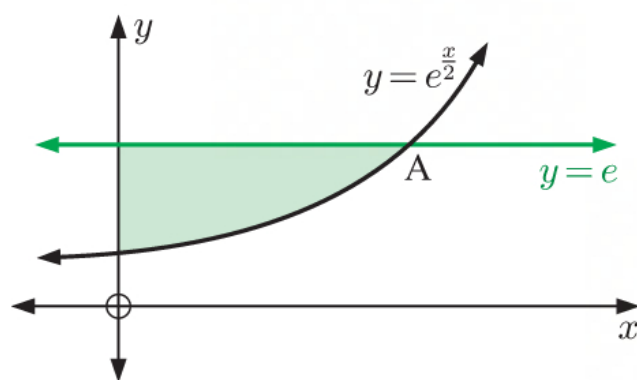
1 a



The graphs meet where  $4 - x^2 = 3$   
 $\therefore x^2 = 1$   
 $\therefore x = \pm 1$   
 $\therefore$  A is at  $(-1, 3)$  and B is at  $(1, 3)$ .

$$\begin{aligned}
 \text{b } V &= \pi \int_{-1}^1 (y_U^2 - y_L^2) dx \\
 &= \pi \int_{-1}^1 ((4 - x^2)^2 - 3^2) dx \\
 &= \pi \int_{-1}^1 (16 - 8x^2 + x^4 - 9) dx \\
 &= \pi \int_{-1}^1 (x^4 - 8x^2 + 7) dx \\
 &= \pi \left[ \frac{x^5}{5} - \frac{8x^3}{3} + 7x \right]_{-1}^1 \\
 &= \pi \left[ \left( \frac{1}{5} - \frac{8}{3} + 7 \right) - \left( -\frac{1}{5} + \frac{8}{3} - 7 \right) \right] \\
 &= \frac{136\pi}{15} \text{ units}^3
 \end{aligned}$$

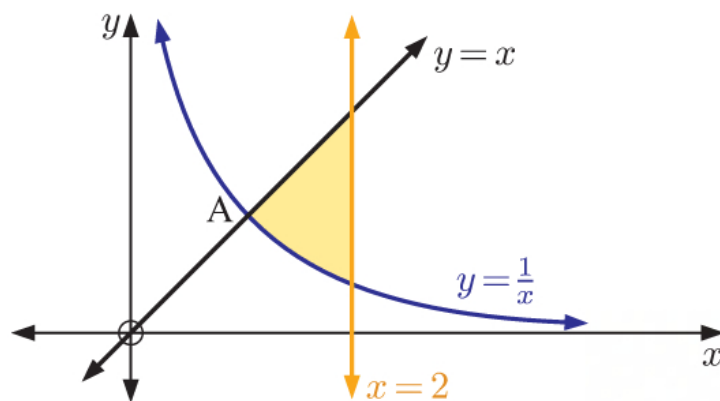
2 a



The graphs meet where  $e^{\frac{x}{2}} = e$   
 $\therefore e^{\frac{x}{2}} = e^1$   
 $\therefore \frac{x}{2} = 1$   
 $\therefore x = 2$   
 $\therefore$  A is at  $(2, e)$ .

$$\begin{aligned}
 \text{b } V &= \pi \int_0^2 (y_U^2 - y_L^2) dx \\
 &= \pi \int_0^2 (e^2 - (e^{\frac{x}{2}})^2) dx \\
 &= \pi \int_0^2 (e^2 - e^x) dx \\
 &= \pi [e^2 x - e^x]_0^2 \\
 &= \pi [(2e^2 - e^2) - (0 - 1)] \\
 &= \pi(e^2 + 1) \text{ units}^3
 \end{aligned}$$

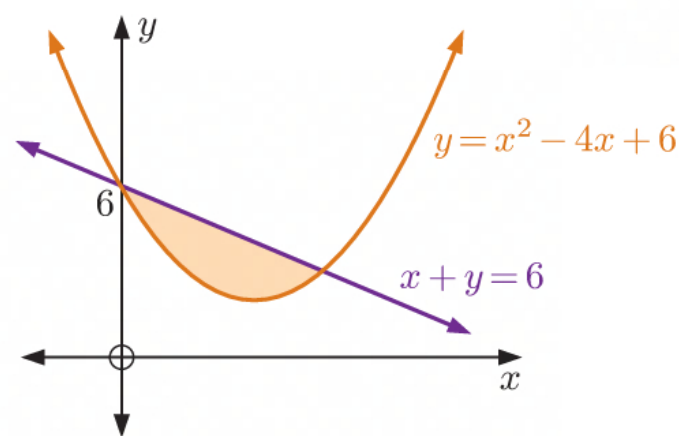
3 a



The graphs meet where  $x = \frac{1}{x}$   
 $\therefore x^2 = 1$   
 $\therefore x = \pm 1$   
 $\therefore x = 1$  {as  $x > 0$ }  
 $\therefore$  A is at  $(1, 1)$ .

$$\begin{aligned}
 \text{b } V &= \pi \int_1^2 (y_U^2 - y_L^2) dx \\
 &= \pi \int_1^2 \left( x^2 - \left( \frac{1}{x} \right)^2 \right) dx \\
 &= \pi \int_1^2 (x^2 - x^{-2}) dx \\
 &= \pi \left[ \frac{x^3}{3} + x^{-1} \right]_1^2 \\
 &= \pi \left[ \left( \frac{8}{3} + \frac{1}{2} \right) - \left( \frac{1}{3} + 1 \right) \right] \\
 &= \frac{11\pi}{6} \text{ units}^3
 \end{aligned}$$

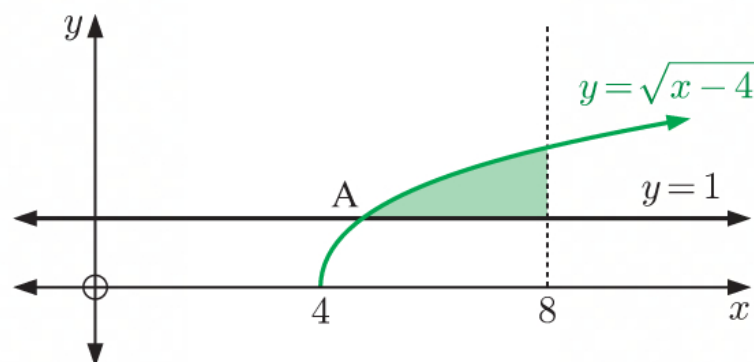
4 The graphs meet where  $x^2 - 4x + 6 = 6 - x$   
 $\therefore x^2 - 3x = 0$   
 $\therefore x(x - 3) = 0$   
 $\therefore x = 0 \text{ or } 3$



$\therefore$  volume

$$\begin{aligned} &= \pi \int_0^3 [(6 - x)^2 - (x^2 - 4x + 6)^2] dx \\ &= \pi \int_0^3 [(36 - 12x + x^2) - (x^4 - 4x^3 + 6x^2 - 4x^3 + 16x^2 - 24x + 6x^2 - 24x + 36)] dx \\ &= \pi \int_0^3 (-x^4 + 8x^3 - 27x^2 + 36x) dx \\ &= \pi \left[ -\frac{x^5}{5} + 2x^4 - 9x^3 + 18x^2 \right]_0^3 \\ &= \pi \left[ \left( -\frac{3^5}{5} + 2(3^4) - 9(27) + 18(9) \right) - 0 \right] \\ &= \frac{162\pi}{5} \text{ units}^3 \end{aligned}$$

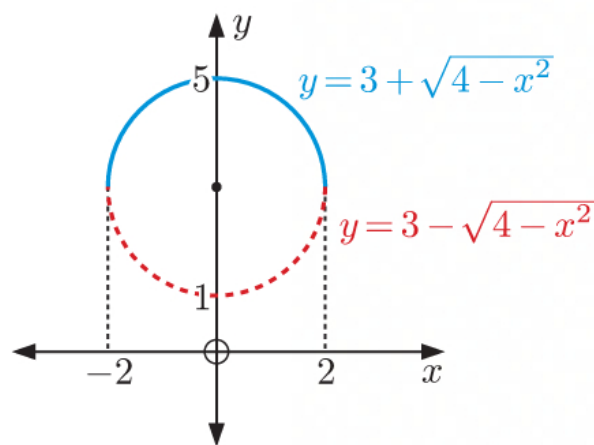
5 a The graphs meet where  $\sqrt{x - 4} = 1$   
 $\therefore x - 4 = 1$   
 $\therefore x = 5$   
 $\therefore$  A is at (5, 1).



b  $V = \pi \int_5^8 ((\sqrt{x - 4})^2 - 1^2) dx$   
 $= \pi \int_5^8 (x - 4 - 1) dx$   
 $= \pi \int_5^8 (x - 5) dx$   
 $= \pi \left[ \frac{x^2}{2} - 5x \right]_5^8$   
 $= \pi \left[ (32 - 40) - \left( \frac{25}{2} - 25 \right) \right]$   
 $= \frac{9\pi}{2} \text{ units}^3$



$$\begin{aligned}
 6 \quad x^2 + (y-3)^2 &= 4 \\
 \therefore (y-3)^2 &= 4 - x^2 \\
 \therefore y-3 &= \pm \sqrt{4-x^2} \\
 \therefore y &= 3 \pm \sqrt{4-x^2}
 \end{aligned}$$



$$\begin{aligned}
 \therefore \text{volume} &= \pi \int_{-2}^2 \left( \left( 3 + \sqrt{4-x^2} \right)^2 - \left( 3 - \sqrt{4-x^2} \right)^2 \right) dx \\
 &= 2\pi \int_0^2 \left( \left( 3 + \sqrt{4-x^2} \right)^2 - \left( 3 - \sqrt{4-x^2} \right)^2 \right) dx \\
 &= 2\pi \int_0^2 \left( \left( 9 + 6\sqrt{4-x^2} + 4 - x^2 \right) - \left( 9 - 6\sqrt{4-x^2} + 4 - x^2 \right) \right) dx \\
 &= 2\pi \int_0^2 12\sqrt{4-x^2} \, dx \\
 &= 24\pi \int_0^2 \sqrt{4-x^2} \, dx
 \end{aligned}$$

Let  $x = 2 \sin u \quad \therefore \frac{dx}{du} = 2 \cos u$

When  $x = 0$ ,  $u = 0$

When  $x = 2$ ,  $u = \frac{\pi}{2}$

$$\begin{aligned}
 \therefore \text{volume} &= 24\pi \int_0^2 \sqrt{4 - (2 \sin u)^2} \, dx \\
 &= 24\pi \int_0^{\frac{\pi}{2}} \sqrt{4 - 4 \sin^2 u} \frac{dx}{du} \, du \\
 &= 48\pi \int_0^{\frac{\pi}{2}} \sqrt{1 - \sin^2 u} (2 \cos u) \, du \\
 &= 48\pi \int_0^{\frac{\pi}{2}} 2 \cos^2 u \, du \quad \{ \sqrt{1 - \sin^2 u} = \cos u \} \\
 &= 48\pi \int_0^{\frac{\pi}{2}} (1 + \cos 2u) \, du \\
 &= 48\pi \left[ u + \frac{1}{2} \sin 2u \right]_0^{\frac{\pi}{2}} \\
 &= 48\pi \left( \frac{\pi}{2} + \frac{1}{2}(0) - 0 \right) \\
 &= 24\pi^2 \text{ units}^3 \quad (\approx 237 \text{ units}^3)
 \end{aligned}$$

- 7 a** Since the chord is parallel to the  $y$ -axis,  
the  $y$ -coordinate of P is  $\frac{r}{2}$ .

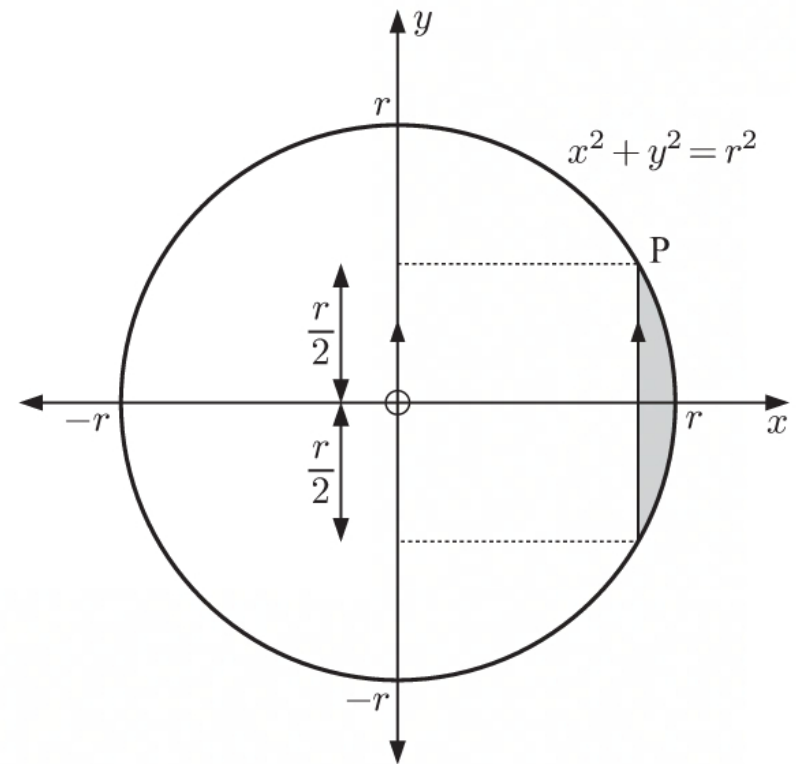
$$\begin{aligned}\text{When } y = \frac{r}{2}, \quad x^2 + \left(\frac{r}{2}\right)^2 &= r^2 \\ \therefore x^2 &= \frac{3}{4}r^2 \\ \therefore x &= \pm \frac{\sqrt{3}}{2}r\end{aligned}$$

$\therefore$  the coordinates of P are  $\left(\frac{\sqrt{3}}{2}r, \frac{r}{2}\right)$ .

$$\begin{aligned}\text{The “upper” relation is } x^2 + y^2 &= r^2 \\ \therefore x^2 &= r^2 - y^2\end{aligned}$$

The “lower” relation is  $x = \frac{\sqrt{3}}{2}r$

$$\begin{aligned}\therefore V &= \pi \int_{-\frac{r}{2}}^{\frac{r}{2}} \left( (r^2 - y^2) - \left(\frac{\sqrt{3}}{2}r\right)^2 \right) dy \\ &= 2\pi \int_0^{\frac{r}{2}} \left( (r^2 - y^2) - \frac{3}{4}r^2 \right) dy \\ &= 2\pi \left[ r^2y - \frac{y^3}{3} - \frac{3}{4}r^2y \right]_0^{\frac{r}{2}} \\ &= 2\pi \left( \frac{r^3}{2} - \frac{r^3}{24} - \frac{3r^3}{8} - 0 \right) \\ &= 2\pi \left( \frac{r^3}{12} \right) = \frac{\pi r^3}{6} \text{ units}^3\end{aligned}$$



- b** If the chord has length 6 units, it extends 3 units above and below the  $x$ -axis.

$$\begin{aligned}\text{When } y = 3, \quad x^2 + 3^2 &= r^2 \\ \therefore x^2 &= r^2 - 9 \\ \therefore x &= \pm \sqrt{r^2 - 9}\end{aligned}$$

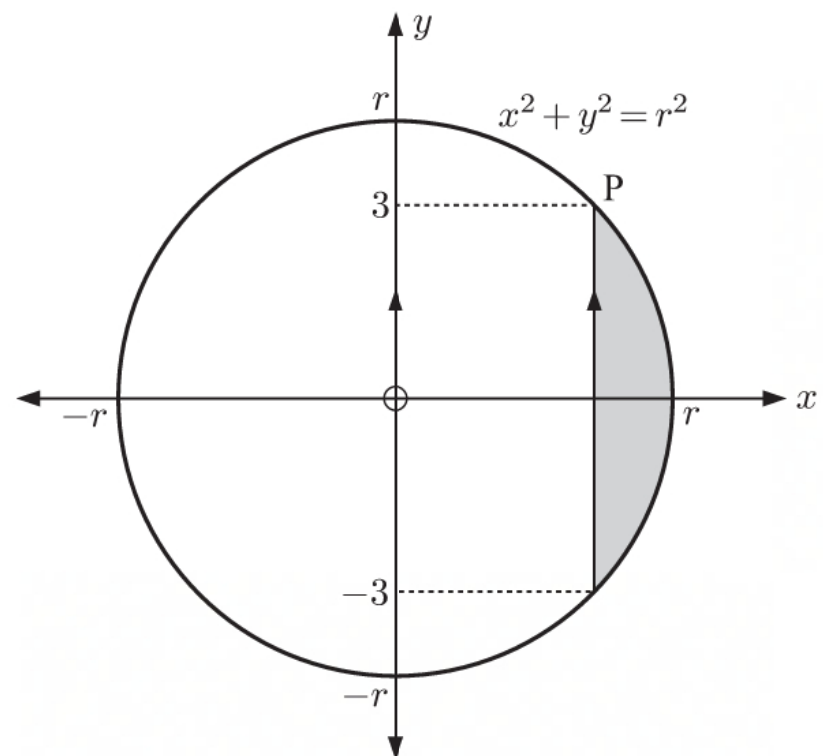
$\therefore$  the coordinates of P are  $(\sqrt{r^2 - 9}, 3)$ .

The “upper” relation is  $x^2 = r^2 - y^2$ .

The “lower” relation is  $x = \sqrt{r^2 - 9}$ .

$$\begin{aligned}\therefore V &= \pi \int_{-3}^3 \left( (r^2 - y^2) - (\sqrt{r^2 - 9})^2 \right) dy \\ &= 2\pi \int_0^3 \left( (r^2 - y^2) - (r^2 - 9) \right) dy \\ &= 2\pi \int_0^3 (9 - y^2) dy \\ &= 2\pi \left[ 9y - \frac{y^3}{3} \right]_0^3 \\ &= 2\pi(27 - 9 - 0) \\ &= 36\pi \text{ units}^3, \text{ no matter what the value of } r \text{ is.}\end{aligned}$$

$\therefore$  the volume is independent of  $r$ .



## EXERCISE 22H

- 1 a 8:20 am is 20 minutes after 8 am.

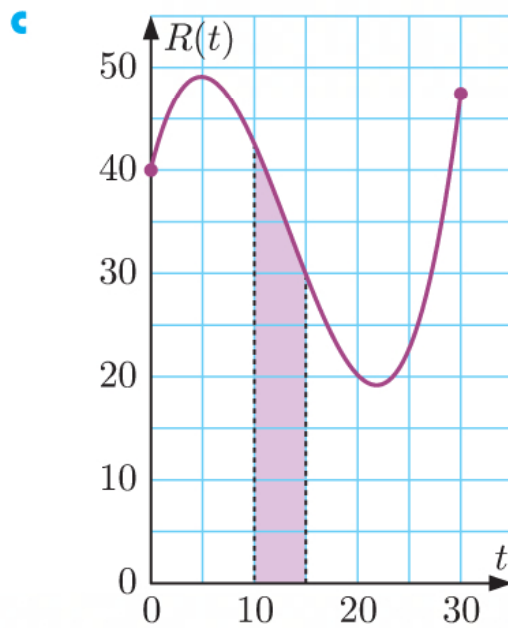
$$R(t) = \frac{t^3}{80} - \frac{t^2}{2} + 4t + 40$$

$$\begin{aligned}\therefore R(20) &= \frac{20^3}{80} - \frac{20^2}{2} + 4(20) + 40 \\ &= 20\end{aligned}$$

So, the rate of traffic flow at 8:20 am was 20 cars per minute.

- b The traffic flow was greatest when  $R(t)$  was a maximum. Looking at the graph, the maximum value of  $R(t)$  occurred when  $t \approx 5$ .

$\therefore$  the traffic flow was greatest at about 5 minutes after 8 am, that is, about 8:05 am.



$\int_{10}^{15} R(t) dt$  represents the total number of cars going past the pedestrian crossing from 8:10 am to 8:15 am.

- d 8 am is 0 minutes after 8 am and 8:30 am is 30 minutes after 8 am.

Total number of cars which passed the crossing between 8 am and 8:30 am

$$\begin{aligned}&= \int_0^{30} \left( \frac{t^3}{80} - \frac{t^2}{2} + 4t + 40 \right) dt \\ &= \left[ \frac{1}{320}t^4 - \frac{1}{6}t^3 + 2t^2 + 40t \right]_0^{30} \\ &= 1031.25 \\ &\approx 1031 \text{ cars}\end{aligned}$$

- 2 a  $R_1(t) = 5 - 5e^{-0.2t}$ ,  $R_2(t) = 6 - 6e^{-0.1t}$

i  $R_1(2) = 5 - 5e^{-0.2(2)}$   
 $\approx 1.65$  litres per minute

ii  $R_2(2) = 6 - 6e^{-0.1(2)}$   
 $\approx 1.09$  litres per minute

- b The rate of water leaking into the kayak is greater than the rate of water being bailed from the kayak after 2 minutes. So, the amount of water in the kayak is increasing after 2 minutes.

$$\begin{aligned}
 \text{c i } \int_0^3 R_1(t) dt &= \int_0^3 (5 - 5e^{-0.2t}) dt \\
 &= [5t + 25e^{-0.2t}]_0^3 \\
 &= (15 + 25e^{-0.6}) - (0 + 25) \\
 &\approx 3.72
 \end{aligned}$$

About 3.72 litres of water have leaked into the kayak in the first 3 minutes.

$$\begin{aligned}
 \text{ii } \int_2^5 R_2(t) dt &= \int_2^5 (6 - 6e^{-0.1t}) dt \\
 &= [6t + 60e^{-0.1t}]_2^5 \\
 &= (30 + 60e^{-0.5}) - (12 + 60e^{-0.2}) \\
 &\approx 5.27
 \end{aligned}$$

About 5.27 litres of water have been bailed out of the kayak from  $t = 2$  minutes to  $t = 5$  minutes.

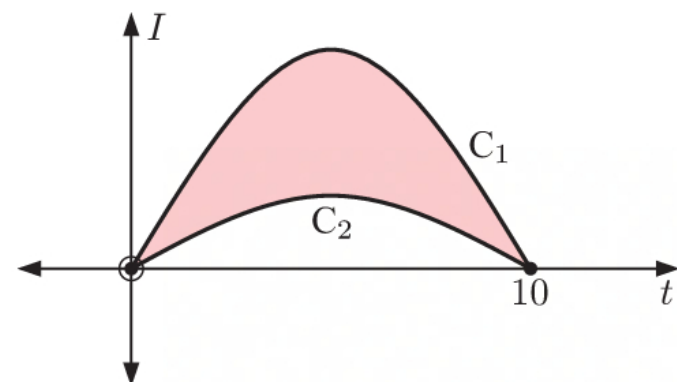
$$\begin{aligned}
 \text{iii } \int_0^8 [R_1(t) - R_2(t)] dt &= \int_0^8 (5 - 5e^{-0.2t} - (6 - 6e^{-0.1t})) dt \\
 &= \int_0^8 (-1 - 5e^{-0.2t} + 6e^{-0.1t}) dt \\
 &= [-t + 25e^{-0.2t} - 60e^{-0.1t}]_0^8 \\
 &= (-8 + 25e^{-1.6} - 60e^{-0.8}) - (0 + 25 - 60) \\
 &\approx 5.09
 \end{aligned}$$

There are about 5.09 litres of water in the kayak 8 minutes after striking the rock.

$$\begin{aligned}
 \text{d } \int_0^{10} [R_1(t) - R_2(t)] dt &= \int_0^{10} (5 - 5e^{-0.2t} - (6 - 6e^{-0.1t})) dt \\
 &= \int_0^{10} (-1 - 5e^{-0.2t} + 6e^{-0.1t}) dt \\
 &= [-t + 25e^{-0.2t} - 60e^{-0.1t}]_0^{10} \\
 &= (-10 + 25e^{-2} - 60e^{-1}) - (0 + 25 - 60) \\
 &\approx 6.31
 \end{aligned}$$

There are about 6.31 litres of water in the kayak 10 minutes after striking the rock.

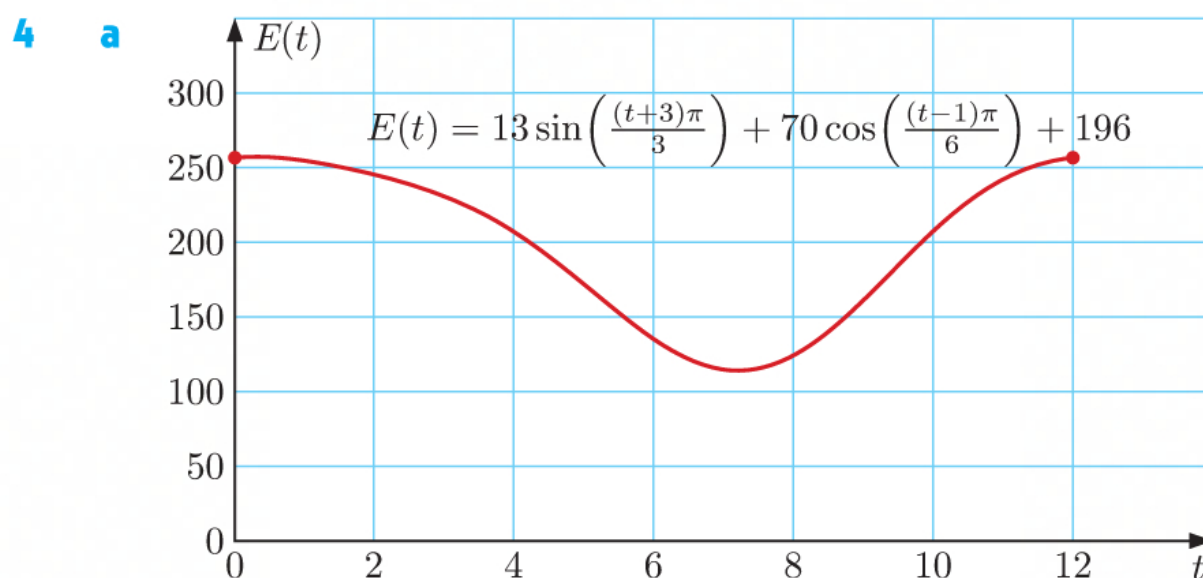
- 3 a  $y = 3 \sin \frac{\pi t}{10}$  has amplitude 3, which is larger than the amplitude of  $y = \sin \frac{\pi t}{10}$  which has amplitude 1.  
 $\therefore C_1$  is  $y = 3 \sin \frac{\pi t}{10}$  and  
 $C_2$  is  $y = \sin \frac{\pi t}{10}$





$$\begin{aligned}
 \text{b Area} &= \int_0^{10} \left( 3 \sin \frac{\pi t}{10} - \sin \frac{\pi t}{10} \right) dt \\
 &= \int_0^{10} 2 \sin \frac{\pi t}{10} dt \\
 &= \left[ -\frac{20}{\pi} \cos \frac{\pi t}{10} \right]_0^{10} \\
 &= -\frac{20}{\pi} \cos \pi + \frac{20}{\pi} \cos 0 \\
 &= \frac{20}{\pi} + \frac{20}{\pi} \\
 &= \frac{40}{\pi} \text{ units}
 \end{aligned}$$

- c The area in b represents the total amount of energy that enters the greenhouse in the first 10 hours.



b  $E(t) = 13 \sin\left(\frac{(t+3)\pi}{3}\right) + 70 \cos\left(\frac{(t-1)\pi}{6}\right) + 196$  TWh per month

$$\begin{aligned}
 \therefore \int E(t) dt &= \int \left( 13 \sin\left(\frac{(t+3)\pi}{3}\right) + 70 \cos\left(\frac{(t-1)\pi}{6}\right) + 196 \right) dt \\
 &= 13 \left( -\cos\left(\frac{(t+3)\pi}{3}\right) \right) \left( \frac{3}{\pi} \right) + 70 \sin\left(\frac{(t-1)\pi}{6}\right) \left( \frac{6}{\pi} \right) + 196t + c \\
 &= -\frac{39}{\pi} \cos\left(\frac{(t+3)\pi}{3}\right) + \frac{420}{\pi} \sin\left(\frac{(t-1)\pi}{6}\right) + 196t + c
 \end{aligned}$$

i  $\int_3^4 E(t) dt = \left( -\frac{39}{\pi} \cos \frac{7\pi}{3} + \frac{420}{\pi} \sin \frac{\pi}{2} + 784 \right) - \left( -\frac{39}{\pi} \cos 2\pi + \frac{420}{\pi} \sin \frac{\pi}{3} + 588 \right)$   
 $\approx 220.12$  TWh

The power consumption of the United Kingdom in April is about 220.12 TWh.

ii  $\int_5^8 E(t) dt = \left( -\frac{39}{\pi} \cos \frac{11\pi}{3} + \frac{420}{\pi} \sin \frac{7\pi}{6} + 1568 \right) - \left( -\frac{39}{\pi} \cos \frac{8\pi}{3} + \frac{420}{\pi} \sin \frac{2\pi}{3} + 980 \right)$   
 $\approx 392.96$  TWh

The power consumption of the United Kingdom for June 1st to September 1st is about 392.96 TWh.

iii  $\int_0^{12} E(t) dt = \left( -\frac{39}{\pi} \cos 5\pi + \frac{420}{\pi} \sin \frac{11\pi}{6} + 2352 \right) - \left( -\frac{39}{\pi} \cos \pi + \frac{420}{\pi} \sin \left(-\frac{\pi}{6}\right) \right)$   
 $= 2352$  TWh

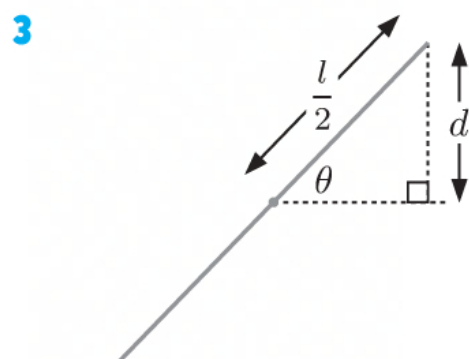
The yearly power consumption of the United Kingdom is 2352 TWh.

**ACTIVITY 3****BUFFON'S NEEDLE PROBLEM****CASE 1: THE SHORT NEEDLE**

**1 a**  $0 \leq \theta \leq \pi$

**b**  $0 \leq D \leq \frac{w}{2}$

- 2** Assuming that the needle toss is “random”, that is, it is tossed vertically rather than being cast on a particular orientation, it is reasonable to assume that  $\theta$  and  $D$  will take values in their ranges with equal probability.



$$\sin \theta = \frac{d}{\left(\frac{l}{2}\right)}$$

$$\therefore d = \frac{l}{2} \sin \theta$$

The needle will lie on a line if  $D \leq d$

$$\therefore D \leq \frac{l}{2} \sin \theta$$

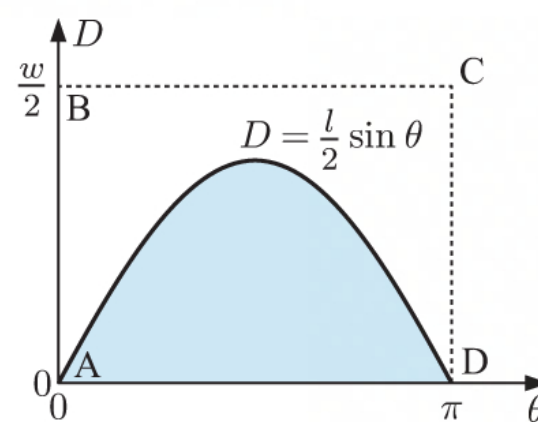
- 4**  $\theta$  is on the horizontal axis. The length of the rectangle, AD, covers the range of values for  $\theta$  found in **1 a**.

$D$  is on the vertical axis. The height of the rectangle, AB, covers the range of values for  $D$  found in **1 b**.

So, any toss of the needle can be described by a unique point  $(\theta, D)$  which lies in the rectangle. Under the assumption in **2**, each possible outcome  $(\theta, D)$  is equally likely.

Assuming  $l \leq w$ , the shaded area describes the set of points for which  $D \leq \frac{l}{2} \sin \theta$ . From **3**, this is the set of points for which the needle will lie on a line.

$$\therefore P(\text{needle lies on a line}) = \frac{\text{shaded area}}{\text{area of rectangle ABCD}}$$



**5** Shaded area  $= \int_0^{\pi} \frac{l}{2} \sin \theta \, d\theta$

$$= \frac{l}{2} \int_0^{\pi} \sin \theta \, d\theta$$

$$= \frac{l}{2} [-\cos \theta]_0^{\pi}$$

$$= \frac{l}{2} (-\cos \pi + \cos 0)$$

$$= l$$

**6** Using **4** and **5**,  $P(\text{needle lies on a line}) = \frac{l}{\pi(\frac{w}{2})}$

$$= \frac{2l}{w\pi}$$

### CASE 2: THE LONG NEEDLE

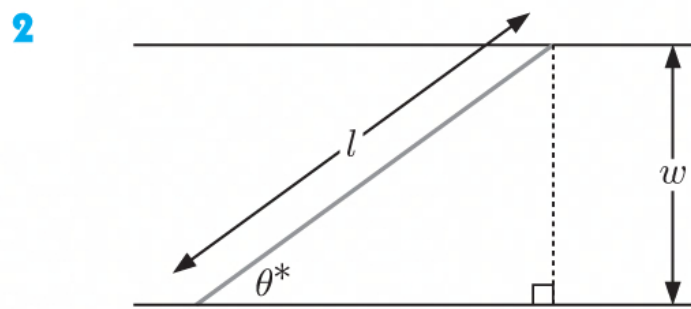
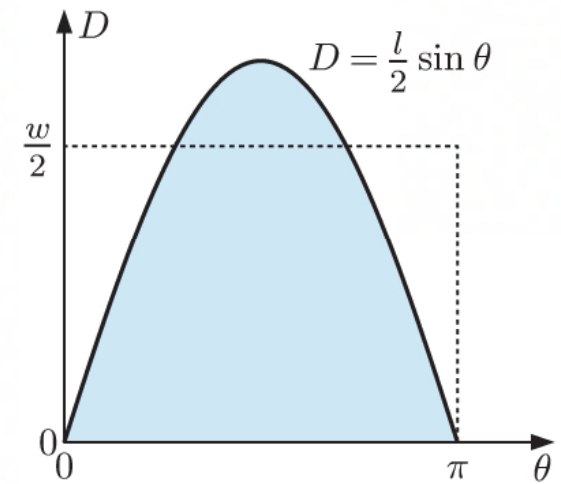
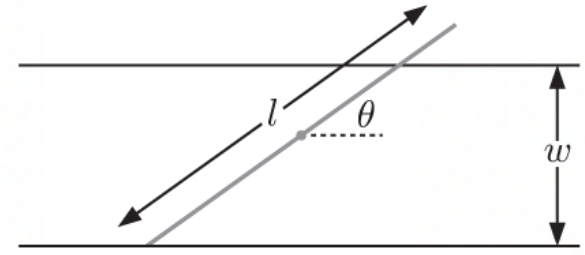
- 1 For a needle with length  $l > w$ , the curve  $D = \frac{l}{2} \sin \theta$  leaves the rectangle.

Physically, this means that there is a range of angles  $\theta^* < \theta < \pi - \theta^*$  for which the needle is *certain* to lie on a line.

Since any toss of the needle is still described by the set of points in the rectangle, the part of the shaded area *outside* the rectangle is impossible.

So, for  $l > w$ ,

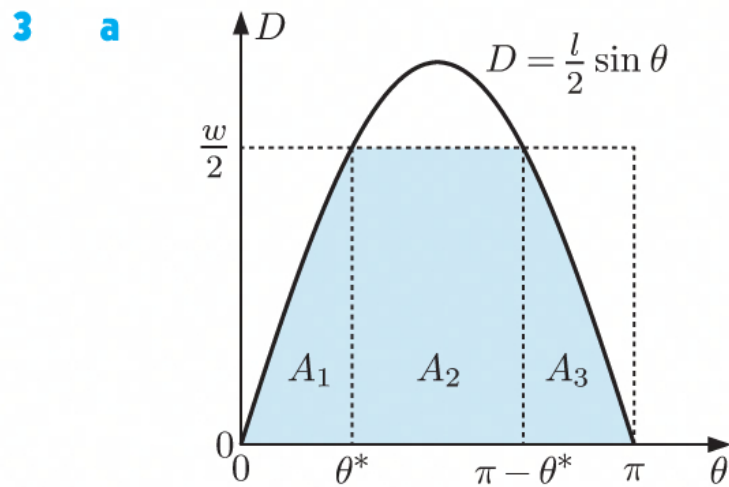
$$P(\text{needle lies on a line}) = \frac{\text{shaded area within rectangle}}{\text{area of rectangle}}.$$



$\theta^*$  is the critical value of  $\theta$  for which the needle *exactly* fits between two lines.

$$\sin \theta^* = \frac{w}{l}$$

$$\therefore \theta^* = \sin^{-1}\left(\frac{w}{l}\right)$$



$$A_1 = \int_0^{\theta^*} \frac{l}{2} \sin \theta \, d\theta$$

$$\begin{aligned} A_2 &= (\pi - \theta^* - \theta^*) \frac{w}{2} \\ &= (\pi - 2\theta^*) \frac{w}{2} \end{aligned}$$

$$A_3 = A_1 \quad \{\text{by symmetry}\}$$

$$\therefore P(\text{needle lies on a line})$$

$$= \frac{A_1 + A_2 + A_3}{\pi\left(\frac{w}{2}\right)}$$

$$= \frac{(\pi - 2\theta^*) \frac{w}{2} + 2 \int_0^{\theta^*} \left(\frac{l}{2} \sin \theta\right) d\theta}{\frac{w\pi}{2}}$$

**b** Continuing from **a**,  $P(\text{needle lies on a line})$

$$\begin{aligned}
 &= \frac{\pi - 2\theta^*}{\pi} + \frac{4}{w\pi} \frac{l}{2} \int_0^{\theta^*} \sin \theta \, d\theta \\
 &= 1 - \frac{2}{\pi} \theta^* + \frac{2l}{w\pi} [-\cos \theta]_0^{\theta^*} \\
 &= 1 - \frac{2}{\pi} \theta^* + \frac{2l}{w\pi} (-\cos \theta^* + \cos 0) \\
 &= 1 - \frac{2}{\pi} \theta^* + \frac{2l}{w\pi} (1 - \cos \theta^*) \\
 &= 1 - \frac{2}{\pi} \theta^* + \frac{2l}{w\pi} \left(1 - \sqrt{1 - \sin^2 \theta^*}\right) \quad \{\text{since } 0 < \theta^* \leq \frac{\pi}{2}\} \\
 &= 1 - \frac{2}{\pi} \sin^{-1}\left(\frac{w}{l}\right) + \frac{2l}{w\pi} \left(1 - \sqrt{1 - \left(\sin(\sin^{-1}(\frac{w}{l}))\right)^2}\right) \quad \{\text{using } \mathbf{2}\} \\
 &= 1 - \frac{2}{\pi} \sin^{-1}\left(\frac{w}{l}\right) + \frac{2l}{w\pi} \left(1 - \sqrt{1 - \frac{w^2}{l^2}}\right) \\
 &= 1 - \frac{2}{\pi} \sin^{-1}\left(\frac{w}{l}\right) + \frac{2l}{w\pi} \left(1 - \frac{\sqrt{l^2 - w^2}}{l}\right)
 \end{aligned}$$

**4** For the boundary case  $l = w$ , the formula in **3 b** gives

$$\begin{aligned}
 P(\text{needle lies on a line}) &= 1 - \frac{2}{\pi} \sin^{-1}(1) + \frac{2l}{w\pi} \left(1 - \frac{\sqrt{0}}{l}\right) \\
 &= 1 - \frac{2}{\pi} \times \frac{\pi}{2} + \frac{2l}{w\pi} \\
 &= \frac{2l}{w\pi}
 \end{aligned}$$

which agrees with the formula for the short needle.

## INVESTIGATION 2

$$\int_1^{\infty} x^p \, dx$$

**1** It is necessary that  $\lim_{x \rightarrow \infty} f(x) = 0$  for  $\int_a^{\infty} f(x) \, dx$  to exist.

If  $p = 0$ , then  $x^p = 1$  and  $\lim_{x \rightarrow \infty} x^p = 1 \neq 0$ .

If  $p > 0$ , then  $x^p$  is increasing, and  $\lim_{x \rightarrow \infty} x^p$  does not exist.

$\therefore \int_1^{\infty} x^p \, dx$  does not exist for  $p \geq 0$ .



$$\begin{aligned}
 \mathbf{2} \quad \mathbf{a} \quad \int_1^\infty x^{-2} dx &= \lim_{b \rightarrow \infty} \int_1^b x^{-2} dx \\
 &= \lim_{b \rightarrow \infty} \left[ -\frac{1}{x} \right]_1^b \\
 &= \lim_{b \rightarrow \infty} \left( -\frac{1}{b} - (-1) \right) \\
 &= \lim_{b \rightarrow \infty} \left( 1 - \frac{1}{b} \right) \\
 &= 1
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad \int_1^\infty x^{-\frac{3}{2}} dx &= \lim_{b \rightarrow \infty} \int_1^b x^{-\frac{3}{2}} dx \\
 &= \lim_{b \rightarrow \infty} \left[ \frac{x^{-\frac{1}{2}}}{-\frac{1}{2}} \right]_1^b \\
 &= \lim_{b \rightarrow \infty} \left[ -\frac{2}{\sqrt{x}} \right]_1^b \\
 &= \lim_{b \rightarrow \infty} \left( -\frac{2}{\sqrt{b}} - (-2) \right) \\
 &= \lim_{b \rightarrow \infty} \left( 2 - \frac{2}{\sqrt{b}} \right) \\
 &= 2
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad \int_1^\infty x^{-1} dx &= \lim_{b \rightarrow \infty} \int_1^b x^{-1} dx \\
 &= \lim_{b \rightarrow \infty} [\ln |x|]_1^b \\
 &= \lim_{b \rightarrow \infty} (\ln b - \ln 1) \\
 &= \lim_{b \rightarrow \infty} \ln b \quad \text{which does not exist.}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad \int_1^\infty x^{-\frac{1}{2}} dx &= \lim_{b \rightarrow \infty} \int_1^b x^{-\frac{1}{2}} dx \\
 &= \lim_{b \rightarrow \infty} \left[ \frac{x^{\frac{1}{2}}}{\frac{1}{2}} \right]_1^b \\
 &= \lim_{b \rightarrow \infty} [2\sqrt{x}]_1^b \\
 &= \lim_{b \rightarrow \infty} (2\sqrt{b} - 2) \quad \text{which does not exist.}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{3} \quad \mathbf{a} \quad \int_1^b x^p dx &= \left[ \frac{x^{p+1}}{p+1} \right]_1^b \\
 &= \frac{b^{p+1}}{p+1} - \frac{1}{p+1} \\
 &= \frac{b^{p+1} - 1}{p+1}, \quad p \neq -1
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad \int_1^\infty x^p dx &= \lim_{b \rightarrow \infty} \int_1^b x^p dx \\
 &= \lim_{b \rightarrow \infty} \frac{b^{p+1} - 1}{p+1}
 \end{aligned}$$

The limit does not exist for:  $p \geq 0$  {from **1**}  
 $0 < p+1 < 1$  or  $-1 < p < 0$   $\{b^{p+1} \text{ increasing}\}$   
and  $p = -1$  {from **2 c**}

$$\therefore \int_1^\infty x^p dx \text{ exists} \Leftrightarrow p < -1$$

## EXERCISE 22I

$$\begin{aligned}
 \text{1 a } \int_2^{\infty} \frac{1}{x^4} dx &= \lim_{b \rightarrow \infty} \int_2^b x^{-4} dx \\
 &= \lim_{b \rightarrow \infty} \left[ \frac{x^{-3}}{-3} \right]_2^b \\
 &= \lim_{b \rightarrow \infty} \left[ -\frac{1}{3x^3} \right]_2^b \\
 &= \lim_{b \rightarrow \infty} \left( -\frac{1}{3b^3} - \left( -\frac{1}{24} \right) \right) \\
 &= \lim_{b \rightarrow \infty} \left( \frac{1}{24} - \frac{1}{3b^3} \right) \\
 &= \frac{1}{24}
 \end{aligned}$$

$$\begin{aligned}
 \text{b } \lim_{x \rightarrow \infty} x^2 &\text{ does not exist} \\
 \therefore \int_1^{\infty} x^2 dx &\text{ does not exist.}
 \end{aligned}$$

$$\begin{aligned}
 \text{c } \int_8^{\infty} \frac{1}{\sqrt[3]{x}} dx &= \lim_{b \rightarrow \infty} \int_8^b x^{-\frac{1}{3}} dx \\
 &= \lim_{b \rightarrow \infty} \left[ \frac{x^{\frac{2}{3}}}{\frac{2}{3}} \right]_8^b \\
 &= \lim_{b \rightarrow \infty} \left[ \frac{3}{2} x^{\frac{2}{3}} \right]_8^b \\
 &= \lim_{b \rightarrow \infty} \left( \frac{3}{2} b^{\frac{2}{3}} - \frac{3}{2} (8)^{\frac{2}{3}} \right) \\
 &= \lim_{b \rightarrow \infty} \left( \frac{3}{2} b^{\frac{2}{3}} - 6 \right) \text{ which does not exist.}
 \end{aligned}$$

$$\begin{aligned}
 \text{d } \int_3^{\infty} \left( \frac{1}{x^2} - \frac{1}{x^3} \right) dx &= \lim_{b \rightarrow \infty} \int_3^b (x^{-2} - x^{-3}) dx \\
 &= \lim_{b \rightarrow \infty} \left[ \frac{x^{-1}}{-1} - \frac{x^{-2}}{-2} \right]_3^b \\
 &= \lim_{b \rightarrow \infty} \left[ -\frac{1}{x} + \frac{1}{2x^2} \right]_3^b \\
 &= \lim_{b \rightarrow \infty} \left[ \left( -\frac{1}{b} + \frac{1}{2b^2} \right) - \left( -\frac{1}{3} + \frac{1}{18} \right) \right] \\
 &= \lim_{b \rightarrow \infty} \left( \frac{5}{18} - \frac{1}{b} + \frac{1}{2b^2} \right) \\
 &= \frac{5}{18}
 \end{aligned}$$

$$\begin{aligned}
 \text{e } \int_1^{\infty} e^{-x} dx &= \lim_{b \rightarrow \infty} \int_1^b e^{-x} dx \\
 &= \lim_{b \rightarrow \infty} [-e^{-x}]_1^b \\
 &= \lim_{b \rightarrow \infty} [-e^{-b} - (-e^{-1})] \\
 &= \lim_{b \rightarrow \infty} \left[ \frac{1}{e} - \frac{1}{e^b} \right] \\
 &= \frac{1}{e}
 \end{aligned}$$

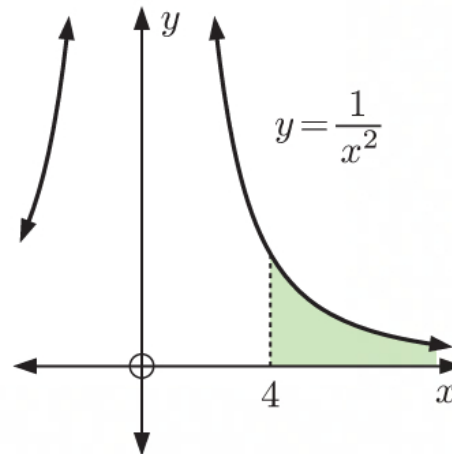
**f** Let  $u = -x^2$ ,  $\frac{du}{dx} = -2x$

$$\begin{aligned}\therefore \int x e^{-x^2} dx &= \int e^u \left( -\frac{1}{2} \frac{du}{dx} \right) dx \\ &= -\frac{1}{2} \int e^u du \\ &= -\frac{1}{2} e^u + c \\ &= -\frac{1}{2} e^{-x^2} + c\end{aligned}$$

$$\begin{aligned}\text{So, } \int_0^\infty x e^{-x^2} dx &= \lim_{b \rightarrow \infty} \int_0^b x e^{-x^2} dx \\ &= \lim_{b \rightarrow \infty} \left[ -\frac{1}{2} e^{-x^2} \right]_0^b \\ &= \lim_{b \rightarrow \infty} \left( -\frac{1}{2} e^{-b^2} - \left( -\frac{1}{2} \right) \right) \\ &= \lim_{b \rightarrow \infty} \left( \frac{1}{2} - \frac{1}{2e^{b^2}} \right) \\ &= \frac{1}{2}\end{aligned}$$

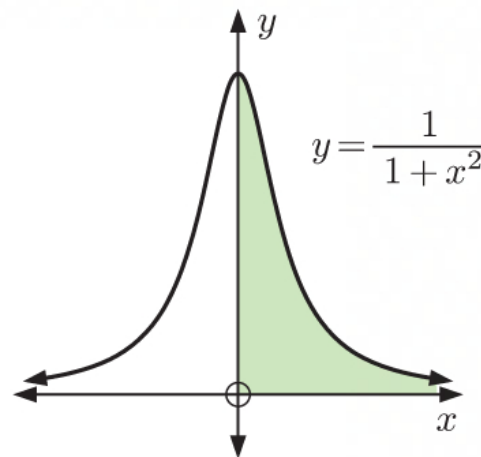
**2 a** Area  $= \int_4^\infty \frac{1}{x^2} dx$

$$\begin{aligned}&= \lim_{b \rightarrow \infty} \int_4^b x^{-2} dx \\ &= \lim_{b \rightarrow \infty} \left[ \frac{x^{-1}}{-1} \right]_4^b \\ &= \lim_{b \rightarrow \infty} \left[ -\frac{1}{x} \right]_4^b \\ &= \lim_{b \rightarrow \infty} \left( -\frac{1}{b} - \left( -\frac{1}{4} \right) \right) \\ &= \lim_{b \rightarrow \infty} \left( \frac{1}{4} - \frac{1}{b} \right) \\ &= \frac{1}{4} \text{ units}^2\end{aligned}$$



**b** Area  $= \int_0^\infty \frac{1}{1+x^2} dx$

$$\begin{aligned}&= \lim_{b \rightarrow \infty} \int_0^b \frac{1}{1+x^2} dx \\ &= \lim_{b \rightarrow \infty} [\arctan x]_0^b \\ &= \lim_{b \rightarrow \infty} (\arctan b - \arctan 0) \\ &= \frac{\pi}{2} \text{ units}^2\end{aligned}$$



$$\text{c Area} = - \int_0^{\infty} -3^{-x} dx$$

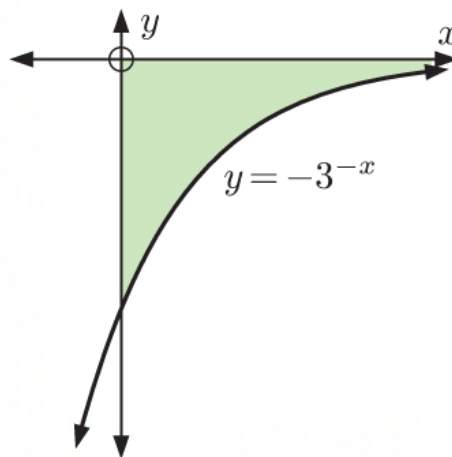
$$= \lim_{b \rightarrow \infty} \int_0^b 3^{-x} dx$$

$$= \lim_{b \rightarrow \infty} \left[ -\frac{3^{-x}}{\ln 3} \right]_0^b$$

$$= \lim_{b \rightarrow \infty} \left[ -\frac{3^{-b}}{\ln 3} - \left( -\frac{1}{\ln 3} \right) \right]$$

$$= \lim_{b \rightarrow \infty} \left( \frac{1}{\ln 3} - \frac{1}{3^b \ln 3} \right)$$

$$= \frac{1}{\ln 3} \text{ units}^2$$



$$\begin{aligned} 3 \quad \text{a} \quad \int_1^{\infty} \left( \frac{1}{\sqrt{x}} - \frac{1}{\sqrt{x+3}} \right) dx &= \lim_{b \rightarrow \infty} \int_1^b [x^{-\frac{1}{2}} - (x+3)^{-\frac{1}{2}}] dx \\ &= \lim_{b \rightarrow \infty} \left[ \frac{x^{\frac{1}{2}}}{\frac{1}{2}} - \frac{(x+3)^{\frac{1}{2}}}{\frac{1}{2}} \right]_1^b \\ &= \lim_{b \rightarrow \infty} [2\sqrt{x} - 2\sqrt{x+3}]_1^b \\ &= \lim_{b \rightarrow \infty} \left[ (2\sqrt{b} - 2\sqrt{b+3}) - (2 - 4) \right] \\ &= \lim_{b \rightarrow \infty} (2\sqrt{b} - 2\sqrt{b+3} + 2) \\ &= 2 \quad \{\text{as } b \rightarrow \infty, \sqrt{b+3} \rightarrow \sqrt{b}\} \end{aligned}$$

$$\text{b Let } u = \frac{1}{x}, \quad \frac{du}{dx} = -\frac{1}{x^2}$$

$$\therefore \int \frac{1}{x^2} \sin \frac{1}{x} dx = \int \sin u \left( -\frac{du}{dx} \right) dx$$

$$= \int -\sin u du$$

$$= \cos u + c$$

$$= \cos \frac{1}{x} + c$$

$$\therefore \int_{\frac{1}{\pi}}^{\infty} \frac{1}{x^2} \sin \frac{1}{x} dx = \lim_{b \rightarrow \infty} \int_{\frac{1}{\pi}}^b \frac{1}{x^2} \sin \frac{1}{x} dx$$

$$= \lim_{b \rightarrow \infty} \left[ \cos \frac{1}{x} \right]_{\frac{1}{\pi}}^b$$

$$= \lim_{b \rightarrow \infty} \left( \cos \frac{1}{b} - \cos \frac{1}{\left(\frac{1}{\pi}\right)} \right)$$

$$= \lim_{b \rightarrow \infty} \left( \cos \frac{1}{b} - \cos \pi \right)$$

$$= \cos 0 - \cos \pi$$

$$= 1 - (-1)$$

$$= 2$$



• Let  $u = \ln x$ ,  $\frac{du}{dx} = \frac{1}{x}$

$$\begin{aligned}\therefore \int \frac{1}{x(\ln x)^2} dx &= \int \frac{1}{u^2} \frac{du}{dx} dx \\ &= \int u^{-2} du \\ &= \frac{u^{-1}}{-1} + c \\ &= -\frac{1}{u} + c \\ &= -\frac{1}{\ln x} + c\end{aligned}$$

$$\begin{aligned}\therefore \int_2^\infty \frac{1}{x(\ln x)^2} dx &= \lim_{b \rightarrow \infty} \int_2^b \frac{1}{x(\ln x)^2} dx \\ &= \lim_{b \rightarrow \infty} \left[ -\frac{1}{\ln x} \right]_2^b \\ &= \lim_{b \rightarrow \infty} \left( -\frac{1}{\ln b} - \left( -\frac{1}{\ln 2} \right) \right) \\ &= \lim_{b \rightarrow \infty} \left( \frac{1}{\ln 2} - \frac{1}{\ln b} \right) \\ &= \frac{1}{\ln 2}\end{aligned}$$

4  $\int \frac{1}{e^x + e^{-x}} dx = \int \frac{e^x}{e^{2x} + 1} dx$

$$\begin{aligned}&= \int \frac{1}{u^2 + 1} \frac{du}{dx} dx \quad \{u = e^x, \frac{du}{dx} = e^x\} \\ &= \int \frac{1}{u^2 + 1} du \\ &= \arctan u + c \\ &= \arctan(e^x) + c\end{aligned}$$

$$\begin{aligned}\therefore \int_{\ln \sqrt{3}}^\infty \frac{1}{e^x + e^{-x}} dx &= \lim_{b \rightarrow \infty} \int_{\ln \sqrt{3}}^b \frac{1}{e^x + e^{-x}} dx \\ &= \lim_{b \rightarrow \infty} [\arctan(e^x)]_{\ln \sqrt{3}}^b \\ &= \lim_{b \rightarrow \infty} \left( \arctan(e^b) - \arctan(e^{\ln \sqrt{3}}) \right) \\ &= \lim_{b \rightarrow \infty} \left( \arctan(e^b) - \arctan \sqrt{3} \right) \\ &= \frac{\pi}{2} - \frac{\pi}{3} \\ &= \frac{\pi}{6}\end{aligned}$$

$$\begin{aligned}
 \text{5 a If } n = 0, \quad \int_0^\infty e^{-x} dx &= \lim_{b \rightarrow \infty} \left( \int_0^b e^{-x} dx \right) \\
 &= \lim_{b \rightarrow \infty} [-e^{-x}]_0^b \\
 &= \lim_{b \rightarrow \infty} \left( -\frac{1}{e^b} - (-1) \right) \\
 &= \lim_{b \rightarrow \infty} \left( 1 - \frac{1}{e^b} \right) \\
 &= 1 - 0 \\
 &= 1
 \end{aligned}$$

If  $n = 1$ , we need to find  $\int_0^\infty x e^{-x} dx$

Using integration by parts with  $u' = e^{-x}$ ,  $v = x$   
 $u = -e^{-x}$ ,  $v' = 1$

$$\begin{aligned}
 \int x e^{-x} dx &= -x e^{-x} - \int -e^{-x} dx \\
 &= -x e^{-x} + (-e^{-x}) + c \\
 &= -e^{-x}(x + 1) + c
 \end{aligned}$$

$$\begin{aligned}
 \therefore \int_0^\infty x e^{-x} dx &= \lim_{b \rightarrow \infty} \int_0^b x e^{-x} dx \\
 &= \lim_{b \rightarrow \infty} [-e^{-x}(x + 1)]_0^b \\
 &= \lim_{b \rightarrow \infty} \left( -\frac{b+1}{e^b} - (-1) \right) \\
 &= \lim_{b \rightarrow \infty} \left( 1 - \frac{b+1}{e^b} \right) \\
 &= 1
 \end{aligned}$$

If  $n = 2$ , we need to find  $\int_0^\infty x^2 e^{-x} dx$

Using integration by parts with  $u' = e^{-x}$ ,  $v = x^2$   
 $u = -e^{-x}$ ,  $v' = 2x$

$$\begin{aligned}
 \therefore \int x^2 e^{-x} dx &= -x^2 e^{-x} - \int -2x e^{-x} dx \\
 &= -x^2 e^{-x} + 2 \int x e^{-x} dx \\
 &= -x^2 e^{-x} + 2(-e^{-x}(x + 1)) + d \\
 &= -e^{-x}(x^2 + 2x + 2) + d
 \end{aligned}$$

$$\begin{aligned}
\therefore \int_0^\infty x^2 e^{-x} dx &= \lim_{b \rightarrow \infty} \left[ -\frac{x^2 + 2x + 2}{e^x} \right]_0^b \\
&= \lim_{b \rightarrow \infty} \left( -\frac{b^2 + 2b + 2}{e^b} - (-2) \right) \\
&= \lim_{b \rightarrow \infty} \left( 2 - \frac{b^2 + 2b + 2}{e^b} \right) \\
&= 2
\end{aligned}$$

If  $n = 3$ , we need to find  $\int_0^\infty x^3 e^{-x} dx$

Using integration by parts with  $u' = e^{-x}$ ,  $v = x^3$   
 $u = -e^{-x}$ ,  $v' = 3x^2$

$$\begin{aligned}
\therefore \int x^3 e^{-x} dx &= -x^3 e^{-x} - \int -3x^2 e^{-x} dx \\
&= -x^3 e^{-x} + 3(-e^{-x}(x^2 + 2x + 2)) + f \\
&= -e^{-x}(x^3 + 3x^2 + 6x + 6) + f
\end{aligned}$$

$$\begin{aligned}
\therefore \int_0^\infty x^3 e^{-x} dx &= \lim_{b \rightarrow \infty} \left[ -\frac{x^3 + 3x^2 + 6x + 6}{e^x} \right]_0^b \\
&= \lim_{b \rightarrow \infty} \left( -\frac{b^3 + 3b^2 + 6b + 6}{e^b} - (-6) \right) \\
&= \lim_{b \rightarrow \infty} \left( 6 - \frac{b^3 + 3b^2 + 6b + 6}{e^b} \right) \\
&= 6
\end{aligned}$$

**b**  $\int_0^\infty x^0 e^{-x} dx = 1 = 0!$

$$\int_0^\infty x^1 e^{-x} dx = 1 = 1!$$

$$\int_0^\infty x^2 e^{-x} dx = 2 = 2!$$

$$\int_0^\infty x^3 e^{-x} dx = 6 = 3!$$

So, we predict  $\int_0^\infty x^n e^{-x} dx = n!$  for all  $n \in \mathbb{Z}$ ,  $n \geq 0$ .

**c**  $P_n$  is:  $\int_0^\infty x^n e^{-x} dx = n!$  for all  $n \in \mathbb{Z}$ ,  $n \geq 0$ .

**Proof:** (By the principle of mathematical induction)

(1)  $P_0$  was proved in **a**.

(2) If  $P_k$  is true, then  $\int_0^\infty x^k e^{-x} dx = k!$

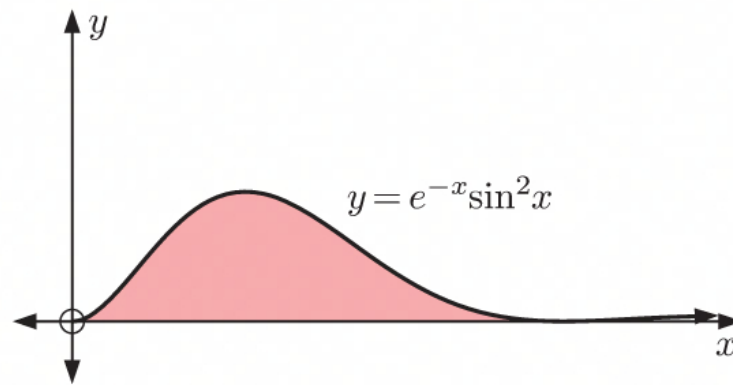
$$\begin{aligned}
\text{Now } & \int_0^\infty x^{k+1} e^{-x} dx \\
&= \lim_{b \rightarrow \infty} \int_0^b x^{k+1} e^{-x} dx \\
&= \lim_{b \rightarrow \infty} \left( [-e^{-x} x^{k+1}]_0^b - \int_0^b -e^{-x} (k+1) x^k dx \right) \quad \begin{cases} u' = e^{-x} & v = x^{k+1} \\ u = -e^{-x} & v' = (k+1)x^k \end{cases} \\
&= \lim_{b \rightarrow \infty} \left( \frac{-b^{k+1}}{e^b} - (0) + (k+1) \int_0^b e^{-x} x^k dx \right) \\
&= 0 + (k+1) \int_0^\infty e^{-x} x^k dx \\
&= (k+1)k! \quad \{\text{using } P_k\} \\
&= (k+1)!
\end{aligned}$$

Since  $P_0$  is true and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all  $n \in \mathbb{Z}$ ,  $n \geq 0$ . {principle of mathematical induction}

6

$$\begin{aligned}
& \int e^{-x} \sin^2 x dx \\
&= \int e^{-x} \left( \frac{1}{2} - \frac{1}{2} \cos 2x \right) dx \\
&= \frac{1}{2} \int (e^{-x} - e^{-x} \cos 2x) dx \\
&= \frac{1}{2} \int e^{-x} dx - \underbrace{\frac{1}{2} \int e^{-x} \cos 2x dx}_{(*)}
\end{aligned}$$



$$\begin{aligned}
(*) : & \int e^{-x} \cos 2x dx \\
&= -e^{-x} \cos 2x - \int -e^{-x} (-2 \sin 2x) dx \quad \leftarrow \begin{cases} u = \cos 2x & v' = e^{-x} \\ u' = -2 \sin 2x & v = -e^{-x} \end{cases} \\
&= -e^{-x} \cos 2x - 2 \int e^{-x} \sin 2x dx \\
&= -e^{-x} \cos 2x - 2 \left( -e^{-x} \sin 2x - \int -e^{-x} (2 \cos 2x) dx \right) \quad \leftarrow \begin{cases} u = \sin 2x & v' = e^{-x} \\ u' = 2 \cos 2x & v = -e^{-x} \end{cases} \\
\therefore & \int e^{-x} \cos 2x dx = -e^{-x} \cos 2x + 2e^{-x} \sin 2x - 4 \int e^{-x} \cos 2x dx \\
\therefore & 5 \int e^{-x} \cos 2x dx = -e^{-x} \cos 2x + 2e^{-x} \sin 2x + c \\
\therefore & \int e^{-x} \cos 2x dx = -\frac{1}{5} e^{-x} \cos 2x + \frac{2}{5} e^{-x} \sin 2x + c
\end{aligned}$$



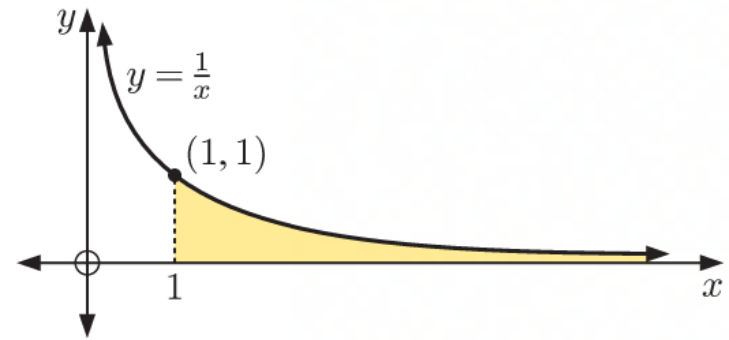
$$\begin{aligned}
\text{So, } \int e^{-x} \sin^2 x \, dx &= \frac{1}{2} \int e^{-x} \, dx - \frac{1}{2} \int e^{-x} \cos 2x \, dx \\
&= -\frac{1}{2}e^{-x} - \frac{1}{2} \left( -\frac{1}{5}e^{-x} \cos 2x + \frac{2}{5}e^{-x} \sin 2x \right) + c \\
&= -\frac{1}{2}e^{-x} + \frac{1}{10}e^{-x} \cos 2x - \frac{1}{5}e^{-x} \sin 2x + c \\
&= -\frac{1}{10}e^{-x}(5 - \cos 2x + 2 \sin 2x) + c
\end{aligned}$$

$$\begin{aligned}
\text{Now } \int_0^\infty e^{-x} \sin^2 x \, dx &= \lim_{b \rightarrow \infty} \int_0^b e^{-x} \sin^2 x \, dx \\
&= \lim_{b \rightarrow \infty} \left[ -\frac{1}{10}e^{-x}(5 - \cos 2x + 2 \sin 2x) \right]_0^b \\
&= \lim_{b \rightarrow \infty} \left[ -\frac{1}{10}e^{-b}(5 - \cos 2b + 2 \sin 2b) - \left( -\frac{1}{10}(5 - \cos 0 + 2 \sin 0) \right) \right] \\
&= \frac{1}{10}(5 - 1) \\
&= \frac{2}{5}
\end{aligned}$$

$\therefore$  total area =  $\frac{2}{5}$  units<sup>2</sup>

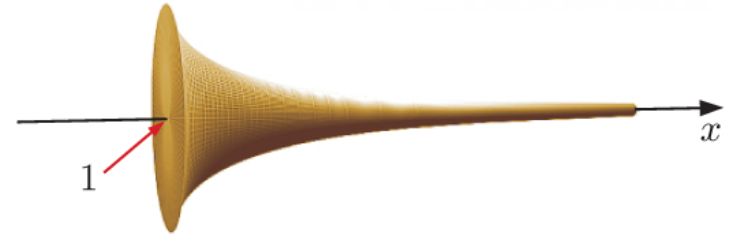
**7** The shaded area =  $\int_1^\infty \frac{1}{x} \, dx$

$$\begin{aligned}
&= \lim_{b \rightarrow \infty} \int_1^b \frac{1}{x} \, dx \\
&= \lim_{b \rightarrow \infty} [\ln(x)]_1^b, \quad x > 0 \\
&= \lim_{b \rightarrow \infty} \ln b, \quad \text{which is infinite}
\end{aligned}$$



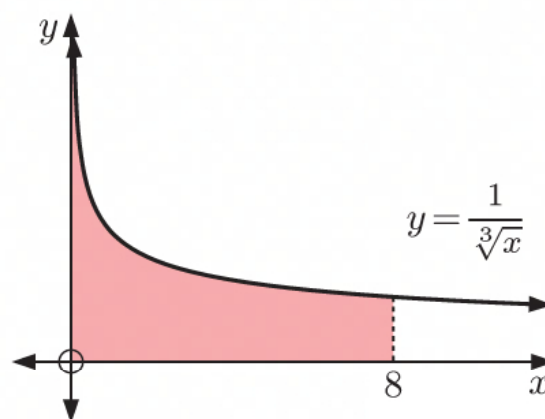
The volume of revolution =  $\pi \int_1^\infty \left(\frac{1}{x}\right)^2 \, dx$

$$\begin{aligned}
&= \pi \lim_{b \rightarrow \infty} \int_1^b x^{-2} \, dx \\
&= \pi \lim_{b \rightarrow \infty} \left[ -\frac{1}{x} \right]_1^b \\
&= \pi \lim_{b \rightarrow \infty} \left( -\frac{1}{b} + 1 \right) \\
&= \pi, \quad \text{which is finite}
\end{aligned}$$

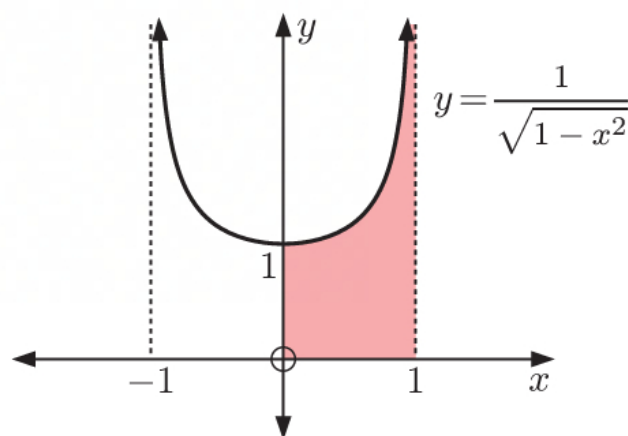


# ACTIVITY 4 SINGULARITIES

$$\begin{aligned}
 \mathbf{1} \quad \mathbf{a} \quad \int_0^8 \frac{1}{\sqrt[3]{x}} dx &= \lim_{a \rightarrow 0^+} \int_a^8 x^{-\frac{1}{3}} dx \\
 &= \lim_{a \rightarrow 0^+} \left[ \frac{3}{2} x^{\frac{2}{3}} \right]_a^8 \\
 &= \lim_{a \rightarrow 0^+} \left( \frac{3}{2} (8)^{\frac{2}{3}} - \frac{3}{2} a^{\frac{2}{3}} \right) \\
 &= \lim_{a \rightarrow 0^+} \left( 6 - \frac{3}{2} a^{\frac{2}{3}} \right) \\
 &= 6
 \end{aligned}$$



$$\begin{aligned}
 \mathbf{b} \quad \int_0^1 \frac{1}{\sqrt{1-x^2}} dx &= \lim_{b \rightarrow 1^-} \int_0^b \frac{1}{\sqrt{1-x^2}} dx \\
 &= \lim_{b \rightarrow 1^-} [\arcsin x]_0^b \\
 &= \lim_{b \rightarrow 1^-} (\arcsin b - \arcsin 0) \\
 &= \arcsin 1 \\
 &= \frac{\pi}{2}
 \end{aligned}$$



$$\begin{aligned}
 \mathbf{c} \quad \int \frac{1}{\sqrt{x}(x+1)} dx &= \int \frac{2}{u^2+1} \frac{du}{dx} dx \quad \left\{ u = \sqrt{x} = x^{\frac{1}{2}}, \quad \frac{du}{dx} = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}} \right\} \\
 &= 2 \int \frac{1}{u^2+1} du \\
 &= 2 \arctan u + c \\
 &= 2 \arctan \sqrt{x} + c \\
 \therefore \int_0^1 \frac{1}{\sqrt{x}(x+1)} dx &= \lim_{a \rightarrow 0^+} \int_a^1 \frac{1}{\sqrt{x}(x+1)} dx \\
 &= \lim_{a \rightarrow 0^+} [2 \arctan \sqrt{x}]_a^1 \\
 &= \lim_{a \rightarrow 0^+} (2 \arctan 1 - 2 \arctan \sqrt{a}) \\
 &= 2 \arctan 1 - 2 \arctan 0 \\
 &= \frac{\pi}{2}
 \end{aligned}$$

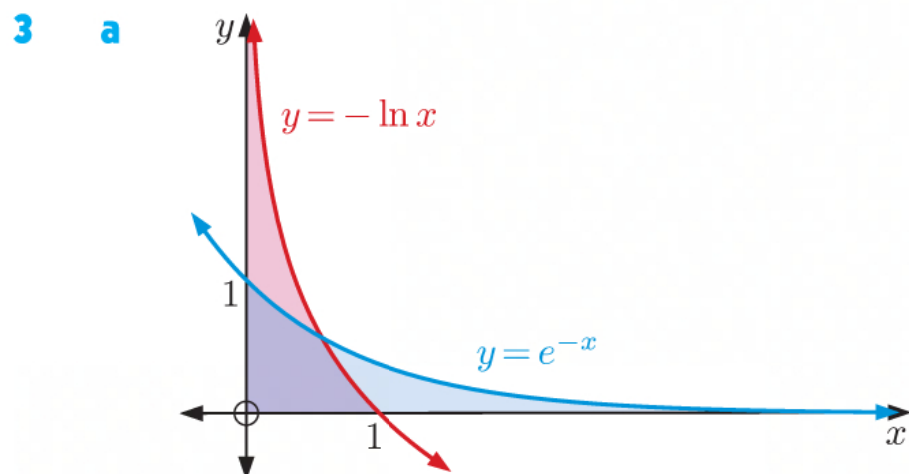
**2 a**  $f(x) = x^{-\frac{2}{3}}$

$$\begin{aligned}\therefore f(-x) &= (-x)^{-\frac{2}{3}} \\ &= ((-x)^2)^{-\frac{1}{3}} \\ &= (x^2)^{-\frac{1}{3}} \\ &= x^{-\frac{2}{3}} \\ &= f(x)\end{aligned}$$

$\therefore f(x) = x^{-\frac{2}{3}}$  is even.

**b**  $\int_{-1}^1 f(x) dx = 2 \int_0^1 x^{-\frac{2}{3}} dx$

$$\begin{aligned}&= 2 \times \lim_{a \rightarrow 0^+} \int_a^1 x^{-\frac{2}{3}} dx \\ &= 2 \times \lim_{a \rightarrow 0^+} \left[ \frac{x^{\frac{1}{3}}}{\frac{1}{3}} \right]_a^1 \\ &= 2 \times \lim_{a \rightarrow 0^+} \left[ 3x^{\frac{1}{3}} \right]_a^1 \\ &= 2 \times \lim_{a \rightarrow 0^+} (3 - 3a^{\frac{1}{3}}) \\ &= 2 \times 3 \\ &= 6\end{aligned}$$



**b** The functions  $e^{-x}$  and  $-\ln x$  are inverses of one another. So, their graphs are symmetric about the line  $y = x$ . Thus the shaded region corresponding to  $\int_0^1 (-\ln x) dx$  is a reflection of the shaded region corresponding to  $\int_0^\infty e^{-x} dx$  in the line  $y = x$ .

$\therefore$  the shaded areas are equal in size.

**c**  $\int_0^\infty e^{-x} dx = \lim_{b \rightarrow \infty} \int_0^b e^{-x} dx$

$$\begin{aligned}&= \lim_{b \rightarrow \infty} [-e^{-x}]_0^b \\ &= \lim_{b \rightarrow \infty} (-e^{-b} - (-1)) \\ &= \lim_{b \rightarrow \infty} (1 - e^{-b}) \\ &= 1\end{aligned}$$

$$\begin{aligned}
\int_0^1 (-\ln x) dx &= \lim_{a \rightarrow 0^+} \int_a^1 (-\ln x) dx \\
&= \lim_{a \rightarrow 0^+} - \int_a^1 \ln x dx \\
&= \lim_{a \rightarrow 0^+} - [x \ln x - x]_a^1 \\
&= \lim_{a \rightarrow 0^+} -((\ln 1 - 1) - (a \ln a - a)) \\
&= \lim_{a \rightarrow 0^+} -(-1 - a \ln a + a) \\
&= \lim_{a \rightarrow 0^+} (1 + a \ln a - a) \\
&= 1 \\
\therefore \int_0^\infty e^{-x} dx &= \int_0^1 (-\ln x) dx = 1, \text{ the shaded areas are equal.}
\end{aligned}$$

## REVIEW SET 22A

$$\begin{aligned}
1 \quad a \quad \int_{-2}^0 (1 - 3x) dx &= \left[ x - \frac{3}{2}x^2 \right]_{-2}^0 \\
&= 0 - \left( -2 - \frac{3}{2}(4) \right) \\
&= 0 - (-8) \\
&= 8
\end{aligned}$$

$$\begin{aligned}
c \quad \int_1^2 (x^2 + 1)^2 dx &= \int_1^2 (x^4 + 2x^2 + 1) dx \\
&= \left[ \frac{1}{5}x^5 + \frac{2}{3}x^3 + x \right]_1^2 \\
&= \left( \frac{32}{5} + \frac{16}{3} + 2 \right) - \left( \frac{1}{5} + \frac{2}{3} + 1 \right) \\
&= \frac{178}{15} = 11\frac{13}{15}
\end{aligned}$$

$$\begin{aligned}
2 \quad a \quad \int_0^b (x - b)^2 dx &= 9 \\
\therefore \int_0^b (x^2 - 2bx + b^2) dx &= 9 \\
\therefore \left[ \frac{1}{3}x^3 - bx^2 + b^2x \right]_0^b &= 9 \\
\therefore \left( \frac{1}{3}b^3 - \cancel{b^3} + \cancel{b^3} \right) - 0 &= 9 \\
\therefore \frac{1}{3}b^3 &= 9 \\
\therefore b^3 &= 27 \\
\therefore b &= 3
\end{aligned}$$

$$\begin{aligned}
b \quad \int_0^{\frac{1}{2}} (x - \sqrt{x}) dx &= \int_0^{\frac{1}{2}} (x - x^{\frac{1}{2}}) dx \\
&= \left[ \frac{1}{2}x^2 - \frac{x^{\frac{3}{2}}}{\frac{3}{2}} \right]_0^{\frac{1}{2}} \\
&= \left[ \frac{1}{2}x^2 - \frac{2}{3}x^{\frac{3}{2}} \right]_0^{\frac{1}{2}} \\
&= \frac{1}{2} \left( \frac{1}{4} \right) - \frac{2}{3} \left( \frac{1}{2\sqrt{2}} \right) - 0 \\
&= \frac{1}{8} - \frac{1}{3\sqrt{2}}
\end{aligned}$$



$$\begin{aligned}
 \text{b} \quad & \int_0^b \left(x^2 + \frac{1}{2}x\right) dx = 3 \\
 & \therefore \left[\frac{1}{3}x^3 + \frac{1}{4}x^2\right]_0^b = 3 \\
 & \therefore \left(\frac{1}{3}b^3 + \frac{1}{4}b^2\right) - 0 = 3 \\
 & \therefore \frac{1}{3}b^3 + \frac{1}{4}b^2 = 3 \\
 & \therefore \frac{1}{3}b^3 + \frac{1}{4}b^2 - 3 = 0 \\
 & \therefore b \approx 1.86 \\
 & \text{\{using technology\}}
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & \int_0^b e^{1-2x} dx = \frac{e}{4} \\
 & \therefore \left[\left(\frac{1}{-2}\right) e^{1-2x}\right]_0^b = \frac{e}{4} \\
 & \therefore \left[-\frac{1}{2}e^{1-2x}\right]_0^b = \frac{e}{4} \\
 & \therefore -\frac{1}{2}e^{1-2b} - \left(-\frac{1}{2}e^1\right) = \frac{e}{4} \\
 & \therefore -\frac{1}{2}e^{1-2b} + \frac{1}{2}e = \frac{e}{4} \\
 & \therefore -2e^{1-2b} + 2e = e \\
 & \therefore 2e^{1-2b} = e \\
 & \therefore e^{1-2b} = \frac{e}{2} \\
 & \therefore e^{-2b} = \frac{1}{2} \\
 & \therefore -2b = \ln\left(\frac{1}{2}\right) \\
 & \therefore -2b = -\ln 2 \\
 & \therefore b = \frac{1}{2} \ln 2
 \end{aligned}$$

$$\begin{aligned}
 \text{3 a} \quad & \int_{-5}^{-1} \sqrt{1-3x} dx = \int_{-5}^{-1} (1-3x)^{\frac{1}{2}} dx \\
 & = \left[\left(\frac{1}{-3}\right) \frac{(1-3x)^{\frac{3}{2}}}{\frac{3}{2}}\right]_{-5}^{-1} \\
 & = \left[-\frac{2}{9}(1-3x)^{\frac{3}{2}}\right]_{-5}^{-1} \\
 & = -\frac{2}{9}(4)^{\frac{3}{2}} - \left(-\frac{2}{9}(16)^{\frac{3}{2}}\right) \\
 & = -\frac{16}{9} + \frac{128}{9} \\
 & = \frac{112}{9} \\
 & = 12\frac{4}{9}
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & \int_0^{\frac{\pi}{2}} \cos \frac{x}{2} dx = \left[2 \sin \frac{x}{2}\right]_0^{\frac{\pi}{2}} \\
 & = 2 \sin \frac{\pi}{4} - 2 \sin 0 \\
 & = \sqrt{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & \int_2^6 \frac{2}{x} dx = \left[2 \ln |x|\right]_2^6 \\
 & = 2 \ln 6 - 2 \ln 2 \\
 & = 2(\ln 6 - \ln 2) \\
 & = 2 \ln \left(\frac{6}{2}\right) \\
 & = 2 \ln 3
 \end{aligned}$$

$$\mathbf{d} \quad \sin^2 x = \frac{1}{2} - \frac{1}{2} \cos 2x$$

$$\therefore \sin^2\left(\frac{x}{2}\right) = \frac{1}{2} - \frac{1}{2} \cos x$$

$$\begin{aligned} \therefore \int_0^{\frac{\pi}{6}} \sin^2\left(\frac{x}{2}\right) dx &= \int_0^{\frac{\pi}{6}} \left(\frac{1}{2} - \frac{1}{2} \cos x\right) dx \\ &= \left[\frac{1}{2}x - \frac{1}{2} \sin x\right]_0^{\frac{\pi}{6}} \\ &= \left(\frac{\pi}{12} - \frac{1}{4}\right) - 0 \\ &= \frac{\pi}{12} - \frac{1}{4} \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} 2 \sec^2\left(\frac{x}{2}\right) dx &= \left[4 \tan \frac{x}{2}\right]_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \\ &= \left(4 \tan \frac{\pi}{6} - 4 \tan\left(-\frac{\pi}{6}\right)\right) \\ &= \frac{4}{\sqrt{3}} + \frac{4}{\sqrt{3}} \\ &= \frac{8}{\sqrt{3}} \end{aligned}$$

$$\begin{aligned} \mathbf{4} \quad \frac{d}{dx} (e^{-2x} \sin x) &= -2e^{-2x} \sin x + e^{-2x} (\cos x) \\ &= e^{-2x} (\cos x - 2 \sin x) \end{aligned}$$

$$\begin{aligned} \therefore \int_0^{\frac{\pi}{2}} [e^{-2x} (\cos x - 2 \sin x)] dx &= \left[e^{-2x} \sin x\right]_0^{\frac{\pi}{2}} \\ &= (e^{-\pi} \sin \frac{\pi}{2}) - 0 \\ &= e^{-\pi} \end{aligned}$$

$$\mathbf{5} \quad \mathbf{a} \quad \text{Let } u = 2x + 1 \quad \therefore \frac{du}{dx} = 2$$

$$\text{When } x = 3, \quad u = 2(3) + 1 = 7$$

$$\text{When } x = 4, \quad u = 2(4) + 1 = 9$$

$$\begin{aligned} \therefore \int_3^4 \frac{x}{\sqrt{2x+1}} dx &= \int_7^9 \frac{\frac{u-1}{2}}{\sqrt{u}} \left(\frac{1}{2} \frac{du}{dx}\right) dx \\ &= \int_7^9 \frac{u-1}{4\sqrt{u}} du \\ &= \frac{1}{4} \int_7^9 (u^{\frac{1}{2}} - u^{-\frac{1}{2}}) du \\ &= \frac{1}{4} \left[ \frac{2}{3} u^{\frac{3}{2}} - 2u^{\frac{1}{2}} \right]_7^9 \\ &= \frac{1}{4} \left[ \left( \frac{2}{3} (9)^{\frac{3}{2}} - 2(9)^{\frac{1}{2}} \right) - \left( \frac{2}{3} (7)^{\frac{3}{2}} - 2(7)^{\frac{1}{2}} \right) \right] \\ &= \frac{1}{4} \left( 18 - 6 - \frac{14\sqrt{7}}{3} + 2\sqrt{7} \right) \\ &= \frac{1}{4} \left( 12 - \frac{8\sqrt{7}}{3} \right) \\ &= 3 - \frac{2\sqrt{7}}{3} \end{aligned}$$

$$\begin{aligned}
\text{b } \int x^2 e^{x+1} dx &= x^2 e^{x+1} - \int 2x e^{x+1} dx \quad \leftarrow \begin{cases} u = x^2 & v' = e^{x+1} \\ u' = 2x & v = e^{x+1} \end{cases} \\
&= x^2 e^{x+1} - \left( 2x e^{x+1} - \int 2e^{x+1} dx \right) \quad \leftarrow \begin{cases} u = 2x & v' = e^{x+1} \\ u' = 2 & v = e^{x+1} \end{cases} \\
&= x^2 e^{x+1} - 2x e^{x+1} + \int 2e^{x+1} dx \\
&= x^2 e^{x+1} - 2x e^{x+1} + 2e^{x+1} + c \\
&= e^{x+1}(x^2 - 2x + 2) + c \\
\therefore \int_0^1 x^2 e^{x+1} dx &= [e^{x+1}(x^2 - 2x + 2)]_0^1 \\
&= e^2(1 - 2 + 2) - e^1(0 - 0 + 2) \\
&= e^2 - 2e \\
&= e(e - 2)
\end{aligned}$$

$$\text{c } \text{Let } u = \sin x \quad \therefore \frac{du}{dx} = \cos x$$

$$\text{When } x = \frac{\pi}{6}, \quad u = \sin \frac{\pi}{6} = \frac{1}{2}$$

$$\text{When } x = \frac{\pi}{2}, \quad u = \sin \frac{\pi}{2} = 1$$

$$\begin{aligned}
\therefore \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cot \theta d\theta &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos \theta}{\sin \theta} d\theta \\
&= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1}{u} \frac{du}{dx} dx \\
&= \int_{\frac{1}{2}}^1 \frac{1}{u} du \\
&= \left[ \ln |u| \right]_{\frac{1}{2}}^1 \\
&= \ln 1 - \ln \frac{1}{2} \\
&= \ln 2
\end{aligned}$$

$$\text{d } \text{Let } u = \tan x \quad \therefore \frac{du}{dx} = \sec^2 x$$

$$\text{When } x = \frac{\pi}{4}, \quad u = \tan \frac{\pi}{4} = 1$$

$$\text{When } x = \frac{\pi}{6}, \quad u = \tan \frac{\pi}{6} = \frac{1}{\sqrt{3}}$$

$$\begin{aligned}
\therefore \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\sec^2 x}{\tan x} dx &= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{1}{u} \frac{du}{dx} dx \\
&= \int_{\frac{1}{\sqrt{3}}}^1 \frac{1}{u} du \\
&= \left[ \ln |u| \right]_{\frac{1}{\sqrt{3}}}^1 \\
&= \ln 1 - \ln \frac{1}{\sqrt{3}} \\
&= -(\ln 1 - \ln 3^{\frac{1}{2}}) \\
&= \ln 3^{\frac{1}{2}} \\
&= \frac{1}{2} \ln 3
\end{aligned}$$

**e** Let  $u = 1 + \sin x \quad \therefore \frac{du}{dx} = \cos x$

When  $x = \frac{\pi}{6}$ ,  $u = 1 + \sin \frac{\pi}{6} = \frac{3}{2}$

When  $x = 0$ ,  $u = 1 + \sin 0 = 1$

$$\begin{aligned} \therefore \int_0^{\frac{\pi}{6}} \frac{\cos x}{(1 + \sin x)^3} dx &= \int_1^{\frac{3}{2}} \frac{1}{u^3} \frac{du}{dx} dx \\ &= \int_1^{\frac{3}{2}} u^{-3} du \\ &= \left[ \frac{u^{-2}}{-2} \right]_1^{\frac{3}{2}} \\ &= \left[ -\frac{1}{2u^2} \right]_1^{\frac{3}{2}} \\ &= -\frac{1}{2\left(\frac{3}{2}\right)^2} - \left( -\frac{1}{2(1)^2} \right) \\ &= -\frac{2}{9} + \frac{1}{2} \\ &= \frac{5}{18} \end{aligned}$$

**f** Let  $\frac{60}{x^2 - 16} = \frac{A}{x + 4} + \frac{B}{x - 4}$   
 $\therefore 60 = A(x - 4) + B(x + 4)$

Substituting  $x = 4$ :  $60 = B(4 + 4)$

$$\therefore 60 = 8B$$

$$\therefore B = \frac{15}{2}$$

Substituting  $x = -4$ :  $60 = A(-4 - 4)$

$$\therefore 60 = -8A$$

$$\therefore A = -\frac{15}{2}$$

$$\therefore \frac{60}{x^2 - 16} = -\frac{15}{2(x + 4)} + \frac{15}{2(x - 4)}$$

$$\begin{aligned} \therefore \int_{-2}^2 \frac{60}{x^2 - 16} dx &= \int_{-2}^2 \left( -\frac{15}{2(x + 4)} + \frac{15}{2(x - 4)} \right) dx \\ &= \left[ -\frac{15}{2} \ln |x + 4| + \frac{15}{2} \ln |x - 4| \right]_{-2}^2 \\ &= \left( -\frac{15}{2} \ln 6 + \frac{15}{2} \ln 2 \right) - \left( -\frac{15}{2} \ln 2 + \frac{15}{2} \ln 6 \right) \\ &= -\frac{15}{2} \ln 6 + \frac{15}{2} \ln 2 + \frac{15}{2} \ln 2 - \frac{15}{2} \ln 6 \\ &= 15(\ln 2 - \ln 6) \\ &= 15 \ln \frac{1}{3} \\ &= 15 \ln(3^{-1}) \\ &= -15 \ln 3 \end{aligned}$$



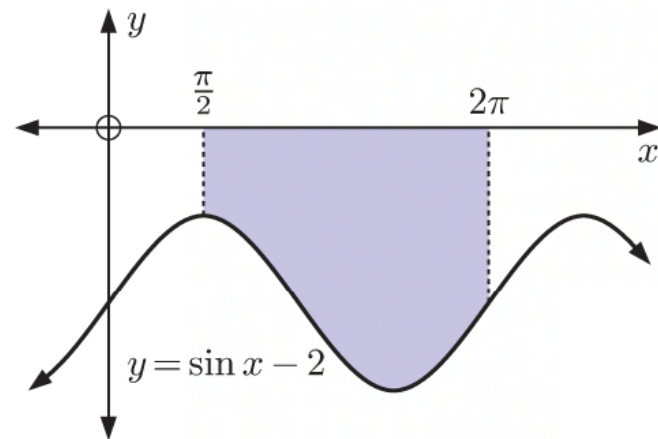
$$6 \quad \int_1^4 f(x) \, dx = 3$$

$$\begin{aligned} \text{a} \quad & \int_1^4 (f(x) + 1) \, dx \\ &= \int_1^4 f(x) \, dx + \int_1^4 1 \, dx \\ &= 3 + [x]_1^4 \\ &= 3 + (4 - 1) \\ &= 3 + 3 \\ &= 6 \end{aligned}$$

$$\begin{aligned} \text{c} \quad & \int_4^1 k f(x) \, dx = 5 \\ \therefore -k \int_1^4 f(x) \, dx &= 5 \\ \therefore -3k &= 5 \\ \therefore k &= -\frac{5}{3} \end{aligned}$$

$$\begin{aligned} \text{b} \quad & \int_1^2 f(x) \, dx - \int_4^2 f(x) \, dx \\ &= \int_1^2 f(x) \, dx + \int_2^4 f(x) \, dx \\ &= \int_1^4 f(x) \, dx \\ &= 3 \end{aligned}$$

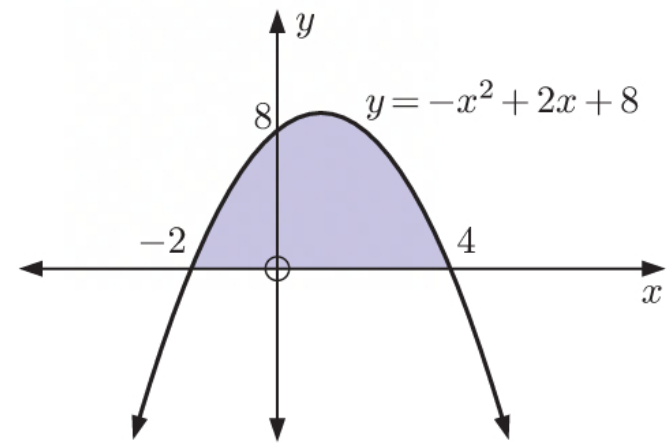
$$\begin{aligned} 7 \quad \text{a} \quad \text{Area} &= - \int_{\frac{\pi}{2}}^{2\pi} (\sin x - 2) \, dx \\ &= - \left[ -\cos x - 2x \right]_{\frac{\pi}{2}}^{2\pi} \\ &= -[(-1 - 4\pi) - (0 - \pi)] \\ &= (3\pi + 1) \text{ units}^2 \end{aligned}$$



$$\begin{aligned} \text{b} \quad & \text{The curve cuts the } x\text{-axis when } y = 0 \\ & \therefore -x^2 + 2x + 8 = 0 \\ & \therefore x^2 - 2x - 8 = 0 \\ & \therefore (x + 2)(x - 4) = 0 \\ & \therefore x = -2 \text{ or } 4 \end{aligned}$$

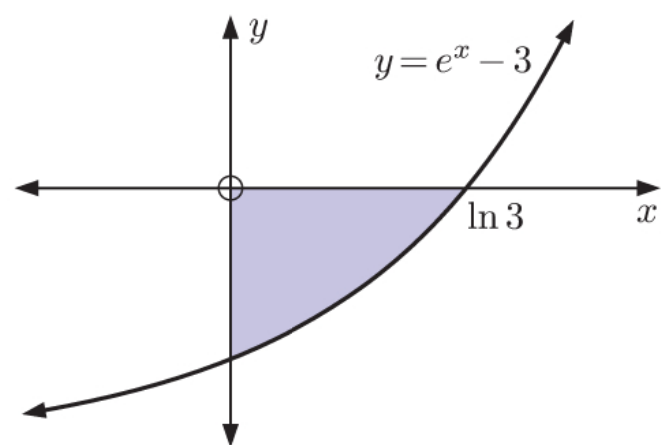
$\therefore$  the  $x$ -intercepts are  $-2$  and  $4$ .

$$\begin{aligned} \text{Area} &= \int_{-2}^4 (-x^2 + 2x + 8) \, dx \\ &= \left[ -\frac{1}{3}x^3 + x^2 + 8x \right]_{-2}^4 \\ &= \left( -\frac{64}{3} + 16 + 32 \right) - \left( \frac{8}{3} + 4 - 16 \right) \\ &= \frac{80}{3} - \left( -\frac{28}{3} \right) \\ &= 36 \text{ units}^2 \end{aligned}$$



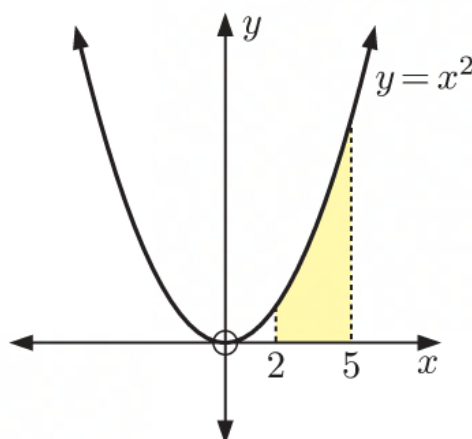
- The curve cuts the  $x$ -axis when  $y = 0$   
 $\therefore e^x - 3 = 0$   
 $\therefore e^x = 3$   
 $\therefore x = \ln 3$

$$\begin{aligned}\text{Area} &= - \int_0^{\ln 3} (e^x - 3) dx \\ &= -[e^x - 3x]_0^{\ln 3} \\ &= -[(3 - 3 \ln 3) - 1] \\ &= (3 \ln 3 - 2) \text{ units}^2\end{aligned}$$



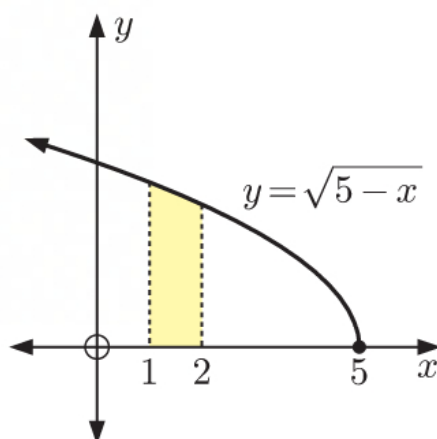
8 a  $\text{Area} = \int_2^5 x^2 dx$

$$\begin{aligned}&= \left[ \frac{1}{3} x^3 \right]_2^5 \\ &= \frac{125}{3} - \frac{8}{3} \\ &= 39 \text{ units}^2\end{aligned}$$



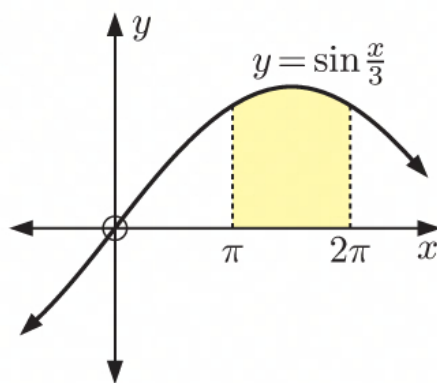
b  $\text{Area} = \int_1^2 \sqrt{5-x} dx$

$$\begin{aligned}&= \int_1^2 (5-x)^{\frac{1}{2}} dx \\ &= \left[ \left( \frac{1}{-\frac{1}{2}} \right) \frac{(5-x)^{\frac{3}{2}}}{\frac{3}{2}} \right]_1^2 \\ &= \left[ -\frac{2}{3} (5-x)^{\frac{3}{2}} \right]_1^2 \\ &= -\frac{2}{3} (3)^{\frac{3}{2}} - \left( -\frac{2}{3} (4)^{\frac{3}{2}} \right) \\ &= \left( \frac{16}{3} - 2\sqrt{3} \right) \text{ units}^2\end{aligned}$$



•  $\text{Area} = \int_{\pi}^{2\pi} \sin \frac{x}{3} dx$

$$\begin{aligned}&= \left[ -3 \cos \frac{x}{3} \right]_{\pi}^{2\pi} \\ &= -3 \cos \frac{2\pi}{3} - \left( -3 \cos \frac{\pi}{3} \right) \\ &= \frac{3}{2} - \left( -\frac{3}{2} \right) \\ &= 3 \text{ units}^2\end{aligned}$$



9  $y = x^2 + 4x + 1$  meets  $y = 3x + 3$

where  $x^2 + 4x + 1 = 3x + 3$

$$\therefore x^2 + x - 2 = 0$$

$$\therefore (x + 2)(x - 1) = 0$$

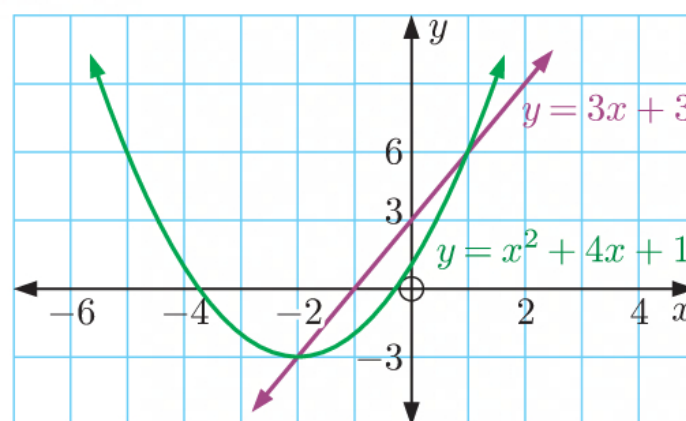
$$\therefore x = -2 \text{ or } 1$$

When  $x = -2$ ,  $y = 3(-2) + 3$   
 $= -3$

When  $x = 1$ ,  $y = 3(1) + 3$   
 $= 6$

$\therefore$  the graphs meet at the points  $(-2, -3)$  and  $(1, 6)$ .

$$\begin{aligned} \therefore \text{area} &= \int_{-2}^1 [(3x + 3) - (x^2 + 4x + 1)] dx \\ &= \int_{-2}^1 (-x^2 - x + 2) dx \\ &= \left[ -\frac{1}{3}x^3 - \frac{1}{2}x^2 + 2x \right]_{-2}^1 \\ &= \left( -\frac{1}{3} - \frac{1}{2} + 2 \right) - \left( \frac{8}{3} - 2 - 4 \right) \\ &= \frac{7}{6} - \left( -\frac{10}{3} \right) \\ &= 4\frac{1}{2} \text{ units}^2 \end{aligned}$$



10  $y = x^2$  meets  $y = k$  where  $x^2 = k$

$$\therefore x = \pm\sqrt{k}$$

Now, the area  $= 5\frac{1}{3}$

$$\therefore \int_{-\sqrt{k}}^{\sqrt{k}} (k - x^2) dx = 5\frac{1}{3}$$

$$\therefore \left[ kx - \frac{x^3}{3} \right]_{-\sqrt{k}}^{\sqrt{k}} = 5\frac{1}{3}$$

$$\therefore \left( k\sqrt{k} - \frac{k\sqrt{k}}{3} \right) - \left( -k\sqrt{k} - \left( -\frac{k\sqrt{k}}{3} \right) \right) = \frac{16}{3}$$

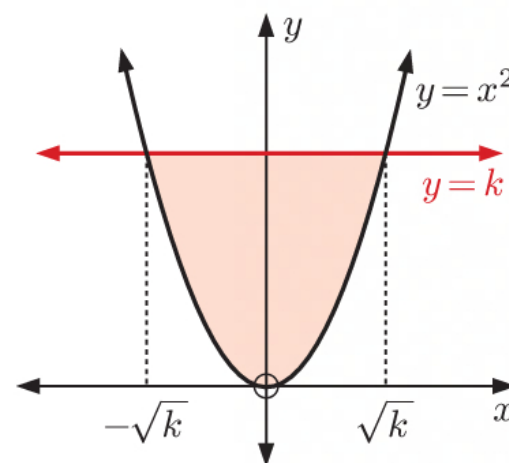
$$\therefore 2k\sqrt{k} - \frac{2k\sqrt{k}}{3} = \frac{16}{3}$$

$$\therefore \frac{4}{3}k^{\frac{3}{2}} = \frac{16}{3}$$

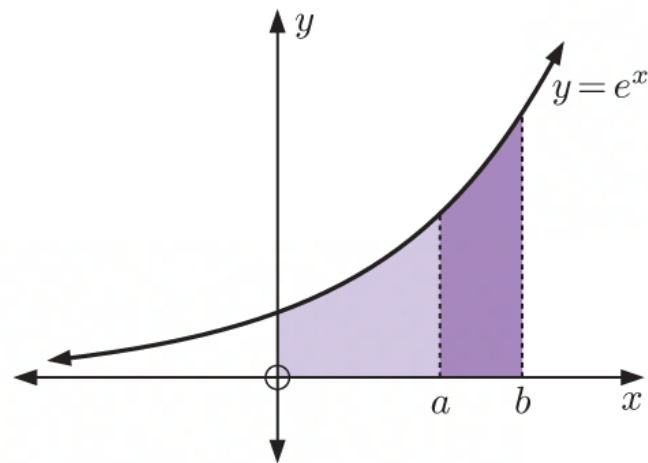
$$\therefore k^{\frac{3}{2}} = 4$$

$$\therefore k^3 = 16$$

$$\therefore k = \sqrt[3]{16}$$



**11 a** Area from  $x = 0$  to  $x = a$  is  $\int_0^a e^x dx = 2$   
 $\therefore [e^x]_0^a = 2$   
 $\therefore e^a - 1 = 2$   
 $\therefore e^a = 3$   
 $\therefore a = \ln 3$



**b** Area from  $x = a$  to  $x = b$  is  $\int_{\ln 3}^b e^x dx = 2$   
 $\therefore [e^x]_{\ln 3}^b = 2$   
 $\therefore e^b - e^{\ln 3} = 2$   
 $\therefore e^b - 3 = 2$   
 $\therefore e^b = 5$   
 $\therefore b = \ln 5$

**12** The curves meet where

$$x^3 + x^2 + 2x + 6 = 7x^2 - x - 4$$

$$\therefore x^3 - 6x^2 + 3x + 10 = 0$$

$$\therefore (x + 1)(x^2 - 7x + 10) = 0$$

$$\therefore (x + 1)(x - 2)(x - 5) = 0$$

$$\therefore x = -1, 2, \text{ or } 5$$

$\therefore$  enclosed area

$$= \int_{-1}^2 ((x^3 + x^2 + 2x + 6) - (7x^2 - x - 4)) dx$$

$$+ \int_2^5 ((7x^2 - x - 4) - (x^3 + x^2 + 2x + 6)) dx$$

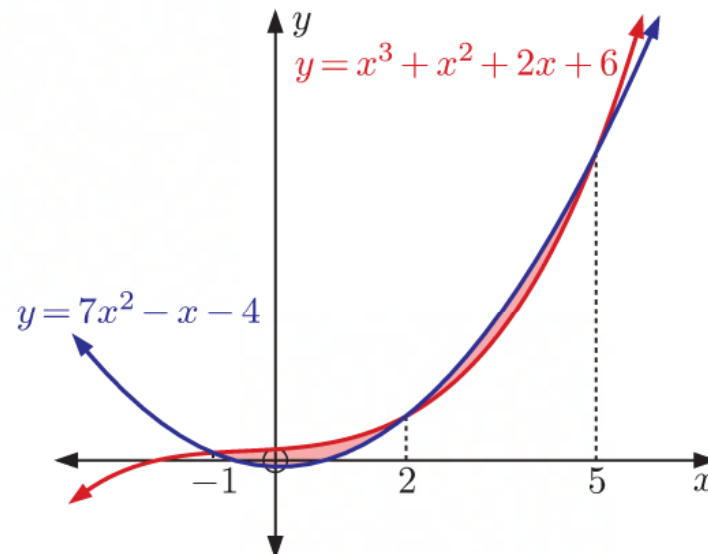
$$= \int_{-1}^2 (x^3 - 6x^2 + 3x + 10) dx + \int_2^5 (-x^3 + 6x^2 - 3x - 10) dx$$

$$= \left[ \frac{1}{4}x^4 - 2x^3 + \frac{3}{2}x^2 + 10x \right]_{-1}^2 + \left[ -\frac{1}{4}x^4 + 2x^3 - \frac{3}{2}x^2 - 10x \right]_2^5$$

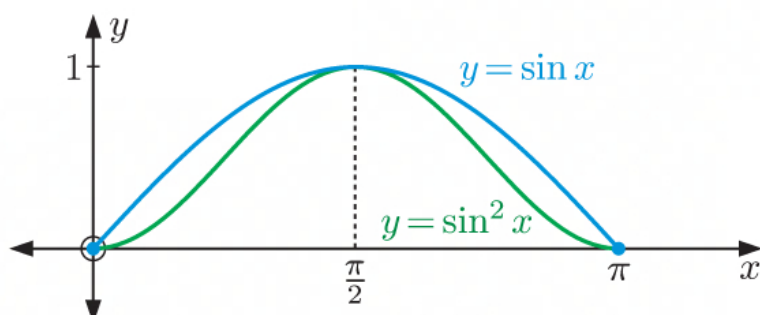
$$= \left[ (4 - 16 + 6 + 20) - \left( \frac{1}{4} + 2 + \frac{3}{2} - 10 \right) \right]$$

$$+ \left[ \left( -\frac{625}{4} + 250 - \frac{75}{2} - 50 \right) - (-4 + 16 - 6 - 20) \right]$$

$$= 40\frac{1}{2} \text{ units}^2$$

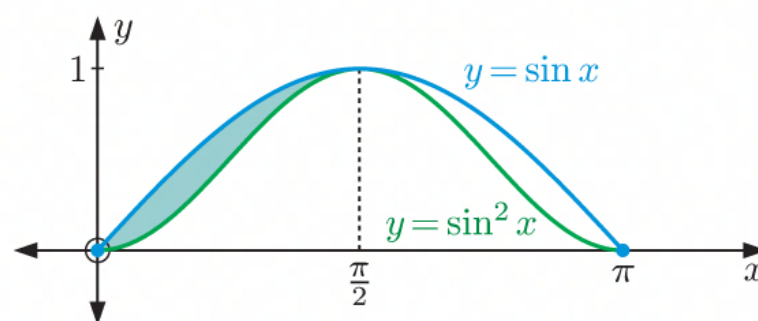


**13 a**





$$\begin{aligned}
 \text{b Area} &= \int_0^{\frac{\pi}{2}} (\sin x - \sin^2 x) dx \\
 &= \int_0^{\frac{\pi}{2}} \left( \sin x - \left( \frac{1}{2} - \frac{1}{2} \cos 2x \right) \right) dx \\
 &= \int_0^{\frac{\pi}{2}} \left( \sin x + \frac{1}{2} \cos 2x - \frac{1}{2} \right) dx \\
 &= \left[ -\cos x + \frac{1}{4} \sin 2x - \frac{1}{2} x \right]_0^{\frac{\pi}{2}} \\
 &= \left( 0 + \frac{1}{4}(0) - \frac{\pi}{4} \right) - \left( -1 + 0 - 0 \right) \\
 &= \left( 1 - \frac{\pi}{4} \right) \text{ units}^2
 \end{aligned}$$

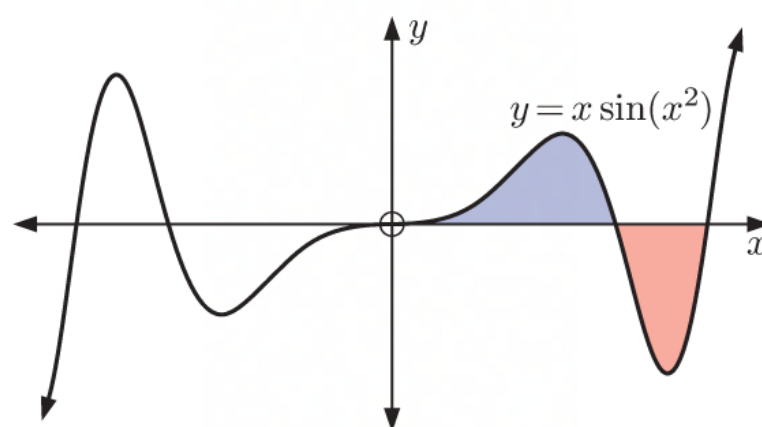


**14**  $y = x \sin(x^2)$  cuts the  $x$ -axis when  $x = 0$  or  $\sin(x^2) = 0$

$\therefore x = 0$ , and the smallest positive  $x$ -intercepts occur when

$$x^2 = \pi \text{ or } 2\pi$$

$$\therefore x = \sqrt{\pi} \text{ or } \sqrt{2\pi} \quad \{x > 0\}$$



$$\begin{aligned}
 \int x \sin(x^2) dx &= \frac{1}{2} \int 2x \sin(x^2) dx \\
 &= \frac{1}{2} \int \sin u \frac{du}{dx} dx \quad \{u = x^2, \frac{du}{dx} = 2x\} \\
 &= \frac{1}{2} \int \sin u du \\
 &= \frac{1}{2} (-\cos u) + c \\
 &= -\frac{1}{2} \cos(x^2) + c
 \end{aligned}$$

$$\begin{aligned}
 \text{Purple area} &= \int_0^{\sqrt{\pi}} x \sin(x^2) dx \\
 &= \left[ -\frac{1}{2} \cos(x^2) \right]_0^{\sqrt{\pi}} \\
 &= -\frac{1}{2} \cos \pi - \left( -\frac{1}{2} \cos 0 \right) \\
 &= 1 \text{ unit}^2
 \end{aligned}$$

$$\begin{aligned}
 \text{Red area} &= - \int_{\sqrt{\pi}}^{\sqrt{2\pi}} x \sin(x^2) dx \\
 &= - \left[ -\frac{1}{2} \cos(x^2) \right]_{\sqrt{\pi}}^{\sqrt{2\pi}} \\
 &= \frac{1}{2} \cos 2\pi - \frac{1}{2} \cos \pi \\
 &= 1 \text{ unit}^2
 \end{aligned}$$

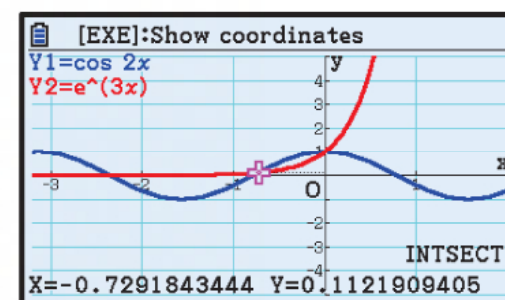
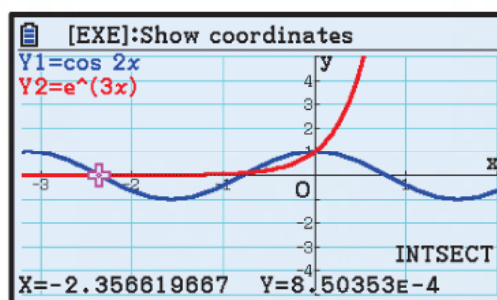
$\therefore$  the shaded regions have equal area.

**15 a** The curves meet where  $\cos 2x = e^{3x}$

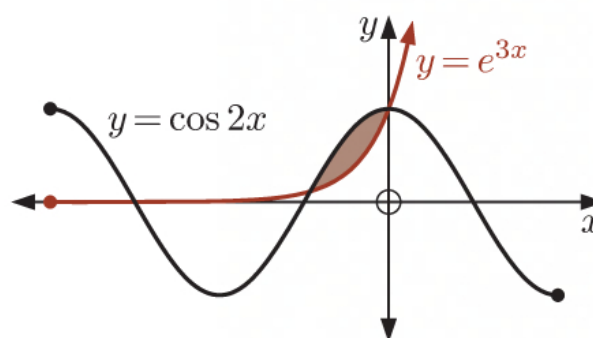
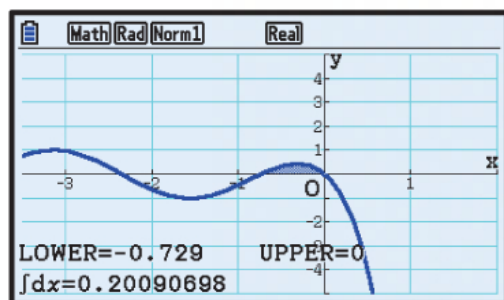
Using technology,

$x \approx -2.3566, -0.7292$ , or  $0$ .

These are the  $x$ -coordinates of the points of intersection.

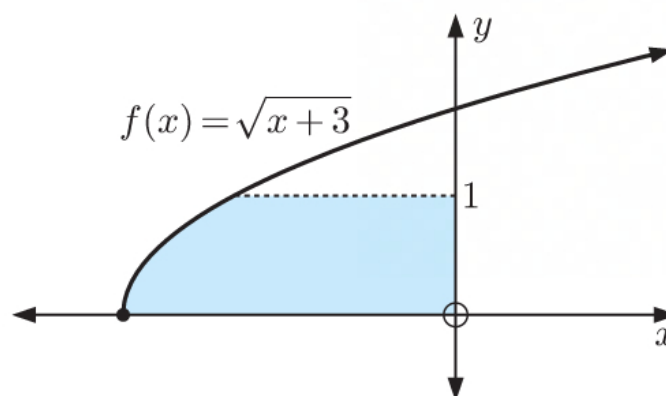


**b** Area  $\approx \int_{-0.7292}^0 (\cos 2x - e^{3x}) dx$   
 $\approx 0.2009$  {using technology}



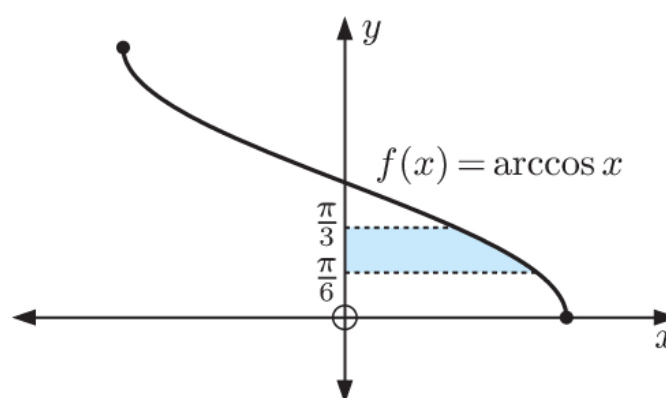
**16 a**

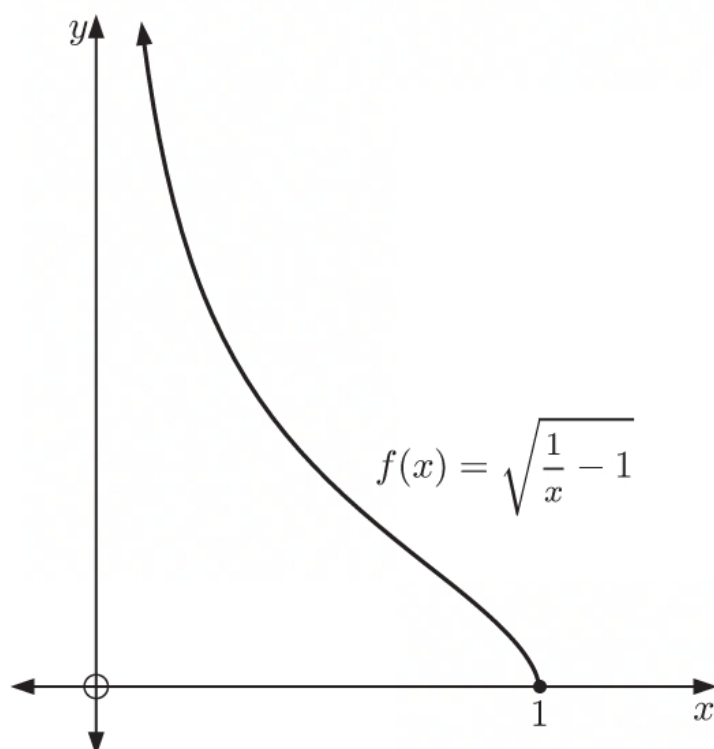
$$\begin{aligned} y &= \sqrt{x+3} \\ \therefore y^2 &= x+3 \\ \therefore x &= y^2-3 \\ \therefore f^{-1}(y) &= y^2-3 \\ \therefore \text{area} &= -\int_0^1 (y^2-3) dy \\ &= -\left[\frac{1}{3}y^3 - 3y\right]_0^1 \\ &= -\left[\left(\frac{1}{3} - 3\right) - 0\right] \\ &= \frac{8}{3} \\ &= 2\frac{2}{3} \text{ units}^2 \end{aligned}$$



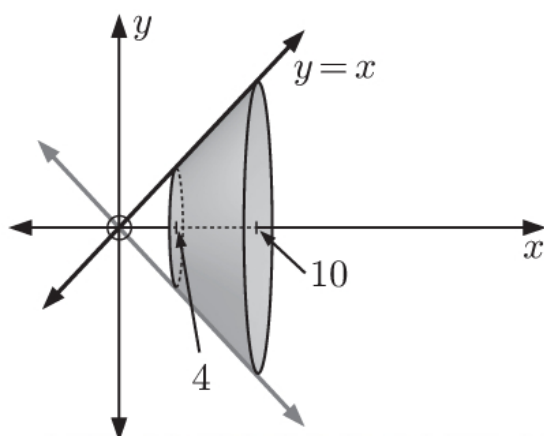
**b**

$$\begin{aligned} y &= \arccos x \\ \therefore x &= \cos y \\ \therefore f^{-1}(y) &= \cos y \\ \therefore \text{area} &= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos y dy \\ &= \left[\sin y\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} \\ &= \sin \frac{\pi}{3} - \sin \frac{\pi}{6} \\ &= \left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right) \\ &= \left(\frac{\sqrt{3}-1}{2}\right) \text{ units}^2 \end{aligned}$$

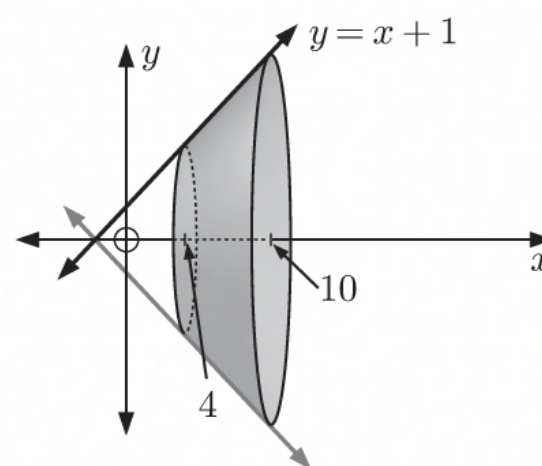


**17 a****b**

$$\begin{aligned}
 y &= \sqrt{\frac{1}{x} - 1} \\
 \therefore y^2 &= \frac{1}{x} - 1 \\
 \therefore xy^2 &= 1 - x \\
 \therefore xy^2 + x &= 1 \\
 \therefore x(y^2 + 1) &= 1 \\
 \therefore x &= \frac{1}{y^2 + 1} \\
 \therefore f^{-1}(y) &= \frac{1}{y^2 + 1} \\
 \therefore \text{area} &= \int_1^{\sqrt{3}} \frac{1}{y^2 + 1} dy \\
 &= [\arctan y]_1^{\sqrt{3}} \\
 &= \arctan \sqrt{3} - \arctan 1 \\
 &= \frac{\pi}{3} - \frac{\pi}{4} \\
 &= \frac{\pi}{12} \text{ units}^2
 \end{aligned}$$

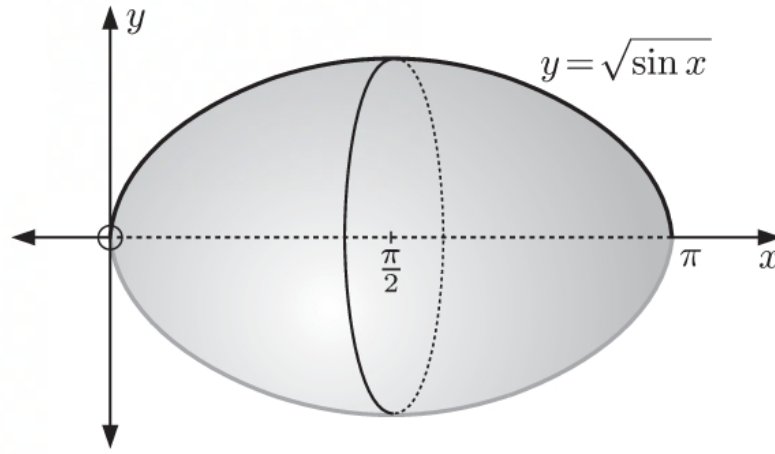
**18 a**

$$\begin{aligned}
 \text{Volume} &= \pi \int_4^{10} y^2 dx \\
 &= \pi \int_4^{10} x^2 dx \\
 &= \pi \left[ \frac{x^3}{3} \right]_4^{10} \\
 &= \pi \left( \frac{1000}{3} - \frac{64}{3} \right) \\
 &= \frac{936\pi}{3} \\
 &= 312\pi \text{ units}^3
 \end{aligned}$$

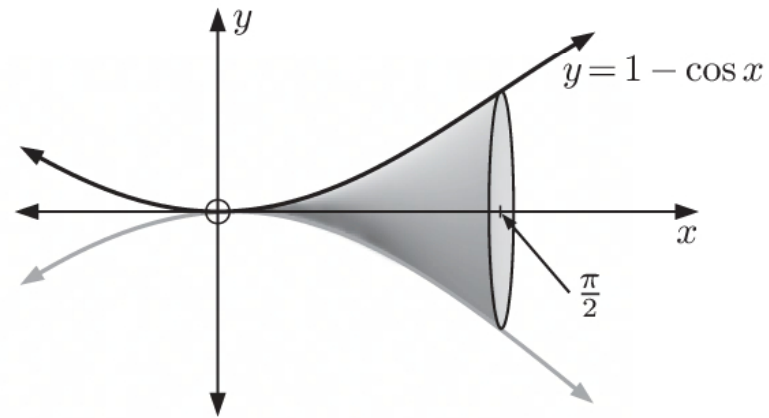
**b**

$$\begin{aligned}
 \text{Volume} &= \pi \int_4^{10} y^2 dx \\
 &= \pi \int_4^{10} (x + 1)^2 dx \\
 &= \pi \left[ \frac{(x + 1)^3}{3} \right]_4^{10} \\
 &= \pi \left( \frac{11^3}{3} - \frac{5^3}{3} \right) \\
 &= \frac{1206\pi}{3} \\
 &= 402\pi \text{ units}^3
 \end{aligned}$$

$$\begin{aligned}
 \text{c Volume} &= \pi \int_0^{\pi} y^2 dx \\
 &= \pi \int_0^{\pi} (\sqrt{\sin x})^2 dx \\
 &= \pi \int_0^{\pi} \sin x dx \\
 &= \pi [-\cos x]_0^{\pi} \\
 &= \pi(1 - (-1)) \\
 &= 2\pi \text{ units}^3
 \end{aligned}$$

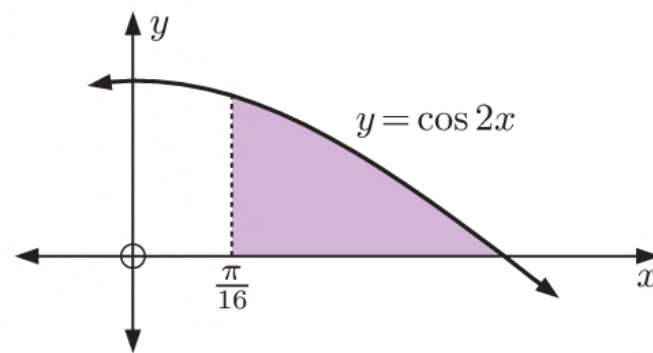


$$\begin{aligned}
 \text{d Volume} &= \pi \int_0^{\frac{\pi}{2}} y^2 dx \\
 &= \pi \int_0^{\frac{\pi}{2}} (1 - \cos x)^2 dx \\
 &= \pi \int_0^{\frac{\pi}{2}} (1 - 2\cos x + \cos^2 x) dx \\
 &= \pi \int_0^{\frac{\pi}{2}} \left(1 - 2\cos x + \frac{1}{2} + \frac{1}{2}\cos 2x\right) dx \\
 &= \pi \left[\frac{3}{2}x - 2\sin x + \frac{1}{4}\sin 2x\right]_0^{\frac{\pi}{2}} \\
 &= \pi \left[\left(\frac{3}{2}\left(\frac{\pi}{2}\right) - 2\sin \frac{\pi}{2} + \frac{1}{4}\sin \pi\right) - \left(\frac{3}{2}(0) - 2\sin 0 + \frac{1}{4}\sin 0\right)\right] \\
 &= \pi\left(\frac{3\pi}{4} - 2\right) \text{ units}^3 \\
 &= \pi\left(\frac{3\pi-8}{4}\right) \text{ units}^3
 \end{aligned}$$



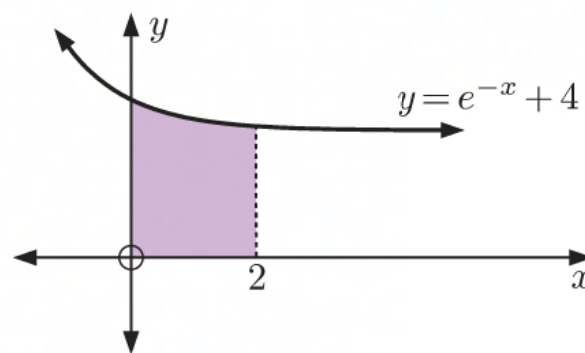
19 a  $y = \cos 2x$  meets the  $x$ -axis where  $2x = \frac{\pi}{2}$ , or  $x = \frac{\pi}{4}$ .

$$\begin{aligned}
 \therefore \text{volume} &= \pi \int_{\frac{\pi}{16}}^{\frac{\pi}{4}} y^2 dx \\
 &= \pi \int_{\frac{\pi}{16}}^{\frac{\pi}{4}} \cos^2 2x dx \\
 &= \pi \int_{\frac{\pi}{16}}^{\frac{\pi}{4}} \left(\frac{1}{2} + \frac{1}{2}\cos 4x\right) dx \\
 &= \pi \left[\frac{1}{2}x + \frac{1}{8}\sin 4x\right]_{\frac{\pi}{16}}^{\frac{\pi}{4}} \\
 &= \pi \left[\left(\frac{\pi}{8} + \frac{1}{8}\sin \pi\right) - \left(\frac{\pi}{32} + \frac{1}{8}\sin \frac{\pi}{4}\right)\right] \\
 &= \pi \left(\frac{\pi}{8} - \frac{\pi}{32} - \frac{1}{8}\left(\frac{1}{\sqrt{2}}\right)\right) \\
 &= \pi \left(\frac{3\pi}{32} - \frac{1}{8\sqrt{2}}\right) \text{ units}^3
 \end{aligned}$$

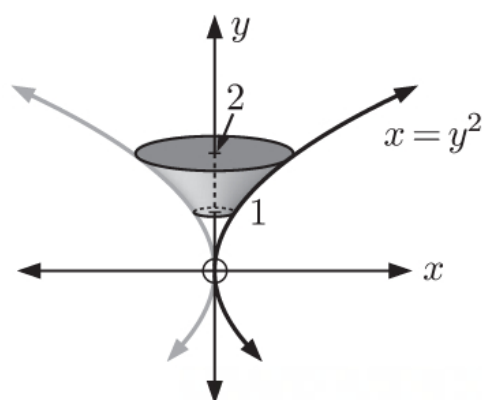




$$\begin{aligned}
 \text{b Volume} &= \pi \int_0^2 y^2 dx \\
 &= \pi \int_0^2 (e^{-x} + 4)^2 dx \\
 &= \pi \int_0^2 (e^{-2x} + 8e^{-x} + 16) dx \\
 &= \pi \left[ \frac{1}{-2} e^{-2x} + \frac{8}{-1} e^{-x} + 16x \right]_0^2 \\
 &= \pi \left[ \left( -\frac{1}{2} e^{-4} - 8e^{-2} + 32 \right) - \left( -\frac{1}{2} - 8 \right) \right] \\
 &= \pi \left( \frac{81}{2} - \frac{1}{2e^4} - \frac{8}{e^2} \right) \text{ units}^3 \approx 124 \text{ units}^3
 \end{aligned}$$



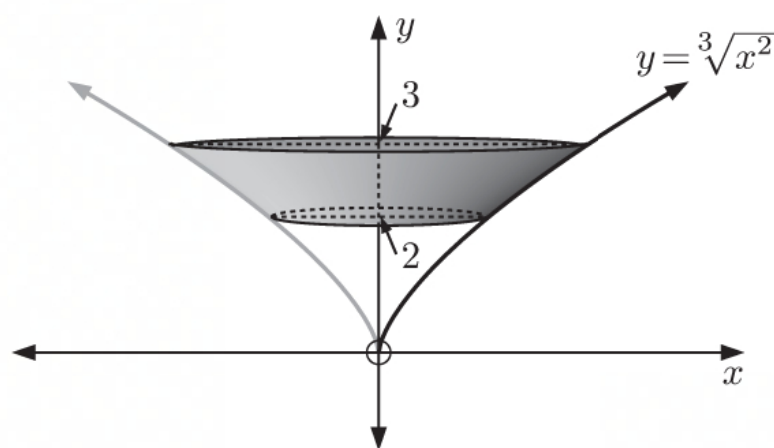
20 a



$$y = \sqrt{x} \quad \therefore x^2 = y^4$$

$$\begin{aligned}
 \text{Volume} &= \pi \int_1^2 x^2 dy \\
 &= \pi \int_1^2 y^4 dy \\
 &= \pi \left[ \frac{y^5}{5} \right]_1^2 \\
 &= \pi \left( \frac{32}{5} - \frac{1}{5} \right) \\
 &= \frac{31\pi}{5} \text{ units}^3
 \end{aligned}$$

b



$$y = \sqrt[3]{x^2} \quad \therefore x^2 = y^3$$

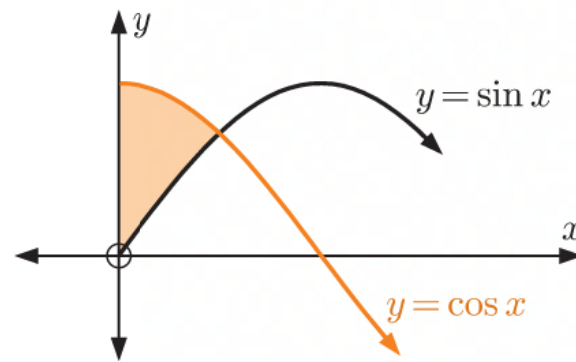
$$\begin{aligned}
 \text{Volume} &= \pi \int_2^3 x^2 dy \\
 &= \pi \int_2^3 y^3 dy \\
 &= \pi \left[ \frac{y^4}{4} \right]_2^3 \\
 &= \pi \left( \frac{81}{4} - \frac{16}{4} \right) \\
 &= \frac{65\pi}{4} \text{ units}^3
 \end{aligned}$$

**21**  $y = \sin x$  and  $y = \cos x$  meet where  $\sin x = \cos x$

$$\therefore \frac{\sin x}{\cos x} = 1$$

$$\therefore \tan x = 1$$

$$\therefore x = \frac{\pi}{4}$$



$$\begin{aligned} \text{Volume} &= \pi \int_0^{\frac{\pi}{4}} (y_U^2 - y_L^2) dx \\ &= \pi \int_0^{\frac{\pi}{4}} (\cos^2 x - \sin^2 x) dx \\ &= \pi \int_0^{\frac{\pi}{4}} \cos 2x dx \\ &= \pi \left[ \frac{1}{2} \sin 2x \right]_0^{\frac{\pi}{4}} \\ &= \pi \left( \frac{1}{2} \sin \frac{\pi}{2} - \frac{1}{2} \sin 0 \right) \\ &= \pi \left( \frac{1}{2} - 0 \right) \\ &= \frac{\pi}{2} \text{ units}^3 \end{aligned}$$

**22**  $R_1(t) = 6.4$ ,  $R_2(t) = 2.5 - 1.25e^{-0.2t}$

$$\begin{aligned} \text{a} \quad \text{i} \quad \int_0^{\frac{1}{2}} R_2(t) dt &= \int_0^{\frac{1}{2}} (2.5 - 1.25e^{-0.2t}) dt \\ &= \left[ 2.5t - 1.25 \left( \frac{1}{-0.2} \right) e^{-0.2t} \right]_0^{\frac{1}{2}} \\ &= \left[ 2.5t + 6.25e^{-0.2t} \right]_0^{\frac{1}{2}} \\ &= (2.5(\frac{1}{2}) + 6.25e^{-0.2(\frac{1}{2})}) - (0 + 6.25) \\ &\approx 0.655 \end{aligned}$$

About 655 millilitres of water leak from the watering can in the first 30 seconds.

$$\begin{aligned} \text{ii} \quad \int_0^1 [R_1(t) - R_2(t)] dt &= \int_0^1 [6.4 - (2.5 - 1.25e^{-0.2t})] dt \\ &= \int_0^1 (3.9 + 1.25e^{-0.2t}) dt \\ &= \left[ 3.9t + 1.25 \left( \frac{1}{-0.2} \right) e^{-0.2t} \right]_0^1 \\ &= \left[ 3.9t - 6.25e^{-0.2t} \right]_0^1 \\ &= (3.9 - 6.25e^{-0.2}) - (0 - 6.25) \\ &\approx 5.03 \end{aligned}$$

There are about 5.03 litres of water in the watering can after 1 minute.

- b** Suppose it takes  $x$  seconds for the watering can to be full.

We find  $x$  such that  $\int_0^x [R_1(t) - R_2(t)] dt = 16$

$$\therefore [3.9t - 6.25e^{-0.2t}]_0^x = 16 \quad \{\text{using a ii}\}$$

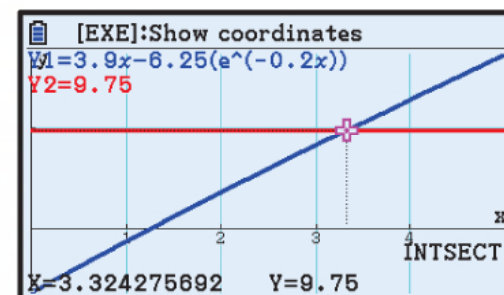
$$\therefore (3.9x - 6.25e^{-0.2x}) - (0 - 6.25) = 16$$

$$\therefore 3.9x - 6.25e^{-0.2x} + 6.25 = 16$$

$$\therefore 3.9x - 6.25e^{-0.2x} = 9.75$$

$$\therefore x \approx 3.324$$

{using technology}



$$3.324 \text{ minutes} \approx 3.324 \times 60 \text{ seconds}$$

$$\approx 199 \text{ seconds}$$

$\therefore$  it will take about 199 seconds for the watering can to be full.

**23 a**  $\int_4^\infty \frac{1}{\sqrt{x}} dx = \lim_{b \rightarrow \infty} \int_4^b x^{-\frac{1}{2}} dx$

$$= \lim_{b \rightarrow \infty} \left[ \frac{x^{\frac{1}{2}}}{\frac{1}{2}} \right]_4^b$$

$$= \lim_{b \rightarrow \infty} [2\sqrt{x}]_4^b$$

$$= \lim_{b \rightarrow \infty} (2\sqrt{b} - 4) \quad \text{which does not exist}$$

**b** Let  $u = x^2 - 1 \quad \therefore \frac{du}{dx} = 2x$

$$\therefore \int \frac{x}{\sqrt{x^2 - 1}} dx = \int \frac{1}{\sqrt{u}} \left( \frac{1}{2} \frac{du}{dx} \right) dx$$

$$= \int \frac{1}{2\sqrt{u}} du$$

$$= \int \frac{1}{2} u^{-\frac{1}{2}} du$$

$$= \frac{1}{2} \left( \frac{u^{\frac{1}{2}}}{\frac{1}{2}} \right) + c$$

$$= \sqrt{u} + c$$

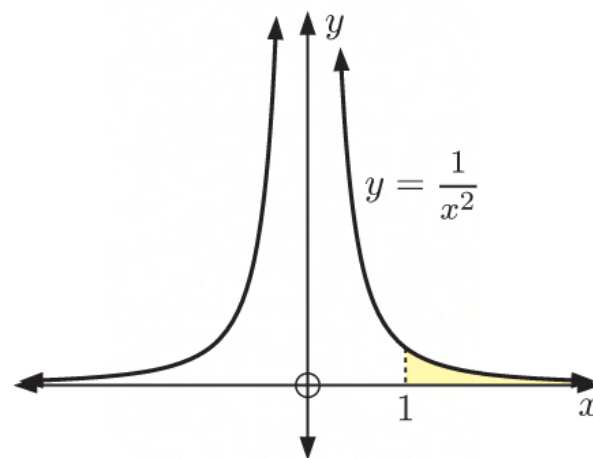
$$\therefore \int_2^\infty \frac{x}{\sqrt{x^2 - 1}} dx = \lim_{b \rightarrow \infty} \int_2^b \frac{x}{\sqrt{x^2 - 1}} dx$$

$$= \lim_{b \rightarrow \infty} [\sqrt{u}]_2^b$$

$$= \lim_{b \rightarrow \infty} (\sqrt{b} - \sqrt{2}) \quad \text{which does not exist}$$

$$\begin{aligned}
 \text{c } \int_{-1}^{\infty} 2^{-x} dx &= \lim_{b \rightarrow \infty} \int_{-1}^b 2^{-x} dx \\
 &= \lim_{b \rightarrow \infty} \left[ -\frac{2^{-x}}{\ln 2} \right]_{-1}^b \\
 &= \lim_{b \rightarrow \infty} \left( -\frac{2^{-b}}{\ln 2} - \left( -\frac{2}{\ln 2} \right) \right) \\
 &= \lim_{b \rightarrow \infty} \left( \frac{2}{\ln 2} - \frac{1}{2^b \ln 2} \right) \\
 &= \frac{2}{\ln 2}
 \end{aligned}$$

$$\begin{aligned}
 \text{24 Volume of revolution} &= \pi \int_1^{\infty} y^2 dx \\
 &= \lim_{b \rightarrow \infty} \pi \int_1^b \frac{1}{x^4} dx \\
 &= \pi \lim_{b \rightarrow \infty} \int_1^b x^{-4} dx \\
 &= \pi \lim_{b \rightarrow \infty} \left[ \frac{x^{-3}}{-3} \right]_1^b \\
 &= \pi \lim_{b \rightarrow \infty} \left[ -\frac{1}{3x^3} \right]_1^b \\
 &= \pi \lim_{b \rightarrow \infty} \left( -\frac{1}{3b^3} - \left( -\frac{1}{3} \right) \right) \\
 &= \pi \lim_{b \rightarrow \infty} \left( \frac{1}{3} - \frac{1}{3b^3} \right) \\
 &= \pi \times \frac{1}{3} \\
 &= \frac{\pi}{3} \text{ units}^3
 \end{aligned}$$



**25 a**  $f(x)$  is even, so

$$\begin{aligned}
 V &= \pi \int_{-\infty}^{\infty} [f(x)]^2 dx \\
 &= 2\pi \int_0^{\infty} [f(x)]^2 dx
 \end{aligned}$$

$$\begin{aligned}
 \text{b } V &= 2\pi \int_0^{\infty} [f(x)]^2 dx \\
 &= 2\pi \int_0^{\infty} \frac{1}{\pi^2} \left( \frac{a}{x^2 + a^2} \right) dx \\
 &= \frac{2}{\pi} \int_0^{\infty} \frac{a}{x^2 + a^2} dx \\
 &= \frac{2}{\pi} \lim_{b \rightarrow \infty} \int_0^b \frac{a}{x^2 + a^2} dx \\
 &= \frac{2}{\pi} \lim_{b \rightarrow \infty} \left[ \arctan \frac{x}{a} \right]_0^b \\
 &= \frac{2}{\pi} \lim_{b \rightarrow \infty} \left( \arctan \frac{b}{a} - \arctan 0 \right) \\
 &= \frac{2}{\pi} \times \frac{\pi}{2} \\
 &= 1 \text{ unit}^3
 \end{aligned}$$



## REVIEW SET 22B

$$\begin{aligned}
 1 \quad a \quad \int_2^3 \frac{1}{\sqrt{3x}} dx &= \int_2^3 (3x)^{-\frac{1}{2}} dx \\
 &= \left[ \left(\frac{1}{3}\right) \frac{(3x)^{\frac{1}{2}}}{\frac{1}{2}} \right]_2^3 \\
 &= \left[ \frac{2}{3} \sqrt{3x} \right]_2^3 \\
 &= \frac{2}{3} \sqrt{9} - \frac{2}{3} \sqrt{6} \\
 &= 2 - \frac{2}{3} \sqrt{2} \sqrt{3} \\
 &= 2 - \frac{2\sqrt{2}}{\sqrt{3}}
 \end{aligned}$$

$$\begin{aligned}
 b \quad \int_1^4 \left(x - \frac{1}{2}x^2\right) dx &= \left[\frac{1}{2}x^2 - \frac{1}{6}x^3\right]_1^4 \\
 &= \left(\frac{1}{2}(16) - \frac{1}{6}(64)\right) - \left(\frac{1}{2} - \frac{1}{6}\right) \\
 &= -\frac{8}{3} - \frac{1}{3} \\
 &= -3
 \end{aligned}$$

$$\begin{aligned}
 c \quad \int_0^1 \left(x^2 + \frac{1}{3}\right)^2 dx &= \int_0^1 \left(x^4 + \frac{2}{3}x^2 + \frac{1}{9}\right) dx \\
 &= \left[\frac{1}{5}x^5 + \frac{2}{9}x^3 + \frac{1}{9}x\right]_0^1 \\
 &= \left(\frac{1}{5} + \frac{2}{9} + \frac{1}{9}\right) - 0 \\
 &= \frac{8}{15}
 \end{aligned}$$

$$\begin{aligned}
 2 \quad a \quad \int_0^a \left(x^2 - \frac{1}{2}x\right) dx &= \frac{9}{16} \\
 \therefore \left[\frac{1}{3}x^3 - \frac{1}{4}x^2\right]_0^a &= \frac{9}{16} \\
 \therefore \frac{1}{3}a^3 - \frac{1}{4}a^2 &= \frac{9}{16} \\
 \therefore 16a^3 - 12a^2 &= 27 \\
 \therefore 16a^3 - 12a^2 - 27 &= 0 \\
 \therefore a &= \frac{3}{2} \\
 &\text{\{using technology\}}
 \end{aligned}$$

Math Rad Norm1 d/c Real  
 $aX^3 + bX^2 + cX + d = 0$   

a	b	c	d
16	-12	0	-27

  
 -27  
 SOLVE DELETE CLEAR EDIT

Math Rad Norm1 d/c Real  
 $aX^3 + bX^2 + cX + d = 0$   
 X1 1.5  
 3/2  
 REPEAT

$$\begin{aligned}
 b \quad \int_0^a \cos x dx &= \frac{1}{\sqrt{2}}, \quad 0 < a < \pi \\
 \therefore [\sin x]_0^a &= \frac{1}{\sqrt{2}} \\
 \therefore \sin a - 0 &= \frac{1}{\sqrt{2}} \\
 \therefore \sin a &= \frac{1}{\sqrt{2}} \\
 \therefore a &= \frac{\pi}{4} \text{ or } \frac{3\pi}{4}
 \end{aligned}$$

$$\begin{aligned}
 \text{3 a } \int_2^3 \frac{1}{\sqrt{3x-4}} dx &= \int_2^3 (3x-4)^{-\frac{1}{2}} dx \\
 &= \left[ \left(\frac{1}{3}\right) \frac{(3x-4)^{\frac{1}{2}}}{\frac{1}{2}} \right]_2^3 \\
 &= \left[ \frac{2}{3} \sqrt{3x-4} \right]_2^3 \\
 &= \frac{2}{3} \sqrt{5} - \frac{2}{3} \sqrt{2} \\
 &= \frac{2}{3} (\sqrt{5} - \sqrt{2})
 \end{aligned}$$

$$\begin{aligned}
 \text{b } \int_{2e}^{3e} \frac{4}{x+e} dx &= \left[ 4 \ln |x+e| \right]_{2e}^{3e} \\
 &= 4 \ln 4e - 4 \ln 3e \\
 &= 4(\ln 4e - \ln 3e) \\
 &= 4 \ln \left( \frac{4e}{3e} \right) \\
 &= 4 \ln \left( \frac{4}{3} \right)
 \end{aligned}$$

$$\begin{aligned}
 \text{c } \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (2 \sin x + 1) dx &= \left[ -2 \cos x + x \right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} \\
 &= \left( -2 \cos \frac{\pi}{2} + \frac{\pi}{2} \right) - \left( -2 \cos \frac{\pi}{4} + \frac{\pi}{4} \right) \\
 &= \frac{\pi}{2} - \left( -\sqrt{2} + \frac{\pi}{4} \right) \\
 &= \frac{\pi}{2} + \sqrt{2} - \frac{\pi}{4} \\
 &= \frac{\pi}{4} + \sqrt{2}
 \end{aligned}$$

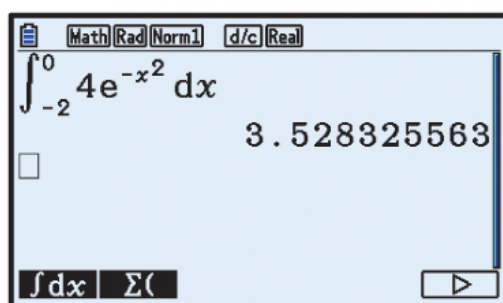
$$\begin{aligned}
 \text{d } \cos^2 x &= \frac{1}{2} + \frac{1}{2} \cos 2x \\
 \therefore \cos^2 \left( \frac{x}{2} \right) &= \frac{1}{2} + \frac{1}{2} \cos x \\
 \therefore \int_0^{\frac{\pi}{3}} \cos^2 \left( \frac{x}{2} \right) dx &= \int_0^{\frac{\pi}{3}} \left( \frac{1}{2} + \frac{1}{2} \cos x \right) dx \\
 &= \left[ \frac{1}{2} x + \frac{1}{2} \sin x \right]_0^{\frac{\pi}{3}} \\
 &= \left( \frac{1}{2} \left( \frac{\pi}{3} \right) + \frac{1}{2} \sin \frac{\pi}{3} \right) - \left( \frac{1}{2} (0) + \frac{1}{2} \sin 0 \right) \\
 &= \left( \frac{\pi}{6} + \frac{1}{2} \times \frac{\sqrt{3}}{2} \right) - 0 \\
 &= \frac{\pi}{6} + \frac{\sqrt{3}}{4}
 \end{aligned}$$

$$\begin{aligned}
 \text{e } \int_0^{\frac{\pi}{4}} \tan x dx &= \left[ -\ln |\cos x| \right]_0^{\frac{\pi}{4}} \\
 &= -\ln \left| \cos \frac{\pi}{4} \right| - (-\ln |\cos 0|) \\
 &= -\ln \frac{1}{\sqrt{2}} + \ln 1 \\
 &= \ln \left( (2^{-\frac{1}{2}})^{-1} \right) \\
 &= \ln(2^{\frac{1}{2}}) \\
 &= \frac{1}{2} \ln 2
 \end{aligned}$$

$$\begin{aligned} \text{f } (e^x + 2)^3 &= (e^x)^3 + 3(e^x)^2(2) + 3(e^x)(2)^2 + 2^3 \\ &= e^{3x} + 6e^{2x} + 12e^x + 8 \end{aligned}$$

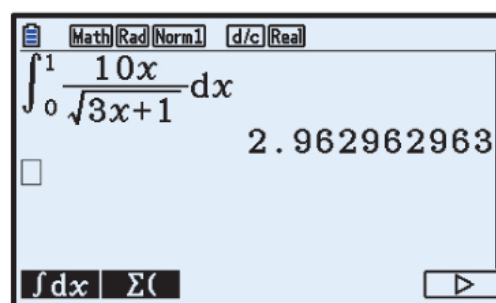
$$\begin{aligned} \therefore \int_0^1 (e^x + 2)^3 dx &= \int_0^1 (e^{3x} + 6e^{2x} + 12e^x + 8) dx \\ &= \left[ \frac{1}{3}e^{3x} + 3e^{2x} + 12e^x + 8x \right]_0^1 \\ &= \left( \frac{1}{3}e^3 + 3e^2 + 12e^1 + 8 \right) - \left( \frac{1}{3}e^0 + 3e^0 + 12e^0 + 0 \right) \\ &= \frac{1}{3}e^3 + 3e^2 + 12e + 8 - \left( \frac{1}{3} + 3 + 12 \right) \\ &= \frac{1}{3}e^3 + 3e^2 + 12e + 8 - \frac{46}{3} \\ &= \frac{1}{3}e^3 + 3e^2 + 12e - \frac{22}{3} \end{aligned}$$

4 a



$$\therefore \int_{-2}^0 4e^{-x^2} dx \approx 3.528$$

b



$$\therefore \int_0^1 \frac{10x}{\sqrt{3x+1}} dx \approx 2.963$$

5

$$\begin{aligned} \text{a } (\sin \theta - \cos \theta)^2 &= \sin^2 \theta - 2 \sin \theta \cos \theta + \cos^2 \theta \\ &= 1 - \sin 2\theta \quad \{ \sin^2 \theta + \cos^2 \theta = 1, \quad \sin 2\theta = 2 \sin \theta \cos \theta \} \end{aligned}$$

$$\begin{aligned} \text{b } \int_0^{\frac{\pi}{4}} (\sin \theta - \cos \theta)^2 d\theta &= \int_0^{\frac{\pi}{4}} (1 - \sin 2\theta) d\theta \\ &= \left[ \theta + \frac{1}{2} \cos 2\theta \right]_0^{\frac{\pi}{4}} \\ &= \left( \frac{\pi}{4} + \frac{1}{2} \cos \frac{\pi}{2} \right) - \left( 0 + \frac{1}{2} \cos 0 \right) \\ &= \frac{\pi}{4} - \frac{1}{2} \end{aligned}$$

6 a

$$\begin{aligned} &\frac{4}{x-2} + \frac{3}{x+1} \\ &= \frac{4(x+1)}{(x-2)(x+1)} + \frac{3(x-2)}{(x-2)(x+1)} \\ &= \frac{4x+4+3x-6}{(x-2)(x+1)} \\ &= \frac{7x-2}{(x-2)(x+1)} \end{aligned}$$

b

$$\begin{aligned} &\int_3^4 \frac{7x-2}{x^2-x-2} dx \\ &= \int_3^4 \frac{7x-2}{(x-2)(x+1)} dx \\ &= \int_3^4 \left( \frac{4}{x-2} + \frac{3}{x+1} \right) dx \\ &= [4 \ln |x-2| + 3 \ln |x+1|]_3^4 \\ &= (4 \ln 2 + 3 \ln 5) - (4 \ln 1 + 3 \ln 4) \\ &= 4 \ln 2 + 3 \ln 5 - 6 \ln 2 \\ &= 3 \ln 5 - 2 \ln 2 \end{aligned}$$

**7 a** Let  $u = x^3 + 2 \quad \therefore \frac{du}{dx} = 3x^2$

When  $x = 0, \quad u = 2$

When  $x = 1, \quad u = 3$

$$\begin{aligned} \therefore \int_0^1 \frac{4x^2}{(x^3 + 2)^3} dx &= \int_0^1 \frac{1}{u^3} \left( \frac{4}{3} \frac{du}{dx} \right) dx \\ &= \frac{4}{3} \int_2^3 u^{-3} du \\ &= \frac{4}{3} \left[ \frac{u^{-2}}{-2} \right]_2^3 \\ &= -\frac{2}{3} \left[ \frac{1}{u^2} \right]_2^3 \\ &= -\frac{2}{3} \left( \frac{1}{9} - \frac{1}{4} \right) \\ &= \frac{5}{54} \end{aligned}$$

**b** Let  $u = \sin x \quad \therefore \frac{du}{dx} = \cos x$

When  $x = \frac{\pi}{4}, \quad u = \frac{1}{\sqrt{2}}$

When  $x = \frac{\pi}{3}, \quad u = \frac{\sqrt{3}}{2}$

$$\begin{aligned} \therefore \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sin^5 x \cos x dx &= \int_{\frac{1}{\sqrt{2}}}^{\frac{\sqrt{3}}{2}} u^5 \frac{du}{dx} dx \\ &= \int_{\frac{1}{\sqrt{2}}}^{\frac{\sqrt{3}}{2}} u^5 du \\ &= \left[ \frac{u^6}{6} \right]_{\frac{1}{\sqrt{2}}}^{\frac{\sqrt{3}}{2}} \\ &= \frac{1}{6} \left( \left( \frac{\sqrt{3}}{2} \right)^6 - \left( \frac{1}{\sqrt{2}} \right)^6 \right) \\ &= \frac{19}{384} \end{aligned}$$

**c** Let  $u = x^3 - 1 \quad \therefore \frac{du}{dx} = 3x^2$

When  $x = 1, \quad u = 0$

When  $x = 2, \quad u = 7$

$$\begin{aligned} \therefore \int_1^2 3x^2 \sqrt{x^3 - 1} dx &= \int_0^7 u^{\frac{1}{2}} \frac{du}{dx} dx \\ &= \int_0^7 u^{\frac{1}{2}} du \\ &= \left[ \frac{2}{3} u^{\frac{3}{2}} \right]_0^7 \\ &= \frac{2}{3} (7\sqrt{7} - 0) \\ &= \frac{14\sqrt{7}}{3} \end{aligned}$$

**d** Let  $u = 1 + x^2 \quad \therefore \frac{du}{dx} = 2x$

When  $x = -1, \quad u = 2$

When  $x = 2, \quad u = 5$

$$\begin{aligned} \therefore \int_{-1}^2 -x(1 + x^2)^3 dx &= \int_2^5 u^3 \left( -\frac{1}{2} \frac{du}{dx} \right) dx \\ &= -\frac{1}{2} \int_2^5 u^3 du \\ &= -\frac{1}{2} \left[ \frac{1}{4} u^4 \right]_2^5 \\ &= -\frac{1}{2} \left( \frac{625}{4} - 4 \right) \\ &= -\frac{609}{8} \end{aligned}$$



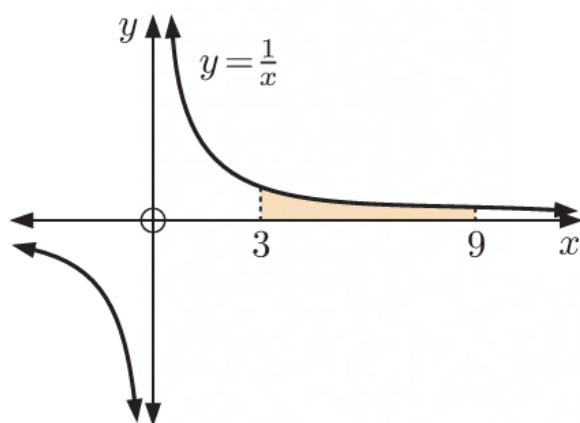
**e** Let  $u = 1 - x^3 \quad \therefore \frac{du}{dx} = -3x^2$

When  $x = 0$ ,  $u = 1$

When  $x = 1$ ,  $u = 0$

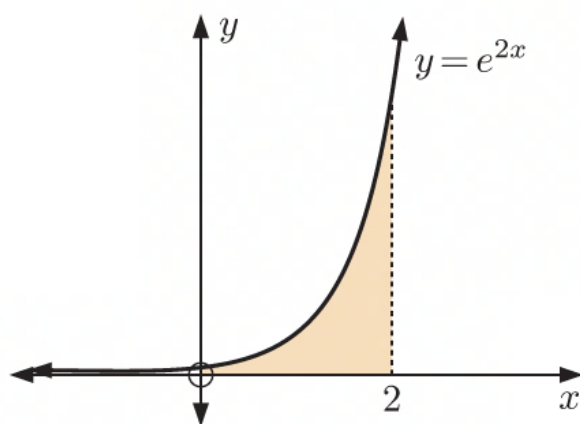
$$\begin{aligned} \therefore \int_0^1 x^2 e^{1-x^3} dx &= \int_0^1 e^u \left( -\frac{1}{3} \frac{du}{dx} \right) dx \\ &= -\frac{1}{3} \int_1^0 e^u du \\ &= \frac{1}{3} \int_0^1 e^u du \\ &= \frac{1}{3} [e^u]_0^1 \\ &= \frac{1}{3}(e - 1) \\ &= \frac{e - 1}{3} \end{aligned}$$

**8 a**



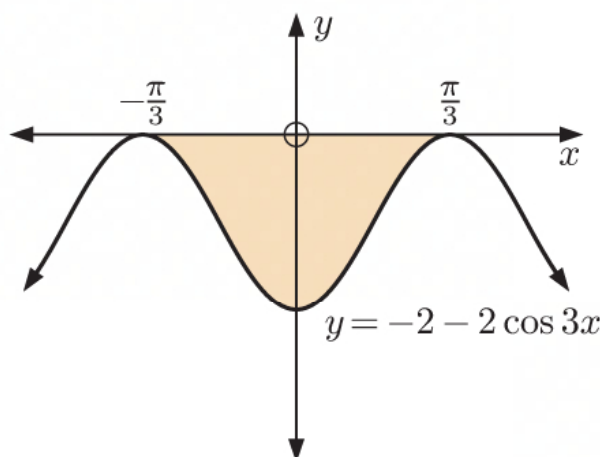
$$\begin{aligned} \text{Area} &= \int_3^9 \frac{1}{x} dx \\ &= [\ln |x|]_3^9 \\ &= \ln 9 - \ln 3 \\ &= \ln\left(\frac{9}{3}\right) \\ &= \ln 3 \text{ units}^2 \end{aligned}$$

**b**



$$\begin{aligned} \text{Area} &= \int_0^2 e^{2x} dx \\ &= \left[ \frac{1}{2} e^{2x} \right]_0^2 \\ &= \frac{1}{2} e^4 - \frac{1}{2} \\ &= \frac{e^4 - 1}{2} \text{ units}^2 \end{aligned}$$

**c**



$$\begin{aligned} \text{Area} &= - \int_{-\pi/3}^{\pi/3} (-2 - 2 \cos 3x) dx \\ &= - \left[ -2x - \frac{2}{3} \sin 3x \right]_{-\pi/3}^{\pi/3} \\ &= - \left[ \left( -\frac{2\pi}{3} - \frac{2}{3} \sin \pi \right) - \left( \frac{2\pi}{3} - \frac{2}{3} \sin(-\pi) \right) \right] \\ &= - \left( -\frac{2\pi}{3} - \frac{2\pi}{3} \right) \\ &= \frac{4\pi}{3} \text{ units}^2 \end{aligned}$$

9  $y = x^2 - 5x + 5$  meets  $y = 2x - 5$

where  $x^2 - 5x + 5 = 2x - 5$

$$\therefore x^2 - 7x + 10 = 0$$

$$\therefore (x - 2)(x - 5) = 0$$

$$\therefore x = 2 \text{ or } 5$$

$$\text{Area} = \int_2^5 [(2x - 5) - (x^2 - 5x + 5)] dx$$

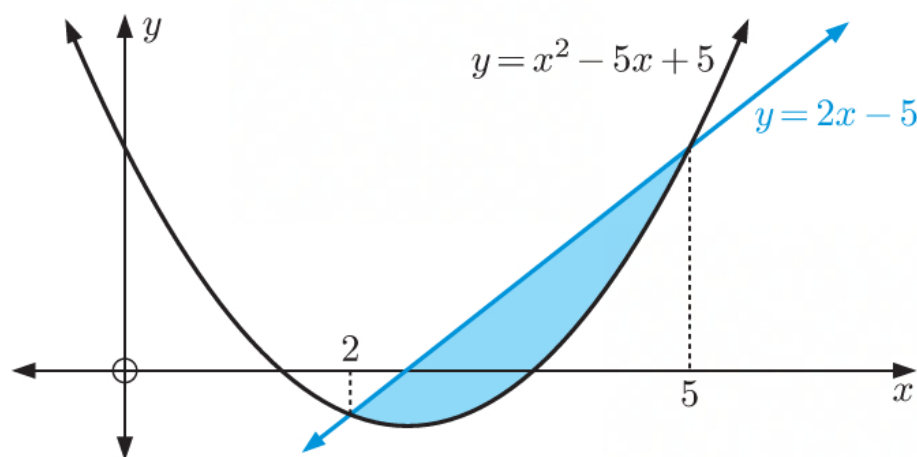
$$= \int_2^5 (-x^2 + 7x - 10) dx$$

$$= \left[ -\frac{1}{3}x^3 + \frac{7}{2}x^2 - 10x \right]_2^5$$

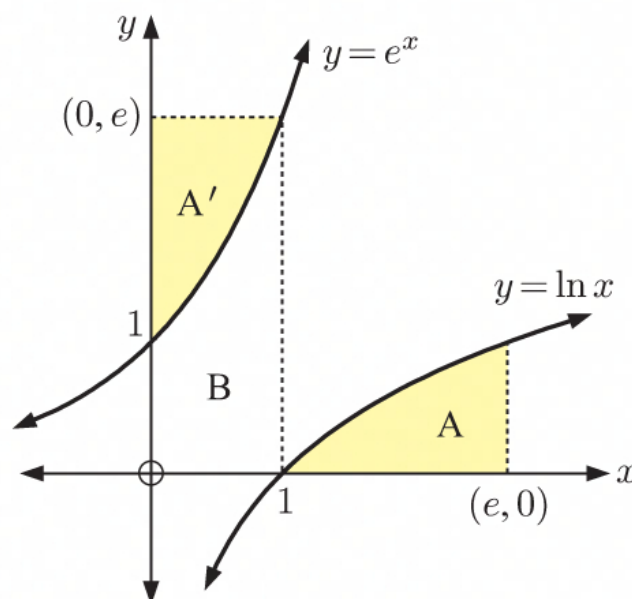
$$= \left( -\frac{125}{3} + \frac{175}{2} - 50 \right) - \left( -\frac{8}{3} + 14 - 20 \right)$$

$$= -\frac{25}{6} - \left( -\frac{26}{3} \right)$$

$$= 4\frac{1}{2} \text{ units}^2$$



10



$y = e^x$  and  $y = \ln x$  are inverse functions, so they are symmetric about  $y = x$

$$\therefore \text{area A} = \text{area A'}$$

But  $\text{area A'} + \text{area B} = \text{area of rectangle}$

$$\therefore \text{area A} + \text{area B} = e \times 1 = e$$

$$\text{Since } \text{area A} = \int_1^e \ln x dx$$

$$\text{and } \text{area B} = \int_0^1 e^x dx,$$

$$\int_0^1 e^x dx + \int_1^e \ln x dx = e$$

11 a The curve cuts the  $x$ -axis when  $y = 0$

$$\therefore 4e^x - 1 = 0$$

$$\therefore 4e^x = 1$$

$$\therefore e^x = \frac{1}{4}$$

$$\therefore x = \ln\left(\frac{1}{4}\right)$$

$$= \ln(4^{-1})$$

$$= -\ln 4$$

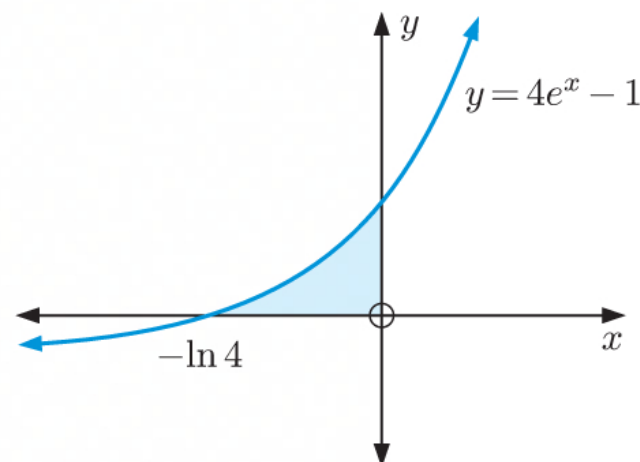
$$\text{Area} = \int_{-\ln 4}^0 (4e^x - 1) dx$$

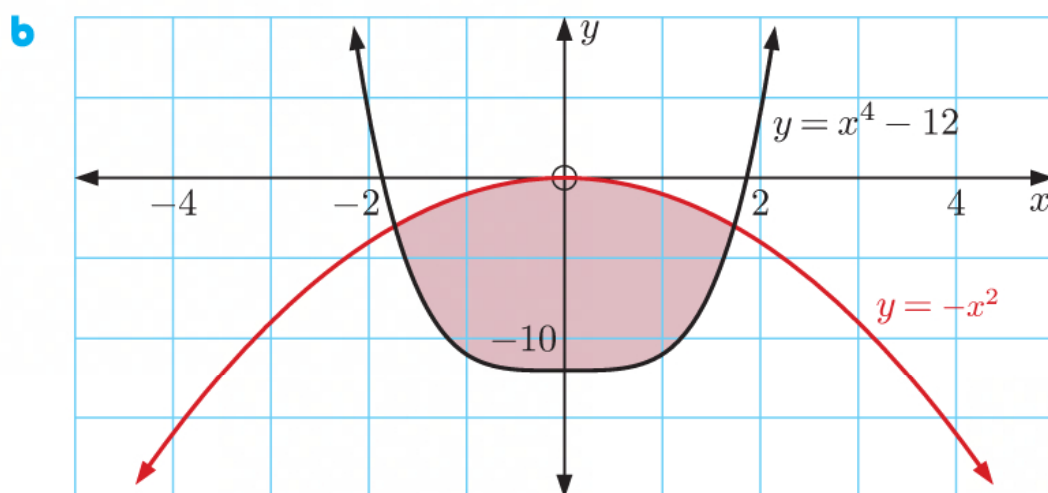
$$= [4e^x - x]_{-\ln 4}^0$$

$$= 4 - (4e^{-\ln 4} + \ln 4)$$

$$= 4 - (1 + \ln 4)$$

$$= (3 - \ln 4) \text{ units}^2$$





The graphs meet where  $x^4 - 12 = -x^2$

$$\therefore x^4 + x^2 - 12 = 0$$

$$\therefore (x^2 + 4)(x^2 - 3) = 0$$

$$\therefore x^2 - 3 = 0 \quad \{x^2 + 4 > 0 \text{ for all } x\}$$

$$\therefore x^2 = 3$$

$$\therefore x = \pm\sqrt{3}$$

$$\begin{aligned} \text{When } x = \sqrt{3}, \quad y &= -(\sqrt{3})^2 \\ &= -3 \end{aligned}$$

$$\begin{aligned} \text{When } x = -\sqrt{3}, \quad y &= -(-\sqrt{3})^2 \\ &= -3 \end{aligned}$$

$\therefore$  the graphs meet at  $(-\sqrt{3}, -3)$  and  $(\sqrt{3}, -3)$ .

$$\begin{aligned} \text{Area} &= \int_{-\sqrt{3}}^{\sqrt{3}} (-x^2 - (x^4 - 12)) \, dx \\ &= \int_{-\sqrt{3}}^{\sqrt{3}} (-x^4 - x^2 + 12) \, dx \\ &= \left[ -\frac{1}{5}x^5 - \frac{1}{3}x^3 + 12x \right]_{-\sqrt{3}}^{\sqrt{3}} \\ &= \left( -\frac{1}{5}(\sqrt{3})^5 - \frac{1}{3}(\sqrt{3})^3 + 12\sqrt{3} \right) - \left( -\frac{1}{5}(-\sqrt{3})^5 - \frac{1}{3}(-\sqrt{3})^3 - 12\sqrt{3} \right) \\ &= -\frac{9}{5}\sqrt{3} - \sqrt{3} + 12\sqrt{3} - \left( \frac{9}{5}\sqrt{3} + \sqrt{3} - 12\sqrt{3} \right) \\ &= -\frac{9}{5}\sqrt{3} - \sqrt{3} + 12\sqrt{3} - \frac{9}{5}\sqrt{3} - \sqrt{3} + 12\sqrt{3} \\ &= \frac{92\sqrt{3}}{5} \text{ units}^2 \end{aligned}$$

**12** B has coordinates  $(2, 2^2 + k)$ , or  $(2, 4 + k)$ .

$\therefore$  the horizontal line from A to B is  $y = 4 + k$ .

Now, upper area  $U$  = lower area  $L$

$$\therefore \int_0^2 [(4 + k) - (x^2 + k)] dx = \int_0^2 (x^2 + k) dx$$

$$\therefore \int_0^2 (-x^2 + 4) dx = \int_0^2 (x^2 + k) dx$$

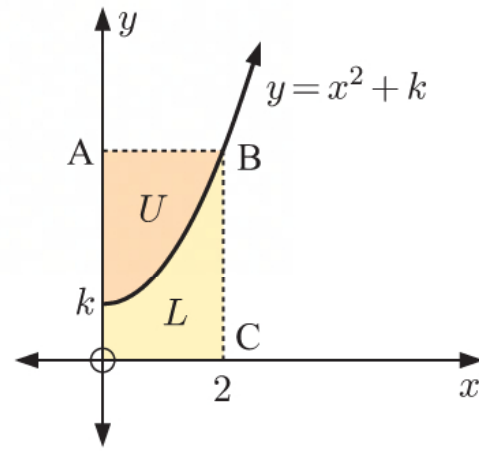
$$\therefore \left[-\frac{1}{3}x^3 + 4x\right]_0^2 = \left[\frac{1}{3}x^3 + kx\right]_0^2$$

$$\therefore \left(-\frac{8}{3} + 8\right) - 0 = \left(\frac{8}{3} + 2k\right) - 0$$

$$\therefore \frac{16}{3} = \frac{8}{3} + 2k$$

$$\therefore \frac{8}{3} = 2k$$

$$\therefore k = \frac{4}{3}$$



**13** Area =  $\int_0^m \sin x dx = \frac{1}{2}$

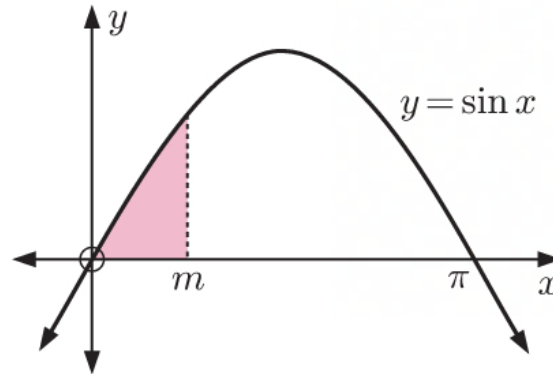
$$\therefore [-\cos x]_0^m = \frac{1}{2}$$

$$\therefore -\cos m - (-1) = \frac{1}{2}$$

$$\therefore -\cos m = -\frac{1}{2}$$

$$\therefore \cos m = \frac{1}{2}$$

$$\therefore m = \frac{\pi}{3}$$



**14 a** Area =  $\int_0^2 ax(x - 2) dx = 4$

$$\therefore \int_0^2 (ax^2 - 2ax) dx = 4$$

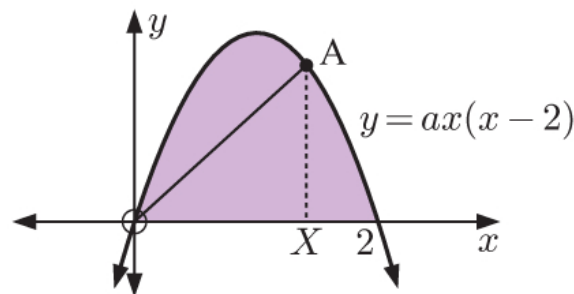
$$\therefore \left[\frac{1}{3}ax^3 - ax^2\right]_0^2 = 4$$

$$\therefore \left(\frac{8}{3}a - 4a\right) - 0 = 4$$

$$\therefore \frac{8}{3}a - 4a = 4$$

$$\therefore -\frac{4}{3}a = 4$$

$$\therefore a = -3$$



**b** Let A have  $x$ -coordinate  $X$ , then A has  $y$ -coordinate  $-3X(X - 2)$ .

If [OA] divides the shaded region into equal parts, then each region has area 2 units<sup>2</sup>.

We consider the area of the region bounded by [OA],  $y = -3x(x - 2)$ , and the  $x$ -axis.

This is the area between [OA] and the  $x$ -axis from O to  $X$ , and the area between  $y = -3x(x - 2)$  and the  $x$ -axis from  $X$  to 2.



$$\text{So, } \frac{1}{2}(X)(-3X(X-2)) + \int_X^2 -3x(x-2) dx = 2$$

$$\therefore -\frac{3}{2}X^2(X-2) + \int_X^2 (-3x^2 + 6x) dx = 2$$

$$\therefore -\frac{3}{2}X^3 + 3X^2 + [-x^3 + 3x^2]_X^2 = 2$$

$$\therefore -\frac{3}{2}X^3 + 3X^2 + ((-8 + 12) - (-X^3 + 3X^2)) = 2$$

$$\therefore -\frac{3}{2}X^3 + 3X^2 + (4 + X^3 - 3X^2) = 2$$

$$\therefore -\frac{3}{2}X^3 + \cancel{3X^2} + 4 + X^3 - \cancel{3X^2} = 2$$

$$\therefore -\frac{1}{2}X^3 = -2$$

$$\therefore X^3 = 4$$

$$\therefore X = \sqrt[3]{4}$$

$\therefore$  A has  $x$ -coordinate  $\sqrt[3]{4}$ .

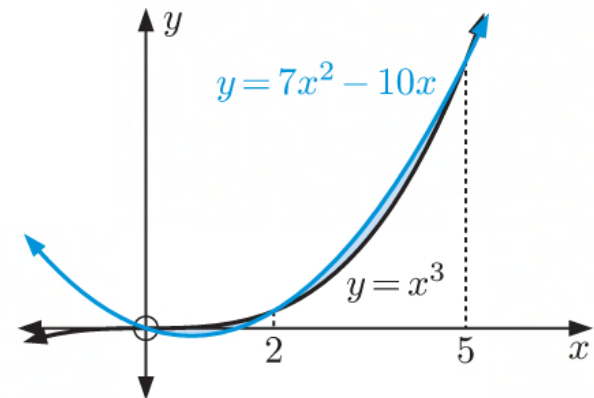
**15 a** The curves meet where  $x^3 = 7x^2 - 10x$

$$\therefore x^3 - 7x^2 + 10x = 0$$

$$\therefore x(x^2 - 7x + 10) = 0$$

$$\therefore x(x-2)(x-5) = 0$$

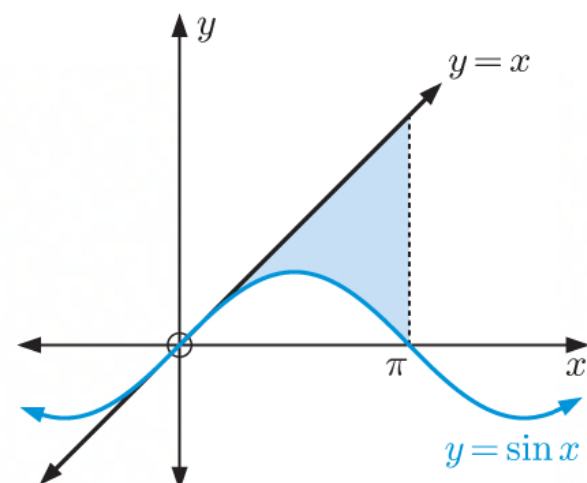
$$\therefore x = 0, 2, \text{ or } 5$$



$$\begin{aligned} \therefore \text{area} &= \int_0^2 (x^3 - (7x^2 - 10x)) dx + \int_2^5 (7x^2 - 10x - x^3) dx \\ &= \int_0^2 (x^3 - 7x^2 + 10x) dx + \int_2^5 (-x^3 + 7x^2 - 10x) dx \\ &= \left[ \frac{1}{4}x^4 - \frac{7}{3}x^3 + 5x^2 \right]_0^2 + \left[ -\frac{1}{4}x^4 + \frac{7}{3}x^3 - 5x^2 \right]_2^5 \\ &= \left( \left( 4 - \frac{56}{3} + 20 \right) - 0 \right) + \left( \left( -\frac{625}{4} + \frac{875}{3} - 125 \right) - \left( -4 + \frac{56}{3} - 20 \right) \right) \\ &= \frac{16}{3} + \frac{63}{4} \\ &= \frac{253}{12} \\ &= 21\frac{1}{12} \text{ units}^2 \end{aligned}$$

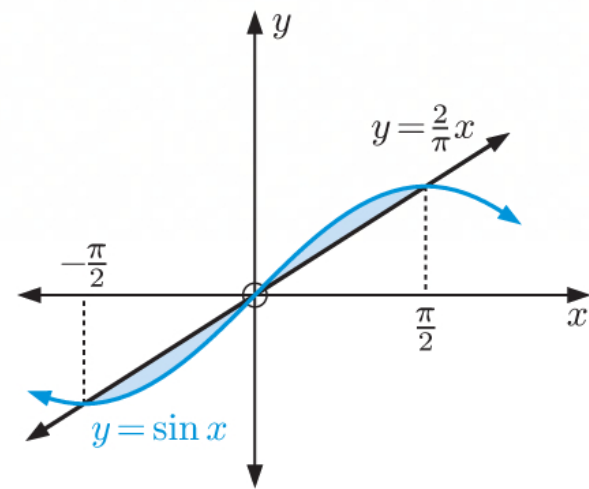
**b** Area =  $\int_0^\pi (x - \sin x) dx$

$$\begin{aligned} &= \left[ \frac{1}{2}x^2 + \cos x \right]_0^\pi \\ &= \left( \frac{1}{2}\pi^2 + \cos \pi \right) - (0 + \cos 0) \\ &= \frac{\pi^2}{2} - 1 - 1 \\ &= \left( \frac{\pi^2}{2} - 2 \right) \text{ units}^2 \end{aligned}$$



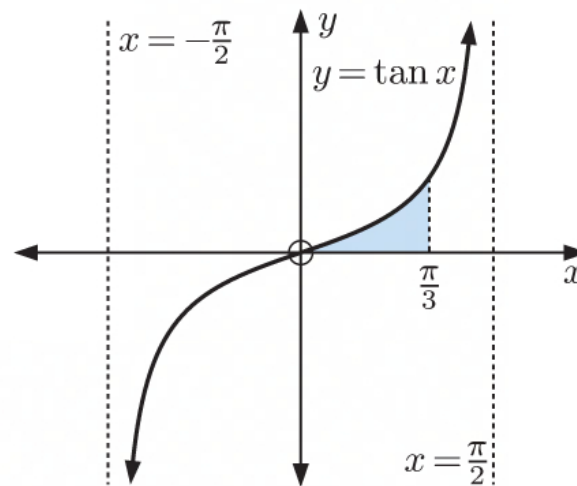
- c The curves meet where  $\frac{2}{\pi}x = \sin x$   
 $\therefore x = -\frac{\pi}{2}, 0, \text{ or } \frac{\pi}{2}$

$$\begin{aligned}\therefore \text{area} &= \int_{-\frac{\pi}{2}}^0 \left( \frac{2}{\pi}x - \sin x \right) dx + \int_0^{\frac{\pi}{2}} \left( \sin x - \frac{2}{\pi}x \right) dx \\ &= 2 \int_0^{\frac{\pi}{2}} \left( \sin x - \frac{2}{\pi}x \right) dx \quad \{\text{symmetry}\} \\ &= 2 \left[ -\cos x - \frac{1}{\pi}x^2 \right]_0^{\frac{\pi}{2}} \\ &= 2 \left[ \left( -\cos \frac{\pi}{2} - \frac{1}{\pi} \left( \frac{\pi}{2} \right)^2 \right) - \left( -\cos 0 - 0 \right) \right] \\ &= 2 \left( -\frac{\pi}{4} + 1 \right) \\ &= \left( 2 - \frac{\pi}{2} \right) \text{ units}^2\end{aligned}$$

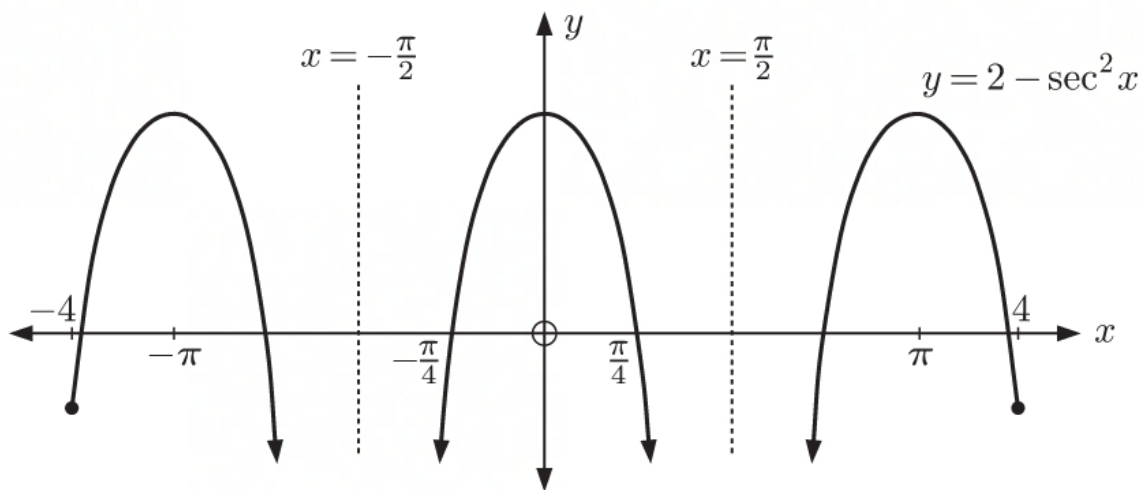


d Area =  $\int_0^{\frac{\pi}{3}} \tan x \, dx$

$$\begin{aligned}&= \left[ -\ln |\cos x| \right]_0^{\frac{\pi}{3}} \\ &= -\ln \left| \cos \frac{\pi}{3} \right| - (-\ln |\cos 0|) \\ &= -\ln \left( \frac{1}{2} \right) + \ln 1 \\ &= \ln((2^{-1})^{-1}) \\ &= \ln 2 \text{ units}^2\end{aligned}$$



16 a



b  $f(x) = 2 - \sec^2 x, \quad -4 \leq x \leq 4$

$$= 2 - \frac{1}{\cos^2 x}$$

which is undefined when  $\cos^2 x = 0$

$$\therefore \cos x = 0$$

$$\therefore x = -\frac{\pi}{2} \text{ or } \frac{\pi}{2} \quad \{-4 \leq x \leq 4\}$$

$\therefore$  the vertical asymptotes of  $f(x)$  are  $x = -\frac{\pi}{2}$  and  $x = \frac{\pi}{2}$ .

c When  $x = 0$ ,  $y = 2 - \sec^2 0$

$$= 2 - \frac{1}{\cos^2 0}$$

$$= 2 - 1$$

$$= 1$$

$\therefore$  the  $y$ -intercept is 1.

When  $y = 0$ ,  $2 - \sec^2 x = 0$

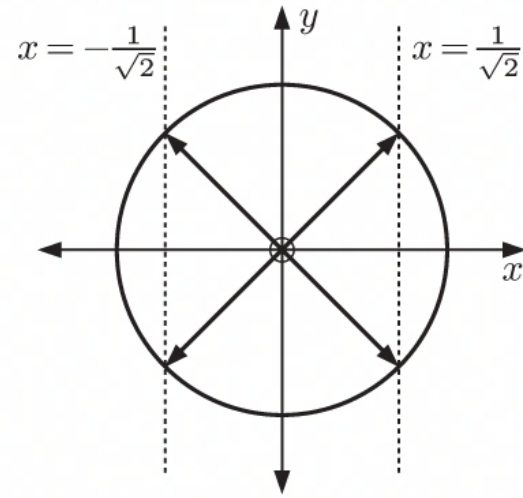
$$\therefore \sec^2 x = 2$$

$$\therefore \cos^2 x = \frac{1}{2}$$

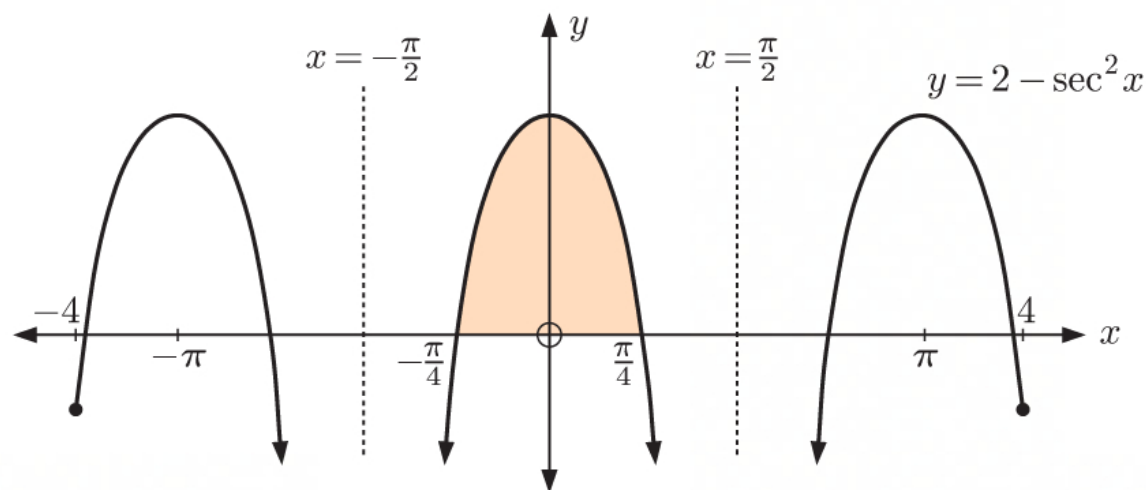
$$\therefore \cos x = \pm \frac{1}{\sqrt{2}}$$

$$\therefore x = -\frac{5\pi}{4}, -\frac{3\pi}{4}, -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}, \text{ or } \frac{5\pi}{4} \quad \{-4 \leq x \leq 4\}$$

$\therefore$  the  $x$ -intercepts are  $-\frac{5\pi}{4}$ ,  $-\frac{3\pi}{4}$ ,  $-\frac{\pi}{4}$ ,  $\frac{\pi}{4}$ ,  $\frac{3\pi}{4}$ , and  $\frac{5\pi}{4}$ .



d



Area bounded by one arch  $= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (2 - \sec^2 x) dx$

$$= \left[ 2x - \tan x \right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$= \left( \frac{\pi}{2} - \tan \frac{\pi}{4} \right) - \left( -\frac{\pi}{2} - \tan \left( -\frac{\pi}{4} \right) \right)$$

$$= \frac{\pi}{2} - 1 + \frac{\pi}{2} - 1$$

$$= (\pi - 2) \text{ units}$$

**17 a**

$$x = \ln\left(\frac{y+3}{2}\right)$$

$$\therefore f^{-1}(y) = \ln\left(\frac{y+3}{2}\right)$$

$$\text{Now } \int \ln\left(\frac{y+3}{2}\right) dy$$

$$= \int \ln \frac{u}{2} du \quad \left\{ u = y + 3, \frac{du}{dy} = 1 \right\}$$

$$= u \ln \frac{u}{2} - \int 1 du \quad \leftarrow \begin{cases} a = \ln \frac{u}{2} & b' = 1 \\ a' = \frac{1}{u} & b = u \end{cases}$$

$$= u \ln \frac{u}{2} - u + c$$

$$= (y+3) \ln \frac{y+3}{2} - (y+3)$$

$$\therefore \text{area} = \int_0^3 \ln\left(\frac{y+3}{2}\right) dy$$

$$= \left[ (y+3) \ln\left(\frac{y+3}{2}\right) - (y+3) \right]_0^3$$

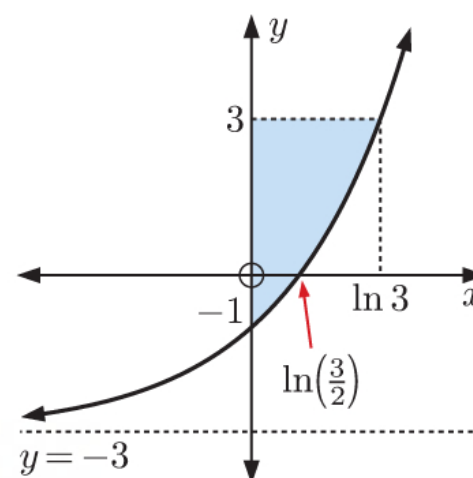
$$= (6 \ln 3 - 6) - (3 \ln\left(\frac{3}{2}\right) - 3)$$

$$= 6 \ln 3 - 6 - 3(\ln 3 - \ln 2) + 3$$

$$= 6 \ln 3 - 3 \ln 3 + 3 \ln 2 - 3$$

$$= 3 \ln 3 + 3 \ln 2 - 3$$

$$= 3(\ln 3 + \ln 2 - 1) \text{ units}^2$$

**b**

$$y = x^3 + 2$$

$$\therefore x^3 = y - 2$$

$$\therefore x = \sqrt[3]{y-2}$$

$$\therefore f^{-1}(y) = \sqrt[3]{y-2}$$

$$\therefore \text{area} = \int_3^6 (y-2)^{\frac{1}{3}} dy$$

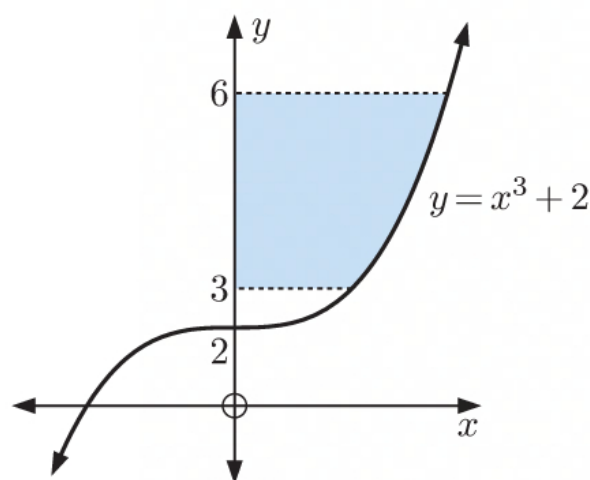
$$= \left[ \frac{(y-2)^{\frac{4}{3}}}{\frac{4}{3}} \right]_3^6$$

$$= \left[ \frac{3}{4}(y-2)^{\frac{4}{3}} \right]_3^6$$

$$= \frac{3}{4}(4)^{\frac{4}{3}} - \frac{3}{4}(1)^{\frac{4}{3}}$$

$$= \frac{3}{4} \times 4 \times \sqrt[3]{4} - \frac{3}{4}$$

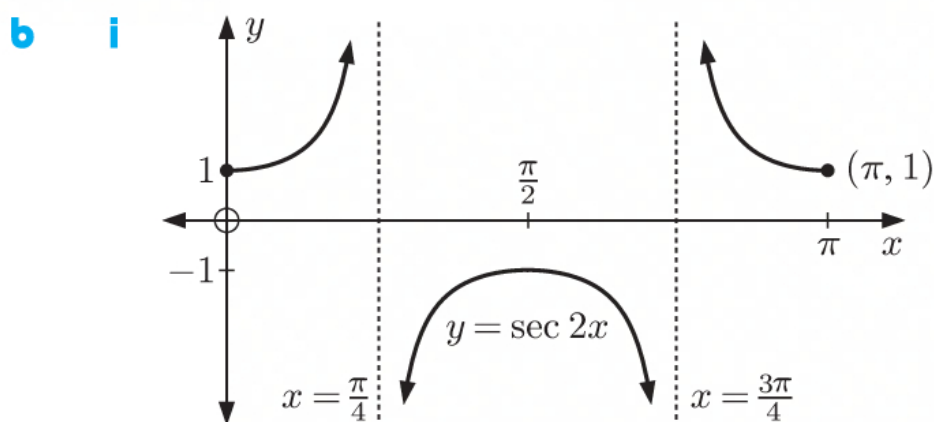
$$= \left( 3\sqrt[3]{4} - \frac{3}{4} \right) \text{ units}^2$$





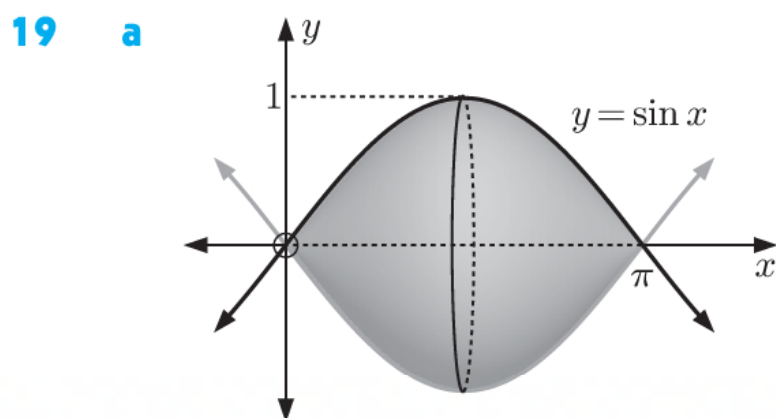
$$\begin{aligned}
 18 \quad a \quad \frac{d}{dx} [\ln(\tan x + \sec x)] &= \frac{\frac{d}{dx}(\tan x + \sec x)}{\tan x + \sec x} \\
 &= \frac{\sec^2 x + \sec x \tan x}{\tan x + \sec x} \\
 &= \frac{\sec x(\sec x + \tan x)}{\cancel{\tan x + \sec x}} \\
 &= \sec x
 \end{aligned}$$

$$\therefore \int \sec x \, dx = \ln |\tan x + \sec x| + c$$

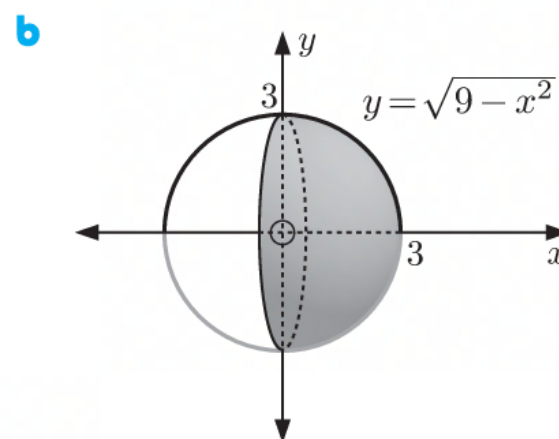


**ii**  $y = \sec 2x$  and  $y = 3$  meet when  $\sec 2x = 3$   
 $\therefore \cos 2x = \frac{1}{3}$   
 $\therefore x \approx 0.615$  {using technology}

$$\begin{aligned}
 \therefore \text{total area} &\approx (\text{rectangle area}) - \int_0^{0.615} \sec 2x \, dx \\
 &\approx 0.615 \times 3 - \left[ \frac{1}{2} \ln |\tan 2x + \sec 2x| \right]_0^{0.615} \\
 &\approx 1.846 - \left[ \frac{1}{2} \ln |\tan(1.23) + \sec(1.23)| - \frac{1}{2} \ln |0 + 1| \right] \\
 &\approx 0.965 \text{ units}^2
 \end{aligned}$$



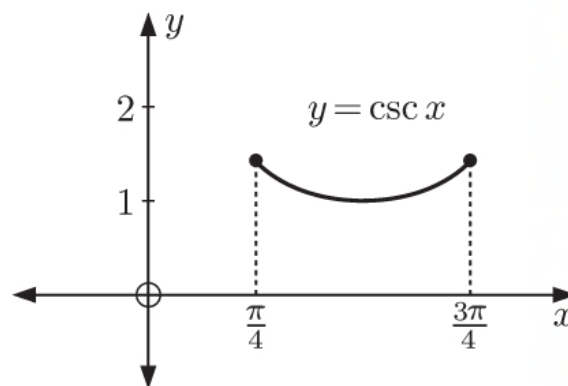
$$\begin{aligned}
 \text{Volume} &= \pi \int_0^\pi \sin^2 x \, dx \\
 &= \pi \int_0^\pi \left( \frac{1}{2} - \frac{1}{2} \cos 2x \right) dx \\
 &= \pi \left[ \frac{1}{2}x - \frac{1}{2} \left( \frac{1}{2} \right) \sin 2x \right]_0^\pi \\
 &= \pi \left( \frac{1}{2}\pi - \frac{1}{4} \sin 2\pi - 0 \right) \\
 &= \frac{\pi^2}{2} \text{ units}^3
 \end{aligned}$$



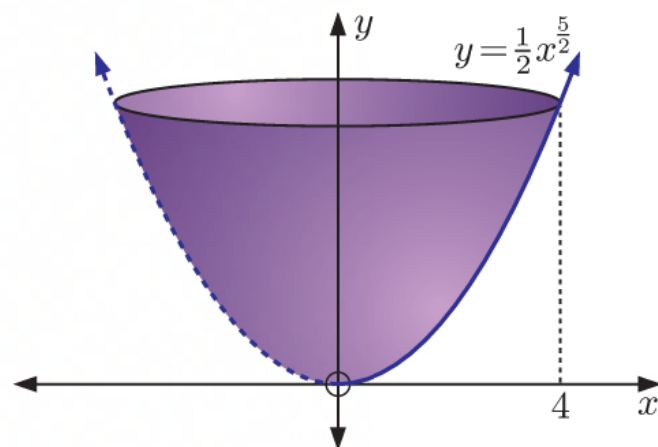
$$\begin{aligned}
 \text{Volume} &= \pi \int_0^3 (9 - x^2) \, dx \\
 &= \pi \left[ 9x - \frac{x^3}{3} \right]_0^3 \\
 &= \pi \left( 27 - \frac{27}{3} - 0 \right) \\
 &= 18\pi \text{ units}^3
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{20} \quad \mathbf{a} \quad \frac{d}{dx}(\cot x) &= \frac{d}{dx} \left( \frac{\cos x}{\sin x} \right) \\
 &= \frac{-\sin x \sin x - \cos x \cos x}{\sin^2 x} \quad \{\text{quotient rule}\} \\
 &= \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x} \\
 &= \frac{-1}{\sin^2 x} \\
 &= -\operatorname{cosec}^2 x
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad \text{Volume} &= \pi \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} y^2 dx \\
 &= \pi \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \operatorname{cosec}^2 x dx \\
 &= \pi \left[ -\cot x \right]_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \\
 &= \pi \left( -\cot \frac{3\pi}{4} + \cot \frac{\pi}{4} \right) \\
 &= \pi(-(-1) + 1) \\
 &= 2\pi \text{ units}^3
 \end{aligned}$$



$$\mathbf{21} \quad \mathbf{a} \quad \text{The height of the vase is } y = \frac{1}{2}(4)^{\frac{5}{2}} = 16 \text{ cm.}$$



$$\begin{aligned}
 \mathbf{b} \quad y &= \frac{1}{2}x^{\frac{5}{2}} \quad \therefore x^2 = (2y)^{\frac{4}{5}} \\
 \text{Volume} &= \pi \int_0^{16} x^2 dy \\
 &= \pi \int_0^{16} (2y)^{\frac{4}{5}} dy \\
 &= 2^{\frac{4}{5}} \pi \left[ \frac{5}{9} y^{\frac{9}{5}} \right]_0^{16} \\
 &= 2^{\frac{4}{5}} \pi \left( \frac{5}{9} \times 2^{\frac{36}{5}} - 0 \right) \\
 &= 2^8 \pi \times \frac{5}{9} \\
 &= \frac{1280\pi}{9} \text{ cm}^3
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad \text{Volume} &= \pi \int_0^{10} (2y)^{\frac{4}{5}} dy \\
 &= 2^{\frac{4}{5}} \pi \left[ \frac{5}{9} y^{\frac{9}{5}} \right]_0^{10} \\
 &= 2^{\frac{4}{5}} \pi \left( \frac{5}{9} \times 10^{\frac{9}{5}} - 0 \right) \\
 &\approx 192
 \end{aligned}$$

$\therefore$  the vase contains about 192 mL of water.

**22**  $E(t) = 2 \sin\left(\frac{t-5}{5}\right) + \frac{1}{2} \sin\left(\frac{t-5}{4}\right) \text{ kW}$

$$\begin{aligned} \therefore \int E(t) dt &= \int \left(2 \sin\left(\frac{t-5}{5}\right) + \frac{1}{2} \sin\left(\frac{t-5}{4}\right)\right) dt \\ &= 2\left(-\cos\left(\frac{t-5}{5}\right)(5)\right) + \frac{1}{2}\left(-\cos\left(\frac{t-5}{4}\right)(4)\right) + c \\ &= -10 \cos\left(\frac{t-5}{5}\right) - 2 \cos\left(\frac{t-5}{4}\right) + c \end{aligned}$$

**a** 
$$\begin{aligned} \int_5^{12} E(t) dt &= \left[-10 \cos\left(\frac{t-5}{5}\right) - 2 \cos\left(\frac{t-5}{4}\right)\right]_5^{12} \\ &= \left(-10 \cos \frac{7}{5} - 2 \cos \frac{7}{4}\right) - (-10 \cos 0 - 2 \cos 0) \\ &\approx 10.66 \end{aligned}$$

The solar energy transferred into Callum's solar panels from 5 am to 12 pm is about 10.7 kWh.

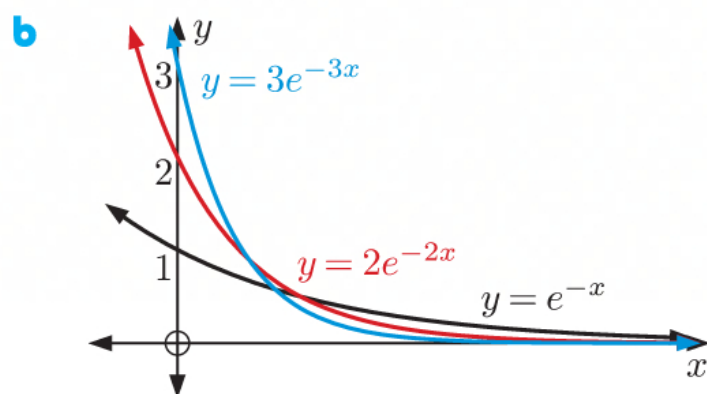
**b** 
$$\begin{aligned} \int_{12}^{20} E(t) dt &= \left[-10 \cos\left(\frac{t-5}{5}\right) - 2 \cos\left(\frac{t-5}{4}\right)\right]_{12}^{20} \\ &= \left(-10 \cos 3 - 2 \cos \frac{15}{4}\right) - \left(-10 \cos \frac{7}{5} - 2 \cos \frac{7}{4}\right) \\ &\approx 12.88 \end{aligned}$$

The solar energy transferred into Callum's solar panels from 12 pm to 8 pm is about 12.9 kWh.

**c** 
$$\begin{aligned} \int_5^{20} E(t) dt &= \int_5^{12} E(t) dt + \int_{12}^{20} E(t) dt \\ &\approx 10.66 + 12.88 \quad \{\text{using a and b}\} \\ &\approx 23.54 \end{aligned}$$

The solar energy transferred into Callum's solar panels from 5 am to 8 pm is about 23.5 kWh.

**23 a** 
$$\begin{aligned} &\int_0^{\infty} a e^{-ax} dx, \quad a > 0 \\ &= \lim_{b \rightarrow \infty} \int_0^b a e^{-ax} dx \\ &= \lim_{b \rightarrow \infty} [-e^{-ax}]_0^b \\ &= \lim_{b \rightarrow \infty} (-e^{-ab} - (-1)) \\ &= \lim_{b \rightarrow \infty} \left(1 - \frac{1}{e^{ab}}\right) \\ &= 1 \quad \text{for all } a > 0 \end{aligned}$$



The area between the curves and the  $x$ -axis appears to remain constant.

**24 a**  $P_n$  is:  $\lim_{x \rightarrow \infty} x^n e^{-x} = 0$  for all  $n \in \mathbb{Z}^+$

**Proof:** (By the principle of mathematical induction)

$$\begin{aligned} (1) \quad \text{If } n = 1, \quad \lim_{x \rightarrow \infty} x e^{-x} &= \lim_{x \rightarrow \infty} \frac{x}{e^x} \\ &= \lim_{x \rightarrow \infty} \frac{1}{e^x} \quad \{\text{l'Hôpital's rule}\} \end{aligned}$$

$\therefore P_1$  is true.

$$(2) \quad \text{If } P_k \text{ is true, then } \lim_{x \rightarrow \infty} x^k e^{-x} = 0$$

$$\begin{aligned} \text{Now } \lim_{x \rightarrow \infty} x^{k+1} e^{-x} &= \lim_{x \rightarrow \infty} \frac{x^{k+1}}{e^x} \\ &= \lim_{x \rightarrow \infty} \frac{(k+1)x^k}{e^x} \quad \{\text{l'Hôpital's rule}\} \\ &= (k+1) \lim_{x \rightarrow \infty} x^k e^{-x} \\ &= (k+1) \times 0 \quad \{\text{using } P_k\} \\ &= 0 \end{aligned}$$

$\therefore P_{k+1}$  is also true.

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

**b i**  $\Gamma(1) = \int_0^\infty x^{1-1} e^{-x} dx$

$$\begin{aligned} &= \lim_{b \rightarrow \infty} \int_0^b e^{-x} dx \\ &= \lim_{b \rightarrow \infty} [-e^{-x}]_0^b \\ &= \lim_{b \rightarrow \infty} (-e^{-b} - (-1)) \\ &= \lim_{b \rightarrow \infty} \left(1 - \frac{1}{e^b}\right) \\ &= 1 \end{aligned}$$



$$\begin{aligned}
\text{ii} \quad \Gamma(n) &= \int_0^\infty x^{n-1} e^{-x} dx \\
&= \lim_{b \rightarrow \infty} \int_0^b x^{n-1} e^{-x} dx \\
&= \lim_{b \rightarrow \infty} \left( \left[ -x^{n-1} e^{-x} \right]_0^b - \int_0^b (n-1)x^{n-2}(-e^{-x}) dx \right) \\
&\quad \begin{array}{l} \nearrow \left\{ \begin{array}{ll} u = x^{n-1} & v' = e^{-x} \\ u' = (n-1)x^{n-2} & v = -e^{-x} \end{array} \right. \end{array} \\
&= \lim_{b \rightarrow \infty} \left( -b^{n-1} e^{-b} + 0 + (n-1) \int_0^b x^{n-2} e^{-x} dx \right) \\
&= -\lim_{b \rightarrow \infty} b^{n-1} e^{-b} + (n-1) \lim_{b \rightarrow \infty} \int_0^b x^{n-2} e^{-x} dx \\
&= 0 + (n-1) \int_0^\infty x^{(n-1)-1} e^{-x} dx \quad \{\text{using a}\} \\
&= (n-1)\Gamma(n-1)
\end{aligned}$$

$$\begin{aligned}
\text{iii} \quad \Gamma(n) &= (n-1)\Gamma(n-1) && \{\text{from ii}\} \\
&= (n-1)(n-2)\Gamma(n-2) && \{\text{using ii again}\} \\
&\quad \vdots \\
&= (n-1)(n-2)(n-3)\dots \times 2 \times 1 \times \Gamma(1) \\
&= (n-1)!\Gamma(1) \\
&= (n-1)! \times 1 && \{\text{using i}\} \\
&= (n-1)! \quad \text{for all } n \in \mathbb{Z}^+
\end{aligned}$$

# Chapter 23

## KINEMATICS

### EXERCISE 23A

1  $s(t) = 5 - t$  cm,  $0 \leq t \leq 10$  s

a  $s(0) = 5 - 0 = 5$  cm

$\therefore$  the initial displacement of the object is 5 cm to the right of the origin.

b i  $s(3) = 5 - 3 = 2$  cm

At time  $t = 3$  seconds, the object is 2 cm to the right of the origin.

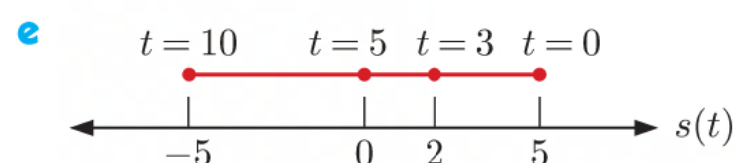
ii  $s(10) = 5 - 10 = -5$  cm

At time  $t = 10$  seconds, the object is 5 cm to the left of the origin.

c  $s(t) = 5 - t = 0$  when  $t = 5$

$\therefore$  the object reaches the origin at time  $t = 5$  seconds.

d No, the displacement function  $s(t)$  is linear, so it has no turning points.

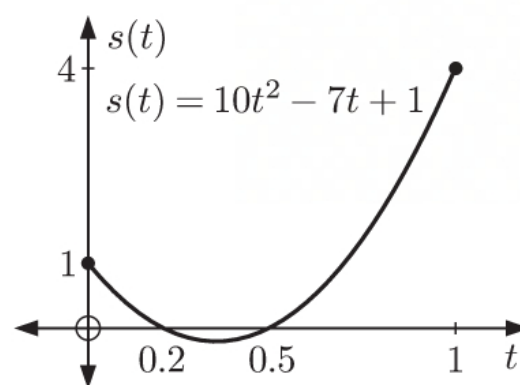


2  $s(t) = 10t^2 - 7t + 1$  m,  $0 \leq t \leq 1$  s

a  $s(0) = 1$  m

$\therefore$  the initial displacement of the object is 1 m to the right of the origin.

b  $s(t) = 10t^2 - 7t + 1$   
 $= (5t - 1)(2t - 1)$



c The object changes direction at the turning point of  $s(t)$ .

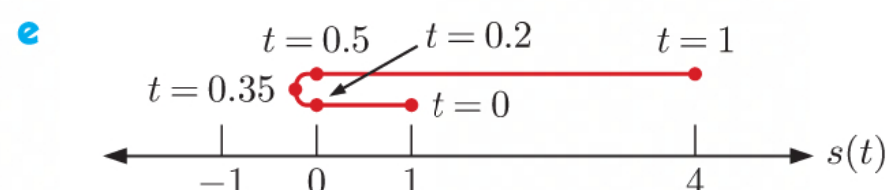
$$\begin{aligned} \text{This occurs when } t &= \frac{-(-7)}{2(10)} \\ &= \frac{7}{20} = 0.35 \text{ s} \end{aligned}$$

$$\begin{aligned} s(0.35) &= 10(0.35)^2 - 7(0.35) + 1 \\ &= -0.225 \end{aligned}$$

$\therefore$  the object changes direction after 0.35 seconds, when it is 0.225 m to the left of the origin.

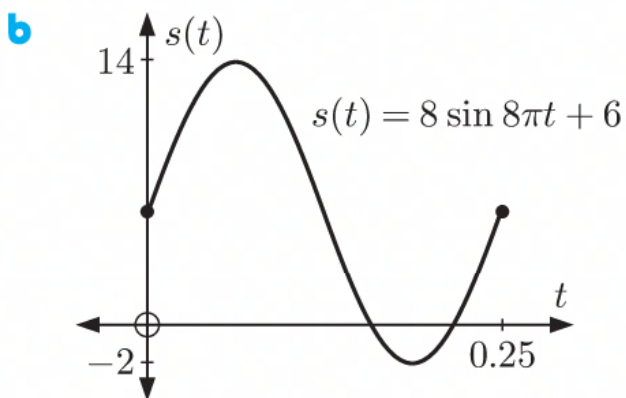
d The object is to the right of the origin when  $s(t) > 0$ .

This occurs for  $0 \leq t < 0.2$  s and  $0.5 \text{ s} < t \leq 1$  s.



3  $s(t) = 8 \sin 8\pi t + 6 \text{ cm}, 0 \leq t \leq 0.25 \text{ s}$

a  $s(t) = 6$   
 $\therefore 8 \sin 8\pi t + 6 = 6$   
 $\therefore 8 \sin 8\pi t = 0$   
 $\therefore \sin 8\pi t = 0$   
 $\therefore 8\pi t = 0, \pi, 2\pi, \dots$  {since  $t \geq 0$ }  
 $\therefore t = 0 \text{ s}, 0.125 \text{ s}, \text{ or } 0.25 \text{ s}$   $\{0 \leq t \leq 0.25 \text{ s}\}$



c The mass changes direction at the turning points of  $s(t)$ .

This occurs when  $\sin 8\pi t = 1$  or  $\sin 8\pi t = -1, 0 \leq t \leq 0.25$

$$\therefore 8\pi t = \frac{\pi}{2} \quad \therefore \sin 8\pi t = \frac{3\pi}{2}$$

$$\therefore t = \frac{1}{16} \quad \therefore t = \frac{3}{16}$$

$$= 0.0625$$

$$= 0.1875$$

When  $t = 0.0625$ ,  $\sin 8\pi t = 1$

$$\therefore 8 \sin 8\pi t + 6 = 8(1) + 6$$

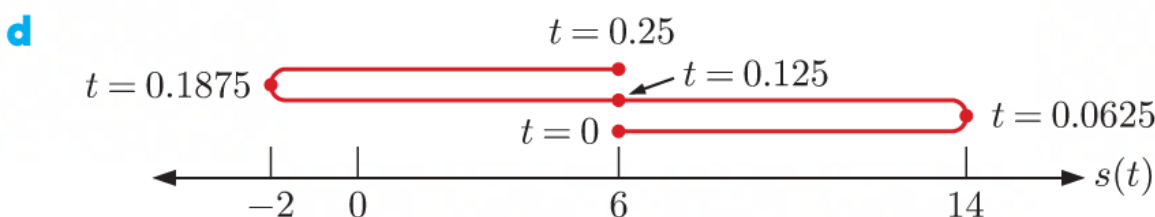
$$= 14$$

When  $t = 0.1875$ ,  $\sin 8\pi t = -1$

$$\therefore 8 \sin 8\pi t + 6 = 8(-1) + 6$$

$$= -2$$

$\therefore$  the mass changes direction 14 cm to the right of the origin, at  $t = 0.0625$  seconds, and 2 cm to the left of the origin, at  $t = 0.1875$  seconds.



## EXERCISE 23B.1

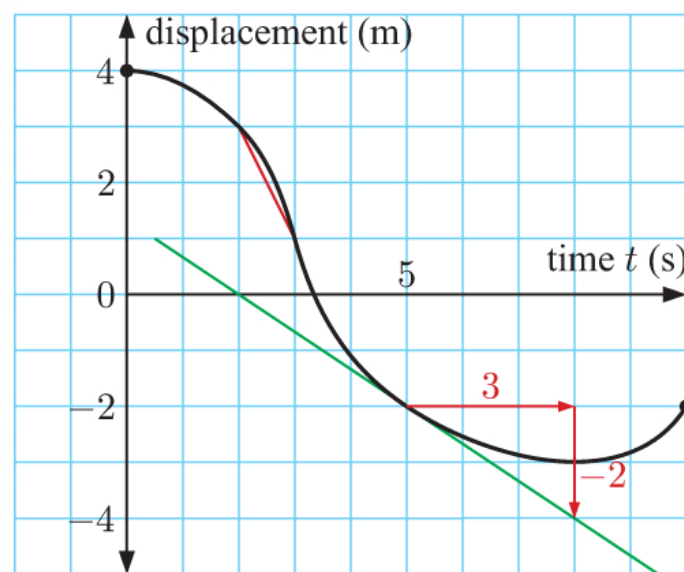
1 a i At  $t = 2$  seconds, the displacement is 3 m.

ii At  $t = 8$  seconds, the displacement is  $-3$  m.

b average velocity  $= \frac{s(3) - s(2)}{3 - 2}$   
 $= \frac{1 - 3}{1}$   
 $= -2 \text{ m s}^{-1}$

c The gradient of the tangent at  $t = 5$  seconds is  $-\frac{2}{3}$ .

$\therefore$  the instantaneous velocity at  $t = 5$  seconds is  $-\frac{2}{3} \text{ m s}^{-1}$ .



**2**  $s(t) = t^2 - 6t + 1 \text{ m}, t \geq 0 \text{ s}$

**a**  $s(1) = (1)^2 - 6(1) + 1 = 1 - 6 + 1 = -4 \text{ m}$        $s(3) = (3)^2 - 6(3) + 1 = 9 - 18 + 1 = -8 \text{ m}$

$$\begin{aligned} \text{average velocity} &= \frac{s(3) - s(1)}{3 - 1} \\ &= \frac{-8 - (-4)}{2} \\ &= \frac{-4}{2} \\ &= -2 \text{ m s}^{-1} \end{aligned}$$

**b**  $v(t) = s'(t) = 2t - 6 \text{ m s}^{-1}$

**c i**  $v(1) = 2(1) - 6 = 2 - 6 = -4 \text{ m s}^{-1}$

$\therefore$  the instantaneous velocity at  $t = 1$  second is  $-4 \text{ m s}^{-1}$ .

**ii**  $v(5) = 2(5) - 6 = 10 - 6 = 4 \text{ m s}^{-1}$

$\therefore$  the instantaneous velocity at  $t = 5$  seconds is  $4 \text{ m s}^{-1}$ .

**3 a** At time  $t = 0$  seconds, the displacement of the object is  $-2 \text{ cm}$ .

$\therefore$  the object is initially  $2 \text{ cm}$  to the left of the origin.

**b** The displacement of the object is  $0 \text{ cm}$  at time  $t = 6$  seconds.

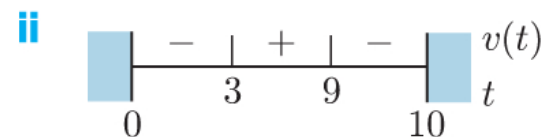
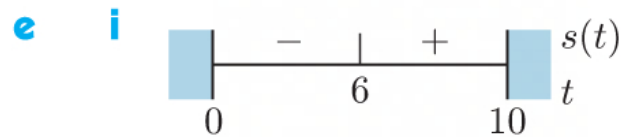
$\therefore$  the object is at the origin when  $t = 6$  seconds.

**c** At time  $t = 5$  seconds, the object has negative displacement, but this value is increasing.

$\therefore$  the object is moving to the right when  $t = 5$  seconds.

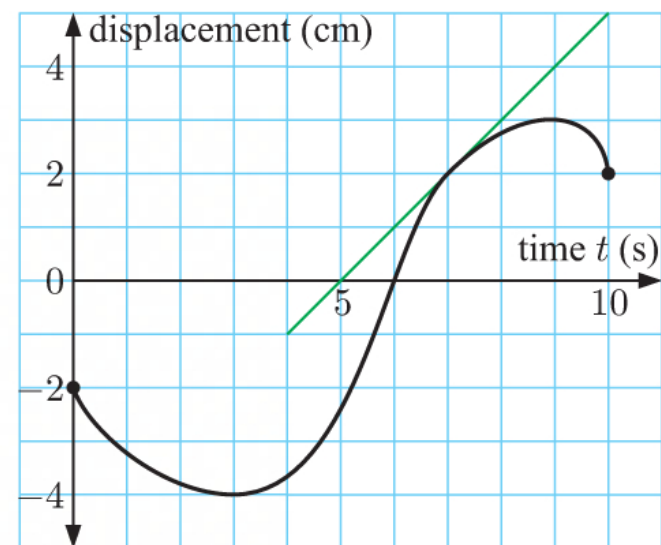
**d** The object changes direction at the turning points of the displacement graph.

$\therefore$  the object changes direction at times  $t = 3$  seconds and  $t = 9$  seconds.



**f** The gradient of the tangent at  $t = 7$  is  $1$ .

$\therefore$  the instantaneous velocity at  $t = 7$  seconds is  $1 \text{ cm s}^{-1}$ .





4  $s(t) = 2\sqrt{t} + 3$  cm,  $t \geq 0$  s

a  $s(1) = 2\sqrt{1} + 3 = 2 + 3 = 5$  cm  
 $s(4) = 2\sqrt{4} + 3 = 4 + 3 = 7$  cm

average velocity  $= \frac{s(4) - s(1)}{4 - 1}$   
 $= \frac{7 - 5}{3}$   
 $= \frac{2}{3} \text{ cm s}^{-1}$

b  $s(0) = 2\sqrt{0} + 3 = 3$  cm  
 $\therefore$  the initial position of the object is 3 cm to the right of the origin.

d i  $v(4) = \frac{1}{\sqrt{4}} = \frac{1}{2} \text{ cm s}^{-1}$   
 $\therefore$  the instantaneous velocity at  $t = 4$  seconds is  $\frac{1}{2} \text{ cm s}^{-1}$ .

c  $s(t) = 2t^{\frac{1}{2}} + 3$   
 $\therefore v(t) = s'(t) = t^{-\frac{1}{2}} = \frac{1}{\sqrt{t}} \text{ cm s}^{-1}$

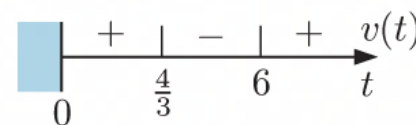
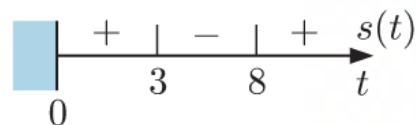
ii  $v(16) = \frac{1}{\sqrt{16}} = \frac{1}{4} \text{ cm s}^{-1}$   
 $\therefore$  the instantaneous velocity at  $t = 16$  seconds is  $\frac{1}{4} \text{ cm s}^{-1}$ .

5  $s(t) = t^3 - 11t^2 + 24t$  m,  $t \geq 0$  s

a  $v(t) = s'(t) = 3t^2 - 22t + 24 \text{ m s}^{-1}$

b  $s(0) = 0$  m,  $v(0) = 24 \text{ m s}^{-1}$   
 $\therefore$  the object is initially at the origin, moving to the right at  $24 \text{ m s}^{-1}$ .

c  $s(t)$  has sign diagram:  $v(t)$  has sign diagram:



d  $s(t) = 0$  when  $t = 0, 3$ , or  $8$  {from c}  
 $\therefore$  the object is at O at  $t = 0$  seconds, 3 seconds, and 8 seconds.

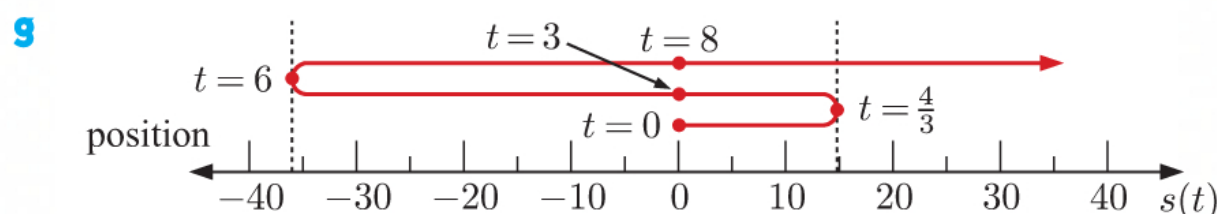
e The object changes direction when  $v(t)$  changes sign.

This occurs when  $t = \frac{4}{3}$  and  $t = 6$  {from c}

$s(\frac{4}{3}) = (\frac{4}{3})^3 - 11(\frac{4}{3})^2 + 24(\frac{4}{3}) \approx 14.8$  m  
 $s(6) = (6)^3 - 11(6)^2 + 24(6) = -36$  m

$\therefore$  the object changes direction at  $t = \frac{4}{3}$  seconds when it is about 14.8 m to the right of the origin, and at  $t = 6$  seconds, when it is 36 m to the left of the origin.

f The object starts at O, and moves towards the right at  $24 \text{ m s}^{-1}$ . Its velocity is decreasing. After  $\frac{4}{3}$  seconds, when it is about 14.8 m to the right of O, it changes direction and moves to the left, passing O after 3 seconds. After 6 seconds, when it is 36 m to the left of O, it changes direction again and moves towards the right, passing O once more after 8 seconds.



6  $s(t) = bt - 4.9t^2$  m,  $t \geq 0$  s

a  $v(t) = s'(t) = b - 9.8t$

$$\therefore v(0) = b - 9.8(0) \\ = b \text{ m s}^{-1}$$

$\therefore$  the initial velocity is  $b \text{ m s}^{-1}$  upwards.

b i The shell reaches its maximum height after 7.1 seconds.

$\therefore$  the velocity of the shell at  $t = 7.1$  seconds is zero.

$$\therefore v(7.1) = 0$$

$$\therefore b - 9.8(7.1) = 0$$

$$\therefore b - 69.58 = 0$$

$$\therefore b = 69.58$$

$\therefore$  the initial velocity of the shell is  $69.58 \text{ m s}^{-1}$  upwards.

ii  $s(t) = 69.58t - 4.9t^2$

The shell reaches its maximum height after 7.1 seconds.

$$s(7.1) = 69.58(7.1) - 4.9(7.1)^2 \\ \approx 247 \text{ m}$$

$\therefore$  the shell reached a maximum height of about 247 m.

## EXERCISE 23B.2

1 Total distance travelled

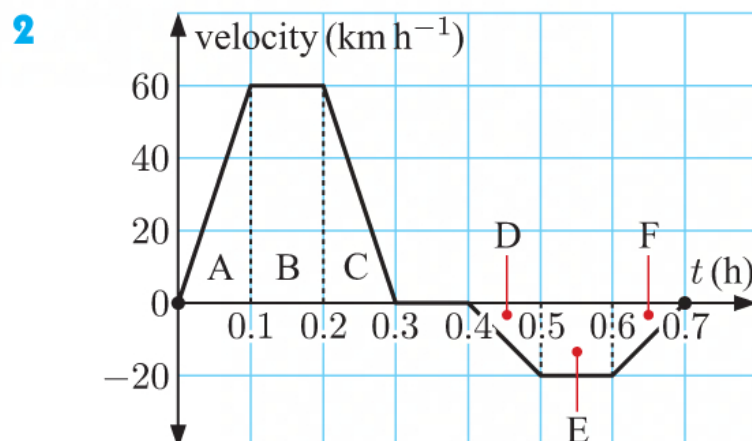
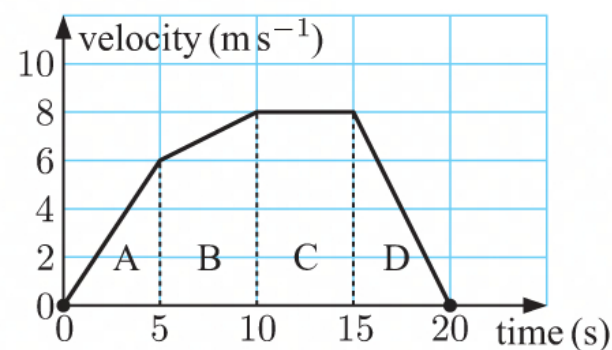
= total area under the graph

= area A + area B + area C + area D

$$= \frac{1}{2}(5)(6) + \left(\frac{6+8}{2}\right)(5) + (5)(8) + \frac{1}{2}(5)(8)$$

$$= 15 + 35 + 40 + 20$$

$$= 110 \text{ m}$$



a i When the graph is above the  $t$ -axis, the car is travelling forwards.

ii When the graph is below the  $t$ -axis, the car is travelling backwards (in the opposite direction).

b Total distance travelled

= total area between the graph and the  $t$ -axis

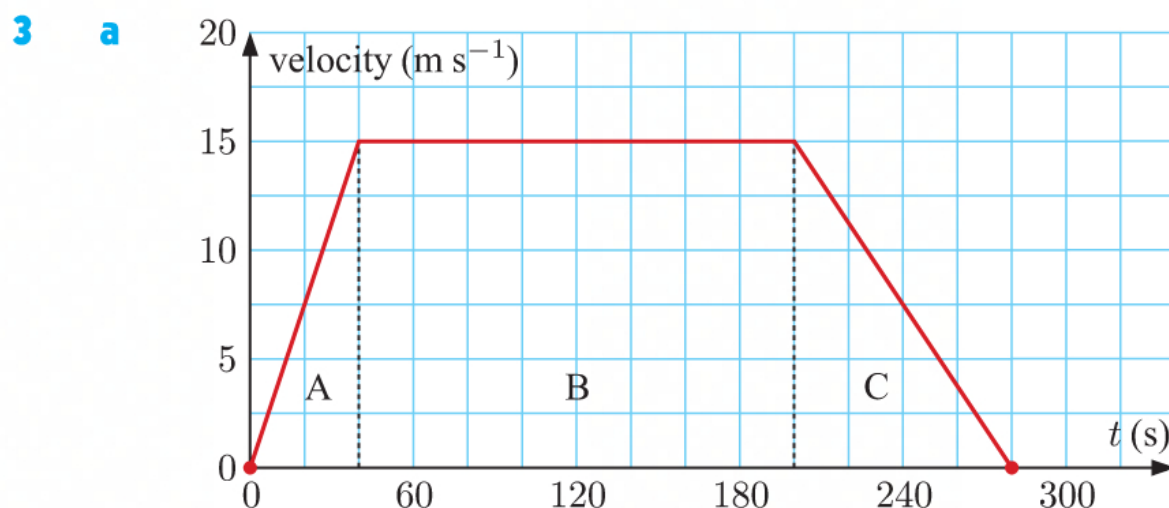
= area A + area B + area C + area D + area E + area F

$$= \frac{1}{2}(0.1)(60) + (0.1)(60) + \frac{1}{2}(0.1)(60) + \frac{1}{2}(0.1)(20) + (0.1)(20) + \frac{1}{2}(0.1)(20)$$

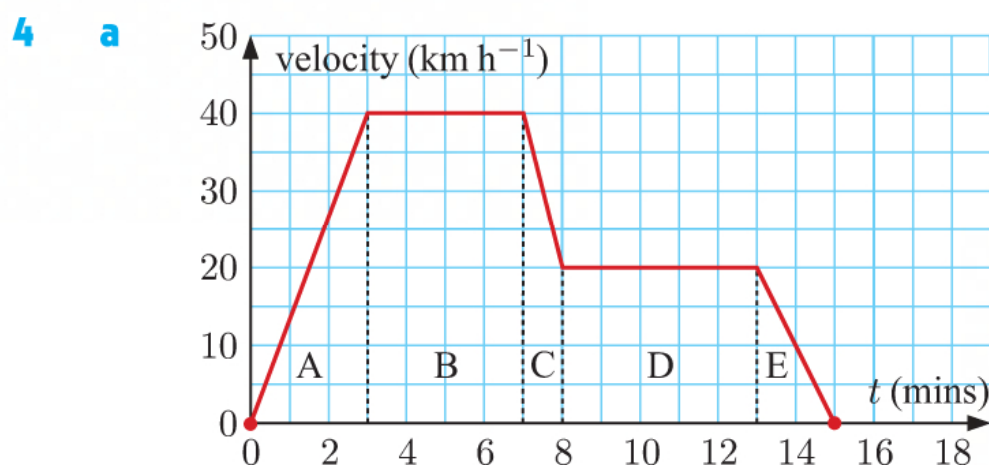
$$= 3 + 6 + 3 + 1 + 2 + 1$$

$$= 16 \text{ km}$$

- c** Displacement = forward distance travelled – backward distance travelled  
 = area A + area B + area C – area D – area E – area F  
 =  $3 + 6 + 3 - 1 - 2 - 1$   
 = 8 km from the starting point (on positive side)



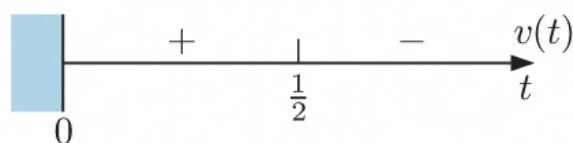
- b** Total distance travelled = total area under graph  
 = area A + area B + area C  
 =  $\frac{1}{2}(40)(15) + (160)(15) + \frac{1}{2}(80)(15)$   
 =  $300 + 2400 + 600$   
 = 3300 m  
 = 3.3 km



- b** Total distance travelled  
 = total area under graph  
 = area A + area B + area C + area D + area E  
 =  $\frac{1}{2}\left(\frac{3}{60}\right)(40) + \left(\frac{4}{60}\right)(40) + \left(\frac{40+20}{2}\right)\left(\frac{1}{60}\right) + \left(\frac{5}{60}\right)(20) + \frac{1}{2}\left(\frac{2}{60}\right)(20)$      $\{t \text{ min} = \frac{t}{60} \text{ hours}\}$   
 =  $1 + \frac{8}{3} + \frac{1}{2} + \frac{5}{3} + \frac{1}{3}$   
 =  $6\frac{1}{6}$  km

- 5 a**  $v(t) = s'(t) = 1 - 2t$

$\therefore$  the sign diagram of  $v$  is:

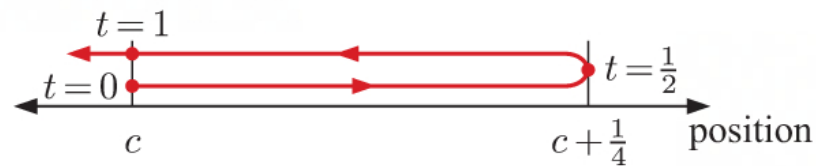


Since the sign changes, the particle changes direction at  $t = \frac{1}{2}$  second.

$$\begin{aligned} \text{b } s(t) &= \int (1 - 2t) dt \\ &= t - t^2 + c \end{aligned}$$

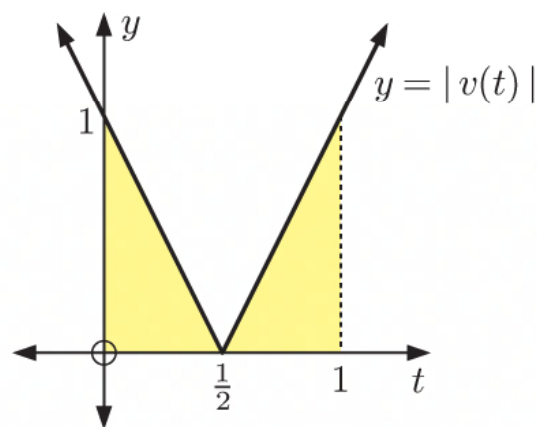
$$\begin{aligned} \text{Hence } s(0) &= c & s\left(\frac{1}{2}\right) &= \frac{1}{2} - \frac{1}{4} + c & s(1) &= 1 - 1 + c \\ & & &= c + \frac{1}{4} & &= c \end{aligned}$$

Motion diagram:



$$\begin{aligned} \therefore \text{ total distance travelled} &= (c + \frac{1}{4} - c) + (c + \frac{1}{4} - c) \\ &= \frac{1}{2} \text{ cm} \end{aligned}$$

$$\text{Now, } |v(t)| = |1 - 2t|$$



$$\begin{aligned} \int_0^1 |v(t)| dt &= \frac{1}{2} \left(\frac{1}{2}\right) (1) + \frac{1}{2} \left(\frac{1}{2}\right) (1) \\ &= \frac{1}{4} + \frac{1}{4} \\ &= \frac{1}{2} \quad \checkmark \end{aligned}$$

$$\begin{aligned} \text{c Displacement} &= \text{final position} - \text{original position} \\ &= s(1) - s(0) \\ &= c - c \\ &= 0 \text{ cm} \end{aligned}$$

So, the particle returned to its original position after one second.

$$\text{6 a } v(t) = s'(t) = t^2 - t - 2$$

$$\begin{aligned} \text{Now } s(t) &= \int (t^2 - t - 2) dt \\ &= \frac{1}{3}t^3 - \frac{1}{2}t^2 - 2t + c \end{aligned}$$

The particle is initially at the origin.

$$\therefore s(0) = 0$$

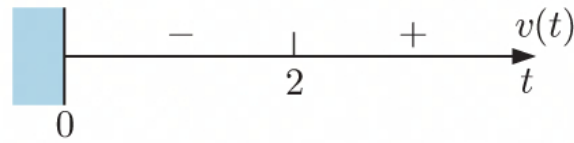
$$\therefore c = 0$$

$$\therefore s(t) = \frac{1}{3}t^3 - \frac{1}{2}t^2 - 2t \text{ cm}$$



$$\begin{aligned} \text{b } v(t) &= t^2 - t - 2 \\ &= (t+1)(t-2) \end{aligned}$$

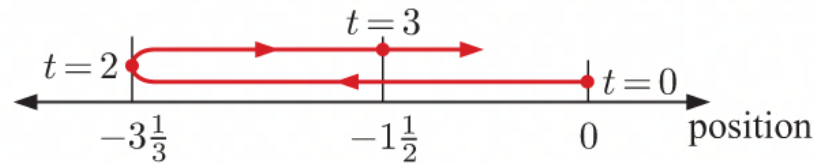
$\therefore$  the sign diagram of  $v$  is:



Since the sign changes, the particle changes direction at  $t = 2$  seconds.

$$\begin{aligned} \text{Hence } s(0) &= 0 & s(2) &= \frac{8}{3} - 2 - 4 & s(3) &= 9 - \frac{9}{2} - 6 \\ & & &= -3\frac{1}{3} & &= -1\frac{1}{2} \end{aligned}$$

Motion diagram:



$$\begin{aligned} \therefore \text{ total distance travelled} &= (0 - (-3\frac{1}{3})) + (-1\frac{1}{2} - (-3\frac{1}{3})) \\ &= 5\frac{1}{6} \text{ cm} \end{aligned}$$

$$\begin{aligned} \text{c Displacement} &= \text{final position} - \text{original position} \\ &= s(3) - s(0) \\ &= -1\frac{1}{2} - 0 \\ &= -1\frac{1}{2} \end{aligned}$$

So, the particle's displacement is  $1\frac{1}{2}$  cm left of its starting position.

$$\begin{aligned} \text{7 a } v(t) &= s'(t) = 29.4 - 9.8t \\ \therefore s(t) &= \int (29.4 - 9.8t) dt \\ &= 29.4t - 4.9t^2 + c \end{aligned}$$

The ball is initially 1 metre above ground level.

$$\begin{aligned} \therefore s(0) &= 1 \\ \therefore c &= 1 \\ \therefore s(t) &= 29.4t - 4.9t^2 + 1 \text{ m} \end{aligned}$$

b The maximum height reached by the ball occurs when its velocity equals 0.

$$\begin{aligned} \therefore 29.4 - 9.8t &= 0 \\ \therefore 9.8t &= 29.4 \\ \therefore t &= 3 \end{aligned}$$

So, the maximum height is reached at  $t = 3$  seconds.

$$\begin{aligned} s(3) &= 29.4(3) - 4.9(3)^2 + 1 \\ &= 45.1 \text{ m} \end{aligned}$$

$\therefore$  the maximum height reached by the ball is 45.1 m.

**8 a**  $v(t) = s'(t) = 32 + 4t$

$$\begin{aligned}\therefore s(t) &= \int (32 + 4t) dt \\ &= 32t + 2t^2 + c\end{aligned}$$

Now  $s(0) = 16$

$$\therefore c = 16$$

$$\therefore s(t) = 32t + 2t^2 + 16 \text{ m}$$

**b** The moving object changes direction when  $v(t) = 0$

$$\therefore 32 + 4t = 0$$

$$\therefore t = -8$$

But  $t \geq 0$ , so there is no change of direction.

$$\therefore \text{displacement} = \text{total distance travelled} = s(\tau) - s(0)$$

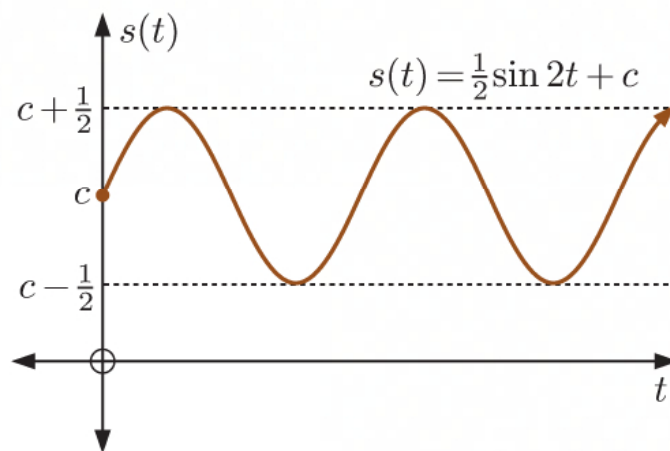
$$= \int_0^\tau (32 + 4t) dt$$

**c** Total distance travelled in first 4 seconds  $= \int_0^4 (32 + 4t) dt$

$$\begin{aligned}&= [32t + 2t^2]_0^4 \\ &= (128 + 32) - 0 \\ &= 160 \text{ m}\end{aligned}$$

**9 a**  $v(t) = s'(t) = \cos 2t$

$$\begin{aligned}\therefore s(t) &= \int \cos 2t dt \\ &= \frac{1}{2} \sin 2t + c\end{aligned}$$



The graph shows that the particle oscillates between positions  $c + \frac{1}{2}$  and  $c - \frac{1}{2}$ .

$$\begin{aligned}\text{Distance} &= (c + \tfrac{1}{2}) - (c - \tfrac{1}{2}) \\ &= 1 \text{ m}\end{aligned}$$

**b**  $s(\frac{\pi}{4}) = 1, \therefore \frac{1}{2} \sin \frac{\pi}{2} + c = 1$

$$\therefore \frac{1}{2}(1) + c = 1$$

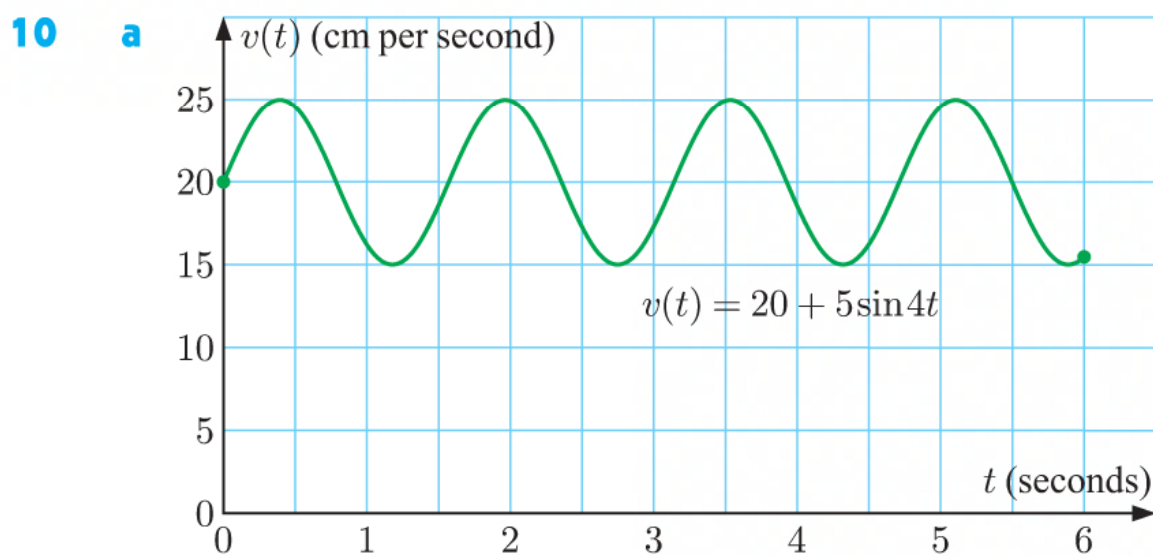
$$\therefore c = \frac{1}{2}$$

$$\therefore s(t) = \frac{1}{2} \sin 2t + \frac{1}{2}$$

$$\therefore s(\frac{\pi}{3}) = \frac{1}{2} \sin \frac{2\pi}{3} + \frac{1}{2}$$

$$= \frac{\sqrt{3}}{4} + \frac{1}{2}$$

$$= \frac{\sqrt{3} + 2}{4} \text{ m}$$



**b**  $v(4.5) = 20 + 5 \sin(4 \times 4.5)$   
 $= 20 + 5 \sin 18$   
 $\approx 16.2 \text{ cm s}^{-1}$

$\therefore$  the pendulum's velocity after 4.5 seconds is about  $16.2 \text{ cm s}^{-1}$ .

**c** Distance travelled by the tip of the pendulum in the first 2 seconds

$$\begin{aligned} &= \int_0^2 |v(t)| dt \\ &= \int_0^2 v(t) dt \\ &= \int_0^2 (20 + 5 \sin 4t) dt \\ &= \left[ 20t - \frac{5}{4} \cos 4t \right]_0^2 \\ &= \left( 40 - \frac{5}{4} \cos 8 \right) - \left( 0 - \frac{5}{4} \right) \\ &\approx 41.4 \text{ cm} \end{aligned}$$

**11 a**  $v(t) = s'(t) = -4 + t^{\frac{1}{2}}$   
 $\therefore s(t) = \int (-4 + t^{\frac{1}{2}}) dt$   
 $= -4t + \frac{2}{3} t^{\frac{3}{2}} + c$   
 $s(0) = 0, \therefore c = 0$   
 $\therefore s(t) = -4t + \frac{2}{3} t^{\frac{3}{2}} \text{ m}$

**b** The object changes direction when  $v(t) = 0$   
 $\therefore -4 + \sqrt{t} = 0$   
 $\therefore \sqrt{t} = 4$   
 $\therefore t = 16$

$\therefore$  the object changes direction at  $t = 16$  seconds.

**c** Change in displacement  $= s(30) - s(0)$   
 $= -4(30) + \frac{2}{3}(30)^{\frac{3}{2}} - 0$   
 $\approx -10.5 \text{ m}$

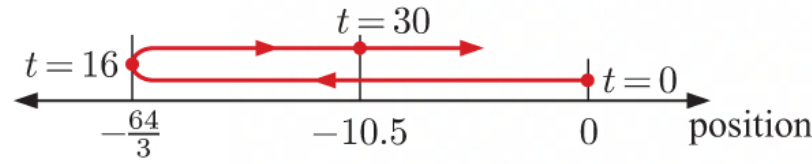
$\therefore$  after the first 30 seconds the particle is about 10.5 m to the left of the origin.

**d**  $s(0) = 0$

$$\begin{aligned} s(16) &= -4(16) + \frac{2}{3}(16)^{\frac{3}{2}} \\ &= -\frac{64}{3} \end{aligned}$$

$$s(30) \approx -10.5$$

Motion diagram:



$$\begin{aligned} \therefore \text{total distance travelled} &\approx (0 - (-\frac{64}{3})) + (-10.5 - (-\frac{64}{3})) \\ &\approx 32.2 \text{ m} \end{aligned}$$

**12 a**  $v(t) = 10\sqrt{t} \text{ m s}^{-1}$

**i**  $\begin{aligned} v(1) &= 10\sqrt{1} \\ &= 10 \end{aligned}$

So the motorcyclist's velocity after 1 second is  $10 \text{ m s}^{-1}$ .

**ii**  $v(2) = 10\sqrt{2}$

So the motorcyclist's velocity after 2 seconds is  $10\sqrt{2} \text{ m s}^{-1}$ .

**b**  $\begin{aligned} s(t) &= \int v(t) dt \\ &= \int 10\sqrt{t} dt \\ &= \int 10t^{\frac{1}{2}} dt \\ &= \frac{20}{3}t^{\frac{3}{2}} + c \text{ m} \end{aligned}$

We assume that  $s(0) = 0$ ,  $c = 0$

$$\therefore s(t) = \frac{20}{3}t^{\frac{3}{2}} \text{ m}$$

**d i**  $v(t) = 20$  when  $10\sqrt{t} = 20$   
 $\therefore \sqrt{t} = 2$   
 $\therefore t = 4$

It will take 4 seconds for the motorcyclist to reach a speed of  $20 \text{ m s}^{-1}$ .

**ii** Distance travelled in first 4 seconds  $= \int_0^4 v(t) dt$   
 $= \left[ \frac{20}{3}t^{\frac{3}{2}} \right]_0^4$   
 $= \frac{20}{3}(4^{\frac{3}{2}}) - 0$   
 $= 53\frac{1}{3} \text{ m}$

$\therefore$  yes, the motorcyclist has given himself enough distance as he only needs  $53\frac{1}{3} \text{ m}$  to reach the required speed.

**c**  $\begin{aligned} \int_0^2 v(t) dt &= \left[ \frac{20}{3}t^{\frac{3}{2}} \right]_0^2 \\ &= \frac{20}{3}(2^{\frac{3}{2}}) - 0 \\ &\approx 18.9 \end{aligned}$

The motorcyclist travels about 18.9 m in the first 2 seconds.



**13 a**  $v(t) = -54(1 - e^{-\frac{t}{6}}) \text{ m s}^{-1}$

$$\int_0^{15} |v(t)| dt = \int_0^{15} \left| -54(1 - e^{-\frac{t}{6}}) \right| dt$$

$$\approx 513 \quad \{\text{using technology}\}$$

$\therefore$  the skydiver travels a total distance of about 513 m in the first 15 seconds.

Math Deg Norm1 ab/c Real

$$\int_0^{15} \left| -54 \left( 1 - e^{-\frac{x}{6}} \right) \right| dx$$

512.5955396

JUMP DELETE ▶ MAT MATH

**b**  $v(t) = e^{-t} \cos 16t \text{ cm s}^{-1}$

$$\int_0^{10} |v(t)| dt = \int_0^{10} |e^{-t} \cos 16t| dt$$

$$\approx 0.637$$

$\therefore$  the mass on the spring travels a total distance of about 0.637 cm in the first 10 seconds.

Math Rad Norm1 ab/c Real

$$\int_0^{10} |e^{-x} \cos (16x)| dx$$

0.6369870327

JUMP DELETE ▶ MAT MATH

## EXERCISE 23C

**1 a**  $v(t) = 10t - t^2 \text{ cm s}^{-1}, \quad t \geq 0 \text{ s}$

$$\begin{aligned} v(2) &= 10(2) - (2)^2 \\ &= 20 - 4 \\ &= 16 \text{ cm s}^{-1} \end{aligned}$$

$\therefore$  the velocity of the particle at  $t = 2$  seconds is  $16 \text{ cm s}^{-1}$ .

**b** average acceleration  $= \frac{v(3) - v(1)}{3 - 1}$

$$\begin{aligned} &= \frac{(10(3) - (3)^2) - (10(1) - (1)^2)}{3 - 1} \\ &= \frac{(30 - 9) - (10 - 1)}{2} \\ &= \frac{21 - 9}{2} \\ &= 6 \text{ cm s}^{-2} \end{aligned}$$

$\therefore$  the average acceleration from  $t = 1$  to  $t = 3$  seconds is  $6 \text{ cm s}^{-2}$ .

**c**  $a(t) = v'(t) = 10 - 2t \text{ cm s}^{-2}$

**d**  $a(3) = 10 - 2(3)$

$$= 4 \text{ cm s}^{-2}$$

$\therefore$  the instantaneous acceleration of the particle at  $t = 3$  seconds is  $4 \text{ cm s}^{-2}$ .

**2 a**  $s(t) = t^3 - t^2 - 5 \text{ m}, \quad t \geq 0 \text{ s}$

$$\therefore v(t) = s'(t) = 3t^2 - 2t \text{ m s}^{-1}$$

$$\therefore a(t) = v'(t) = 6t - 2 \text{ m s}^{-2}$$

$$\begin{aligned} \therefore s(2) &= 2^3 - 2^2 - 5 & v(2) &= 3(2)^2 - 2(2) & a(2) &= 6(2) - 2 \\ &= 8 - 4 - 5 & &= 12 - 4 & &= 12 - 2 \\ &= -1 \text{ m} & &= 8 \text{ m s}^{-1} & &= 10 \text{ m s}^{-2} \end{aligned}$$

At  $t = 2$  seconds, the object has displacement  $-1 \text{ m}$ , velocity  $8 \text{ m s}^{-1}$ , and acceleration  $10 \text{ m s}^{-2}$ .

**b**  $a(t) = 0$  when  $6t - 2 = 0$

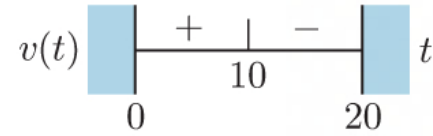
$$\therefore 6t = 2$$

$$\therefore t = \frac{1}{3}$$

$\therefore$  the object has zero acceleration at  $t = \frac{1}{3}$  seconds.

**3 a**  $s(t) = 98t - 4.9t^2 \text{ m}$

$\therefore v(t) = s'(t) = 98 - 9.8t \text{ m s}^{-1}$  which has sign diagram:



$\therefore a(t) = v'(t) = -9.8 \text{ m s}^{-2}$  which has sign diagram:



**b**  $s(0) = 0 \text{ m}$   $v(0) = 98 \text{ m s}^{-1}$

The stone is initially  $0 \text{ m}$  above the ground, moving upward with velocity  $98 \text{ m s}^{-1}$ .

**c**  $s(5) = 98(5) - 4.9(5)^2$   $v(5) = 98 - 9.8(5)$   $a(5) = -9.8 \text{ m s}^{-2}$   
 $= 367.5 \text{ m}$   $= 49 \text{ m s}^{-1}$

At  $t = 5$  seconds, the stone is  $367.5 \text{ m}$  above the ground and moving upward at  $49 \text{ m s}^{-1}$ . It has acceleration  $-9.8 \text{ m s}^{-2}$ .

$$\begin{aligned} s(12) &= 98(12) - 4.9(12)^2 & v(12) &= 98 - 9.8(12) & a(12) &= -9.8 \text{ m s}^{-2} \\ &= 470.4 \text{ m} & &= -19.6 \text{ m s}^{-1} \end{aligned}$$

At  $t = 12$  seconds, the stone is  $470.4 \text{ m}$  above the ground and moving downward at  $19.6 \text{ m s}^{-1}$ . It has acceleration  $-9.8 \text{ m s}^{-2}$ .

**d** The maximum height reached by the stone occurs when its upward velocity equals  $0$ .

$$\therefore 98 - 9.8t = 0$$

$$\therefore 9.8t = 98$$

$$\therefore t = 10$$

So, the maximum height is reached at  $t = 10$  seconds.

$$\begin{aligned} s(10) &= 98(10) - 4.9(10)^2 \\ &= 490 \text{ m} \end{aligned}$$

$\therefore$  the maximum height reached by the stone is  $490 \text{ m}$ .

- e The stone is on the ground when its displacement equals 0.

$$\therefore 98t - 4.9t^2 = 0$$

$$\therefore t(98 - 4.9t) = 0$$

$$\therefore t = 0 \text{ or } 98 - 4.9t = 0$$

$$4.9t = 98$$

$$t = 20$$

After it is fired from the catapult, it takes 20 seconds for the stone to hit the ground.

4 a  $s(t) = 100t + 200e^{-\frac{t}{5}} \text{ cm}$

$$\therefore v(t) = 100 + 200\left(-\frac{1}{5}\right)e^{-\frac{t}{5}} \quad \{v(t) = s'(t)\}$$

$$= 100 - 40e^{-\frac{t}{5}} \text{ cm s}^{-1}$$

and  $a(t) = -40\left(-\frac{1}{5}\right)e^{-\frac{t}{5}} \quad \{a(t) = v'(t)\}$

$$= 8e^{-\frac{t}{5}} \text{ cm s}^{-2}$$

b When  $t = 0$ ,  $s(0) = 200 \text{ cm}$   
 $v(0) = 60 \text{ cm s}^{-1}$   
 $a(0) = 8 \text{ cm s}^{-2}$

$\therefore$  the particle is initially 200 cm to the right of the origin, moving to the right at  $60 \text{ cm s}^{-1}$ , and has acceleration  $8 \text{ cm s}^{-2}$ .

c As  $t \rightarrow \infty$ ,  $v(t) \rightarrow 100$   
 $\therefore$  the velocity of P approaches  $100 \text{ cm s}^{-1}$  as  $t \rightarrow \infty$ .

d As  $t \rightarrow \infty$ ,  $a(t) \rightarrow 0$   
 $\therefore$  the acceleration of P approaches  $0 \text{ cm s}^{-2}$  as  $t \rightarrow \infty$ .

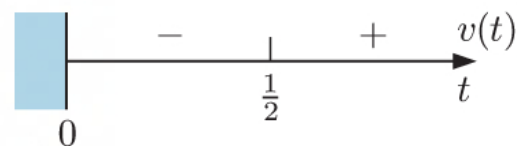
5 a  $s(t) = t - \ln(2t + 1) \text{ cm}$   
 $\therefore s(0) = 0 - \ln(2(0) + 1)$   
 $= -\ln(1)$   
 $= 0$

$\therefore$  the object is initially at the origin. ✓

b  $v(t) = 1 - \frac{2}{2t+1} \text{ cm s}^{-1} \quad \{v(t) = s'(t)\}$

c  $v(t) = 1 - \frac{2}{2t+1}$   
 $= \frac{2t+1-2}{2t+1}$

$= \frac{2t-1}{2t+1}$  which has sign diagram:

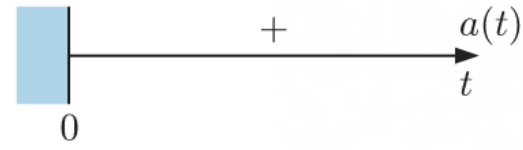


- i The object is moving to the right when the velocity is positive.  
 $\therefore$  the object is moving to the right for  $t > \frac{1}{2}$  second.
- ii The object is moving to the left when the velocity is negative.  
 $\therefore$  the object is moving to the left for  $0 \leq t < \frac{1}{2}$  second.

**d**  $v(t) = 1 - 2(2t + 1)^{-1} \text{ cm s}^{-1}$

$\therefore a(t) = 2(2t + 1)^{-2}(2) \quad \{\text{chain rule}\}$

$= \frac{4}{(2t + 1)^2} \text{ cm s}^{-2} \quad \text{which has sign diagram:}$



$\therefore$  the object's acceleration is positive for all  $t \geq 0$ .

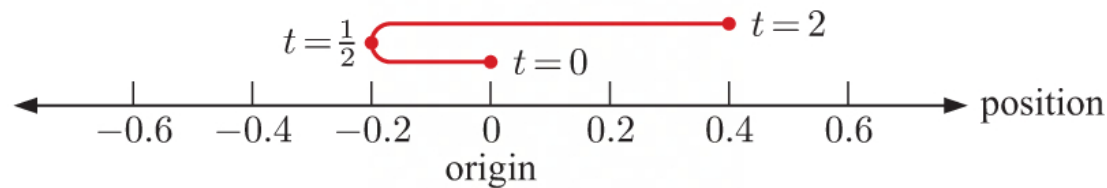
**e**  $a(2) = \frac{4}{(2(2) + 1)^2}$   
 $= \frac{4}{25} \text{ cm s}^{-2}$

$\therefore$  the acceleration of the object after 2 seconds is  $\frac{4}{25} \text{ cm s}^{-2}$ .

**f**  $v(t)$  changes sign when  $t = \frac{1}{2}$ , so this is where the object changes direction.

$$\begin{aligned} s\left(\frac{1}{2}\right) &= \frac{1}{2} - \ln\left(2\left(\frac{1}{2}\right) + 1\right) & \text{and} & & s(2) &= 2 - \ln(2(2) + 1) \\ &= \frac{1}{2} - \ln 2 \text{ cm} & & & &= 2 - \ln 5 \text{ cm} \end{aligned}$$

The motion diagram of P is:



$\therefore$  the total distance travelled by the object in the first 2 seconds

$$\begin{aligned} &= \left(0 - \left(\frac{1}{2} - \ln 2\right)\right) + \left(2 - \ln 5 - \left(\frac{1}{2} - \ln 2\right)\right) \\ &= -\frac{1}{2} + \ln 2 + 2 - \ln 5 - \frac{1}{2} + \ln 2 \\ &= 1 + 2\ln 2 - \ln 5 \\ &= 1 + \ln 4 - \ln 5 \\ &= 1 + \ln\left(\frac{4}{5}\right) \approx 0.777 \text{ cm} \end{aligned}$$

**6**  $v(t) = 50 - 10e^{-0.5t} \text{ m s}^{-1}$

**a**  $v(0) = 50 - 10e^0$   
 $= 40$

So, the initial velocity is  $40 \text{ m s}^{-1}$ .

**b**  $v(3) = 50 - 10e^{-0.5(3)}$   
 $\approx 47.8$

So, the velocity after 3 seconds is about  $47.8 \text{ m s}^{-1}$ .

**c**  $v(t) = 45$  when  $50 - 10e^{-0.5t} = 45$   
 $\therefore -10e^{-0.5t} = -5$   
 $\therefore e^{-0.5t} = \frac{1}{2}$   
 $\therefore -0.5t = \ln\left(\frac{1}{2}\right)$   
 $\therefore t = 2\ln 2 \approx 1.39$

So, it will take about 1.39 seconds for the particle's velocity to reach  $45 \text{ m s}^{-1}$ .

**d** As  $t \rightarrow \infty$ ,  $10e^{-0.5t} \rightarrow 0$  from above,  
 thus  $v(t) \rightarrow 50 \text{ m s}^{-1}$  from below.

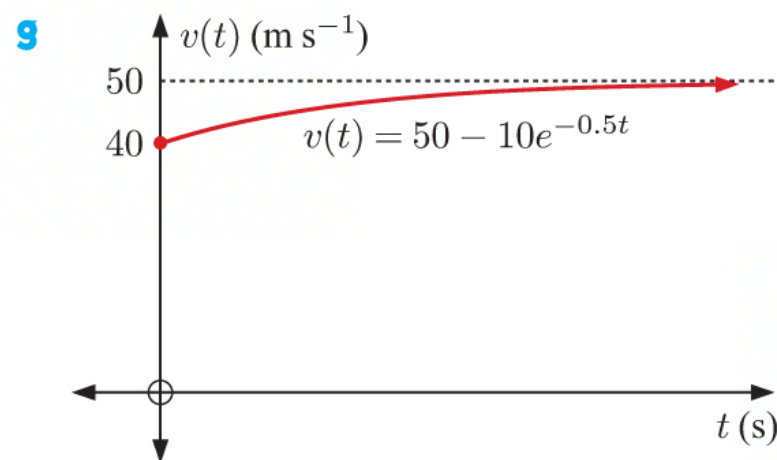
**e**  $a(t) = v'(t)$   
 $= 5e^{-0.5t}$

And as  $e^x > 0$  for all  $x$ ,

then  $a(t) = 5e^{-0.5t} > 0$  for all  $t$ . ✓



$$\begin{aligned}
 \text{f } a(t) = 2 \text{ when } 5e^{-0.5t} &= 2 \\
 \therefore e^{-0.5t} &= \frac{2}{5} \\
 \therefore -0.5t &= \ln\left(\frac{2}{5}\right) \\
 \therefore -0.5t &= -\ln\left(\frac{5}{2}\right) \\
 \therefore t &= 2\ln\left(\frac{5}{2}\right) \text{ s}
 \end{aligned}$$



**h** The particle does not change direction.

$$\begin{aligned}
 \therefore \text{total distance travelled in first 3 seconds} &= \int_0^3 v(t) dt \\
 &= \int_0^3 (50 - 10e^{-0.5t}) dt \\
 &= [50t + 20e^{-0.5t}]_0^3 \\
 &= 150 + 20e^{-1.5} - (20) \\
 &\approx 134 \text{ m}
 \end{aligned}$$

**7 a**

$$\begin{aligned}
 v(t) &= \int a(t) dt \\
 &= \int \left(\frac{t}{10} - 3\right) dt \\
 &= \frac{t^2}{20} - 3t + c
 \end{aligned}$$

Now  $v(0) = 45$   
 $\therefore c = 45$   
 $\therefore v(t) = \frac{t^2}{20} - 3t + 45 \text{ m s}^{-1}$

**b** The train does not change direction.

$$\begin{aligned}
 \therefore \text{distance travelled} &= \int_0^{60} v(t) dt \\
 &= \int_0^{60} \left(\frac{t^2}{20} - 3t + 45\right) dt \\
 &= \left[\frac{t^3}{60} - \frac{3}{2}t^2 + 45t\right]_0^{60} \\
 &= (3600 - 5400 + 2700) - 0 \\
 &= 900
 \end{aligned}$$

The train travels a total of 900 m in the first 60 seconds.

**8 a**

$$\begin{aligned}
 v(t) &= \int a(t) dt \\
 &= \int 4e^{-\frac{t}{20}} dt \\
 &= -80e^{-\frac{t}{20}} + c
 \end{aligned}$$

Now  $v(0) = 20$   
 $\therefore -80 + c = 20$   
 $\therefore c = 100$   
 $\therefore v(t) = 100 - 80e^{-\frac{t}{20}} \text{ m s}^{-1}$

As  $t \rightarrow \infty$ ,  $80e^{-\frac{t}{20}} \rightarrow 0$ ,  
 and thus  $v(t) \rightarrow 100 \text{ m s}^{-1}$ .

**b** The object does not change direction.

$$\begin{aligned}
 \therefore \text{total distance travelled in first 10 seconds} &= \int_0^{10} v(t) dt \\
 &= \int_0^{10} (100 - 80e^{-\frac{t}{20}}) dt \\
 &= \left[100t + 1600e^{-\frac{t}{20}}\right]_0^{10} \\
 &= (1000 + 1600e^{-\frac{1}{2}}) - 1600 \\
 &\approx 370 \text{ m}
 \end{aligned}$$

$$9 \quad a \quad v(t) = \int a(t) dt$$

$$= \int 2(t+1)^{-3} dt$$

$$= -(t+1)^{-2} + c$$

At  $t = 0$  the particle is stationary

$$\therefore v(0) = 0$$

$$\therefore -1^{-2} + c = 0$$

$$\therefore c = 1$$

$$\therefore v(t) = -\frac{1}{(t+1)^2} + 1 \text{ m s}^{-1}$$

$$b \quad s(t) = \int v(t) dt$$

$$= \int (-(t+1)^{-2} + 1) dt$$

$$= (t+1)^{-1} + t + c$$

At  $t = 0$  the particle is at the origin

$$\therefore s(0) = 0$$

$$\therefore 1^{-1} + 0 + c = 0$$

$$\therefore c = -1$$

$$\therefore s(t) = \frac{1}{t+1} + t - 1 \text{ m}$$

$$c \quad a(2) = \frac{2}{(2+1)^3}$$

$$= \frac{2}{27} \text{ m s}^{-2}$$

$$v(2) = -\frac{1}{(2+1)^2} + 1$$

$$= -\frac{1}{9} + 1$$

$$= \frac{8}{9} \text{ m s}^{-1}$$

$$s(2) = \frac{1}{2+1} + 2 - 1$$

$$= \frac{1}{3} + 1$$

$$= \frac{4}{3} \text{ m}$$

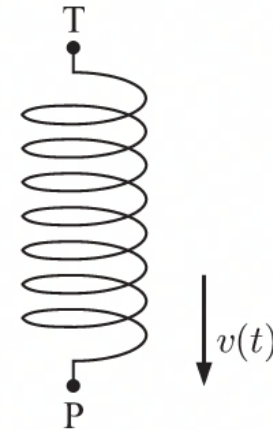
$\therefore$  at  $t = 2$ , the particle is  $\frac{4}{3}$  metres to the right of the origin, moving to the right at  $\frac{8}{9} \text{ m s}^{-1}$ , and accelerating at  $\frac{2}{27} \text{ m s}^{-2}$ .

$$10 \quad v(t) = 10e^{-2t} \sin 5t - 5e^{-2t} \cos 5t, \quad t \geq 0 \text{ s}$$

$$a \quad v(0.3) = 10e^{-0.6} \sin 1.5 - 5e^{-0.6} \cos 1.5$$

$$\approx 5.28 \text{ cm s}^{-1}$$

$\therefore$  the velocity of P relative to T after 0.3 seconds is about  $5.28 \text{ cm s}^{-1}$ .



$$b \quad v(t) = 10e^{-2t} \sin 5t - 5e^{-2t} \cos 5t$$

$$\therefore a(t) = v'(t) = -20e^{-2t} \sin 5t + 50e^{-2t} \cos 5t + 10e^{-2t} \cos 5t + 25e^{-2t} \sin 5t$$

$$= 5e^{-2t}(-4 \sin 5t + 10 \cos 5t + 2 \cos 5t + 5 \sin 5t)$$

$$= 5e^{-2t}(12 \cos 5t + \sin 5t)$$

$$\therefore a(0.3) = 5e^{-0.6}(12 \cos 1.5 + \sin 1.5)$$

$$\approx 5.07 \text{ cm s}^{-2}$$

$\therefore$  the acceleration of P relative to T after 0.3 seconds is about  $5.07 \text{ cm s}^{-2}$ .

$$\text{c } v(t) = 10e^{-2t} \sin 5t - 5e^{-2t} \cos 5t$$

$$\begin{aligned} \therefore s(t) &= \int (10e^{-2t} \sin 5t - 5e^{-2t} \cos 5t) dt \quad \dots (*) \\ &= 10 \int e^{-2t} \sin 5t dt - 5 \int e^{-2t} \cos 5t dt \\ &= 10 \left( -\frac{1}{2}e^{-2t} \sin 5t - \int -\frac{1}{2}e^{-2t} 5 \cos 5t dt \right) \leftarrow \begin{cases} u = \sin 5t & v' = e^{-2t} \\ u' = 5 \cos 5t & v = -\frac{1}{2}e^{-2t} \end{cases} \\ &\quad - 5 \left( -\frac{1}{2}e^{-2t} \cos 5t - \int -\frac{1}{2}e^{-2t} (-5 \sin 5t) dt \right) \leftarrow \begin{cases} u = \cos 5t & v' = e^{-2t} \\ u' = -5 \sin 5t & v = -\frac{1}{2}e^{-2t} \end{cases} \\ &= -5e^{-2t} \sin 5t + \frac{5}{2}e^{-2t} \cos 5t + 25 \int e^{-2t} \cos 5t dt + \frac{25}{2} \int e^{-2t} \sin 5t dt \\ &= -5e^{-2t} \sin 5t + \frac{5}{2}e^{-2t} \cos 5t \\ &\quad + 25 \left( -\frac{1}{2}e^{-2t} \cos 5t - \int -\frac{1}{2}e^{-2t} (-5 \sin 5t) dt \right) \leftarrow \begin{cases} u = \cos 5t & v' = e^{-2t} \\ u' = -5 \sin 5t & v = -\frac{1}{2}e^{-2t} \end{cases} \\ &\quad + \frac{25}{2} \left( -\frac{1}{2}e^{-2t} \sin 5t - \int -\frac{1}{2}e^{-2t} 5 \cos 5t dt \right) \leftarrow \begin{cases} u = \sin 5t & v' = e^{-2t} \\ u' = 5 \cos 5t & v = -\frac{1}{2}e^{-2t} \end{cases} \\ &= -5e^{-2t} \sin 5t + \frac{5}{2}e^{-2t} \cos 5t \\ &\quad - \frac{25}{2}e^{-2t} \cos 5t - \frac{25}{4}e^{-2t} \sin 5t \\ &\quad - \frac{125}{2} \int e^{-2t} \sin 5t dt + \frac{125}{4} \int e^{-2t} \cos 5t dt \\ &= -\frac{45}{4}e^{-2t} \sin 5t - 10e^{-2t} \cos 5t \\ &\quad - \frac{25}{4} \int (10e^{-2t} \sin 5t - 5e^{-2t} \cos 5t) dt \quad \dots (**) \end{aligned}$$

Comparing (\*) and (\*\*):

$$\begin{aligned} &\int (10e^{-2t} \sin 5t - 5e^{-2t} \cos 5t) dt \\ &= -\frac{45}{4}e^{-2t} \sin 5t - 10e^{-2t} \cos 5t - \frac{25}{4} \int (10e^{-2t} \sin 5t - 5e^{-2t} \cos 5t) dt \\ \therefore \frac{29}{4} \int (10e^{-2t} \sin 5t - 5e^{-2t} \cos 5t) dt &= -\frac{45}{4}e^{-2t} \sin 5t - 10e^{-2t} \cos 5t + c \\ \therefore \int (10e^{-2t} \sin 5t - 5e^{-2t} \cos 5t) dt &= -\frac{45}{29}e^{-2t} \sin 5t - \frac{40}{29}e^{-2t} \cos 5t + c \\ \therefore s(t) &= -\frac{5}{29}e^{-2t}(9 \sin 5t + 8 \cos 5t) + c \end{aligned}$$

$$\text{Now } s(0) = -\frac{5}{29}e^0(9 \sin 0 + 8 \cos 0) + c = 10$$

$$\therefore -\frac{40}{29} + c = 10$$

$$\therefore c = \frac{330}{29}$$

$$\therefore s(t) = -\frac{5}{29}e^{-2t}(9 \sin 5t + 8 \cos 5t) + \frac{330}{29} \text{ cm}$$

$$\begin{aligned} \text{d } s(0.1) &= -\frac{5}{29}e^{-0.2}(9 \sin 0.5 + 8 \cos 0.5) + \frac{330}{29} \\ &\approx 9.779 \end{aligned}$$

$\therefore$  after 0.1 seconds, the suspension has contracted by about  $10 - 9.779 \approx 0.221$  cm.

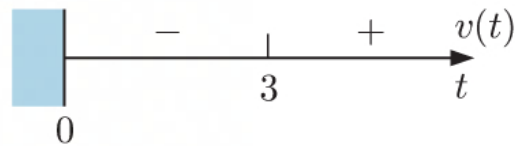
## EXERCISE 23D

1 a  $s(t) = t^2 - 6t + 7$  m

$$\therefore v(t) = 2t - 6 \quad \{v(t) = s'(t)\}$$

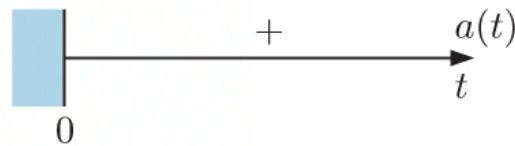
$$= 2(t - 3) \text{ m s}^{-1}$$

which has sign diagram:



and  $a(t) = 2 \text{ m s}^{-2} \quad \{a(t) = v'(t)\}$

which has sign diagram:



b When  $t = 0$ ,  $s(0) = 7$  m

$$v(0) = -6 \text{ m s}^{-1}$$

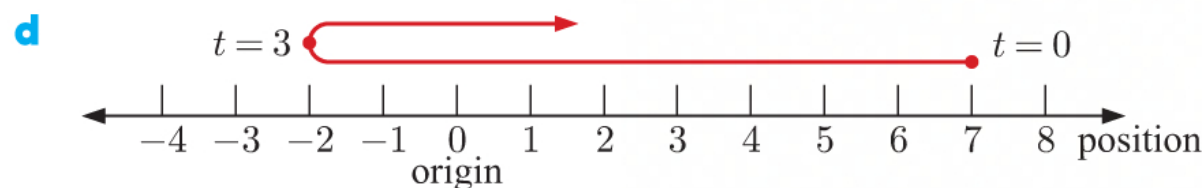
$$a(0) = 2 \text{ m s}^{-2}$$

$\therefore$  the object is initially 7 m to the right of O, moving to the left at  $6 \text{ m s}^{-1}$ , with acceleration  $2 \text{ m s}^{-2}$ .

c  $v(t)$  changes sign when  $t = 3$ , so this is when the object changes direction.

$$s(3) = 9 - 18 + 7 = -2$$

So, the object changes direction when it is 2 m to the left of O.



e The object's speed is decreasing when  $v(t)$  and  $a(t)$  have opposite signs. This occurs for  $0 \leq t \leq 3$ .

2  $s(t) = 1.2 + 28.1t - 4.9t^2$  m

a When the ball was first released,  $t = 0$ .

$$s(0) = 1.2$$

So, the ball was released 1.2 m above ground level.

b  $s'(t) = 28.1 - 9.8t \text{ m s}^{-1}$  which is the instantaneous velocity of the ball  $t$  seconds after being released.

c  $s'(t) = 0$  when  $28.1 - 9.8t = 0$

$$\therefore 28.1 = 9.8t$$

$$\therefore t = \frac{28.1}{9.8}$$

So, the ball has reached its maximum height after  $\frac{28.1}{9.8}$  seconds, and is instantaneously at rest.

$$s\left(\frac{28.1}{9.8}\right) = 1.2 + 28.1\left(\frac{28.1}{9.8}\right) - 4.9\left(\frac{28.1}{9.8}\right)^2$$

$$\approx 41.5$$

So, the maximum height reached by the ball is about 41.5 m.



**d**  $s'(t) = 28.1 - 9.8t \text{ m s}^{-1}$

**i**  $s'(0) = 28.1$

The ball's speed when released is  $28.1 \text{ m s}^{-1}$ .

**ii**  $s'(2) = 28.1 - 9.8(2)$   
 $= 8.5$

The ball's speed at  $t = 2$  seconds is  $8.5 \text{ m s}^{-1}$ .

**iii**  $s'(5) = 28.1 - 9.8(5)$   
 $= -20.9$

The ball's speed at  $t = 5$  seconds is  $20.9 \text{ m s}^{-1}$ .

**3 a**  $s(t) = 12t - 2t^3 - 1 \text{ cm}$

$\therefore v(t) = 12 - 6t^2 \text{ cm s}^{-1} \quad \{v(t) = s'(t)\}$

$\therefore a(t) = -12t \text{ cm s}^{-2} \quad \{a(t) = v'(t)\}$

**b**  $s(0) = -1 \text{ cm}, \quad v(0) = 12 \text{ cm s}^{-1}, \quad a(0) = 0 \text{ cm s}^{-2}$

The particle is initially 1 cm to the left of the origin, travelling to the right at a constant speed of  $12 \text{ cm s}^{-1}$ .

**c**  $v(t) = 12 - 6t^2$   
 $= 6(2 - t^2)$

$= 6(\sqrt{2} + t)(\sqrt{2} - t) \text{ cm s}^{-1}$  which has sign diagram:

$v(t)$  changes sign when  $t = \sqrt{2}$  seconds, so this is when the particle changes direction.

$$\begin{aligned} s(\sqrt{2}) &= 12(\sqrt{2}) - 2(\sqrt{2})^3 - 1 \\ &= 12\sqrt{2} - 4\sqrt{2} - 1 \\ &= 8\sqrt{2} - 1 \end{aligned}$$

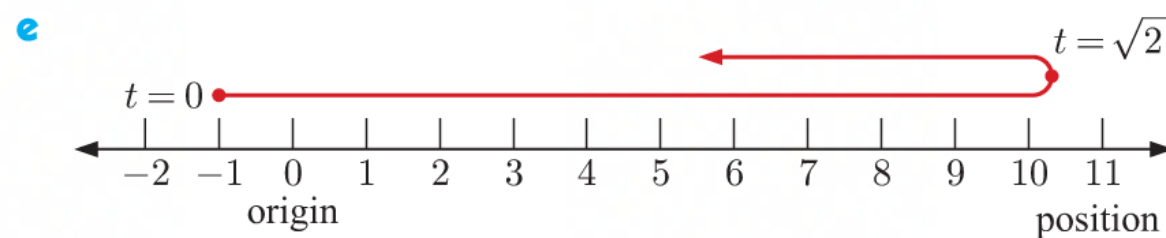
So, the particle changes direction when it is  $(8\sqrt{2} - 1) \text{ cm}$  to the right of O.

**d**  $a(t) = -12t$  has sign diagram:



**i** The particle's speed is increasing when  $v(t)$  and  $a(t)$  have the same sign. This occurs for  $t \geq \sqrt{2}$ .

**ii** The particle's velocity is never increasing.



**4 a**  $s(t) = 4 - \sqrt{t+1} = 4 - (t+1)^{\frac{1}{2}} \text{ m}$

$$\therefore v(t) = -\frac{1}{2}(t+1)^{-\frac{1}{2}} \quad \{v(t) = s'(t)\}$$

$$= -\frac{1}{2\sqrt{t+1}} \text{ m s}^{-1}$$

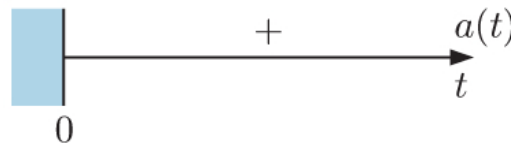
which has sign diagram:



and  $a(t) = \frac{1}{4}(t+1)^{-\frac{3}{2}} \quad \{a(t) = v'(t)\}$

$$= \frac{1}{4(t+1)^{\frac{3}{2}}} \text{ m s}^{-2}$$

which has sign diagram:



**b**  $s(0) = 3 \text{ m}, \quad v(0) = -\frac{1}{2} \text{ m s}^{-1}, \quad a(0) = \frac{1}{4} \text{ m s}^{-2}$

Initially, the particle is 3 m to the right of O, moving to the left at  $\frac{1}{2} \text{ m s}^{-1}$  with acceleration  $\frac{1}{4} \text{ m s}^{-2}$ .

**c**  $s(3) = 4 - \sqrt{4} = 2 \text{ m}, \quad v(3) = -\frac{1}{2\sqrt{4}} = -\frac{1}{4} \text{ m s}^{-1}, \quad a(3) = \frac{1}{4(4)^{\frac{3}{2}}} = \frac{1}{32} \text{ m s}^{-2}$

After 3 seconds, the particle is 2 m to the right of O, moving to the left at  $\frac{1}{4} \text{ m s}^{-1}$ , with acceleration  $\frac{1}{32} \text{ m s}^{-2}$ .

**d** The particle's speed is continuously decreasing.

**5 a** When the device reaches the water,

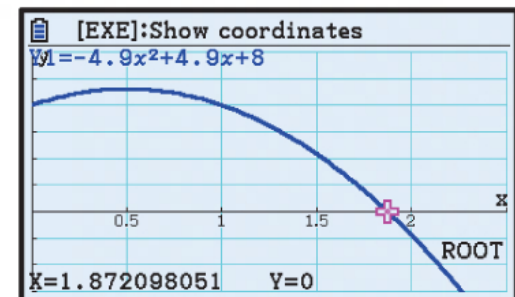
$$s(t) = 0$$

$$\therefore -4.9t^2 + 4.9t + 8 = 0$$

$$\therefore t \approx 1.87 \quad \{\text{using technology, } t > 0\}$$

So, the device takes approximately 1.87 seconds to reach the water.

$$\therefore k \approx 1.87$$

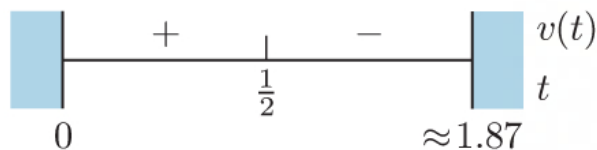


**b**  $s(t) = -4.9t^2 + 4.9t + 8 \text{ m}$

$$\therefore v(t) = -9.8t + 4.9 \text{ m s}^{-1} \quad \{v(t) = s'(t)\}$$

$$= 4.9(1 - 2t)$$

which has sign diagram:



and  $a(t) = -9.8 \text{ m s}^{-2} \quad \{a(t) = v'(t)\}$

which has sign diagram:



- c i** Looking at the sign diagrams for  $v(t)$  and  $a(t)$ ,  $v(0.2) > 0$  and  $a(0.2) < 0$   
 $\therefore v(0.2)$  and  $a(0.2)$  have opposite sign.  
 $\therefore$  the speed of the device is decreasing after 0.2 seconds.

- ii  $v(1) < 0$  and  $a(1) < 0$   
 $\therefore v(1)$  and  $a(1)$  have the same sign.  
 $\therefore$  the speed of the device is increasing after 1 second.

**6 a**  $x(t) = 1 - 2 \cos t \text{ cm}$

$\therefore v(t) = 2 \sin t \text{ cm s}^{-1} \quad \{v(t) = x'(t)\}$

$\therefore a(t) = 2 \cos t \text{ cm s}^{-2} \quad \{a(t) = v'(t)\}$

When  $t = 0$ ,  $x(0) = 1 - 2 \cos 0 = -1 \text{ cm}$

$v(0) = 2 \sin 0 = 0 \text{ cm s}^{-1}$

$a(0) = 2 \cos 0 = 2 \text{ cm s}^{-2}$

$\therefore$  P is initially 1 cm to the left of the origin, instantaneously at rest, and accelerating at  $2 \text{ cm s}^{-2}$ .

**b**  $x\left(\frac{\pi}{4}\right) = 1 - 2 \cos \frac{\pi}{4} = 1 - \sqrt{2} = -(\sqrt{2} - 1) \text{ cm}$ ,  $v\left(\frac{\pi}{4}\right) = 2 \sin \frac{\pi}{4} = \sqrt{2} \text{ cm s}^{-1}$ ,

$a\left(\frac{\pi}{4}\right) = 2 \cos \frac{\pi}{4} = \sqrt{2} \text{ cm s}^{-2}$

$\therefore$  at  $t = \frac{\pi}{4}$  seconds, the particle is  $(\sqrt{2} - 1) \text{ cm}$  left of O, moving to the right at  $\sqrt{2} \text{ cm s}^{-1}$ , with acceleration  $\sqrt{2} \text{ cm s}^{-2}$ .

**c** The particle reverses direction when  $v(t) = 0$

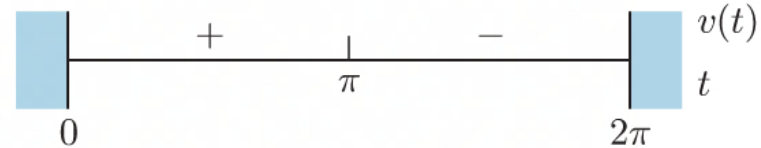
$\therefore 2 \sin t = 0$

$\therefore t = \pi \quad \{0 < t < 2\pi\}$

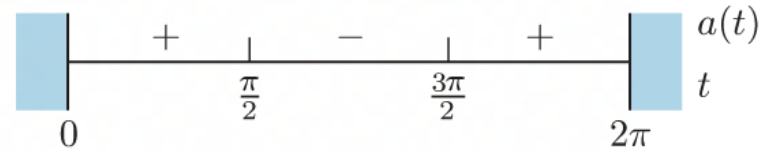
$x(\pi) = 1 - 2 \cos \pi = 3 \text{ cm}$

$\therefore$  the particle reverses direction at  $t = \pi$  seconds, 3 cm to the right of the origin.

**d**  $v(t) = 2 \sin t$  has sign diagram:



$a(t) = 2 \cos t$  has sign diagram:



The particle's speed is increasing when  $v(t)$  and  $a(t)$  have the same sign.

This occurs for  $0 \leq t \leq \frac{\pi}{2}$  seconds and  $\pi \leq t \leq \frac{3\pi}{2}$  seconds.

**7 a**  $s(t) = 8 \sin \frac{t}{2} \text{ m}$

i  $s(3) = 8 \sin \frac{3}{2}$   
 $\approx 7.98 > 0$

$\therefore$  after 3 seconds, the dog is to the right of its kennel.

ii  $s(7) = 8 \sin \frac{7}{2}$   
 $\approx -2.81 < 0$

$\therefore$  after 7 seconds, the dog is to the left of its kennel.

**b**  $s(t) = 8 \sin \frac{t}{2} \text{ m}$

$\therefore v(t) = 8\left(\frac{1}{2}\right) \cos \frac{t}{2} \quad \{v(t) = s'(t)\}$   
 $= 4 \cos \frac{t}{2} \text{ m s}^{-1}$



$$\text{c i } v(4) = 4 \cos 2 \\ \approx -1.66 < 0$$

$\therefore$  after 4 seconds, the dog is moving to the left.

$$\text{ii } v(10) = 4 \cos 5 \\ \approx 1.13 > 0$$

$\therefore$  after 10 seconds, the dog is moving to the right.

$$\text{d } a(t) = 4\left(-\frac{1}{2}\right) \sin \frac{t}{2} \quad \{a(t) = v'(t)\} \\ = -2 \sin \frac{t}{2} \text{ m s}^{-2}$$

$$\text{e } v(2) = 4 \cos 1 \quad a(2) = -2 \sin 1 \\ \approx 2.16 \text{ m s}^{-1} \quad \approx -1.68 \text{ m s}^{-2}$$

$$v(2) > 0 \text{ and } a(2) < 0$$

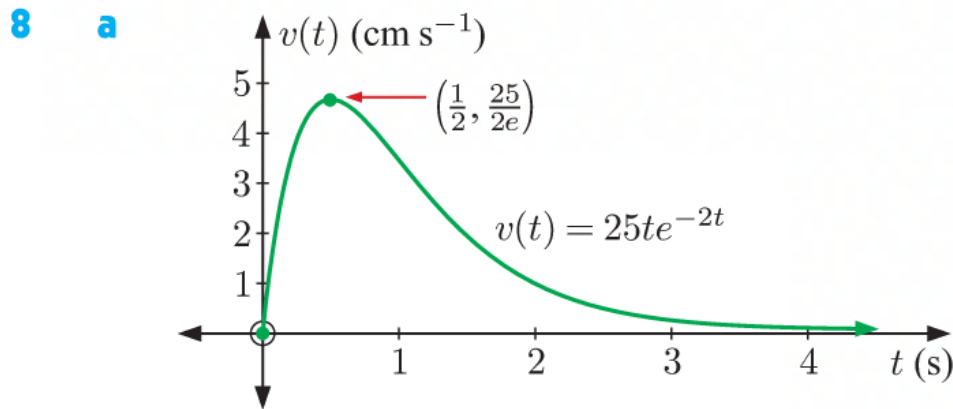
$\therefore v(2)$  and  $a(2)$  have opposite sign.

$\therefore$  the dog's speed is decreasing after 2 seconds.

$$\text{f } |v(t)| \text{ is a maximum when } v'(t) = 0 \\ \therefore -4\left(\frac{1}{2}\right) \sin \frac{t}{2} = 0 \\ \therefore -2 \sin \frac{t}{2} = 0 \\ \therefore \frac{t}{2} = k\pi \quad \{k \in \mathbb{Z}\} \\ \therefore t = 2k\pi$$

$$\text{Now } s(2k\pi) = 8 \sin k\pi \\ = 0$$

$\therefore$  the dog's speed is maximised when it is moving past its kennel.



$$\text{b } v(t) = 25te^{-2t} \text{ cm s}^{-1}, \quad t \geq 0 \\ \therefore a(t) = v'(t) = 25e^{-2t} + 25t(-2)e^{-2t} \quad \{\text{product rule}\} \\ = 25e^{-2t} - 50te^{-2t} \\ = 25(1 - 2t)e^{-2t} \text{ cm s}^{-2}, \quad t \geq 0 \quad \checkmark$$

$$\text{c } a(t) = 25(1 - 2t)e^{-2t} \text{ has sign diagram:}$$

The acceleration is positive and hence the velocity is increasing for  $0 \leq t \leq \frac{1}{2}$  second.

$$\text{d } v(t) = 25te^{-2t} \text{ has sign diagram:}$$

The speed of the object is decreasing when  $v(t)$  and  $a(t)$  have opposite signs.

This occurs when  $t \geq \frac{1}{2}$  second.

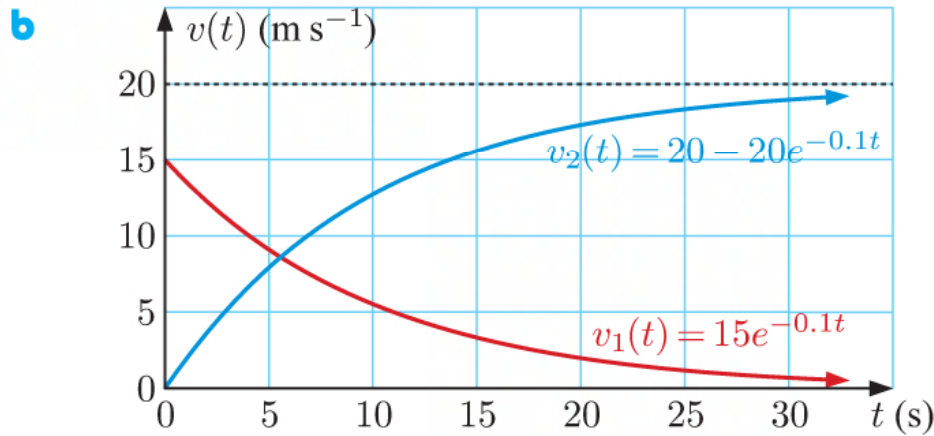


9 Lion:  $v_1(t) = 15e^{-0.1t} \text{ m s}^{-1}$

Zebra:  $v_2(t) = 20 - 20e^{-0.1t} \text{ m s}^{-1}$

a After 1 second, speed of lion =  $v_1(1) = 15e^{-0.1(1)}$   
 $\approx 13.6 \text{ m s}^{-1}$

speed of zebra =  $v_2(1) = 20 - 20e^{-0.1(1)}$   
 $\approx 1.90 \text{ m s}^{-1}$



As shown by the graph, the lion's speed  $v_1(t)$  decreases over time whereas the zebra's speed  $v_2(t)$  increases over time.

c 
$$\begin{aligned} \int_0^3 v_1(t) dt &= \int_0^3 15e^{-0.1t} dt \\ &= [-150e^{-0.1t}]_0^3 \\ &= -150e^{-0.1(3)} - (-150) \\ &= 150 - 150e^{-0.3} \\ &\approx 38.9 \end{aligned}$$

The lion has travelled a total distance of about 38.9 metres in the first 3 seconds.

d 
$$\begin{aligned} \int_0^3 [v_1(t) - v_2(t)] dt &= \int_0^3 [15e^{-0.1t} - (20 - 20e^{-0.1t})] dt \\ &= \int_0^3 (35e^{-0.1t} - 20) dt \\ &= [-350e^{-0.1t} - 20t]_0^3 \\ &= (-350e^{-0.3} - 60) - (-350) \\ &= 290 - 350e^{-0.3} \\ &\approx 30.7 \end{aligned}$$

In the first 3 seconds, the lion has gained about 30.7 metres on the zebra.

- e At the time when  $v_1(t) = v_2(t)$ , the lion and the zebra will be moving at the same speed. Since the lion's speed decreases over time and the zebra's speed increases over time, the zebra will be faster than the lion after that time. So, they will be closest at the point when their speeds are equal.

$$\begin{aligned}
 \text{f } v_1(t) &= v_2(t) \text{ when } 15e^{-0.1t} = 20 - 20e^{-0.1t} \\
 \therefore 35e^{-0.1t} &= 20 \\
 \therefore e^{-0.1t} &= \frac{20}{35} \\
 \therefore -0.1t &= \ln\left(\frac{20}{35}\right) \\
 \therefore t &= -10 \ln\left(\frac{4}{7}\right) \\
 \therefore t &\approx 5.60
 \end{aligned}$$

So,  $v_1(t) = v_2(t)$  after about 5.60 seconds.

g From e and f, the lion is closest to the zebra after about 5.60 seconds.

$$\begin{aligned}
 \int_0^{5.60} [v_1(t) - v_2(t)] dt &\approx [-350e^{-0.1t} - 20t]_0^{5.60} \\
 &\approx (-350e^{-0.560} - 20 \times 5.60) - (-350) \\
 &\approx 38.08
 \end{aligned}$$

So, after about 5.60 seconds, the lion has travelled about 38.08 metres more than the zebra has travelled. The zebra was however initially 40 metres ahead of the lion, so at their closest point, the zebra will still be about  $40 \text{ m} - 38.08 \text{ m} \approx 1.92 \text{ m}$  ahead of the lion.

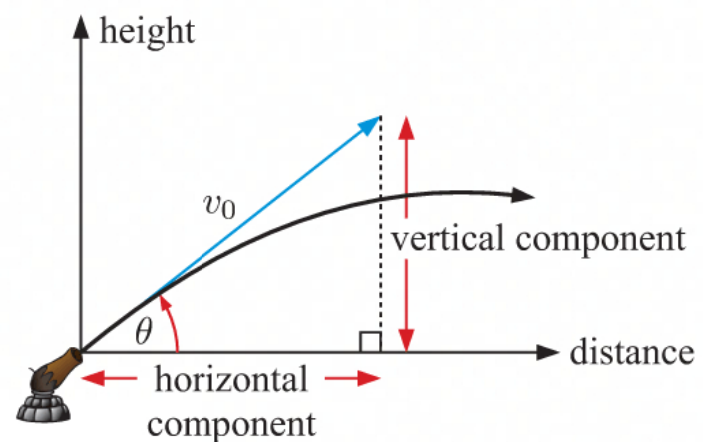
So, the lion did not catch the zebra but was about 1.92 m from the zebra at their closest point.

## INVESTIGATION

## PROJECTILE MOTION

1 a If the cannon is fired from ground level, the initial vertical height of the cannonball is 0 m.

$$\begin{aligned}
 \text{b } \sin \theta &= \frac{\text{vertical component}}{v_0} \\
 \therefore \text{vertical component} &= v_0 \sin \theta \\
 \therefore \text{the initial vertical velocity of the} \\
 &\text{cannonball is } v_0 \sin \theta \text{ m s}^{-1}.
 \end{aligned}$$



c After the cannonball is fired, the only force acting on the cannonball is the downward force of gravity, which has acceleration  $9.8 \text{ m s}^{-2}$ .

Since the positive direction is *upwards*, the vertical acceleration of the cannonball is given by  $a(t) = -9.8 \text{ m s}^{-2}$ .

$$\begin{aligned}
 \text{d } s(t) &= -4.9t^2 + [v_0 \sin \theta]t \\
 \therefore v(t) &= -9.8t + v_0 \sin \theta \quad \{v(t) = s'(t)\} \\
 \therefore a(t) &= -9.8
 \end{aligned}$$

$$s(0) = 0$$

So, the initial vertical height of the cannonball is 0 m.

$$v(0) = v_0 \sin \theta$$

So, the initial vertical velocity of the cannonball is  $v_0 \sin \theta \text{ m s}^{-1}$ .

$$a(t) = -9.8$$

So, the vertical acceleration of the cannonball is a constant  $-9.8 \text{ m s}^{-2}$ .

All of these satisfy the properties in a, b, and c.

**e** When the cannonball hits the ground,  $s(t) = 0$

$$\therefore -4.9t^2 + [v_0 \sin \theta]t = 0$$

$$\therefore t(v_0 \sin \theta - 4.9t) = 0$$

$$\therefore t = 0 \text{ or } v_0 \sin \theta - 4.9t = 0$$

$$\therefore 4.9t = v_0 \sin \theta$$

$$\therefore t = \frac{v_0 \sin \theta}{4.9}$$

$\therefore$  the cannonball takes  $\frac{v_0 \sin \theta}{4.9}$  seconds to hit the ground.

**2 a**  $\cos \theta = \frac{\text{horizontal component}}{v_0}$

$$\therefore \text{horizontal component} = v_0 \cos \theta$$

$\therefore$  the horizontal velocity of the cannonball is  $v_0 \cos \theta \text{ m s}^{-1}$ .

**b** Horizontal distance travelled = horizontal velocity  $\times$  time of flight

$$= v_0 \cos \theta \text{ m s}^{-1} \times \frac{v_0 \sin \theta}{4.9} \text{ s} \quad \{\text{from 1 e}\}$$

$$= \frac{v_0^2 \cos \theta \sin \theta}{4.9} \text{ m}$$

$$= \frac{v_0^2 \frac{1}{2} \sin 2\theta}{4.9} \text{ m}$$

$$= \frac{v_0^2 \sin 2\theta}{9.8} \text{ m}$$

**c i**  $v_0 = 200 \text{ m s}^{-1}$ ,  $\theta = 20^\circ$

Horizontal distance travelled

$$= \frac{(200)^2 \sin 40^\circ}{9.8}$$

$$\approx 2623.62 \text{ m}$$

**ii**  $v_0 = 200 \text{ m s}^{-1}$ ,  $\theta = 50^\circ$

Horizontal distance travelled

$$= \frac{(200)^2 \sin 100^\circ}{9.8}$$

$$\approx 4019.62 \text{ m}$$

**iii**  $v_0 = 200 \text{ m s}^{-1}$ ,  $\theta = 80^\circ$

$$\text{Horizontal distance travelled} = \frac{(200)^2 \sin 160^\circ}{9.8}$$

$$\approx 1396.00 \text{ m}$$

**d**  $\frac{v_0^2 \sin 2\theta}{9.8}$  is maximised when  $\sin 2\theta = 1$   
 $\therefore 2\theta = 90^\circ \quad \{0^\circ \leq 2\theta \leq 180^\circ\}$   
 $\therefore \theta = 45^\circ$

$\therefore$  the angle which maximises the range of the cannonball is  $\theta = 45^\circ$ .

## REVIEW SET 23A

**1 a**  $s(t) = 12 - 2t \text{ m}$ ,  $0 \leq t \leq 10 \text{ s}$

$$s(0) = 12 - 2(0)$$

$$= 12 \text{ m}$$

$\therefore$  the initial displacement of the object is 12 m to the right of the origin.



$$\begin{aligned} \text{b i } s(1) &= 12 - 2(1) \\ &= 12 - 2 \\ &= 10 \text{ m} \end{aligned}$$

$\therefore$  the displacement of the object at  $t = 1$  second is 10 m to the right of the origin.

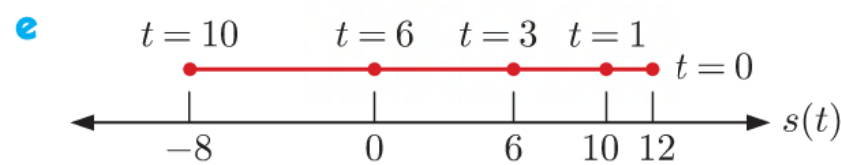
$$\begin{aligned} \text{ii } s(3) &= 12 - 2(3) \\ &= 12 - 6 \\ &= 6 \text{ m} \end{aligned}$$

$\therefore$  the displacement of the object at  $t = 3$  seconds is 6 m to the right of the origin.

$$\begin{aligned} \text{c The object is at the origin when } s(t) &= 0 \\ \therefore 12 - 2t &= 0 \\ \therefore 2t &= 12 \\ \therefore t &= 6 \end{aligned}$$

$\therefore$  the object reaches the origin at  $t = 6$  seconds.

d No, the displacement function is linear, so it has no turning points.



$$\begin{aligned} \text{2 a } s(t) &= 2t^2 + t - 5 \text{ cm}, \quad t \geq 0 \text{ s} \\ s(1) &= 2(1)^2 + 1 - 5 & s(5) &= 2(5)^2 + 5 - 5 \\ &= 2 + 1 - 5 & &= 50 + 5 - 5 \\ &= -2 \text{ cm} & &= 50 \text{ cm} \end{aligned}$$

$$\begin{aligned} \text{average velocity} &= \frac{s(5) - s(1)}{5 - 1} \\ &= \frac{50 - (-2)}{5 - 1} \\ &= \frac{52}{4} \\ &= 13 \text{ cm s}^{-1} \end{aligned}$$

$\therefore$  the average velocity from  $t = 1$  to  $t = 5$  seconds is  $13 \text{ cm s}^{-1}$ .

$$\text{b } v(t) = s'(t) = 4t + 1 \text{ cm s}^{-1}$$

$$\begin{aligned} \text{i } v(2) &= 4(2) + 1 \\ &= 8 + 1 \\ &= 9 \text{ cm s}^{-1} \end{aligned}$$

$\therefore$  the instantaneous velocity at  $t = 2$  seconds is  $9 \text{ cm s}^{-1}$ .

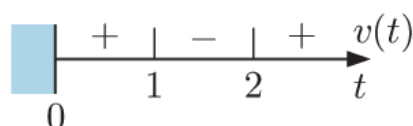
$$\begin{aligned} \text{ii } v(4) &= 4(4) + 1 \\ &= 16 + 1 \\ &= 17 \text{ cm s}^{-1} \end{aligned}$$

$\therefore$  the instantaneous velocity at  $t = 4$  seconds is  $17 \text{ cm s}^{-1}$ .

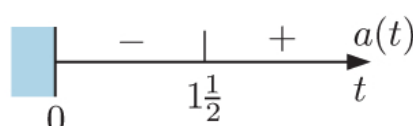
$$\text{c } a(t) = v'(t) = 4 \text{ cm s}^{-2}$$

$$\begin{aligned} \text{3 a } s(t) &= 2t^3 - 9t^2 + 12t - 5 \text{ cm}, \quad t \geq 0 \text{ s} \\ \therefore v(t) &= 6t^2 - 18t + 12 \text{ cm s}^{-1} & \{v(t) &= s'(t)\} \\ \therefore a(t) &= 12t - 18 \text{ cm s}^{-2} & \{a(t) &= v'(t)\} \end{aligned}$$

$v(t)$  has sign diagram:



$a(t)$  has sign diagram:





**b**  $s(0) = -5 \text{ cm}$   
 $v(0) = 12 \text{ cm s}^{-1}$   
 $a(0) = -18 \text{ cm s}^{-2}$

The particle P is initially 5 cm to the left of the origin, moving to the right at  $12 \text{ cm s}^{-1}$ , with acceleration  $-18 \text{ cm s}^{-2}$  (decreasing speed).

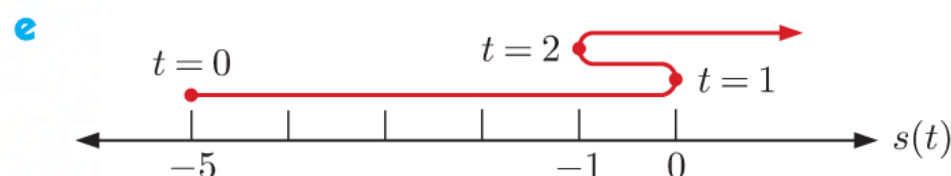
**c**  $s(2) = 2(2)^3 - 9(2)^2 + 12(2) - 5$   $v(2) = 6(2)^2 - 18(2) + 12$   
 $= 16 - 36 + 24 - 5$   $= 24 - 36 + 12$   
 $= -1 \text{ cm}$   $= 0 \text{ cm s}^{-1}$   
 $a(2) = 12(2) - 18$   
 $= 24 - 18$   
 $= 6 \text{ cm s}^{-2}$

At  $t = 2$ , the particle is 1 cm to the left of the origin, is instantaneously stationary, and is beginning to accelerate.

- d** The particle changes direction when  $v(t)$  changes sign.  
 From the sign diagram in **a**, this occurs at  $t = 1$  second and  $t = 2$  seconds.

$$\begin{array}{ll} s(1) = 2 - 9 + 12 - 5 & s(2) = 2(2)^3 - 9(2)^2 + 12(2) - 5 \\ = 0 & = 16 - 36 + 24 - 5 \\ & = -1 \end{array}$$

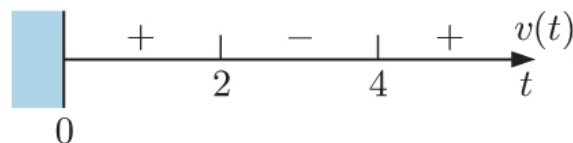
$\therefore$  the particle changes direction at  $t = 1$  second when it is at the origin, and at  $t = 2$  seconds when it is 1 cm to the left of the origin.



- f** The particle's speed is increasing when  $v(t)$  and  $a(t)$  have the same sign.  
 From the sign diagrams in **a**, this occurs for  $1 \leq t \leq 1\frac{1}{2}$  and  $t \geq 2$  seconds.

**4 a**  $v(t) = s'(t) = t^2 - 6t + 8$   
 $= (t - 2)(t - 4)$

$\therefore$  the sign diagram of  $v(t)$  is:



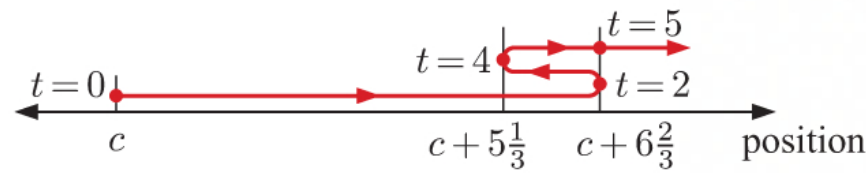
- b** Since the signs change, the particle reverses direction at  $t = 2$  and  $t = 4$  seconds.

Now  $s(t) = \int v(t) dt$   
 $= \int (t^2 - 6t + 8) dt$   
 $= \frac{1}{3}t^3 - 3t^2 + 8t + c$

Hence  $s(0) = c$

$$\begin{aligned} s(2) &= \frac{1}{3}(2)^3 - 3(2)^2 + 8(2) + c = \frac{8}{3} - 12 + 16 + c = c + 6\frac{2}{3} \\ s(4) &= \frac{1}{3}(4)^3 - 3(4)^2 + 8(4) + c = \frac{64}{3} - 48 + 32 + c = c + 5\frac{1}{3} \\ s(5) &= \frac{1}{3}(5)^3 - 3(5)^2 + 8(5) + c = \frac{125}{3} - 75 + 40 + c = c + 6\frac{2}{3} \end{aligned}$$

Motion diagram:



The particle initially moves in the positive direction, then at  $t = 2$ ,  $6\frac{2}{3}$  m from its starting point, it changes direction. It changes direction again at  $t = 4$ ,  $5\frac{1}{3}$  m from its starting point, and at  $t = 5$ , it is  $6\frac{2}{3}$  m from its starting point again.

- c** After 5 seconds, the particle is  $6\frac{2}{3}$  metres from its original position.
- d** Total distance travelled  $= (c + 6\frac{2}{3} - c) + (c + 6\frac{2}{3} - [c + 5\frac{1}{3}]) + (c + 6\frac{2}{3} - [c + 5\frac{1}{3}])$   
 $= 6\frac{2}{3} + 1\frac{1}{3} + 1\frac{1}{3}$   
 $= 9\frac{1}{3}$  metres

**5 a**  $v(t) = 2.75 - t + 0.5t^{1.2} \text{ m s}^{-1}, \quad 0 \leq t \leq 6 \text{ s}$

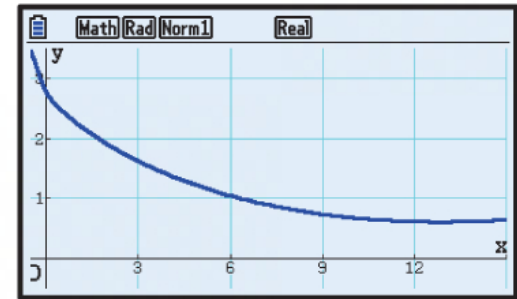
**i**  $v(0) = 2.75 \text{ m s}^{-1}$

$\therefore$  the velocity of the kayak after the kayaker stops paddling is  $2.75 \text{ m s}^{-1}$ .

**ii**  $v(3) = 2.75 - 3 + 0.5(3)^{1.2}$   
 $\approx 1.62$

$\therefore$  the velocity of the kayak after 3 seconds is about  $1.62 \text{ m s}^{-1}$ .

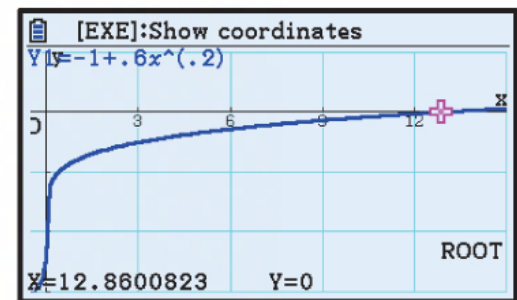
**b**  $v(t) > 0$  for all  $t$  {using technology}



So,  $v(t)$  has sign diagram:

$a(t) = -1 + 0.5(1.2)t^{0.2} \quad \{a(t) = v'(t)\}$   
 $= -1 + 0.6t^{0.2} \text{ m s}^{-2}$

$a(t) = 0$  when  $t \approx 12.9$  {using technology}



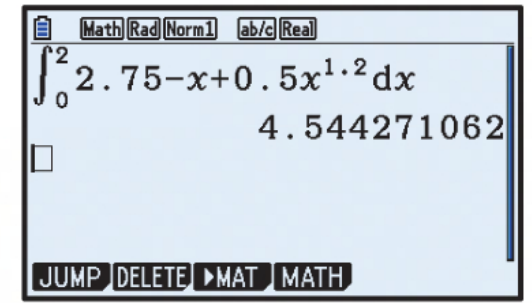
So,  $a(t)$  has sign diagram:

Now, our model only considers the interval  $0 \leq t \leq 6$  seconds, where  $v(t)$  and  $a(t)$  have opposite sign.

$\therefore$  the kayak's speed is decreasing during the 6 second period.

$$\begin{aligned} \text{c} \quad \int_0^2 v(t) dt &= \int_0^2 (2.75 - t + 0.5t^{1.2}) dt \\ &\approx 4.54 \quad \{\text{using technology}\} \end{aligned}$$

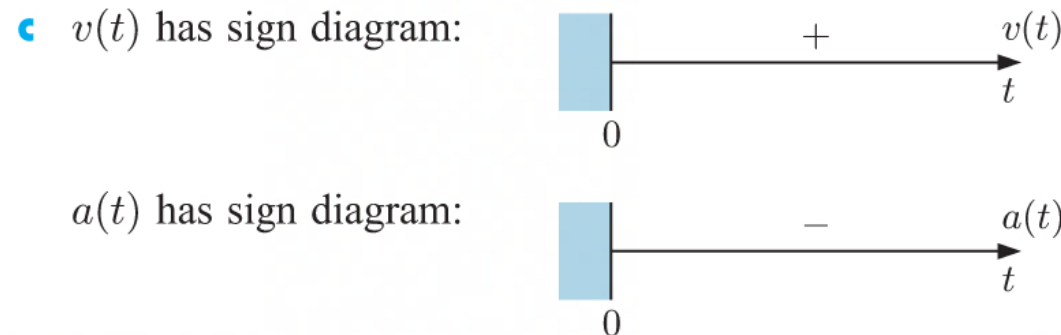
The kayak travels approximately 4.54 m in the first 2 seconds after the kayaker stops paddling.



$$\begin{aligned} \text{6 a} \quad s(t) &= 15t - \frac{60}{(t+1)^2} \text{ cm}, \quad t \geq 0 \text{ s} \\ &= 15t - 60(t+1)^{-2} \\ \therefore v(t) &= 15 + 120(t+1)^{-3} \quad \{v(t) = s'(t)\} \\ &= 15 + \frac{120}{(t+1)^3} \text{ cm s}^{-1} \\ \therefore a(t) &= -360(t+1)^{-4} \quad \{a(t) = v'(t)\} \\ &= -\frac{360}{(t+1)^4} \text{ cm s}^{-2} \end{aligned}$$

$$\begin{aligned} \text{b} \quad s(3) &= 15(3) - \frac{60}{(3+1)^2} & v(3) &= 15 + \frac{120}{(3+1)^3} & a(3) &= -\frac{360}{(3+1)^4} \\ &= 41.25 \text{ cm} & &\approx 16.9 \text{ cm s}^{-1} & &\approx -1.41 \text{ cm s}^{-2} \end{aligned}$$

So, at time  $t = 3$  seconds, the particle is 41.25 cm to the right of O, moving to the right at about  $16.9 \text{ cm s}^{-1}$ , with decreasing speed ( $a(3) \approx -1.41 \text{ cm s}^{-2}$ ).



The signs of  $v(t)$  and  $a(t)$  are never the same.  
 $\therefore$  the particle's speed is never increasing.

$$\begin{aligned} \text{7 a} \quad x(t) &= 3 + 2 \sin \pi t \text{ m}, \quad t \geq 0 \text{ s} \\ \therefore v(t) &= 2\pi \cos \pi t \quad \{v(t) = x'(t)\} \\ \therefore a(t) &= -2\pi^2 \sin \pi t \quad \{a(t) = v'(t)\} \\ x(0) &= 3 + 2 \sin 0 & v(0) &= 2\pi \cos 0 & a(0) &= -2\pi^2 \sin 0 \\ &= 3 \text{ m} & &= 2\pi \text{ m s}^{-1} & &= 0 \text{ m s}^{-2} \end{aligned}$$

The object is initially 3 m to the right of the origin, moving to the right at  $2\pi \text{ m s}^{-1}$ , and has acceleration  $0 \text{ m s}^{-2}$ .



**b**  $v(t) = 0$  when  $2\pi \cos \pi t = 0$

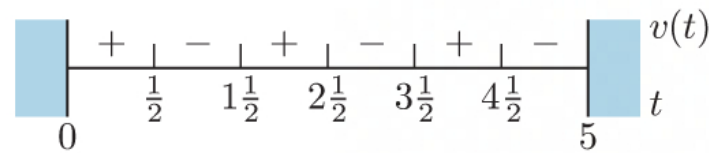
$$\therefore \cos \pi t = 0$$

$$\therefore \pi t = \left(k + \frac{1}{2}\right) \pi, \quad k \in \mathbb{Z}$$

$$\therefore t = k + \frac{1}{2}, \quad k \in \mathbb{Z}$$

$$\therefore t = \frac{1}{2}, 1\frac{1}{2}, 2\frac{1}{2}, 3\frac{1}{2}, 4\frac{1}{2} \text{ seconds} \quad \{0 \leq t \leq 5\}$$

$v(t)$  has sign diagram:

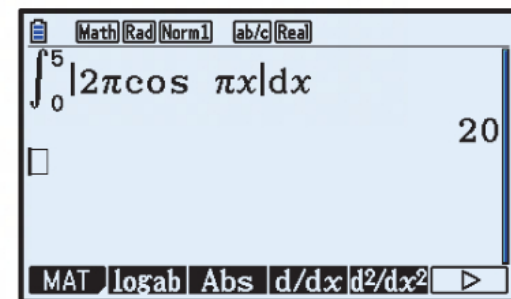


So, the spotlight changes direction at  $t = \frac{1}{2}, 1\frac{1}{2}, 2\frac{1}{2}, 3\frac{1}{2}$ , and  $4\frac{1}{2}$  seconds during the first 5 seconds.

**c** 
$$\int_0^5 |v(t)| dt = \int_0^5 |2\pi \cos \pi t| dt$$

$$= 20 \quad \{\text{using technology}\}$$

$\therefore$  the total distance travelled by the spotlight in the first 5 seconds is 20 m.



**8 a**  $v(t) = s'(t) = 2 \cos 4t \text{ m s}^{-1}$

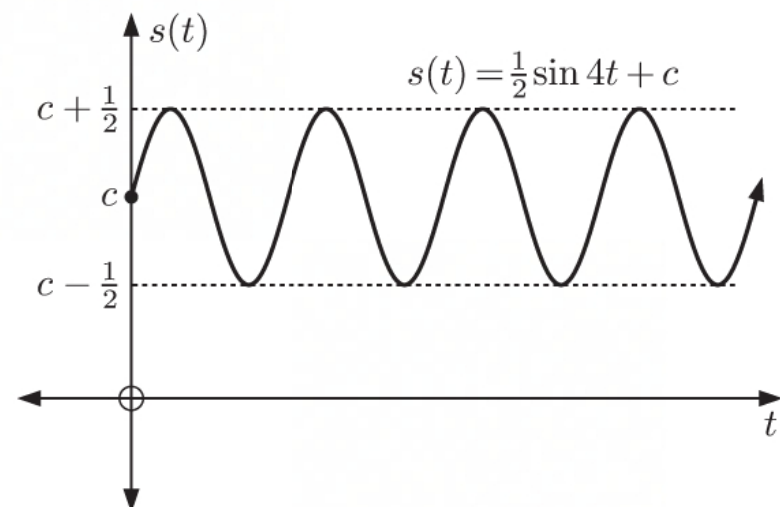
$$\therefore s(t) = \int 2 \cos 4t dt$$

$$= \frac{1}{2} \sin 4t + c \text{ m}$$

The graph shows that the particle oscillates between positions  $c + \frac{1}{2}$  and  $c - \frac{1}{2}$ .

$$\text{Distance} = \left(c + \frac{1}{2}\right) - \left(c - \frac{1}{2}\right)$$

$$= 1 \text{ m}$$



**b**  $s(t) = \frac{1}{2} \sin 4t + c$

$$s\left(\frac{\pi}{12}\right) = 6$$

$$\therefore \frac{1}{2} \sin \frac{\pi}{3} + c = 6$$

$$\therefore \frac{\sqrt{3}}{4} + c = 6$$

$$\therefore c = 6 - \frac{\sqrt{3}}{4}$$

$$s\left(\frac{\pi}{6}\right) = \frac{1}{2} \sin \frac{2\pi}{3} + 6 - \frac{\sqrt{3}}{4}$$

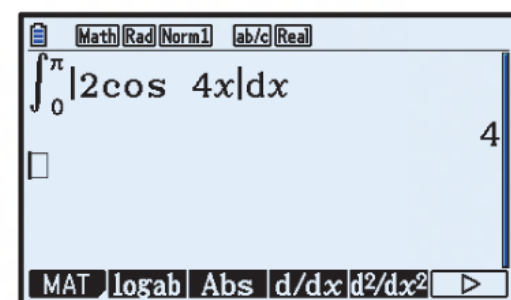
$$= \frac{\sqrt{3}}{4} + 6 - \frac{\sqrt{3}}{4}$$

$$= 6 \text{ m}$$

**c** 
$$\int_0^\pi |v(t)| dt = \int_0^\pi |2 \cos 4t| dt$$

$$= 4 \quad \{\text{using technology}\}$$

$\therefore$  the total distance travelled by the particle in the first  $\pi$  seconds is 4 m.





**9 a**  $a(t) = -2 \text{ m s}^{-2}$

$$\begin{aligned}\therefore v(t) &= \int a(t) dt \\ &= \int -2 dt \\ &= -2t + c \text{ m s}^{-1}\end{aligned}$$

But  $v(0) = 65$

$$\therefore c = 65$$

$$\therefore v(t) = -2t + 65 \text{ m s}^{-1}$$

**c i**  $v(t) = 3$  when  $-2t + 65 = 3$   
 $\therefore 2t = 62$   
 $\therefore t = 31$

$\therefore$  it will take 31 seconds for the plane to reduce its speed to  $3 \text{ m s}^{-1}$ .

**ii**  $\int_0^{31} v(t) dt = \int_0^{31} (-2t + 65) dt$   
 $= [-t^2 + 65t]_0^{31}$   
 $= (-31^2 + 65(31)) - 0$   
 $= -961 + 2015$   
 $= 1054$

$\therefore$  the plane will have travelled 1054 m along the runway after 31 seconds.

**10**  $v(t) = \frac{100}{(t+2)^2} \text{ m s}^{-1}$

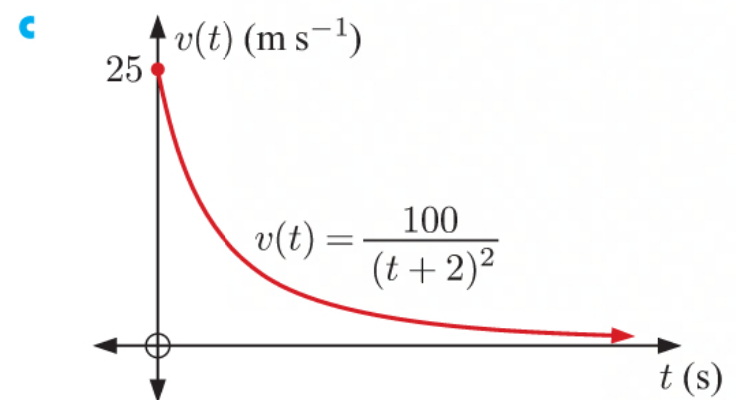
**a**  $v(0) = \frac{100}{2^2} = \frac{100}{4} = 25$

The initial velocity of the boat was  $25 \text{ m s}^{-1}$ .

**b** As  $t \rightarrow \infty$ ,  $(t+2)^2 \rightarrow \infty$   
 $\therefore v(t) \rightarrow 0$  from above

$v(3) = \frac{100}{5^2} = \frac{100}{25} = 4$

The velocity of the boat after 3 seconds was  $4 \text{ m s}^{-1}$ .



**d**  $\int_0^2 v(t) dt = \int_0^2 100(t+2)^{-2} dt$   
 $= [-100(t+2)^{-1}]_0^2$   
 $= -\frac{100}{4} - \left(-\frac{100}{2}\right)$   
 $= 25$

The boat travels a total distance of 25 m in the first 2 seconds after its engine is turned off.

e Suppose the boat travels 30 m after  $T$  seconds, then

$$\int_0^T (100(t+2)^{-2}) dt = 30$$

$$\therefore \left[ -100(t+2)^{-1} \right]_0^T = 30$$

$$\therefore \left( -\frac{100}{T+2} \right) - \left( -\frac{100}{2} \right) = 30$$

$$\therefore -\frac{100}{T+2} = -20$$

$$\therefore 20T + 40 = 100$$

$$\therefore 20T = 60$$

$$\therefore T = 3$$

So it will take 3 seconds for the boat to travel 30 metres.

## REVIEW SET 23B

1 a  $s(t) = t^2 + 4t + 1$  m,  $t \geq 0$  s  
 $\therefore s(0) = 1$  m  
 $\therefore$  the object is initially 1 m to the right of the origin.

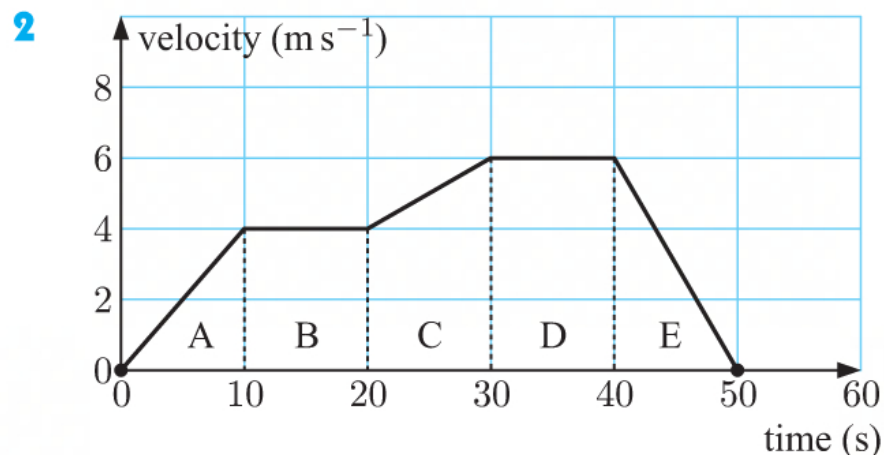
b average velocity =  $\frac{s(3) - s(1)}{3 - 1}$   
 $= \frac{3^2 + 4(3) + 1 - (1^2 + 4(1) + 1)}{2}$   
 $= \frac{22 - 6}{2}$   
 $= 8 \text{ m s}^{-1}$

$\therefore$  the average velocity from  $t = 1$  to  $t = 3$  seconds is  $8 \text{ m s}^{-1}$ .

c  $v(t) = s'(t) = 2t + 4 \text{ m s}^{-1}$

d  $v(1) = 2(1) + 4$   
 $= 2 + 4$   
 $= 6 \text{ m s}^{-1}$

$\therefore$  the instantaneous velocity at  $t = 1$  second is  $6 \text{ m s}^{-1}$ .



Total distance travelled = total area under graph

$$= \text{area A} + \text{area B} + \text{area C} + \text{area D} + \text{area E}$$

$$= \frac{1}{2}(10)(4) + (10)(4) + \left(\frac{6+4}{2}\right)(10) + (10)(6) + \frac{1}{2}(10)(6)$$

$$= 20 + 40 + 50 + 60 + 30$$

$$= 200 \text{ metres}$$

$$\begin{aligned}
 \mathbf{3} \quad \mathbf{a} \quad v(t) &= \int a(t) \, dt \\
 &= \int (6t - 30) \, dt \\
 &= 3t^2 - 30t + c
 \end{aligned}$$

$$\begin{aligned}
 \text{Now } v(0) &= 27 \\
 \therefore c &= 27 \\
 \therefore v(t) &= 3t^2 - 30t + 27 \text{ cm s}^{-1}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad s(t) &= \int v(t) \, dt \\
 &= \int (3t^2 - 30t + 27) \, dt \\
 &= t^3 - 15t^2 + 27t + c
 \end{aligned}$$

$$\begin{aligned}
 \text{Now } s(0) &= 0 \\
 \therefore c &= 0 \\
 \therefore s(t) &= t^3 - 15t^2 + 27t \text{ cm} \\
 \therefore s(6) &= 6^3 - 15(6)^2 + 27(6) \\
 &= 216 - 540 + 162 \\
 &= -162
 \end{aligned}$$

$\therefore$  the particle is 162 cm to the left of the origin after 6 seconds.

$$\begin{aligned}
 \mathbf{4} \quad \mathbf{a} \quad x(t) &= 3t - t\sqrt{t} \text{ cm}, \quad t \geq 0 \text{ s} \\
 &= 3t - t^{\frac{3}{2}}
 \end{aligned}$$

$$\begin{aligned}
 \therefore v(t) &= 3 - \frac{3}{2}t^{\frac{1}{2}} \text{ cm s}^{-1} & \{v(t) = x'(t)\} \\
 &= 3 - \frac{3}{2}\sqrt{t} \text{ cm s}^{-1}
 \end{aligned}$$

$$\begin{aligned}
 \therefore a(t) &= -\frac{3}{4}t^{-\frac{1}{2}} \text{ cm s}^{-2} & \{a(t) = v'(t)\} \\
 &= -\frac{3}{4\sqrt{t}} \text{ cm s}^{-2}
 \end{aligned}$$

$v(t)$  has sign diagram:



$a(t)$  has sign diagram:



$$\mathbf{b} \quad x(0) = 0 \text{ cm} \quad v(0) = 3 \text{ cm s}^{-1}$$

The particle is initially at the origin, moving to the right at  $3 \text{ cm s}^{-1}$ .

$$\begin{aligned}
 \mathbf{c} \quad x(2) &= 3(2) - 2\sqrt{2} & v(2) &= 3 - \frac{3}{2}\sqrt{2} & a(2) &= -\frac{3}{4\sqrt{2}} \\
 &= 6 - 2\sqrt{2} & &\approx 0.879 \text{ cm s}^{-1} & &\approx -0.530 \text{ cm s}^{-2} \\
 &\approx 3.17 \text{ cm}
 \end{aligned}$$

So, at time  $t = 2$  seconds, the particle is about 3.17 cm to the right of the origin, travelling to the right at about  $0.879 \text{ cm s}^{-1}$ , with decreasing speed ( $a(2) \approx -0.530 \text{ cm s}^{-2}$ ).

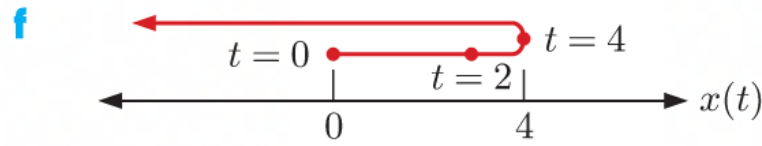
$\mathbf{d}$  The particle reverses direction when the sign of  $v(t)$  changes.

This occurs at  $t = 4$  seconds.

$$\begin{aligned}
 x(4) &= 3(4) - 4\sqrt{4} \\
 &= 12 - 8 \\
 &= 4 \text{ cm}
 \end{aligned}$$

$\therefore$  the particle reverses direction at  $t = 4$  seconds, when it is 4 cm to the right of the origin.

- e** The particle's speed is decreasing when  $v(t)$  and  $a(t)$  have opposite sign.  
From the sign diagrams in **a**, this occurs when  $0 \leq t \leq 4$  seconds.



**g** 
$$\begin{aligned} x(6) &= 3(6) - 6\sqrt{6} \\ &= 18 - 6\sqrt{6} \\ &\approx 3.30 \text{ cm} \end{aligned}$$

Total distance travelled in first 6 seconds  
 $\approx 4 + (4 - 3.30)$   
 $\approx 4.70 \text{ cm}$

**5 a**  $v(t) = 4.8t^2 - 0.8t^3 \text{ m s}^{-1}, \quad 0 \leq t \leq 6 \text{ s}$   
 $a(t) = 9.6t - 2.4t^2 \text{ m s}^{-2} \quad \{a(t) = v'(t)\}$

**i** 
$$\begin{aligned} a(1) &= 9.6 - 2.4 \\ &= 7.2 \text{ m s}^{-2} \end{aligned}$$

$\therefore$  the acceleration of the human cannonball after 1 second is  $7.2 \text{ m s}^{-2}$ .

**iii** 
$$\begin{aligned} a(4) &= 9.6(4) - 2.4(4)^2 \\ &= 38.4 - 38.4 \\ &= 0 \text{ m s}^{-2} \end{aligned}$$

$\therefore$  the acceleration of the human cannonball after 4 seconds is  $0 \text{ m s}^{-2}$ .

**ii** 
$$\begin{aligned} a(2) &= 9.6(2) - 2.4(2)^2 \\ &= 19.2 - 9.6 \\ &= 9.6 \text{ m s}^{-2} \end{aligned}$$

$\therefore$  the acceleration of the human cannonball after 2 seconds is  $9.6 \text{ m s}^{-2}$ .

**iv** 
$$\begin{aligned} a(5) &= 9.6(5) - 2.4(5)^2 \\ &= 48 - 60 \\ &= -12 \text{ m s}^{-2} \end{aligned}$$

$\therefore$  the acceleration of the human cannonball after 5 seconds is  $-12 \text{ m s}^{-2}$ .

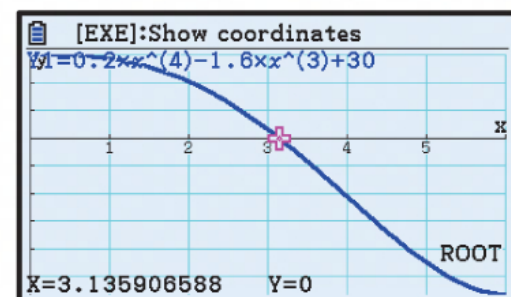
**b** 
$$\begin{aligned} \int_0^3 v(t) dt &= \int_0^3 (4.8t^2 - 0.8t^3) dt \\ &= [1.6t^3 - 0.2t^4]_0^3 \\ &= (1.6(3)^3 - 0.2(3)^4) - 0 \\ &= 27 \end{aligned}$$

The human cannonball travels 27 m in the first 3 seconds.

- c** Suppose the human cannonball has travelled 30 m after  $T$  seconds, then

$$\begin{aligned} \int_0^T v(t) dt &= 30 \\ \therefore \int_0^T (4.8t^2 - 0.8t^3) dt &= 30 \\ \therefore [1.6t^3 - 0.2t^4]_0^T &= 30 \\ \therefore 1.6T^3 - 0.2T^4 &= 30 \\ \therefore 0.2T^4 - 1.6T^3 + 30 &= 0 \\ \therefore T &\approx 3.14 \quad \{0 \leq T \leq 6\} \\ &\quad \{\text{using technology}\} \end{aligned}$$

$\therefore$  it takes about 3.14 seconds for the human cannonball to travel 30 m.





**6 a**  $s(t) = 80e^{-\frac{t}{10}} - 40t \text{ m}, t \geq 0 \text{ s}$

$$\therefore v(t) = -8e^{-\frac{t}{10}} - 40 \text{ m s}^{-1} \quad \{v(t) = s'(t)\}$$

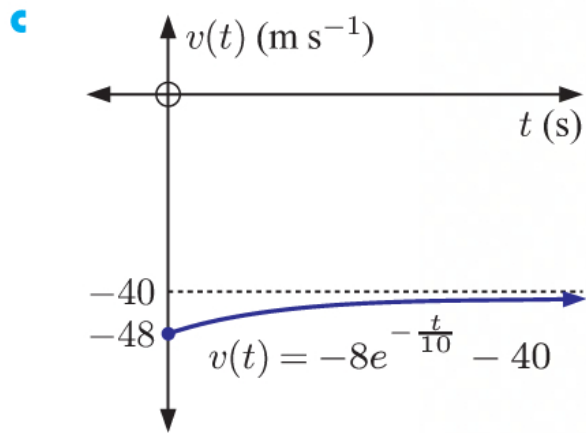
$$\therefore a(t) = \frac{8}{10}e^{-\frac{t}{10}} = \frac{4}{5}e^{-\frac{t}{10}} \text{ m s}^{-2} \quad \{a(t) = v'(t)\}$$

**b** When  $t = 0$ ,  $s(0) = 80 \text{ m}$

$$v(0) = -8 - 40 = -48 \text{ m s}^{-1}$$

$$a(0) = \frac{4}{5} \text{ m s}^{-2}$$

$\therefore$  the particle is initially 80 m to the right of the origin, moving to the left at  $48 \text{ m s}^{-1}$  with acceleration  $0.8 \text{ m s}^{-2}$ .



**d** When  $v = -44$ ,  $-8e^{-\frac{t}{10}} - 40 = -44$

$$\therefore -8e^{-\frac{t}{10}} = -4$$

$$\therefore e^{-\frac{t}{10}} = \frac{1}{2}$$

$$\therefore -\frac{t}{10} = \ln\left(\frac{1}{2}\right)$$

$$\therefore t = 10 \ln 2$$

$\therefore$  the particle P has velocity  $-44 \text{ m s}^{-1}$  at  $t = 10 \ln 2$  seconds.

**7 a**  $s(t) = 30 + \cos \pi t \text{ cm}, t \geq 0 \text{ s}$

$$\therefore v(t) = -\pi \sin \pi t \text{ cm s}^{-1} \quad \{v(t) = s'(t)\}$$

$$v(0) = -\pi \sin 0 = 0 \text{ cm s}^{-1},$$

$$v\left(\frac{1}{2}\right) = -\pi \sin \frac{\pi}{2} = -\pi \text{ cm s}^{-1},$$

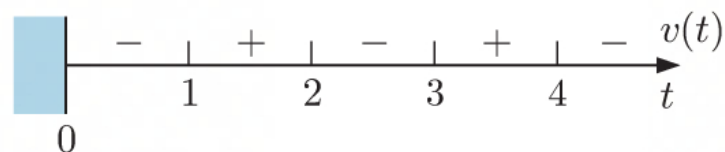
$$v(1) = -\pi \sin \pi = 0 \text{ cm s}^{-1},$$

$$v\left(1\frac{1}{2}\right) = -\pi \sin \frac{3\pi}{2} = \pi \text{ cm s}^{-1}$$

$$v(2) = -\pi \sin 2\pi = 0 \text{ cm s}^{-1}$$

**b** The cork is falling when its velocity is negative.

$v(t)$  has sign diagram:



$v(t)$  is negative when  $0 \leq t \leq 1$ ,  $2 \leq t \leq 3$ ,  $4 \leq t \leq 5$ , and so on.

So, the cork is falling when  $2n \leq t \leq 2n + 1$ ,  $n \in \{0, 1, 2, 3, \dots\}$

$$\begin{aligned} \mathbf{8} \quad \mathbf{a} \quad v(t) &= \frac{(t^{1.1} + 3t)^{1.5}}{10} \text{ m s}^{-1} \\ \therefore v(4) &= \frac{(4^{1.1} + 3(4))^{1.5}}{10} \\ &\approx 6.76 \text{ m s}^{-1} \end{aligned}$$

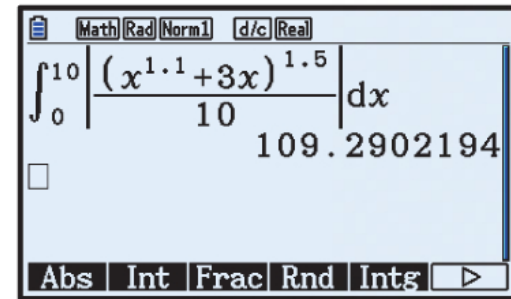
$$\begin{aligned} \mathbf{b} \quad a(t) &= v'(t) \\ &= \frac{1}{10} \times 1.5(t^{1.1} + 3t)^{0.5} (1.1t^{0.1} + 3) \quad \{\text{chain rule}\} \\ &= 0.15(t^{1.1} + 3t)^{0.5} (1.1t^{0.1} + 3) \text{ m s}^{-2} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad a(2) &= 0.15(2^{1.1} + 3(2))^{0.5} (1.1(2)^{0.1} + 3) \\ &\approx 1.79 \text{ m s}^{-2} \end{aligned}$$

$\therefore$  the acceleration of the skier after 2 seconds is about  $1.79 \text{ m s}^{-2}$ .

$\mathbf{d}$  Total distance travelled in first 10 seconds

$$\begin{aligned} &= \int_0^{10} |v(t)| dt \\ &= \int_0^{10} \left| \frac{(t^{1.1} + 3t)^{1.5}}{10} \right| dt \\ &\approx 109 \text{ m} \quad \{\text{using technology}\} \end{aligned}$$



$$\mathbf{9} \quad \mathbf{a} \quad v(t) = -\frac{1}{24}t^3 - \frac{1}{12}t \text{ m s}^{-1}$$

$$\begin{aligned} s(t) &= \int v(t) dt \\ &= \int \left( -\frac{1}{24}t^3 - \frac{1}{12}t \right) dt \\ &= -\frac{1}{96}t^4 - \frac{1}{24}t^2 + c \text{ m} \end{aligned}$$

$$\text{But } s(0) = 2$$

$$\therefore c = 2$$

$$\therefore s(t) = -\frac{1}{96}t^4 - \frac{1}{24}t^2 + 2 \text{ m}$$

$\mathbf{b}$  The feather is on the ground when

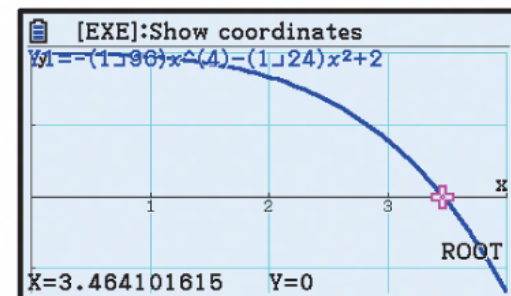
$$s(t) = 0$$

$$\therefore -\frac{1}{96}t^4 - \frac{1}{24}t^2 + 2 = 0$$

$$\therefore t \approx 3.46 \text{ or } -3.46 \quad \{\text{using technology}\}$$

$$\therefore t \approx 3.46 \quad \{t \geq 0\}$$

$\therefore$  it takes about 3.46 seconds for the feather to reach the ground.



$$\begin{aligned} \mathbf{10} \quad \mathbf{a} \quad \text{After 2 seconds, } v_1(2) &= 10(1 - e^{-1.25(2)}) & v_2(2) &= 10.5(1 - e^{-2}) \\ &\approx 9.18 \text{ m s}^{-1} & &\approx 9.08 \text{ m s}^{-1} \end{aligned}$$

$\therefore$  Tyson is running faster after 2 seconds.

$$\begin{aligned}
 \text{b } \int_0^5 v_1(t) dt &= \int_0^5 10(1 - e^{-1.25t}) dt \\
 &= \int_0^5 (10 - 10e^{-1.25t}) dt \\
 &= [10t + 8e^{-1.25t}]_0^5 \\
 &\approx (50 + 0.015) - (8) \\
 &\approx 42.0
 \end{aligned}$$

Tyson has travelled about 42.0 m in the first 5 seconds of the race.

$$\begin{aligned}
 \text{c } s_1(t) &= \int v_1(t) dt & s_2(t) &= \int v_2(t) dt \\
 &= \int 10(1 - e^{-1.25t}) dt & &= \int 10.5(1 - e^{-t}) dt \\
 &= \int (10 - 10e^{-1.25t}) dt & &= \int (10.5 - 10.5e^{-t}) dt \\
 &= 10t + 8e^{-1.25t} + c & &= 10.5t + 10.5e^{-t} + c
 \end{aligned}$$

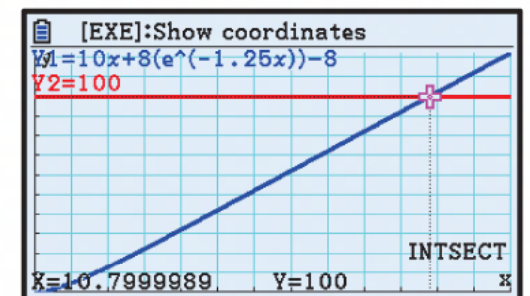
Now  $s_1(0) = 0$  Now  $s_2(0) = 0$   
 $\therefore 0 + 8e^0 + c = 0$   $\therefore 0 + 10.5e^0 + c = 0$   
 $\therefore 8 + c = 0$   $\therefore 10.5 + c = 0$   
 $\therefore c = -8$   $\therefore c = -10.5$   
 $\therefore s_1(t) = 10t + 8e^{-1.25t} - 8 \text{ m}$   $\therefore s_2(t) = 10.5t + 10.5e^{-t} - 10.5 \text{ m}$

$$\begin{aligned}
 \text{d } \text{After 3 seconds, } s_1(3) &= 10(3) + 8e^{-1.25(3)} - 8 \approx 22.2 \text{ m} \\
 s_2(3) &= 10.5(3) + 10.5e^{-3} - 10.5 \approx 21.5 \text{ m}
 \end{aligned}$$

$\therefore$  Tyson is winning the race after 3 seconds.

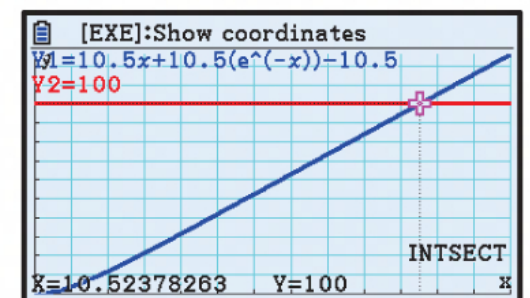
e When Tyson completes the race,

$$\begin{aligned}
 s_1(t) &= 100 \\
 \therefore 10t + 8e^{-1.25t} - 8 &= 100 \\
 \therefore t &\approx 10.8 \quad \{\text{using technology}\} \\
 \therefore \text{Tyson completes the race in approximately} & \\
 10.8 \text{ seconds.} &
 \end{aligned}$$



f Likewise, when Maurice completes the race,

$$\begin{aligned}
 s_2(t) &= 100 \\
 \therefore 10.5t + 10.5e^{-t} - 10.5 &= 100 \\
 \therefore t &\approx 10.5 \quad \{\text{using technology}\} \\
 \therefore \text{Maurice completes the race in approximately} & \\
 10.5 \text{ seconds, so Maurice wins the race.} &
 \end{aligned}$$



# Chapter 24

## MACLAURIN SERIES

### INVESTIGATION 1

### TAYLOR SERIES

$$\begin{aligned}f(x) &= \sum_{k=0}^{\infty} c_k(x-a)^k \\&= c_0 + c_1(x-a) + c_2(x-a)^2 + \dots\end{aligned}$$

$$\begin{aligned}1 \quad f(a) &= c_0 + c_1(a-a) + c_2(a-a)^2 + \dots \\&= c_0 + c_1(0) + c_2(0)^2 + \dots \\&= c_0 \\ \therefore c_0 &= f(a)\end{aligned}$$

$$\begin{aligned}2 \quad f'(x) &= \frac{d}{dx} \sum_{k=0}^{\infty} c_k(x-a)^k \\&= \sum_{k=1}^{\infty} k c_k(x-a)^{k-1} \\&= \sum_{j=0}^{\infty} (j+1) c_{j+1}(x-a)^j \quad \{j = k-1\} \\&= c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \dots \\ \therefore f'(a) &= c_1 + 2c_2(a-a) + 3c_3(a-a)^2 + \dots \\&= c_1 + 2c_2(0) + 3c_3(0)^2 + \dots \\&= c_1 \\ \therefore c_1 &= f'(a)\end{aligned}$$

So, the first two terms of the Taylor series expansion are  $f(a) + f'(a)(x-a)$ . The graph of  $y = f(a) + f'(a)(x-a)$  is the tangent to the graph of  $y = f(x)$  at  $x = a$ .

$$\begin{aligned}3 \quad f''(x) &= \frac{d}{dx} \sum_{k=0}^{\infty} (k+1) c_{k+1}(x-a)^k \quad \{\text{using 2, replacing } j \text{ with } k\} \\&= \sum_{k=1}^{\infty} (k+1) k c_{k+1}(x-a)^{k-1} \\&= \sum_{j=0}^{\infty} (j+2)(j+1) c_{j+2}(x-a)^j \quad \{j = k-1\} \\&= (2 \times 1) c_2 + (3 \times 2) c_3(x-a) + (4 \times 3) c_4(x-a)^2 + \dots \\&= 2c_2 + 6c_3(x-a) + 12c_4(x-a)^2 + \dots \\ \therefore f''(a) &= 2c_2 + 6c_3(a-a) + 12c_4(a-a)^2 + \dots \\&= 2c_2 + 6c_3(0) + 12c_4(0)^2 + \dots \\&= 2c_2 \\ \therefore c_2 &= \frac{f''(a)}{2}\end{aligned}$$



$$\begin{aligned}
 4 \quad \mathbf{a} \quad P_n \text{ is: } f^{(n)}(x) &= n! c_n + (n+1)! c_{n+1}(x-a) + \frac{(n+2)!}{2!} c_{n+2}(x-a)^2 + \dots \\
 &= \sum_{k=0}^{\infty} \frac{(n+k)!}{k!} c_{n+k}(x-a)^k \quad \text{for } n \in \mathbb{Z}^+.
 \end{aligned}$$

**Proof:** (By the principle of mathematical induction)

$$\begin{aligned}
 (1) \quad \text{If } n=0, \quad \text{LHS} &= f(x) & \text{and} \quad \text{RHS} &= \sum_{k=0}^{\infty} \frac{(0+k)!}{k!} c_{0+k}(x-a)^k \\
 &= \sum_{k=0}^{\infty} c_k(x-a)^k & &= \sum_{k=0}^{\infty} c_k(x-a)^k
 \end{aligned}$$

$\therefore P_0$  is true.

$$(2) \quad \text{If } P_m \text{ is true, then } f^{(m)}(x) = \sum_{k=0}^{\infty} \frac{(m+k)!}{k!} c_{m+k}(x-a)^k.$$

$$\begin{aligned}
 \text{Now, } f^{(m+1)}(x) &= \frac{d}{dx} (f^{(m)}(x)) \\
 &= \frac{d}{dx} \sum_{k=0}^{\infty} \frac{(m+k)!}{k!} c_{m+k}(x-a)^k && \{\text{using } P_m\} \\
 &= \sum_{k=1}^{\infty} k \times \frac{(m+k)!}{k!} c_{m+k}(x-a)^{k-1} \\
 &= \sum_{j=0}^{\infty} (j+1) \frac{(m+j+1)!}{(j+1)!} c_{m+1+j}(x-a)^j && \{j = k-1\} \\
 &= \sum_{j=0}^{\infty} \frac{(m+1+j)!}{j!} c_{m+1+j}(x-a)^j \\
 &= \sum_{k=0}^{\infty} \frac{(m+1+k)!}{k!} c_{m+1+k}(x-a)^k && \{\text{replacing } j \text{ with } k\}
 \end{aligned}$$

$\therefore P_{m+1}$  is also true.

Since  $P_0$  is true, and  $P_{m+1}$  is true whenever  $P_m$  is true,

$P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

$$\mathbf{b} \quad \text{Using } \mathbf{a}, \quad f^{(n)}(x) = n! c_n + (n+1)! c_{n+1}(x-a) + \frac{(n+2)!}{2!} c_{n+2}(x-a)^2 + \dots$$

$$\therefore f^{(n)}(a) = n! c_n + (n+1)! c_{n+1}(a-a) + \frac{(n+2)!}{2!} c_{n+2}(a-a)^2 + \dots$$

$$= n! c_n + (n+1)! c_{n+1}(0) + \frac{(n+2)!}{2!} c_{n+2}(0)^2 + \dots$$

$$= n! c_n$$

$$\therefore c_n = \frac{f^{(n)}(a)}{n!}$$

$$\begin{aligned}
 \mathbf{c} \quad f(x) &= \sum_{k=0}^{\infty} c_k(x-a)^k \\
 &= \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k && \{\text{using } \mathbf{b}\}
 \end{aligned}$$

**5 a** Information about the curve at another point where  $x = b$  can be established by substituting  $x = b$  into the formula and evaluating the sum.

- b**
- i** There are no polynomials with horizontal asymptotes, so it is not reasonable to expect that the sum of polynomial terms can exactly produce a horizontal asymptote.
  - ii** Similarly, there are no polynomials with a vertical asymptote (polynomials are defined for all real numbers), so it is not reasonable to expect that the sum of polynomial terms can exactly produce a vertical asymptote.
- c** Since we do not expect the sum of polynomial terms to exactly produce horizontal or vertical asymptotes, we therefore do not expect that a Taylor series expansion will converge for every value of  $x$  in the domain of every function.

## EXERCISE 24A

**1**  $f(x) = e^x$

**a**  $f(x) = e^x$

$$\therefore f'(x) = e^x$$

$$\vdots$$

$$f^{(n)}(x) = e^x$$

$$\therefore f^{(n)}(0) = e^0 = 1 \quad \text{for all } n \in \mathbb{Z}^+.$$

Since  $f(0) = e^0 = 1$ , the Maclaurin series representation for  $f(x)$  is

$$f(x) = 1 + \sum_{k=1}^{\infty} \frac{1}{k!} x^k$$

$$\therefore e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k = 1 + x + \frac{x^2}{2!} + \dots$$

**b**  $M_1(x) = 1 + x$  and  $M_2(x) = 1 + x + \frac{x^2}{2!}$

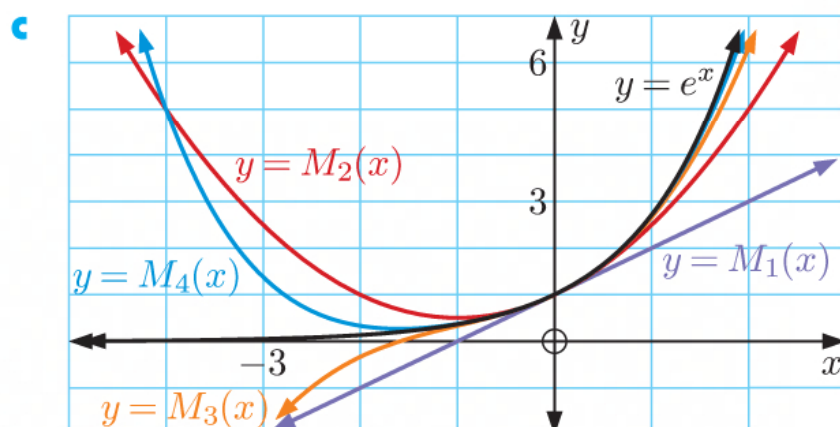
The next four Maclaurin polynomial approximations are:

$$M_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

$$M_4(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$$

$$M_5(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}$$

$$M_6(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!}$$



The accuracy of the approximation is improved by the addition of extra terms.

<b>d</b>	$n$	1	2	3	4	5	6
	$M_n(1)$	2	2.5	2.667	2.708	2.717	2.718

As  $n$  increases,  $M_n(1)$  gets closer to  $f(1) = e \approx 2.718282$ .

**2 a** Let  $f(x) = \frac{1}{1-x} = (1-x)^{-1}$

$$\therefore f'(x) = (-1)(1-x)^{-2}(-1) = 1!(1-x)^{-2}$$

$$\therefore f''(x) = (-1)(-2)(1-x)^{-3}(-1)^2 = 2!(1-x)^{-3}$$

$$\vdots$$

$$f^{(k)}(x) = k!(1-x)^{-(k+1)}$$

$$\therefore f^{(k)}(0) = k! \quad \text{for all } k \in \mathbb{Z}^+.$$

Since  $f(0) = 1$ , the Maclaurin series representation for  $f(x)$  is

$$f(x) = 1 + \sum_{k=1}^{\infty} \frac{k!}{k!} x^k$$

$$= \sum_{k=0}^{\infty} \frac{k!}{k!} x^k$$

$$\therefore \frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \dots$$

**b** Let  $f(x) = \ln(1+x)$

$$\therefore f'(x) = \frac{1}{1+x} = (1+x)^{-1}$$

$$\therefore f''(x) = (-1)(1+x)^{-2}$$

$$\therefore f'''(x) = (-1)(-2)(1+x)^{-3}$$

$$\vdots$$

$$f^{(k)}(x) = (-1)^{k-1}(k-1)!(1+x)^{-k}$$

$$\therefore f^{(k)}(0) = (-1)^{k-1}(k-1)! \quad \text{for all } k \in \mathbb{Z}^+$$

Since  $f(0) = \ln 1 = 0$ , the Maclaurin series representation for  $f(x)$  is

$$f(x) = 0 + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(k-1)!}{k!} x^k$$

$$\therefore \ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} x^k = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots$$

**3** Let  $f(x) = \ln x$

$$\therefore f'(x) = \frac{1}{x} = x^{-1}$$

$$\therefore f''(x) = (-1)x^{-2}$$

$$\therefore f'''(x) = (-1)(-2)x^{-3}$$

$$\vdots$$

$$f^{(k)}(x) = (-1)^{k-1}(k-1)!x^{-k} \quad \text{for all } k \in \mathbb{Z}^+.$$

But,  $\ln x$  and all its derivatives are undefined at  $x = 0$ .

$\therefore$  there is no Maclaurin series representation for  $\ln x$ .



$$\begin{aligned}
4 \quad a \quad \frac{1}{x+1} &= (1+x)^{-1} \\
&= \sum_{k=0}^{\infty} \binom{-1}{k} (1)^{-1-k} x^k \quad \{\text{binomial theorem}\} \\
&= \sum_{k=0}^{\infty} \binom{-1}{k} x^k \\
&= 1 + (-1)x + \frac{(-1)(-2)}{2!} x^2 + \frac{(-1)(-2)(-3)}{3!} x^3 + \dots \\
&= 1 - x + x^2 - x^3 + \dots
\end{aligned}$$

$$\begin{aligned}
b \quad \text{Let } f(x) &= \frac{1}{x+1} = (x+1)^{-1} \\
\therefore f'(x) &= (-1)(x+1)^{-2} \\
\therefore f''(x) &= (-1)(-2)(x+1)^{-3} \\
&\vdots \\
f^{(k)}(x) &= (-1)^k k! (x+1)^{-(k+1)} \\
\therefore f^{(k)}(0) &= (-1)^k k! \quad \text{for all } k \in \mathbb{Z}^+.
\end{aligned}$$

Since  $f(0) = 1$ , the Maclaurin series representation for  $f(x)$  is

$$\begin{aligned}
f(x) &= 1 + \sum_{k=1}^{\infty} \frac{(-1)^k k!}{k!} x^k \\
\therefore \frac{1}{x+1} &= \sum_{k=0}^{\infty} (-1)^k x^k = 1 - x + x^2 - x^3 + \dots
\end{aligned}$$

$$\begin{aligned}
5 \quad a \quad \text{Let } f(x) &= \cos x \\
\therefore f'(x) &= -\sin x \\
\therefore f''(x) &= -\cos x \\
\therefore f'''(x) &= \sin x \\
\therefore f^{(4)}(x) &= \cos x \\
&\vdots \\
f^{(2k)}(x) &= (-1)^k \cos x \quad \text{and} \quad f^{(2k+1)}(x) = (-1)^{k+1} \sin x \\
\therefore f^{(2k)}(0) &= (-1)^k \quad \text{and} \quad f^{(2k+1)}(0) = 0 \quad \text{for all } k \in \mathbb{Z}^+.
\end{aligned}$$

Since  $f(0) = 1$ , the Maclaurin series representation for  $f(x)$  is

$$\begin{aligned}
f(x) &= 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} \quad \{\text{as } f^{(2k+1)}(0) = 0\} \\
\therefore \cos x &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots
\end{aligned}$$

b There is no 3rd order Maclaurin polynomial for  $f(x) = \cos x$  because  $f^{(3)}(0) = 0$ , and so the coefficient of  $x^3$  is 0.

$\cos x$  is an even function, so its Maclaurin series representation will only contain even powers of  $x$ .

$$\begin{aligned}
c \quad M_4(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \\
\therefore M_4\left(\frac{\pi}{3}\right) &\approx 0.502 \quad \text{which is a reasonable approximation to } \cos \frac{\pi}{3} = \frac{1}{2} = 0.5
\end{aligned}$$



**6 a** Let  $f(x) = \sin x$   
 $\therefore f'(x) = \cos x$   
 $\therefore f''(x) = -\sin x$   
 $\therefore f'''(x) = -\cos x$   
 $\therefore f^{(4)}(x) = \sin x$   
 $\vdots$

$$f^{(2k)}(x) = (-1)^k \sin x \quad \text{and} \quad f^{(2k+1)}(x) = (-1)^k \cos x$$

$$\therefore f^{(2k)}(0) = 0 \quad \text{and} \quad f^{(2k+1)}(0) = (-1)^k \quad \text{for all } k \in \mathbb{Z}^+.$$

Since  $f(0) = 0$ , the Maclaurin series representation for  $f(x)$  is

$$f(x) = 0 + \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} \quad \{\text{as } f^{(2k)}(0) = 0\}$$

$$\therefore \sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

- b** There is no 2nd order Maclaurin polynomial for  $f(x) = \sin x$  because  $f^{(2)}(0) = 0$ , and so the coefficient of  $x^2$  is 0.  
 $\sin x$  is an odd function, so its Maclaurin series representation will only contain odd powers of  $x$ .

**c**  $M_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$

$$\therefore M_5\left(\frac{\pi}{6}\right) \approx 0.500\,002 \quad \text{which is a reasonable approximation to } \sin \frac{\pi}{6} = \frac{1}{2} = 0.5$$

**7** Let  $f(x) = \ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x)$   
 $\therefore f'(x) = \frac{1}{1+x} + \frac{1}{1-x} = (1+x)^{-1} + (1-x)^{-1}$   
 $\therefore f''(x) = (-1)(1+x)^{-2} + (-1)(1-x)^{-2}(-1)$   
 $\therefore f'''(x) = (-1)(-2)(1+x)^{-3} + (-1)(-2)(1-x)^{-3}(-1)^2$   
 $\vdots$   
 $f^{(k)}(x) = (-1)^{k-1}(k-1)!(1+x)^{-k} + (k-1)!(1-x)^{-k}$   
 $\therefore f^{(k)}(0) = (-1)^{k-1}(k-1)! + (k-1)!$   
 $= [(-1)^{k-1} + 1](k-1)!$

$$\therefore f^{(2k-1)}(0) = 2(2k-2)! \quad \text{and} \quad f^{(2k)}(0) = 0 \quad \text{for all } k \in \mathbb{Z}^+.$$

Since  $f(0) = \ln 1 = 0$ , the Maclaurin series representation for  $f(x)$  is

$$f(x) = 0 + \sum_{k=1}^{\infty} \frac{2(2k-2)!}{(2k-1)!} x^{2k-1} \quad \{\text{as } f^{(2k)}(0) = 0\}$$

$$\therefore \ln\left(\frac{1+x}{1-x}\right) = \sum_{k=1}^{\infty} \frac{2}{2k-1} x^{2k-1}$$

**Note:** Question 8 has replaced question 7 b from the initial print.

$$\mathbf{8} \quad \mathbf{a} \quad P_n \text{ is: } \frac{d^n}{dx^n} (\arctan x) = \frac{i(-1)^{n-1}(n-1)!}{2} \left( \frac{1}{(x+i)^n} - \frac{1}{(x-i)^n} \right) \text{ for all } n \in \mathbb{Z}^+.$$

**Proof:** (By the principle of mathematical induction)

$$(1) \quad \text{If } n = 1, \quad \text{LHS} = \frac{d}{dx} (\arctan x)$$

$$= \frac{1}{1+x^2}$$

$$\text{and RHS} = \frac{i(-1)^0 \times 0!}{2} \left( \frac{1}{x+i} - \frac{1}{x-i} \right)$$

$$= \frac{i}{2} \left( \frac{x-i - (x+i)}{(x+i)(x-i)} \right)$$

$$= \frac{i}{2} \left( \frac{-2i}{x^2+1} \right)$$

$$= \frac{1}{x^2+1}$$

$\therefore P_1$  is true.

$$(2) \quad \text{If } P_k \text{ is true, then } \frac{d^k}{dx^k} (\arctan x) = \frac{i(-1)^{k-1}(k-1)!}{2} \left( \frac{1}{(x+i)^k} - \frac{1}{(x-i)^k} \right).$$

$$\begin{aligned} \text{Now, } \frac{d^{k+1}}{dx^{k+1}} (\arctan x) &= \frac{d}{dx} \left( \frac{d^k}{dx^k} (\arctan x) \right) \\ &= \frac{d}{dx} \left[ \frac{i(-1)^{k-1}(k-1)!}{2} \left( \frac{1}{(x+i)^k} - \frac{1}{(x-i)^k} \right) \right] \quad \{\text{using } P_k\} \\ &= \frac{i(-1)^{k-1}(k-1)!}{2} \frac{d}{dx} \left( \frac{1}{(x+i)^k} - \frac{1}{(x-i)^k} \right) \\ &= \frac{i(-1)^{k-1}(k-1)!}{2} \left( \frac{-k}{(x+i)^{k+1}} - \frac{-k}{(x-i)^{k+1}} \right) \\ &= \frac{i(-1)^{k-1}(k-1)!(-k)}{2} \left( \frac{1}{(x+i)^{k+1}} - \frac{1}{(x-i)^{k+1}} \right) \\ &= \frac{i(-1)^k k!}{2} \left( \frac{1}{(x+i)^{k+1}} - \frac{1}{(x-i)^{k+1}} \right) \end{aligned}$$

$\therefore P_{k+1}$  is also true.

Since  $P_1$  is true, and  $P_{k+1}$  is true whenever  $P_k$  is true,

$P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

**b** Let  $f(x) = \arctan x$

$$\therefore f^{(n)}(x) = \frac{i(-1)^{n-1}(n-1)!}{2} \left( \frac{1}{(x+i)^n} - \frac{1}{(x-i)^n} \right) \quad \text{for all } n \in \mathbb{Z}^+ \quad \{\text{using a}\}$$

$$\begin{aligned} \therefore f^{(n)}(0) &= \frac{i(-1)^{n-1}(n-1)!}{2} \left( \frac{1}{i^n} - \frac{1}{(-i)^n} \right) \\ &= \frac{i(-1)^{n-1}(n-1)!}{2} \left( \frac{1 - (-1)^n}{i^n} \right) \\ &= \frac{(-1)^{n-1}(n-1)!}{2i^{n-1}} (1 - (-1)^n) \quad \text{for all } n \in \mathbb{Z}^+. \end{aligned}$$

$$\begin{aligned} \therefore f^{(2k)}(0) &= \frac{(-1)^{2k-1}(2k-1)!}{2i^{2k-1}} (1 - (-1)^{2k}) \\ &= \frac{(-1)^{2k-1}(2k-1)!}{2i^{2k-1}} (1 - 1) \\ &= 0 \quad \text{for all } k \in \mathbb{Z}^+. \end{aligned}$$

$$\begin{aligned} \text{and } f^{(2k-1)}(0) &= \frac{(-1)^{2k-2}(2k-2)!}{2i^{2k-2}} (1 - (-1)^{2k-1}) \\ &= \frac{(2k-2)!}{2(-1)^{k-1}} (1 - (-1)) \\ &= (-1)^{1-k}(2k-2)! \\ &= (-1)^{k-1}(2k-2)! \quad \text{for all } k \in \mathbb{Z}^+. \end{aligned}$$

Since  $f(0) = \arctan 0 = 0$ , the Maclaurin series representation for  $f(x)$  is

$$f(x) = 0 + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(2k-2)!}{(2k-1)!} x^{2k-1} \quad \{\text{as } f^{(2k)}(0) = 0\}$$

$$\therefore \arctan x = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2k-1} x^{2k-1}$$

## INVESTIGATION 2

$$\int x^k dx$$

$$\begin{aligned} \mathbf{1} \quad \mathbf{a} \quad \left(1 + \frac{x}{n}\right)^n &= \sum_{i=0}^n \binom{n}{i} (1)^{n-i} \left(\frac{x}{n}\right)^i \quad \{\text{binomial theorem}\} \\ &= \sum_{i=0}^n \binom{n}{i} \times \frac{1}{n^i} \times x^i \end{aligned}$$

$$\begin{aligned} \therefore \text{the coefficient of } x^i \text{ is } \binom{n}{i} \times \frac{1}{n^i} &= \frac{n(n-1)\dots(n-i+1)}{i!} \times \frac{1}{n^i} \\ &= \frac{1}{i!} \times \frac{n}{n} \times \frac{n-1}{n} \times \dots \times \frac{n-i+1}{n} \end{aligned}$$

**b** Using l'Hôpital's rule, the terms  $\frac{n}{n}, \frac{n-1}{n}, \dots, \frac{n-i+1}{n}$  all approach 1 as  $n \rightarrow \infty$ .

$\therefore$  as  $n \rightarrow \infty$ , the limit of the coefficient of  $x^i$  is  $\frac{1}{i!}$ .

**c** Using **a** and **b**,  $\lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n = \sum_{i=0}^{\infty} \frac{1}{i!} x^i$ , which is identical to the Maclaurin expansion of  $e^x$ .

**2**  $y = \frac{x^{k+1} - 1}{k+1}$  for  $x > 0$

**a** Let  $n = \frac{1}{k+1}$ , then  $k+1 = \frac{1}{n}$

$$\therefore y = \frac{x^{\frac{1}{n}} - 1}{\frac{1}{n}}$$

$$\therefore \frac{y}{n} = x^{\frac{1}{n}} - 1$$

$$\therefore x^{\frac{1}{n}} = 1 + \frac{y}{n}$$

$$\therefore x = \left(1 + \frac{y}{n}\right)^n$$

**b**  $\lim_{n \rightarrow \infty} \left(1 + \frac{y}{n}\right)^n = \sum_{i=0}^{\infty} \frac{1}{i!} y^i \quad \{\text{using 1 c}\}$   
 $= e^y$

$\therefore$  in the limit as  $n \rightarrow \infty$ ,  $x = e^y$ .

**c** Now  $n = \frac{1}{k+1}$

$$\therefore k+1 = \frac{1}{n}$$

$$\therefore k = \frac{1}{n} - 1$$

$\therefore$  as  $n \rightarrow \infty$ ,  $k \rightarrow -1$ .

So,  $\lim_{k \rightarrow -1} \frac{x^{k+1} - 1}{k+1} = \lim_{n \rightarrow \infty} y$   
 $= \lim_{n \rightarrow \infty} \ln(e^y)$   
 $= \ln \left( \lim_{n \rightarrow \infty} e^y \right) \quad \{f(t) = \ln t \text{ is continuous for all } t > 0\}$   
 $= \ln x \quad \{\text{using b}\}$

**3** The formula  $\int x^k dx = \frac{x^{k+1}}{k+1} + c$  is undefined for  $k = -1$ , and the limit as  $k \rightarrow -1$  does not exist.

The formula  $\int x^k dx = \frac{x^{k+1} - 1}{k+1} + c$  is also undefined for  $k = -1$ , but might be considered “better” as the limit as  $k \rightarrow -1$  *does* exist as in **2 c**, and agrees with the formula

$$\int x^{-1} dx = \ln x + c.$$



**ACTIVITY****THE REMAINDER TERM**

$$1 \quad \sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

$$a \quad M_3(x) = x - \frac{x^3}{3!}$$

$$\begin{aligned} \therefore \sin \frac{\pi}{5} &\approx M_3\left(\frac{\pi}{5}\right) \\ &\approx \frac{\pi}{5} - \frac{\left(\frac{\pi}{5}\right)^3}{3!} \\ &\approx 0.587 \end{aligned}$$

Now, let  $f(x) = \sin x$

$$\therefore f'(x) = \cos x$$

$$\therefore f''(x) = -\sin x$$

$$\therefore f'''(x) = -\cos x$$

$$\therefore f^{(4)}(x) \approx \sin x$$

So, the remainder term of the estimate is

$$\begin{aligned} R_3\left(\frac{\pi}{5}\right) &= \frac{f^{(4)}(c)}{4!} \left(\frac{\pi}{5}\right)^4 \quad \text{where } 0 < c < \frac{\pi}{5} \\ &= \frac{\sin c}{4!} \left(\frac{\pi}{5}\right)^4 \end{aligned}$$

$$\therefore |R_3\left(\frac{\pi}{5}\right)| = \frac{\sin c}{4!} \left(\frac{\pi}{5}\right)^4 \quad \{\sin c > 0 \text{ as } 0 < c < \frac{\pi}{5}\}$$

$$\therefore \left| M_3\left(\frac{\pi}{5}\right) - \sin \frac{\pi}{5} \right| < \frac{\left(\frac{\pi}{5}\right)^4}{4!} \approx 0.00649 \quad \{\sin c < 1 \text{ as } 0 < c < \frac{\pi}{5}\}$$

$$\therefore \left| M_3\left(\frac{\pi}{5}\right) - \sin \frac{\pi}{5} \right| < 0.0065$$

Thus we have  $-0.0065 < M_3\left(\frac{\pi}{5}\right) - \sin \frac{\pi}{5} < 0.0065$

$$\therefore -0.0065 < 0.587 - \sin \frac{\pi}{5} < 0.0065$$

$$\therefore -0.5935 < -\sin \frac{\pi}{5} < -0.5805$$

$$\therefore 0.5805 < \sin \frac{\pi}{5} < 0.5935$$

So, the approximation is accurate to 1 decimal place only.

**b** Now,  $f^{(4)}(x) = \sin x$   
 $\therefore f^{(5)}(x) = \cos x$   
 $\therefore f^{(6)}(x) = -\sin x$

So, the remainder term obtained by approximating  $\sin x$  using  $M_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$  on the interval  $-0.3 \leq x \leq 0.3$  is

$$\begin{aligned} R_5(x) &= \frac{f^{(6)}(c)}{6!} x^6 \quad \text{where } c \text{ is between } 0 \text{ and } x \\ &= -\frac{\sin c}{6!} x^6 \\ \therefore |R_5(x)| &= \left| -\frac{\sin c}{6!} x^6 \right| \\ &= \frac{|\sin c|}{6!} |x|^6 \end{aligned}$$

But  $|x| \leq 0.3$  and  $|\sin c| < \sin(0.3)$  as  $-0.3 \leq x \leq 0.3$  and  $c$  is between 0 and  $x$ .

$$\therefore |R_5(x)| < \frac{\sin(0.3)}{6!} (0.3)^6 \approx 2.992 \times 10^{-7}$$

$$\therefore |R_5(x)| < 3.00 \times 10^{-7}$$

So, an upper bound for the error in using the approximation  $\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!}$  on the interval  $-0.3 \leq x \leq 0.3$  is  $2.99 \times 10^{-7}$ .

**c**  $3^\circ = \frac{3}{180} \times \pi = \frac{\pi}{60}$

Now, the remainder term in estimating  $\sin 3^\circ$  with the  $n$ th order Maclaurin polynomial approximation is

$$R_n\left(\frac{\pi}{60}\right) = \frac{f^{(n+1)}(c)}{(n+1)!} \left(\frac{\pi}{60}\right)^{n+1} \quad \text{where } 0 < c < \frac{\pi}{60}$$

But  $f^{(n+1)}(c) = \pm \sin c$  or  $\pm \cos c$   
 $\therefore |f^{(n+1)}(c)| = |\sin c|$  or  $|\cos c|$   
 $\therefore |f^{(n+1)}(c)| \leq 1 \quad \{\text{as } |\sin c| \leq 1 \text{ and } |\cos c| \leq 1\}$

$$\begin{aligned} \text{So, } |R_n\left(\frac{\pi}{60}\right)| &= \left| \frac{f^{(n+1)}(c)}{(n+1)!} \left(\frac{\pi}{60}\right)^{n+1} \right| \\ &= \frac{|f^{(n+1)}(c)|}{(n+1)!} \left(\frac{\pi}{60}\right)^{n+1} \\ \therefore |R_n\left(\frac{\pi}{60}\right)| &\leq \frac{\left(\frac{\pi}{60}\right)^{n+1}}{(n+1)!} \quad \{|f^{(n+1)}(c)| \leq 1\} \end{aligned}$$

To compute  $\sin 3^\circ$  correct to 5 decimal places, we require

$$|R_n\left(\frac{\pi}{60}\right)| < 0.000\,005$$

So, we need  $n$  such that  $\frac{\left(\frac{\pi}{60}\right)^{n+1}}{(n+1)!} < 0.000\,005$   
 $\therefore n \geq 3 \quad \{\text{using technology}\}$

X	Y1
1	1.3E-3
2	2.3E-5
3	3.1E-7
4	3.2E-9

3.13172232E-07

$$\begin{aligned}\text{Using } n = 3, \quad \sin 3^\circ &= \sin \frac{\pi}{60} \\ &\approx \frac{\pi}{60} - \frac{(\frac{\pi}{60})^3}{3!} \\ &\approx 0.05234\end{aligned}$$

*Check:* Using technology,  $\sin 3^\circ \approx 0.05234$  ✓

- 2** Let  $f(x) = \cos x$ . Using **Exercise 24A** question **5 a**,  $f^{(n)}(x) = \pm \cos x$  or  $\pm \sin x$ , and the Maclaurin series representation of  $f(x)$  is  $f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$ .

Now, the remainder term in estimating  $\cos(0.2)$  with the  $n$ th order Maclaurin polynomial approximation is

$$R_n(0.2) = \frac{f^{(n+1)}(c)}{(n+1)!} (0.2)^{n+1} \quad \text{where } 0 < c < 0.2$$

But  $f^{(n+1)}(c) = \pm \cos c$  or  $\pm \sin c$

$$\therefore |f^{(n+1)}(c)| = |\cos c| \quad \text{or} \quad |\sin c|$$

$$\therefore |f^{(n+1)}(c)| \leq 1 \quad \{\text{as } |\cos c| \leq 1 \text{ and } |\sin c| \leq 1\}$$

$$\begin{aligned}\text{So, } |R_n(0.2)| &= \left| \frac{f^{(n+1)}(c)}{(n+1)!} (0.2)^{n+1} \right| \\ &= \frac{|f^{(n+1)}(c)|}{(n+1)!} (0.2)^{n+1}\end{aligned}$$

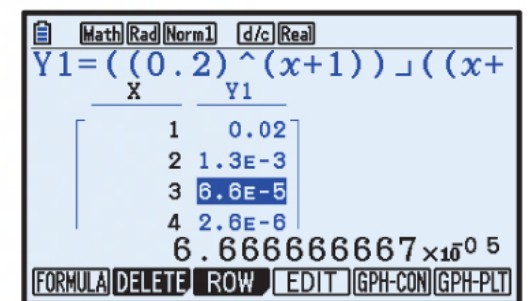
$$\therefore |R_n(0.2)| < \frac{(0.2)^{n+1}}{(n+1)!} \quad \{|f^{(n+1)}(c)| \leq 1\}$$

To compute  $\cos(0.2)$  with error  $< 0.001$ , we require

$$|R_n(0.2)| < 0.001$$

$$\text{So, we need } n \text{ such that } \frac{(0.2)^{n+1}}{(n+1)!} < 0.001$$

$$\therefore n \geq 3 \quad \{\text{using technology}\}$$



x	Y1
1	0.02
2	1.3E-3
3	6.6E-6
4	2.6E-6

6.666666667 × 10<sup>-5</sup>

Since  $\cos x$  is even, the third order Maclaurin polynomial approximation is equivalent to the second.

$$\begin{aligned}\text{So, using } n = 2, \quad \cos(0.2) &\approx 1 - \frac{(0.2)^2}{2!} \\ &\approx 0.98\end{aligned}$$

*Check:* Using technology,  $\cos(0.2) \approx 0.9801$  ✓

**3 a** Let  $f(x) = e^x$ . Using **Exercise 24A** question **1 a**,  $f^{(n)}(x) = e^x$ , and the Maclaurin series representation for  $f(x)$  is  $f(x) = \sum_{k=0}^{\infty} \frac{1}{k!} x^k$ .

Now, the remainder term in estimating  $e^3$  with the  $n$ th order Maclaurin polynomial approximation is

$$\begin{aligned} R_n(3) &= \frac{f^{(n+1)}(c)}{(n+1)!} (3)^{n+1} \quad \text{where } 0 < c < 3 \\ &= \frac{e^c}{(n+1)!} (3)^{n+1} \end{aligned}$$

Now, if  $0 < c < 3$

then  $e^0 < e^c < e^3 < 2.72^3 \approx 20.1$  {since  $e^x$  is an increasing function}

$$\begin{aligned} \text{So, } |R_n(3)| &= \left| \frac{e^c}{(n+1)!} (3)^{n+1} \right| \\ &= \frac{e^c}{(n+1)!} (3)^{n+1} \quad \{\text{as } e^c > 0\} \\ \therefore |R_n(3)| &< \frac{20.1 \times 3^{n+1}}{(n+1)!} \quad \{e^c < 20.1\} \end{aligned}$$

To estimate  $e^3$  with error  $< 0.0001$ , we require

$$|R_n(3)| < 0.0001$$

So, we need  $n$  such that  $\frac{20.1 \times 3^{n+1}}{(n+1)!} < 0.0001$

$$\therefore n \geq 15$$

{using technology}

X	Y1
12	5.1E-3
13	1.1E-3
14	2.2E-4
15	4.1E-5

4.135390628 × 10<sup>-5</sup>

So, using  $n = 15$ ,  $e^3 \approx \sum_{k=0}^{15} \frac{3^k}{k!}$   
 $\approx 20.08553$

Check: Using technology,  $e^3 \approx 20.08554$  ✓



**b** Let  $f(x) = \sin x$ .

Using a similar argument to **1 c**, the remainder term in estimating  $\sin(0.1)$  with the  $n$ th order Maclaurin polynomial approximation is

$$R_n(0.1) = \frac{f^{(n+1)}(c)}{(n+1)!} (0.1)^{n+1} \quad \text{where } 0 < c < 0.1$$

$$\text{and } |R_n(0.1)| \leq \frac{(0.1)^{n+1}}{(n+1)!}$$

To estimate  $\sin(0.1)$  with error  $< 0.0001$ , we require

$$|R_n(0.1)| < 0.0001$$

$$\text{So, we need } n \text{ such that } \frac{(0.1)^{n+1}}{(n+1)!} < 0.0001$$

$$\therefore n \geq 3$$

{using technology}

X	Y1
1	5E-3
2	1.6E-4
3	4.1E-6
4	8.3E-8

4.166666667E-6

$$\begin{aligned} \text{So, using } n = 3, \quad \sin(0.1) &\approx 0.1 - \frac{(0.1)^3}{3!} \\ &\approx 0.0998333 \end{aligned}$$

*Check:* Using technology,  $\sin(0.1) \approx 0.0998334$  ✓

**4** Let  $f(x) = \ln(x+1)$ . Using **Exercise 24A** question **2 b**,  $f^{(n)}(x) = (-1)^{n-1}(n-1)!(x+1)^{-n}$ , and the Maclaurin series representation for  $f(x)$  is  $f(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(k-1)!}{k!} x^k$ .

Now, the remainder term in estimating  $\ln 2$  with the  $n$ th order Maclaurin polynomial approximation is

$$\begin{aligned} R_n(1) &= \frac{f^{(n+1)}(c)}{(n+1)!} (1)^{n+1} \\ &= \frac{f^{(n+1)}(c)}{(n+1)!} \quad \text{where } 0 < c < 1 \end{aligned}$$

Now, if  $0 < c < 1$

$$\text{then } 1 < c+1 < 2$$

$$\therefore \frac{1}{2} < \frac{1}{c+1} < 1$$

$$\therefore \left(\frac{1}{2}\right)^{n+1} < \left(\frac{1}{c+1}\right)^{n+1} < 1$$

$$\begin{aligned} \text{So, } |R_n(1)| &= \left| \frac{f^{(n+1)}(c)}{(n+1)!} \right| \\ &= \frac{|f^{(n+1)}(c)|}{(n+1)!} \\ &= \frac{n!(c+1)^{-(n+1)}}{(n+1)!} \\ &= \frac{1}{n+1} \times \left(\frac{1}{c+1}\right)^{n+1} \end{aligned}$$

$$\therefore |R_n(1)| < \frac{1}{n+1} \quad \left\{ \left(\frac{1}{c+1}\right)^{n+1} < 1 \right\}$$

To estimate  $\ln 2$  correct to 4 decimal places, we require

$$|R_n(1)| < 0.000\,05$$

So, we need  $n$  such that  $\frac{1}{n+1} < 0.000\,05$

$$\therefore n+1 > 20\,000$$

$$\therefore n > 19\,999$$

$$\begin{aligned} \text{So, using } n = 20\,000, \quad \ln 2 &\approx \sum_{k=1}^{20\,000} \frac{(-1)^{k-1}}{k} (1)^k \\ &\approx \sum_{k=1}^{20\,000} \frac{(-1)^{k-1}}{k} \\ &\approx 0.6931 \end{aligned}$$

Check: Using technology,  $\ln 2 \approx 0.6931$  ✓

## EXERCISE 24B

- 1 a** For any  $x$ , we can find a value of  $n$  for which  $n!$  grows faster than  $x^n$ , and  $\frac{x^n}{n!}$  will reduce in size.

$$\therefore \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \quad \text{converges for all } x \in \mathbb{R}.$$

- b** For any  $x$ , we can find a value of  $n$  for which  $n!$  grows faster than  $x^n$ , and  $\frac{x^n}{n!}$  will reduce in size.

$$\therefore \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \quad \text{converges for all } x \in \mathbb{R}.$$

- c** For any  $|x| > 1$ , the terms *increase* in size.

For any  $|x| \leq 1$ , the terms *decrease* in size.

$$\therefore \arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \quad \text{converges for } |x| \leq 1.$$

- d** For any  $|x| > 1$ , the terms *increase* in size.

For any  $|x| < 1$ , the terms *decrease* in size.

The series does not converge for  $x = \pm 1$  where  $\ln\left(\frac{1+x}{1-x}\right)$  is undefined.

$$\therefore \ln\left(\frac{1+x}{1-x}\right) = 2x + \frac{2}{3}x^3 + \frac{2}{5}x^5 + \frac{2}{7}x^7 + \dots \quad \text{converges for } |x| < 1.$$

**2 a** Let  $f(x) = (1+x)^p$ ,  $p \in \mathbb{R}$

$$\therefore f'(x) = p(1+x)^{p-1}$$

$$\therefore f''(x) = p(p-1)(1+x)^{p-2}$$

$$\vdots$$

$$f^{(k)}(x) = p(p-1)\dots(p-k+1)(1+x)^{p-k}$$

$$\therefore f^{(k)}(0) = p(p-1)\dots(p-k+1) \quad \text{for all } k \in \mathbb{Z}^+.$$

Since  $f(0) = 1 = \binom{p}{0}$ , the Maclaurin series representation of  $f(x)$  is

$$f(x) = \binom{p}{0} + \sum_{k=1}^{\infty} \frac{p(p-1)\dots(p-k+1)}{k!} x^k$$

$$\therefore (1+x)^p = \sum_{k=0}^{\infty} \frac{p(p-1)\dots(p-k+1)}{k!} x^k$$

Yes, this agrees with the binomial theorem which states:

$$\begin{aligned} (1+x)^p &= \sum_{k=0}^{\infty} \binom{p}{k} (1)^{p-k} x^k \\ &= \sum_{k=0}^{\infty} \frac{p(p-1)\dots(p-k+1)}{k!} x^k \end{aligned}$$

**b** In the coefficients of  $x^k$ , the product of the  $k$  terms in the numerator counteracts the effect of the  $k!$  in the denominator, so we require  $|x| < 1$  for the series to converge.

**3 a** Let  $f(x) = \tan x$

$$\therefore f'(x) = \sec^2 x = (\cos x)^{-2}$$

$$\therefore f''(x) = (-2)(\cos x)^{-3}(-\sin x)$$

$$= 2 \tan x (\cos x)^{-2}$$

$$= 2 f(x) f'(x)$$

$$\therefore f^{(3)}(x) = 2[f'(x)]^2 + 2 f(x) f''(x) \quad \{\text{product rule}\}$$

$$\therefore f^{(4)}(x) = 4 f'(x) f''(x) + 2 f'(x) f''(x) + 2 f(x) f^{(3)}(x) \quad \{\text{product rule and chain rule}\}$$

$$= 6 f'(x) f''(x) + 2 f(x) f^{(3)}(x)$$

$$\therefore f^{(5)}(x) = 6[f''(x)]^2 + 6 f'(x) f^{(3)}(x) + 2 f'(x) f^{(3)}(x) + 2 f(x) f^{(4)}(x)$$

\{\text{product rule and chain rule}\}

$$= 6[f''(x)]^2 + 8 f'(x) f^{(3)}(x) + 2 f(x) f^{(4)}(x)$$

$$\text{So, } f(0) = 0$$

$$f'(0) = 1$$

$$f''(0) = 2(0)(1) = 0$$

$$f^{(3)}(0) = 2(1)^2 + 2(0)(0) = 2$$

$$f^{(4)}(0) = 6(1)(0) + 2(0)(2) = 0$$

$$\text{and } f^{(5)}(0) = 6(0)^2 + 8(1)(2) + 2(0)(0) = 16$$

Thus, the Maclaurin series representation for  $f(x)$  is

$$f(x) = x + \frac{2}{3!} x^3 + \frac{16}{5!} x^5 + \dots$$

$$\therefore \tan x = x + \frac{1}{3} x^3 + \frac{2}{15} x^5 + \dots$$

**b** The interval of convergence needs to include 0, and *not* include  $-\frac{\pi}{2}$  and  $\frac{\pi}{2}$  where  $\tan x$  is undefined.

$\therefore$  the Maclaurin series in **a** is only convergent for  $-\frac{\pi}{2} < x < \frac{\pi}{2}$ .



**4 a** Let  $f(x) = \frac{1}{1-2x} = (1-2x)^{-1}$

$$\therefore f'(x) = (-1)(1-2x)^{-2}(-2)$$

$$\therefore f''(x) = (-1)(-2)(1-2x)^{-3}(-2)^2$$

$$\vdots$$

$$\therefore f^{(k)}(x) = k!(1-2x)^{-(k+1)}2^k$$

$$\therefore f^{(k)}(0) = k!2^k \quad \text{for all } k \in \mathbb{Z}^+.$$

Since  $f(0) = 1$ , the Maclaurin series representation for  $f(x)$  is

$$f(x) = 1 + \sum_{k=1}^{\infty} \frac{k!2^k}{k!} x^k$$

$$\therefore \frac{1}{1-2x} = \sum_{k=0}^{\infty} 2^k x^k = 1 + 2x + 4x^2 + 8x^3 + 16x^4 + \dots$$

Thus, the  $n$ th order Maclaurin polynomial for  $\frac{1}{1-2x}$  is

$$M_n(x) = \sum_{k=0}^n 2^k x^k$$

$$= 1 + 2x + 4x^2 + 8x^3 + 16x^4 + \dots + 2^n x^n$$

**b i**  $M_n(1) = \sum_{k=0}^n 2^k = 1 + 2 + 4 + 8 + 16 + \dots + 2^n$

$$\therefore$$

$n$	1	2	3	4	5	6
$M_n(1)$	3	7	15	31	63	127

- ii** The difference between consecutive terms is increasing, so it does not seem like the Maclaurin series will converge when  $x = 1$ .

In fact,  $\sum_{k=0}^{\infty} 2^k$  is a geometric series with common ratio  $r = 2$ , which we know does not converge as  $|r| \geq 1$ .

**c i**  $M_n(0.1) = \sum_{k=0}^n 2^k (0.1)^k = \sum_{k=0}^n (0.2)^k$

$$\therefore$$

$n$	1	2	3	4	5	6
$M_n(0.1)$	1.2	1.24	1.248	1.2496	1.24992	1.249984

- ii** The terms appear to be approaching  $1.25 = \frac{1}{1-2(0.1)}$ , so it seems like the Maclaurin series will converge when  $x = 0.1$ .

In fact,  $\sum_{k=0}^{\infty} (0.2)^k$  is a geometric series with common ratio  $r = 0.2$ , which we know does converge as  $|r| < 1$ .



- d** The Maclaurin series  $\frac{1}{1-2x} = \sum_{k=0}^{\infty} 2^k x^k = \sum_{k=0}^{\infty} (2x)^k$  is a geometric series with common ratio  $r = 2x$ . This series will therefore converge when  $|r| < 1$   
 $\therefore |2x| < 1$   
 $\therefore |x| < \frac{1}{2}$

- 5 a** Let  $f(x) = \sqrt{x+1} = (x+1)^{\frac{1}{2}}$ . Using **2 a** with  $p = \frac{1}{2}$ , the Maclaurin series representation for  $f(x)$  is

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right) \left(\frac{1}{2} - 1\right) \dots \left(\frac{1}{2} - k + 1\right)}{k!} x^k \\ &= \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right) \left(-\frac{1}{2}\right) \dots \left(\frac{3}{2} - k\right)}{k!} x^k \end{aligned}$$

$$\therefore \sqrt{x+1} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \frac{7}{256}x^5 - \dots$$

For  $|x| \leq 1$ , the terms *decrease* in size.

$\therefore$  the maximum domain on which this series might converge is  $|x| \leq 1$ .

**b** 
$$M_n(1) = \sum_{k=0}^n \frac{\left(\frac{1}{2}\right) \left(-\frac{1}{2}\right) \dots \left(\frac{3}{2} - k\right)}{k!}$$

$$= 1 + \frac{1}{2} - \frac{1}{8} + \frac{1}{16} - \frac{5}{128} + \frac{7}{256} - \dots + \frac{\left(\frac{1}{2}\right) \left(-\frac{1}{2}\right) \dots \left(\frac{3}{2} - n\right)}{n!}$$

$$\therefore$$

$n$	1	2	3	4	5
$M_n(1)$	1.5	1.375	1.438	1.398	1.426

The series appears to be convergent for  $x = 1$ , as the values approach  $\sqrt{2} \approx 1.414$ .

- c** Let  $g(x) = \sqrt{x+1.8} = (x+1.8)^{\frac{1}{2}}$
- $$\therefore g'(x) = \frac{1}{2}(x+1.8)^{-\frac{1}{2}}$$
- $$\therefore g''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)(x+1.8)^{-\frac{3}{2}}$$
- $$\therefore g'''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(x+1.8)^{-\frac{5}{2}}$$
- $$\vdots$$
- $$\therefore g^{(k)}(x) = \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\dots\left(\frac{1}{2} - k + 1\right)(x+1.8)^{\frac{1}{2}-k}$$
- $$\therefore g^{(k)}(0) = \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\dots\left(\frac{3}{2} - k\right)(1.8)^{\frac{1}{2}-k} \quad \text{for all } k \in \mathbb{Z}^+.$$

Since  $g(0) = (1.8)^{\frac{1}{2}}$ , the Maclaurin series representation for  $g(x)$  is

$$\begin{aligned}
 g(x) &= (1.8)^{\frac{1}{2}} + \sum_{k=1}^{\infty} \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\dots\left(\frac{3}{2}-k\right)}{k!} (1.8)^{\frac{1}{2}-k} x^k \\
 \therefore \sqrt{x+1.8} &= (1.8)^{\frac{1}{2}} + \frac{\frac{1}{2}(1.8)^{-\frac{1}{2}}}{1!} x + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)(1.8)^{-\frac{3}{2}}}{2!} x^2 \\
 &\quad + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(1.8)^{-\frac{5}{2}}}{3!} x^3 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)(1.8)^{-\frac{7}{2}}}{4!} x^4 \\
 &\quad + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)(1.8)^{-\frac{9}{2}}}{5!} x^5 - \dots \\
 &= (1.8)^{\frac{1}{2}} + \frac{1}{2(1.8)^{\frac{1}{2}}} x - \frac{1}{8(1.8)^{\frac{3}{2}}} x^2 + \frac{1}{16(1.8)^{\frac{5}{2}}} x^3 - \frac{5}{128(1.8)^{\frac{7}{2}}} x^4 \\
 &\quad + \frac{7}{256(1.8)^{\frac{9}{2}}} x^5 - \dots
 \end{aligned}$$

$$\text{Now, } M_n(0.2) = (1.8)^{\frac{1}{2}} + \frac{0.2}{2(1.8)^{\frac{1}{2}}} - \frac{(0.2)^2}{8(1.8)^{\frac{3}{2}}} + \frac{(0.2)^3}{16(1.8)^{\frac{5}{2}}} - \frac{5(0.2)^4}{128(1.8)^{\frac{7}{2}}} + \frac{7(0.2)^5}{256(1.8)^{\frac{9}{2}}} - \dots$$

$$\therefore$$

$n$	1	2	3	4	5
$M_n(0.2)$	1.416 18	1.414 11	1.414 22	1.414 21	1.414 21

The Maclaurin series for  $\sqrt{x+1.8}$  approximated  $\sqrt{2}$  much more quickly and accurately than the Maclaurin series for  $\sqrt{x+1}$  (0.2 is closer to 0 than 1 is to 0).

## EXERCISE 24C

**1**  $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$

**a**  $\frac{1}{1-x^2} = \sum_{k=0}^{\infty} (x^2)^k$   
 $= \sum_{k=0}^{\infty} x^{2k}$   
 $= 1 + x^2 + x^4 + \dots$

**b**  $\frac{1}{1+x} = \frac{1}{1-(-x)}$   
 $= \sum_{k=0}^{\infty} (-x)^k$   
 $= \sum_{k=0}^{\infty} (-1)^k x^k$   
 $= 1 - x + x^2 - x^3 + x^4 - \dots$

**c**  $\frac{1}{1-2x} = \sum_{k=0}^{\infty} (2x)^k$   
 $= \sum_{k=0}^{\infty} 2^k x^k$   
 $= 1 + 2x + 4x^2 + 8x^3 + 16x^4 + \dots$

**d**  $\frac{2}{1+x} = 2 \sum_{k=0}^{\infty} (-1)^k x^k$  {using **b**}  
 $= 2 - 2x + 2x^2 - 2x^3 + 2x^4 - \dots$

$$\mathbf{2} \quad \cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$$

$$\begin{aligned} \mathbf{a} \quad \cos 2x &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} (2x)^{2k} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} 4^k x^{2k} \\ &= \sum_{k=0}^{\infty} \frac{(-4)^k x^{2k}}{(2k)!} \\ &= 1 - \frac{4x^2}{2!} + \frac{16x^4}{4!} - \frac{64x^6}{6!} + \dots \\ &= 1 - 2x^2 + \frac{2x^4}{3} - \frac{4x^6}{45} + \dots \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \cos(-x) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} (-x)^{2k} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} (-1)^{2k} x^{2k} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} \\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \\ &= 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \dots \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad \cos(x^2) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} (x^2)^{2k} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k x^{4k}}{(2k)!} \\ &= 1 - \frac{x^4}{2!} + \dots \\ &= 1 - \frac{x^4}{2} + \dots \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad x \cos x &= x \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k)!} \\ &= x - \frac{x^3}{2!} + \frac{x^5}{4!} - \dots \\ &= x - \frac{x^3}{2} + \frac{x^5}{24} - \dots \end{aligned}$$

$$\mathbf{3} \quad \sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

$$\begin{aligned} \mathbf{a} \quad \sin(-x) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} (-x)^{2k+1} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} (-1)^{2k+1} x^{2k+1} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} (-1)(-1)^{2k} x^{2k+1} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^{k+1} x^{2k+1}}{(2k+1)!} \end{aligned}$$

$$\mathbf{b} \quad \sin 3x = \sum_{k=0}^{\infty} \frac{(-1)^k (3x)^{2k+1}}{(2k+1)!}$$

$$\begin{aligned} \mathbf{c} \quad \sin(x^2) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} (x^2)^{2k+1} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k x^{4k+2}}{(2k+1)!} \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad x \sin x &= x \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+2}}{(2k+1)!} \end{aligned}$$

$$4 \quad e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$\begin{aligned} a \quad e^{3x} &= \sum_{k=0}^{\infty} \frac{(3x)^k}{k!} \\ &= \sum_{k=0}^{\infty} \frac{3^k x^k}{k!} \\ &= 1 + 3x + \frac{9x^2}{2!} + \frac{27x^3}{3!} + \frac{81x^4}{4!} + \dots \\ &= 1 + 3x + \frac{9x^2}{2} + \frac{9x^3}{2} + \frac{27x^4}{8} + \dots \end{aligned}$$

$$\begin{aligned} b \quad e^{x^2} &= \sum_{k=0}^{\infty} \frac{(x^2)^k}{k!} \\ &= \sum_{k=0}^{\infty} \frac{x^{2k}}{k!} \\ &= 1 + x^2 + \frac{x^4}{2!} + \dots \\ &= 1 + x^2 + \frac{x^4}{2} + \dots \end{aligned}$$

$$\begin{aligned} c \quad 2^x &= e^{\ln(2^x)} \\ &= e^{x \ln 2} \\ &= \sum_{k=0}^{\infty} \frac{(x \ln 2)^k}{k!} \\ &= \sum_{k=0}^{\infty} \frac{x^k (\ln 2)^k}{k!} \\ &= 1 + x \ln 2 + \frac{x^2 (\ln 2)^2}{2!} + \frac{x^3 (\ln 2)^3}{3!} + \frac{x^4 (\ln 2)^4}{4!} + \dots \\ &= 1 + x \ln 2 + \frac{x^2 (\ln 2)^2}{2} + \frac{x^3 (\ln 2)^3}{6} + \frac{x^4 (\ln 2)^4}{24} + \dots \end{aligned}$$

$$\begin{aligned} 5 \quad a \quad \sin x &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} \\ \therefore \sin\left(x + \frac{\pi}{2}\right) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \left(x + \frac{\pi}{2}\right)^{2k+1} \end{aligned}$$



$$\begin{aligned}
& \mathbf{b} \quad \sin\left(x + \frac{\pi}{2}\right) \\
&= \left(\frac{\pi}{2} + x\right) - \frac{1}{3!} \left(\frac{\pi}{2} + x\right)^3 + \frac{1}{5!} \left(\frac{\pi}{2} + x\right)^5 - \frac{1}{7!} \left(\frac{\pi}{2} + x\right)^7 + \dots \\
&= \frac{\pi}{2} + x \\
&\quad - \frac{1}{3!} \left( \left(\frac{\pi}{2}\right)^3 + 3\left(\frac{\pi}{2}\right)^2 x + 3\left(\frac{\pi}{2}\right) x^2 + x^3 \right) \\
&\quad + \frac{1}{5!} \left( \left(\frac{\pi}{2}\right)^5 + 5\left(\frac{\pi}{2}\right)^4 x + 10\left(\frac{\pi}{2}\right)^3 x^2 + 10\left(\frac{\pi}{2}\right)^2 x^3 + 5\left(\frac{\pi}{2}\right) x^4 + x^5 \right) \\
&\quad - \frac{1}{7!} \left( \left(\frac{\pi}{2}\right)^7 + 7\left(\frac{\pi}{2}\right)^6 x + 21\left(\frac{\pi}{2}\right)^5 x^2 + 35\left(\frac{\pi}{2}\right)^4 x^3 + 35\left(\frac{\pi}{2}\right)^3 x^4 + 21\left(\frac{\pi}{2}\right)^2 x^5 + 7\left(\frac{\pi}{2}\right) x^6 + x^7 \right) \\
&\quad + \dots \\
&= \left( \frac{\pi}{2} - \frac{1}{3!} \left(\frac{\pi}{2}\right)^3 + \frac{1}{5!} \left(\frac{\pi}{2}\right)^5 - \frac{1}{7!} \left(\frac{\pi}{2}\right)^7 + \dots \right) \\
&\quad + \left( 1 - \frac{1}{2!} \left(\frac{\pi}{2}\right)^2 + \frac{1}{4!} \left(\frac{\pi}{2}\right)^4 - \frac{1}{6!} \left(\frac{\pi}{2}\right)^6 + \dots \right) x \\
&\quad - \frac{1}{2!} \left( \frac{\pi}{2} - \frac{1}{3!} \left(\frac{\pi}{2}\right)^3 + \frac{1}{5!} \left(\frac{\pi}{2}\right)^5 - \dots \right) x^2 \\
&\quad - \frac{1}{3!} \left( 1 - \frac{1}{2!} \left(\frac{\pi}{2}\right)^2 + \frac{1}{4!} \left(\frac{\pi}{2}\right)^4 - \dots \right) x^3 \\
&\quad + \frac{1}{4!} \left( \frac{\pi}{2} - \frac{1}{3!} \left(\frac{\pi}{2}\right)^3 + \dots \right) x^4 \\
&\quad + \dots
\end{aligned}$$

$$\text{Now, } \frac{\pi}{2} - \frac{1}{3!} \left(\frac{\pi}{2}\right)^3 + \frac{1}{5!} \left(\frac{\pi}{2}\right)^5 - \frac{1}{7!} \left(\frac{\pi}{2}\right)^7 + \dots = \sin \frac{\pi}{2} = 1$$

$$\text{and } 1 - \frac{1}{2!} \left(\frac{\pi}{2}\right)^2 + \frac{1}{4!} \left(\frac{\pi}{2}\right)^4 - \frac{1}{6!} \left(\frac{\pi}{2}\right)^6 + \dots = \cos \frac{\pi}{2} = 0$$

$$\therefore \sin\left(x + \frac{\pi}{2}\right) = 1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 - \dots$$

which is the Maclaurin series for  $\cos x$ .

This result agrees with the identity  $\sin\left(x + \frac{\pi}{2}\right) = \cos x$ .

$$\mathbf{6} \quad \mathbf{a} \quad e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\begin{aligned}
\mathbf{i} \quad e^{-x} &= \sum_{k=0}^{\infty} \frac{(-x)^k}{k!} \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k!} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots
\end{aligned}$$

$$\mathbf{ii} \quad e^{-3x} = \sum_{k=0}^{\infty} \frac{(-1)^k (3x)^k}{k!} = 1 - 3x + \frac{9x^2}{2!} - \frac{27x^3}{3!} + \dots$$

$$\begin{aligned}
\mathbf{iii} \quad e^{-(2n+1)x} &= \sum_{k=0}^{\infty} \frac{(-1)^k [(2n+1)x]^k}{k!} \\
&= 1 - (2n+1)x + \frac{(2n+1)^2 x^2}{2!} - \frac{(2n+1)^3 x^3}{3!} + \dots \quad \text{where } n \in \mathbb{Z}^+
\end{aligned}$$

$$\begin{aligned}
 \text{b} \quad \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \\
 \therefore \sin(e^{-x}) &= e^{-x} - \frac{e^{-3x}}{3!} + \frac{e^{-5x}}{5!} - \frac{e^{-7x}}{7!} + \dots \\
 &= \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots\right) - \frac{1}{3!} \left(1 - 3x + \frac{9x^2}{2!} - \frac{27x^3}{3!} + \dots\right) \\
 &\quad + \dots + \frac{(-1)^k}{(2k+1)!} \left(1 + (2k+1)x + \frac{(2k+1)^2 x^2}{2!} - \dots\right) \quad \{\text{using a}\} \\
 &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} + \sum_{k=0}^{\infty} \frac{(-1)^{k+1}(2k+1)}{(2k+1)!} x + \dots \\
 &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} + \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(2k)!} x + \dots
 \end{aligned}$$

c For  $x$  close to 0, higher order terms involving  $x^2, x^3, \dots$  tend to 0.

$$\therefore \sin(e^{-x}) \approx \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} - \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x$$

$$\text{Now } \sin(1) = 1 - \frac{1}{3!} + \frac{1}{5!} - \dots = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \quad \{\text{using known series for } \cos x \text{ and } \sin x\}$$

$$\text{and } \cos(1) = 1 - \frac{1}{2!} + \frac{1}{4!} - \dots = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}$$

$$\therefore \sin(e^{-x}) \approx \sin(1) - \cos(1)x$$

$$\begin{aligned}
 \text{d} \quad \text{Let } f(x) &= \sin(e^{-x}) & \therefore f(0) &= \sin(1) \\
 \therefore f'(x) &= \cos(e^{-x}) \times (-e^{-x}) & \therefore f'(0) &= -\cos(1) \\
 \therefore f''(x) &= -\sin(e^{-x}) \times (-e^{-x})^2 + \cos(e^{-x})e^{-x} & \therefore f''(0) &= -\sin(1) + \cos(1) \\
 \therefore M_2(x) &= \sin(1) - \cos(1)x + (\cos(1) - \sin(1)) \frac{x^2}{2!}
 \end{aligned}$$

## EXERCISE 24D

$$1 \quad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \quad \text{and} \quad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\text{a} \quad \sin x + \cos x = 1 + x - \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} - \frac{x^6}{6!} - \frac{x^7}{7!} + \dots$$

$$\begin{aligned}
 \text{b} \quad \sin 2x &= 2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \frac{(2x)^7}{7!} + \dots \\
 &= 2x - \frac{8x^3}{3!} + \frac{32x^5}{5!} - \frac{128x^7}{7!} + \dots \\
 \therefore \sin x + \sin 2x &= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots\right) + \left(2x - \frac{8x^3}{3!} + \frac{32x^5}{5!} - \frac{128x^7}{7!} + \dots\right) \\
 &= 3x - \frac{9x^3}{3!} + \frac{33x^5}{5!} - \frac{129x^7}{7!} + \dots \\
 &= 3x - \frac{3x^3}{2} + \frac{11x^5}{40} - \frac{43x^7}{1680} + \dots
 \end{aligned}$$

$$\text{c} \quad \sin(x^2) = x^2 - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} - \frac{(x^2)^7}{7!} + \dots$$

$$= x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \dots$$

$$\cos 3x = 1 - \frac{(3x)^2}{2!} + \frac{(3x)^4}{4!} - \frac{(3x)^6}{6!} + \dots$$

$$= 1 - \frac{9x^2}{2!} + \frac{81x^4}{4!} - \frac{729x^6}{6!} + \dots$$

$$\begin{aligned} \therefore \sin(x^2) - \cos 3x &= \left( x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \dots \right) - \left( 1 - \frac{9x^2}{2!} + \frac{81x^4}{4!} - \frac{729x^6}{6!} + \dots \right) \\ &= -1 + \left( 1 + \frac{9}{2!} \right) x^2 - \frac{81x^4}{4!} + \left( -\frac{1}{3!} + \frac{729}{6!} \right) x^6 - \dots \\ &= -1 + \frac{11x^2}{2} - \frac{27x^4}{8} + \frac{203x^6}{240} - \dots \end{aligned}$$

$$\begin{aligned} \text{2 a} \quad e^x &= \sum_{k=0}^{\infty} \frac{x^k}{k!} \\ \therefore e^{-x} &= \sum_{k=0}^{\infty} \frac{(-x)^k}{k!} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k!} \end{aligned}$$

$$\begin{aligned} \text{b} \quad e^x - e^{-x} &= \sum_{k=0}^{\infty} \frac{x^k}{k!} - \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k!} \quad \{\text{using a}\} \\ &= \sum_{k=0}^{\infty} \frac{(1 - (-1)^k) x^k}{k!} \end{aligned}$$

Now, when  $k$  is even,  $1 - (-1)^k = 1 - 1 = 0$

and when  $k$  is odd,  $1 - (-1)^k = 1 + 1 = 2$

$$\therefore e^x - e^{-x} = \sum_{k=0}^{\infty} \frac{2x^{2k+1}}{(2k+1)!}$$

$$\begin{aligned} \text{So, } \sinh x &= \frac{e^x - e^{-x}}{2} \\ &= \frac{1}{2} \sum_{k=0}^{\infty} \frac{2x^{2k+1}}{(2k+1)!} \\ &= \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!} \end{aligned}$$

$$\begin{aligned} \text{c} \quad e^x + e^{-x} &= \sum_{k=0}^{\infty} \frac{x^k}{k!} + \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k!} \quad \{\text{using a}\} \\ &= \sum_{k=0}^{\infty} \frac{(1 + (-1)^k) x^k}{k!} \end{aligned}$$

Now, when  $k$  is even,  $1 + (-1)^k = 1 + 1 = 2$

and when  $k$  is odd,  $1 + (-1)^k = 1 - 1 = 0$

$$\therefore e^x + e^{-x} = \sum_{k=0}^{\infty} \frac{2x^{2k}}{(2k)!}$$

$$\begin{aligned} \text{So, } \cosh x &= \frac{e^x + e^{-x}}{2} \\ &= \frac{1}{2} \sum_{k=0}^{\infty} \frac{2x^{2k}}{(2k)!} \\ &= \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} \end{aligned}$$

$$\begin{aligned}
 3 \quad e^x &= \sum_{k=0}^{\infty} \frac{x^k}{k!} \\
 \therefore e^{i\theta} &= \sum_{k=0}^{\infty} \frac{(i\theta)^k}{k!} \\
 &= \sum_{k=0}^{\infty} \frac{i^k \theta^k}{k!}
 \end{aligned}$$

Now, for  $k \in \mathbb{Z}$ ,  $k \geq 0$ ,  $i^{2k} = (i^2)^k = (-1)^k$   
 and  $i^{2k+1} = i^{2k} \times i = (-1)^k i$

$$\begin{aligned}
 \therefore e^{i\theta} &= \sum_{k=0}^{\infty} \frac{(-1)^k \theta^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{(-1)^k i \theta^{2k+1}}{(2k+1)!} \\
 &= \sum_{k=0}^{\infty} \frac{(-1)^k \theta^{2k}}{(2k)!} + i \sum_{k=0}^{\infty} \frac{(-1)^k \theta^{2k+1}}{(2k+1)!}
 \end{aligned}$$

$$\text{But } \cos \theta = \sum_{k=0}^{\infty} \frac{(-1)^k \theta^{2k}}{(2k)!}$$

$$\text{and } \sin \theta = \sum_{k=0}^{\infty} \frac{(-1)^k \theta^{2k+1}}{(2k+1)!} \quad \{\text{using known series for } \cos \theta \text{ and } \sin \theta\}$$

So,  $e^{i\theta} = \cos \theta + i \sin \theta$  as required.

## EXERCISE 24E

1 The Maclaurin series for  $e^x$  is  $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ .

$$\begin{aligned}
 \text{Now, } \frac{d}{dx} \left( \sum_{k=0}^{\infty} \frac{x^k}{k!} \right) &= \frac{d}{dx} \left( 1 + \sum_{k=1}^{\infty} \frac{x^k}{k!} \right) \\
 &= \frac{d}{dx} (1) + \sum_{k=1}^{\infty} \frac{1}{k!} \frac{d}{dx} (x^k) \\
 &= \sum_{k=1}^{\infty} \frac{1}{k!} k x^{k-1} \\
 &= \sum_{k=1}^{\infty} \frac{x^{k-1}}{(k-1)!} \\
 &= \sum_{j=0}^{\infty} \frac{x^j}{j!} \quad \{j = k-1\}
 \end{aligned}$$

which is the Maclaurin series for  $e^x$ .



$$\mathbf{2} \quad \sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

$$\mathbf{a} \quad \frac{d}{dx} (\sin x) = \cos x$$

$$\begin{aligned} \therefore \cos x &= \frac{d}{dx} \left( \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} \right) \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \frac{d}{dx} (x^{2k+1}) \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} (2k+1) x^{2k} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} \end{aligned}$$

$$\mathbf{b} \quad \int \sin x \, dx = -\cos x + c$$

$$\begin{aligned} \therefore \int_0^x \sin t \, dt &= [-\cos t]_0^x \\ &= -\cos x + 1 \end{aligned}$$

$$\begin{aligned} \therefore \cos x &= 1 - \int_0^x \sin t \, dt \\ &= 1 - \int_0^x \left( \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} t^{2k+1} \right) dt \\ &= 1 - \sum_{k=0}^{\infty} \left( \int_0^x \frac{(-1)^k}{(2k+1)!} t^{2k+1} dt \right) \\ &= 1 - \sum_{k=0}^{\infty} \left[ \frac{(-1)^k}{(2k+2)!} t^{2k+2} \right]_0^x \\ &= 1 - \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+2)!} x^{2k+2} \\ &= 1 - \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{(2j)!} x^{2j} \quad \{j = k+1\} \\ &= \frac{(-1)^0}{(2(0))!} + \sum_{j=1}^{\infty} \frac{(-1)^j}{(2j)!} x^{2j} \\ &= \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j)!} x^{2j} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} \quad \{\text{replacing } j \text{ with } k\} \end{aligned}$$

$$\mathbf{3} \quad \mathbf{a} \quad \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots$$

$$\frac{d}{dx}(\ln(1+x)) = \frac{1}{1+x}$$

$$\begin{aligned} \therefore \frac{1}{1+x} &= \frac{d}{dx} \left( x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots \right) \\ &= 1 - x + x^2 - x^3 + x^4 - \dots \end{aligned}$$

$$\mathbf{b} \quad \frac{1}{1+x^2} = 1 - x^2 + (x^2)^2 - (x^2)^3 + (x^2)^4 - \dots \quad \{\text{using } \mathbf{a}\}$$

$$= 1 - x^2 + x^4 - x^6 + x^8 - \dots$$

$$\mathbf{c} \quad \int \frac{1}{1+x^2} dx = \arctan x + c$$

$$\begin{aligned} \therefore \int_0^x \frac{1}{1+t^2} dt &= [\arctan t]_0^x \\ &= \arctan x - \arctan 0 \\ &= \arctan x \\ \therefore \arctan x &= \int_0^x (1 - t^2 + t^4 - t^6 + t^8 - \dots) dt \\ &= \left[ t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \frac{t^9}{9} - \dots \right]_0^x \\ &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \end{aligned}$$

$$\mathbf{4} \quad \mathbf{a} \quad \sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

$$\begin{aligned} \therefore \sin(x^2) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} (x^2)^{2k+1} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{4k+2} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \int_0^1 \sin(x^2) dx &= \int_0^1 \left( \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{4k+2} \right) dx \quad \{\text{using } \mathbf{a}\} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \left( \int_0^1 x^{4k+2} dx \right) \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \left[ \frac{x^{4k+3}}{4k+3} \right]_0^1 \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!(4k+3)} \\ &= \frac{1}{3} - \frac{1}{42} + \frac{1}{1320} - \frac{1}{75600} + \dots \quad \text{where } \frac{1}{75600} \approx 0.0000132 \end{aligned}$$

$$\text{Using the first 3 terms, } \int_0^1 \sin(x^2) dx \approx 0.3103$$

$$\text{Check: Using technology, } \int_0^1 \sin(x^2) dx \approx 0.3103 \quad \checkmark$$

**5 a**

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$\begin{aligned} \therefore \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} &= \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{\left(-\frac{x^2}{2}\right)^k}{k!} \\ &= \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^k k!} \end{aligned}$$

**b i**  $\int_{-1}^1 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = \int_{-1}^1 \left( \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^k k!} \right) dx \quad \{\text{using a}\}$

$$\begin{aligned} &= \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k k!} \left( \int_{-1}^1 x^{2k} dx \right) \\ &= \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k k!} \left[ \frac{x^{2k+1}}{2k+1} \right]_{-1}^1 \\ &= \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k k!} \left( \frac{1}{2k+1} - \frac{(-1)^{2k+1}}{2k+1} \right) \\ &= \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k k!} \left( \frac{1}{2k+1} + \frac{1}{2k+1} \right) \\ &= \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k k!} \left( \frac{2}{2k+1} \right) \\ &= \frac{2}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k k! (2k+1)} \\ &= \frac{2}{\sqrt{2\pi}} \left( 1 - \frac{1}{6} + \frac{1}{40} - \frac{1}{336} + \frac{1}{3456} - \frac{1}{42\,240} + \dots \right) \\ &= \frac{2}{\sqrt{2\pi}} - \frac{1}{3\sqrt{2\pi}} + \frac{1}{20\sqrt{2\pi}} - \frac{1}{168\sqrt{2\pi}} + \frac{1}{1728\sqrt{2\pi}} - \frac{1}{21\,120\sqrt{2\pi}} + \dots \end{aligned}$$

where  $\frac{1}{21\,120\sqrt{2\pi}} \approx 0.000\,018\,9$

Using the first 5 terms,  $\int_{-1}^1 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \approx 0.6827$

Check: Using technology,  $\int_{-1}^1 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \approx 0.6827 \quad \checkmark$

$$\begin{aligned}
 \text{ii} \quad \int_{-2}^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx &= \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k k!} \left[ \frac{x^{2k+1}}{2k+1} \right]_{-2}^2 \quad \{\text{using i}\} \\
 &= \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k k!} \left( \frac{2^{2k+1}}{2k+1} - \frac{(-2)^{2k+1}}{2k+1} \right) \\
 &= \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k k!} \left( \frac{2^{2k+1}}{2k+1} + \frac{2^{2k+1}}{2k+1} \right) \\
 &= \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k k!} \times \frac{2 \times 2^{2k+1}}{2k+1} \\
 &= \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k \times 2^{k+2}}{k!(2k+1)}
 \end{aligned}$$

Using technology, we find that the first 4 decimal places of the sum stop changing after the 10th term.

Using the first 10 terms,  $\int_{-2}^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \approx 0.9545$

Check: Using technology,  $\int_{-2}^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \approx 0.9545$  ✓

$$\begin{aligned}
 \text{iii} \quad \int_{-3}^3 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx &= \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k k!} \left[ \frac{x^{2k+1}}{2k+1} \right]_{-3}^3 \quad \{\text{using i}\} \\
 &= \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k k!} \left( \frac{3^{2k+1}}{2k+1} - \frac{(-3)^{2k+1}}{2k+1} \right) \\
 &= \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k k!} \left( \frac{3^{2k+1}}{2k+1} + \frac{3^{2k+1}}{2k+1} \right) \\
 &= \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k \times 2 \times 3^{2k+1}}{2^k k! (2k+1)} \\
 &= \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k \times 3^{2k+1}}{2^{k-1} k! (2k+1)}
 \end{aligned}$$

Using technology, we find that the first 4 decimal places of the sum stop changing after the 17th term.

Using the first 17 terms,  $\int_{-3}^3 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \approx 0.9973$

Check: Using technology,  $\int_{-3}^3 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \approx 0.9973$  ✓



$$\begin{aligned}
 \text{6 a} \quad \frac{1}{1-x} &= \sum_{k=0}^{\infty} x^k \\
 \therefore \frac{1}{1+x^3} &= \frac{1}{1-(-x^3)} \\
 &= \sum_{k=0}^{\infty} (-x^3)^k \\
 &= \sum_{k=0}^{\infty} (-1)^k x^{3k}
 \end{aligned}$$

Since the Maclaurin series for  $\frac{1}{1-x}$  converges for  $|x| < 1$ , we expect the Maclaurin series for  $\frac{1}{1+x^3}$  to converge for  $|-x^3| < 1$

$$\therefore |x|^3 < 1$$

$$\therefore |x| < 1$$

$$\begin{aligned}
 \text{b} \quad \int_0^{\frac{1}{3}} \frac{1}{1+x^3} dx &= \int_0^{\frac{1}{3}} \left( \sum_{k=0}^{\infty} (-1)^k x^{3k} \right) dx \quad \{\text{using a}\} \\
 &= \sum_{k=0}^{\infty} (-1)^k \left( \int_0^{\frac{1}{3}} x^{3k} dx \right) \\
 &= \sum_{k=0}^{\infty} (-1)^k \left[ \frac{x^{3k+1}}{3k+1} \right]_0^{\frac{1}{3}} \\
 &= \sum_{k=0}^{\infty} \frac{(-1)^k}{3^{3k+1}(3k+1)} \\
 &= \frac{1}{3} - \frac{1}{324} + \frac{1}{15\,309} - \frac{1}{590\,490} + \dots \quad \text{where } \frac{1}{590\,490} \approx 1.69 \times 10^{-6}
 \end{aligned}$$

Using the first 3 terms,  $\int_0^{\frac{1}{3}} \frac{1}{1+x^3} dx \approx 0.3303$

Check: Using technology,  $\int_0^{\frac{1}{3}} \frac{1}{1+x^3} dx \approx 0.3303$  ✓

$$\text{7 a} \quad (1+x)^p = \sum_{k=0}^{\infty} \frac{p(p-1)\dots(p-k+1)}{k!} x^k \quad \text{for all } |x| < 1$$

$$\begin{aligned}
 \therefore (1+x^2)^{-1} &= 1 + \sum_{k=1}^{\infty} \frac{(-1)(-2)\dots(-k)}{k!} (x^2)^k \\
 &= 1 + \sum_{k=1}^{\infty} \frac{(-1)^k k!}{k!} x^{2k} \\
 &= (-1)^0 + \sum_{k=1}^{\infty} (-1)^k x^{2k}
 \end{aligned}$$

$$\begin{aligned}
 \therefore (1+x^2)^{-1} &= \sum_{k=0}^{\infty} (-1)^k x^{2k} \\
 &= 1 - x^2 + x^4 - \dots \quad \text{for all } |x^2| < 1, \text{ which is for all } |x| < 1.
 \end{aligned}$$

$$\begin{aligned}
\text{b } \arctan x - \arctan(0) &= \int_0^x \frac{1}{1+t^2} dt \\
\therefore \arctan x &= \int_0^x (1+t^2)^{-1} dt \\
&= \int_0^x \left( \sum_{k=0}^{\infty} (-1)^k t^{2k} \right) dt \quad \{\text{using a}\} \\
&= \sum_{k=0}^{\infty} (-1)^k \left( \int_0^x t^{2k} dt \right) \\
&= \sum_{k=0}^{\infty} (-1)^k \left[ \frac{t^{2k+1}}{2k+1} \right]_0^x \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1} \\
&= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots
\end{aligned}$$

This is valid provided  $|x| < 1$ . These values of  $x$  all lie in the domain of  $\arctan x$ , so the series is valid for  $-1 < x < 1$ .

$$\begin{aligned}
\text{c } \arctan x &\approx x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} \quad \{\text{using the first 4 terms}\} \\
\therefore \arctan(x^2) &\approx x^2 - \frac{(x^2)^3}{3} + \frac{(x^2)^5}{5} - \frac{(x^2)^7}{7} \\
&\approx x^2 - \frac{x^6}{3} + \frac{x^{10}}{5} - \frac{x^{14}}{7} \quad \text{for } |x| < 1 \\
\therefore \int_0^1 \arctan(x^2) dx &\approx \int_0^1 \left( x^2 - \frac{x^6}{3} + \frac{x^{10}}{5} - \frac{x^{14}}{7} \right) dx \\
&\approx \left[ \frac{x^3}{3} - \frac{x^7}{21} + \frac{x^{11}}{55} - \frac{x^{15}}{105} \right]_0^1 \\
&\approx 0.2944
\end{aligned}$$

**8 a** From **Example 6** part **a**, the Maclaurin series for  $(1 - x^2)^{-\frac{1}{2}}$  is

$$(1 - x^2)^{-\frac{1}{2}} = \sum_{k=0}^{\infty} \frac{(2k)!}{4^k (k!)^2} x^{2k}.$$

$$\begin{aligned} \text{Now, } \arcsin x - \arcsin(0) &= \int_0^x \frac{1}{\sqrt{1-t^2}} dt \\ \therefore \arcsin x &= \int_0^x (1-t^2)^{-\frac{1}{2}} dt \\ &= \int_0^x \left( \sum_{k=0}^{\infty} \frac{(2k)!}{4^k (k!)^2} t^{2k} \right) dt \\ &= \sum_{k=0}^{\infty} \frac{(2k)!}{4^k (k!)^2} \left( \int_0^x t^{2k} dt \right) \\ &= \sum_{k=0}^{\infty} \frac{(2k)!}{4^k (k!)^2} \left[ \frac{t^{2k+1}}{2k+1} \right]_0^x \\ &= \sum_{k=0}^{\infty} \frac{(2k)!}{4^k (k!)^2 (2k+1)} x^{2k+1} \end{aligned}$$

This is valid provided  $|x| < 1$ . These values all lie in the domain of  $\arcsin x$ , so the series is valid for  $-1 < x < 1$ .

**b**  $\arcsin \frac{1}{2} = \sum_{k=0}^{\infty} \frac{(2k)!}{4^k (k!)^2 (2k+1)} \left(\frac{1}{2}\right)^{2k+1} \quad \text{since } \left|\frac{1}{2}\right| < 1$

$$\begin{aligned} \therefore \frac{\pi}{6} &= \sum_{k=0}^{\infty} \frac{(2k)!}{2^{2k} (k!)^2 (2k+1)} \times \frac{1}{2^{2k+1}} \\ &= \sum_{k=0}^{\infty} \frac{(2k)!}{2^{4k+1} (k!)^2 (2k+1)} \\ &= \frac{(2(0))!}{2^{4(0)+1} (0!)^2 (2(0)+1)} + \sum_{k=1}^{\infty} \frac{(2k)!}{2^{4k+1} (k!)^2 (2k+1)} \\ \therefore \frac{\pi}{6} &= \frac{1}{2} + \sum_{k=1}^{\infty} \frac{(2k)!}{2^{4k+1} (k!)^2 (2k+1)} \end{aligned}$$

**EXERCISE 24F****1** The Maclaurin series expansion for:

- $e^x$  is  $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots$

- $\sin x$  is  $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$

$$\begin{aligned}
 \therefore e^x \sin x &= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots\right) \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right) \\
 &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \\
 &\quad + x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} - \dots \\
 &\quad + \frac{x^3}{2!} - \frac{x^5}{2! \times 3!} + \dots \\
 &\quad + \frac{x^4}{3!} - \frac{x^6}{3! \times 3!} + \dots \\
 &\quad + \frac{x^5}{4!} - \dots \\
 &\quad + \frac{x^6}{5!} - \dots \\
 &= x + x^2 + \frac{1}{3}x^3 - \frac{1}{30}x^5 - \frac{1}{90}x^6 - \dots
 \end{aligned}$$

**2 a**  $\sin x = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \dots$  and  $\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \dots$

$$\begin{aligned}
 \therefore 2 \sin x \cos x &= 2 \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \dots\right) \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \dots\right) \\
 &= \left(2x - \frac{1}{3}x^3 + \frac{1}{60}x^5 - \dots\right) \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \dots\right) \\
 &= 2x - x^3 + \frac{1}{12}x^5 - \dots \\
 &\quad - \frac{1}{3}x^3 + \frac{1}{6}x^5 - \dots \\
 &\quad + \frac{1}{60}x^5 - \dots \\
 &= 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 - \dots
 \end{aligned}$$

**b**  $\sin 2x = 2x - \frac{1}{6}(2x)^3 + \frac{1}{120}(2x)^5 - \dots$   
 $= 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 - \dots$

which agrees with our result in **a**  $\{\sin 2x = 2 \sin x \cos x\}$



$$\mathbf{3} \quad \mathbf{a} \quad e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots \quad \text{and} \quad \arccos x = \frac{\pi}{2} - x - \frac{1}{6}x^3 - \dots, \quad |x| < 1$$

$$\begin{aligned} \therefore e^x \arccos x &= \left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots\right) \left(\frac{\pi}{2} - x - \frac{1}{6}x^3 - \dots\right) \\ &= \frac{\pi}{2} - x - \frac{1}{6}x^3 - \dots \\ &\quad + \frac{\pi}{2}x - x^2 + \dots \\ &\quad + \frac{\pi}{4}x^2 - \frac{1}{2}x^3 + \dots \\ &\quad + \frac{\pi}{12}x^3 - \dots \\ &= \frac{\pi}{2} + \left(\frac{\pi}{2} - 1\right)x + \left(\frac{\pi}{4} - 1\right)x^2 + \left(\frac{\pi}{12} - \frac{2}{3}\right)x^3 + \dots \end{aligned}$$

**b** The Maclaurin series for  $e^x \arccos x$  converges when *both* the Maclaurin series for  $e^x$  and  $\arccos x$  converge.

Since the Maclaurin series for  $e^x$  converges for all  $x \in \mathbb{R}$ , and the Maclaurin series for  $\arccos x$  converges for  $|x| < 1$ , the Maclaurin series for  $e^x \arccos x$  converges for  $|x| < 1$ .

$$\begin{aligned} \mathbf{4} \quad \mathbf{a} \quad e^x &= \sum_{k=0}^{\infty} \frac{x^k}{k!} \\ \therefore e^{-x^3} &= \sum_{k=0}^{\infty} \frac{(-x^3)^k}{k!} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k x^{3k}}{k!} \\ &= 1 - x^3 + \frac{x^6}{2!} - \frac{x^9}{3!} + \dots \end{aligned}$$

$$\mathbf{b} \quad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots$$

$$\begin{aligned} \therefore e^{-x^3} \cos x &= \left(1 - x^3 + \frac{x^6}{2!} - \frac{x^9}{3!} + \dots\right) \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots\right) \\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots \\ &\quad - x^3 + \frac{x^5}{2!} - \frac{x^7}{4!} + \frac{x^9}{6!} - \dots \\ &\quad + \frac{x^6}{2!} - \frac{x^8}{2! \times 2!} + \dots \\ &\quad - \frac{x^9}{3!} + \dots \\ &= 1 - \frac{1}{2}x^2 - x^3 + \frac{1}{24}x^4 + \frac{1}{2}x^5 + \frac{359}{720}x^6 - \frac{1}{24}x^7 - \frac{10\,079}{40\,320}x^8 - \frac{199}{720}x^9 - \dots \end{aligned}$$

**EXERCISE 24G**

1  $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$

Let  $\frac{x}{\cos x} = \sum_{k=0}^{\infty} a_k x^k$

$$\begin{aligned} \therefore x &= \left( \sum_{k=0}^{\infty} a_k x^k \right) \cos x \\ &= (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \dots) \left( 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \right) \\ &= a_0 + a_1 x + \left( a_2 - \frac{a_0}{2} \right) x^2 + \left( a_3 - \frac{a_1}{2} \right) x^3 + \left( a_4 - \frac{a_2}{2} + \frac{a_0}{24} \right) x^4 \\ &\quad + \left( a_5 - \frac{a_3}{2} + \frac{a_1}{24} \right) x^5 + \dots \end{aligned}$$

Equating coefficients,  $a_0 = 0$ ,  $a_1 = 1$ ,  $a_2 = 0$ ,  $a_3 = \frac{1}{2}$ ,  $a_4 = 0$ ,  $a_5 = \frac{5}{24}$

$\therefore \frac{x}{\cos x} = x + \frac{1}{2}x^3 + \frac{5}{24}x^5 + \dots$

2 The Maclaurin series for:

- $\sin x$  is  $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$

- $\cos x$  is  $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$

Let  $\tan x = \sum_{k=0}^{\infty} a_k x^k$

$\therefore \frac{\sin x}{\cos x} = \sum_{k=0}^{\infty} a_k x^k$

$\therefore \sin x = \left( \sum_{k=0}^{\infty} a_k x^k \right) \cos x$

$$\begin{aligned} \therefore x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots &= (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \dots) \left( 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \right) \\ &= a_0 + a_1 x + \left( a_2 - \frac{a_0}{2} \right) x^2 + \left( a_3 - \frac{a_1}{2} \right) x^3 + \left( a_4 - \frac{a_2}{2} + \frac{a_0}{24} \right) x^4 \\ &\quad + \left( a_5 - \frac{a_3}{2} + \frac{a_1}{24} \right) x^5 + \dots \end{aligned}$$

Equating coefficients,  $a_0 = 0$ ,  $a_1 = 1$ ,  $a_2 = 0$ ,  $a_3 = \frac{1}{3}$ ,  $a_4 = 0$ ,  $a_5 = \frac{2}{15}$

$\therefore \tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \dots$

**3 a i**

$$\begin{aligned}
 e^x &= \sum_{k=0}^{\infty} \frac{x^k}{k!} \\
 \therefore e^{2x} &= \sum_{k=0}^{\infty} \frac{(2x)^k}{k!} \\
 &= \sum_{k=0}^{\infty} \frac{2^k x^k}{k!} \\
 &= 1 + \sum_{k=1}^{\infty} \frac{2^k x^k}{k!} \\
 \therefore e^{2x} - 1 &= \sum_{k=1}^{\infty} \frac{2^k x^k}{k!} = 2x + 2x^2 + \frac{4}{3}x^3 + \dots
 \end{aligned}$$

**ii**

$$\begin{aligned}
 \sin x &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} \\
 \therefore \sin 4x &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} (4x)^{2k+1} \\
 &= \sum_{k=0}^{\infty} \frac{(-1)^k 4^{2k+1} x^{2k+1}}{(2k+1)!} \\
 &= 4x - \frac{32}{3}x^3 + \frac{128}{15}x^5 - \dots
 \end{aligned}$$

**b**

$$\begin{aligned}
 \text{Let } \frac{e^{2x} - 1}{\sin 4x} &= \sum_{k=0}^{\infty} a_k x^k \\
 \therefore e^{2x} - 1 &= \left( \sum_{k=0}^{\infty} a_k x^k \right) \sin 4x
 \end{aligned}$$

$$\begin{aligned}
 \therefore 2x + 2x^2 + \frac{4}{3}x^3 + \dots &= (a_0 + a_1x + a_2x^2 + \dots) \left( 4x - \frac{32}{3}x^3 + \dots \right) \\
 &= 4a_0x + 4a_1x^2 + \left( 4a_2 - \frac{32}{3}a_0 \right) x^3 + \dots
 \end{aligned}$$

Equating coefficients,  $a_0 = \frac{1}{2}$ ,  $a_1 = \frac{1}{2}$ ,  $a_2 = \frac{5}{3}$

$$\therefore \frac{e^{2x} - 1}{\sin 4x} = \frac{1}{2} + \frac{1}{2}x + \frac{5}{3}x^2 + \dots$$

$$\therefore \lim_{x \rightarrow 0} \frac{e^{2x} - 1}{\sin 4x} = \frac{1}{2}$$

**4 a** The Maclaurin series for:

- $\sin x$  is  $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$
- $\ln(1+x)$  is  $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \dots$

$$\begin{aligned}
 \therefore \frac{\sin x}{\ln(1+x)} &= \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots}{x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \dots} \\
 &= \frac{x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \dots}{x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \dots}
 \end{aligned}$$

$$\begin{aligned}
 \text{b } \lim_{x \rightarrow 0} \frac{\sin x}{\ln(1+x)} &= \lim_{x \rightarrow 0} \frac{x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \dots}{x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \dots} \\
 &= \lim_{x \rightarrow 0} \frac{1 - \frac{1}{2}x^2 + \frac{1}{40}x^4 - \dots}{1 - x + x^2 - \dots} \quad \{\text{l'Hôpital's rule}\} \\
 &= \frac{1}{1} = 1
 \end{aligned}$$

**5** The Maclaurin series expansion for:

- $f(x)$  is  $f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots$
- $g(x)$  is  $g(x) = g(0) + g'(0)x + \frac{g''(0)}{2!}x^2 + \dots$

$$\begin{aligned} \text{Now } \lim_{x \rightarrow 0} \frac{f(x)}{g(x)} &= \lim_{x \rightarrow 0} \frac{f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots}{g(0) + g'(0)x + \frac{g''(0)}{2!}x^2 + \dots} && \{\text{form } \frac{0}{0} \text{ as } f(0) = g(0) = 0\} \\ &= \lim_{x \rightarrow 0} \frac{f'(0) + f''(0)x + \dots}{g'(0) + g''(0)x + \dots} && \{\text{l'Hôpital's rule}\} \\ &= \frac{f'(0)}{g'(0)} && \{\text{as } g'(0) \neq 0\} \end{aligned}$$

**6 a** Let  $f(x) = e^{\cos x}$

$$\begin{aligned} \therefore f'(x) &= (-\sin x)e^{\cos x} = -\sin x f(x) \\ \therefore f''(x) &= -\cos x f(x) - \sin x f'(x) && \{\text{product rule}\} \\ \therefore f^{(3)}(x) &= \sin x f(x) - \cos x f'(x) - \cos x f'(x) - \sin x f''(x) \\ &= -f'(x) - 2\cos x f'(x) - \sin x f''(x) \\ &= -(1 + 2\cos x) f'(x) - \sin x f''(x) \\ \therefore f^{(4)}(x) &= -(-2\sin x) f'(x) - (1 + 2\cos x) f''(x) - \cos x f''(x) - \sin x f^{(3)}(x) \\ &= 2\sin x f'(x) - (1 + 3\cos x) f''(x) - \sin x f^{(3)}(x) \end{aligned}$$

$$\begin{aligned} \text{So, } f(0) &= e^1 = e \\ f'(0) &= -(0)(e) = 0 \\ f''(0) &= -(1)(e) - (0)(0) = -e \\ f^{(3)}(0) &= -(1 + 2(1))(0) - (0)(-e) = 0 \\ f^{(4)}(0) &= 2(0)(0) - (1 + 3(1))(-e) - (0)(0) = 4e \end{aligned}$$

Thus, the Maclaurin series representation for  $f(x)$  is

$$\begin{aligned} f(x) &= e - \frac{e}{2!}x^2 + \frac{4e}{4!}x^4 - \dots \\ \therefore e^{\cos x} &= e - \frac{e}{2}x^2 + \frac{e}{6}x^4 - \dots \end{aligned}$$

**b** The Maclaurin series for:

- $\sin x$  is  $\sin x = x - \frac{x^3}{3!} + \dots$
- $e^x$  is  $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$



$$\text{Now, } e^{2x} = 1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \frac{(2x)^4}{4!} + \dots$$

$$= 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + \dots$$

$$\therefore e^{2x} - 1 = 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + \dots$$

$$\therefore (e^{2x} - 1)^2 = (2x + 2x^2 + \dots)(2x + 2x^2 + \dots) \\ = 4x^2 + 8x^3 + \dots$$

$$\therefore (e^{2x} - 1)^3 = (4x^2 + 8x^3 + \dots)(2x + 2x^2 + \dots) \\ = 8x^3 + 24x^4 + \dots$$

$$\text{So, } \sin(e^{2x} - 1) = (e^{2x} - 1) - \frac{(e^{2x} - 1)^3}{3!} + \dots \\ = \left(2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + \dots\right) - \frac{1}{3!}(8x^3 + 24x^4 + \dots) + \dots \\ = 2x + 2x^2 - \frac{10}{3}x^4 - \dots$$

$$\begin{aligned} \text{c } \lim_{x \rightarrow 0} \frac{e^{\cos x} - e}{\sin(e^{2x} - 1)} &= \lim_{x \rightarrow 0} \frac{-\frac{e}{2}x^2 + \frac{e}{6}x^4 - \dots}{2x + 2x^2 - \frac{10}{3}x^4 - \dots} \\ &= \lim_{x \rightarrow 0} \frac{-ex + \frac{2e}{3}x^3 - \dots}{2 + 4x - \frac{40}{3}x^3 - \dots} \quad \{\text{l'Hôpital's rule}\} \\ &= \frac{0}{2} \\ &= 0 \end{aligned}$$

**7 a**  $\sin x = 0$  when  $x = n\pi$ ,  $n \in \mathbb{Z}$

$\therefore \sin x$  has zeros when  $x = n\pi$ ,  $n \in \mathbb{Z}$

$\frac{\sin x}{x} = 0$  when  $x = n\pi$ ,  $n \in \mathbb{Z}$ ,  $n \neq 0$

$\therefore \frac{\sin x}{x}$  has zeros when  $x = n\pi$ ,  $n \in \mathbb{Z}$ ,  $n \neq 0$

**b**  $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$  for all  $x \in \mathbb{R}$

$\therefore \frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots$  for all  $x \in \mathbb{R}$ ,  $x \neq 0$

**c**  $\left(1 - \frac{x}{\pi}\right) \left(1 + \frac{x}{\pi}\right) \left(1 - \frac{x}{2\pi}\right) \left(1 + \frac{x}{2\pi}\right) \dots$  has zeros when  $x = n\pi$ ,  $n \in \mathbb{Z}$ ,  $n \neq 0$ .

**d** Now  $\left(1 - \frac{x}{\pi}\right) \left(1 + \frac{x}{\pi}\right) = 1 - \frac{x^2}{\pi^2}$

$$\left(1 - \frac{x}{2\pi}\right) \left(1 + \frac{x}{2\pi}\right) = 1 - \frac{x^2}{4\pi^2}$$

$$\left(1 - \frac{x}{3\pi}\right) \left(1 + \frac{x}{3\pi}\right) = 1 - \frac{x^2}{9\pi^2}, \text{ and so on}$$

$$\therefore \left(1 - \frac{x}{\pi}\right) \left(1 + \frac{x}{\pi}\right) \left(1 - \frac{x}{2\pi}\right) \left(1 + \frac{x}{2\pi}\right) \dots = \left(1 - \frac{x^2}{\pi^2}\right) \left(1 - \frac{x^2}{4\pi^2}\right) \left(1 - \frac{x^2}{9\pi^2}\right) \dots$$

Since  $\frac{\sin x}{x}$  and  $\left(1 - \frac{x}{\pi}\right) \left(1 + \frac{x}{\pi}\right) \left(1 - \frac{x}{2\pi}\right) \left(1 + \frac{x}{2\pi}\right) \dots$  both have zeros when

$x = n\pi$ ,  $n \in \mathbb{Z}$ ,  $n \neq 0$ , this supports Euler's claim that

$$1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots = \left(1 - \frac{x^2}{\pi^2}\right) \left(1 - \frac{x^2}{4\pi^2}\right) \left(1 - \frac{x^2}{9\pi^2}\right) \dots \quad (*)$$

**e** Now,  $\left(1 - \frac{x^2}{\pi^2}\right) \left(1 - \frac{x^2}{4\pi^2}\right) \left(1 - \frac{x^2}{9\pi^2}\right) \dots = 1 - \left(\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \dots\right) x^2 + \dots$

So, equating coefficients of  $x^2$  in (\*):

$$-\frac{1}{3!} = -\left(\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \dots\right)$$

$$\therefore \frac{1}{6} = \frac{1}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{k^2}$$

$$\therefore \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$$

**f**

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \sum_{r=1}^{\infty} \frac{1}{(2r)^2} + \sum_{r=1}^{\infty} \frac{1}{(2r-1)^2}$$

$$\therefore \frac{\pi^2}{6} = \sum_{r=1}^{\infty} \frac{1}{4r^2} + \sum_{r=1}^{\infty} \frac{1}{(2r-1)^2} \quad \{\text{using e}\}$$

$$\therefore \frac{\pi^2}{6} = \frac{1}{4} \sum_{r=1}^{\infty} \frac{1}{r^2} + \sum_{r=1}^{\infty} \frac{1}{(2r-1)^2}$$

$$\therefore \frac{\pi^2}{6} = \frac{1}{4} \left(\frac{\pi^2}{6}\right) + \sum_{r=1}^{\infty} \frac{1}{(2r-1)^2} \quad \{\text{using e}\}$$

$$\therefore \sum_{r=1}^{\infty} \frac{1}{(2r-1)^2} = \frac{\pi^2}{8}$$

## REVIEW SET 24A

**1 a** Let  $f(x) = \frac{1}{3-x} = (3-x)^{-1}$

$$\therefore f'(x) = (-1)(3-x)^{-2}(-1)$$

$$\therefore f''(x) = (-1)(-2)(3-x)^{-3}(-1)^2$$

$$\vdots$$

$$f^{(k)}(x) = k!(3-x)^{-(k+1)}$$

$$\therefore f^{(k)}(0) = k! 3^{-(k+1)} \quad \text{for all } k \in \mathbb{Z}, k \geq 0$$

So, the Maclaurin series representation for  $f(x)$  is

$$f(x) = \sum_{k=0}^{\infty} \frac{k! 3^{-(k+1)}}{k!} x^k$$

$$\therefore \frac{1}{3-x} = \sum_{k=0}^{\infty} \frac{x^k}{3^{k+1}}$$

**b**

$$M_3(x) = \frac{1}{3} + \frac{1}{9}x + \frac{1}{27}x^2 + \frac{1}{81}x^3$$

$$\therefore M_3(1) = \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} = \frac{40}{81} \approx 0.494$$

which is a reasonable approximation to  $f(1) = \frac{1}{2} = 0.5$ .

$$\begin{aligned}
 \text{2} \quad \cos x &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} \\
 \therefore \cos \frac{\pi}{90} &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \left(\frac{\pi}{90}\right)^{2k} \\
 &= \sum_{k=0}^{\infty} \frac{(-1)^k \pi^{2k}}{90^{2k} (2k)!} \\
 &= 1 - \frac{\pi^2}{16\,200} + \frac{\pi^4}{1\,574\,640\,000} - \dots \quad \text{where } \frac{\pi^4}{1\,574\,640\,000} \approx 6.19 \times 10^{-8}
 \end{aligned}$$

Using the first 2 terms,  $\cos \frac{\pi}{90} \approx 0.999\,39$

Check: Using technology,  $\cos \frac{\pi}{90} \approx 0.999\,39$  ✓

$$\text{3} \quad \arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

For any  $|x| \leq 1$ , the terms *decrease* in size.

$\therefore$  the series converges for  $|x| \leq 1$ .

$$\text{4} \quad \text{a} \quad \text{Let } f(x) = \ln(2-x)$$

$$\therefore f'(x) = \frac{-1}{2-x} = -(2-x)^{-1}$$

$$\therefore f''(x) = -(-1)(2-x)^{-2}(-1)$$

$$\therefore f'''(x) = -(-1)(-2)(2-x)^{-3}(-1)^2$$

$$\vdots$$

$$f^{(k)}(x) = -(k-1)!(2-x)^{-k}$$

$$\therefore f^{(k)}(0) = -(k-1)!2^{-k} \quad \text{for all } k \in \mathbb{Z}^+.$$

Since  $f(0) = \ln 2$ , the Maclaurin series representation for  $f(x)$  is

$$f(x) = \ln 2 + \sum_{k=1}^{\infty} \frac{-(k-1)!2^{-k}}{k!} x^k$$

$$\begin{aligned}
 \therefore \ln(2-x) &= \ln 2 - \sum_{k=1}^{\infty} \frac{1}{k 2^k} x^k \\
 &= \ln 2 - \sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{x}{2}\right)^k
 \end{aligned}$$

For any  $\left|\frac{x}{2}\right| > 1$ , or  $|x| > 2$ , the terms will certainly increase.

For the case  $\frac{x}{2} = 1$ , or  $x = 2$ , the series does not converge which is reasonable since  $\ln 0$  is undefined.

For the case  $\frac{x}{2} = -1$ , or  $x = -2$ , the series does converge.

$\therefore$  we expect the series to converge for  $-2 \leq x < 2$ .

$$\text{b} \quad \ln(2-1) = \ln 2 - \sum_{k=1}^{\infty} \frac{1}{k 2^k} (1)^k \quad \text{since } -2 \leq 1 < 2$$

$$\therefore \ln 1 = \ln 2 - \sum_{k=1}^{\infty} \frac{1}{k 2^k}$$

$$\therefore \sum_{k=1}^{\infty} \frac{1}{k 2^k} = \ln 2$$

$$\begin{aligned}
 \text{5 a} \quad \ln(1+x) &= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} x^k \\
 \therefore \ln(1+3x) &= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} (3x)^k \\
 &= \sum_{k=1}^{\infty} \frac{(-1)^{k-1} 3^k x^k}{k}
 \end{aligned}$$

**b**

$n$	1	2	3	4	5
$M_n(1)$	3	-1.5	7.5	-12.75	35.85
$M_n(-\frac{1}{4})$	-0.75	-1.0313	-1.1719	-1.2510	-1.2984

**i** When  $x = 1$ , the difference between consecutive terms is increasing. So, it appears that the Maclaurin series is not convergent for  $x = 1$ .

**ii** When  $x = -\frac{1}{4}$ , the terms appear to be approaching  $\ln(1 + 3(-\frac{1}{4})) \approx -1.39$ .

So, it appears that the Maclaurin series is convergent for  $x = -\frac{1}{4}$ .

**c** The Maclaurin series for  $\ln(1+x)$  converges for  $-1 < x \leq 1$ , so we expect the Maclaurin series for  $\ln(1+3x)$  to converge for  $-1 < 3x \leq 1$

$$\therefore -\frac{1}{3} < x \leq \frac{1}{3}$$

$$\begin{aligned}
 \text{6} \quad e^x &= \sum_{k=0}^{\infty} \frac{x^k}{k!} \\
 \therefore e^{x^2} &= \sum_{k=0}^{\infty} \frac{(x^2)^k}{k!} \\
 &= \sum_{k=0}^{\infty} \frac{x^{2k}}{k!} \\
 \therefore \int_0^1 e^{x^2} dx &= \int_0^1 \left( \sum_{k=0}^{\infty} \frac{x^{2k}}{k!} \right) dx \\
 &= \sum_{k=0}^{\infty} \frac{1}{k!} \left( \int_0^1 x^{2k} dx \right) \\
 &= \sum_{k=0}^{\infty} \frac{1}{k!} \left[ \frac{x^{2k+1}}{2k+1} \right]_0^1 \\
 &= \sum_{k=0}^{\infty} \frac{1}{k!(2k+1)} \\
 &= 1 + \frac{1}{3} + \frac{1}{10} + \frac{1}{42} + \frac{1}{216} + \frac{1}{1320} + \frac{1}{9360} + \dots \quad \text{where } \frac{1}{9360} \approx 0.000107
 \end{aligned}$$

Using the first 6 terms,  $\int_0^1 e^{x^2} dx \approx 1.463$

Check: Using technology,  $\int_0^1 e^{x^2} dx \approx 1.463$  ✓



$$\mathbf{7} \quad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

$$\text{Let } \sec x = \sum_{k=0}^{\infty} a_k x^k$$

$$\therefore \frac{1}{\cos x} = \sum_{k=0}^{\infty} a_k x^k$$

$$\therefore 1 = \left( \sum_{k=0}^{\infty} a_k x^k \right) \cos x$$

$$= (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots) \left( 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \right)$$

$$= a_0 + a_1 x + \left( a_2 - \frac{a_0}{2} \right) x^2 + \left( a_3 - \frac{a_1}{2} \right) x^3 + \left( a_4 - \frac{a_2}{2} + \frac{a_0}{24} \right) x^4 + \dots$$

$$\text{Equating coefficients, } a_0 = 1, \quad a_1 = 0, \quad a_2 = \frac{1}{2}, \quad a_3 = 0, \quad a_4 = \frac{5}{24}$$

$$\therefore \sec x = 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \dots$$

$$\mathbf{8} \quad \mathbf{a} \quad \frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$$

$$\therefore \frac{1}{1+x^3} = \frac{1}{1-(-x^3)}$$

$$= \sum_{k=0}^{\infty} (-x^3)^k$$

$$= \sum_{k=0}^{\infty} (-1)^k x^{3k}$$

$$= 1 - x^3 + x^6 - x^9 + \dots$$

$$\begin{aligned} \mathbf{b} \quad \frac{x^2}{(1+x^3)^2} &= x^2 \left( \frac{1}{1+x^3} \right)^2 \\ &= x^2 (1 - x^3 + x^6 - \dots)(1 - x^3 + x^6 - \dots) \quad \{\text{using } \mathbf{a}\} \\ &= x^2 (1 - 2x^3 + 3x^6 - \dots) \\ &= x^2 - 2x^5 + 3x^8 - \dots \end{aligned}$$

$$\mathbf{9} \quad \mathbf{a} \quad e^x = 1 + x + \frac{x^2}{2!} + \dots \quad \text{and} \quad \tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \dots$$

$$\begin{aligned} \mathbf{i} \quad \tan 2x &= 2x + \frac{1}{3}(2x)^3 + \frac{2}{15}(2x)^5 + \dots \\ &= 2x + \frac{8}{3}x^3 + \frac{64}{15}x^5 + \dots \end{aligned}$$

$$\begin{aligned} \mathbf{ii} \quad \tan^2 x &= (x + \frac{1}{3}x^3 + \dots)(x + \frac{1}{3}x^3 + \dots) \\ &= x^2 + \frac{2}{3}x^4 + \dots \end{aligned}$$

$$\therefore e^{\tan x} = 1 + \tan x + \frac{1}{2!} \tan^2 x + \dots$$

$$= 1 + (x + \frac{1}{3}x^3 + \dots) + \frac{1}{2!} (x^2 + \frac{2}{3}x^4 + \dots)$$

$$= 1 + x + \frac{x^2}{2} + \dots$$

$$\begin{aligned}
 \text{b } \lim_{x \rightarrow 0} \frac{\tan 2x}{e^{\tan x} - 1} &= \lim_{x \rightarrow 0} \frac{2x + \frac{8}{3}x^3 + \frac{64}{15}x^5 + \dots}{x + \frac{1}{2!}x^2 + \dots} \quad \{\text{using a}\} \\
 &= \lim_{x \rightarrow 0} \frac{2 + 8x^2 + \frac{64}{3}x^4 + \dots}{1 + x + \dots} \quad \{\text{l'Hôpital's rule}\} \\
 &= \frac{2}{1} = 2
 \end{aligned}$$

## REVIEW SET 24B

$$\begin{aligned}
 \text{1 } e^x &= \sum_{k=0}^{\infty} \frac{x^k}{k!} \\
 \therefore xe^x &= x \left( \sum_{k=0}^{\infty} \frac{x^k}{k!} \right) \\
 &= \sum_{k=0}^{\infty} \frac{x^{k+1}}{k!} \\
 &= x + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} + \dots \\
 &= x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4 + \dots
 \end{aligned}$$

$$\text{2 } \frac{1}{x^2 - 4} = -\frac{1}{4} - \frac{x^2}{16} - \frac{x^4}{64} - \dots = \sum_{k=0}^{\infty} -\frac{x^{2k}}{4^{k+1}}$$

<b>a</b>	$n$	2	4	6
	$M_n(0.1)$	-0.2506	-0.2506	-0.2506
	$M_n(1)$	-0.3125	-0.3281	-0.3320
	$M_n(2)$	-0.5	-0.75	-1

$$\text{b } \sum_{k=0}^{\infty} \frac{x^{2k}}{4^{k+1}} = \frac{1}{4} \sum_{k=0}^{\infty} \left( \frac{x^2}{4} \right)^k \quad \text{which is a geometric series with common ratio } r = \frac{x^2}{4}.$$

Thus, we expect the Maclaurin series to converge when  $|r| < 1$

$$\therefore \left| \frac{x^2}{4} \right| < 1$$

$$\therefore |x^2| < 4$$

$$\therefore |x| < 2$$

$$\text{3 } \arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

$$\begin{aligned}
 \text{a } \arctan \frac{x}{2} &= \frac{x}{2} - \frac{\left(\frac{x}{2}\right)^3}{3} + \frac{\left(\frac{x}{2}\right)^5}{5} - \frac{\left(\frac{x}{2}\right)^7}{7} + \dots \\
 &= \frac{x}{2} - \frac{x^3}{24} + \frac{x^5}{160} - \frac{x^7}{896} + \dots
 \end{aligned}$$

$$\begin{aligned}\text{b } \arctan(x^2) &= x^2 - \frac{(x^2)^3}{3} + \frac{(x^2)^5}{5} - \frac{(x^2)^7}{7} + \dots \\ &= x^2 - \frac{x^6}{3} + \frac{x^{10}}{5} - \frac{x^{14}}{7} + \dots\end{aligned}$$

$$\begin{aligned}\text{c } \frac{d}{dx}(\arctan x) &= \frac{1}{1+x^2} \\ \therefore \frac{1}{1+x^2} &= \frac{d}{dx} \left( x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \right) \\ &= 1 - x^2 + x^4 - x^6 + \dots\end{aligned}$$

**4** The Maclaurin series for:

$$\bullet \quad e^x \quad \text{is} \quad e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\bullet \quad (1+x)^p \quad \text{is} \quad (1+x)^p = \sum_{k=0}^{\infty} \frac{p(p-1)\dots(p-k+1)}{k!} x^k$$

$$\begin{aligned}\text{a } (1+x)^{\frac{3}{2}} &= \sum_{k=0}^{\infty} \frac{\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\dots\left(\frac{5}{2}-k\right)}{k!} x^k \\ &= \frac{1}{0!} + \frac{\left(\frac{3}{2}\right)}{1!} x + \frac{\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)}{2!} x^2 + \frac{\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{3!} x^3 + \dots \\ &= 1 + \frac{3}{2}x + \frac{3}{8}x^2 - \frac{1}{16}x^3 + \dots\end{aligned}$$

**b**  $(1-x)e^{1-x} = e(1-x)e^{-x}$

$$\begin{aligned}\text{Now, } e^{-x} &= \sum_{k=0}^{\infty} \frac{(-x)^k}{k!} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k!}\end{aligned}$$

$$\begin{aligned}\therefore e(1-x)e^{-x} &= e(1-x) \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k!} \\ &= e \sum_{k=0}^{\infty} \frac{(-1)^k (1-x)x^k}{k!} \\ &= e \sum_{k=0}^{\infty} \frac{(-1)^k (x^k - x^{k+1})}{k!} \\ &= e \sum_{k=0}^{\infty} \frac{(-1)^k x^k + (-1)^{k+1} x^{k+1}}{k!} \\ &= e \left( \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k!} + \sum_{k=0}^{\infty} \frac{(-1)^{k+1} x^{k+1}}{k!} \right) \\ &= e \left( \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k!} + \sum_{j=1}^{\infty} \frac{(-1)^j x^j}{(j-1)!} \right) \quad \{j = k+1 \text{ in the second series}\} \\ &= e \left( 1 + \sum_{k=1}^{\infty} \frac{(-1)^k x^k}{k!} + \sum_{k=1}^{\infty} \frac{(-1)^k x^k}{(k-1)!} \right) \quad \{\text{replacing } j \text{ with } k\} \\ &= e \left( 1 + \sum_{k=1}^{\infty} (-1)^k \left( \frac{1}{k!} + \frac{1}{(k-1)!} \right) x^k \right) \\ &= e + \sum_{k=1}^{\infty} (-1)^k \left( \frac{1}{(k-1)!} + \frac{1}{k!} \right) ex^k \\ &= e - 2ex + \frac{3}{2}ex^2 - \frac{2}{3}ex^3 + \dots\end{aligned}$$

**c**  $e^{2x} = 1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \dots$       and       $e^{4x} = 1 + 4x + \frac{(4x)^2}{2!} + \frac{(4x)^3}{3!} + \dots$   
 $= 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \dots$        $= 1 + 4x + 8x^2 + \frac{32}{3}x^3 + \dots$

$$\begin{aligned}\therefore (1 + e^{2x})^2 &= 1 + 2e^{2x} + e^{4x} \\ &= 1 + 2(1 + 2x + 2x^2 + \frac{4}{3}x^3 + \dots) + (1 + 4x + 8x^2 + \frac{32}{3}x^3 + \dots) \\ &= 4 + 8x + 12x^2 + \frac{40}{3}x^3 + \dots\end{aligned}$$

$$\begin{aligned}\text{Let } \frac{1}{(1 + e^{2x})^2} &= \sum_{k=0}^{\infty} a_k x^k \\ \therefore 1 &= \left( \sum_{k=0}^{\infty} a_k x^k \right) (1 + e^{2x})^2 \\ &= (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots)(4 + 8x + 12x^2 + \frac{40}{3}x^3 + \dots) \\ &= 4a_0 + (4a_1 + 8a_0)x + (4a_2 + 8a_1 + 12a_0)x^2 \\ &\quad + (4a_3 + 8a_2 + 12a_1 + \frac{40}{3}a_0)x^3 + \dots\end{aligned}$$

Equating coefficients,  $a_0 = \frac{1}{4}$ ,  $a_1 = -\frac{1}{2}$ ,  $a_2 = \frac{1}{4}$ ,  $a_3 = \frac{1}{6}$

$$\therefore \frac{1}{(1 + e^{2x})^2} = \frac{1}{4} - \frac{1}{2}x + \frac{1}{4}x^2 + \frac{1}{6}x^3 - \dots$$



$$\begin{aligned}
 \text{5} \quad \sin x &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} \\
 \therefore \sin 2x &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} (2x)^{2k+1} \\
 &= \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k+1}}{(2k+1)!} x^{2k+1} \\
 \therefore \int_0^1 \sin 2x \, dx &= \int_0^1 \left( \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k+1}}{(2k+1)!} x^{2k+1} \right) dx \\
 &= \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k+1}}{(2k+1)!} \left( \int_0^1 x^{2k+1} \, dx \right) \\
 &= \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k+1}}{(2k+1)!} \left[ \frac{x^{2k+2}}{2k+2} \right]_0^1 \\
 &= \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k+1}}{(2k+2)!} \\
 &= 1 - \frac{1}{3} + \frac{2}{45} - \frac{1}{315} + \frac{2}{14175} + \dots \quad \text{where } \frac{2}{14175} \approx 0.000141
 \end{aligned}$$

Using the first 4 terms,  $\int_0^1 \sin 2x \, dx \approx 0.708$

Check: Using technology,  $\int_0^1 \sin 2x \, dx \approx 0.708$  ✓

$$\text{6 a} \quad \text{Let } \sqrt{1+x} = \sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots \quad \dots (*)$$

$$\therefore (\sqrt{1+x})^2 = (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots)(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots)$$

$$\therefore 1+x = a_0^2 + 2a_0 a_1 x + (2a_0 a_2 + a_1^2) x^2 + (2a_0 a_3 + 2a_1 a_2) x^3 + \dots$$

$$\text{Equating coefficients, } a_0^2 = 1, \quad 2a_0 a_1 = 1, \quad 2a_0 a_2 + a_1^2 = 0, \quad 2a_0 a_3 + 2a_1 a_2 = 0$$

Now, substituting  $x = 0$  into (\*) gives  $a_0 = 1$

$$\therefore a_1 = \frac{1}{2}, \quad a_2 = -\frac{1}{8}, \quad a_3 = \frac{1}{16}$$

$$\therefore \sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \dots$$

$$\text{b} \quad (1+x)^p = \sum_{k=0}^{\infty} \frac{p(p-1)\dots(p-k+1)}{k!} x^k$$

$$\therefore (1+x)^{\frac{1}{2}} = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\dots\left(\frac{3}{2}-k\right)}{k!} x^k$$

$$= 1 + \frac{\left(\frac{1}{2}\right)}{1!} x + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!} x^2 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!} x^3 - \dots$$

$$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \dots$$

which agrees with our answer in a.

$$\begin{aligned}
 7 \quad a \quad e^x &= \sum_{k=0}^{\infty} \frac{x^k}{k!} \\
 \therefore e^{-x} &= \sum_{k=0}^{\infty} \frac{(-x)^k}{k!} \\
 &= \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k!} \\
 &= 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots
 \end{aligned}$$

$$\begin{aligned}
 \text{Let } \frac{e^{-x}}{x^2 + 1} &= \sum_{k=0}^{\infty} a_k x^k \\
 \therefore e^{-x} &= \left( \sum_{k=0}^{\infty} a_k x^k \right) (x^2 + 1)
 \end{aligned}$$

$$\begin{aligned}
 \therefore 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots &= (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots)(x^2 + 1) \\
 &= a_0 + a_1 x + (a_2 + a_0)x^2 + (a_3 + a_1)x^3 + \dots
 \end{aligned}$$

Equating coefficients,  $a_0 = 1$ ,  $a_1 = -1$ ,  $a_2 = -\frac{1}{2}$ ,  $a_3 = \frac{5}{6}$

$$\begin{aligned}
 b \quad \text{Let } f(x) &= \frac{e^{-x}}{x^2 + 1} = e^{-x}(x^2 + 1)^{-1} \\
 \therefore f'(x) &= -e^{-x}(x^2 + 1)^{-1} + e^{-x}(-1)(x^2 + 1)^{-2}(2x) \\
 &= -f(x) - 2x(x^2 + 1)^{-1} f(x) \\
 &= -f(x)[1 + 2x(x^2 + 1)^{-1}] \\
 \therefore f''(x) &= -f'(x)[1 + 2x(x^2 + 1)^{-1}] - f(x)[2(x^2 + 1)^{-1} + 2x(-1)(x^2 + 1)^{-2}(2x)] \\
 &= -f'(x)[1 + 2x(x^2 + 1)^{-1}] - f(x)[2(x^2 + 1)^{-1} - 4x^2(x^2 + 1)^{-2}] \\
 \therefore f^{(3)}(x) &= -f''(x)[1 + 2x(x^2 + 1)^{-1}] - f'(x)[2(x^2 + 1)^{-1} + 2x(-1)(x^2 + 1)^{-2}(2x)] \\
 &\quad - f'(x)[2(x^2 + 1)^{-1} - 4x^2(x^2 + 1)^{-2}] \\
 &\quad - f(x)[2(-1)(x^2 + 1)^{-2}(2x) - 8x(x^2 + 1)^{-2} - 4x^2(-2)(x^2 + 1)^{-3}(2x)] \\
 &= -f''(x)[1 + 2x(x^2 + 1)^{-1}] - f'(x)[2(x^2 + 1)^{-1} - 4x^2(x^2 + 1)^{-2}] \\
 &\quad - f'(x)[2(x^2 + 1)^{-1} - 4x^2(x^2 + 1)^{-2}] \\
 &\quad - f(x)[-4x(x^2 + 1)^{-2} - 8x(x^2 + 1)^{-2} + 16x^3(x^2 + 1)^{-3}] \\
 &= -f''(x)[1 + 2x(x^2 + 1)^{-1}] - 2f'(x)[2(x^2 + 1)^{-1} - 4x^2(x^2 + 1)^{-2}] \\
 &\quad - f(x)[16x^3(x^2 + 1)^{-3} - 12x(x^2 + 1)^{-2}]
 \end{aligned}$$

$$\text{So, } f(0) = \frac{e^0}{1} = 1$$

$$f'(0) = -(1) \left( 1 + \frac{2(0)}{1} \right) = -1$$

$$f''(0) = -(-1) \left( 1 + \frac{2(0)}{1} \right) - (1) \left( \frac{2}{1} - \frac{4(0)^2}{1^2} \right) = -1$$

$$\text{and } f^{(3)}(0) = -(-1) \left( 1 + \frac{2(0)}{1} \right) - 2(-1) \left( \frac{2}{1} - \frac{4(0)^2}{1^2} \right) - (1) \left( \frac{16(0)^3}{1^3} - \frac{12(0)}{1^2} \right) = 5$$

Thus, the first 4 coefficients in the Maclaurin series for  $f(x)$

are  $1$ ,  $-1$ ,  $-\frac{1}{2!}$ , and  $\frac{5}{3!}$

which is  $1$ ,  $-1$ ,  $-\frac{1}{2}$ , and  $\frac{5}{6}$ .

This agrees with our answer in **a**.

**8 a** The Maclaurin series for:

- $\cos x$  is  $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$

- $\sin x$  is  $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$

**i**  $\cos^2 x = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\right) \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\right)$

$$= 1 + \left(-\frac{1}{2!} - \frac{1}{2!}\right)x^2 + \left(\frac{1}{4!} + \frac{1}{2! \times 2!} + \frac{1}{4!}\right)x^4$$

$$+ \left(-\frac{1}{6!} - \frac{1}{2! \times 4!} - \frac{1}{4! \times 2!} - \frac{1}{6!}\right)x^6 + \dots$$

$$= 1 - x^2 + \frac{1}{3}x^4 - \frac{2}{45}x^6 + \dots$$

**ii**  $\sin^2 x = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right) \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right)$

$$= x^2 + \left(-\frac{1}{3!} - \frac{1}{3!}\right)x^4 + \left(\frac{1}{5!} + \frac{1}{3! \times 3!} + \frac{1}{5!}\right)x^6 + \dots$$

$$= x^2 - \frac{1}{3}x^4 + \frac{2}{45}x^6 - \dots$$

**b i**  $\cos^2 x + \sin^2 x = (1 - x^2 + \frac{1}{3}x^4 - \frac{2}{45}x^6 + \dots) + (x^2 - \frac{1}{3}x^4 + \frac{2}{45}x^6 - \dots)$

$$= 1$$

which is consistent with the identity  $\cos^2 x + \sin^2 x = 1$ .

**ii**  $\cos 2x = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \dots$

$$= 1 - 2x^2 + \frac{2}{3}x^4 - \frac{4}{45}x^6 + \dots$$

and  $\cos^2 x - \sin^2 x = (1 - x^2 + \frac{1}{3}x^4 - \frac{2}{45}x^6 + \dots) - (x^2 - \frac{1}{3}x^4 + \frac{2}{45}x^6 - \dots)$

$$= 1 - 2x^2 + \frac{2}{3}x^4 - \frac{4}{45}x^6 + \dots$$

which is consistent with the identity  $\cos^2 x - \sin^2 x = \cos 2x$ .

# Chapter 25

## DIFFERENTIAL EQUATIONS

### EXERCISE 25A

1 a If  $y = x^4$ , then

$$\frac{dy}{dx} = 4x^3 \quad \text{as required.}$$

b If  $y = 5e^{2x}$ , then

$$\begin{aligned}\frac{dy}{dx} &= 10e^{2x} \\ &= 2(5e^{2x}) \\ &= 2y \quad \text{as required.}\end{aligned}$$

c If  $y = \sqrt{x^2 + 1} = (x^2 + 1)^{\frac{1}{2}}$ , then

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{2}(x^2 + 1)^{-\frac{1}{2}}(2x) \\ &= \frac{x}{\sqrt{x^2 + 1}} \\ &= \frac{x}{y} \quad \text{as required.}\end{aligned}$$

d If  $y = -\frac{1}{x} = -x^{-1}$ , then

$$\begin{aligned}\frac{dy}{dx} &= x^{-2} \\ &= (-x^{-1})^2 \\ &= y^2 \quad \text{as required.}\end{aligned}$$

e If  $y = 3e^{\frac{x^2}{2} + x}$ , then  $\frac{dy}{dx} = 3\left(\frac{1}{2}(2x) + 1\right)e^{\frac{x^2}{2} + x}$

$$\begin{aligned}&= (x + 1)3e^{\frac{x^2}{2} + x} \\ &= (x + 1)y \\ &= xy + y\end{aligned}$$

$$\therefore \frac{dy}{dx} - y = xy \quad \text{as required.}$$

2 If  $y = \sqrt[3]{x^2 + 1} = (x^2 + 1)^{\frac{1}{3}}$ , then  $\frac{dy}{dx} = \frac{1}{3}(x^2 + 1)^{-\frac{2}{3}} \times 2x$

$$\begin{aligned}&= \frac{2x}{3\left(\sqrt[3]{x^2 + 1}\right)^2} \\ &= \frac{2x}{3y^2}\end{aligned}$$

If  $y = \sqrt{x + 3} = (x + 3)^{\frac{1}{2}}$ , then  $\frac{dy}{dx} = \frac{1}{2}(x + 3)^{-\frac{1}{2}}$

$$\begin{aligned}&= \frac{1}{2\sqrt{x + 3}} \\ &= \frac{1}{2y}\end{aligned}$$

If  $y = \frac{1}{x^2} = x^{-2}$ , then  $\frac{dy}{dx} = -2x^{-3}$

$$\begin{aligned}&= -\frac{2}{x^3} \\ &= -\frac{2}{x} \times \frac{1}{x^2} \\ &= -\frac{2y}{x}\end{aligned}$$

$\therefore$  a B      b C      c A



3 a If  $y = x^3 + c$ , then

$$\frac{dy}{dx} = 3x^2 \quad \text{as required.}$$

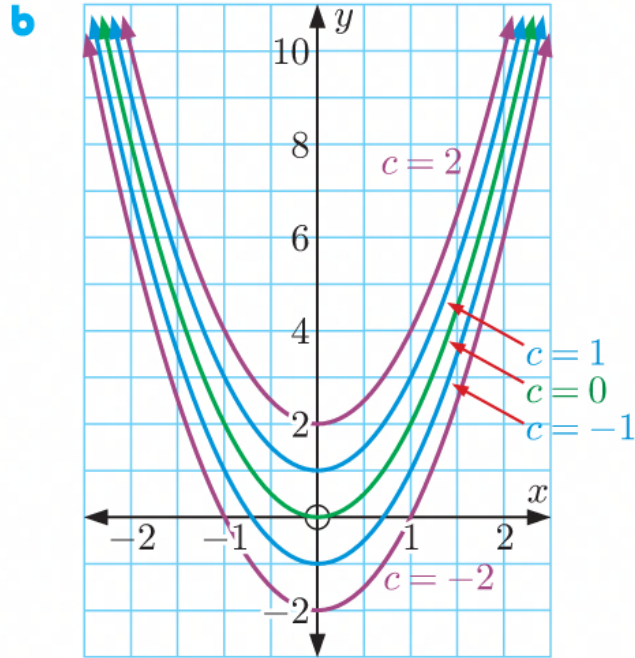
b If  $y = ce^{-x}$ , then

$$\begin{aligned} \frac{dy}{dx} &= -ce^{-x} \\ &= -y \quad \text{as required.} \end{aligned}$$

c If  $y = -\frac{2}{x^2 + c} = -2(x^2 + c)^{-1}$ , then

$$\begin{aligned} \frac{dy}{dx} &= 2(x^2 + c)^{-2}(2x) \\ &= \frac{4x}{(x^2 + c)^2} \\ &= x \left( -\frac{2}{x^2 + c} \right)^2 = xy^2 \quad \text{as required.} \end{aligned}$$

4 a If  $y = 2x^2 + c$ , then  $\frac{dy}{dx} = 4x$  for any constant  $c$  as required.



c From a,  $y = 2x^2 + c$  is a general solution to the differential equation.

The particular solution passes through  $(1, \frac{1}{2})$ , so

$$\frac{1}{2} = 2(1)^2 + c$$

$$\therefore c = \frac{1}{2} - 2 = -\frac{3}{2}$$

$$\therefore \text{the particular solution is } y = 2x^2 - \frac{3}{2}$$

d  $\frac{dy}{dx} = 4x$

$$\therefore \text{at the point } (1, \frac{1}{2}), \quad \frac{dy}{dx} = 4(1) = 4$$

$$\therefore \text{the gradient of the tangent to the particular solution } y = 2x^2 - \frac{3}{2} \text{ at } (1, \frac{1}{2}), \text{ is } 4.$$

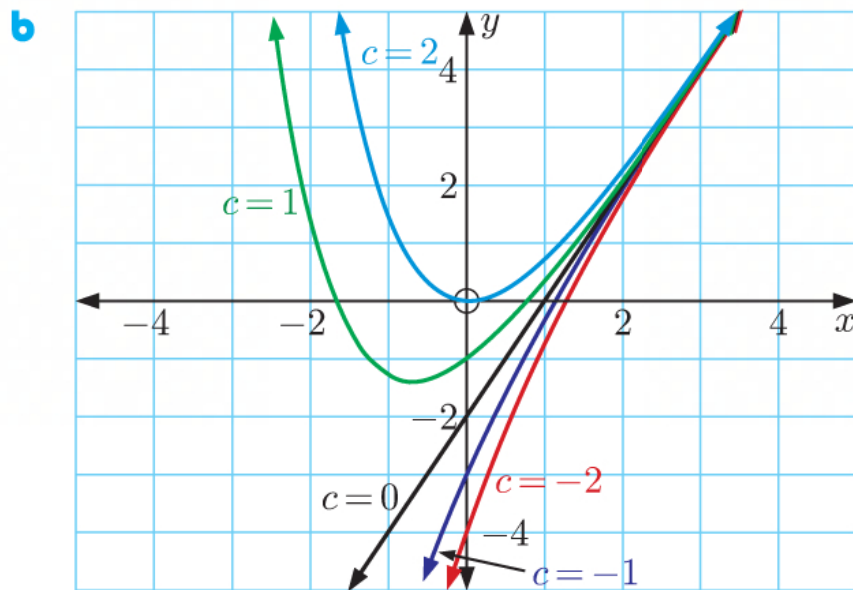
$$\therefore \text{the equation of the tangent is } y = 4(x - 1) + \frac{1}{2}$$

$$\therefore y = 4x - \frac{7}{2}$$

5 a If  $y = 2x - 2 + ce^{-x}$ , then  $\frac{dy}{dx} = 2 - ce^{-x}$

$$= 2x - (2x - 2 + ce^{-x})$$

$$= 2x - y \quad \text{for any constant } c \text{ as required.}$$



c From a,  $y = 2x - 2 + ce^{-x}$  is a general solution to the differential equation.

The particular solution passes through  $(0, 1)$ , so  $1 = -2 + c$

$$\therefore c = 3$$

$\therefore$  the particular solution is

$$y = 2x - 2 + 3e^{-x}$$

**d**  $\frac{dy}{dx} = 2x - y$

$\therefore$  at the point  $(0, 1)$ ,  $\frac{dy}{dx} = 0 - 1 = -1$

$\therefore$  the gradient of the tangent to the particular solution  $y = 2x - 2 + 3e^{-x}$  at  $(0, 1)$ , is  $-1$ .

$\therefore$  the equation of the tangent is  $y = -(x - 0) + 1$

$\therefore y = -x + 1$

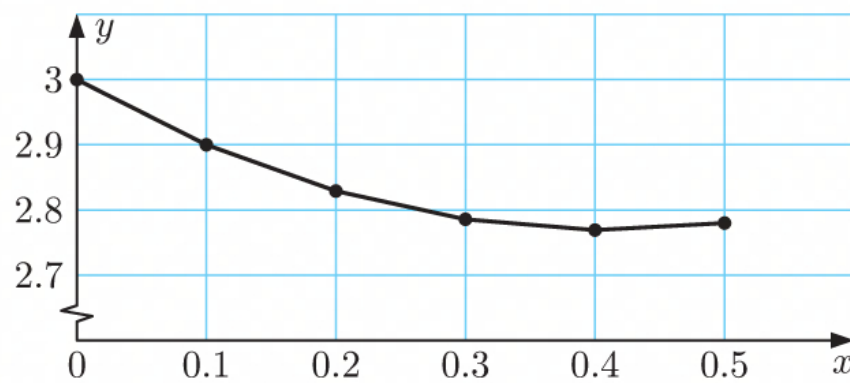
## EXERCISE 25B

**1**  $\frac{dy}{dx} = xy - 1$  with initial point  $(0, 3)$ .

**a**

Iteration	$x_{i-1}$	$y_{i-1}$	$\frac{dy}{dx}$	$x_i$	$y_i$
1	0	3	-1	0.1	2.9
2	0.1	2.9	-0.71	0.2	2.829
3	0.2	2.829	-0.4342	0.3	2.7856
4	0.3	2.7856	-0.1643	0.4	2.7691
5	0.4	2.7691	0.1077	0.5	2.7799

**b**



**2**  $\frac{dy}{dx} = 3e^{2x} - 1$ ,  $y(0) = 2$

$y(0) = 2$  gives us  $x_0 = 0$  and  $y_0 = 2$ .

**a**

**i**

Iteration	$x_{i-1}$	$y_{i-1}$	$\frac{dy}{dx}$	$x_i$	$y_i$
1	0	2	2	0.25	2.5
2	0.25	2.5	3.9462	0.5	3.4865

$\therefore y(0.5) \approx 3.4865$

**ii**

Iteration	$x_{i-1}$	$y_{i-1}$	$\frac{dy}{dx}$	$x_i$	$y_i$
1	0	2	2	0.1	2.2
2	0.1	2.2	2.6642	0.2	2.4664
3	0.2	2.4664	3.4755	0.3	2.8140
4	0.3	2.8140	4.4664	0.4	3.2606
5	0.4	3.2606	5.6766	0.5	3.8283

$\therefore y(0.5) \approx 3.8283$

**b** Using the Fundamental Theorem of Calculus,

$$\begin{aligned}
 y(0.5) &= y(0) + \int_0^{0.5} \frac{dy}{dx} dx \\
 &= 2 + \int_0^{0.5} (3e^{2x} - 1) dx \\
 &= 2 + \left[ \frac{3}{2}e^{2x} - x \right]_0^{0.5} \\
 &= 2 + \left( \frac{3}{2}e - 0.5 \right) - \left( \frac{3}{2} - 0 \right) \\
 &= \frac{3}{2}e \\
 &\approx 4.0774
 \end{aligned}$$

The accuracy of Euler's method was improved by decreasing the step size.

**3**  $\frac{dy}{dx} = 1 + 2x - 3y, \quad y(0) = 1$

$y(0) = 1$  gives us  $x_0 = 0$  and  $y_0 = 1$ .

Iteration	$x_{i-1}$	$y_{i-1}$	$\frac{dy}{dx}$	$x_i$	$y_i$
1	0	1	-2	0.2	0.6
2	0.2	0.6	-0.4	0.4	0.52
3	0.4	0.52	0.24	0.6	0.568
4	0.6	0.568	0.496	0.8	0.6672
5	0.8	0.6672	0.5984	1	0.7869

$\therefore y(1) \approx 0.7869$

**4**  $\frac{dy}{dx} = -\cos x, \quad y(0) = 0$

$y(0) = 0$  gives us  $x_0 = 0$  and  $y_0 = 0$ .

**a**

Iteration	$x_{i-1}$	$y_{i-1}$	$\frac{dy}{dx}$	$x_i$	$y_i$
1	0	0	-1	0.1	-0.1
2	0.1	-0.1	-0.9950	0.2	-0.1995
3	0.2	-0.1995	-0.9801	0.3	-0.2975
4	0.3	-0.2975	-0.9553	0.4	-0.3930
5	0.4	-0.3930	-0.9211	0.5	-0.4851

$\therefore y(0.5) \approx -0.4851$

**b** Using technology to estimate  $y(0.5)$  using Euler's method with step size 0.001 for 500 steps, we get  $y(0.5) \approx -0.4795$ .

- Using the Fundamental Theorem of Calculus,

$$\begin{aligned}
 y(0.5) &= y(0) + \int_0^{0.5} \frac{dy}{dx} dx \\
 &= 0 + \int_0^{0.5} (-\cos x) dx \\
 &= [-\sin x]_0^{0.5} \\
 &= -\sin(0.5) + 0 \\
 &\approx -0.4794
 \end{aligned}$$

The accuracy of Euler's method was improved by decreasing the step size.

- 5  $\frac{dy}{dx} = x \cos y$  with initial point  $(0, 0)$

a

Iteration	$x_{i-1}$	$y_{i-1}$	$\frac{dy}{dx}$	$x_i$	$y_i$
1	0	0	0	1	0
2	1	0	1	2	1

$$\therefore y(2) \approx 1$$

- b Using technology to estimate  $y(2)$  using Euler's method with step size 0.1 for 20 steps, we get  $y(2) \approx 1.3021$ .
- c Using technology to estimate  $y(2)$  using Euler's method with step size 0.01 for 200 steps, we get  $y(2) \approx 1.3018$ .

## EXERCISE 25C

- 1 a  $\frac{dy}{dx} = 4x^3$   
 $\therefore y = \int 4x^3 dx$   
 $\therefore y = x^4 + c$
- b  $\frac{dy}{dx} = x^2 + 6x$   
 $\therefore y = \int (x^2 + 6x) dx$   
 $\therefore y = \frac{1}{3}x^3 + 3x^2 + c$
- c  $\frac{dy}{dx} = e^{3x} + 4$   
 $\therefore y = \int (e^{3x} + 4) dx$   
 $\therefore y = \frac{1}{3}e^{3x} + 4x + c$
- d  $\frac{dy}{dx} = \cos x + \sin 2x$   
 $\therefore y = \int (\cos x + \sin 2x) dx$   
 $\therefore y = \sin x - \frac{1}{2} \cos 2x + c$
- e  $\frac{dx}{dt} = \cos^2 t = \frac{1}{2} + \frac{1}{2} \cos 2t$   
 $\therefore x = \int \left( \frac{1}{2} + \frac{1}{2} \cos 2t \right) dt$   
 $\therefore x = \frac{1}{2}t + \frac{1}{4} \sin 2t + c$
- f  $\frac{dy}{dx} = \frac{1}{x+4} - \frac{2}{3x-5}$   
 $\therefore y = \int \left( \frac{1}{x+4} - \frac{2}{3x-5} \right) dx$   
 $\therefore y = \ln|x+4| - \frac{2}{3} \ln|3x-5| + c$
- g  $\frac{dM}{dt} = \frac{3t^2}{t^3-4}$   
 $\therefore M = \int \frac{3t^2}{t^3-4} dt$   
 $\therefore M = \ln|t^3-4| + c$



$$\begin{aligned}
 \text{h} \quad \frac{dy}{dx} &= \frac{x}{\sqrt{25-x^2}} \\
 \therefore y &= \int \frac{x}{\sqrt{25-x^2}} dx \\
 \therefore y &= \int \frac{1}{\sqrt{u}} \left(-\frac{1}{2} \frac{du}{dx}\right) dx \\
 &\quad \{u = 25 - x^2, \quad \frac{du}{dx} = -2x\} \\
 \therefore y &= -\frac{1}{2} \int u^{-\frac{1}{2}} du \\
 \therefore y &= -\frac{1}{2}(2u^{\frac{1}{2}} + c) \\
 \therefore y &= -\sqrt{25-x^2} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{j} \quad \frac{dS}{dt} &= \frac{\sqrt{\ln t}}{t} \\
 \therefore S &= \int \frac{\sqrt{\ln t}}{t} dt \\
 \therefore S &= \int \sqrt{u} \frac{du}{dt} dt \\
 &\quad \{u = \ln t, \quad \frac{du}{dt} = \frac{1}{t}\} \\
 \therefore S &= \int u^{\frac{1}{2}} du \\
 \therefore S &= \frac{2}{3}u^{\frac{3}{2}} + c \\
 \therefore S &= \frac{2}{3}(\ln t)^{\frac{3}{2}} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{l} \quad \frac{dy}{dx} + \frac{2}{x} &= \sqrt{x} \\
 \therefore \frac{dy}{dx} &= \sqrt{x} - \frac{2}{x} \\
 \therefore y &= \int \left(x^{\frac{1}{2}} - \frac{2}{x}\right) dx \\
 \therefore y &= \frac{2}{3}x^{\frac{3}{2}} - 2 \ln|x| + c
 \end{aligned}$$

$$\begin{aligned}
 \text{2 a} \quad \frac{dy}{dx} &= 3x - 2 \\
 \therefore y &= \int (3x - 2) dx \\
 \therefore y &= \frac{3}{2}x^2 - 2x + c \\
 \text{Now } y(0) &= 5 \\
 \therefore 5 &= 0 - 0 + c \\
 \therefore c &= 5
 \end{aligned}$$

So, the solution is  $y = \frac{3}{2}x^2 - 2x + 5$ .

$$\begin{aligned}
 \text{i} \quad f'(t) &= te^{-t^2+1} + 2 \\
 \therefore f(t) &= \int (te^{-t^2+1} + 2) dt \\
 \therefore f(t) &= \int te^{-t^2+1} dt + \int 2 dt \\
 \therefore f(t) &= \int e^u \left(-\frac{1}{2} \frac{du}{dt}\right) dt + \int 2 dt \\
 &\quad \{u = -t^2 + 1, \quad \frac{du}{dt} = -2t\} \\
 \therefore f(t) &= -\frac{1}{2} \int e^u du + \int 2 dt \\
 \therefore f(t) &= -\frac{1}{2}e^u + 2t + c \\
 \therefore f(t) &= -\frac{1}{2}e^{-t^2+1} + 2t + c
 \end{aligned}$$

$$\begin{aligned}
 \text{k} \quad \frac{dy}{dx} + \cos 3x &= 1 \\
 \therefore \frac{dy}{dx} &= 1 - \cos 3x \\
 \therefore y &= \int (1 - \cos 3x) dx \\
 \therefore y &= x - \frac{1}{3} \sin 3x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad \frac{dy}{dx} &= e^{3x} + 1 \\
 \therefore y &= \int (e^{3x} + 1) dx \\
 \therefore y &= \frac{1}{3}e^{3x} + x + c \\
 \text{Now } y(0) &= 0 \\
 \therefore 0 &= \frac{1}{3}e^0 + 0 + c \\
 \therefore c &= -\frac{1}{3}
 \end{aligned}$$

So, the solution is  $y = \frac{1}{3}e^{3x} + x - \frac{1}{3}$ .

$$\text{c} \quad \frac{dy}{dx} = \frac{1}{x}$$

$$\therefore y = \int \frac{1}{x} dx$$

$$\therefore y = \ln |x| + c$$

$$\text{Now } y(2) = \ln 12$$

$$\therefore \ln 12 = \ln |2| + c$$

$$\therefore c = \ln 12 - \ln 2 = \ln 6$$

So, the solution is  $y = \ln |x| + \ln 6$ .

$$\text{3 a} \quad \frac{dy}{dt} = 2e^{2t} - e^{-t}$$

$$\therefore y = \int (2e^{2t} - e^{-t}) dt$$

$$\therefore y = e^{2t} + e^{-t} + c$$

$$\text{Now } y(\ln 2) = 2.5$$

$$\therefore 2.5 = e^{2 \ln 2} + e^{-\ln 2} + c$$

$$\therefore 2.5 = 4 + 0.5 + c$$

$$\therefore c = -2$$

So, the solution is  $y = e^{2t} + e^{-t} - 2$ .

$$\text{b} \quad \frac{dM}{d\alpha} = \cos 2\alpha - 3 \sin \alpha$$

$$\therefore M = \int (\cos 2\alpha - 3 \sin \alpha) d\alpha$$

$$\therefore M = \frac{1}{2} \sin 2\alpha + 3 \cos \alpha + c$$

$$\text{Now } M\left(\frac{\pi}{2}\right) = 2$$

$$\therefore 2 = \frac{1}{2} \sin \pi + 3 \cos \frac{\pi}{2} + c$$

$$\therefore c = 2$$

So, the solution is

$$M = \frac{1}{2} \sin 2\alpha + 3 \cos \alpha + 2.$$

$$\text{c} \quad \frac{dP}{dx} = 2x \cos x$$

$$\therefore P = \int 2x \cos x dx$$

$$\therefore P = 2x \sin x - \int 2 \sin x dx \quad \begin{cases} u = 2x, & v' = \cos x \\ u' = 2, & v = \sin x \end{cases}$$

$$\therefore P = 2x \sin x + 2 \cos x + c$$

$$\text{Now } P\left(\frac{\pi}{6}\right) = \sqrt{3}$$

$$\therefore \sqrt{3} = \frac{\pi}{3} \sin \frac{\pi}{6} + 2 \cos \frac{\pi}{6} + c$$

$$\therefore \sqrt{3} = \frac{\pi}{6} + \sqrt{3} + c$$

$$\therefore c = -\frac{\pi}{6}$$

So, the solution is  $P = 2x \sin x + 2 \cos x - \frac{\pi}{6}$ .

$$\text{4} \quad f'(x) = 2x - 5$$

$$\therefore f(x) = \int (2x - 5) dx$$

$$= x^2 - 5x + c$$

$$\text{Now } f(2) = -18$$

$$\therefore -18 = 2^2 - 5(2) + c$$

$$\therefore c = -12$$

$$\therefore f(x) = x^2 - 5x - 12$$

$$\therefore f(-2) = (-2)^2 - 5(-2) - 12$$

$$= 4 + 10 - 12$$

$$= 2$$

**5 a**  $\frac{dy}{dx} = e^x - e^{-x}$

$$\therefore y = \int (e^x - e^{-x}) dx$$

$$\therefore y = e^x + e^{-x} + c$$

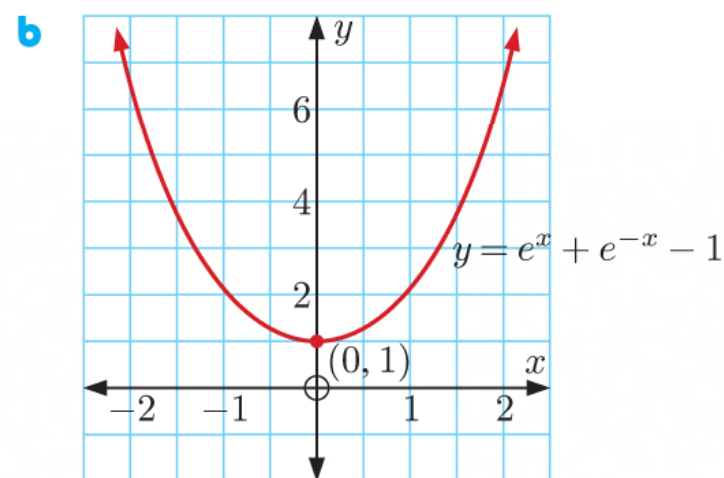
Now  $y(0) = 1$

$$\therefore 1 = e^0 + e^0 + c$$

$$\therefore 1 = 2 + c$$

$$\therefore c = -1$$

So, the solution is  $y = e^x + e^{-x} - 1$ .



**c** When  $x = \ln 2$ ,  $\frac{dy}{dx} = e^{\ln 2} - e^{-\ln 2}$  and  $y = e^{\ln 2} + e^{-\ln 2} - 1$

$$= 2 - \frac{1}{2} = 2 + \frac{1}{2} - 1 = \frac{3}{2}$$

$$\therefore \text{the equation of the tangent is } y = \frac{3}{2}(x - \ln 2) + \frac{3}{2}$$

$$\therefore 2y = 3x - 3\ln 2 + 3$$

$$\therefore 3x - 2y = 3\ln 2 - 3$$

**6**  $f'(x) = ax + bx^{-2}$

$$\therefore f(x) = \int (ax + bx^{-2}) dx$$

$$= \frac{a}{2}x^2 - bx^{-1} + c$$

Now  $f(-1) = -2$  and  $f(1) = 0$

$$\therefore \frac{a}{2}(-1)^2 - b(-1)^{-1} + c = -2$$

$$\therefore \frac{a}{2} + b + c = -2 \quad \dots (1)$$

and  $\frac{a}{2}(1)^2 - b(1)^{-1} + c = 0$

$$\therefore \frac{a}{2} - b + c = 0 \quad \dots (2)$$

Also,  $f'(1) = 0$

$$\therefore a(1) + b(1)^{-2} = 0$$

$$\therefore a + b = 0$$

$$\therefore a = -b \quad \dots (3)$$

Substituting (3) into (2):

$$\frac{-b}{2} - b + c = 0$$

$$\therefore c = \frac{3b}{2} \quad \dots (4)$$

Substituting (3) and (4) into (1):

$$-\frac{b}{2} + b + \frac{3b}{2} = -2$$

$$\therefore 2b = -2$$

$$\therefore b = -1$$

$$\therefore a = 1 \quad \{\text{using (3)}\}$$

$$\text{and } c = -\frac{3}{2} \quad \{\text{using (4)}\}$$

$$\therefore f(x) = \frac{1}{2}x^2 + \frac{1}{x} - \frac{3}{2}$$

$$7 \quad \frac{dy}{dx} = \frac{2x}{x^2 + k}, \quad k > 0$$

$$\therefore y = \int \frac{2x}{x^2 + k} dx$$

$$\therefore y = \int \frac{1}{u} \frac{du}{dx} dx \quad \{u = x^2 + k, \quad \frac{du}{dx} = 2x\}$$

$$\therefore y = \int \frac{1}{u} du$$

$$\therefore y = \ln |u| + c$$

$$\therefore y = \ln |x^2 + k| + c$$

$$\therefore y = \ln(x^2 + k) + c \quad \{x^2 + k > 0 \text{ for all } x\}$$

$$\text{Now, } y(0) = \ln 6 \quad \text{and} \quad y(2) = \ln 18$$

$$\therefore \ln 6 = \ln k + c \quad \therefore \ln 18 = \ln(2^2 + k) + c$$

$$\therefore c = \ln 6 - \ln k \quad \therefore c = \ln 18 - \ln(4 + k)$$

$$= \ln\left(\frac{6}{k}\right) \quad \dots (1) \quad = \ln\left(\frac{18}{4 + k}\right) \quad \dots (2)$$

$$\text{Equating (1) and (2), } \ln\left(\frac{6}{k}\right) = \ln\left(\frac{18}{4 + k}\right)$$

$$\therefore \frac{6}{k} = \frac{18}{4 + k}$$

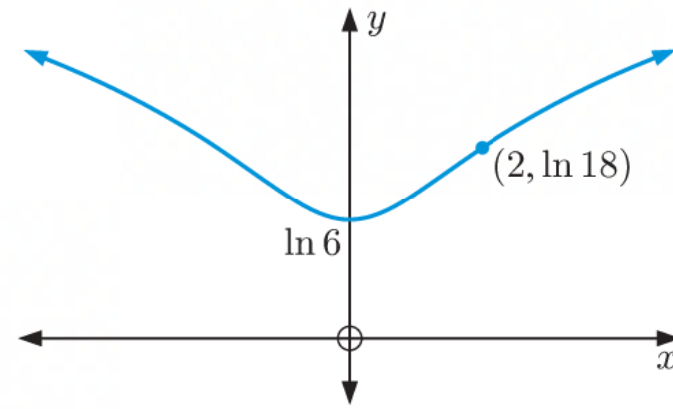
$$\therefore 4 + k = 3k$$

$$\therefore 2k = 4$$

$$\therefore k = 2$$

$$\text{Substituting } k = 2 \text{ into (1), } c = \ln\left(\frac{6}{2}\right) = \ln 3$$

$$\therefore \text{the equation of the curve is } y = \ln(x^2 + 2) + \ln 3.$$



$$8 \quad a \quad f''(x) = 6x - 4$$

$$\therefore f'(x) = \int (6x - 4) dx = 3x^2 - 4x + c$$

$$\therefore f(x) = \int (3x^2 - 4x + c) dx = x^3 - 2x^2 + cx + d$$

$$\text{Now, } f'(1) = 3 \quad \text{and} \quad f(2) = 7$$

$$\therefore 3 = 3(1)^2 - 4(1) + c \quad \therefore 7 = 2^3 - 2(2)^2 + 4(2) + d$$

$$\therefore c = 4 \quad \therefore d = -1$$

$$\text{So, the solution is } f(x) = x^3 - 2x^2 + 4x - 1.$$



**b**  $\frac{d^2y}{dx^2} = \sin 2x$

$$\therefore \frac{dy}{dx} = \int \sin 2x \, dx = -\frac{1}{2} \cos 2x + c$$

$$\therefore y = \int \left(-\frac{1}{2} \cos 2x + c\right) dx = -\frac{1}{4} \sin 2x + cx + d$$

Now,  $y(0) = 0$

and  $y\left(\frac{\pi}{2}\right) = 2\pi$

$$\therefore 0 = -\frac{1}{4} \sin 0 + c(0) + d$$

$$\therefore 2\pi = -\frac{1}{4} \sin 2\left(\frac{\pi}{2}\right) + c\left(\frac{\pi}{2}\right)$$

$$\therefore d = 0$$

$$\therefore c = 4$$

So, the solution is  $y = -\frac{1}{4} \sin 2x + 4x$ .

- 9** The marginal cost is  $C'(x) = 3.15 + 0.004x$  pounds per gadget

$$\begin{aligned} \therefore C(x) &= \int (3.15 + 0.004x) \, dx \\ &= 3.15x + 0.002x^2 + c \text{ pounds} \end{aligned}$$

But  $C(0) = 450$  pounds, so  $c = 450$

$$\therefore C(x) = 3.15x + 0.002x^2 + 450 \text{ pounds}$$

$$\begin{aligned} \therefore C(800) &= 3.15(800) + 0.002(800)^2 + 450 \text{ pounds} \\ &= 4250 \text{ pounds} \end{aligned}$$

$\therefore$  the total cost is £4250.

- 10 a** The marginal profit is  $P'(x) = 15 - 0.03x$  pounds per plate

$$\begin{aligned} \therefore P(x) &= \int (15 - 0.03x) \, dx \\ &= 15x - 0.015x^2 + c \text{ pounds} \end{aligned}$$

But  $P(0) = -650$  pounds, so  $c = -650$

$$\therefore P(x) = 15x - 0.015x^2 - 650 \text{ pounds}$$

**b**  $P''(x) = -0.03 < 0$

$\therefore$  the maximum profit occurs when  $P'(x) = 0$

$$\therefore 0 = 15 - 0.03x$$

$$\therefore 0.03x = 15$$

$$\therefore x = 500$$

Now  $P(500) = 15(500) - 0.015(500)^2 - 650 = 3100$

$\therefore$  the maximum profit per week is £3100 when 500 plates are made.

- c** In order for a profit to be made,  $P(x)$  must be greater than 0

$$\therefore 15x - 0.015x^2 - 650 > 0$$

Using technology, the  $x$ -intercepts of  $P(x)$  are  $x_1 \approx 45.39$  and  $x_2 \approx 954.6$ .

Since we cannot produce part of a plate, a profit is made when between 46 and 954 plates inclusive are made per week.

**11** Since  $\frac{dT}{dr} = -\frac{q}{2\pi kr}$ ,  $T = -\frac{q}{2\pi k} \int \frac{1}{r} dr$   $\{q, k \text{ constant}\}$

$$\therefore T = -\frac{680}{2\pi(0.2)} \ln r + c \quad \{r > 0\}$$

$$\therefore T = -\frac{1700}{\pi} \ln r + c$$

But when  $r = 0.02$ ,  $T = 600$

$$\therefore 600 = -\frac{1700}{\pi} \ln(0.02) + c$$

$$\therefore 600 = \frac{1700}{\pi} \ln 50 + c$$

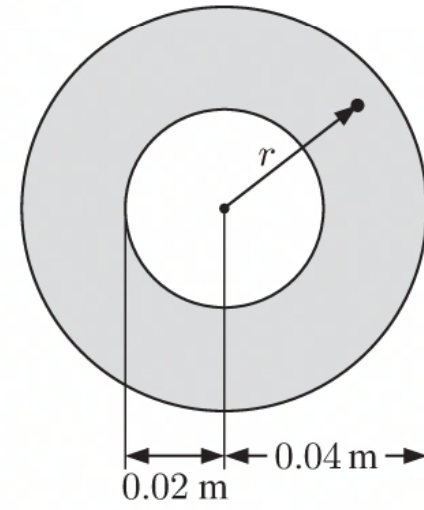
$$\therefore c = 600 - \frac{1700}{\pi} \ln 50$$

Thus  $T = -\frac{1700}{\pi} \ln r + 600 - \frac{1700}{\pi} \ln 50$

$$\therefore T = 600 - \frac{1700}{\pi} \ln 50r$$

When  $r = 0.04$ ,  $T = 600 - \frac{1700}{\pi} \ln 2 \approx 225$

The external temperature of the tube is about  $225^\circ\text{C}$ .



**12** Since  $\frac{dT}{dr} = -\frac{q}{2\pi kr}$ ,  $T(r) = -\frac{q}{2\pi k} \int \frac{1}{r} dr$   $\{q, k \text{ constant}\}$

$$\therefore T(r) = -\frac{60}{2\pi k} \ln r + c \quad \{r > 0\}$$

$$\therefore T(r) = -\frac{30}{\pi k} \ln r + c$$

**a** For the pipe,  $k = 19$ .

$$T(r_1) = T(0.14) = 400$$

$$\therefore 400 = -\frac{30}{19\pi} \ln(0.14) + c$$

$$\therefore c = 400 + \frac{30}{19\pi} \ln(0.14)$$

$$\begin{aligned} \text{Thus for the pipe, } T(r) &= -\frac{30}{19\pi} \ln r + 400 + \frac{30}{19\pi} \ln(0.14) \\ &= 400 + \frac{30}{19\pi} \ln\left(\frac{0.14}{r}\right) \end{aligned}$$

$$T(r_2) = T(0.20) = 400 + \frac{30}{19\pi} \ln(0.7) \approx 400$$

The temperature on the outer surface of the pipe is about  $400^\circ\text{C}$ .

**b** For the insulation,  $k = 0.018$ .

$$T(r_2) = T(0.20) = 400 + \frac{30}{19\pi} \ln(0.7) \quad \{\text{from a}\}$$

$$\therefore 400 + \frac{30}{19\pi} \ln(0.7) = -\frac{30}{\pi(0.018)} \ln(0.20) + c$$

$$\therefore c = 400 + \frac{30}{19\pi} \ln(0.7) + \frac{5000}{3\pi} \ln(0.20)$$

$$\text{Thus for the insulation, } T(r) = -\frac{5000}{3\pi} \ln r + 400 + \frac{30}{19\pi} \ln(0.7) + \frac{5000}{3\pi} \ln(0.20)$$

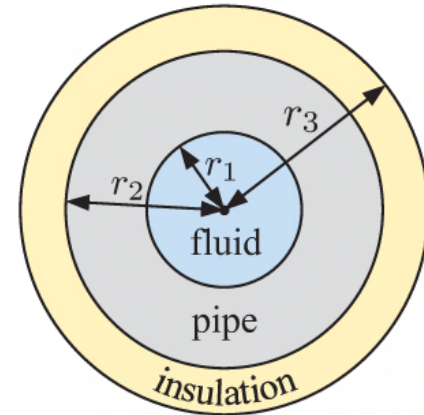
$$\therefore \frac{5000}{3\pi} \ln r = 400 + \frac{30}{19\pi} \ln(0.7) + \frac{5000}{3\pi} \ln(0.20) - T$$

$$\therefore \ln r = \frac{6\pi}{25} + \frac{9}{9500} \ln(0.7) + \ln(0.20) - \frac{3\pi}{5000} T$$

$$T(r_3) = 50 \quad \therefore \ln r_3 = \frac{6\pi}{25} + \frac{9}{9500} \ln(0.7) + \frac{3\pi}{5000} \ln(0.20) - \frac{3\pi}{100}$$

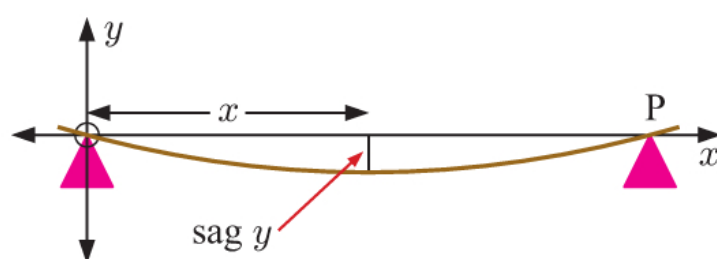
$$\therefore r_3 \approx 0.387$$

The outer radius of the insulation is  $r_3 \approx 0.387$  m.



- c Thickness of insulation needed =  $r_3 - r_2 \approx 0.387 - 0.20 \approx 0.187$  m.

13



a Since  $\frac{d^2y}{dx^2} = 0.01 \left( 2x - \frac{x^2}{2} \right)$ ,  $\frac{dy}{dx} = 0.01 \int \left( 2x - \frac{x^2}{2} \right) dx$

$$= \frac{1}{100}x^2 - \frac{1}{600}x^3 + c$$

and so,  $y(x) = \int \left( \frac{1}{100}x^2 - \frac{1}{600}x^3 + c \right) dx$

$$= \frac{1}{300}x^3 - \frac{1}{2400}x^4 + cx + d$$

But  $y(0) = 0$

$$\therefore 0 = 0 - 0 + 0 + d$$

$$\therefore d = 0$$

and  $y(4) = 0$

$$\therefore 0 = \frac{1}{300}(4)^3 - \frac{1}{2400}(4)^4 + 4c$$

$$\therefore 4c = \frac{1}{2400}(4)^4 - \frac{1}{300}(4)^3$$

$$\therefore c = \frac{1}{2400}(4)^3 - \frac{1}{300}(4)^2$$

$$\therefore c = -\frac{2}{75}$$

So, the sag  $y(x) = \left( \frac{1}{300}x^3 - \frac{1}{2400}x^4 - \frac{2}{75}x \right)$  metres.

b  $\frac{dy}{dx} = \frac{1}{100}x^2 - \frac{1}{600}x^3 - \frac{2}{75}$  {from a}

The maximum sag occurs when  $\frac{dy}{dx} = 0$

$$\therefore 0 = \frac{1}{100}x^2 - \frac{1}{600}x^3 - \frac{2}{75}$$

$$\therefore 6x^2 - x^3 - 16 = 0$$

Using technology, the three solutions are  $x \approx -1.464$ ,  $2$ , and  $\approx 5.464$ .

But the maximum lies between  $0$  and  $4$ , so it must occur when  $x = 2$ .

When  $x = 2$ ,  $y = \frac{1}{300}(2)^3 - \frac{1}{2400}(2)^4 - \frac{2}{75}(2) \approx -0.0333$

$\therefore$  the maximum sag from the horizontal is about  $0.0333$  m which is about  $3.33$  cm.

Yes, it seems reasonable that the maximum sag occurs when  $x = 2$ , as it is the middle point of the plank.

- c At the point  $1$  m from P,  $x = 3$ .

$$y(3) = \frac{1}{300}(3)^3 - \frac{1}{2400}(3)^4 - \frac{2}{75}(3) = -0.02375$$

$\therefore$  the sag  $1$  m away from P is  $0.02375$  m which is  $2.375$  cm.

d When  $x = 3$ ,  $\frac{dy}{dx} = \frac{1}{100}(3)^2 - \frac{1}{600}(3)^3 - \frac{2}{75} = \frac{11}{600}$

$\therefore$  the angle  $\theta$  that the plank makes with the horizontal satisfies  $\tan \theta = \frac{11}{600}$

$$\therefore \theta = \tan^{-1} \left( \frac{11}{600} \right) \approx 1.05^\circ$$



## EXERCISE 25D

1 a

$$\begin{aligned}\frac{dy}{dx} &= \frac{x}{y^2} \\ \therefore y^2 \frac{dy}{dx} &= x \\ \therefore \int y^2 \frac{dy}{dx} dx &= \int x dx \\ \therefore \int y^2 dy &= \int x dx \\ \therefore \frac{1}{3}y^3 &= \frac{1}{2}x^2 + c \\ \therefore y^3 &= \frac{3}{2}x^2 + c \\ \therefore y &= \sqrt[3]{\frac{3}{2}x^2 + c}\end{aligned}$$

b

$$\begin{aligned}\frac{dy}{dx} &= \frac{2x}{e^y} \\ \therefore e^y \frac{dy}{dx} &= 2x \\ \therefore \int e^y \frac{dy}{dx} dx &= \int 2x dx \\ \therefore \int e^y dy &= \int 2x dx \\ \therefore e^y &= x^2 + c \\ \therefore y &= \ln(x^2 + c)\end{aligned}$$

c

$$\begin{aligned}\frac{dy}{dx} &= 3xy \\ \therefore \frac{1}{y} \frac{dy}{dx} &= 3x \\ \therefore \int \frac{1}{y} \frac{dy}{dx} dx &= \int 3x dx \\ \therefore \int \frac{1}{y} dy &= \int 3x dx \\ \therefore \ln|y| &= \frac{3}{2}x^2 + c \\ \therefore y &= \pm e^{\frac{3}{2}x^2 + c} \\ \therefore y &= Ae^{\frac{3}{2}x^2} \quad \{A = \pm e^c\}\end{aligned}$$

d

$$\begin{aligned}\frac{dy}{dx} &= 2x\sqrt{y} \\ \therefore \frac{1}{\sqrt{y}} \frac{dy}{dx} &= 2x \\ \therefore \int \frac{1}{\sqrt{y}} \frac{dy}{dx} dx &= \int 2x dx \\ \therefore \int y^{-\frac{1}{2}} dy &= \int 2x dx \\ \therefore 2y^{\frac{1}{2}} &= x^2 + c \\ \therefore \sqrt{y} &= \frac{1}{2}x^2 + c \\ \therefore y &= \left(\frac{x^2}{2} + c\right)^2\end{aligned}$$

e

$$\begin{aligned}\frac{dy}{dx} &= y \sin x \\ \therefore \frac{1}{y} \frac{dy}{dx} &= \sin x \\ \therefore \int \frac{1}{y} \frac{dy}{dx} dx &= \int \sin x dx \\ \therefore \int \frac{1}{y} dy &= \int \sin x dx \\ \therefore \ln|y| &= -\cos x + c \\ \therefore y &= \pm e^{-\cos x + c} \\ \therefore y &= Ae^{-\cos x} \quad \{A = \pm e^c\}\end{aligned}$$

f

$$\begin{aligned}\frac{dy}{dx} &= -x\sqrt{y+1} \\ \therefore \frac{1}{\sqrt{y+1}} \frac{dy}{dx} &= -x \\ \therefore \int \frac{1}{\sqrt{y+1}} \frac{dy}{dx} dx &= \int -x dx \\ \therefore \int (y+1)^{-\frac{1}{2}} dy &= \int -x dx \\ \therefore 2(y+1)^{\frac{1}{2}} &= -\frac{1}{2}x^2 + c \\ \therefore \sqrt{y+1} &= -\frac{1}{4}x^2 + c \\ \therefore y+1 &= \left(-\frac{1}{4}x^2 + c\right)^2 \\ \therefore y &= \left(-\frac{1}{4}x^2 + c\right)^2 - 1\end{aligned}$$

g

$$\begin{aligned}\frac{dy}{dx} &= \frac{y}{x} \\ \therefore \frac{1}{y} \frac{dy}{dx} &= \frac{1}{x} \\ \therefore \int \frac{1}{y} \frac{dy}{dx} dx &= \int \frac{1}{x} dx \\ \therefore \int \frac{1}{y} dy &= \int \frac{1}{x} dx \\ \therefore \ln|y| &= \ln|x| + c \\ \therefore \ln|y| - \ln|x| &= c \\ \therefore \ln\left|\frac{y}{x}\right| &= c \\ \therefore \frac{y}{x} &= \pm e^c \\ \therefore y &= Ax \quad \{A = \pm e^c\}\end{aligned}$$



$$\begin{aligned}
 \mathbf{h} \quad & \frac{dy}{dx} = 3x^2 e^y \\
 & \therefore e^{-y} \frac{dy}{dx} = 3x^2 \\
 & \therefore \int e^{-y} \frac{dy}{dx} dx = \int 3x^2 dx \\
 & \therefore \int e^{-y} dy = \int 3x^2 dx \\
 & \therefore -e^{-y} = x^3 + c \\
 & \therefore e^{-y} = -x^3 + c \\
 & \therefore -y = \ln(c - x^3) \\
 & \therefore y = -\ln(c - x^3)
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{i} \quad & \frac{dy}{dx} = \frac{y+2}{x-1} \\
 & \therefore \frac{1}{y+2} \frac{dy}{dx} = \frac{1}{x-1} \\
 & \therefore \int \frac{1}{y+2} \frac{dy}{dx} dx = \int \frac{1}{x-1} dx \\
 & \therefore \int \frac{1}{y+2} dy = \int \frac{1}{x-1} dx \\
 & \therefore \ln|y+2| = \ln|x-1| + c \\
 & \therefore \ln|y+2| - \ln|x-1| = c \\
 & \therefore \ln\left|\frac{y+2}{x-1}\right| = c \\
 & \therefore \frac{y+2}{x-1} = \pm e^c \\
 & \therefore y+2 = A(x-1) \\
 & \qquad \qquad \qquad \{A = \pm e^c\} \\
 & \therefore y = A(x-1) - 2
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{2} \quad \mathbf{a} \quad & \frac{dy}{dx} = y \\
 & \therefore \frac{1}{y} \frac{dy}{dx} = 1 \\
 & \therefore \int \frac{1}{y} \frac{dy}{dx} dx = \int 1 dx \\
 & \therefore \int \frac{1}{y} dy = \int 1 dx \\
 & \therefore \ln|y| = x + c \\
 & \therefore y = \pm e^{x+c} \\
 & \therefore y = Ae^x \quad \{A = \pm e^c\}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \frac{dy}{dx} = \frac{1}{y} \\
 & \therefore y \frac{dy}{dx} = 1 \\
 & \therefore \int y \frac{dy}{dx} dx = \int 1 dx \\
 & \therefore \int y dy = \int 1 dx \\
 & \therefore \frac{1}{2}y^2 = x + c \\
 & \therefore y^2 = 2x + c \\
 & \therefore y = \pm\sqrt{2x+c}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & \frac{dy}{dt} = y - 4 \\
 & \therefore \frac{1}{y-4} \frac{dy}{dt} = 1 \\
 & \therefore \int \frac{1}{y-4} \frac{dy}{dt} dt = \int 1 dt \\
 & \therefore \int \frac{1}{y-4} dy = \int 1 dt \\
 & \therefore \ln|y-4| = t + c \\
 & \therefore y-4 = \pm e^{t+c} \\
 & \therefore y = Ae^t + 4 \quad \{A = \pm e^c\}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad & \frac{dP}{dt} = 3\sqrt{P} \\
 & \therefore \frac{1}{\sqrt{P}} \frac{dP}{dt} = 3 \\
 & \therefore \int \frac{1}{\sqrt{P}} \frac{dP}{dt} dt = \int 3 dt \\
 & \therefore \int P^{-\frac{1}{2}} dP = \int 3 dt \\
 & \therefore 2P^{\frac{1}{2}} = 3t + c \\
 & \therefore \sqrt{P} = \frac{3}{2}t + c \\
 & \therefore P = \left(\frac{3}{2}t + c\right)^2
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & \frac{dQ}{dt} = 2Q + 3 \\
 & \therefore \frac{1}{2Q+3} \frac{dQ}{dt} = 1 \\
 & \therefore \int \frac{1}{2Q+3} \frac{dQ}{dt} dt = \int 1 dt \\
 & \therefore \int \frac{1}{2Q+3} dQ = \int 1 dt \\
 & \therefore \frac{1}{2} \ln |2Q+3| = t + c \\
 & \therefore \ln |2Q+3| = 2t + c \\
 & \therefore 2Q+3 = \pm e^{2t+c} \\
 & \therefore 2Q = Ae^{2t} - 3 \quad \{A = \pm e^c\} \\
 & \therefore Q = Ae^{2t} - \frac{3}{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & \frac{dQ}{dt} = \frac{1}{2Q+3} \\
 & \therefore (2Q+3) \frac{dQ}{dt} = 1 \\
 & \therefore \int (2Q+3) \frac{dQ}{dt} dt = \int 1 dt \\
 & \therefore \int (2Q+3) dQ = \int 1 dt \\
 & \therefore Q^2 + 3Q = t + c \\
 & \therefore Q^2 + 3Q + \left(\frac{3}{2}\right)^2 = t + c \\
 & \therefore \left(Q + \frac{3}{2}\right)^2 = t + c \\
 & \therefore Q + \frac{3}{2} = \pm \sqrt{t+c} \\
 & \therefore Q = -\frac{3}{2} \pm \sqrt{t+c}
 \end{aligned}$$

$$\begin{aligned}
 \text{3 a} \quad & \frac{dy}{dx} = \frac{y}{3x+1} \\
 & \therefore \frac{1}{y} \frac{dy}{dx} = \frac{1}{3x+1} \\
 & \therefore \int \frac{1}{y} \frac{dy}{dx} dx = \int \frac{1}{3x+1} dx \\
 & \therefore \int \frac{1}{y} dy = \int \frac{1}{3x+1} dx \\
 & \therefore \ln |y| = \frac{1}{3} \ln |3x+1| + c \\
 & \therefore \ln |y| = \ln \left| \sqrt[3]{3x+1} \right| + c \\
 & \therefore \ln |y| - \ln \left| \sqrt[3]{3x+1} \right| = c \\
 & \therefore \ln \left| \frac{y}{\sqrt[3]{3x+1}} \right| = c \\
 & \therefore \frac{y}{\sqrt[3]{3x+1}} = \pm e^c \\
 & \therefore y = A \sqrt[3]{3x+1} \quad \{A = \pm e^c\}
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & 4 + \frac{dy}{dx} = 2y \\
 & \therefore \frac{dy}{dx} = 2y - 4 \\
 & \therefore \frac{1}{y-2} \frac{dy}{dx} = 2 \\
 & \therefore \int \frac{1}{y-2} \frac{dy}{dx} dx = \int 2 dx \\
 & \therefore \int \frac{1}{y-2} dy = \int 2 dx \\
 & \therefore \ln |y-2| = 2x + c \\
 & \therefore y-2 = \pm e^{2x+c} \\
 & \therefore y = Ae^{2x} + 2 \quad \{A = \pm e^c\}
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & (x^2 + 5) \frac{dy}{dx} = \frac{2x}{y^2} \\
 & \therefore y^2 \frac{dy}{dx} = \frac{2x}{x^2 + 5} \\
 & \therefore \int y^2 \frac{dy}{dx} dx = \int \frac{2x}{x^2 + 5} dx \\
 & \therefore \int y^2 dy = \int \frac{2x}{x^2 + 5} dx \\
 & \therefore \frac{1}{3}y^3 = \ln|x^2 + 5| + c \\
 & \therefore y^3 = 3\ln(x^2 + 5) + c \\
 & \quad \{x^2 + 5 > 0 \text{ for all } x\} \\
 & \therefore y = \sqrt[3]{3\ln(x^2 + 5) + c}
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad & \sqrt{4-x} \frac{dy}{dx} = 1-y \\
 & \therefore \frac{1}{1-y} \frac{dy}{dx} = \frac{1}{\sqrt{4-x}} \\
 & \therefore \int \frac{1}{1-y} \frac{dy}{dx} dx = \int \frac{1}{\sqrt{4-x}} dx \\
 & \therefore \int \frac{1}{1-y} dy = \int (4-x)^{-\frac{1}{2}} dx \\
 & \therefore -\ln|1-y| = -2(4-x)^{\frac{1}{2}} + c \\
 & \therefore \ln|1-y| = 2\sqrt{4-x} + c \\
 & \therefore 1-y = \pm e^{2\sqrt{4-x}+c} \\
 & \therefore y = 1 \pm e^{2\sqrt{4-x}+c} \\
 & \therefore y = 1 + Ae^{2\sqrt{4-x}} \\
 & \quad \{A = \pm e^c\}
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & \frac{dy}{dx} = xy^2 - 2y^2 \\
 & \therefore \frac{dy}{dx} = y^2(x-2) \\
 & \therefore \frac{1}{y^2} \frac{dy}{dx} = x-2 \\
 & \therefore \int \frac{1}{y^2} \frac{dy}{dx} dx = \int (x-2) dx \\
 & \therefore \int y^{-2} dy = \int (x-2) dx \\
 & \therefore -y^{-1} = \frac{1}{2}x^2 - 2x + c \\
 & \therefore y = \frac{1}{-\frac{1}{2}x^2 + 2x + c}
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & y \frac{dy}{dx} = \frac{6x\sqrt{y}}{x^2+5} \\
 & \therefore \frac{y}{\sqrt{y}} \frac{dy}{dx} = \frac{6x}{x^2+5} \\
 & \therefore \int \sqrt{y} \frac{dy}{dx} dx = \int \frac{6x}{x^2+5} dx \\
 & \therefore \int y^{\frac{1}{2}} dy = 3 \int \frac{2x}{x^2+5} dx \\
 & \therefore \frac{2}{3}y^{\frac{3}{2}} = 3\ln|x^2+5| + c \\
 & \therefore y^{\frac{3}{2}} = \frac{9}{2}\ln(x^2+5) + c \\
 & \quad \{x^2+5 > 0 \text{ for all } x\} \\
 & \therefore y = \left(\frac{9}{2}\ln(x^2+5) + c\right)^{\frac{2}{3}}
 \end{aligned}$$

$$\begin{aligned}
 \text{4} \quad & \frac{dy}{dx} = y^2 \\
 & \therefore \frac{1}{y^2} \frac{dy}{dx} = 1 \\
 & \therefore \int \frac{1}{y^2} \frac{dy}{dx} dx = \int 1 dx \\
 & \therefore \int y^{-2} dy = \int 1 dx \\
 & \therefore -y^{-1} = x + c \\
 & \therefore y = \frac{1}{c-x} \quad \text{which is defined for } x \neq c.
 \end{aligned}$$

**5 a**

$$\begin{aligned}\frac{dy}{dx} &= \frac{3x}{y^2} \\ \therefore y^2 \frac{dy}{dx} &= 3x \\ \therefore \int y^2 \frac{dy}{dx} dx &= \int 3x dx \\ \therefore \int y^2 dy &= \int 3x dx \\ \therefore \frac{1}{3}y^3 &= \frac{3}{2}x^2 + c \\ \therefore y^3 &= \frac{9}{2}x^2 + c \\ \therefore y &= \sqrt[3]{\frac{9}{2}x^2 + c}\end{aligned}$$

But  $y(0) = 1$ , so  $1 = \sqrt[3]{c}$   
 $\therefore c = 1$

The particular solution is

$$y = \sqrt[3]{\frac{9}{2}x^2 + 1}.$$

**b**

$$\begin{aligned}\frac{dy}{dx} &= \frac{\sqrt{y}}{3} \\ \therefore \frac{1}{\sqrt{y}} \frac{dy}{dx} &= \frac{1}{3} \\ \therefore \int \frac{1}{\sqrt{y}} \frac{dy}{dx} dx &= \int \frac{1}{3} dx \\ \therefore \int y^{-\frac{1}{2}} dy &= \int \frac{1}{3} dx \\ \therefore 2y^{\frac{1}{2}} &= \frac{1}{3}x + c \\ \therefore y^{\frac{1}{2}} &= \frac{1}{6}x + c \\ \therefore \sqrt{y} &= \frac{1}{6}x + c\end{aligned}$$

But  $y(44) = 9$ , so  $\sqrt{9} = \frac{44}{6} + c$   
 $\therefore 3 = \frac{22}{3} + c$   
 $\therefore c = -\frac{13}{3}$

So,  $\sqrt{y} = \frac{1}{6}x - \frac{13}{3}$

The particular solution is

$$\begin{aligned}y &= \left(\frac{1}{6}x - \frac{13}{3}\right)^2 \\ &= \left(\frac{1}{6}x - \frac{26}{6}\right)^2 \\ &= \frac{1}{36}(x - 26)^2\end{aligned}$$

**c**

$$\begin{aligned}\frac{dy}{dx} &= y + yx^2 = y(1 + x^2) \\ \therefore \frac{1}{y} \frac{dy}{dx} &= 1 + x^2 \\ \therefore \int \frac{1}{y} \frac{dy}{dx} dx &= \int (1 + x^2) dx \\ \therefore \int \frac{1}{y} dy &= \int (1 + x^2) dx \\ \therefore \ln|y| &= x + \frac{1}{3}x^3 + c \\ \therefore y &= \pm e^{x + \frac{1}{3}x^3 + c} \\ \therefore y &= Ae^{x + \frac{1}{3}x^3} \quad \{A = \pm e^c\}\end{aligned}$$

But  $y(0) = 1$ , so  $1 = Ae^0$   
 $\therefore A = 1$

The particular solution is  $y = e^{x + \frac{1}{3}x^3}$ .**d**

$$\begin{aligned}\frac{dy}{dx} &= \frac{3x}{\cos y} \\ \therefore \cos y \frac{dy}{dx} &= 3x \\ \therefore \int \cos y \frac{dy}{dx} dx &= \int 3x dx \\ \therefore \int \cos y dy &= \int 3x dx \\ \therefore \sin y &= \frac{3}{2}x^2 + c\end{aligned}$$

But  $y(1) = 0$ , so  $\sin 0 = \frac{3}{2} + c$   
 $\therefore 0 = \frac{3}{2} + c$   
 $\therefore c = -\frac{3}{2}$

So,  $\sin y = \frac{3}{2}x^2 - \frac{3}{2}$

The particular solution is

$$y = \arcsin\left(\frac{3}{2}x^2 - \frac{3}{2}\right).$$



$$\begin{aligned}
 \text{e} \quad & \frac{dy}{dx} = \frac{6 \cos 2x}{\sqrt{y}} \\
 & \therefore \sqrt{y} \frac{dy}{dx} = 6 \cos 2x \\
 & \therefore \int \sqrt{y} \frac{dy}{dx} dx = \int 6 \cos 2x dx \\
 & \therefore \int y^{\frac{1}{2}} dy = \int 6 \cos 2x dx \\
 & \therefore \frac{2}{3} y^{\frac{3}{2}} = 3 \sin 2x + c \\
 & \therefore y^{\frac{3}{2}} = \frac{9}{2} \sin 2x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{But } y(0) = 3, \text{ so } 3^{\frac{3}{2}} &= c \\
 \therefore c &= 3\sqrt{3}
 \end{aligned}$$

$$\text{So, } y^{\frac{3}{2}} = \frac{9}{2} \sin 2x + 3\sqrt{3}$$

$$\text{The particular solution is } y = \left( \frac{9}{2} \sin 2x + 3\sqrt{3} \right)^{\frac{2}{3}}.$$

$$\begin{aligned}
 \text{f} \quad & e^y(2x^2 + 4x + 1) \frac{dy}{dx} = (x + 1)(e^y + 3) \\
 & \therefore \frac{e^y}{e^y + 3} \frac{dy}{dx} = \frac{x + 1}{2x^2 + 4x + 1} \\
 & \therefore \int \frac{e^y}{e^y + 3} \frac{dy}{dx} dx = \int \frac{x + 1}{2x^2 + 4x + 1} dx \\
 & \therefore \int \frac{e^y}{e^y + 3} dy = \frac{1}{4} \int \frac{4x + 4}{2x^2 + 4x + 1} dx \\
 & \therefore \ln |e^y + 3| = \frac{1}{4} \ln |2x^2 + 4x + 1| + c \\
 & \therefore \ln(e^y + 3) = \frac{1}{4} \ln(2x^2 + 4x + 1) + c \quad \begin{array}{l} \{e^y + 3 > 0 \text{ for all } y, \\ 2x^2 + 4x + 1 > 0 \text{ for all } x\} \end{array} \\
 & \therefore e^y + 3 = e^c (2x^2 + 4x + 1)^{\frac{1}{4}} \\
 & \therefore e^y + 3 = A \sqrt[4]{2x^2 + 4x + 1} \quad \{A = e^c\} \\
 & \therefore e^y = A \sqrt[4]{2x^2 + 4x + 1} - 3 \\
 & \therefore y = \ln \left[ A \sqrt[4]{2x^2 + 4x + 1} - 3 \right]
 \end{aligned}$$

$$\text{But } y(0) = 2, \text{ so } 2 = \ln(A - 3)$$

$$\therefore e^2 = A - 3$$

$$\therefore A = e^2 + 3$$

$$\text{The particular solution is } y = \ln \left[ \sqrt[4]{2x^2 + 4x + 1} (e^2 + 3) - 3 \right].$$

$$\begin{aligned}
 6 \quad & \frac{dy}{dx} = 2\sqrt{y} \\
 & \therefore \frac{1}{\sqrt{y}} \frac{dy}{dx} = 2 \\
 & \therefore \int \frac{1}{\sqrt{y}} \frac{dy}{dx} dx = \int 2 dx \\
 & \therefore \int y^{-\frac{1}{2}} dy = \int 2 dx \\
 & \therefore 2y^{\frac{1}{2}} = 2x + c \\
 & \therefore y^{\frac{1}{2}} = x + c \\
 & \therefore \sqrt{y} = x + c
 \end{aligned}$$

But  $y(0) = 9$ , so  $\sqrt{9} = c$   
 $\therefore c = 3$

So,  $\sqrt{y} = x + 3$  which is defined for  $x \geq -3$ .

The particular solution is  $y = (x + 3)^2$ , defined for  $x \geq -3$ .

$$\begin{aligned}
 7 \quad a \quad & \frac{1}{x-1} - \frac{2}{x+1} = \frac{x+1-2(x-1)}{(x-1)(x+1)} \\
 & = \frac{x+1-2x+2}{x^2-1} \\
 & = \frac{3-x}{x^2-1} \quad \text{as required.}
 \end{aligned}$$

$$\begin{aligned}
 b \quad & \frac{dy}{dx} = \frac{3y-xy}{x^2-1} = \frac{y(3-x)}{x^2-1} \\
 & \therefore \frac{1}{y} \frac{dy}{dx} = \frac{3-x}{x^2-1} \\
 & \therefore \int \frac{1}{y} \frac{dy}{dx} dx = \int \frac{3-x}{x^2-1} dx \\
 & \therefore \int \frac{1}{y} dy = \int \left( \frac{1}{x-1} - \frac{2}{x+1} \right) dx \quad \{\text{using } a\} \\
 & \therefore \ln|y| = \ln|x-1| - 2\ln|x+1| + c \\
 & \therefore \ln|y| - \ln|x-1| + 2\ln|x+1| = c \\
 & \therefore \ln \left| \frac{y(x+1)^2}{x-1} \right| = c \\
 & \therefore \frac{y(x+1)^2}{x-1} = \pm e^c \\
 & \therefore y = \frac{A(x-1)}{(x+1)^2} \quad \{A = \pm e^c\}
 \end{aligned}$$

But  $y(0) = 1$ , so  $1 = \frac{A(-1)}{(1)^2}$

$$\therefore A = -1$$

The particular solution is  $y = \frac{-(x-1)}{(x+1)^2} = \frac{1-x}{(x+1)^2}$ .

**8 a**  $x^2 - x - 12 = (x - 4)(x + 3)$

Let  $\frac{8x + 3}{x^2 - x - 12} = \frac{A}{x - 4} + \frac{B}{x + 3}$

$$\therefore 8x + 3 = A(x + 3) + B(x - 3)$$

Substituting  $x = -3$ ,  $-21 = -7B$

$$\therefore B = 3$$

Substituting  $x = 4$ ,  $35 = 7A$

$$\therefore A = 5$$

$$\therefore \frac{8x + 3}{x^2 - x - 12} = \frac{5}{x - 4} + \frac{3}{x + 3}$$

**b**  $\frac{dy}{dx} = \frac{8x\sqrt{y} + 3\sqrt{y}}{x^2 - x - 12} = \frac{\sqrt{y}(8x + 3)}{x^2 - x - 12}$

$$\therefore \frac{1}{\sqrt{y}} \frac{dy}{dx} = \frac{8x + 3}{x^2 - x - 12}$$

$$\therefore \int \frac{1}{\sqrt{y}} \frac{dy}{dx} dx = \int \frac{8x + 3}{x^2 - x - 12} dx$$

$$\therefore \int y^{-\frac{1}{2}} dy = \int \left( \frac{5}{x - 4} + \frac{3}{x + 3} \right) dx \quad \{\text{using a}\}$$

$$\therefore 2y^{\frac{1}{2}} = 5 \ln |x - 4| + 3 \ln |x + 3| + c$$

$$\therefore y^{\frac{1}{2}} = \frac{1}{2}(5 \ln |x - 4| + 3 \ln |x + 3| + c)$$

$$\therefore y = \frac{1}{4}(5 \ln |x - 4| + 3 \ln |x + 3| + c)^2$$

**9**  $x^2 + x - 2 = (x + 2)(x - 1)$

Let  $\frac{5x + 4}{x^2 + x - 2} = \frac{A}{x + 2} + \frac{B}{x - 1}$

$$\therefore 5x + 4 = A(x - 1) + B(x + 2)$$

Substituting  $x = 1$ ,  $9 = 3B$

$$\therefore B = 3$$

Substituting  $x = -2$ ,  $-6 = -3A$

$$\therefore A = 2$$

$$\therefore \frac{5x + 4}{x^2 + x - 2} = \frac{2}{x + 2} + \frac{3}{x - 1}$$

Now,  $\frac{dy}{dx} = \frac{5xy^2 + 4y^2}{x^2 + x - 2} = \frac{y^2(5x + 4)}{x^2 + x - 2}$

$$\therefore \frac{1}{y^2} \frac{dy}{dx} = \frac{5x + 4}{x^2 + x - 2}$$

$$\therefore \int \frac{1}{y^2} \frac{dy}{dx} dx = \int \frac{5x + 4}{x^2 + x - 2} dx$$

$$\therefore \int y^{-2} dy = \int \left( \frac{2}{x + 2} + \frac{3}{x - 1} \right) dx$$

$$\therefore -y^{-1} = 2 \ln |x + 2| + 3 \ln |x - 1| + c$$

$$\therefore y = \frac{1}{c - 2 \ln |x + 2| - 3 \ln |x - 1|}$$

$$\begin{aligned}\text{But } y(0) &= -\frac{1}{2}, \text{ so } -\frac{1}{2} = \frac{1}{c - 2 \ln 2} \\ \therefore c - 2 \ln 2 &= -2 \\ \therefore c &= 2 \ln 2 - 2\end{aligned}$$

$$\begin{aligned}\text{The particular solution is } y &= \frac{1}{2 \ln 2 - 2 - 2 \ln |x + 2| - 3 \ln |x - 1|} \\ &= \frac{1}{2 \ln \left| \frac{2}{x+2} \right| - 3 \ln |x - 1| - 2} \\ &= \frac{1}{\ln \left| \frac{4}{(x+2)^2(x-1)^3} \right| - 2}\end{aligned}$$

$$\begin{aligned}10 \quad \frac{dy}{dx} &= \frac{x^2 y + y}{x^2 - 1} = y \left( \frac{x^2 + 1}{x^2 - 1} \right) \\ \therefore \frac{1}{y} \frac{dy}{dx} &= \frac{(x^2 - 1) + 2}{x^2 - 1} = 1 + \frac{2}{x^2 - 1}\end{aligned}$$

$$\text{Now } x^2 - 1 = (x - 1)(x + 1)$$

$$\begin{aligned}\text{Let } \frac{2}{x^2 - 1} &= \frac{A}{x - 1} + \frac{B}{x + 1} \\ \therefore 2 &= A(x + 1) + B(x - 1)\end{aligned}$$

$$\begin{aligned}\text{Substituting } x &= -1, \quad 2 = -2B \\ \therefore B &= -1\end{aligned}$$

$$\begin{aligned}\text{Substituting } x &= 1, \quad 2 = 2A \\ \therefore A &= 1\end{aligned}$$

$$\therefore \frac{2}{x^2 - 1} = \frac{1}{x - 1} - \frac{1}{x + 1}$$

$$\text{So, } \frac{1}{y} \frac{dy}{dx} = 1 + \frac{1}{x - 1} - \frac{1}{x + 1}$$

$$\therefore \int \frac{1}{y} \frac{dy}{dx} dx = \int \left( 1 + \frac{1}{x - 1} - \frac{1}{x + 1} \right) dx$$

$$\therefore \int \frac{1}{y} dy = \int \left( 1 + \frac{1}{x - 1} - \frac{1}{x + 1} \right) dx$$

$$\therefore \ln |y| = x + \ln |x - 1| - \ln |x + 1| + c$$

$$\therefore \ln |y| - \ln |x - 1| + \ln |x + 1| = x + c$$

$$\therefore \ln \left| \frac{y(x + 1)}{(x - 1)} \right| = x + c$$

$$\therefore y \left( \frac{x + 1}{x - 1} \right) = \pm e^{x+c}$$

$$\therefore y = A e^x \left( \frac{x - 1}{x + 1} \right) \quad \{A = \pm e^c\}$$



$$\begin{aligned}
 11 \quad a \quad & \frac{dP}{dt} = \frac{1}{2}P \\
 & \therefore \frac{1}{P} \frac{dP}{dt} = \frac{1}{2} \\
 & \therefore \int \frac{1}{P} \frac{dP}{dt} dt = \int \frac{1}{2} dt \\
 & \therefore \int \frac{1}{P} dP = \int \frac{1}{2} dt \\
 & \therefore \ln|P| = \frac{1}{2}t + c \\
 & \therefore P = e^{\frac{t}{2}+c} \quad \{\text{since } P \geq 0\} \\
 & \therefore P = Ae^{\frac{t}{2}} \quad \{A = e^c\}
 \end{aligned}$$

$$\begin{aligned}
 \text{When } t = 0, \quad P &= 40 \\
 \therefore 40 &= Ae^0 \quad \text{and so } A = 40 \\
 \therefore P &= 40e^{\frac{t}{2}}
 \end{aligned}$$

$$\begin{aligned}
 b \quad \text{When } t = 6, \quad P &= 40e^3 \\
 \therefore P &\approx 803
 \end{aligned}$$

After 6 months, there were approximately 803 rabbits.

$$\begin{aligned}
 12 \quad a \quad & \frac{dI}{dt} = -0.4I \\
 & \therefore \frac{1}{I} \frac{dI}{dt} = -0.4 \\
 & \therefore \int \frac{1}{I} \frac{dI}{dt} dt = \int -0.4 dt \\
 & \therefore \int \frac{1}{I} dI = \int -0.4 dt \\
 & \therefore \ln|I| = -0.4t + c \\
 & \therefore I = \pm e^{-0.4t+c} \\
 & \therefore I = Ae^{-0.4t} \quad \{A = \pm e^c\}
 \end{aligned}$$

$$\begin{aligned}
 \text{When } t = 0, \quad I &= 350 \\
 \therefore 350 &= Ae^0 \quad \text{and so } A = 350 \\
 \therefore I &= 350e^{-0.4t}
 \end{aligned}$$

$$\begin{aligned}
 b \quad \text{When } t = 5, \quad I &= 350e^{-2} \\
 \therefore I &\approx 47.4
 \end{aligned}$$

After 5 milliseconds, the current is approximately 47.4 milliamps.

$$\begin{aligned}
 c \quad \text{When } I = 20, \text{ we have } 20 &= 350e^{-0.4t} \\
 \therefore e^{-0.4t} &= \frac{2}{35} \\
 \therefore -0.4t &= \ln\left(\frac{2}{35}\right) \\
 \therefore t &= -\frac{5}{2} \ln\left(\frac{2}{35}\right) \\
 \therefore t &\approx 7.16
 \end{aligned}$$

It takes approximately 7.16 milliseconds for the current to fall to 20 milliamps.

**13**

$$\begin{aligned}
& \frac{dC_A}{dt} = -kC_A \\
& \therefore \frac{1}{C_A} \frac{dC_A}{dt} = -k \\
& \therefore \int \frac{1}{C_A} \frac{dC_A}{dt} dt = - \int k dt \\
& \therefore \int \frac{1}{C_A} dC_A = - \int k dt \\
& \therefore \ln |C_A| = -kt + c \\
& \therefore C_A = e^{-kt+c} \quad \{\text{since } C_A \geq 0\} \\
& \therefore C_A = Ae^{-0.31t} \quad \{A = e^c\}
\end{aligned}$$

When  $t = 0$ ,  $C_A = 1$ 

$$\begin{aligned}
& \therefore 1 = Ae^0 \quad \text{and so } A = 1 \\
& \therefore C_A = e^{-0.31t}
\end{aligned}$$

When  $C_A = 1 - 0.8 = 0.2$ ,  $0.2 = e^{-0.31t}$ 

$$\begin{aligned}
& \therefore \ln(0.2) = -0.31t \\
& \therefore t = -\frac{1}{0.31} \ln(0.2) \\
& \therefore t \approx 5.19
\end{aligned}$$

It will take approximately 5.19 minutes for 80% of the ethylene oxide to be used up.

**14** The rate of change in the weight  $w$  of raw sugar is directly proportional to weight  $w$ .

$$\begin{aligned}
& \text{So, } \frac{dw}{dt} = kw \quad \text{for some constant } k \neq 0 \\
& \therefore \frac{1}{w} \frac{dw}{dt} = k \\
& \therefore \int \frac{1}{w} \frac{dw}{dt} dt = \int k dt \\
& \therefore \int \frac{1}{w} dw = \int k dt \\
& \therefore \ln |w| = kt + c \\
& \therefore w = e^{kt+c} \quad \{\text{since } w \geq 0\} \\
& \therefore w = Ae^{kt} \quad \{A = e^c\}
\end{aligned}$$

Let  $w_0$  be the initial weight of the sugar.When  $t = 0$ ,  $w = w_0$ 

$$\begin{aligned}
& \therefore w_0 = Ae^0 \quad \text{and so } A = w_0 \\
& \therefore w = w_0 e^{kt}
\end{aligned}$$

When  $t = 10$ ,  $w = (1 - 0.8)w_0$ 

$$\begin{aligned}
& \therefore 0.2w_0 = w_0 e^{10k} \\
& \therefore 0.2 = e^{10k} \\
& \therefore e^k = (0.2)^{\frac{1}{10}} \\
& \therefore w = w_0 (0.2)^{\frac{t}{10}}
\end{aligned}$$

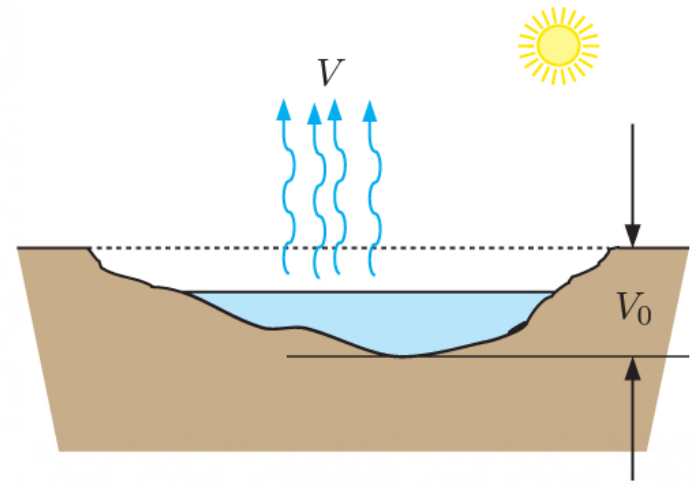
$$\begin{aligned}
& \text{When } t = 30, \quad w = w_0 (0.2)^{\frac{30}{10}} \\
& \quad = w_0 (0.2)^3 \\
& \quad = 0.008w_0
\end{aligned}$$

 $\therefore$  0.8% of raw sugar remains after 30 hours.

- 15 a** The amount of water remaining in the lake is  $(V_0 - V)$ .

The water evaporates at a rate proportional to the volume of water remaining.

$$\therefore \frac{dV}{dt} = k(V_0 - V) \text{ for some constant } k.$$



**b**

$$\frac{dV}{dt} = k(V_0 - V)$$

$$\therefore \frac{1}{V_0 - V} \frac{dV}{dt} = k$$

$$\therefore \int \frac{1}{V_0 - V} \frac{dV}{dt} dt = \int k dt$$

$$\therefore \int \frac{1}{V_0 - V} dV = \int k dt$$

$$\therefore -\ln |V_0 - V| = kt + c$$

$$\therefore \ln |V_0 - V| = -kt + c$$

$$\therefore V_0 - V = e^{-kt+c} \quad \{\text{since } V_0 - V \geq 0\}$$

$$\therefore V = V_0 - Ae^{-kt} \quad \{A = e^c\}$$

When  $t = 0$ ,  $V = 0$

$$\therefore 0 = V_0 - Ae^0 \text{ and so } A = V_0$$

$$\therefore V = V_0 - V_0e^{-kt}$$

When  $t = 20$ ,  $V = \frac{1}{2}V_0$

$$\therefore \frac{1}{2}V_0 = V_0 - V_0e^{-20k}$$

$$\therefore \frac{1}{2} = 1 - e^{-20k}$$

$$\therefore e^{-20k} = \frac{1}{2}$$

$$\therefore e^{-k} = \left(\frac{1}{2}\right)^{\frac{1}{20}}$$

$$\therefore e^{-kt} = \left(\frac{1}{2}\right)^{\frac{t}{20}}$$

$$\therefore V = V_0 - V_0\left(\frac{1}{2}\right)^{\frac{t}{20}}$$

When  $t = 50$ ,  $V = V_0 - V_0\left(\frac{1}{2}\right)^{\frac{50}{20}}$

$$= V_0 - 2^{-\frac{5}{2}}V_0$$

$$= V_0(1 - 2^{-\frac{5}{2}}) \text{ is the amount of water that has evaporated.}$$

$$\therefore \text{the amount of water remaining} = V_0 - V_0(1 - 2^{-\frac{5}{2}})$$

$$= 2^{-\frac{5}{2}}V_0$$

$$\approx 0.177V_0$$

$\therefore$  approximately 17.7% of the original water remains after 50 days without rain.

**16**

$$\frac{dT}{dt} \propto (T - T_m)$$

$$\therefore \frac{dT}{dt} = k(T - T_m) \quad \text{for some constant } k.$$

$$\therefore \frac{1}{T - T_m} \frac{dT}{dt} = k$$

$$\therefore \int \frac{1}{T - T_m} \frac{dT}{dt} dt = \int k dt$$

$$\therefore \int \frac{1}{T - T_m} dT = \int k dt$$

$$\therefore \ln |T - T_m| = kt + c$$

$$\therefore T - T_m = \pm e^{kt+c}$$

$$\therefore T = T_m + Ae^{kt} \quad \{A = \pm e^c\}$$

**a**  $T_m = 5$ When  $t = 0$ ,  $T = 100$ 

$$\therefore 100 = 5 + Ae^0 \quad \text{and so } A = 95$$

$$\therefore T = 5 + 95e^{kt}$$

When  $t = 1$ ,  $T = 80$ 

$$\therefore 80 = 5 + 95e^k$$

$$\therefore 75 = 95e^k$$

$$\therefore e^k = \frac{75}{95} = \frac{15}{19}$$

$$\therefore T = 5 + 95\left(\frac{15}{19}\right)^t$$

When  $T = 10$ ,  $10 = 5 + 95\left(\frac{15}{19}\right)^t$ 

$$\therefore \left(\frac{15}{19}\right)^t = \frac{5}{95} = \frac{1}{19}$$

$$\therefore t \ln\left(\frac{15}{19}\right) = \ln\left(\frac{1}{19}\right)$$

$$\therefore t = \frac{\ln\left(\frac{1}{19}\right)}{\ln\left(\frac{15}{19}\right)} \approx 12.5$$

It will take about 12.5 minutes for the temperature of the object to drop to  $10^\circ\text{C}$ .**b** Let  $t$  be the time in hours since 6 am, and  $T_m = 5$ .When  $t = 0$ ,  $T = 13$ 

$$\therefore 13 = 5 + Ae^0 \quad \text{and so } A = 8$$

$$\therefore T = 5 + 8e^{kt}$$

When  $t = 3$ ,  $T = 9$ 

$$\therefore 9 = 5 + 8e^{3k}$$

$$\therefore e^{3k} = \frac{4}{8} = \frac{1}{2}$$

$$\therefore e^k = \left(\frac{1}{2}\right)^{\frac{1}{3}}$$

$$\therefore T = 5 + 8\left(\frac{1}{2}\right)^{\frac{t}{3}}$$

When  $T = 37$ ,  $37 = 5 + 8\left(\frac{1}{2}\right)^{\frac{t}{3}}$ 

$$\therefore \left(\frac{1}{2}\right)^{\frac{t}{3}} = \frac{32}{8} = 4$$

$$\therefore 2^{-\frac{t}{3}} = 2^2$$

$$\therefore -\frac{t}{3} = 2$$

$$\therefore t = -6$$

 $\therefore$  the person died 6 hours before 6 am. $\therefore$  the time of death was 12 am which is midnight.



**17 a**  $L \frac{dI}{dt} + RI = E$   
 $\therefore 0.3 \frac{dI}{dt} + 10I = 20$   
 $\therefore \frac{3}{10} \frac{dI}{dt} = 20 - 10I$   
 $\therefore \frac{dI}{dt} = \frac{10}{3}(-10)(I - 2)$   
 $\therefore \int \frac{1}{I-2} \frac{dI}{dt} = -\frac{100}{3}$   
 $\therefore \int \frac{1}{I-2} \frac{dI}{dt} dt = \int -\frac{100}{3} dt$   
 $\therefore \int \frac{1}{I-2} dI = \int -\frac{100}{3} dt$   
 $\therefore \ln|I-2| = -\frac{100}{3}t + c$   
 $\therefore I-2 = \pm e^{-\frac{100}{3}t+c}$   
 $\therefore I(t) = Ae^{-\frac{100}{3}t} + 2 \quad \{A = \pm e^c\}$

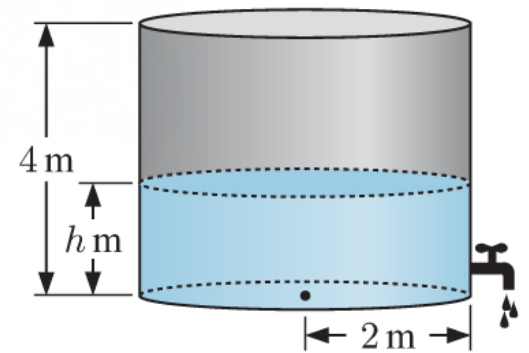
**b**  $I(0) = 0$   
 $\therefore Ae^0 + 2 = 0$   
 $\therefore A = -2$   
The particular solution is  
 $I(t) = 2 - 2e^{-\frac{100}{3}t}$   
**c** As  $t \rightarrow \infty$ ,  $e^{-\frac{100}{3}t} \rightarrow 0$   
 $\therefore I(t) \rightarrow 2$   
The limiting current is 2 amps.

**d** When  $I = 0.99 \times 2 = 1.98$ ,  $1.98 = 2 - 2e^{-\frac{100}{3}t}$   
 $\therefore 2e^{-\frac{100}{3}t} = 0.02$   
 $\therefore e^{-\frac{100}{3}t} = \frac{1}{100}$   
 $\therefore -\frac{100}{3}t = -\ln 100$   
 $\therefore t = \frac{3}{100} \ln 100$   
 $\therefore t \approx 0.138$

It will take about 0.138 seconds for the current to reach 99% of its limiting value.

**18** We are given that

$\frac{dV}{dt} \propto \sqrt{h}$  where  $h$  is the depth of the water.  
 $\therefore \frac{dV}{dt} = k\sqrt{h}$  where  $k$  is a constant.  
 $\therefore \frac{dV}{dh} \frac{dh}{dt} = k\sqrt{h}$  {chain rule}  
 $\therefore 4\pi \frac{dh}{dt} = k\sqrt{h}$   $\{V = \pi r^2 h = 4\pi h \therefore \frac{dV}{dh} = 4\pi\}$   
 $\therefore \frac{4\pi}{\sqrt{h}} \frac{dh}{dt} = k$   
 $\therefore \int 4\pi h^{-\frac{1}{2}} \frac{dh}{dt} dt = \int k dt$   
 $\therefore 4\pi \int h^{-\frac{1}{2}} dh = \int k dt$   
 $\therefore 4\pi \frac{h^{\frac{1}{2}}}{\frac{1}{2}} = kt + c$   
 $\therefore 8\pi\sqrt{h} = kt + c$



Now when  $t = 0$ ,  $h = 4$

$$\therefore 8\pi\sqrt{4} = c$$

$$\therefore c = 16\pi$$

$$\therefore 8\pi\sqrt{h} = kt + 16\pi$$

Also, when  $t = 2$ ,  $h = 1$

$$\therefore 8\pi\sqrt{1} = 2k + 16\pi$$

$$\therefore 2k = -8\pi$$

$$\therefore k = -4\pi$$

So, the equation connecting the depth of the water and the time  $t$  is  $8\pi\sqrt{h} = -4\pi t + 16\pi$ .

The tank is empty when  $h = 0$ . This occurs when  $4\pi t = 16\pi$

$$\therefore t = 4$$

The tank empties in 4 hours.

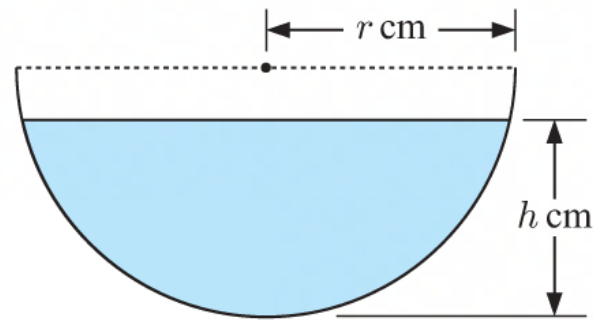
**19 a**  $V = \frac{1}{3}\pi h^2(3r - h) = \pi r h^2 - \frac{1}{3}\pi h^3$

$$\therefore \frac{dV}{dt} = \frac{dV}{dh} \frac{dh}{dt}$$

$$= (2\pi r h - \pi h^2) \frac{dh}{dt}$$

Now  $\frac{dV}{dt} = -r^2$ , so  $-r^2 = (2\pi r h - \pi h^2) \frac{dh}{dt}$

$$\begin{aligned} \therefore \frac{dh}{dt} &= \frac{-r^2}{2\pi r h - \pi h^2} \\ &= \frac{r^2}{\pi h^2 - 2\pi r h} \end{aligned}$$



**b i**  $(\pi h^2 - 2\pi r h) \frac{dh}{dt} = r^2$  {using **a**}

$$\therefore \int (\pi h^2 - 2\pi r h) \frac{dh}{dt} dt = \int r^2 dt$$

$$\therefore \int (\pi h^2 - 2\pi r h) dh = \int r^2 dt$$

$$\therefore \frac{1}{3}\pi h^3 - \pi r h^2 = r^2 t + c$$

$$\therefore \frac{1}{3}\pi h^3 - 10\pi h^2 = 100t + c \quad \{r = 10\}$$

When  $t = 0$ ,  $h = 10$

$$\therefore \frac{1000}{3}\pi - 1000\pi = c$$

$$\therefore c = -\frac{2000}{3}\pi$$

$$\therefore 100t = \frac{1}{3}\pi h^3 - 10\pi h^2 + \frac{2000}{3}\pi$$

$$\therefore t = \frac{\pi}{300}h^3 - \frac{\pi}{10}h^2 + \frac{2000}{300}\pi$$

$$\therefore t = \frac{\pi}{300}(h^3 - 30h^2 + 2000)$$

**ii** When  $h = 5$ ,  $t = \frac{1375\pi}{300}$

$$\therefore t \approx 14.4$$

It will take about 14.4 hours for the depth of the water to fall to 5 cm.

iii When  $h = 0$ ,  $t = \frac{20\pi}{3}$   
 $\therefore t \approx 20.9$

It will take about 20.9 hours for the bowl to empty.

**20 a** We are given that

$$\frac{dr}{dt} \propto h \quad \text{where } h \text{ is the thickness of the patch.}$$

$$\therefore \frac{dr}{dt} = kh \quad \text{where } k \text{ is a constant.}$$

$$\therefore \frac{dr}{dt} = \frac{kV}{\pi r^2} \quad \{V = \pi r^2 h \quad \therefore h = \frac{V}{\pi r^2}\}$$

$$\therefore r^2 \frac{dr}{dt} = \frac{kV}{\pi}$$

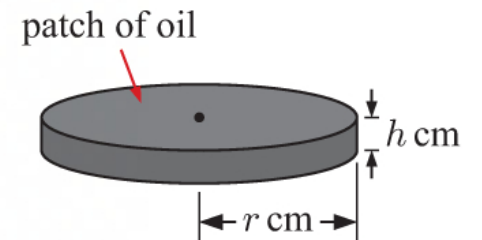
$$\therefore \int r^2 \frac{dr}{dt} dt = \int \frac{kV}{\pi} dt$$

$$\therefore \int r^2 dr = \int \frac{kV}{\pi} dt$$

$$\therefore \frac{1}{3}r^3 = \frac{kV}{\pi} t + c$$

$$\therefore r^3 = \frac{3kV}{\pi} t + c$$

$$\therefore r = \sqrt[3]{\frac{3kV}{\pi} t + c}$$



**b** Since  $V = 1 \text{ L} = 1000 \text{ cm}^3$ ,  $r = \sqrt[3]{\frac{3000k}{\pi} t + c}$ .

When  $t = 0$ ,  $r = 20$

$$\therefore 20 = \sqrt[3]{c}$$

$$\therefore c = 8000$$

$$\therefore r = \sqrt[3]{\frac{3000k}{\pi} t + 8000}$$

When  $t = 2$ ,  $r = 50$

$$\therefore 50 = \sqrt[3]{\frac{6000k}{\pi} + 8000}$$

$$\therefore 125\,000 = \frac{6000k}{\pi} + 8000$$

$$\therefore \frac{6000k}{\pi} = 117\,000$$

$$\therefore \frac{3000k}{\pi} = 58\,500$$

$$\therefore r = \sqrt[3]{58\,500t + 8000}$$

When  $r = 500$ ,  $500 = \sqrt[3]{58\,500t + 8000}$

$$\therefore 125\,000\,000 = 58\,500t + 8000$$

$$\therefore 58\,500t = 124\,992\,000$$

$$\therefore t \approx 2136.6$$

It will take about 2136.6 seconds or 35.6 minutes for the spill radius to reach 5 m.

**21** If P is  $(x, y)$ , then the gradient of [OP] is  $\frac{y}{x}$ .

Since  $\widehat{OPQ} = 90^\circ$ , the gradient of the curve at P is  $-\frac{x}{y}$ .

Hence  $\frac{dy}{dx} = -\frac{x}{y}$

$$\therefore y \frac{dy}{dx} = -x$$

Integrating both sides with respect to  $x$  gives

$$\int y \, dy = - \int x \, dx$$

$$\therefore \frac{1}{2}y^2 = -\frac{1}{2}x^2 + c$$

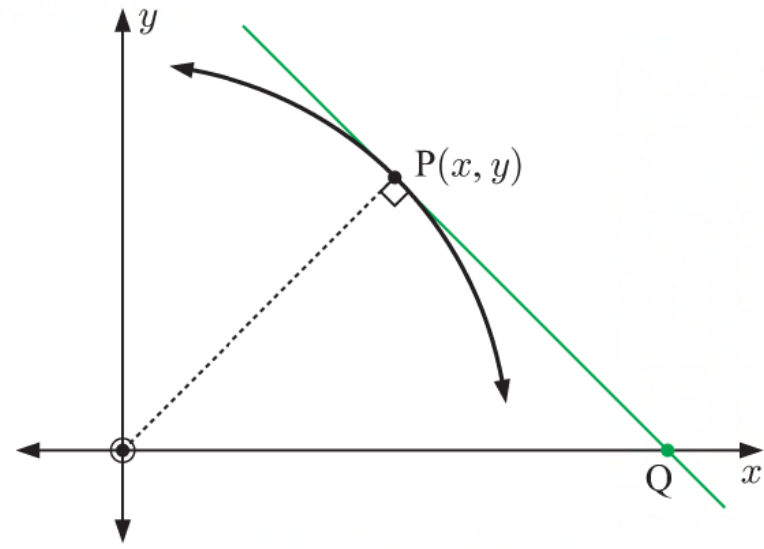
$$\therefore y^2 = -x^2 + c$$

$$\therefore x^2 + y^2 = c$$

Since the curve passes through  $(1, 2)$ ,  $1^2 + 2^2 = c$

$$\therefore c = 5$$

$\therefore$  the equation of the curve is  $x^2 + y^2 = 5$ .



**22** The curve passes through  $(1, 1)$ , so it must be in the first quadrant.

There are two cases to consider:

*Case 1:* P lies on [AB].

AP : PB = 2 : 1, so using similar triangles in the diagram alongside, we see that AX : XO = 2 : 1 and OY : YB = 2 : 1.

$\therefore$  A is  $(3x, 0)$  and B is  $(0, \frac{3}{2}y)$ .

Since (AB) is the tangent to the curve at P, the gradient of the curve at P is the same as the gradient of [AB].

Hence  $\frac{dy}{dx} = \frac{0 - \frac{3}{2}y}{3x - 0} = \frac{-y}{2x}$

$$\therefore \frac{1}{y} \frac{dy}{dx} = -\frac{1}{2x}$$

Integrating both sides with respect to  $x$  gives

$$\int \frac{1}{y} \, dy = -\frac{1}{2} \int \frac{1}{x} \, dx$$

$$\therefore \ln |y| = -\frac{1}{2} \ln |x| + c$$

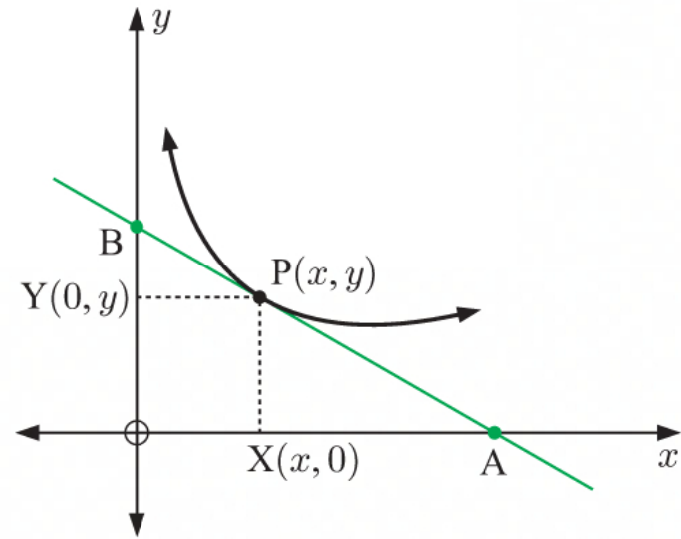
$$\therefore \ln |y| + \frac{1}{2} \ln |x| = c$$

$$\therefore \ln |y\sqrt{x}| = c$$

$$\therefore y\sqrt{x} = \pm e^c = k \quad \text{where } k \text{ is a constant.}$$

Since the curve passes through  $(1, 1)$ ,  $1 = k$

$\therefore$  the equation of the curve is  $y\sqrt{x} = 1$  or  $y = \frac{1}{\sqrt{x}}$ .





Case 2: B lies on [AP].

AP : PB = 2 : 1, so using similar triangles in the diagram alongside, we see that AX : XO = 2 : 1 and OY : YB = 2 : 1.

$\therefore$  A is  $(-x, 0)$  and B is  $(0, \frac{1}{2}y)$ .

Since (AB) is the tangent to the curve at P, the gradient of the curve at P is the same as the gradient of [AB].

$$\text{Hence } \frac{dy}{dx} = \frac{\frac{1}{2}y - 0}{0 - (-x)} = \frac{y}{2x}$$

$$\therefore \frac{1}{y} \frac{dy}{dx} = \frac{1}{2x}$$

Integrating both sides with respect to  $x$  gives

$$\int \frac{1}{y} dy = \frac{1}{2} \int \frac{1}{x} dx$$

$$\therefore \ln |y| = \frac{1}{2} \ln |x| + c$$

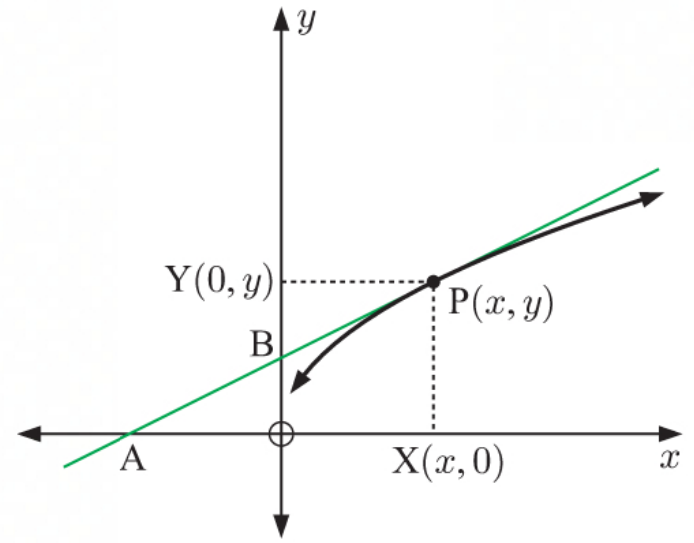
$$\therefore \ln |y| - \frac{1}{2} \ln |x| = c$$

$$\therefore \ln \left| \frac{y}{\sqrt{x}} \right| = c$$

$$\therefore \frac{y}{\sqrt{x}} = \pm e^c = k \quad \text{where } k \text{ is a constant.}$$

Since the curve passes through  $(1, 1)$ ,  $1 = k$

$\therefore$  the equation of the curve is  $\frac{y}{\sqrt{x}} = 1$  or  $y = \sqrt{x}$ .



## EXERCISE 25E

$$\begin{aligned}
 1 \quad a \quad \frac{dP}{dt} &= 0.2P \left( 1 - \frac{P}{200} \right) = 0.2P \left( \frac{200 - P}{200} \right) \\
 \therefore \frac{200}{P(200 - P)} \frac{dP}{dt} &= 0.2 \\
 \therefore \int \frac{200}{P(200 - P)} \frac{dP}{dt} dt &= \int 0.2 dt \\
 \therefore \int \frac{200}{P(200 - P)} dP &= \int 0.2 dt \\
 \therefore \int \left( \frac{1}{P} + \frac{1}{200 - P} \right) dP &= \int 0.2 dt \\
 \therefore \ln |P| + \frac{1}{-1} \ln |200 - P| &= 0.2t + c \\
 \therefore \ln \left| \frac{P}{200 - P} \right| &= 0.2t + c \\
 \therefore \frac{P}{200 - P} &= \pm e^{0.2t + c} \\
 \therefore \frac{200 - P}{P} &= be^{-0.2t} \quad \left\{ \text{letting } b = \pm \frac{1}{e^c} \right\}
 \end{aligned}$$

Now when  $t = 0$ ,  $P = 20$

$$\begin{aligned}
 \therefore \frac{180}{20} &= be^0 \\
 \therefore b &= 9
 \end{aligned}$$

So, we have  $\frac{200 - P}{P} = 9e^{-0.2t}$

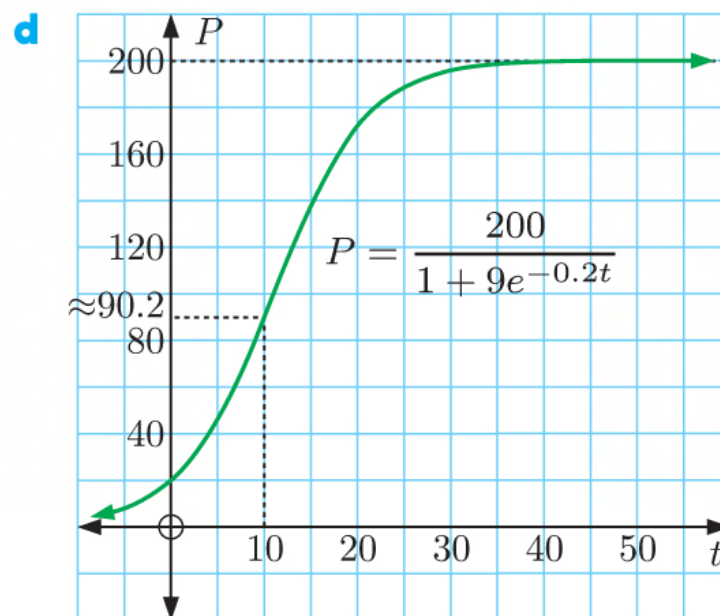
$$\begin{aligned}
 \therefore \frac{200}{P} - 1 &= 9e^{-0.2t} \\
 \therefore \frac{200}{P} &= 1 + 9e^{-0.2t} \\
 \therefore P &= \frac{200}{1 + 9e^{-0.2t}}
 \end{aligned}$$

b When  $t = 10$ ,

$$P = \frac{200}{1 + 9e^{-0.2 \times 10}} \approx 90.2$$

c As  $t \rightarrow \infty$ ,  $e^{-0.2t} \rightarrow 0$

$$\therefore P \rightarrow \frac{200}{1 + 0} = 200$$



2 a The population  $P$  grows logistically over time  $t$ .

b The graph would increase exponentially at first, but then level off to approach the maximum value.

$$\begin{aligned}
\text{c} \quad \frac{dP}{dt} &= 0.1P \left(1 - \frac{P}{3000}\right) = 0.1P \left(\frac{3000 - P}{3000}\right) \\
&\therefore \frac{3000}{P(3000 - P)} \frac{dP}{dt} = 0.1 \\
&\therefore \int \frac{3000}{P(3000 - P)} \frac{dP}{dt} dt = \int 0.1 dt \\
&\therefore \int \frac{3000}{P(3000 - P)} dP = \int 0.1 dt \\
&\therefore \int \left(\frac{1}{P} + \frac{1}{3000 - P}\right) dP = \int 0.1 dt \\
&\therefore \ln|P| + \frac{1}{-1} \ln|3000 - P| = 0.1t + c \\
&\therefore \ln \left| \frac{P}{3000 - P} \right| = 0.1t + c \\
&\therefore \frac{P}{3000 - P} = \pm e^{0.1t+c} \\
&\therefore \frac{3000 - P}{P} = be^{-0.1t} \quad \left\{ \text{letting } b = \pm \frac{1}{e^c} \right\}
\end{aligned}$$

Now when  $t = 0$ ,  $P = 500$

$$\begin{aligned}
&\therefore \frac{2500}{500} = be^0 \\
&\therefore b = 5
\end{aligned}$$

$$\begin{aligned}
\text{So, we have } \frac{3000 - P}{P} &= 5e^{-0.1t} \\
&\therefore \frac{3000}{P} - 1 = 5e^{-0.1t} \\
&\therefore \frac{3000}{P} = 1 + 5e^{-0.1t} \\
&\therefore P = \frac{3000}{1 + 5e^{-0.1t}}
\end{aligned}$$

$$\text{d} \quad \text{i} \quad \text{When } t = 8, \quad P = \frac{3000}{1 + 5e^{-0.1 \times 8}} \approx 924$$

The expected population after 8 years is about 924 rodents.

$$\begin{aligned}
\text{ii} \quad \text{When } P &= 2000, \quad 2000 = \frac{3000}{1 + 5e^{-0.1t}} \\
&\therefore 1 + 5e^{-0.1t} = \frac{3000}{2000} = \frac{3}{2} \\
&\therefore 5e^{-0.1t} = \frac{1}{2} \\
&\therefore e^{-0.1t} = \frac{1}{10} \\
&\therefore -0.1t = \ln\left(\frac{1}{10}\right) \\
&\therefore t = \frac{\ln\left(\frac{1}{10}\right)}{-0.1} \approx 23.0 \text{ years}
\end{aligned}$$

$$\text{iii} \quad \text{As } t \rightarrow \infty, \quad e^{-0.1t} \rightarrow 0$$

$$\therefore P \rightarrow \frac{3000}{1 + 0} = 3000$$

The maximum population which the island can sustain is 3000 rodents.

$$\begin{aligned}
\mathbf{3} \quad \mathbf{a} \quad & \frac{dA}{dt} = 0.1A \left( 1 - \frac{A}{500} \right) = 0.1A \left( \frac{500 - A}{500} \right) \\
& \therefore \frac{500}{A(500 - A)} \frac{dA}{dt} = 0.1 \\
& \therefore \int \frac{500}{A(500 - A)} \frac{dA}{dt} dt = \int 0.1 dt \\
& \therefore \int \frac{500}{A(500 - A)} dA = \int 0.1 dt \\
& \therefore \int \left( \frac{1}{A} + \frac{1}{500 - A} \right) dA = \int 0.1 dt \\
& \therefore \ln |A| + \frac{1}{-1} \ln |500 - A| = 0.1t + c \\
& \therefore \ln \left| \frac{A}{500 - A} \right| = 0.1t + c \\
& \therefore \frac{A}{500 - A} = \pm e^{0.1t + c} \\
& \therefore \frac{500 - A}{A} = be^{-0.1t} \quad \left\{ \text{letting } b = \pm \frac{1}{e^c} \right\}
\end{aligned}$$

Now when  $t = 0$ ,  $A = 20$

$$\therefore \frac{480}{20} = be^0$$

$$\therefore b = 24$$

So, we have  $\frac{500 - A}{A} = 24e^{-0.1t}$

$$\therefore \frac{500}{A} - 1 = 24e^{-0.1t}$$

$$\therefore \frac{500}{A} = 1 + 24e^{-0.1t}$$

$$\therefore A = \frac{500}{1 + 24e^{-0.1t}}$$

$$\mathbf{b} \quad \text{When } t = 20, \quad A = \frac{500}{1 + 24e^{-0.1 \times 20}} \approx 118$$

So the area of the lily after 20 days is approximately 118 cm<sup>2</sup>.

$$\mathbf{c} \quad \text{As } t \rightarrow \infty, \quad e^{-0.1t} \rightarrow 0$$

$$\therefore A \rightarrow \frac{500}{1 + 0} = 500$$

The limiting size of the lily is 500 cm<sup>2</sup>.



**4 a**

$$\frac{dN}{dt} = 0.8N \left( 1 - \frac{N}{600} \right) = 0.8N \left( \frac{600 - N}{600} \right)$$

$$\therefore \frac{600}{N(600 - N)} \frac{dN}{dt} = 0.8$$

$$\therefore \int \frac{600}{N(600 - N)} \frac{dN}{dt} dt = \int 0.8 dt$$

$$\therefore \int \frac{600}{N(600 - N)} dN = \int 0.8 dt$$

$$\therefore \int \left( \frac{1}{N} + \frac{1}{600 - N} \right) dN = \int 0.8 dt$$

$$\therefore \ln |N| + \frac{1}{-1} \ln |600 - N| = 0.8t + c$$

$$\therefore \ln \left| \frac{N}{600 - N} \right| = 0.8t + c$$

$$\therefore \frac{N}{600 - N} = \pm e^{0.8t+c}$$

$$\therefore \frac{600 - N}{N} = be^{-0.8t} \quad \left\{ \text{letting } b = \pm \frac{1}{e^c} \right\}$$

Now when  $t = 0$ ,  $N = 2$

$$\therefore \frac{598}{2} = be^0$$

$$\therefore b = 299$$

So, we have  $\frac{600 - N}{N} = 299e^{-0.8t}$

$$\therefore \frac{600}{N} - 1 = 299e^{-0.8t}$$

$$\therefore \frac{600}{N} = 1 + 299e^{-0.8t}$$

$$\therefore N = \frac{600}{1 + 299e^{-0.8t}}$$

**b** When  $t = 3$ ,  $N = \frac{600}{1 + 299e^{-0.8 \times 3}} \approx 21$

Approximately 21 people have heard the rumour by 11 am.

**c** As  $t \rightarrow \infty$ ,  $e^{-0.8t} \rightarrow 0$

$$\therefore N \rightarrow \frac{600}{1 + 0} = 600$$

Since 600 is the limiting number of people who have heard the rumour, it is likely there are 600 people living in the town.

**d** When  $N = 500$ ,  $500 = \frac{600}{1 + 299e^{-0.8t}}$

$$\therefore 1 + 299e^{-0.8t} = \frac{600}{500} = \frac{6}{5}$$

$$\therefore 299e^{-0.8t} = \frac{1}{5}$$

$$\therefore e^{-0.8t} = \frac{1}{1495}$$

$$\therefore -0.8t = \ln\left(\frac{1}{1495}\right)$$

$$\therefore t = \frac{\ln\left(\frac{1}{1495}\right)}{-0.8} \approx 9.14 \text{ hours} \approx 9 \text{ hours } 8 \text{ minutes}$$

At about 9 hours and 8 minutes after 8 am, or at about 5:08 pm, 500 people would have heard the rumour.

**5 a**

$$\frac{dN}{dt} = kN\left(1 - \frac{N}{10^{30}}\right) = kN\left(\frac{10^{30} - N}{10^{30}}\right)$$

$$\therefore \frac{10^{30}}{N(10^{30} - N)} \frac{dN}{dt} = k$$

$$\therefore \int \frac{10^{30}}{N(10^{30} - N)} \frac{dN}{dt} dt = \int k dt$$

$$\therefore \int \frac{10^{30}}{N(10^{30} - N)} dN = \int k dt$$

$$\therefore \int \left(\frac{1}{N} + \frac{1}{10^{30} - N}\right) dN = \int k dt$$

$$\therefore \ln|N| + \frac{1}{-1} \ln|10^{30} - N| = kt + c$$

$$\therefore \ln\left|\frac{N}{10^{30} - N}\right| = kt + c$$

$$\therefore \frac{N}{10^{30} - N} = \pm e^{kt+c}$$

$$\therefore \frac{10^{30} - N}{N} = be^{-kt} \quad \left\{ \text{letting } b = \pm \frac{1}{e^c} \right\}$$

Now when  $t = 0$ ,  $N = 200$

$$\therefore \frac{10^{30} - 200}{200} = be^0$$

$$\therefore b = \frac{10^{30}}{200} - 1 = 5 \times 10^{27} - 1$$

So, we have  $\frac{10^{30} - N}{N} = (5 \times 10^{27} - 1)e^{-kt}$

$$\therefore \frac{10^{30}}{N} - 1 = (5 \times 10^{27} - 1)e^{-kt}$$

$$\therefore \frac{10^{30}}{N} = 1 + (5 \times 10^{27} - 1)e^{-kt}$$

$$\therefore N = \frac{10^{30}}{1 + (5 \times 10^{27} - 1)e^{-kt}}$$

**b** When  $t = 10^{-5}$ ,  $N = 1.5 \times 10^7$

$$\therefore 1.5 \times 10^7 = \frac{10^{30}}{1 + (5 \times 10^{27} - 1)e^{-(10^{-5})k}}$$

$$\therefore 1 + (5 \times 10^{27} - 1)e^{-(10^{-5})k} = \frac{10^{30}}{1.5 \times 10^7}$$

$$\therefore (5 \times 10^{27} - 1)e^{-(10^{-5})k} = \frac{10^{23}}{1.5} - 1$$

$$\therefore e^{-(10^{-5})k} = \frac{10^{23} - 1.5}{1.5 \times (5 \times 10^{27} - 1)}$$

$$\therefore -(10^{-5})k = \ln\left(\frac{10^{23} - 1.5}{7.5 \times 10^{27} - 1.5}\right)$$

$$\therefore k = -10^5 \times \ln\left(\frac{10^{23} - 1.5}{7.5 \times 10^{27} - 1.5}\right)$$

$$\therefore k \approx 1.12 \times 10^6$$

**c** The reaction is complete when all  $10^{30}$  molecules are “active”.

$$\text{When } N = 0.99 \times 10^{30}, \quad 0.99 \times 10^{30} = \frac{10^{30}}{1 + (5 \times 10^{27} - 1)e^{-kt}}$$

$$\therefore 1 + (5 \times 10^{27} - 1)e^{-kt} = \frac{1}{0.99} = \frac{100}{99}$$

$$\therefore (5 \times 10^{27} - 1)e^{-kt} = \frac{1}{99}$$

$$\therefore e^{-kt} = \frac{1}{495 \times 10^{27} - 99}$$

$$\therefore e^{kt} = 495 \times 10^{27} - 99$$

$$\therefore kt = \ln(495 \times 10^{27} - 99)$$

$$\therefore t = \frac{\ln(495 \times 10^{27} - 99)}{-10^5 \times \ln\left(\frac{10^{23} - 1.5}{7.5 \times 10^{27} - 1.5}\right)} \quad \{\text{from b}\}$$

$$\therefore t \approx 6.09 \times 10^{-5} \text{ seconds}$$

The reaction will be 99% complete after about  $6.09 \times 10^{-5}$  seconds.

**6 a** The population of foxes increased quickly at first, but later levelled off to approach a maximum.

**b i**  $A$  is the limiting population of the foxes.

$$\therefore A = 95\,000$$

$$\begin{aligned}
\text{ii} \quad \frac{dF}{dt} &= kF \left( 1 - \frac{F}{95\,000} \right) = kF \left( \frac{95\,000 - F}{95\,000} \right) \\
\therefore \frac{95\,000}{F(95\,000 - F)} \frac{dF}{dt} &= k \\
\therefore \int \frac{95\,000}{F(95\,000 - F)} \frac{dF}{dt} dt &= \int k dt \\
\therefore \int \frac{95\,000}{F(95\,000 - F)} dF &= \int k dt \\
\therefore \int \left( \frac{1}{F} + \frac{1}{95\,000 - F} \right) dF &= \int k dt \\
\therefore \ln|F| + \frac{1}{-1} \ln|95\,000 - F| &= kt + c \\
\therefore \ln \left| \frac{F}{95\,000 - F} \right| &= kt + c \\
\therefore \frac{F}{95\,000 - F} &= \pm e^{kt+c} \\
\therefore \frac{95\,000 - F}{F} &= be^{-kt} \quad \left\{ \text{letting } b = \pm \frac{1}{e^c} \right\}
\end{aligned}$$

Now when  $t = 0$ ,  $F = 14$

$$\begin{aligned}
\therefore \frac{94\,986}{14} &= be^0 \\
\therefore b &= \frac{47\,493}{7}
\end{aligned}$$

$$\begin{aligned}
\text{So, we have } \frac{95\,000 - F}{F} &= \left( \frac{47\,493}{7} \right) e^{-kt} \\
\therefore \frac{95\,000}{F} - 1 &= \left( \frac{47\,493}{7} \right) e^{-kt} \\
\therefore \frac{95\,000}{F} &= 1 + \left( \frac{47\,493}{7} \right) e^{-kt} \\
\therefore F &= \frac{95\,000}{1 + \left( \frac{47\,493}{7} \right) e^{-kt}}
\end{aligned}$$

In 1900,  $t = 55$  and  $F = 30\,000$

$$\begin{aligned}
\therefore 30\,000 &= \frac{95\,000}{1 + \left( \frac{47\,493}{7} \right) e^{-55k}} \\
\therefore 1 + \left( \frac{47\,493}{7} \right) e^{-55k} &= \frac{95\,000}{30\,000} = \frac{19}{6} \\
\therefore \left( \frac{47\,493}{7} \right) e^{-55k} &= \frac{13}{6} \\
\therefore e^{-55k} &= \frac{91}{284\,958} \\
\therefore -55k &= \ln \left( \frac{91}{284\,958} \right) \\
\therefore k &= -\frac{1}{55} \ln \left( \frac{91}{284\,958} \right) \approx 0.146
\end{aligned}$$

$$\therefore F \approx \frac{95\,000}{1 + \left( \frac{47\,493}{7} \right) e^{-0.146t}}$$



c In 1920,  $t = 75$

$$\therefore F = \frac{95\,000}{1 + \left(\frac{47\,493}{7}\right)e^{-75k}} \approx 85\,100$$

There were approximately 85 100 foxes in 1920.

d

$$F = \frac{95\,000}{1 + \left(\frac{47\,493}{7}\right)e^{-kt}}$$

$$\therefore 1 + \left(\frac{47\,493}{7}\right)e^{-kt} = \frac{95\,000}{F}$$

$$\therefore \left(\frac{47\,493}{7}\right)e^{-kt} = \frac{95\,000}{F} - 1$$

$$\therefore e^{-kt} = \left(\frac{95\,000 - F}{F}\right) \left(\frac{7}{47\,493}\right)$$

$$\therefore e^{kt} = \frac{47\,493F}{7 \times 95\,000 - 7F}$$

$$\therefore kt = \ln\left(\frac{47\,493F}{665\,000 - 7F}\right)$$

$$\therefore t = \frac{\ln\left(\frac{47\,493F}{665\,000 - 7F}\right)}{-\frac{1}{55} \ln\left(\frac{91}{284\,958}\right)} \quad \{\text{from a}\}$$

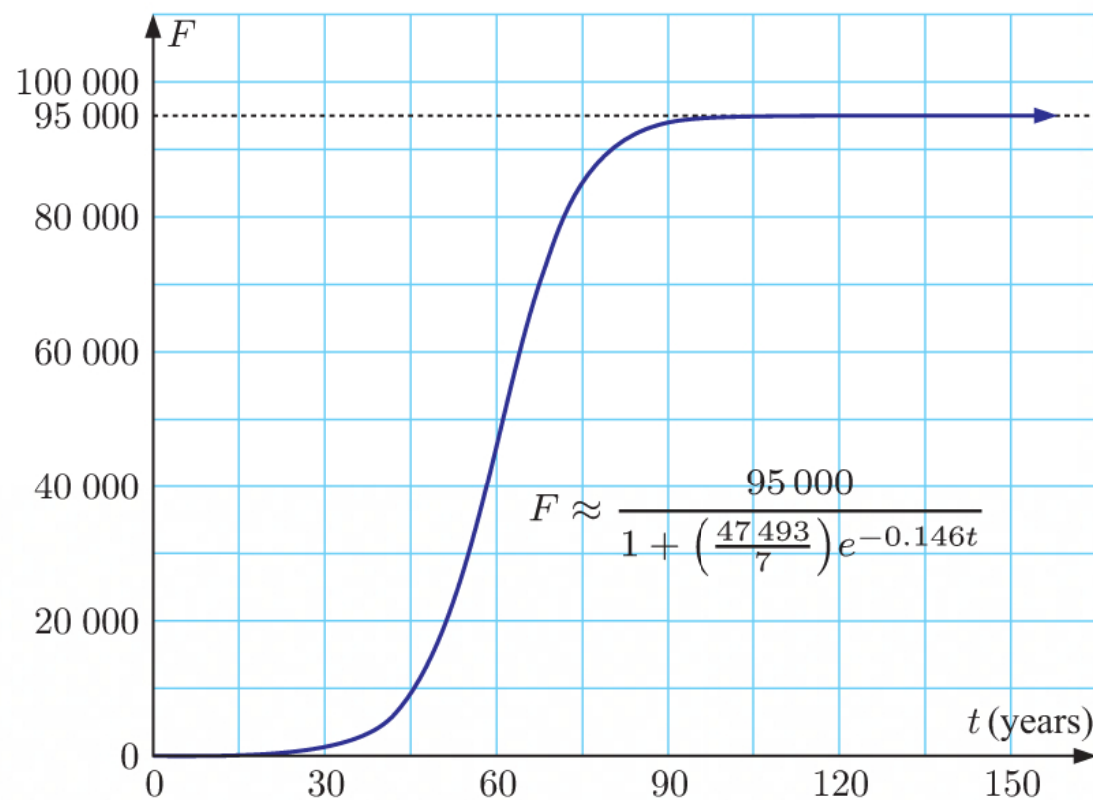
i When  $F = 15\,000$ ,  $t = \frac{\ln\left(\frac{47\,493 \times 15\,000}{665\,000 - 7 \times 15\,000}\right)}{-\frac{1}{55} \ln\left(\frac{91}{284\,958}\right)} \approx 49$  years after 1845

$\therefore$  the fox population was 15 000 around 1894.


ii When  $F = 65\,000$ ,  $t = \frac{\ln\left(\frac{47\,493 \times 65\,000}{665\,000 - 7 \times 65\,000}\right)}{-\frac{1}{55} \ln\left(\frac{91}{284\,958}\right)} \approx 66$  years after 1845

$\therefore$  the fox population was 65 000 around 1911.

e



**f** The population growth rate is a maximum when  $\frac{dF}{dt} = kF\left(1 - \frac{F}{95\,000}\right)$  is maximised.

Notice that  $kF\left(1 - \frac{F}{95\,000}\right) = kF - \frac{kF^2}{95\,000}$  is a quadratic in terms of  $F$  with shape .

$$\begin{aligned}\therefore \frac{dF}{dt} \text{ is maximised when } F &= \frac{-k}{2\left(-\frac{k}{95\,000}\right)} \\ &= \frac{95\,000}{2} \\ &= 47\,500\end{aligned}$$

$$\begin{aligned}\text{When } F = 47\,500, \quad t &= \frac{\ln\left(\frac{47\,493 \times 47\,500}{665\,000 - 7 \times 47\,500}\right)}{-\frac{1}{55} \ln\left(\frac{91}{284\,958}\right)} \quad \{\text{from d}\} \\ &\approx 60 \text{ years after 1845}\end{aligned}$$

$\therefore$  the population growth rate was a maximum when  $F = 47\,500$ , which occurred in 1905. It appears as an inflection on the graph.

**7 a**

$$\begin{aligned}\frac{dP}{dt} &= \frac{1}{10}P\left(1 - \frac{P}{20\,000}\right)(1 - \alpha\sqrt{t}) \\ &= \frac{1}{10}P\left(\frac{20\,000 - P}{20\,000}\right)(1 - \alpha\sqrt{t})\end{aligned}$$

$$\therefore \frac{20\,000}{P(20\,000 - P)} \frac{dP}{dt} = \frac{1}{10}(1 - \alpha\sqrt{t})$$

$$\therefore \int \left(\frac{1}{P} + \frac{1}{20\,000 - P}\right) dP = \frac{1}{10} \int (1 - \alpha\sqrt{t}) dt$$

$$\therefore \ln|P| + \frac{1}{-1} \ln|20\,000 - P| = \frac{1}{10}\left(t - \frac{2}{3}\alpha t^{\frac{3}{2}}\right) + c$$

$$\therefore \ln\left|\frac{P}{20\,000 - P}\right| = 0.1t\left(1 - \frac{2}{3}\alpha\sqrt{t}\right) + c$$

$$\therefore \frac{P}{20\,000 - P} = \pm e^{0.1t(1 - \frac{2}{3}\alpha\sqrt{t}) + c}$$

$$\therefore \frac{20\,000 - P}{P} = be^{-0.1t(1 - \frac{2}{3}\alpha\sqrt{t})} \quad \left\{\text{letting } b = \pm \frac{1}{e^c}\right\}$$

$$\therefore \frac{20\,000}{P} - 1 = be^{-0.1t(1 - \frac{2}{3}\alpha\sqrt{t})}$$

$$\therefore \frac{20\,000}{P} = 1 + be^{-0.1t(1 - \frac{2}{3}\alpha\sqrt{t})}$$

$$\therefore P(t) = \frac{20\,000}{1 + be^{-0.1t(1 - \frac{2}{3}\alpha\sqrt{t})}}$$

**b** The initial population was 12 000, so  $P(0) = 12\,000$

$$\therefore 12\,000 = \frac{20\,000}{1 + be^0}$$

$$\therefore 1 + b = \frac{20\,000}{12\,000} = \frac{5}{3}$$

$$\therefore b = \frac{2}{3}$$

$$\therefore P(t) = \frac{20\,000}{1 + \frac{2}{3}e^{-0.1t(1 - \frac{2}{3}\alpha\sqrt{t})}}$$

20 years ago, the population was 200, so  $P(200) = 200$

$$\therefore 200 = \frac{20\,000}{1 + \frac{2}{3}e^{-0.1(200)(1 - \frac{2}{3}\alpha\sqrt{200})}}$$

$$\therefore 1 + \frac{2}{3}e^{-20 + \frac{40}{3}\sqrt{200}\alpha} = \frac{20\,000}{200} = 100$$

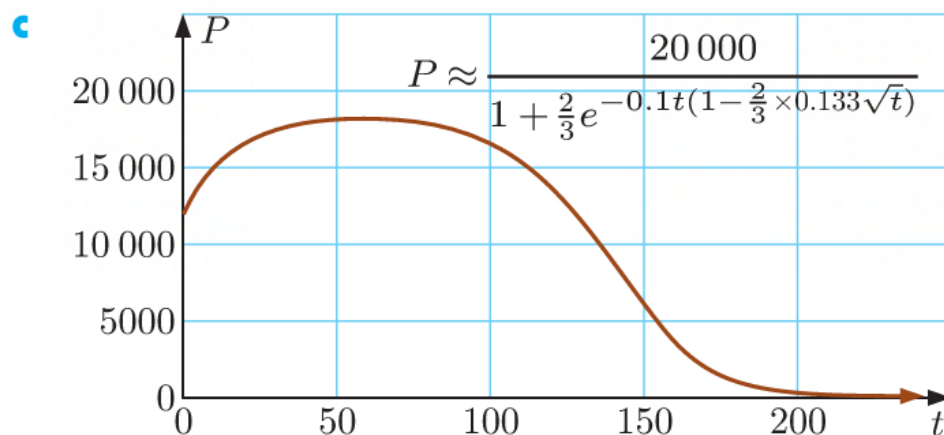
$$\therefore e^{-20 + \frac{40}{3}\sqrt{200}\alpha} = \frac{297}{2}$$

$$\therefore -20 + \frac{40}{3}\sqrt{200}\alpha = \ln\left(\frac{297}{2}\right)$$

$$\therefore \frac{40}{3}\sqrt{200}\alpha = \ln\left(\frac{297}{2}\right) + 20$$

$$\therefore \alpha = \frac{3}{40\sqrt{200}}\left(\ln\left(\frac{297}{2}\right) + 20\right)$$

$$\therefore \alpha \approx 0.133$$



**d** The numbat population is a maximum when  $\frac{dP}{dt} = 0$

$$\therefore \frac{1}{10}P\left(1 - \frac{P}{20\,000}\right)(1 - \alpha\sqrt{t}) = 0$$

$$\therefore 1 - \alpha\sqrt{t} = 0 \quad \{P \neq 0 \text{ or } 20\,000\}$$

$$\therefore \sqrt{t} = \frac{1}{\alpha}$$

$$\therefore t = \left(\frac{1}{\alpha}\right)^2 \approx 56.9 \text{ years}$$

$$\text{When } t = \left(\frac{1}{\alpha}\right)^2, \quad P(t) = \frac{20\,000}{1 + \frac{2}{3}e^{-0.1\left(\frac{1}{\alpha}\right)^2\left(1 - \frac{2}{3}\right)}}$$

$$\approx 18\,200$$

The maximum population was about 18 200 numbats which occurred about  $220 - 56.9 \approx 163$  years ago.

e The current population is 
$$P(220) = \frac{20\,000}{1 + \frac{2}{3}e^{-0.1(220)(1 - \frac{2}{3} \times \alpha \sqrt{220})}}$$
  

$$\approx 32$$

The model predicts that, if nothing had been done, 32 numbats would still exist today.

- f The introduced species will drive the numbat species to extinction, that is,  $P \rightarrow 0$  as  $t \rightarrow \infty$ . Realistically, it is possible for the numbat population to become extinct in a finite time.

## EXERCISE 25F

1 a  $\frac{dy}{dx} = \frac{x-y}{x} = 1 - \frac{y}{x}$  so the differential equation is homogeneous.

Let  $y = vx$ , so  $\frac{dy}{dx} = v + x \frac{dv}{dx}$  {product rule}

Comparing with the differential equation,  $v + x \frac{dv}{dx} = 1 - v$

$$\therefore x \frac{dv}{dx} = 1 - 2v$$

$$\therefore \frac{1}{1-2v} \frac{dv}{dx} = \frac{1}{x}$$

$$\therefore \int \frac{1}{1-2v} dv = \int \frac{1}{x} dx$$

$$\therefore -\frac{1}{2} \ln |1-2v| = \ln |x| + c$$

$$\therefore \ln |1-2v| = c - 2 \ln |x|$$

$$\therefore \ln |1-2v| = \ln \left| \frac{A}{x^2} \right| \quad \text{where } \ln |A| = c$$

$$\therefore 1-2v = \frac{A}{x^2}$$

$$\therefore 1-2\left(\frac{y}{x}\right) = \frac{A}{x^2}$$

$$\therefore 2\left(\frac{y}{x}\right) = 1 + \frac{A}{x^2}$$

$$\therefore y = \frac{x}{2} + \frac{A}{x}, \quad \text{where } A \text{ is a constant.}$$

b Substituting  $y = 2$  and  $x = 1$  into the general solution, we find  $2 = \frac{1}{2} + \frac{A}{1}$

$$\therefore A = \frac{3}{2}$$

$\therefore$  the particular solution is  $y = \frac{x}{2} + \frac{3}{2x}$

$$\therefore y = \frac{x^2 + 3}{2x}$$



**2**  $\frac{dy}{dx} = \frac{x+2y}{4x} = \frac{1}{4} + \frac{1}{2}\left(\frac{y}{x}\right)$  so the differential equation is homogeneous.

Let  $y = vx$ , so  $\frac{dy}{dx} = v + x \frac{dv}{dx}$  {product rule}

Comparing with the differential equation,  $v + x \frac{dv}{dx} = \frac{1}{4} + \frac{1}{2}v$

$$\therefore x \frac{dv}{dx} = \frac{1}{4} - \frac{1}{2}v$$

$$\therefore 4x \frac{dv}{dx} = 1 - 2v$$

$$\therefore \frac{1}{1-2v} \frac{dv}{dx} = \frac{1}{4x}$$

$$\therefore \int \frac{1}{1-2v} dv = \frac{1}{4} \int \frac{1}{x} dx$$

$$\therefore -\frac{1}{2} \ln |1-2v| = \frac{1}{4} \ln |x| + c$$

$$\therefore \ln |1-2v| = c - \frac{1}{2} \ln |x|$$

$$\therefore \ln |1-2v| = \ln \left| \frac{A}{\sqrt{x}} \right| \quad \text{where } \ln |A| = c$$

$$\therefore 1-2v = \frac{A}{\sqrt{x}}$$

$$\therefore 1-2\left(\frac{y}{x}\right) = \frac{A}{\sqrt{x}}$$

$$\therefore 2\left(\frac{y}{x}\right) = 1 + \frac{A}{\sqrt{x}}$$

$$\therefore y = \frac{x}{2} + A\sqrt{x}, \quad \text{where } A \text{ is a constant.}$$

But  $y(1) = 2$ , so  $2 = \frac{1}{2} + A$

$$\therefore A = \frac{3}{2}$$

$\therefore$  the particular solution is  $y = \frac{x}{2} + \frac{3}{2}\sqrt{x}$

$$\therefore y = \frac{x+3\sqrt{x}}{2}$$

$$\begin{aligned}
 \mathbf{3} \quad \mathbf{a} \quad \frac{dy}{dx} &= \frac{xy + y^2}{x^2} \\
 &= \frac{xy}{x^2} + \frac{y^2}{x^2} \\
 &= \frac{y}{x} + \left(\frac{y}{x}\right)^2 \quad \text{so the differential equation is homogeneous.}
 \end{aligned}$$

$$\text{Let } y = vx, \text{ so } \frac{dy}{dx} = v + x \frac{dv}{dx} \quad \{\text{product rule}\}$$

$$\text{Comparing with the differential equation, } v + x \frac{dv}{dx} = v + v^2$$

$$\therefore x \frac{dv}{dx} = v^2$$

$$\therefore \frac{1}{v^2} \frac{dv}{dx} = \frac{1}{x}$$

$$\therefore \int v^{-2} dv = \int \frac{1}{x} dx$$

$$\therefore -v^{-1} = \ln |x| + c$$

$$\therefore -\frac{1}{v} = \ln |Ax| \quad \text{where } \ln |A| = c$$

$$\therefore -\frac{x}{y} = \ln |Ax|$$

$$\therefore y = -\frac{x}{\ln |Ax|}, \quad \text{where } A \text{ is a constant.}$$

$$\mathbf{b} \quad \frac{dy}{dx} = \cos^2\left(\frac{y}{x}\right) + \frac{y}{x} \quad \text{so the differential equation is homogeneous.}$$

$$\text{Let } y = vx, \text{ so } \frac{dy}{dx} = v + x \frac{dv}{dx} \quad \{\text{product rule}\}$$

Comparing with the differential equation,

$$v + x \frac{dv}{dx} = \cos^2 v + v$$

$$\therefore x \frac{dv}{dx} = \cos^2 v$$

$$\therefore \sec^2 v \frac{dv}{dx} = \frac{1}{x}$$

$$\therefore \int \sec^2 v dv = \int \frac{1}{x} dx$$

$$\therefore \tan v = \ln |x| + c$$

$$\therefore \tan v = \ln |Ax| \quad \text{where } \ln |A| = c$$

$$\therefore v = \arctan(\ln |Ax|)$$

$$\therefore \frac{y}{x} = \arctan(\ln |Ax|)$$

$$\therefore y = x \arctan(\ln |Ax|), \quad \text{where } A \text{ is a constant.}$$

$$\begin{aligned}
 \text{c } \frac{dy}{dx} &= \frac{y^2 - x^2}{2xy} \\
 &= \frac{y^2}{2xy} - \frac{x^2}{2xy} \\
 &= \frac{1}{2} \left( \frac{y}{x} \right) - \frac{1}{2} \left( \frac{x}{y} \right) \\
 &= \frac{1}{2} \left( \frac{y}{x} \right) - \frac{1}{2} \left( \frac{1}{\frac{y}{x}} \right) \quad \text{so the differential equation is homogeneous.}
 \end{aligned}$$

$$\text{Let } y = vx, \text{ so } \frac{dy}{dx} = v + x \frac{dv}{dx} \quad \{\text{product rule}\}$$

$$\text{Comparing with the differential equation, } v + x \frac{dv}{dx} = \frac{v}{2} - \frac{1}{2v}$$

$$\therefore x \frac{dv}{dx} = -\frac{v}{2} - \frac{1}{2v}$$

$$\therefore x \frac{dv}{dx} = -\frac{v^2 + 1}{2v}$$

$$\therefore -\frac{2v}{v^2 + 1} \frac{dv}{dx} = \frac{1}{x}$$

$$\therefore -\int \frac{2v}{v^2 + 1} dv = \int \frac{1}{x} dx$$

$$\therefore -\ln |v^2 + 1| = \ln |x| + c$$

$$\therefore \ln \left| \frac{1}{v^2 + 1} \right| = \ln |Ax| \quad \text{where } \ln |A| = c$$

$$\therefore \frac{1}{v^2 + 1} = Ax$$

$$\therefore v^2 + 1 = \frac{A}{x}$$

$$\therefore v^2 = \frac{A}{x} - 1$$

$$\therefore \frac{y^2}{x^2} = \frac{A}{x} - 1$$

$$\therefore y^2 = Ax - x^2$$

$$\therefore y = \pm \sqrt{Ax - x^2}, \text{ where } A \text{ is a constant.}$$

$$\begin{aligned}
 4 \quad \frac{dy}{dx} &= \frac{x+y}{x-y} \times \frac{\frac{1}{x}}{\frac{1}{x}} \\
 &= \frac{1 + \frac{y}{x}}{1 - \frac{y}{x}} \quad \text{so the differential equation is homogeneous.}
 \end{aligned}$$

$$\text{Let } y = vx, \text{ so } \frac{dy}{dx} = v + x \frac{dv}{dx} \quad \{\text{product rule}\}$$

$$\begin{aligned}
 \text{Comparing with the differential equation, } v + x \frac{dv}{dx} &= \frac{1+v}{1-v} \\
 \therefore x \frac{dv}{dx} &= \frac{1+v}{1-v} - v \\
 \therefore x \frac{dv}{dx} &= \frac{1+v-v(1-v)}{1-v} \\
 \therefore x \frac{dv}{dx} &= \frac{v^2+1}{1-v} \\
 \therefore \frac{1-v}{v^2+1} \frac{dv}{dx} &= \frac{1}{x} \\
 \therefore \int \left( \frac{1}{v^2+1} - \frac{v}{v^2+1} \right) dv &= \int \frac{1}{x} dx \\
 \therefore \arctan v - \frac{1}{2} \ln |v^2+1| &= \ln |x| + c
 \end{aligned}$$

When  $x = 1$  and  $y = 1$ ,  $v = 1$ .

$$\begin{aligned}
 \text{So, } \arctan 1 - \frac{1}{2} \ln 2 &= \ln 1 + c \\
 \therefore c &= \frac{\pi}{4} - \ln \sqrt{2}
 \end{aligned}$$

When  $y = 0$ ,  $v = 0$ .

$$\begin{aligned}
 \text{So, } \arctan 0 - \frac{1}{2} \ln 1 &= \ln |x| + \frac{\pi}{4} - \ln \sqrt{2} \\
 \therefore \ln |x| &= \ln \sqrt{2} - \frac{\pi}{4} \\
 \therefore x &= \pm e^{\ln \sqrt{2} - \frac{\pi}{4}} \\
 \therefore x &= \pm \sqrt{2} e^{-\frac{\pi}{4}}
 \end{aligned}$$

$$5 \quad a \quad \frac{dy}{dx} = \frac{y}{x} + f\left(\frac{y}{x}\right) g(x)$$

$$\text{Let } y = vx, \text{ so } \frac{dy}{dx} = v + x \frac{dv}{dx} \quad \{\text{product rule}\}$$

$$\begin{aligned}
 \text{Comparing with the differential equation, } v + x \frac{dv}{dx} &= v + f(v) g(x) \\
 \therefore x \frac{dv}{dx} &= f(v) g(x) \\
 \therefore \frac{dv}{dx} &= f(v) \frac{g(x)}{x} \quad \text{which is separable.}
 \end{aligned}$$

Thus, the substitution  $y = vx$  reduces the differential equation to a separable differential equation.



$$\mathbf{b} \quad x \frac{dy}{dx} = y + e^{\frac{y}{x}}$$

$$\therefore \frac{dy}{dx} = \frac{y}{x} + e^{\frac{y}{x}} \left( \frac{1}{x} \right)$$

$$\therefore \frac{dy}{dx} = \frac{y}{x} + f\left(\frac{y}{x}\right)g(x) \quad \text{where } f\left(\frac{y}{x}\right) = e^{\frac{y}{x}} \text{ and } g(x) = \frac{1}{x}$$

$$\text{Let } y = vx, \text{ so } \frac{dv}{dx} = f(v) \frac{g(x)}{x} \quad \{\text{using } \mathbf{a}\}$$

$$\therefore \frac{dv}{dx} = e^v \left( \frac{1}{x^2} \right)$$

$$\therefore \frac{1}{e^v} \frac{dv}{dx} = \frac{1}{x^2}$$

$$\therefore \int e^{-v} dv = \int x^{-2} dx$$

$$\therefore -e^{-v} = -x^{-1} + c$$

$$\therefore e^{-v} = x^{-1} + c$$

$$\therefore -v = \ln\left(\frac{1}{x} + c\right)$$

$$\therefore -\frac{y}{x} = \ln\left(\frac{1}{x} + c\right)$$

$$\therefore y = -x \ln\left(\frac{1}{x} + c\right)$$

## EXERCISE 25G

$$\mathbf{1} \quad \mathbf{a} \quad \frac{dy}{dx} + 4y = 12$$

The integrating factor is  $I(x) = e^{\int 4 dx} = e^{4x}$ .

Multiplying both sides of the differential equation by  $e^{4x}$  gives

$$e^{4x} \frac{dy}{dx} + 4e^{4x}y = 12e^{4x}$$

$$\therefore \frac{d}{dx}(ye^{4x}) = 12e^{4x}$$

$$\therefore ye^{4x} = \int 12e^{4x} dx$$

$$\therefore ye^{4x} = 3e^{4x} + c$$

$$\therefore y = 3 + ce^{-4x}$$

**b**  $\frac{dy}{dx} - 3y = e^x$

$$\therefore \frac{dy}{dx} + (-3)y = e^x$$

The integrating factor is  $I(x) = e^{\int -3 dx} = e^{-3x}$ .

Multiplying both sides of the differential equation by  $e^{-3x}$  gives

$$e^{-3x} \frac{dy}{dx} - 3e^{-3x}y = e^{-2x}$$

$$\therefore \frac{d}{dx}(ye^{-3x}) = e^{-2x}$$

$$\therefore ye^{-3x} = \int e^{-2x} dx$$

$$\therefore ye^{-3x} = -\frac{1}{2}e^{-2x} + c$$

$$\therefore y = -\frac{1}{2}e^x + ce^{3x}$$

But  $y(0) = 2$ , so  $2 = -\frac{1}{2} + c$

$$\therefore c = \frac{5}{2}$$

The particular solution is  $y = -\frac{1}{2}e^x + \frac{5}{2}e^{3x}$ .

**2**  $\cos x \frac{dy}{dx} = y \sin x + \sin 2x$

$$\therefore \cos x \frac{dy}{dx} = y \sin x + 2 \sin x \cos x$$

$$\therefore \frac{dy}{dx} = y \tan x + 2 \sin x$$

$$\therefore \frac{dy}{dx} + (-\tan x)y = 2 \sin x$$

The integrating factor is  $I(x) = e^{\int -\tan x dx} = e^{\ln(\cos x)} = \cos x$ .

Multiplying both sides of the differential equation by  $\cos x$  gives

$$\cos x \frac{dy}{dx} + (-\tan x \cos x)y = 2 \sin x \cos x$$

$$\therefore \frac{d}{dx}(y \cos x) = \sin 2x$$

$$\therefore y \cos x = \int \sin 2x dx$$

$$\therefore y \cos x = -\frac{1}{2} \cos 2x + c$$

$$\therefore y = \frac{c - \cos 2x}{2 \cos x}$$

But  $y(0) = 1$ , so  $1 = \frac{c - 1}{2}$

$$\therefore c = 3$$

The particular solution is  $y = \frac{3 - \cos 2x}{2 \cos x}$ .

**3 a**  $\frac{dy}{dx} + y = x + e^x$

The integrating factor is  $I(x) = e^{\int 1 \, dx} = e^x$ .

Multiplying both sides of the differential equation by  $e^x$  gives

$$e^x \frac{dy}{dx} + e^x y = xe^x + e^{2x}$$

$$\therefore \frac{d}{dx}(ye^x) = xe^x + e^{2x}$$

$$\therefore ye^x = \int (xe^x + e^{2x}) \, dx$$

$$\therefore ye^x = \int xe^x \, dx + \int e^{2x} \, dx$$

$$\therefore ye^x = xe^x - \int e^x \, dx + \int e^{2x} \, dx \quad \begin{cases} u = x & v' = e^x \\ u' = 1 & v = e^x \end{cases}$$

$$\therefore ye^x = xe^x - e^x + \frac{1}{2}e^{2x} + c$$

$$\therefore y = x - 1 + \frac{1}{2}e^x + ce^{-x}$$

But  $y(1) = 1$ , so  $1 = 1 - 1 + \frac{1}{2}e + ce^{-1}$

$$\therefore ce^{-1} = 1 - \frac{1}{2}e$$

$$\therefore c = e - \frac{1}{2}e^2$$

The particular solution is  $y = x - 1 + \frac{1}{2}e^x + e^{1-x} - \frac{1}{2}e^{2-x}$ .

**b**  $x \frac{dy}{dx} + y = x \cos x$

$$\therefore \frac{dy}{dx} + \left(\frac{1}{x}\right)y = \cos x$$

The integrating factor is  $I(x) = e^{\int \frac{1}{x} \, dx} = e^{\ln x} = x$ .

Multiplying both sides of the differential equation by  $x$  gives

$$x \frac{dy}{dx} + y = x \cos x$$

$$\therefore \frac{d}{dx}(yx) = x \cos x$$

$$\therefore yx = \int x \cos x \, dx$$

$$\therefore yx = x \sin x - \int \sin x \, dx \quad \begin{cases} u = x & v' = \cos x \\ u' = 1 & v = \sin x \end{cases}$$

$$\therefore yx = x \sin x + \cos x + c$$

$$\therefore y = \sin x + \frac{\cos x}{x} + \frac{c}{x}$$

$$\text{c} \quad (x+1)y + x \frac{dy}{dx} = x - x^2$$

$$\therefore \left(\frac{x+1}{x}\right)y + \frac{dy}{dx} = 1 - x$$

$$\therefore \frac{dy}{dx} + \left(1 + \frac{1}{x}\right)y = 1 - x$$

The integrating factor is  $I(x) = e^{\int (1+\frac{1}{x}) dx} = e^{x+\ln x} = xe^x$ .

Multiplying both sides of the differential equation by  $xe^x$  gives

$$xe^x \frac{dy}{dx} + \left(1 + \frac{1}{x}\right)xe^xy = xe^x - x^2e^x$$

$$\therefore \frac{d}{dx}(yxe^x) = xe^x - x^2e^x$$

$$\therefore yxe^x = \int (xe^x - x^2e^x) dx$$

$$\therefore yxe^x = \int xe^x dx - \int x^2e^x dx$$

$$\therefore yxe^x = \int xe^x dx - \left(x^2e^x - \int 2xe^x dx\right) \leftarrow \begin{cases} u = x^2 & v' = e^x \\ u' = 2x & v = e^x \end{cases}$$

$$\therefore yxe^x = 3 \int xe^x dx - x^2e^x$$

$$\therefore yxe^x = 3 \left(xe^x - \int e^x dx\right) - x^2e^x \leftarrow \begin{cases} u = x & v' = e^x \\ u' = 1 & v = e^x \end{cases}$$

$$\therefore yxe^x = 3(xe^x - e^x) - x^2e^x + c$$

$$\therefore yxe^x = 3xe^x - 3e^x - x^2e^x + c$$

$$\therefore y = 3 - \frac{3}{x} - x + \frac{c}{xe^x}$$

## EXERCISE 25H

$$1 \quad \frac{dy}{dx} = \frac{e^{-y}}{3} - 1, \quad y(0) = 0$$

$$\text{a} \quad \frac{dy}{dx} = \frac{1}{3}e^{-y} - 1$$

$$\therefore \frac{d^2y}{dx^2} = -\frac{1}{3}e^{-y} \frac{dy}{dx}$$

$$\therefore \frac{d^3y}{dx^3} = \frac{1}{3}e^{-y} \left(\frac{dy}{dx}\right)^2 - \frac{1}{3}e^{-y} \frac{d^2y}{dx^2}$$

Now  $y(0) = 0$ , so at  $(0, 0)$ :  $\frac{dy}{dx} = \frac{1}{3} - 1 = -\frac{2}{3}$

$$\frac{d^2y}{dx^2} = -\frac{1}{3} \left(-\frac{2}{3}\right) = \frac{2}{9}$$

$$\frac{d^3y}{dx^3} = \frac{1}{3} \left(-\frac{2}{3}\right)^2 - \frac{1}{3} \left(\frac{2}{9}\right) = \frac{2}{27}$$

$$\therefore y \text{ has Maclaurin polynomial } -\frac{2}{3}x + \frac{\left(\frac{2}{9}\right)x^2}{2!} + \frac{\left(\frac{2}{27}\right)x^3}{3!} + \dots$$

$$\text{which is } -\frac{2}{3}x + \frac{1}{9}x^2 + \frac{1}{81}x^3 + \dots$$



**b** Suppose  $y = \ln\left(\frac{1+2e^{-x}}{3}\right)$

$$\therefore e^y = \frac{1}{3} + \frac{2}{3}e^{-x} \quad \dots (*)$$

$$\therefore e^y \frac{dy}{dx} = -\frac{2}{3}e^{-x} \quad \{\text{differentiating both sides by } x\}$$

$$\therefore e^y \frac{dy}{dx} = \frac{1}{3} - e^y \quad \{\text{from } (*)\}$$

$$\therefore \frac{dy}{dx} = \frac{e^{-y}}{3} - 1 \quad \checkmark$$

$$\text{When } x = 0, \quad y = \ln\left(\frac{1+2}{3}\right) = \ln 1 = 0 \quad \checkmark$$

$$\therefore y = \ln\left(\frac{1+2e^{-x}}{3}\right) \text{ is a particular solution.}$$

**2**  $\frac{dy}{dx} = 2x - 2 + \frac{y}{x-1}, \quad \frac{dy}{dx} = 1 \text{ when } x = 0$

**a**  $\frac{dy}{dx} = 2x - 2 + (x-1)^{-1}y$

$$\therefore \frac{d^2y}{dx^2} = 2 - (x-1)^{-2}y + (x-1)^{-1} \frac{dy}{dx}$$

$$\text{When } x = 0, \quad \frac{dy}{dx} = 1$$

$$\therefore 1 = -2 - y$$

$$\therefore y = -3$$

$$\text{So, at } (0, -3), \quad \frac{d^2y}{dx^2} = 2 - (-3) - 1 = 4$$

$$\therefore y \text{ has Maclaurin polynomial } -3 + x + \frac{4x^2}{2!} + \dots$$

$$\text{which is } -3 + x + 2x^2 + \dots$$

**b**

$$\frac{dy}{dx} = 2x - 2 + \frac{y}{x-1}$$

$$\therefore \frac{dy}{dx} - \frac{y}{x-1} = 2x - 2$$

$$\therefore \frac{dy}{dx} + \left(\frac{-1}{x-1}\right)y = 2x - 2$$

The integrating factor is  $I(x) = e^{\int \frac{-1}{x-1} dx} = e^{-\ln(x-1)} = \frac{1}{x-1}$ .

Multiplying both sides of the differential equation by  $\frac{1}{x-1}$  gives

$$\left(\frac{1}{x-1}\right) \frac{dy}{dx} - \frac{y}{(x-1)^2} = \frac{2x-2}{x-1}$$

$$\therefore \frac{d}{dx} \left( y \left( \frac{1}{x-1} \right) \right) = \frac{2(x-1)}{x-1}$$

$$\therefore y \left( \frac{1}{x-1} \right) = \int 2 dx$$

$$\therefore y \left( \frac{1}{x-1} \right) = 2x + c$$

$$\therefore y = (2x + c)(x - 1)$$

$$\therefore y = 2x^2 + (c - 2)x - c$$

$$\text{and so } \frac{dy}{dx} = 4x + c - 2$$

When  $x = 0$ , so  $\frac{dy}{dx} = 1$

$$\therefore 1 = c - 2$$

$$\therefore c = 3$$

The particular solution is  $y = 2x^2 + x - 3$ .

$$\mathbf{3} \quad \cos x \frac{dy}{dx} + y \sin x = 1, \quad y(0) = 2$$

$$\mathbf{a} \quad \cos x \frac{dy}{dx} + y \sin x = 1$$

$$\therefore \cos x \frac{dy}{dx} = 1 - y \sin x$$

$$\therefore \frac{dy}{dx} = \sec x - y \tan x$$

$$\therefore \frac{d^2y}{dx^2} = \tan x \sec x - \frac{dy}{dx} \tan x - y \sec^2 x$$

Now  $y(0) = 2$ , so at  $(0, 2)$ :  $\frac{dy}{dx} = \sec(0) - (2) \tan(0) = 1$

$$\frac{d^2y}{dx^2} = \tan(0) \sec(0) - (1) \tan(0) - (2) \sec^2(0) = -2$$

$\therefore y$  has Maclaurin polynomial  $2 + x - \frac{2x^2}{2!} + \dots$

which is  $2 + x - x^2 + \dots$

**b**  $\cos x \frac{dy}{dx} + y \sin x = 1$

$$\therefore \frac{dy}{dx} + (\tan x)y = \sec x$$

The integrating factor is  $I(x) = e^{\int \tan x \, dx} = e^{-\ln(\cos x)} = \sec x$ .

Multiplying both sides of the differential equation by  $\sec x$  gives

$$\sec x \frac{dy}{dx} + (\sec x \tan x)y = \sec^2 x$$

$$\therefore \frac{d}{dx}(y \sec x) = \sec^2 x$$

$$\therefore y \sec x = \int \sec^2 x \, dx$$

$$\therefore \frac{y}{\cos x} = \tan x + c$$

$$\therefore y = \sin x + c \cos x$$

But  $y(0) = 2$ , so  $2 = c$

$\therefore$  the particular solution is  $y = \sin x + 2 \cos x$ .

**c** The Maclaurin expansion for:

- $\sin x$  is  $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$

- $\cos x$  is  $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$

Hence the Maclaurin expansion for  $y = \sin x + 2 \cos x$  is

$$\begin{aligned} \sin x + 2 \cos x &= \left( x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \right) + 2 \left( 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \right) \\ &= 2 + x - x^2 - \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{1}{120}x^5 - \dots \end{aligned}$$

which is consistent with our answer in **a**.

**4**  $y(x) = \sum_{n=0}^{\infty} \frac{p(p-1)\dots(p-n+1)x^n}{n!}, \quad |x| < 1$

**a** For  $|x| < 1$ ,  $y = \sum_{n=0}^{\infty} \frac{p(p-1)\dots(p-(n-1))x^n}{n!}$

$$= 1 + \sum_{n=1}^{\infty} \frac{p(p-1)\dots(p-(n-1))x^n}{n!}$$

$$\therefore \frac{dy}{dx} = \sum_{n=1}^{\infty} \frac{p(p-1)\dots(p-(n-1))x^{n-1}}{(n-1)!}$$

$$= \sum_{n=0}^{\infty} \frac{p(p-1)\dots(p-n)x^n}{n!} \quad \{\text{replace } n \text{ with } n+1\}$$

and so  $x \frac{dy}{dx} = \sum_{n=0}^{\infty} \frac{p(p-1)\dots(p-n)x^{n+1}}{n!}$

$$= \sum_{n=1}^{\infty} \frac{p(p-1)\dots(p-(n-1))x^n}{(n-1)!} \quad \{\text{replace } n \text{ with } n-1\}$$

$$\begin{aligned}
\text{Thus } (1+x) \frac{dy}{dx} &= \frac{dy}{dx} + x \frac{dy}{dx} \\
&= \sum_{n=0}^{\infty} \frac{p(p-1)\dots(p-n)x^n}{n!} + \sum_{n=1}^{\infty} \frac{p(p-1)\dots(p-(n-1))x^n}{(n-1)!} \\
&= p + \sum_{n=1}^{\infty} \frac{p(p-1)\dots(p-(n-1))(p-n)x^n}{n!} + \sum_{n=1}^{\infty} \frac{p(p-1)\dots(p-(n-1))nx^n}{n!} \\
&= p + \sum_{n=1}^{\infty} \frac{p(p-1)\dots(p-(n-1))(p-n+n)x^n}{n!} \\
&= p + \sum_{n=1}^{\infty} \frac{p^2(p-1)\dots(p-n+1)x^n}{n!} \\
&= p \left( 1 + \sum_{n=1}^{\infty} \frac{p(p-1)\dots(p-n+1)x^n}{n!} \right) \\
&= p \sum_{n=0}^{\infty} \frac{p(p-1)\dots(p-n+1)x^n}{n!} \\
&= py \quad \text{as required}
\end{aligned}$$

**b** Consider  $(1+x) \frac{dy}{dx} = py$

$$\therefore \frac{1}{y} \frac{dy}{dx} = \frac{p}{1+x}$$

$$\therefore \int \frac{1}{y} dy = \int \frac{p}{1+x} dx$$

$$\therefore \ln|y| = p \ln|1+x| + c$$

$$\therefore \ln|y| = p \ln(1+x) + c \quad \{\text{as } |x| < 1\}$$

$$\therefore \ln|y| = \ln((1+x)^p) + c$$

$$\therefore y = \pm e^c (1+x)^p$$

$$\therefore y = A(1+x)^p \quad \text{where } A = \pm e^c \text{ is a constant.}$$

**c**  $y(x)$  is a solution to  $(1+x) \frac{dy}{dx} = py$  for all  $|x| < 1$  {from **a**}

$$\therefore y(x) = A(1+x)^p \quad \text{for some constant } A \quad \{\text{from **b**}\}$$

But  $y(0) = 1$ , so  $1 = A(1+0)^p$

$$\therefore A = 1$$

Hence  $y(x) = (1+x)^p$  for all  $|x| < 1$ .

## ACTIVITY 1

## SIMPLE HARMONIC MOTION

$$\frac{d^2x}{dt^2} = -k^2x$$

**1 a** The function types we have studied where the second derivative of the function is proportional to the function itself are:

- exponential functions of the form  $x = Ce^{\lambda t}$  where  $C, \lambda$  are constants
- cosine functions of the form  $x = A \cos(kt + \alpha)$  where  $A, k, \alpha$  are constants
- sine functions of the form  $x = A \sin(kt + \alpha)$  where  $A, k, \alpha$  are constants.

**b** These function types in **a** are related by Euler's identity  $e^{i\theta} = \cos \theta + i \sin \theta$ .



- 2 a** Let  $x = Ce^{\lambda t}$  where  $C, \lambda$  are constants.

$$\therefore \frac{dx}{dt} = \lambda Ce^{\lambda t}$$

$$\therefore \frac{d^2x}{dt^2} = \lambda^2 Ce^{\lambda t}$$

Now  $x = Ce^{\lambda t}$  is a solution to the differential equation provided  $\frac{d^2x}{dt^2} = -k^2x$

$$\therefore \lambda^2 Ce^{\lambda t} = -k^2 Ce^{\lambda t}$$

$$\therefore \lambda^2 = -k^2$$

$$\therefore \lambda = \pm ik$$

- b** Let  $x = C_1 e^{ikt} + C_2 e^{-ikt}$  where  $C_1, C_2$  are constants.

$$\therefore \frac{dx}{dt} = ikC_1 e^{ikt} - ikC_2 e^{-ikt}$$

$$\begin{aligned} \therefore \frac{d^2x}{dt^2} &= (ik)^2 C_1 e^{ikt} + (ik)^2 C_2 e^{-ikt} \\ &= -k^2 C_1 e^{ikt} - k^2 C_2 e^{-ikt} \\ &= -k^2 (C_1 e^{ikt} + C_2 e^{-ikt}) \\ &= -k^2 x \end{aligned}$$

So,  $x = C_1 e^{ikt} + C_2 e^{-ikt}$  is a general solution to the differential equation.

- c** Using Euler's identity,  $e^{ikt} = \cos kt + i \sin kt$   
and  $e^{-ikt} = \cos(-kt) + i \sin(-kt)$   
 $= \cos kt - i \sin kt$

$$\begin{aligned} \text{Thus, } x &= C_1 e^{ikt} + C_2 e^{-ikt} \\ &= C_1 (\cos kt + i \sin kt) + C_2 (\cos kt - i \sin kt) \\ &= C_1 \cos kt + iC_1 \sin kt + C_2 \cos kt - iC_2 \sin kt \\ &= (C_1 + C_2) \cos kt + i(C_1 - C_2) \sin kt \\ &= B_1 \cos kt + B_2 \sin kt \quad \{B_1 = C_1 + C_2, \quad B_2 = i(C_1 - C_2)\} \end{aligned}$$

- d** Let  $B_1 = A \cos \alpha$  and  $B_2 = -A \sin \alpha$  where  $A, \alpha$  are constants.

$$\begin{aligned} \text{Thus, } x &= B_1 \cos kt + B_2 \sin kt \\ &= A \cos \alpha \cos kt - A \sin \alpha \sin kt \\ &= A(\cos \alpha \cos kt - \sin \alpha \sin kt) \\ &= A \cos(kt + \alpha) \end{aligned}$$

- 3 a**  $v = \frac{dx}{dt}$  and  $a = \frac{dv}{dt}$

$$\begin{aligned} \text{So, } \frac{d}{dx} \left( \frac{1}{2} v^2 \right) &= v \frac{dv}{dx} \\ &= v \frac{dv}{dt} \frac{dt}{dx} \\ &= \frac{dx}{dt} \frac{dt}{dx} a \\ &= \frac{dx}{dx} a \\ &= a \end{aligned}$$

**b**  $\frac{d}{dx} \left( \frac{1}{2} v^2 \right) = -k^2 x \quad \{\text{from a}\}$

$$\therefore \frac{1}{2} v^2 = \int -k^2 x \, dx$$

$$\therefore \frac{1}{2} v^2 = -\frac{1}{2} k^2 x^2 + c$$

$$\therefore v^2 = -k^2 x^2 + c$$

- c** **i** At the endpoints of its motion, the object is stationary.  
 $\therefore$  the velocity  $v = 0$  at these times.

**ii** When  $x = \pm A$ ,  $v = 0$   
 $\therefore 0 = -k^2 A^2 + c$   
 $\therefore c = k^2 A^2$

So,  $v^2 = -k^2 x^2 + k^2 A^2$   
 $= k^2 (A^2 - x^2)$   
 $\therefore v = \pm k \sqrt{A^2 - x^2}$

**d** **i**  $\frac{dx}{dt} = \pm k \sqrt{A^2 - x^2}$

$$\therefore \frac{1}{\sqrt{A^2 - x^2}} \frac{dx}{dt} = \pm k$$

$$\therefore \int \frac{1}{\sqrt{A^2 - x^2}} dx = \int \pm k \, dt$$

$$\therefore \int \frac{1}{\sqrt{A^2 - x^2}} dx = \pm kt + c$$

**ii** Now  $\int \frac{1}{\sqrt{A^2 - x^2}} dx = \int \frac{-A \sin \phi}{\sqrt{A^2 - A^2 \cos^2 \phi}} d\phi \quad \{x = A \cos \phi, \frac{dx}{d\phi} = -A \sin \phi\}$

$$= \int \frac{-A \sin \phi}{A \sqrt{1 - \cos^2 \phi}} d\phi$$

$$= \int \frac{-\sin \phi}{\sin \phi} d\phi$$

$$= \int -1 \, d\phi$$

$$= -\phi + d$$

$$= -\arccos\left(\frac{x}{A}\right) + d$$

**iii**  $\int \frac{1}{\sqrt{A^2 - x^2}} dx = \pm kt + c \quad \{\text{from i}\}$

$$\therefore -\arccos\left(\frac{x}{A}\right) + c_1 = \pm kt + c \quad \{\text{from ii}\}$$

$$\therefore -\arccos\left(\frac{x}{A}\right) = \pm kt + c - d$$

$$\therefore \arccos\left(\frac{x}{A}\right) = \mp kt \mp \alpha \quad \{\alpha = |c - d|\}$$

$$\therefore \frac{x}{A} = \cos(\mp(kt + \alpha))$$

$$\therefore x = A \cos(kt + \alpha) \quad \{\cos(-\theta) = \cos \theta\}$$

- 4** From **3**, the solution can be written in the form  $x = A \cos(kt + \beta)$   $\{\beta \text{ is a constant}\}$   
 which is equivalent to  $x = A \sin\left(kt + \beta + \frac{\pi}{2}\right)$   
 or  $x = A \sin(kt + \alpha)$   $\{\alpha = \beta + \frac{\pi}{2}\}$   
 $\therefore x = A \sin(kt + \alpha)$  is an equivalent solution to the differential equation.

- 5** Since  $x = A \cos(kt + \alpha)$ , the amplitude of the oscillation is  $A$ .

- 6** Since  $x = A \cos(kt + \alpha)$ , the period of the oscillation is  $\frac{2\pi}{k}$ .

- 7** The displacement of the pendulum after  $t$  seconds is  $x = A \cos(kt + \alpha)$  m.

The amplitude of the motion is 0.7 m  $\therefore A = 0.7$  {using **5**}

The period of the motion is 2 seconds  $\therefore \frac{2\pi}{k} = 2$  {using **6**}

$$\therefore k = \pi$$

So,  $x = 0.7 \cos(\pi t + \alpha)$ .

The pendulum is initially at its extreme positive position.

So, when  $t = 0$ ,  $x = 0.7$

$$\therefore 0.7 = 0.7 \cos \alpha$$

$$\therefore \cos \alpha = 1$$

$$\therefore \alpha = 0 + 2\pi m, \quad m \in \mathbb{Z}$$

Hence,  $x = 0.7 \cos \pi t$ .

## ACTIVITY 2

## LAPLACE TRANSFORMS

$$\mathcal{L}\{f(x)\} = \int_0^{\infty} e^{-sx} f(x) dx$$

$$\begin{aligned} \mathbf{1} \quad \mathbf{a} \quad \text{For } s > a, \quad \mathcal{L}\{e^{ax}\} &= \int_0^{\infty} e^{-sx} e^{ax} dx \\ &= \lim_{b \rightarrow \infty} \int_0^b e^{(a-s)x} dx \\ &= \lim_{b \rightarrow \infty} \left[ \frac{1}{a-s} e^{(a-s)x} \right]_0^b \\ &= \frac{1}{a-s} \lim_{b \rightarrow \infty} (e^{(a-s)b} - 1) \\ &= \frac{1}{a-s} (0 - 1) \quad \left\{ \lim_{b \rightarrow \infty} e^{(a-s)b} = 0 \text{ for } s > a \right\} \\ &= \frac{1}{s-a} \end{aligned}$$

**b** For  $s > 0$ ,  $\mathcal{L}\{x\} = \int_0^\infty e^{-sx} x \, dx$

$$\begin{aligned}
&= \lim_{b \rightarrow \infty} \int_0^b x e^{-sx} \, dx \\
&= \lim_{b \rightarrow \infty} \left( \left[ -\frac{x e^{-sx}}{s} \right]_0^b + \int_0^b \frac{1}{s} e^{-sx} \, dx \right) \quad \begin{cases} u = x & v' = e^{-sx} \\ u' = 1 & v = -\frac{1}{s} e^{-sx} \end{cases} \\
&= \lim_{b \rightarrow \infty} \left( -\frac{b e^{-sb}}{s} + \left[ -\frac{e^{-sx}}{s^2} \right]_0^b \right) \\
&= \lim_{b \rightarrow \infty} \left( -\frac{b}{s e^{sb}} - \frac{e^{-sb}}{s^2} + \frac{1}{s^2} \right) \\
&= \frac{1}{s^2} - \frac{1}{s} \lim_{b \rightarrow \infty} \frac{b}{e^{sb}} - \frac{1}{s^2} \lim_{b \rightarrow \infty} e^{-sb} \\
&= \frac{1}{s^2} - \frac{1}{s} \lim_{b \rightarrow \infty} \frac{b}{e^{sb}} \quad \left\{ \lim_{b \rightarrow \infty} e^{-sb} = 0 \text{ for } s > 0 \right\} \\
&= \frac{1}{s^2} - \frac{1}{s} \lim_{b \rightarrow \infty} \frac{1}{s e^{sb}} \quad \{\text{l'Hôpital's rule}\} \\
&= \frac{1}{s^2} - \frac{1}{s^2} \lim_{b \rightarrow \infty} e^{-sb} \\
&= \frac{1}{s^2} \quad \left\{ \lim_{b \rightarrow \infty} e^{-sb} = 0 \text{ for } s > 0 \right\}
\end{aligned}$$

**c** For  $s > 0$ ,  $\mathcal{L}\{\sin(ax)\} = \int_0^\infty e^{-sx} \sin(ax) \, dx$

$$= \lim_{b \rightarrow \infty} \int_0^b e^{-sx} \sin(ax) \, dx$$

Now  $\int_0^b e^{-sx} \sin(ax) \, dx$

$$\begin{aligned}
&= \left[ -\frac{1}{a} e^{-sx} \cos(ax) \right]_0^b - \frac{s}{a} \int_0^b e^{-sx} \cos(ax) \, dx \quad \leftarrow \begin{cases} u = e^{-sx} & v' = \sin(ax) \\ u' = -s e^{-sx} & v = -\frac{1}{a} \cos(ax) \end{cases} \\
&= -\frac{1}{a} e^{-sb} \cos(ab) + \frac{1}{a} - \frac{s}{a} \left( \left[ \frac{1}{a} e^{-sx} \sin(ax) \right]_0^b + \frac{s}{a} \int_0^b e^{-sx} \sin(ax) \, dx \right) \quad \leftarrow \begin{cases} u = e^{-sx} & v' = \cos(ax) \\ u' = -s e^{-sx} & v = \frac{1}{a} \sin(ax) \end{cases} \\
&= \frac{1}{a} - \frac{1}{a} e^{-sb} \cos(ab) - \frac{s}{a} \left( \frac{1}{a} e^{-sb} \sin(ab) \right) - \frac{s^2}{a^2} \int_0^b e^{-sx} \sin(ax) \, dx \\
\therefore \left( \frac{s^2}{a^2} + 1 \right) \int_0^b e^{-sx} \sin(ax) \, dx &= \frac{1}{a} - \frac{1}{a} e^{-sb} \cos(ab) - \frac{s}{a^2} e^{-sb} \sin(ab) \\
\therefore \left( \frac{s^2 + a^2}{a^2} \right) \int_0^b e^{-sx} \sin(ax) \, dx &= \frac{1}{a} - \frac{1}{a} e^{-sb} \cos(ab) - \frac{s}{a^2} e^{-sb} \sin(ab) \\
\therefore \int_0^b e^{-sx} \sin(ax) \, dx &= \left( \frac{a^2}{s^2 + a^2} \right) \left( \frac{1}{a} - \frac{1}{a} e^{-sb} \cos(ab) - \frac{s}{a^2} e^{-sb} \sin(ab) \right)
\end{aligned}$$



Since  $s > 0$ ,  $\lim_{b \rightarrow \infty} e^{-sb} = 0$

$$\therefore \lim_{b \rightarrow \infty} e^{-sb} \cos(ab) = 0 \quad \{-1 \leq \cos(ab) \leq 1\}$$

$$\text{and } \lim_{b \rightarrow \infty} e^{-sb} \sin(ab) = 0 \quad \{-1 \leq \sin(ab) \leq 1\}$$

$$\begin{aligned} \text{So, } \mathcal{L}\{\sin(ax)\} &= \lim_{b \rightarrow \infty} \int_0^b e^{-sx} \sin(ax) dx \\ &= \frac{a^2}{s^2 + a^2} \left( \frac{1}{a} - \frac{1}{a} \lim_{b \rightarrow \infty} e^{-sb} \cos(ab) - \frac{s}{a^2} \lim_{b \rightarrow \infty} e^{-sb} \sin(ab) \right) \\ &= \frac{a^2}{s^2 + a^2} \left( \frac{1}{a} - \frac{1}{a}(0) - \frac{s}{a^2}(0) \right) \\ &= \frac{a}{s^2 + a^2} \end{aligned}$$

**d** For  $s > 0$ ,

$$\begin{aligned} &\mathcal{L}\{\cos(ax)\} \\ &= \int_0^\infty e^{-sx} \cos(ax) dx \\ &= \lim_{b \rightarrow \infty} \int_0^b e^{-sx} \cos(ax) dx \\ &= \lim_{b \rightarrow \infty} \left( \left[ \frac{1}{a} e^{-sx} \sin(ax) \right]_0^b + \frac{s}{a} \int_0^b e^{-sx} \sin(ax) dx \right) \quad \begin{cases} u = e^{-sx} & v' = \cos(ax) \\ u' = -se^{-sx} & v = \frac{1}{a} \sin(ax) \end{cases} \\ &= \lim_{b \rightarrow \infty} \left( \frac{1}{a} e^{-sb} \sin(ab) \right) + \frac{s}{a} \lim_{b \rightarrow \infty} \int_0^b e^{-sx} \sin(ax) dx \\ &= \frac{1}{a} \lim_{b \rightarrow \infty} e^{-sb} \sin(ab) + \frac{s}{a} \int_0^\infty e^{-sx} \sin(ax) dx \\ &= \frac{1}{a}(0) + \frac{s}{a} \mathcal{L}\{\sin(ax)\} \quad \left\{ \lim_{b \rightarrow \infty} e^{-sb} \sin(ab) = 0 \text{ for } s > 0 \text{ from } \text{c} \right\} \\ &= \frac{s}{a} \left( \frac{a}{s^2 + a^2} \right) \quad \{\text{using } \text{c}\} \\ &= \frac{s}{s^2 + a^2} \end{aligned}$$

$$\begin{aligned} \text{2 a } \mathcal{L}\{f'(x)\} &= \int_0^\infty e^{-sx} f'(x) dx \\ &= \lim_{b \rightarrow \infty} \int_0^b e^{-sx} f'(x) dx \\ &= \lim_{b \rightarrow \infty} \left( \left[ e^{-sx} f(x) \right]_0^b + s \int_0^b e^{-sx} f(x) dx \right) \quad \begin{cases} u = e^{-sx} & v' = f'(x) \\ u' = -se^{-sx} & v = f(x) \end{cases} \\ &= \lim_{b \rightarrow \infty} (e^{-sb} f(b) - f(0)) + s \lim_{b \rightarrow \infty} \int_0^b e^{-sx} f(x) dx \\ &= \lim_{b \rightarrow \infty} e^{-sb} f(b) - f(0) + s \int_0^\infty e^{-sx} f(x) dx \\ &= s \mathcal{L}\{f(x)\} - f(0) \quad \left\{ \lim_{b \rightarrow \infty} e^{-sb} f(b) = 0 \right\} \end{aligned}$$

$$\begin{aligned}
\text{b } \mathcal{L}\{f''(x)\} &= \int_0^{\infty} e^{-sx} f''(x) dx \\
&= \lim_{b \rightarrow \infty} \int_0^b e^{-sx} f''(x) dx \\
&= \lim_{b \rightarrow \infty} \left( [e^{-sx} f'(x)]_0^b + s \int_0^b e^{-sx} f'(x) dx \right) \quad \begin{cases} u = e^{-sx} & v' = f''(x) \\ u' = -se^{-sx} & v = f'(x) \end{cases} \\
&= \lim_{b \rightarrow \infty} (e^{-sb} f'(b) - f'(0)) + s \lim_{b \rightarrow \infty} \int_0^b e^{-sx} f'(x) dx \\
&= \lim_{b \rightarrow \infty} e^{-sb} f'(b) - f'(0) + s \int_0^{\infty} e^{-sx} f'(x) dx \\
&= s \mathcal{L}\{f'(x)\} - f'(0) \quad \left\{ \lim_{b \rightarrow \infty} e^{-sb} f(b) = 0 \right\} \\
&= s(s \mathcal{L}\{f(x)\} - f(0)) - f'(0) \quad \{\text{using a}\} \\
&= s^2 \mathcal{L}\{f(x)\} - s f(0) - f'(0)
\end{aligned}$$

$$\text{3 } f''(x) + f(x) = x, \quad f(0) = 0, \quad f'(0) = 2$$

$$\begin{aligned}
\text{a } \mathcal{L}\{f''(x) + f(x)\} &= \mathcal{L}\{f''(x)\} + \mathcal{L}\{f(x)\} \quad \{\mathcal{L}\{g(x) + h(x)\} = \mathcal{L}\{g(x)\} + \mathcal{L}\{h(x)\}\} \\
&= s^2 \mathcal{L}\{f(x)\} - s f(0) - f'(0) + \mathcal{L}\{f(x)\} \quad \{\text{using 2 b}\} \\
&= (s^2 + 1) \mathcal{L}\{f(x)\} - 2
\end{aligned}$$

$$\text{But } \mathcal{L}\{f''(x) + f(x)\} = \mathcal{L}\{x\}$$

$$\therefore (s^2 + 1) \mathcal{L}\{f(x)\} - 2 = \frac{1}{s^2} \quad \{\text{using 1 b}\}$$

$$\therefore (s^2 + 1) \mathcal{L}\{f(x)\} = 2 + \frac{1}{s^2}$$

$$\therefore (s^2 + 1) \mathcal{L}\{f(x)\} = \frac{2s^2 + 1}{s^2}$$

$$\therefore (s^2 + 1) \mathcal{L}\{f(x)\} = \frac{s^2 + 1}{s^2} + \frac{s^2}{s^2}$$

$$\therefore \mathcal{L}\{f(x)\} = \frac{1}{s^2} + \frac{1}{s^2 + 1}$$

$$\text{b } \text{We have } \mathcal{L}\{x\} = \frac{1}{s^2} \quad \{\text{using 1 b}\}$$

$$\text{and } \mathcal{L}\{\sin x\} = \frac{1}{s^2 + 1} \quad \{\text{using 1 c with } a = 1\}$$

$$\begin{aligned}
\text{then } \mathcal{L}\{x + \sin x\} &= \mathcal{L}\{x\} + \mathcal{L}\{\sin x\} \quad \{\mathcal{L}\{g(x) + h(x)\} = \mathcal{L}\{g(x)\} + \mathcal{L}\{h(x)\}\} \\
&= \frac{1}{s^2} + \frac{1}{s^2 + 1} \\
&= \mathcal{L}\{f(x)\} \quad \{\text{from a}\}
\end{aligned}$$

So, the solution function  $f(x) = x + \sin x$ .

$$\text{Now } f'(x) = 1 + \cos x$$

$$\therefore f''(x) = -\sin x$$

Substituting into the original differential equation gives

$$f''(x) + f(x) = -\sin x + x + \sin x = x \quad \checkmark$$

**4**  $f''(t) = -k^2 f(t)$

**a**  $\mathcal{L}\{f''(t)\} = s^2 \mathcal{L}\{f(t)\} - s f(0) - f'(0)$  {using **2 b**}

But  $\mathcal{L}\{f''(t)\} = \mathcal{L}\{-k^2 f(t)\}$

$\therefore s^2 \mathcal{L}\{f(t)\} - s f(0) - f'(0) = -k^2 \mathcal{L}\{f(t)\}$  { $\mathcal{L}\{c g(t)\} = c \mathcal{L}\{g(t)\}$ }

$\therefore (s^2 + k^2) \mathcal{L}\{f(t)\} = s f(0) + f'(0)$

$\therefore \mathcal{L}\{f(t)\} = \frac{s}{s^2 + k^2} f(0) + \frac{1}{s^2 + k^2} f'(0)$

**b** We have  $\mathcal{L}\{\cos kt\} = \frac{s}{s^2 + k^2}$  {using **1 d** with  $a = k$ }

and  $\mathcal{L}\{\sin kt\} = \frac{k}{s^2 + k^2}$  {using **1 c** with  $a = k$ }

Then  $\mathcal{L}\{f(0) \cos kt + \frac{f'(0)}{k} \sin kt\}$

$$= \mathcal{L}\{f(0) \cos kt\} + \mathcal{L}\left\{\frac{f'(0)}{k} \sin kt\right\} \quad \{\mathcal{L}\{g(t) + h(t)\} = \mathcal{L}\{g(t)\} + \mathcal{L}\{h(t)\}\}$$

$$= f(0) \mathcal{L}\{\cos kt\} + \frac{f'(0)}{k} \mathcal{L}\{\sin kt\} \quad \{\mathcal{L}\{c g(t)\} = c \mathcal{L}\{g(t)\}\}$$

$$= \frac{s}{s^2 + k^2} f(0) + \frac{1}{s^2 + k^2} f'(0)$$

$$= \mathcal{L}\{f(t)\} \quad \{\text{from a}\}$$

So,  $f(t) = f(0) \cos kt + \frac{f'(0)}{k} \sin kt$  is the solution function.

$\therefore f(t) = B_1 \cos kt + B_2 \sin kt$  { $B_1 = f(0)$  and  $B_2 = \frac{f'(0)}{k}$ }

## REVIEW SET 25A

**1** If  $y = 4 \ln(x^2 + 3x) + 8$ , then  $\frac{dy}{dx} = \frac{4}{x^2 + 3x} (2x + 3)$

$$= \frac{4(2x + 3)}{x^2 + 3x}$$

$$= \frac{8x + 12}{x^2 + 3x} \quad \text{as required.}$$

**2**  $y = ax + b$  is a solution to  $\frac{dy}{dx} = 4x - 2y$

$$\therefore a = 4x - 2(ax + b) \quad \text{for all } x$$

$$\therefore a = (4 - 2a)x - 2b \quad \text{for all } x$$

$$\therefore 4 - 2a = 0 \quad \text{and} \quad -2b = a$$

$$\therefore 2a = 4 \quad \text{and} \quad b = -\frac{1}{2}a$$

$$\therefore a = 2 \quad \text{and} \quad b = -1$$

3  $\frac{dy}{dx} = x - 2y$  with initial point (1, 2)

Iteration	$x_{i-1}$	$y_{i-1}$	$\frac{dy}{dx}$	$x_i$	$y_i$
1	1	2	-3	1.1	1.7
2	1.1	1.7	-2.3	1.2	1.47
3	1.2	1.47	-1.74	1.3	1.296
4	1.3	1.296	-1.292	1.4	1.1668
5	1.4	1.1668	-0.9336	1.5	1.073 44
6	1.5	1.073 44	-0.646 88	1.6	1.008 752

$\therefore y(1.6) \approx 1.0088$

4 a  $\frac{dy}{dx} = \cos 2x - \sin^2 x$

$$\therefore y = \int (\cos 2x - \sin^2 x) dx$$

$$\therefore y = \int \left( \cos 2x - \left( \frac{1}{2} - \frac{1}{2} \cos 2x \right) \right) dx$$

$$\therefore y = \int \left( \frac{3}{2} \cos 2x - \frac{1}{2} \right) dx$$

$$\therefore y = \frac{3}{4} \sin 2x - \frac{1}{2}x + c$$

c  $\frac{dy}{dx} = \frac{1}{2x+1}$

$$\therefore y = \int \frac{1}{2x+1} dx$$

$$= \frac{1}{2} \ln |2x+1| + c$$

Now  $y(0) = 2$

$$\therefore 2 = \frac{1}{2} \ln 1 + c$$

$$\therefore c = 2$$

So, the solution is  $y = \frac{1}{2} \ln |2x+1| + 2$ .

b  $\frac{dy}{dx} = 3 - e^{-2x}$

$$\therefore y = \int (3 - e^{-2x}) dx$$

$$= 3x + \frac{1}{2}e^{-2x} + c$$

d  $\frac{dy}{dt} - t = te^{t^2}$

$$\therefore \frac{dy}{dt} = t + te^{t^2}$$

$$\therefore y = \int t(1 + e^{t^2}) dt$$

$$\therefore y = \int \frac{1}{2}(1 + e^u) \frac{du}{dt} dt$$

$$\{u = t^2, \quad \frac{du}{dt} = 2t\}$$

$$\therefore y = \frac{1}{2} \int (1 + e^u) du$$

$$\therefore y = \frac{1}{2}(u + e^u) + c$$

$$\therefore y = \frac{1}{2}(t^2 + e^{t^2}) + c$$

Now  $y(1) = 2e$

$$\therefore 2e = \frac{1}{2}(1 + e) + c$$

$$\therefore c = \frac{3}{2}e - \frac{1}{2}$$

So, the solution is

$$y = \frac{1}{2}(t^2 + e^{t^2}) + \frac{3}{2}e - \frac{1}{2}$$

$$\therefore y = \frac{1}{2}(t^2 + e^{t^2} + 3e - 1)$$



- 5 a** The marginal profit is  $P'(x) = 20 - \frac{x}{4}$  pounds per vase

$$\begin{aligned}\therefore P(x) &= \int \left(20 - \frac{x}{4}\right) dx \\ &= 20x - \frac{1}{8}x^2 + c \text{ pounds}\end{aligned}$$

But  $P(0) = -250$  pounds, so  $c = -250$

$$\therefore P(x) = 20x - \frac{1}{8}x^2 - 250 \text{ pounds}$$

**b**  $P''(x) = -\frac{1}{4} < 0$

$\therefore$  the maximum profit occurs when  $P'(x) = 0$

$$\therefore 0 = 20 - \frac{x}{4}$$

$$\therefore \frac{x}{4} = 20$$

$$\therefore x = 80$$

$$\begin{aligned}\text{Now } P(80) &= 20(80) - \frac{1}{8}(80)^2 - 250 \\ &= 1600 - 800 - 250 \\ &= 550\end{aligned}$$

$\therefore$  the maximum profit is £550 per day when 80 vases are made.

- c** In order for a profit to be made,  $P(x)$  must be greater than 0

$$\therefore 20x - \frac{1}{8}x^2 - 250 > 0$$

Using technology, the  $x$ -intercepts of  $P(x)$  are  $x_1 \approx 13.7$  and  $x_2 \approx 146.3$ .

Since we cannot produce part of a vase, a profit is made when between 14 and 146 vases inclusive are produced per day.

**6 a**  $S'(t) = \frac{4000e^{-0.05t}}{(1 + 4e^{-0.05t})^2}$

**i** At noon,  $t = 0$ , so  $S'(0) = \frac{4000e^0}{(1 + 4e^0)^2}$   
 $= 160$

At noon, spectators were entering the stadium at a rate of 160 spectators per minute.

**ii** At 12:30 pm,  $t = 30$ , so  $S'(30) = \frac{4000e^{-1.5}}{(1 + 4e^{-1.5})^2}$   
 $\approx 249$

At 12:30 pm, spectators were entering the stadium at a rate of about 249 spectators per minute.

$$\begin{aligned}
 \text{b } \frac{d}{dt} \left( \frac{1}{1 + 4e^{-0.05t}} \right) &= \frac{d}{dt} \left( (1 + 4e^{-0.05t})^{-1} \right) \\
 &= -(1 + 4e^{-0.05t})^{-2} (-0.2e^{-0.05t}) \quad \{\text{chain rule}\} \\
 &= \frac{0.2e^{-0.05t}}{(1 + 4e^{-0.05t})^2}
 \end{aligned}$$

$$\begin{aligned}
 \text{Hence } \int S'(t) dt &= \int \frac{4000e^{-0.05t}}{(1 + 4e^{-0.05t})^2} dt \\
 &= 20\,000 \int \frac{0.2e^{-0.05t}}{(1 + 4e^{-0.05t})^2} dt \\
 &= \frac{20\,000}{1 + 4e^{-0.05t}} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{c } \int_0^{60} S'(t) dt &= \left[ \frac{20\,000}{1 + 4e^{-0.05t}} \right]_0^{60} \\
 &= \frac{20\,000}{1 + 4e^{-3}} - \frac{20\,000}{1 + 4e^0} \\
 &\approx 12\,700
 \end{aligned}$$

60 minutes after noon is 1 pm. So, about 12 700 spectators in total entered the stadium between noon and 1 pm.

$$\text{d } S(t) = \int S'(t) dt = \frac{20\,000}{1 + 4e^{-0.05t}} + c \quad \{\text{from b}\}$$

$$\text{Now } S(0) = 4000$$

$$\therefore \frac{20\,000}{1 + 4e^0} + c = 4000$$

$$\therefore 4000 + c = 4000$$

$$\therefore c = 0$$

$$\therefore S(t) = \frac{20\,000}{1 + 4e^{-0.05t}}$$

$$\text{At 1:40 pm, } t = 100, \text{ so } S(100) = \frac{20\,000}{1 + 4e^{-0.05 \times 100}} \approx 19\,500$$

There were about 19 500 spectators in the stadium at 1:40 pm.

7 a

$$\begin{aligned}
 \frac{dy}{dx} &= 5x^2y \\
 \therefore \frac{1}{y} \frac{dy}{dx} &= 5x^2 \\
 \therefore \int \frac{1}{y} \frac{dy}{dx} dx &= \int 5x^2 dx \\
 \therefore \int \frac{1}{y} dy &= \int 5x^2 dx \\
 \therefore \ln|y| &= \frac{5}{3}x^3 + c \\
 \therefore y &= \pm e^{\frac{5}{3}x^3 + c} \\
 \therefore y &= Ae^{\frac{5}{3}x^3} \quad \{A = \pm e^c\}
 \end{aligned}$$

b

$$\begin{aligned}
 \frac{dy}{dx} &= 2xy^2 - y^2 = y^2(2x - 1) \\
 \therefore \frac{1}{y^2} \frac{dy}{dx} &= 2x - 1 \\
 \therefore \int \frac{1}{y^2} \frac{dy}{dx} dx &= \int (2x - 1) dx \\
 \therefore \int y^{-2} dy &= \int (2x - 1) dx \\
 \therefore -y^{-1} &= x^2 - x + c \\
 \therefore y &= -\frac{1}{x^2 - x + c}
 \end{aligned}$$

$$\begin{aligned}
\text{c} \quad & \frac{dx}{dt} = \frac{3 \sin 2t}{4\sqrt{x}} \\
& \therefore \sqrt{x} \frac{dx}{dt} = \frac{3}{4} \sin 2t \\
& \therefore \int \sqrt{x} \frac{dx}{dt} dt = \int \frac{3}{4} \sin 2t dt \\
& \therefore \int x^{\frac{1}{2}} dx = \int \frac{3}{4} \sin 2t dt \\
& \therefore \frac{2}{3} x^{\frac{3}{2}} = -\frac{3}{8} \cos 2t + c \\
& \therefore x^{\frac{3}{2}} = -\frac{9}{16} \cos 2t + c \\
& \therefore x = \left(-\frac{9}{16} \cos 2t + c\right)^{\frac{2}{3}}
\end{aligned}$$

$$\begin{aligned}
8 \quad \text{a} \quad & \text{We are given that } \frac{dM}{dt} \propto M \\
& \therefore \frac{dM}{dt} = kM \quad \text{for some constant } k.
\end{aligned}$$

The initial mass is  $M_0$ , so  $M(0) = M_0$ .

$$\begin{aligned}
\text{b} \quad & \frac{dM}{dt} = kM \\
& \therefore \int \frac{1}{M} dM = \int k dt \\
& \therefore \ln |M| = kt + c \\
& \therefore M = \pm e^{kt+c} \\
& \therefore M = Ae^{kt} \quad \{A = \pm e^c\}
\end{aligned}$$

When  $t = 0$ ,  $M = M_0$

$$\therefore M_0 = Ae^0 \quad \text{and so } A = M_0$$

$$\therefore M = M_0 e^{kt}$$

When  $t = 30$ ,  $M = \frac{4}{5}M_0$

$$\therefore \frac{4}{5}M_0 = M_0 e^{30k}$$

$$\therefore \frac{4}{5} = e^{30k}$$

$$\therefore e^k = \left(\frac{4}{5}\right)^{\frac{1}{30}}$$

$$\therefore M = M_0 \left(\frac{4}{5}\right)^{\frac{t}{30}}$$

When  $M = \frac{1}{2}M_0$ ,  $\frac{1}{2}M_0 = M_0 \left(\frac{4}{5}\right)^{\frac{t}{30}}$

$$\therefore \left(\frac{4}{5}\right)^{\frac{t}{30}} = \frac{1}{2}$$

$$\therefore \frac{t}{30} \ln\left(\frac{4}{5}\right) = \ln\left(\frac{1}{2}\right)$$

$$\therefore t = \frac{30 \ln\left(\frac{1}{2}\right)}{\ln\left(\frac{4}{5}\right)} \approx 93.2$$

It will take about 93.2 days for the substance to decay to half its original mass.

$$9 \quad \frac{dy}{dx} = -\frac{3}{e^y}$$

$$\therefore \int e^y dy = \int -3 dx$$

$$\therefore e^y = -3x + c$$

$$\therefore y = \ln(-3x + c)$$

But  $y(0) = 1$ , so  $1 = \ln c$

$$\therefore c = e$$

The particular solution is  $y = \ln(e - 3x)$ .

$$10 \quad a \quad \frac{dC}{dt} = \frac{-VC}{K+C}$$

$$\therefore \frac{K+C}{C} \frac{dC}{dt} = -V$$

$$\therefore \int \frac{K+C}{C} \frac{dC}{dt} dt = \int -V dt$$

$$\therefore \int \left( \frac{K}{C} + 1 \right) dC = \int -V dt$$

$$\therefore K \ln|C| + C = -Vt + c$$

$$\therefore K \ln C + C = -Vt + c \quad \{\text{since } C \geq 0\}$$

When  $t = 0$ ,  $C = 0.1$

$$\therefore K \ln(0.1) + 0.1 = c$$

$$\therefore c = 0.1 - K \ln 10$$

So, we have  $K \ln C + C = -Vt + 0.1 - K \ln 10$

$$\therefore 0.0266 \ln C + C = (-2.66 \times 10^{-4})t + 0.1 - 0.0266 \ln 10$$

$$\therefore (2.66 \times 10^{-4})t = 0.1 - 0.0266 \ln 10 - 0.0266 \ln C - C$$

$$\therefore (2.66 \times 10^{-4})t = -0.0266(\ln 10 + \ln C) + 0.1 - C$$

$$\therefore (2.66 \times 10^{-4})t = -0.0266 \ln(10C) + 0.1 - C$$

$$\therefore t = \frac{(-2.66 \times 10^{-2}) \ln(10C) + 0.1 - C}{2.66 \times 10^{-4}}$$

$$\therefore t = -100 \ln(10C) + \frac{0.1 - C}{2.66 \times 10^{-4}}$$

**b** After 85% of the urea decomposes to ammonia and carbon dioxide, 15% of the urea remains.

$$\therefore 15\% \text{ of the urea} = 0.15 \times 0.1 = 0.015$$

When  $C = 0.015$ ,  $t = -100 \ln(10 \times 0.015) + \frac{0.1 - 0.015}{2.66 \times 10^{-4}} \quad \{\text{using a}\}$

$$\therefore t \approx 509$$

It will take about 509 seconds, or 8 minutes 29 seconds, for 85% of the urea to decompose into ammonia and carbon dioxide.



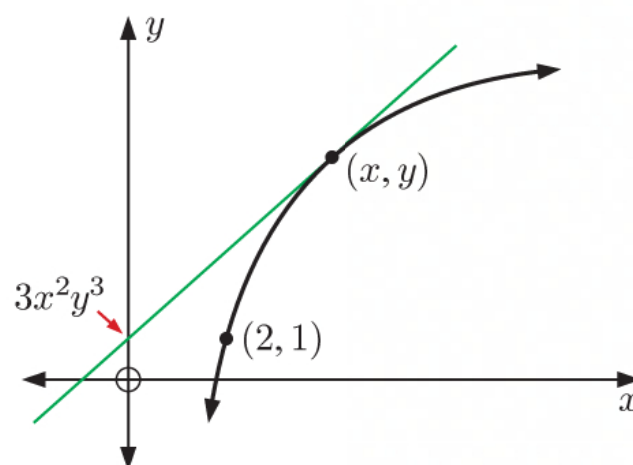
**11** The gradient of the tangent at  $(x, y)$  is  $\frac{dy}{dx}$ .

So, the equation of the tangent at  $(x, y)$  is

$$y = \frac{dy}{dx}(x - 0) + 3x^2y^3$$

$$\therefore x \frac{dy}{dx} = y - 3x^2y^3$$

$$\therefore \frac{dy}{dx} = \frac{y}{x} - 3x^4\left(\frac{y}{x}\right)^3$$



Let  $y = vx$ , so  $\frac{dy}{dx} = v + x \frac{dv}{dx}$  {product rule}

Comparing with the differential equation,  $v + x \frac{dv}{dx} = v - 3x^4v^3$

$$\therefore x \frac{dv}{dx} = -3x^4v^3$$

$$\therefore -\frac{1}{v^3} \frac{dv}{dx} = 3x^3$$

$$\therefore \int -\frac{1}{v^3} \frac{dv}{dx} dx = \int 3x^3 dx$$

$$\therefore \int -v^{-3} dv = \int 3x^3 dx$$

$$\therefore \frac{1}{2}v^{-2} = \frac{3}{4}x^4 + c$$

$$\therefore \frac{1}{v^2} = \frac{3}{2}x^4 + c$$

$$\therefore v^2 = \frac{2}{3x^4 + c}$$

$$\therefore \frac{y^2}{x^2} = \frac{2}{3x^4 + c}$$

$$\therefore y^2 = \frac{2x^2}{3x^4 + c}$$

Since the curve passes through  $(2, 1)$ ,  $1 = \frac{2(2)^2}{3(2)^4 + c}$

$$\therefore c + 48 = 8$$

$$\therefore c = -40$$

$\therefore$  the equation of the curve is  $y^2 = \frac{2x^2}{3x^4 - 40}$ .

**12 a**

$$\frac{dN}{dt} = N \left( 1 - \frac{N}{694} \right) = N \left( \frac{694 - N}{694} \right)$$

$$\therefore \frac{694}{N(694 - N)} \frac{dN}{dt} = 1$$

$$\therefore \int \frac{694}{N(694 - N)} \frac{dN}{dt} dt = \int 1 dt$$

$$\therefore \int \frac{694}{N(694 - N)} dN = \int 1 dt$$

$$\therefore \int \left( \frac{1}{N} + \frac{1}{694 - N} \right) dN = \int 1 dt$$

$$\therefore \ln |N| + \frac{1}{-1} \ln |694 - N| = t + c$$

$$\therefore \ln \left| \frac{N}{694 - N} \right| = t + c$$

$$\therefore \frac{N}{694 - N} = \pm e^{t+c}$$

$$\therefore \frac{694 - N}{N} = be^{-t} \quad \left\{ \text{letting } b = \pm \frac{1}{e^c} \right\}$$

Now when  $t = 0$ ,  $N = 1$

$$\therefore \frac{693}{1} = Ae^0$$

$$\therefore A = 693$$

So, we have  $\frac{694 - N}{N} = 693e^{-t}$

$$\therefore \frac{694}{N} - 1 = 693e^{-t}$$

$$\therefore \frac{694}{N} = 1 + 693e^{-t}$$

$$\therefore N = \frac{694}{1 + 693e^{-t}}$$

**b** When  $t = 4$ ,  $N = \frac{694}{1 + 693e^{-4}} \approx 51$

About 51 people will have had the virus after 4 weeks.

**c** When  $N = 500$ ,  $500 = \frac{694}{1 + 693e^{-t}}$

$$\therefore 1 + 693e^{-t} = \frac{694}{500} = 1.388$$

$$\therefore 693e^{-t} = 0.388$$

$$\therefore e^{-t} = \frac{0.388}{693}$$

$$\therefore -t = \ln \left( \frac{0.388}{693} \right)$$

$$\therefore t = -\ln \left( \frac{0.388}{693} \right) \approx 7.49$$

It will take about 7.49 weeks for the virus to infect 500 people.

**13**  $\frac{dy}{dx} = \frac{x}{y} + \frac{y}{x}$   
 $\therefore \frac{dy}{dx} = \frac{1}{\left(\frac{y}{x}\right)} + \frac{y}{x}$  so the differential equation is homogeneous.

Let  $y = vx$ , so  $\frac{dy}{dx} = v + x \frac{dv}{dx}$  {product rule}

Comparing with the differential equation,  $v + x \frac{dv}{dx} = \frac{1}{v} + v$

$$\therefore x \frac{dv}{dx} = \frac{1}{v}$$

$$\therefore v \frac{dv}{dx} = \frac{1}{x}$$

$$\therefore \int v \frac{dv}{dx} dx = \int \frac{1}{x} dx$$

$$\therefore \int v dv = \int \frac{1}{x} dx$$

$$\therefore \frac{1}{2}v^2 = \ln|x| + c$$

$$\therefore \frac{y^2}{2x^2} = \ln|x| + c$$

$$\therefore y^2 = 2x^2 (\ln|x| + c)$$

**14 a**  $\frac{dy}{dx} - \frac{y}{x} = \sqrt{x}$

$$\therefore \frac{dy}{dx} + \left(\frac{-1}{x}\right)y = \sqrt{x}$$

The integrating factor is  $I(x) = e^{\int -\frac{1}{x} dx} = e^{-\ln x} = \frac{1}{x}$ .

Multiplying both sides of the differential equation by  $\frac{1}{x}$  gives

$$\left(\frac{1}{x}\right) \frac{dy}{dx} - \left(\frac{1}{x^2}\right)y = \frac{\sqrt{x}}{x}$$

$$\therefore \frac{d}{dx} \left(y \left(\frac{1}{x}\right)\right) = x^{-\frac{1}{2}}$$

$$\therefore y \left(\frac{1}{x}\right) = \int x^{-\frac{1}{2}} dx$$

$$\therefore y \left(\frac{1}{x}\right) = 2x^{\frac{1}{2}} + c$$

$$\therefore y = 2x^{\frac{3}{2}} + cx$$

But  $y(4) = 0$ , so  $0 = 2(4)^{\frac{3}{2}} + c(4)$

$$\therefore 4c = -16$$

$$\therefore c = -4$$

The particular solution is  $y = 2x\sqrt{x} - 4x$ .

**b**

$$\frac{dy}{dx} = \cos x - y \cot x$$

$$\therefore \frac{dy}{dx} + (\cot x)y = \cos x$$

The integrating factor is  $I(x) = e^{\int \cot x \, dx} = e^{\int \frac{\cos x}{\sin x} \, dx} = e^{\ln(\sin x)} = \sin x$ .

Multiplying both sides of the differential equation by  $\sin x$  gives

$$\sin x \frac{dy}{dx} + (\sin x \cot x)y = \sin x \cos x$$

$$\therefore \frac{d}{dx}(y \sin x) = \sin x \cos x$$

$$\therefore y \sin x = \frac{1}{2} \int 2 \sin x \cos x \, dx$$

$$\therefore y \sin x = \frac{1}{2} \int \sin 2x \, dx$$

$$\therefore y \sin x = -\frac{1}{4} \cos 2x + c$$

But  $y\left(\frac{\pi}{2}\right) = 0$ , so  $0 = -\frac{1}{4}(-1) + c$

$$\therefore c = -\frac{1}{4}$$

The particular solution is  $y \sin x = -\frac{1}{4} \cos 2x - \frac{1}{4}$

$$\therefore y \sin x = -\frac{1}{2} \left( \frac{1}{2} + \frac{1}{2} \cos 2x \right)$$

$$\therefore y \sin x = -\frac{1}{2} \cos^2 x$$

$$\therefore y = -\frac{1}{2} \cot x \cos x$$



**15**  $\frac{dy}{dx} = 2xe^{-y}, \quad y(0) = 0$

**a**  $\frac{d^2y}{dx^2} = 2e^{-y} - 2xe^{-y} \frac{dy}{dx} = 2e^{-y} - \left(\frac{dy}{dx}\right)^2$

$$\therefore \frac{d^3y}{dx^3} = -2e^{-y} \frac{dy}{dx} - 2 \frac{dy}{dx} \frac{d^2y}{dx^2}$$

$$\therefore \frac{d^4y}{dx^4} = 2e^{-y} \left(\frac{dy}{dx}\right)^2 - 2e^{-y} \frac{d^2y}{dx^2} - 2 \left(\frac{d^2y}{dx^2}\right)^2 - 2 \frac{dy}{dx} \frac{d^3y}{dx^3}$$

$$\begin{aligned} \therefore \frac{d^5y}{dx^5} &= -2e^{-y} \left(\frac{dy}{dx}\right)^3 + 4e^{-y} \frac{dy}{dx} \frac{d^2y}{dx^2} + 2e^{-y} \frac{dy}{dx} \frac{d^2y}{dx^2} - 2e^{-y} \frac{d^3y}{dx^3} - 4 \frac{d^2y}{dx^2} \frac{d^3y}{dx^3} \\ &\quad - 2 \frac{d^2y}{dx^2} \frac{d^3y}{dx^3} - 2 \frac{dy}{dx} \frac{d^4y}{dx^4} \end{aligned}$$

$$= -2e^{-y} \left(\frac{dy}{dx}\right)^3 + 6e^{-y} \frac{dy}{dx} \frac{d^2y}{dx^2} - 2e^{-y} \frac{d^3y}{dx^3} - 6 \frac{d^2y}{dx^2} \frac{d^3y}{dx^3} - 2 \frac{dy}{dx} \frac{d^4y}{dx^4}$$

$$\begin{aligned} \therefore \frac{d^6y}{dx^6} &= 2e^{-y} \left(\frac{dy}{dx}\right)^4 - 6e^{-y} \left(\frac{dy}{dx}\right)^2 \frac{d^2y}{dx^2} - 6e^{-y} \left(\frac{dy}{dx}\right)^2 \frac{d^2y}{dx^2} + 6e^{-y} \left(\frac{d^2y}{dx^2}\right)^2 \\ &\quad + 6e^{-y} \frac{dy}{dx} \frac{d^3y}{dx^3} + 2e^{-y} \frac{dy}{dx} \frac{d^3y}{dx^3} - 2e^{-y} \frac{d^4y}{dx^4} - 6 \left(\frac{d^3y}{dx^3}\right)^2 \\ &\quad - 6 \frac{d^2y}{dx^2} \frac{d^4y}{dx^4} - 2 \frac{d^2y}{dx^2} \frac{d^4y}{dx^4} - 2 \frac{dy}{dx} \frac{d^5y}{dx^5} \\ &= 2e^{-y} \left(\frac{dy}{dx}\right)^4 - 12e^{-y} \left(\frac{dy}{dx}\right)^2 \frac{d^2y}{dx^2} + 6e^{-y} \left(\frac{d^2y}{dx^2}\right)^2 + 8e^{-y} \frac{dy}{dx} \frac{d^3y}{dx^3} - 2e^{-y} \frac{d^4y}{dx^4} \\ &\quad - 6 \left(\frac{d^3y}{dx^3}\right)^2 - 8 \frac{d^2y}{dx^2} \frac{d^4y}{dx^4} - 2 \frac{dy}{dx} \frac{d^5y}{dx^5} \end{aligned}$$

Now  $y(0) = 0$ , so at  $(0, 0)$ :  $\frac{dy}{dx} = 0$

$$\frac{d^2y}{dx^2} = 2 - 0 = 2$$

$$\frac{d^3y}{dx^3} = 0 - 0 = 0$$

$$\frac{d^4y}{dx^4} = 0 - 4 - 8 - 0 = -12$$

$$\frac{d^5y}{dx^5} = 0 + 0 - 0 - 0 - 0 = 0$$

$$\frac{d^6y}{dx^6} = 0 - 0 + 24 + 0 + 24 + 0 + 192 - 0 = 240$$

$\therefore y$  has Maclaurin polynomial  $\frac{2x^2}{2!} - \frac{12x^4}{4!} + \frac{240x^6}{6!} + \dots$

which is  $x^2 - \frac{1}{2}x^4 + \frac{1}{3}x^6 + \dots$

$$\begin{aligned}
 \text{b} \quad & \frac{dy}{dx} = 2xe^{-y} \\
 \therefore e^y \frac{dy}{dx} &= 2x \\
 \therefore \int e^y dy &= \int 2x dx & \text{But } y(0) = 0 \\
 \therefore e^y &= x^2 + c & \therefore \ln c = 0 \\
 \therefore y &= \ln(x^2 + c) & \therefore c = 1 \\
 & & \therefore y = \ln(x^2 + 1)
 \end{aligned}$$

**16**  $x''(t) = -m^2x$ ,  $x(0) = 1$ ,  $x'(0) = 0$

$$\begin{aligned}
 \text{a} \quad & x''(t) = -m^2x(t) \\
 \therefore x'''(t) &= -m^2x'(t) \\
 \therefore x''''(t) &= -m^2x''(t) = m^4x(t) \\
 \therefore x'''''(t) &= m^4x'(t) \\
 & \vdots \\
 x^{(2k)}(t) &= (-1)^k m^{2k} x(t) \quad \text{and} \quad x^{(2k+1)}(t) = (-1)^k m^{2k} x'(t) \\
 \therefore x^{(2k)}(0) &= (-1)^k m^{2k} \quad \text{and} \quad x^{(2k+1)}(0) = 0 \quad \text{for all } k \in \mathbb{Z}, k \geq 0.
 \end{aligned}$$

So, the Maclaurin series for  $x(t)$  is  $x(t) = \sum_{k=0}^{\infty} \frac{(-1)^k m^{2k}}{(2k)!} t^{2k}$  {as  $x^{(2k+1)}(0) = 0$ }

$$\therefore x(t) = \sum_{k=0}^{\infty} \frac{(-1)^k (mt)^{2k}}{(2k)!}$$

**b** The Maclaurin series for  $\cos t$  is  $\cos t = \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{(2k)!}$ .

$$\therefore \cos mt = \sum_{k=0}^{\infty} \frac{(-1)^k (mt)^{2k}}{(2k)!}$$

$\therefore x(t) = \cos mt$  is the solution to the differential equation.

$$\begin{aligned}
 \text{c} \quad & x(t) = \cos mt \\
 \therefore x'(t) &= -m \sin mt \\
 \therefore x''(t) &= -m^2 \cos mt \\
 &= -m^2 x(t) \quad \checkmark
 \end{aligned}$$

Also,  $x(0) = \cos 0 = 1 \quad \checkmark$  and  $x'(0) = -m \sin 0 = 0 \quad \checkmark$

$\therefore x(t) = \cos mt$  is the correct solution to the differential equation.

## REVIEW SET 25B

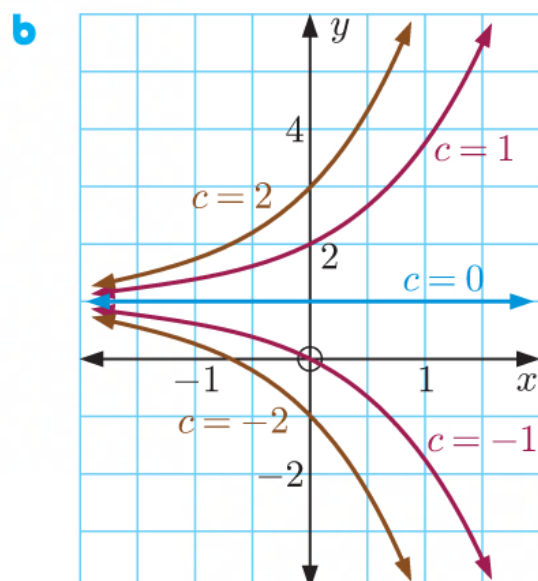
**1** If  $y = 3 \sin x - 2 \cos x$ , then  $\frac{dy}{dx} = 3 \cos x + 2 \sin x$

$$\therefore 3 \frac{dy}{dx} = 9 \cos x + 6 \sin x$$

$$\therefore 3 \frac{dy}{dx} - 2y = 9 \cos x + 6 \sin x - 2(3 \sin x - 2 \cos x)$$

$$\therefore 3 \frac{dy}{dx} - 2y = 13 \cos x \quad \text{as required.}$$

- 2 a** If  $y = ce^x + 1$ , then  $\frac{dy}{dx} = ce^x$   
 $= (ce^x + 1) - 1$   
 $= y - 1$  for any constant  $c$  as required.



- c** From **a**,  $y = ce^x + 1$  is a general solution to the differential equation.

The particular solution passes through  $(0, 4)$ , so

$$4 = ce^0 + 1$$

$$\therefore c = 3$$

$\therefore$  the particular solution is  $y = 3e^x + 1$ .

**d**  $\frac{dy}{dx} = y - 1$

$$\therefore \text{ at the point } (0, 4), \frac{dy}{dx} = 4 - 1 = 3$$

$\therefore$  the gradient of the tangent to the particular solution  $y = 3e^x + 1$  at  $(0, 4)$ , is 3.

$\therefore$  the equation of the tangent is  $y = 3(x - 0) + 4$

$$\therefore y = 3x + 4$$

**3**  $\frac{dy}{dx} = \sin(x + y), \quad y(0) = 0.5$

$y(0) = 0.5$  gives us  $x_0 = 0$  and  $y_0 = 0.5$ .

Iteration	$x_{i-1}$	$y_{i-1}$	$\frac{dy}{dx}$	$x_i$	$y_i$
1	0	0.5	0.4794	0.1	0.5479
2	0.1	0.5479	0.6035	0.2	0.6083
3	0.2	0.6083	0.7231	0.3	0.6806
4	0.3	0.6806	0.8308	0.4	0.7637
5	0.4	0.7637	0.9183	0.5	0.8555

$$\therefore y(0.5) \approx 0.8555$$

**4 a**  $\frac{dy}{dx} = \frac{e^x}{e^x - 2}$

$$\therefore y = \int \frac{e^x}{e^x - 2} dx$$

$$\therefore y = \ln|e^x - 2| + c$$

$$\text{b} \quad \frac{dy}{dx} = \frac{1}{2} \cos\left(\frac{\pi}{3} - 2x\right)$$

$$\begin{aligned} \therefore y &= \int \frac{1}{2} \cos\left(\frac{\pi}{3} - 2x\right) dx \\ &= -\frac{1}{4} \sin\left(\frac{\pi}{3} - 2x\right) + c \end{aligned}$$

$$\text{Now } y\left(\frac{\pi}{2}\right) = 0$$

$$\therefore 0 = -\frac{1}{4} \sin\left(\frac{\pi}{3} - \pi\right) + c$$

$$\therefore c = \frac{1}{4} \sin\left(-\frac{2\pi}{3}\right)$$

$$= -\frac{1}{4} \sin \frac{2\pi}{3}$$

$$= -\frac{1}{4} \left(\frac{\sqrt{3}}{2}\right)$$

$$= -\frac{\sqrt{3}}{8}$$

$$\text{So, the solution is } y = -\frac{1}{4} \sin\left(\frac{\pi}{3} - 2x\right) - \frac{\sqrt{3}}{8}.$$

$$\begin{aligned} \text{5} \quad \text{Since } \frac{dT}{dx} &= \frac{k}{x}, \quad T = \int \frac{k}{x} dx \\ \therefore T &= k \ln x + c \quad \{x > 0\} \end{aligned}$$

$$\text{When } x = r_1, \quad T = T_0$$

$$\therefore T_0 = k \ln r_1 + c$$

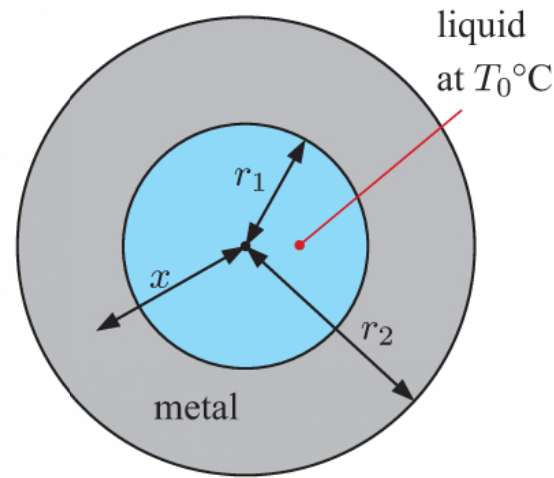
$$\therefore c = T_0 - k \ln r_1$$

$$\text{Thus, } T = k \ln x + T_0 - k \ln r_1$$

$$\therefore T = T_0 + k \ln\left(\frac{x}{r_1}\right)$$

$$\text{When } x = r_2, \quad T = T_0 + k \ln\left(\frac{r_2}{r_1}\right)$$

$$\text{The outer surface has temperature } T_0 = k \ln\left(\frac{r_2}{r_1}\right) ^\circ\text{C}.$$



$$\begin{aligned} \text{6} \quad \text{a} \quad \frac{dy}{dx} &= 2y^4 \\ \therefore \frac{1}{y^4} \frac{dy}{dx} &= 2 \\ \therefore \int \frac{1}{y^4} \frac{dy}{dx} dx &= \int 2 dx \\ \therefore \int y^{-4} dy &= \int 2 dx \\ \therefore -\frac{1}{3} y^{-3} &= 2x + c \\ \therefore y^{-3} &= c - 6x \\ \therefore y^3 &= \frac{1}{c - 6x} \\ \therefore y &= \frac{1}{\sqrt[3]{c - 6x}} \end{aligned}$$

$$\begin{aligned} \text{b} \quad (t^2 + 1) \frac{dP}{dt} &= Pt \\ \therefore \frac{1}{P} \frac{dP}{dt} &= \frac{t}{t^2 + 1} \\ \therefore \int \frac{1}{P} dP &= \frac{1}{2} \int \frac{2t}{t^2 + 1} dt \\ \therefore \ln |P| &= \frac{1}{2} \ln |t^2 + 1| + c \\ \therefore \ln |P| &= \frac{1}{2} \ln(t^2 + 1) + c \quad \{t^2 + 1 > 0\} \\ \therefore P &= \pm e^c \sqrt{t^2 + 1} \\ \therefore P &= A \sqrt{t^2 + 1} \quad \{A = \pm e^c\} \end{aligned}$$



**7 a**

$$\frac{dy}{dx} = \sqrt{y}$$

$$\therefore \frac{1}{\sqrt{y}} \frac{dy}{dx} = 1$$

$$\therefore \int y^{-\frac{1}{2}} dy = \int 1 dx$$

$$\therefore 2y^{\frac{1}{2}} = x + c$$

$$\therefore \sqrt{y} = \frac{1}{2}x + c$$

But  $y(0) = 4$ , so  $\sqrt{4} = \frac{1}{2}(0) + c$

$$\therefore c = 2$$

So,  $\sqrt{y} = \frac{1}{2}x + 2$ .

The particular solution is  $y = \left(\frac{1}{2}x + 2\right)^2$ .

**b**

$$\frac{dy}{dx} = y \cos x$$

$$\therefore \frac{1}{y} \frac{dy}{dx} = \cos x$$

$$\therefore \int \frac{1}{y} dy = \int \cos x dx$$

$$\therefore \ln |y| = \sin x + c$$

$$\therefore y = \pm e^{\sin x + c}$$

$$\therefore y = Ae^{\sin x} \quad \{A = \pm e^c\}$$

But  $y\left(\frac{\pi}{2}\right) = \frac{1}{e^2}$ , so  $\frac{1}{e^2} = Ae^{\sin \frac{\pi}{2}}$

$$\therefore e^{-2} = Ae$$

$$\therefore A = e^{-3}$$

The particular solution is  $y = e^{\sin x - 3}$ .

**8**  $x^2 - 4 = (x + 2)(x - 2)$

Let  $\frac{x-6}{x^2-4} = \frac{A}{x+2} + \frac{B}{x-2}$

$$\therefore x - 6 = A(x - 2) + B(x + 2)$$

Substituting  $x = 2$ ,  $-4 = 4B$

$$\therefore B = -1$$

Substituting  $x = -2$ ,  $-8 = -4A$

$$\therefore A = 2$$

$$\therefore \frac{x-6}{x^2-4} = \frac{2}{x+2} - \frac{1}{x-2}$$

Now,  $\frac{dy}{dx} = \frac{xy-6y}{x^2-4} = \frac{y(x-6)}{x^2-4}$

$$\therefore \frac{1}{y} \frac{dy}{dx} = \frac{x-6}{x^2-4}$$

$$\therefore \int \frac{1}{y} \frac{dy}{dx} dx = \int \frac{x-6}{x^2-4} dx$$

$$\therefore \int \frac{1}{y} dy = \int \left( \frac{2}{x+2} - \frac{1}{x-2} \right) dx$$

$$\therefore \ln |y| = 2 \ln |x+2| - \ln |x-2| + c$$

$$\therefore \ln |y| - 2 \ln |x+2| + \ln |x-2| = c$$

$$\therefore \ln \left| \frac{y(x-2)}{(x+2)^2} \right| = c$$

$$\therefore \frac{y(x-2)}{(x+2)^2} = \pm e^c$$

$$\therefore y = \frac{A(x+2)^2}{x-2} \quad \{A = \pm e^c\}$$

But  $y(3) = 1$ , so  $1 = 25A$

$$\therefore A = \frac{1}{25}$$

The particular solution is  $y = \frac{(x+2)^2}{25(x-2)}$ .

**9 a**

$$\frac{dT}{dt} = -0.1(T - 20)$$

$$\therefore \frac{1}{T - 20} \frac{dT}{dt} = -0.1$$

$$\therefore \int \frac{1}{T - 20} dt = \int -0.1 dt$$

$$\therefore \ln|T - 20| = -0.1t + c$$

$$\therefore T - 20 = \pm e^{-0.1t+c}$$

$$\therefore T = Ae^{-0.1t} + 20 \quad \{A = \pm e^c\}$$

Now when  $t = 0$ ,  $T = 85$

$$\therefore 85 = Ae^0 + 20$$

$$\therefore A = 65$$

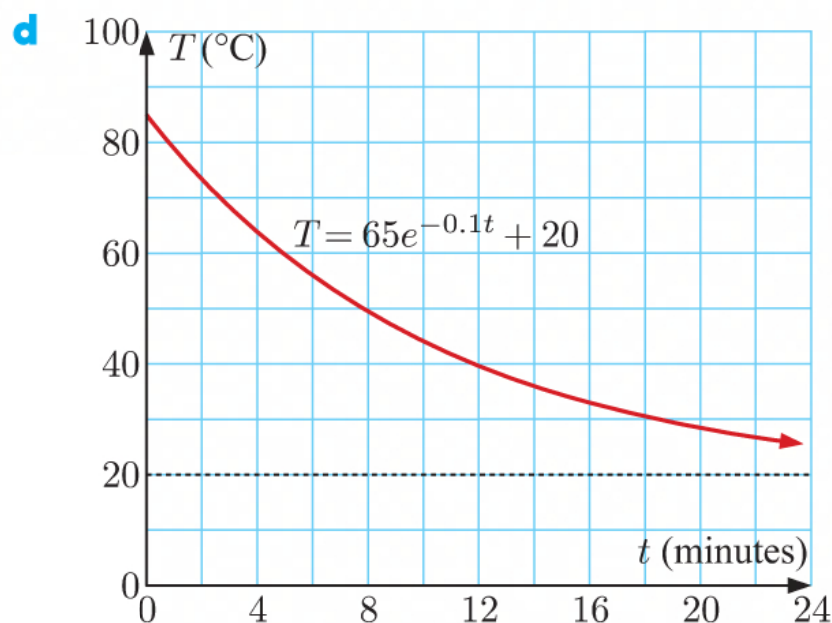
So, we have  $T = 65e^{-0.1t} + 20$ .

**b** When  $t = 4$ ,  $T = 65e^{-0.4} + 20 \approx 63.6$

After 4 minutes, the temperature of the tea is about  $63.6^\circ\text{C}$ .

**c** As  $t \rightarrow \infty$ ,  $e^{-0.1t} \rightarrow 0$

$$\therefore T \rightarrow 20$$



**e**  $T = 65e^{-0.1t} + 20$  {using **a**}

$$\therefore T - 20 = 65e^{-0.1t}$$

$$\therefore \frac{T - 20}{65} = e^{-0.1t}$$

$$\therefore -0.1t = \ln\left(\frac{T - 20}{65}\right)$$

$$\therefore t = -10 \ln\left(\frac{T - 20}{65}\right)$$

**i** When  $T = 65$ ,  $t = -10 \ln\left(\frac{65 - 20}{65}\right) \approx 3.68$

Brian will have to wait about 3.68 minutes for his tea to cool down.

**ii** When  $T = 40$ ,  $t = -10 \ln\left(\frac{40 - 20}{65}\right) \approx 11.79$

Brian will have about  $11.79 - 3.68 \approx 8.11$  minutes to drink the tea before it gets too cold.

- 10** The gradient of the tangent at P is  $\frac{-\frac{3y}{2}}{3x} = -\frac{y}{2x}$ .

Hence  $\frac{dy}{dx} = -\frac{y}{2x}$

$$\therefore \frac{1}{y} \frac{dy}{dx} = -\frac{1}{2x}$$

Integrating both sides with respect to  $x$  gives

$$\int \frac{1}{y} dy = -\frac{1}{2} \int \frac{1}{x} dx$$

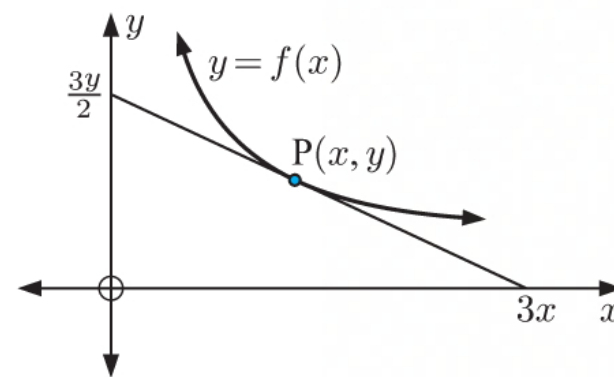
$$\therefore \ln|y| = -\frac{1}{2} \ln x + c \quad \{x > 0\}$$

$$\therefore y = \pm e^{\ln(x^{-\frac{1}{2}}) + c}$$

$$\therefore y = \frac{A}{\sqrt{x}} \quad \{A = \pm e^c\}$$

Since the curve passes through  $(1, 5)$ ,  $5 = A$

$\therefore$  the equation of the curve is  $y = \frac{5}{\sqrt{x}}$ , where  $x > 0$ .



- 11 a** We are given that  $\frac{dV}{dt} \propto \sqrt{h}$  where  $h$  is the depth of the water, and  $V$  is the volume of water in the tank.

$$\therefore \frac{dV}{dt} = k\sqrt{h} \quad \text{where } k \text{ is a constant.}$$

**b**  $V = l \times w \times h = 2 \times 2 \times h = 4h \text{ m}^3$

$$\begin{aligned} \therefore \frac{dV}{dt} &= \frac{dV}{dh} \frac{dh}{dt} \quad \{\text{chain rule}\} \\ &= 4 \frac{dh}{dt} \end{aligned}$$

From **a**,  $\frac{dV}{dt} = k\sqrt{h}$ , so  $k\sqrt{h} = 4 \frac{dh}{dt}$

$$\therefore \frac{dh}{dt} = \frac{k}{4} \sqrt{h}$$

**c**  $\frac{dh}{dt} = \frac{k}{4} \sqrt{h}$

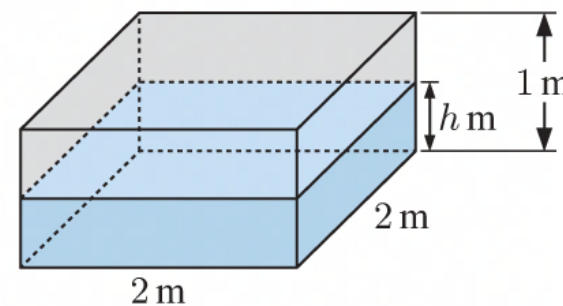
$$\therefore \frac{1}{\sqrt{h}} \frac{dh}{dt} = \frac{k}{4}$$

$$\therefore \int \frac{1}{\sqrt{h}} \frac{dh}{dt} dt = \int \frac{k}{4} dt$$

$$\therefore \int h^{-\frac{1}{2}} dh = \int \frac{k}{4} dt$$

$$\therefore 2h^{\frac{1}{2}} = \frac{k}{4}t + c$$

$$\therefore \sqrt{h} = \frac{k}{8}t + c$$



Now when  $t = 0$ ,  $h = 1$

$$\therefore \sqrt{1} = c$$

$$\therefore c = 1$$

$$\therefore \sqrt{h} = \frac{k}{8}t + 1$$

Also, when  $t = 2$ ,  $h = 1 - 0.19 = 0.81$

$$\therefore \sqrt{0.81} = \frac{k}{8}(2) + 1$$

$$\therefore 0.9 = \frac{k}{4} + 1$$

$$\therefore \frac{k}{4} = -\frac{1}{10}$$

$$\therefore k = -\frac{2}{5}$$

So, the equation connecting the depth of the water and the time  $t$  is  $\sqrt{h} = 1 - \frac{1}{20}t$ .

The tank is empty when  $h = 0$ .

This occurs when  $\frac{t}{20} = 1$

$$\therefore t = 20$$

The tank empties in 20 minutes.

**12**  $\frac{dP}{dt} = kP \left(1 - \frac{P}{A}\right)$

**a**

$$\frac{dP}{dt} = kP \left(\frac{A - P}{A}\right)$$

$$\therefore \frac{A}{P(A - P)} \frac{dP}{dt} = k$$

$$\therefore \int \frac{A}{P(A - P)} \frac{dP}{dt} dt = \int k dt$$

$$\therefore \int \frac{A}{P(A - P)} dP = \int k dt$$

$$\therefore \int \left(\frac{1}{P} + \frac{1}{A - P}\right) dP = \int k dt$$

$$\therefore \ln|P| + \frac{1}{-1} \ln|A - P| = kt + c$$

$$\therefore \ln \left| \frac{P}{A - P} \right| = kt + c$$

$$\therefore \frac{P}{A - P} = \pm e^{kt+c}$$

$$\therefore \frac{A - P}{P} = be^{-kt} \quad \left\{ \text{letting } b = \pm \frac{1}{e^c} \right\}$$

The initial population was 25 000, so  $P(0) = 25\,000$

$$\therefore \frac{200\,000 - 25\,000}{25\,000} = be^0$$

$$\therefore b = 7$$



$$\begin{aligned}
 \text{So, we have } \frac{200\,000 - P}{P} &= 7e^{-kt} \\
 \therefore \frac{200\,000}{P} - 1 &= 7e^{-kt} \\
 \therefore \frac{200\,000}{P} &= 1 + 7e^{-kt} \\
 \therefore P &= \frac{200\,000}{1 + 7e^{-kt}}
 \end{aligned}$$

After 2 years, the population was 31 200, so  $P(2) = 31\,200$


$$\begin{aligned}
 \therefore 31\,200 &= \frac{200\,000}{1 + 7e^{-2k}} \\
 \therefore 1 + 7e^{-2k} &= \frac{200\,000}{31\,200} = \frac{250}{39} \\
 \therefore 7e^{-2k} &= \frac{211}{39} \\
 \therefore e^{-2k} &= \frac{211}{273} \\
 \therefore -2k &= \ln\left(\frac{211}{273}\right) \\
 \therefore k &= -\frac{1}{2} \ln\left(\frac{211}{273}\right) \approx 0.129
 \end{aligned}$$

$$\therefore P(t) \approx \frac{200\,000}{1 + 7e^{-0.129t}}$$

**b** When  $t = 5$ ,  $P = \frac{200\,000}{1 + 7e^{-5k}} \approx 42\,800$

$\therefore$  the ostrich population after 5 years is about 42 800.

**c** The population growth rate is a maximum when  $\frac{dP}{dt} = kP\left(1 - \frac{P}{A}\right)$  is maximised.

Notice that  $kP\left(1 - \frac{P}{A}\right) = kP - \frac{kP^2}{A}$  is a quadratic in terms of  $P$  with shape .

$$\begin{aligned}
 \therefore \frac{dP}{dt} \text{ is maximised when } P &= \frac{-k}{2\left(-\frac{k}{200\,000}\right)} \\
 &= \frac{200\,000}{2} \\
 &= 100\,000
 \end{aligned}$$

When  $P = 100\,000$ ,  $100\,000 = \frac{200\,000}{1 + 7e^{-kt}}$  {from **a**}

$$\therefore 1 + 7e^{-kt} = 2$$

$$\therefore 7e^{-kt} = 1$$

$$\therefore e^{-kt} = \frac{1}{7}$$

$$\therefore e^{kt} = 7$$

$$\therefore kt = \ln 7$$

$$\therefore t = \frac{\ln 7}{-\frac{1}{2} \ln\left(\frac{211}{273}\right)} \quad \text{{from **a**}}$$

$$\therefore t \approx 15.1$$

$\therefore$  the population growth rate is a maximum after about 15.1 years.

**13**  $xy \frac{dy}{dx} = 1 + x + y^2, \quad y(1) = 0$

Let  $y = vx$ , so  $\frac{dy}{dx} = v + x \frac{dv}{dx}$  {product rule}

$$\therefore vx^2 \frac{dy}{dx} = v^2x^2 + vx^3 \frac{dv}{dx}$$

Comparing with the differential equation,  $v^2x^2 + vx^3 \frac{dv}{dx} = 1 + x + v^2x^2$

$$\therefore vx^3 \frac{dv}{dx} = 1 + x$$

$$\therefore v \frac{dv}{dx} = \frac{1}{x^3} + \frac{1}{x^2}$$

$$\therefore \int v \, dv = \int (x^{-3} + x^{-2}) \, dx$$

$$\therefore \frac{1}{2}v^2 = -\frac{1}{2}x^{-2} - x^{-1} + c$$

$$\therefore v^2 = -x^{-2} - 2x^{-1} + c$$

$$\therefore \frac{y^2}{x^2} = -x^{-2} - 2x^{-1} + c$$

$$\therefore y^2 = cx^2 - 2x - 1$$

But  $y(1) = 0$ , so  $0 = c - 2 - 1$

$$\therefore c = 3$$

$\therefore$  the particular solution is  $y^2 = 3x^2 - 2x - 1$ .

**14**  $\frac{dy}{dx} + \frac{3y}{x} = 8x^4$

$$\therefore \frac{dy}{dx} + \left(\frac{3}{x}\right)y = 8x^4$$

The integrating factor is  $I(x) = e^{\int \frac{3}{x} dx} = e^{3 \ln x} = e^{\ln(x^3)} = x^3$ .

Multiplying both sides of the differential equation by  $x^3$  gives

$$x^3 \frac{dy}{dx} + x^3 \left(\frac{3}{x}\right)y = 8x^7$$

$$\therefore \frac{d}{dx}(yx^3) = 8x^7$$

$$\therefore yx^3 = \int 8x^7 \, dx$$

$$\therefore yx^3 = x^8 + c$$

$$\therefore y = x^5 + \frac{c}{x^3}$$

But  $y(1) = 0$ , so  $0 = 1 + c$

$$\therefore c = -1$$

The particular solution is  $y = x^5 - \frac{1}{x^3}$ .

**15**  $\frac{dy}{dx} = y \ln(x+1), \quad y(0) = 1$

**a** 
$$\frac{d^2y}{dx^2} = \frac{dy}{dx} \ln(x+1) + \frac{y}{x+1} = \frac{dy}{dx} \ln(x+1) + y(x+1)^{-1}$$

$$\therefore \frac{d^3y}{dx^3} = \frac{d^2y}{dx^2} \ln(x+1) + \frac{dy}{dx} \left( \frac{1}{x+1} \right) + \frac{dy}{dx} (x+1)^{-1} - y(x+1)^{-2}$$

$$= \frac{d^2y}{dx^2} \ln(x+1) + 2 \frac{dy}{dx} \left( \frac{1}{x+1} \right) - \frac{y}{(x+1)^2}$$

Now  $y(0) = 1$ , so at  $(0, 1)$ :  $\frac{dy}{dx} = 0$

$$\frac{d^2y}{dx^2} = 0 + 1 = 1$$

$$\frac{d^3y}{dx^3} = 0 + 0 - 1 = -1$$

$\therefore y$  has Maclaurin polynomial  $1 + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots$

which is  $1 + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \dots$

**b**  $\frac{dy}{dx} = y \ln(x+1)$

$$\therefore \frac{1}{y} \frac{dy}{dx} = \ln(x+1)$$

$$\therefore \int \frac{1}{y} dy = \int \ln(x+1) dx$$

$$\therefore \ln|y| = x \ln(x+1) - \int \frac{x}{x+1} dx \quad \begin{cases} u = \ln(x+1) & v' = 1 \\ u' = \frac{1}{x+1} & v = x \end{cases}$$

$$\therefore \ln|y| = x \ln(x+1) - \int \frac{x+1-1}{x+1} dx$$

$$\therefore \ln|y| = x \ln(x+1) - \int \left( 1 - \frac{1}{x+1} \right) dx$$

$$\therefore \ln|y| = x \ln(x+1) - (x - \ln|x+1|) + c$$

$$\therefore \ln|y| = x \ln(x+1) - x + \ln(x+1) + c \quad \{x > -1\}$$

$$\therefore \ln|y| = (x+1) \ln(x+1) - x + c$$

$$\therefore y = \pm e^{(x+1) \ln(x+1) - x + c}$$

$$\therefore y = A(x+1)^{x+1} e^{-x} \quad \{A = \pm e^c\}$$

But  $y(0) = 1$ , so  $1 = A$

The particular solution is  $y = (x+1)^{x+1} e^{-x}$ .

- 16 a** Let the tangent at P meet the  $x$ -axis at R.

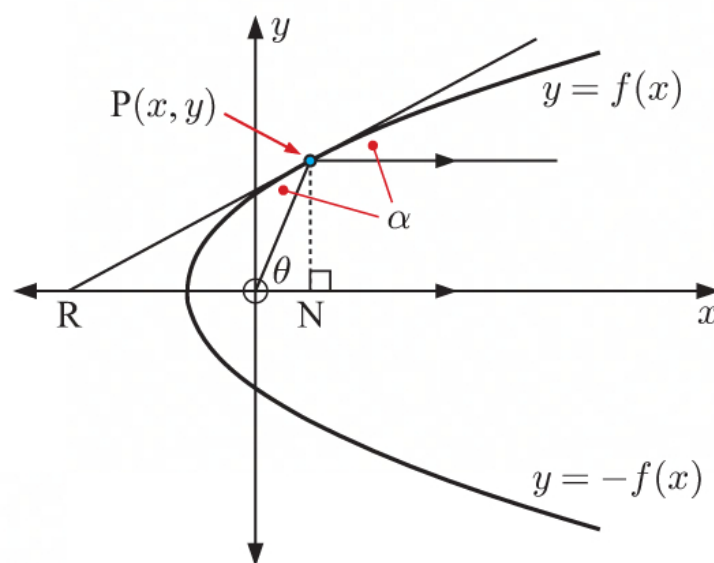
Now,  $\widehat{PRO} = \alpha$  {corresponding angles}

and  $\theta = \alpha + \widehat{PRO}$

{exterior angles in a triangle theorem}

$$\therefore \theta = \alpha + \alpha$$

$$\therefore \theta = 2\alpha$$



- b** Let N be the foot of the perpendicular from P to the  $x$ -axis.

The gradient of the tangent at P is  $\frac{PN}{RN} = \tan \alpha$ .

Hence  $\frac{dy}{dx} = \tan \alpha$ .

- c** In  $\triangle PON$ ,  $\tan \theta = \frac{y}{x}$

$$\therefore \tan 2\alpha = \frac{y}{x} \quad \{\text{using a}\}$$

$$\therefore \frac{2 \tan \alpha}{1 - \tan^2 \alpha} = \frac{y}{x} \quad \left\{ \tan 2\alpha = \frac{2 \tan \alpha}{1 - \tan^2 \alpha} \right\}$$

$$\therefore 2x \tan \alpha = y(1 - \tan^2 \alpha)$$

$$\therefore 2x \tan \alpha = y - y \tan^2 \alpha$$

$$\therefore y \tan^2 \alpha + 2x \tan \alpha - y = 0$$

$$\therefore \tan \alpha = \frac{-2x \pm \sqrt{4x^2 + 4y^2}}{2y}$$

$$= \frac{-2x \pm 2\sqrt{x^2 + y^2}}{2y}$$

$$= \frac{\sqrt{x^2 + y^2} - x}{y} \quad \{\tan \alpha > 0\}$$



$$\mathbf{d} \quad \frac{dy}{dx} = \frac{\sqrt{x^2 + y^2} - x}{y}$$

$$\therefore y \frac{dy}{dx} = \sqrt{x^2 + y^2} - x$$

$$\text{Let } r^2 = x^2 + y^2, \text{ so } 2r \frac{dr}{dx} = 2x + 2y \frac{dy}{dx} \quad \{\text{chain rule}\}$$

$$\therefore 2y \frac{dy}{dx} = 2r \frac{dr}{dx} - 2x$$

$$\therefore y \frac{dy}{dx} = r \frac{dr}{dx} - x$$

$$\text{Comparing with the differential equation, } r \frac{dr}{dx} - x = r - x$$

$$\therefore r \frac{dr}{dx} = r$$

$$\therefore \frac{dr}{dx} = 1$$

$$\therefore \int 1 \, dr = \int 1 \, dx$$

$$\therefore r = x + c$$

$$\therefore \sqrt{x^2 + y^2} = x + c$$

$$\therefore x^2 + y^2 = (x + c)^2$$

$$\therefore y^2 = x^2 + 2cx + c^2 - x^2$$

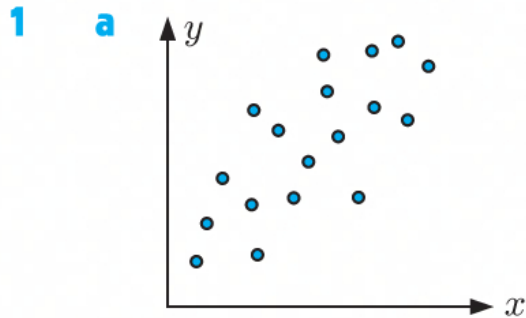
$$\therefore y^2 = 2cx + c^2$$

$\mathbf{e}$  The mirror is a parabola as  $x$  is a quadratic in  $y$ .

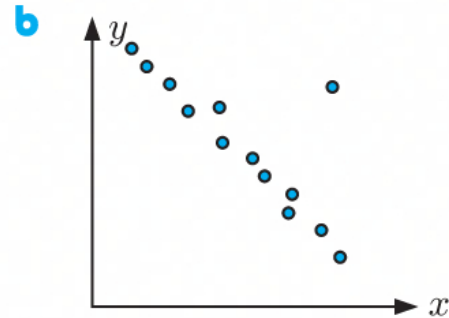
# Chapter 26

## BIVARIATE STATISTICS

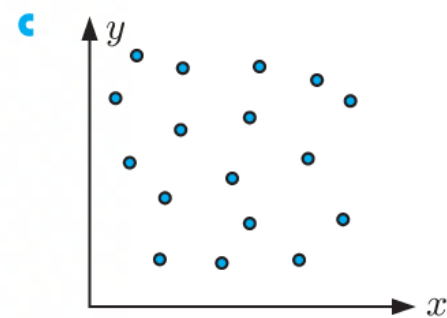
### EXERCISE 26A



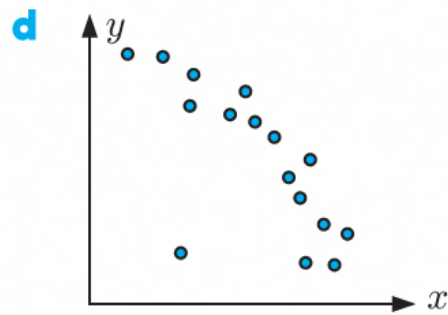
There is a weak, positive, linear correlation with no outliers.



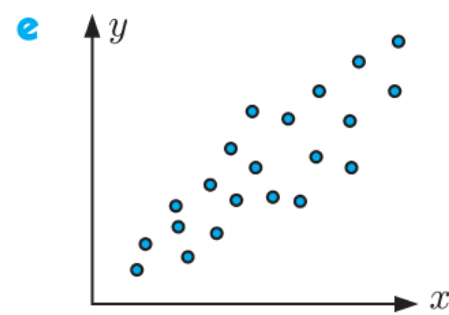
There is a strong, negative, linear correlation with one outlier.



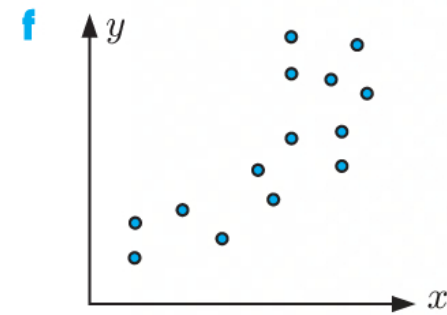
There is no correlation.



There is a strong, negative, non-linear correlation with one outlier.



There is a moderate, positive, linear correlation with no outliers.

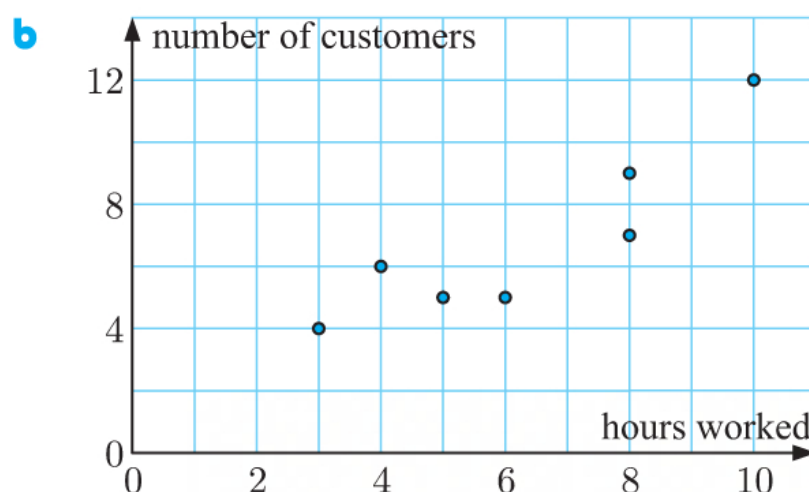


There is a weak, positive, non-linear correlation with no outliers.

**2**

Day	Mon	Tue	Wed	Thu	Fri	Sat	Sun
Hours worked	8	4	5	10	8	3	6
Number of customers	9	6	5	12	7	4	5

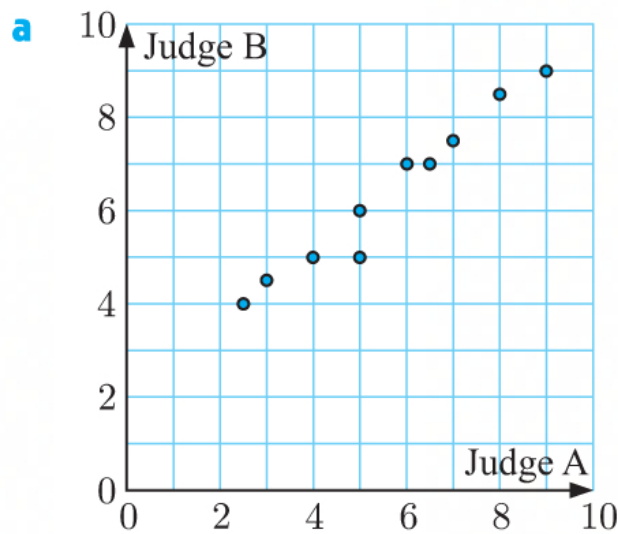
- a** *Hours worked* is the explanatory variable.  
*Number of customers* is the response variable.



- c** **i** Tiffany worked the same number of hours (8 hours) on Monday and Friday.  
**ii** Tiffany had the same number of customers (5 customers) on Wednesday and Sunday.  
**d** The more hours that Tiffany works, the more customers she is likely to have, so we would expect a positive correlation between the variables.

3

Competitor	P	Q	R	S	T	U	V	W	X	Y
Judge A	5	6.5	8	9	4	2.5	7	5	6	3
Judge B	6	7	8.5	9	5	4	7.5	5	7	4.5

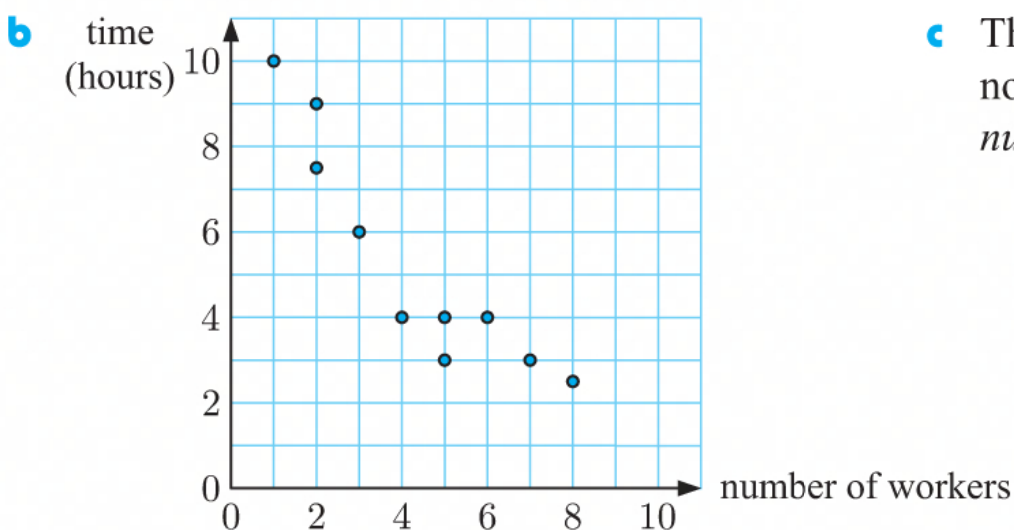


- b There appears to be **strong, positive, linear** correlation between Judge A's scores and Judge B's scores. This means that as Judge A's scores increase, Judge B's scores **increase**.
- c No, an increase in Judge A's scores are not likely to cause an increase in Judge B's scores. It is much more likely that both scores are related to the quality of the ice skaters' performances.

4

Job	A	B	C	D	E	F	G	H	I	J
Number of workers	5	3	8	2	5	6	1	4	2	7
Time (hours)	4	6	2.5	9	3	4	10	4	7.5	3

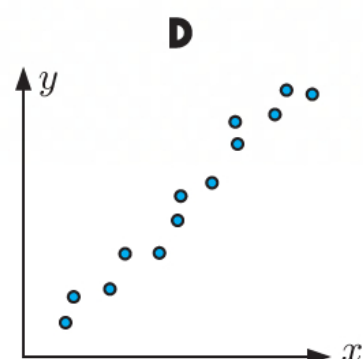
- a
- Job G took the longest to complete (10 hours).
  - Job C involved the most workers (8 workers).



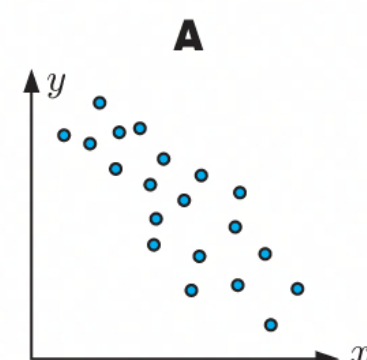
- c There is a strong, negative, non-linear correlation between the *number of workers* and *time*.

5

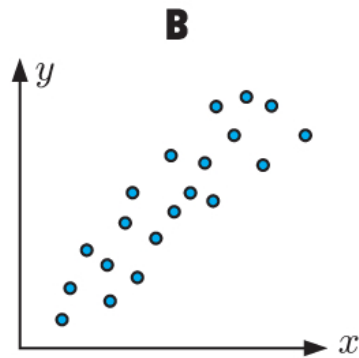
- a  $x$  = the number of apples bought by customers  
 $y$  = the total cost of apples bought  
 We expect strong, positive, linear correlation. This corresponds to **D**.



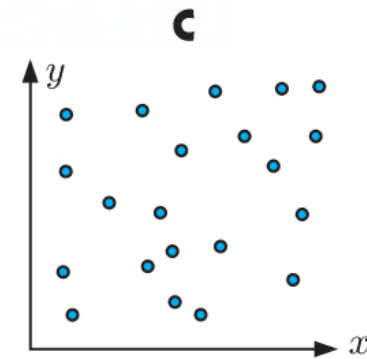
- b  $x$  = the number of pushups a student can perform in one minute  
 $y$  = the time taken for a student to run 100 metres  
 We expect moderate, negative, linear correlation. This corresponds to **A**.



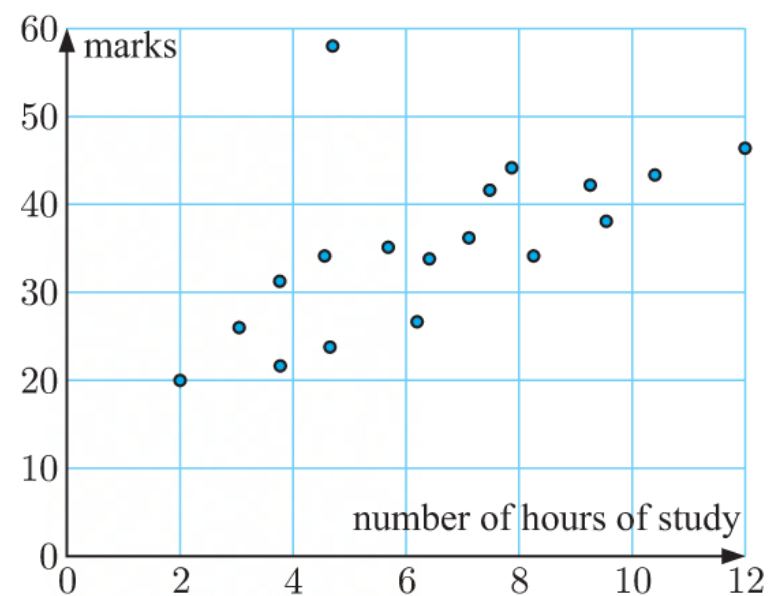
- c**  $x$  = the height of a person  
 $y$  = the weight of the person  
 We expect moderate, positive, linear correlation. This corresponds to **B**.



- d**  $x$  = the distance a student travels to school  
 $y$  = the height of the student's uncle  
 We expect no correlation. This corresponds to **C**.



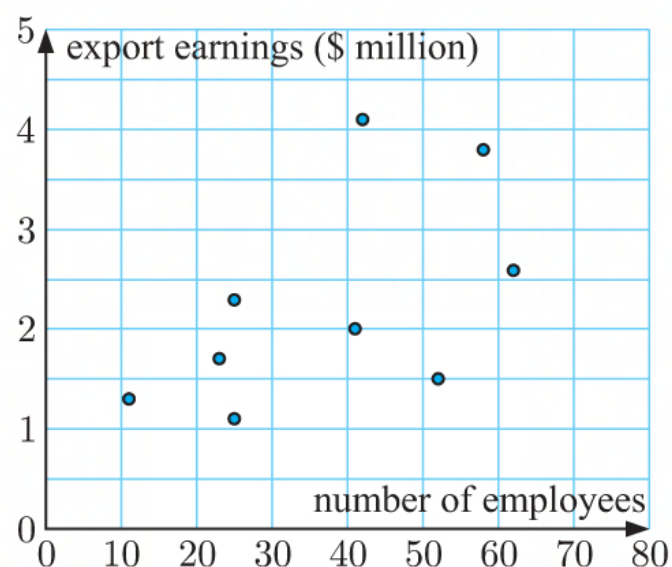
- 6** **a** There is a moderate, positive, linear correlation between the *number of hours of study* and the *marks obtained*.  
**b** As the test is out of 50 marks and the outlier is greater than 50, we can assume it is an error and discard it.  
**c** Yes, this is a causal relationship as spending more time studying for the test is likely to cause a higher mark.



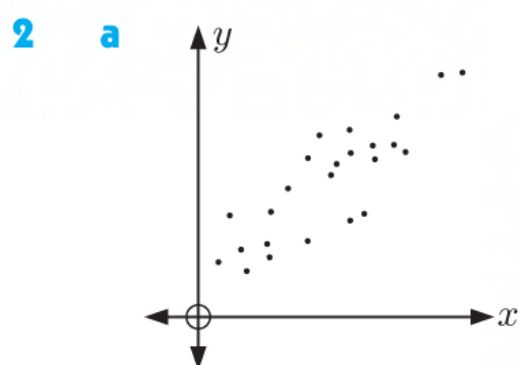
- 7** **a** Not causal, dependent on genetics and/or age.  
**b** Not causal, dependent on the size of the fire.  
**c** Causal, an increase in advertising is likely to cause an increase in sales.  
**d** Causal, the childrens' adult height is determined by the genetics inherited from their parents to a great extent.  
**e** Not causal, dependent on the population of the towns or the number of tourists.

## EXERCISE 26B

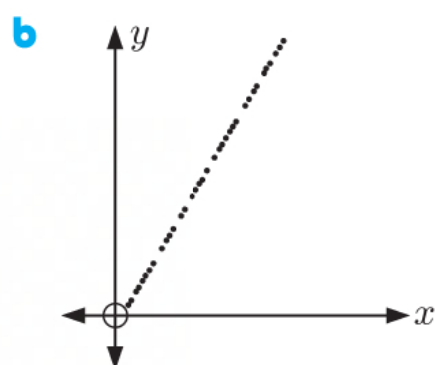
- 1**  $r = 0.556$   
 There is a weak, positive correlation between the *number of employees of a company* and its *export earnings*.



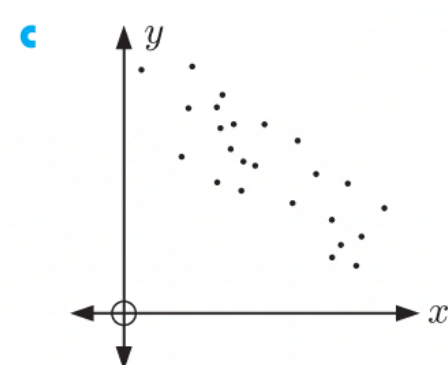




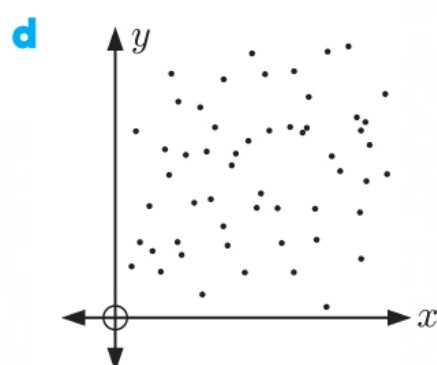
**B**  $r = 0.6$



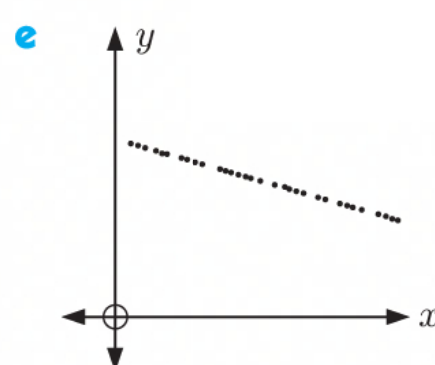
**A**  $r = 1$



**D**  $r = -0.7$



**C**  $r = 0$

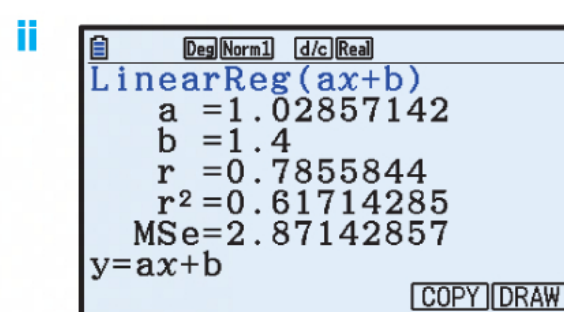
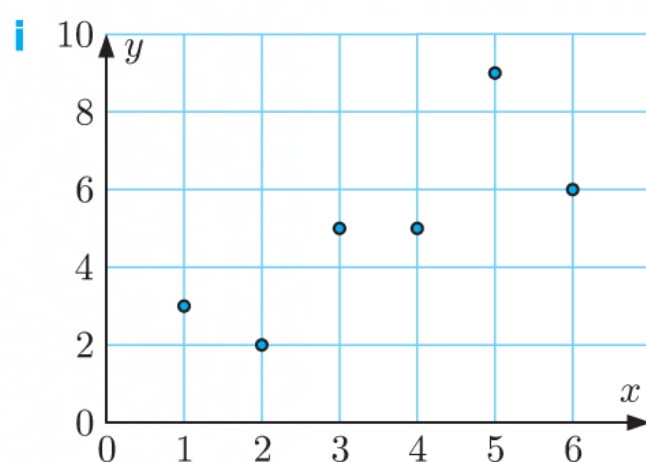
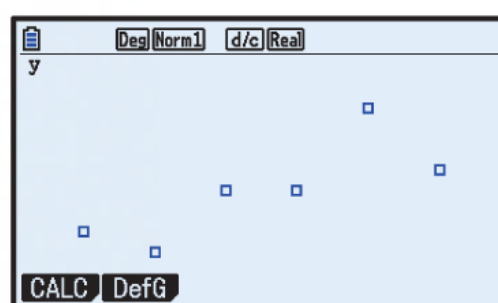


**E**  $r = -1$

**3 a**

$x$	1	2	3	4	5	6
$y$	3	2	5	5	9	6

	List 1	List 2	List 3	List 4
SUB				
1	1	3		
2	2	2		
3	3	5		
4	4	5		



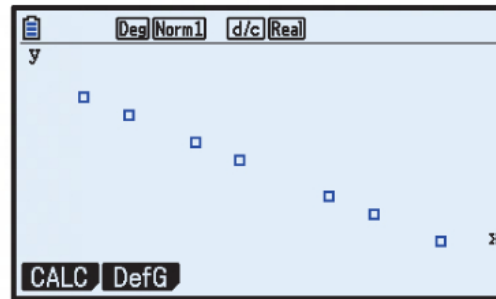
So,  $r \approx 0.786$ .

**iii** There is a moderate, positive correlation between  $x$  and  $y$ .

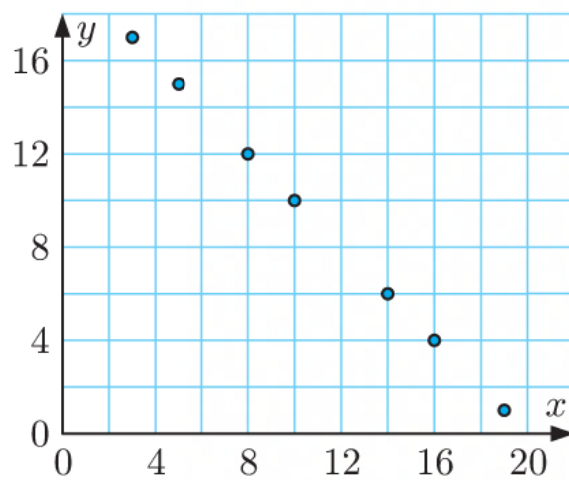
b

$x$	3	8	5	14	19	10	16
$y$	17	12	15	6	1	10	4

	List 1	List 2	List 3	List 4
SUB				
1	3	17		
2	8	12		
3	5	15		
4	14	6		



i



ii

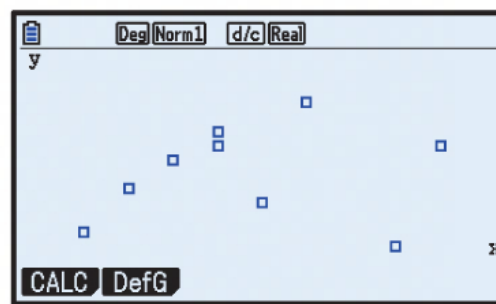
	LinearReg(ax+b)
a	= -1
b	= 20
r	= -1
r <sup>2</sup>	= 1
MSe	= 0
y	= ax+b

So,  $r = -1$ .iii There is a perfect, negative correlation between  $x$  and  $y$ .

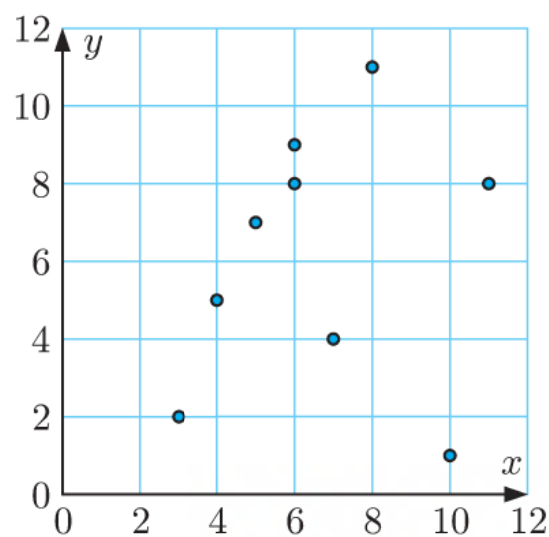
c

$x$	3	6	11	7	5	6	8	10	4
$y$	2	8	8	4	7	9	11	1	5

	List 1	List 2	List 3	List 4
SUB				
1	3	2		
2	6	8		
3	11	8		
4	7	4		



i



ii

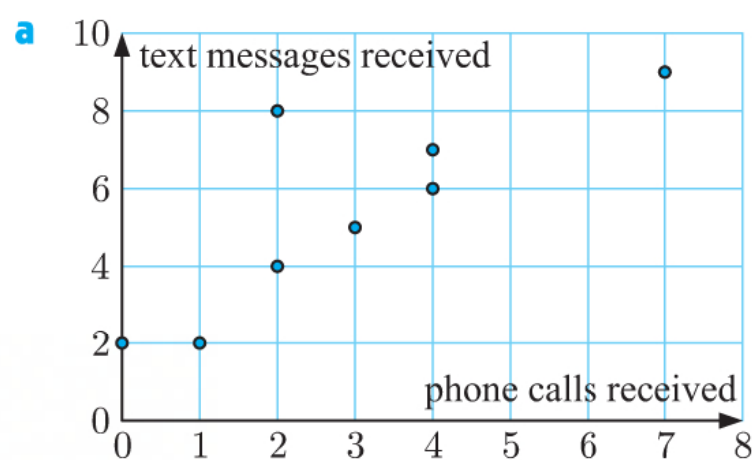
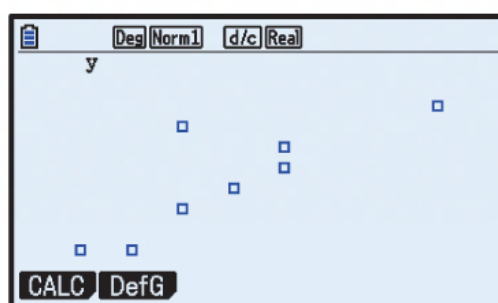
	LinearReg(ax+b)
a	= 0.1845238
b	= 4.88095238
r	= 0.14646123
r <sup>2</sup>	= 0.02145089
MSe	= 12.4260204
y	= ax+b

So,  $r \approx 0.146$ .iii There is a very weak, positive correlation between  $x$  and  $y$ .

4

Student	A	B	C	D	E	F	G	H
Phone calls received	4	7	1	0	3	2	2	4
Text messages received	6	9	2	2	5	8	4	7

	List 1	List 2	List 3	List 4
SUB				
1	4	6		
2	7	9		
3	1	2		
4	0	2		



b

	LinearReg(ax+b)
a	=0.98479087
b	=2.54372623
r	=0.81606077
r <sup>2</sup>	=0.66595518
MSe	=2.66539923
y	=ax+b

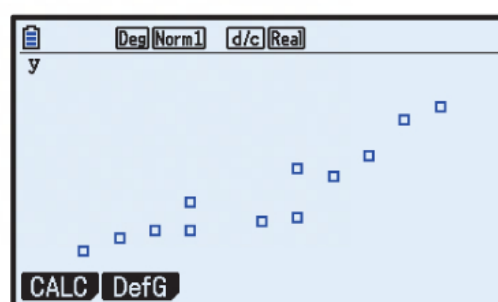
So,  $r \approx 0.816$ .

- c There is a moderate, positive correlation between *phone calls received* and *text messages received*.
- d Those students who receive several phone calls are also likely to receive several text messages and vice versa.

5

Athlete	A	B	C	D	E	F	G	H	I	J	K	L
Age (years)	12	16	16	18	13	19	11	10	20	17	15	13
Distance thrown (m)	20	35	23	38	27	47	18	15	50	33	22	20

	List 1	List 2	List 3	List 4
SUB				
1	12	20		
2	16	35		
3	16	23		
4	18	38		



	LinearReg(ax+b)
a	=3.28947368
b	=-20.342105
r	=0.91730097
r <sup>2</sup>	=0.84144108
MSe	=23.2447368
y	=ax+b

So,  $r \approx 0.917$ .

- b There is a strong, positive correlation between the *age* of the young athlete and the *distance thrown*. In general, the higher the young athlete's age, the further they can throw a discus.

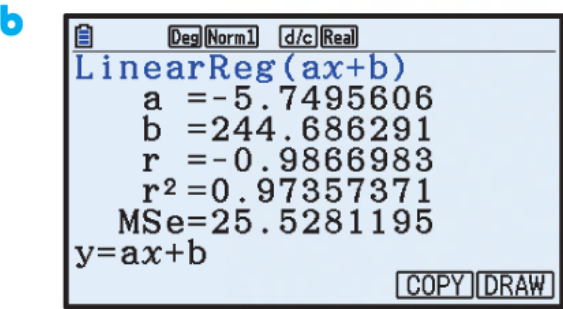
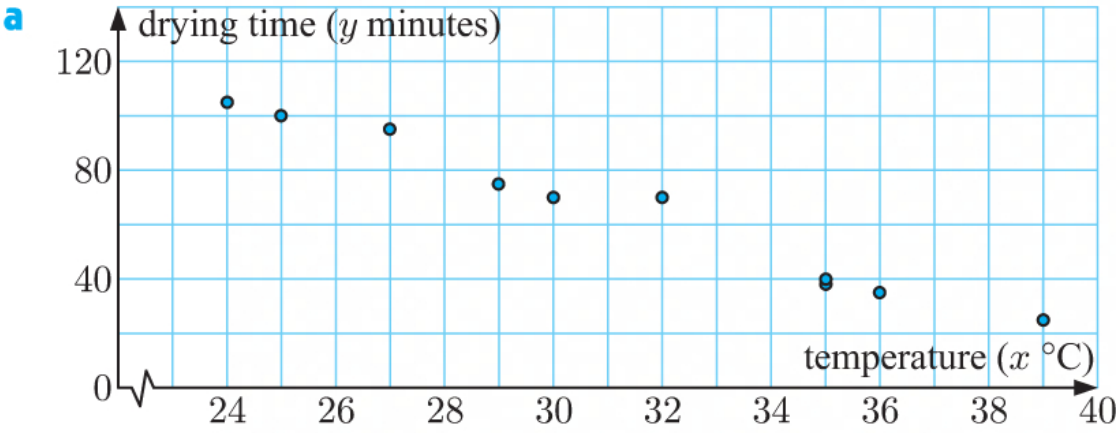
6

Temperature ( $x^{\circ}\text{C}$ )	25	32	27	39	35	24	30	36	29	35
Drying time ( $y$ minutes)	100	70	95	25	38	105	70	35	75	40

	List 1	List 2	List 3	List 4
SUB				
1	25	100		
2	32	70		
3	27	95		
4	39	25		

25

GRAPH CALC TEST INTR DIST



c There is a very strong, negative correlation between *temperature* and *drying time*. In general, the higher the temperature, the lower the drying time.

So,  $r \approx -0.987$ .

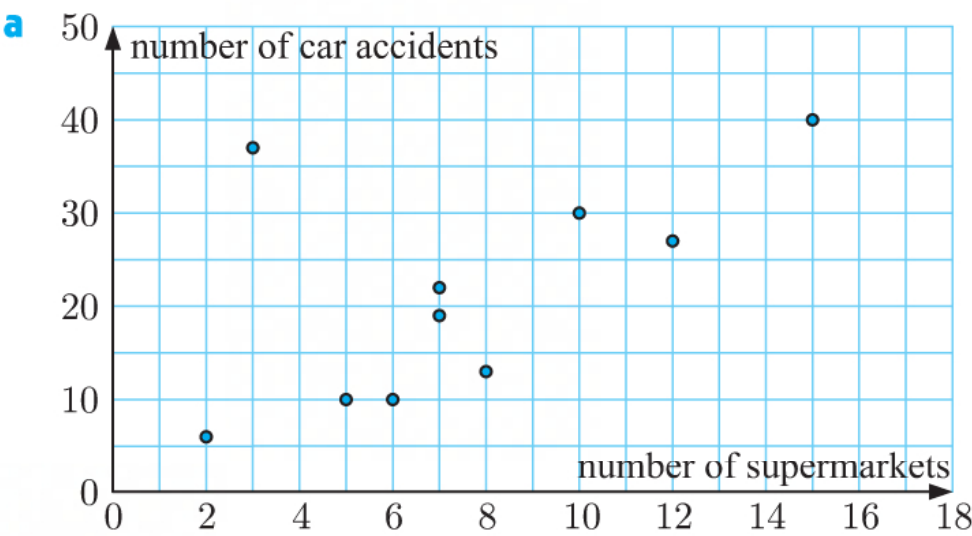
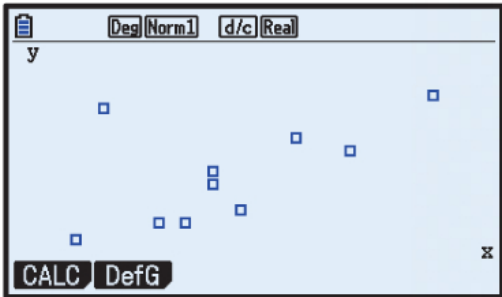
7

Number of supermarkets	5	8	12	7	6	2	15	10	7	3
Number of car accidents	10	13	27	19	10	6	40	30	22	37

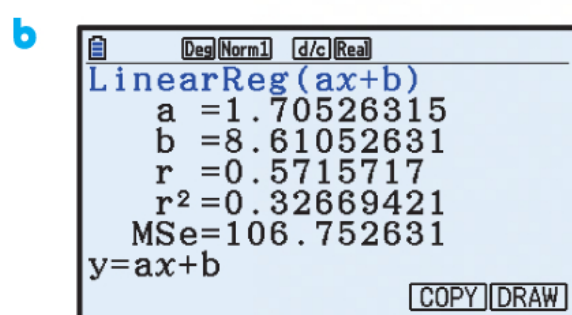
	List 1	List 2	List 3	List 4
SUB				
1	5	10		
2	8	13		
3	12	27		
4	7	19		

19

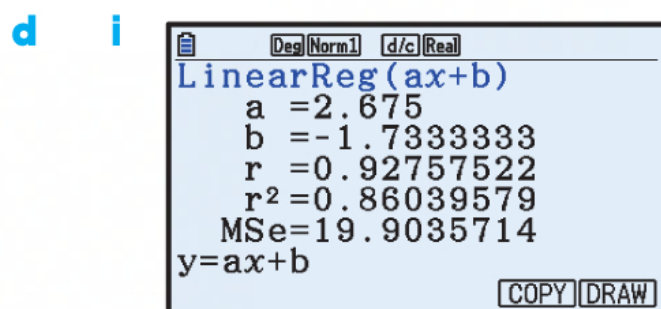
GRAPH CALC TEST INTR DIST







So,  $r \approx 0.572$ .



So,  $r \approx 0.928$ .

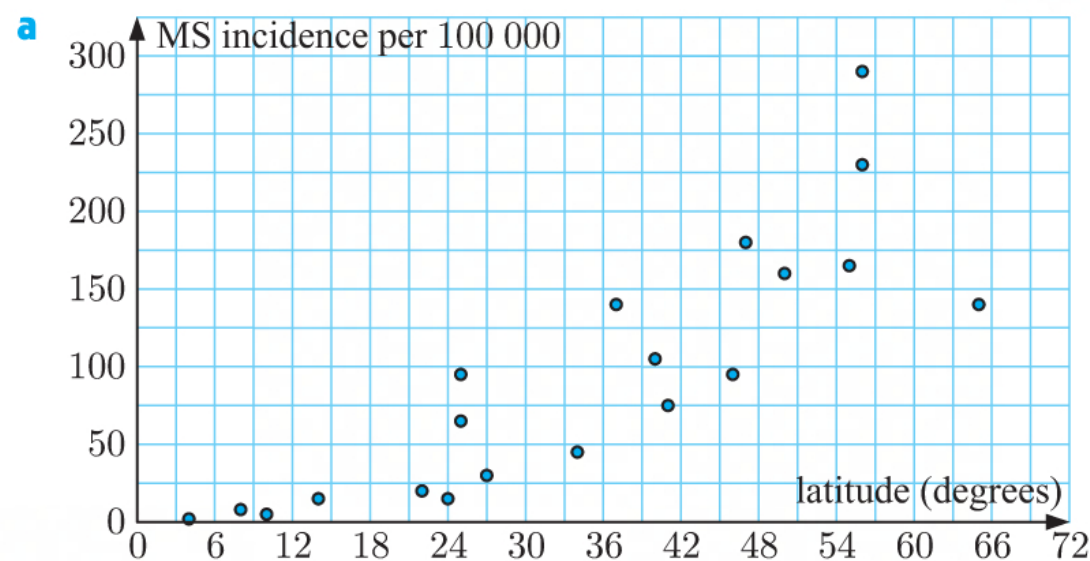
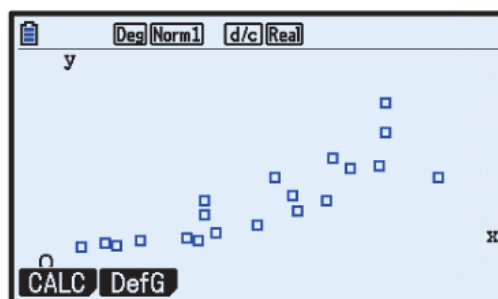
- ii** There is a strong, positive correlation between the *number of supermarkets* and the *number of car accidents*.
- iii** By removing the outlier, the value of  $r$  increased significantly.
- e** No, it is not a causal relationship. Both variables depend on the number of people in each town, not on each other.

**8**

<i>Latitude (degrees)</i>	55	25	41	22	47	37	56	14	34	25
<i>MS incidence per 100 000</i>	165	95	75	20	180	140	230	15	45	65

<i>Latitude (degrees)</i>	27	65	10	24	4	56	46	8	50	40
<i>MS incidence per 100 000</i>	30	140	5	15	2	290	95	8	160	105

	List 1	List 2	List 3	List 4
SUB				
1	55	165		
2	25	95		
3	41	75		
4	22	20		



b

LinearReg(ax+b)
a = 3.90314994
b = -39.878043
r = 0.84940985
r <sup>2</sup> = 0.7214971
MSe = 1972.69789
y = ax + b
<span>COPY</span> <span>DRAW</span>

So,  $r \approx 0.849$ .

d The incidence of MS is higher near the poles.

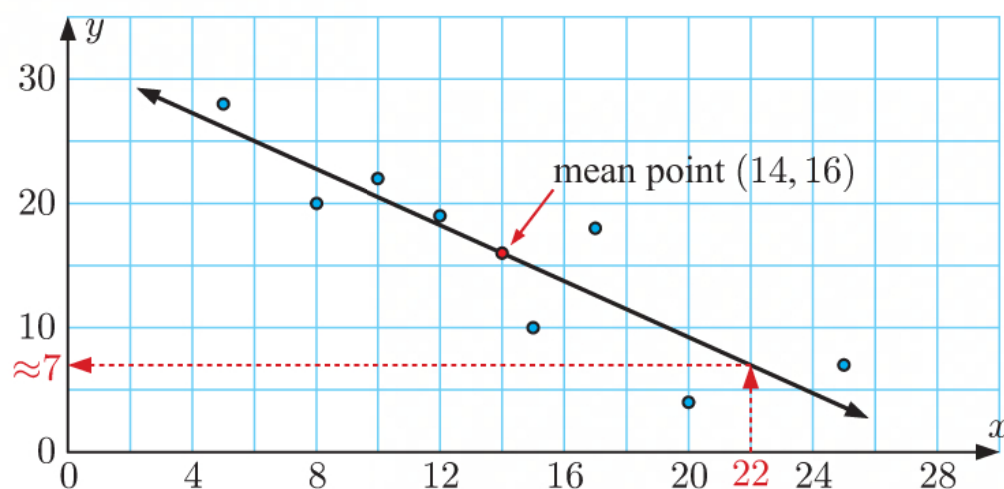
c There is a moderate, positive correlation between *latitude* and *MS incidence*.

## EXERCISE 26C

1

$x$	5	12	20	17	10	8	25	15
$y$	28	19	4	18	22	20	7	10

a, f



b The data appears to be negatively correlated.

c

LinearReg(ax+b)
a = -1.0953947
b = 31.3355263
r = -0.8809647
r <sup>2</sup> = 0.77609882
MSe = 17.5389254
y = ax + b
<span>COPY</span> <span>DRAW</span>

So,  $r \approx -0.881$ .

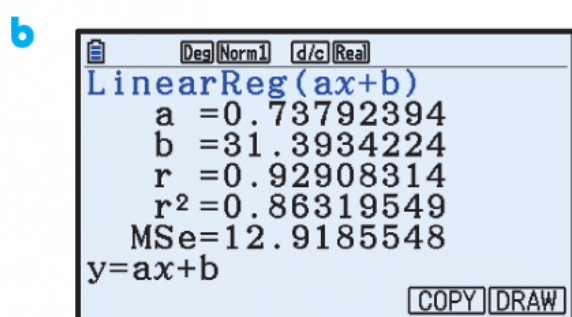
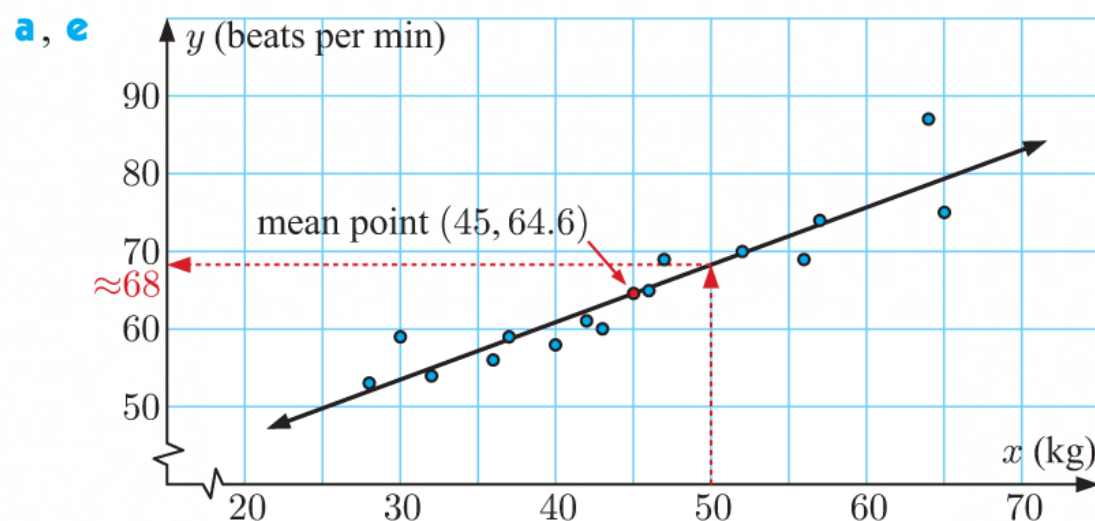
d There is a strong, negative correlation between  $x$  and  $y$ .

$$\begin{aligned} \bar{x} &= \frac{5 + 12 + 20 + 17 + 10 + 8 + 25 + 15}{8}, & \bar{y} &= \frac{28 + 19 + 4 + 18 + 22 + 20 + 7 + 10}{8} \\ &= 14, & &= 16 \end{aligned}$$

So the mean point is  $(14, 16)$ .

g When  $x = 22$ ,  $y \approx 7$ .

<b>2</b>	<i>Weight (<math>x</math> kg)</i>	46	37	32	57	47	64	42	30	52	56	65	43	36	28	40
	<i>Pulse rate (<math>y</math> beats per min)</i>	65	59	54	74	69	87	61	59	70	69	75	60	56	53	58



So,  $r \approx 0.929$ .

**c** There is a strong, positive correlation between the *weight* of a student and their *pulse rate*.

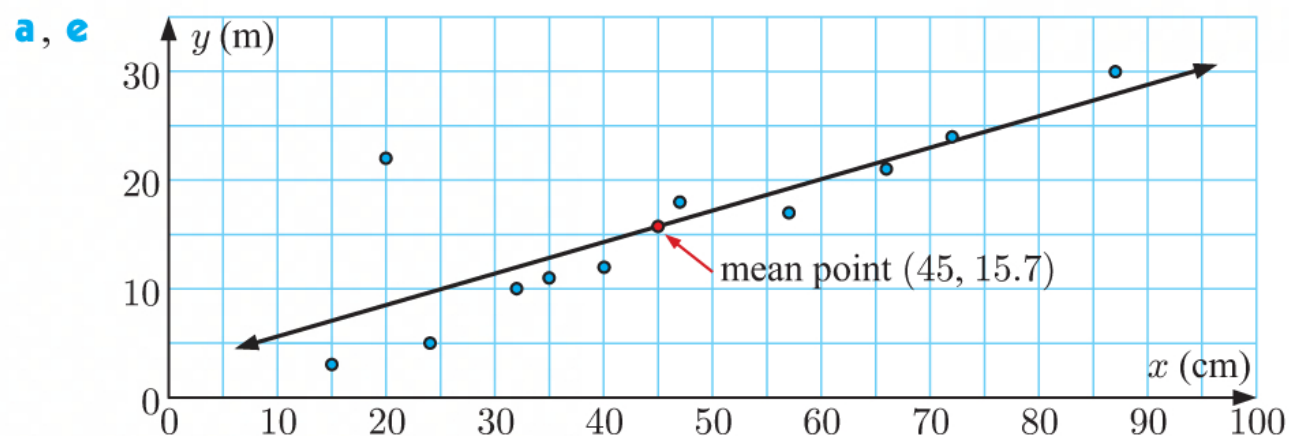
**d**  $\bar{x} = \frac{46 + 37 + \dots + 28 + 40}{15}, \quad \bar{y} = \frac{65 + 59 + \dots + 53 + 58}{15}$   
 $= 45 \qquad \qquad \qquad = 64.6$

So the mean point is  $(45, 64.6)$ .

**f** When  $x = 50$ ,  $y \approx 68$ .

A student who weighs 50 kg will have a pulse rate of approximately 68 beats per minute. This is an interpolation, so the estimate is reliable.

<b>3</b>	<i>Trunk width (<math>x</math> cm)</i>	35	47	72	40	15	87	20	66	57	24	32
	<i>Height (<math>y</math> m)</i>	11	18	24	12	3	30	22	21	17	5	10



**b**  $(20, 22)$  is an outlier as it appears separated from the rest of the data.

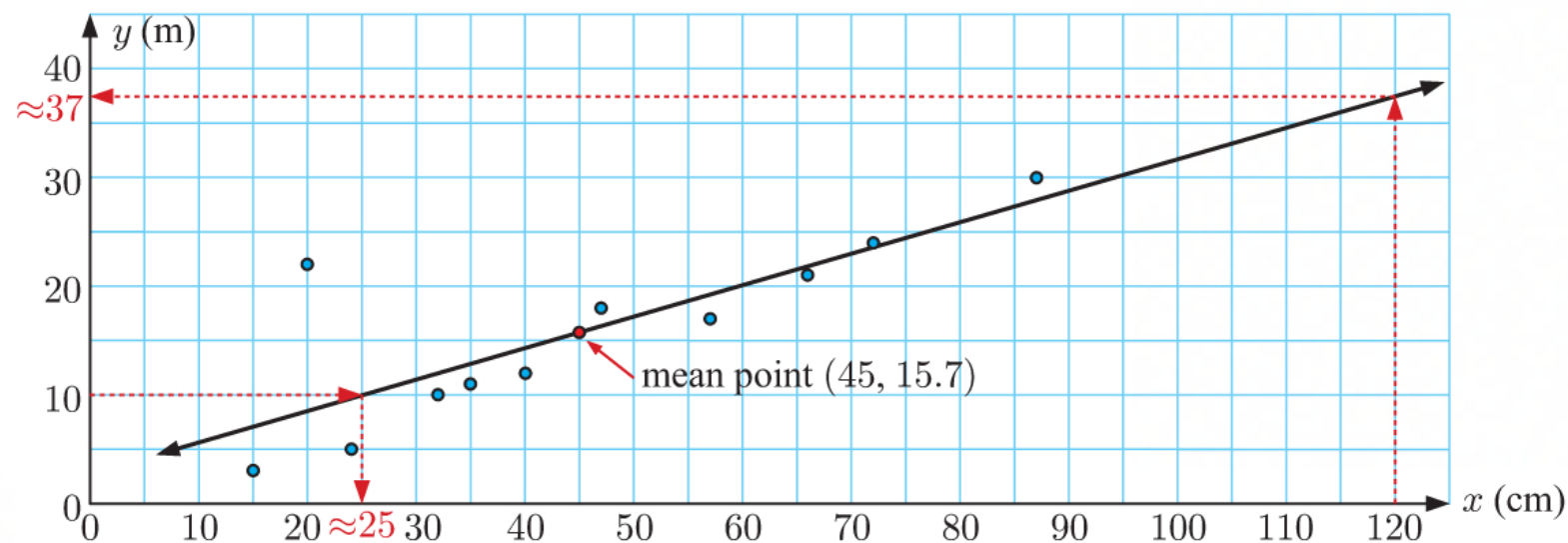
**c** The tree represented by the outlier would be very tall and thin.



$$\begin{aligned} \text{d } \bar{x} &= \frac{35 + 47 + \dots + 24 + 32}{11}, & \bar{y} &= \frac{11 + 18 + \dots + 5 + 10}{11} \\ &= 45 & &\approx 15.7 \end{aligned}$$

So the mean point is (45, 15.7).

f We extend the scatter plot from a to include the value  $x = 120$ :



When  $x = 120$ ,  $y \approx 37$ .

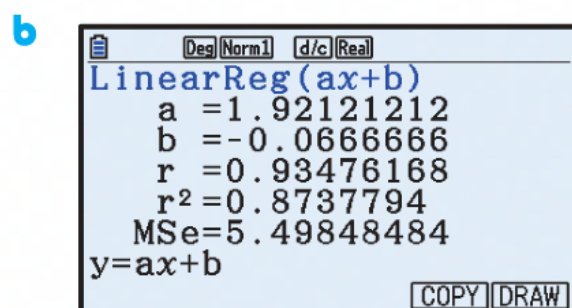
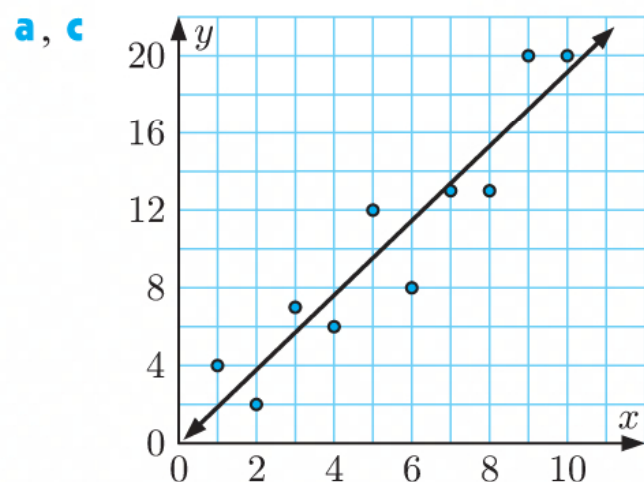
A tree with trunk width 120 cm will have a height of approximately 37 m. This is an extrapolation, so the prediction may not be reliable.

g When  $y = 10$ ,  $x \approx 25$ .

A tree with height 10 m will have a trunk width of approximately 25 cm. This is an interpolation, so the estimate is reliable.

## EXERCISE 26D

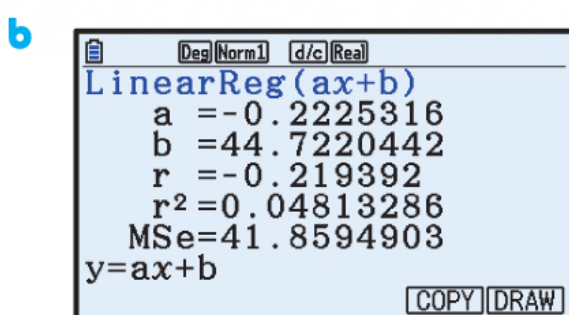
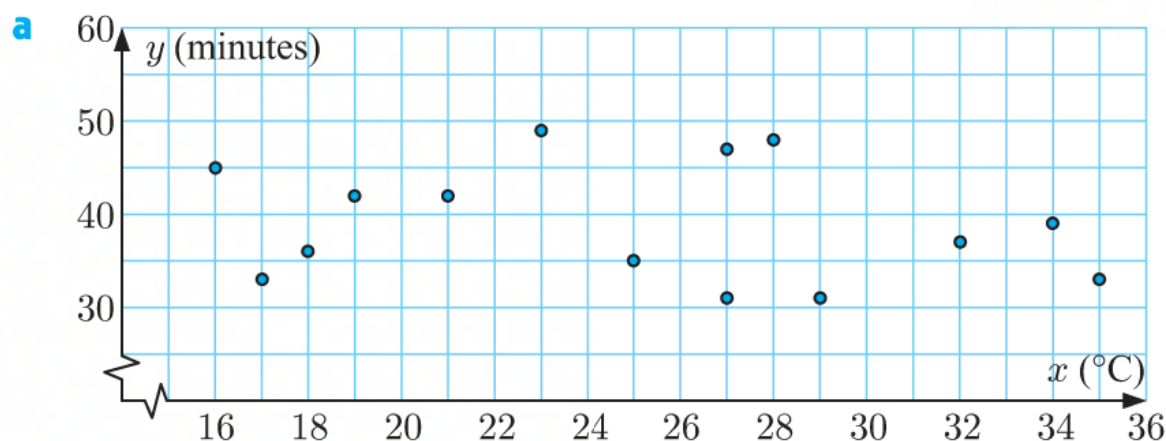
1	$x$	10	4	6	8	9	5	7	1	2	3
	$y$	20	6	8	13	20	12	13	4	2	7



Using technology, the least squares regression line is  $y \approx 1.92x - 0.0667$ .



<b>2</b>	<i>Temperature (<math>x</math> °C)</i>	25	19	23	27	32	35	29	27	21	18	16	17	28	34
	<i>Time (<math>y</math> minutes)</i>	35	42	49	31	37	33	31	47	42	36	45	33	48	39

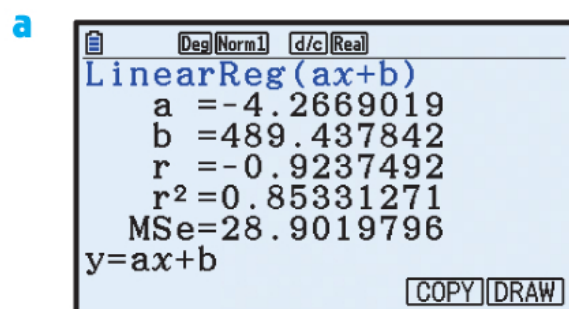


So,  $r \approx -0.219$ .

- c** There is a very weak, negative correlation between *temperature* and *time*.  
**d** No, it is not reasonable to find a line of best fit for this data as there is almost no correlation.

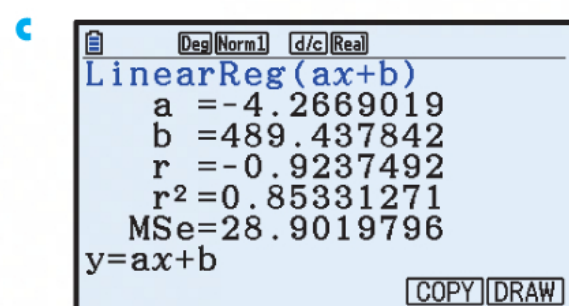
<b>3</b>	<i>Petrol price (<math>x</math> cents per litre)</i>	105.9	106.9	109.9	104.5	104.9	111.9	110.5	112.9
	<i>Number of customers (<math>y</math>)</i>	45	42	25	48	43	15	19	10

<i>Petrol price (<math>x</math> cents per litre)</i>	107.5	108.0	104.9	102.9	110.9	106.9	105.5	109.5
<i>Number of customers (<math>y</math>)</i>	30	23	42	50	12	24	32	17



So,  $r \approx -0.924$ .

- b** There is a strong, negative correlation between *petrol price* and the *number of customers*.



Using technology, the least squares regression line is  $y \approx -4.27x + 489$ .

- d** The gradient of the least squares regression line  $\approx -4.27$ . This means that for every cent per litre the petrol price increases by, the number of customers will decrease by approximately 4.27.

**e** When  $x = 115.9$ ,  $y \approx -4.27(115.9) + 489$   
 $\approx -5.10$

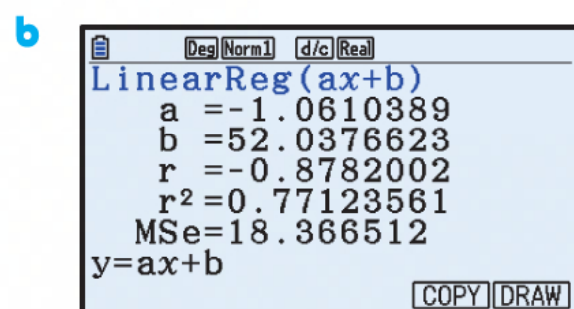
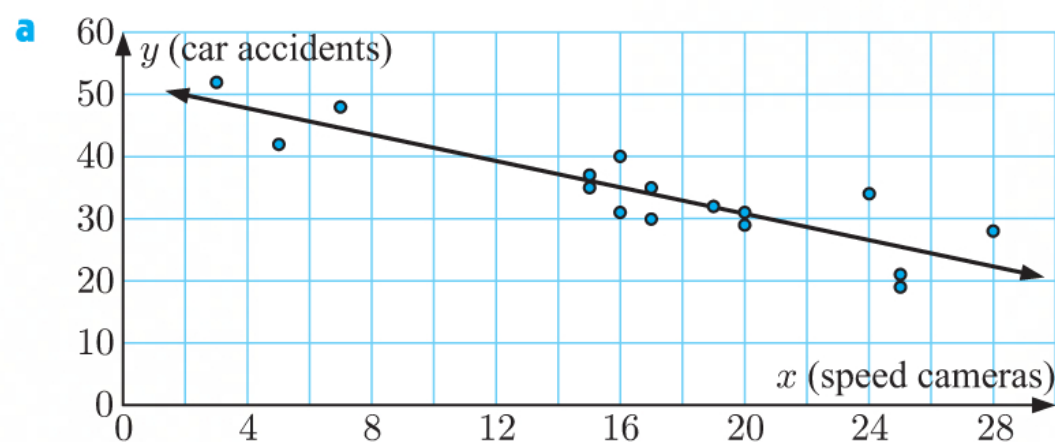
So, when petrol is 115.9 cents per litre, we would expect about  $-5.10$  customers per hour.

**f** When  $y = 40$ ,  $40 \approx -4.27x + 489$   
 $\therefore -449 \approx -4.27x$   
 $\therefore x \approx 105.3$

So, a petrol station which has 40 customers per hour would sell petrol at approximately 105.3 cents per litre.

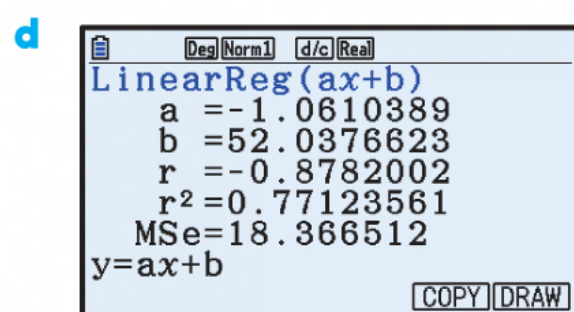
- g** In **e**, it is impossible to have a negative number of customers. This extrapolation is not valid. In **f**, this is an interpolation, so this estimate is likely to be reliable.

<b>4</b>	Number of speed cameras ( $x$ )	7	15	20	3	16	17	28	17	24	25	20	5	16	25	15	19
	Number of car accidents ( $y$ )	48	35	31	52	40	35	28	30	34	19	29	42	31	21	37	32



So,  $r \approx -0.878$ .

- c** There is a strong, negative correlation between the *number of speed cameras* and the *number of car accidents*.



Using technology, the least squares regression line is  $y \approx -1.06x + 52.0$ .

- e The gradient of the least squares regression line  $\approx -1.06$ . This indicates that for every additional speed camera, the number of car accidents per week decreases by an average of 1.06.

The  $y$ -intercept of the least squares regression line  $\approx 52.0$ . This indicates that if there were no speed cameras in a city, an average of 52.0 car accidents would occur each week.

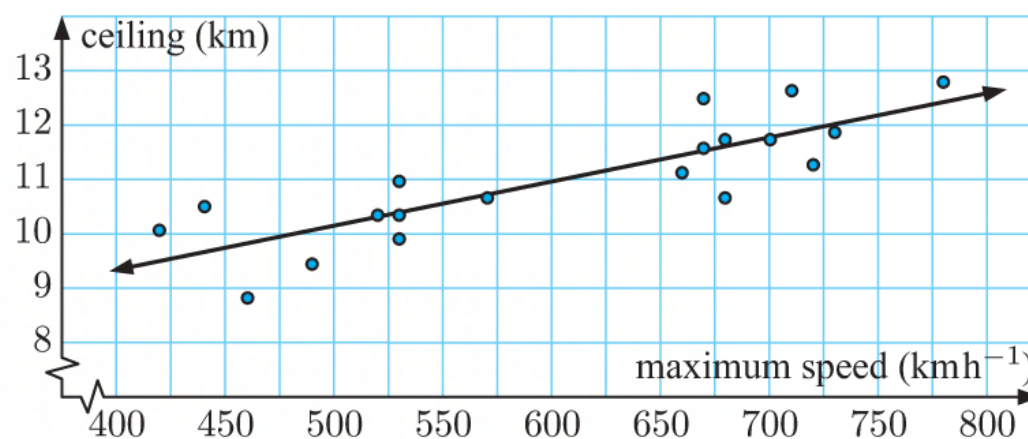
- f When  $x = 10$ ,  $y \approx -1.06(10) + 52.0$   
 $\approx 41.4$

So, there will be approximately 41.4 car accidents per week in a city with 10 speed cameras.

5

Maximum speed	Ceiling	Maximum speed	Ceiling	Maximum speed	Ceiling
460	8.84	680	10.66	670	12.49
420	10.06	720	11.27	570	10.66
530	10.97	710	12.64	440	10.51
530	9.906	660	11.12	670	11.58
490	9.448	780	12.80	700	11.73
530	10.36	730	11.88	520	10.36
680	11.73				

a, d



b

LinearReg(ax+b)
a = 8.1202E-03
b = 6.09013455
r = 0.84010344
r² = 0.70577379
MSe = 0.36102817
y = ax + b
COPY DRAW

So,  $r \approx 0.840$ .

- c There is a moderate, positive, linear correlation between *maximum speed* and *ceiling*.

d

LinearReg(ax+b)
a = 8.1202E-03
b = 6.09013455
r = 0.84010344
r² = 0.70577379
MSe = 0.36102817
y = ax + b
COPY DRAW

Using technology, the least squares regression line is  $y \approx 0.00812x + 6.09$ .

- e The gradient of the least squares regression line  $\approx 0.00812$ . This indicates that for each additional  $\text{km h}^{-1}$ , the ceiling increases by an average of approximately 0.00812 km or 8.12 m.



- f** When  $x = 600$ ,  $y \approx 0.00812(600) + 6.09$   
 $\approx 11.0$

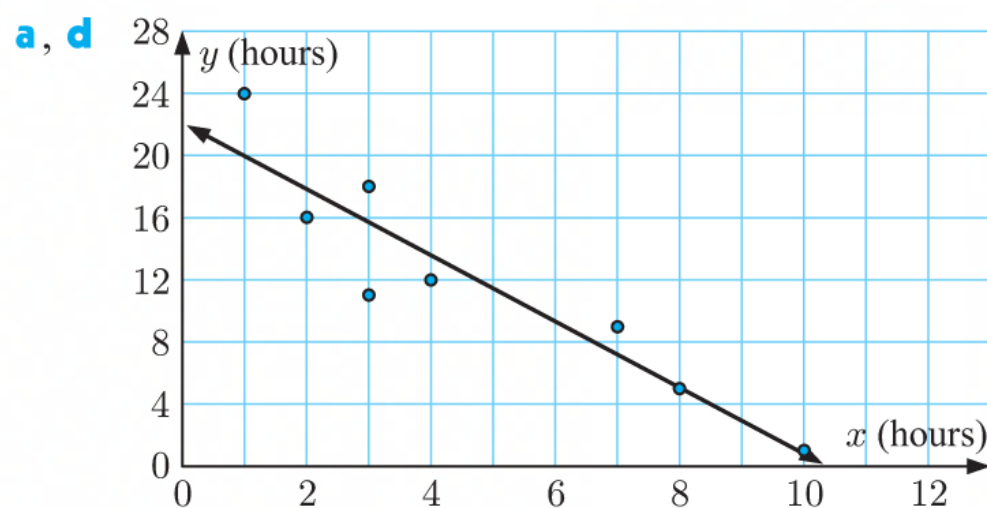
So, a fighter plane with maximum speed  $600 \text{ km h}^{-1}$  would have a ceiling of approximately 11.0 km.

- g** When  $y = 11$ ,  $11 \approx 0.00812x + 6.09$   
 $\therefore 4.91 \approx 0.00812x$   
 $\therefore x \approx 605$

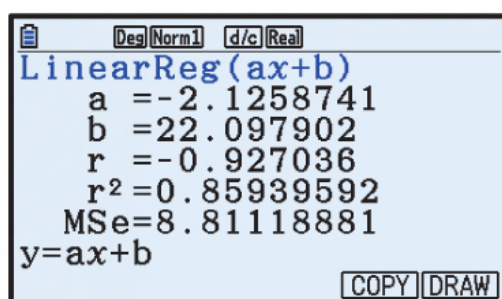
So, a fighter plane with a ceiling of 11 km would have maximum speed of approximately  $605 \text{ km h}^{-1}$ .

6

Exercise ( $x$ hours per week)	4	1	8	7	10	3	3	2
Television ( $y$ hours per week)	12	24	5	9	1	18	11	16



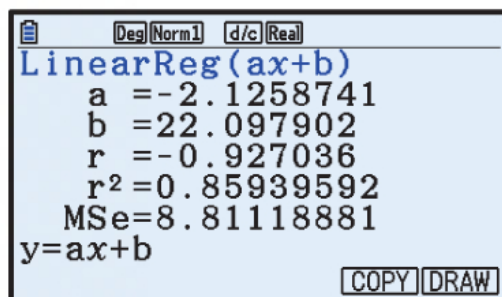
b



So,  $r \approx -0.927$ .

- c** There is a strong, negative, linear correlation between *time exercising* and *time watching television*.

d



Using technology, the least squares regression line is  $y \approx -2.13x + 22.1$ .

- e** The gradient of the least squares regression line  $\approx -2.13$ . This indicates that for each additional hour a child exercises each week, the number of hours they spend watching television each week decreases by 2.13.

The  $y$ -intercept of the least squares regression line  $\approx 22.1$ . This indicates that for children who do not spend time exercising, they would watch television for an average of about 22.1 hours per week.



- f** **i** From the table, the student who exercised for 7 hours each week watched 9 hours of television each week.

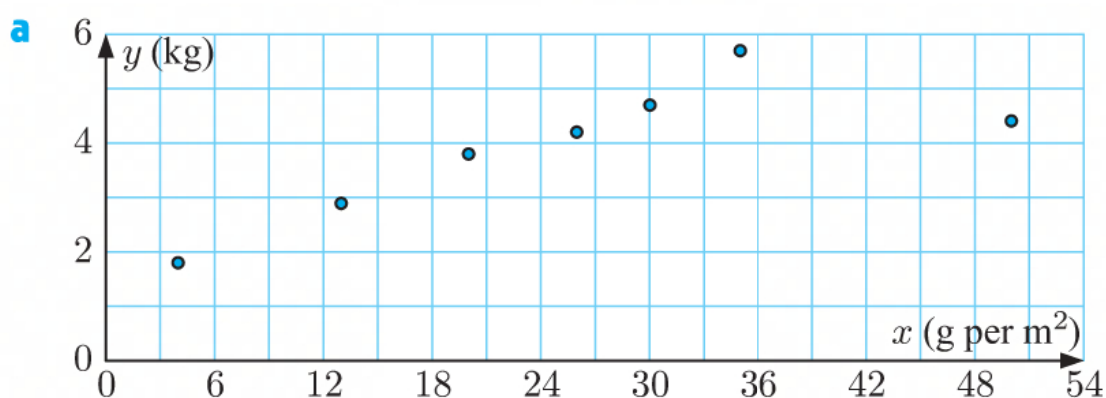
**ii** When  $x = 7$ ,  $y \approx -2.126(7) + 22.1$   
 $\approx 7.22$

Using the least squares regression line, a child who exercises for 7 hours each week watches approximately 7.22 hours of television each week.

- iii** This particular child spent more time watching television than predicted.

7

Fertiliser ( $x$ g per $\text{m}^2$ )	4	13	20	26	30	35	50
Yield ( $y$ kg)	1.8	2.9	3.8	4.2	4.7	5.7	4.4



(50, 4.4) is the outlier.

- b** **i** The outlier reduces the strength of correlation of the data.  
**ii** The outlier decreases the gradient of the least squares regression line.

c

**i**

```

LinearReg(ax+b)
a = 0.06715696
b = 2.22086572
r = 0.79782039
r² = 0.63651738
MSe = 0.70037907
y = ax + b
  
```

So,  $r \approx 0.798$ .

ii

```

LinearReg(ax+b)
a = 0.1187182
b = 1.31734486
r = 0.99257385
r² = 0.98520284
MSe = 0.03468082
y = ax + b
  
```

So,  $r \approx 0.993$ .

d

**i**

```

LinearReg(ax+b)
a = 0.06715696
b = 2.22086572
r = 0.79782039
r² = 0.63651738
MSe = 0.70037907
y = ax + b
  
```

Using technology, the least squares regression line is  $y \approx 0.0672x + 2.22$ .

ii

```

LinearReg(ax+b)
a = 0.1187182
b = 1.31734486
r = 0.99257385
r² = 0.98520284
MSe = 0.03468082
y = ax + b
  
```

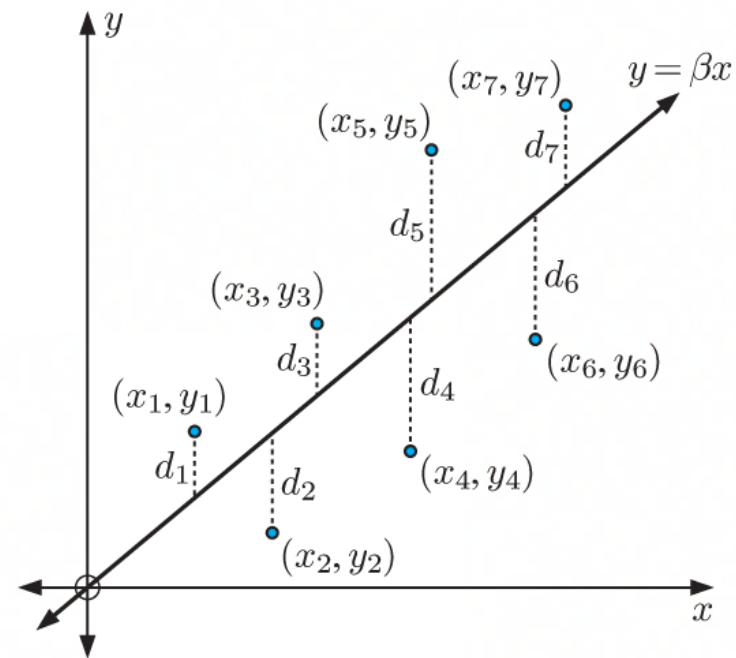
Using technology, the least squares regression line is  $y \approx 0.119x + 1.32$ .

- e** The regression line which excludes the outlier should be used to estimate the yield when 15 g per  $\text{m}^2$  of fertiliser is used. This will be more accurate for an interpolation.
- f** Too much fertiliser often kills the plants. In this case, the outlier should be kept when analysing the data as it is a valid data value. If the outlier is a recording error caused by bad measurement or recording skills, it should be removed before analysing data.

- 8** We are to fit a model of the form  $y = \beta x$  to the bivariate data set  $\{(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)\}$ .

**a** The sum of the squared vertical distances is

$$\begin{aligned}
 D &= \sum_{i=1}^n d_i^2 \\
 &= \sum_{i=1}^n (y_i - \beta x_i)^2 \\
 &= \sum_{i=1}^n (y_i^2 - 2\beta x_i y_i + \beta^2 x_i^2) \\
 &= \sum_{i=1}^n y_i^2 - 2\beta \sum_{i=1}^n x_i y_i + \beta^2 \sum_{i=1}^n x_i^2 \\
 &= \beta^2 \sum_{i=1}^n x_i^2 - 2\beta \sum_{i=1}^n x_i y_i + \sum_{i=1}^n y_i^2
 \end{aligned}$$



**b** From **a**, the equation for  $D$  is a quadratic in terms of  $\beta$ .

$$\begin{aligned}
 \text{Since } \sum_{i=1}^n x_i^2 > 0, \quad D \text{ is minimised when } \beta &= \frac{-\left(-2 \sum_{i=1}^n x_i y_i\right)}{2 \sum_{i=1}^n x_i^2} \\
 &= \frac{\sum_{i=1}^n x_i y_i}{\sum_{i=1}^n x_i^2}
 \end{aligned}$$

## ACTIVITY 2

## ANSCOMBE'S QUARTET

**1** Data set A:

$x$	10	8	13	9	11	14	6	4	12	7	5
$y$	8.04	6.95	7.58	8.81	8.33	9.96	7.24	4.26	10.84	4.82	5.68

1-Variable	
$\bar{x}$	=9
$\Sigma x$	=99
$\Sigma x^2$	=1001
$\sigma x$	=3.16227766
$sx$	=3.31662479
$n$	=11

$$\mu_x = 9, \quad \sigma_x \approx 3.16$$

1-Variable	
$\bar{x}$	=7.50090909
$\Sigma x$	=82.51
$\Sigma x^2$	=660.1763
$\sigma x$	=1.93710869
$sx$	=2.03165673
$n$	=11

$$\mu_y \approx 7.50, \quad \sigma_y \approx 1.94$$

Data set B:

$x$	10	8	13	9	11	14	6	4	12	7	5
$y$	9.14	8.14	8.74	8.77	9.26	8.1	6.13	3.1	9.13	7.26	4.74

$\bar{x}$	=9
$\Sigma x$	=99
$\Sigma x^2$	=1001
$\sigma x$	=3.16227766
$sx$	=3.31662479
$n$	=11

$$\mu_x = 9, \sigma_x \approx 3.16$$

$\bar{x}$	=7.50090909
$\Sigma x$	=82.51
$\Sigma x^2$	=660.1763
$\sigma x$	=1.93710869
$sx$	=2.03165673
$n$	=11

$$\mu_y \approx 7.50, \sigma_y \approx 1.94$$

Data set C:

$x$	10	8	13	9	11	14	6	4	12	7	5
$y$	7.46	6.77	12.74	7.11	7.81	8.84	6.08	5.39	8.15	6.42	5.73

$\bar{x}$	=9
$\Sigma x$	=99
$\Sigma x^2$	=1001
$\sigma x$	=3.16227766
$sx$	=3.31662479
$n$	=11

$$\mu_x = 9, \sigma_x \approx 3.16$$

$\bar{x}$	=7.5
$\Sigma x$	=82.5
$\Sigma x^2$	=659.9762
$\sigma x$	=1.93593294
$sx$	=2.0304236
$n$	=11

$$\mu_y = 7.5, \sigma_y \approx 1.94$$

Data set D:

$x$	8	8	8	8	8	8	8	19	8	8	8
$y$	6.58	5.76	7.71	8.84	8.47	7.04	5.25	12.5	5.56	7.91	6.89

$\bar{x}$	=9
$\Sigma x$	=99
$\Sigma x^2$	=1001
$\sigma x$	=3.16227766
$sx$	=3.31662479
$n$	=11

$$\mu_x = 9, \sigma_x \approx 3.16$$

$\bar{x}$	=7.50090909
$\Sigma x$	=82.51
$\Sigma x^2$	=660.1325
$\sigma x$	=1.93608064
$sx$	=2.03057851
$n$	=11

$$\mu_y \approx 7.50, \sigma_y \approx 1.94$$

- a** In each data set: The mean of  $x$  is 9.  
The mean of  $y$  is 7.5 (or very close to 7.5).
- b** In each data set: The population variance of  $x \approx 3.16$ .  
The population variance of  $y \approx 1.94$ .



**2** Data set A:

```

Rad Norm1 ab/c Real
LinearReg(ax+b)
a =0.5000909
b =3.0000909
r =0.81642051
r²=0.66654245
MSe=1.52918777
y=ax+b
COPY

```

The regression line is  $y \approx 0.500x + 3.00$ .

## Data set C:

```

Rad Norm1 ab/c Real
LinearReg(ax+b)
a =0.49972727
b =3.00245454
r =0.81628673
r²=0.66632404
MSe=1.52846575
y=ax+b
COPY

```

The regression line is  $y \approx 0.500x + 3.00$ .

The regression lines are almost identical for each data set.

## Data set B:

```

Rad Norm1 ab/c Real
LinearReg(ax+b)
a =0.5
b =3.00090909
r =0.8162365
r²=0.66624203
MSe=1.53069898
y=ax+b
COPY

```

The regression line is  $y \approx 0.5x + 3.00$ .

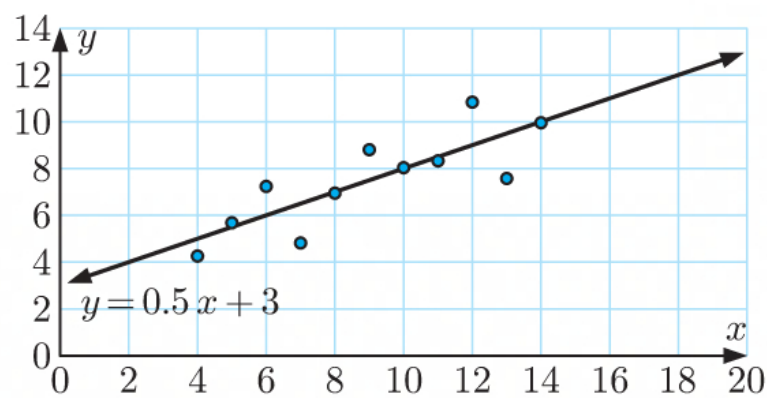
## Data set D:

```

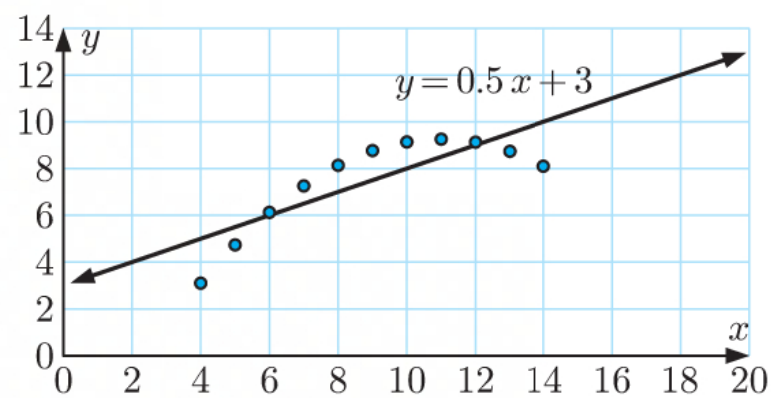
Rad Norm1 ab/c Real
LinearReg(ax+b)
a =0.49990909
b =3.00172727
r =0.81652143
r²=0.66670725
MSe=1.52694333
y=ax+b
COPY

```

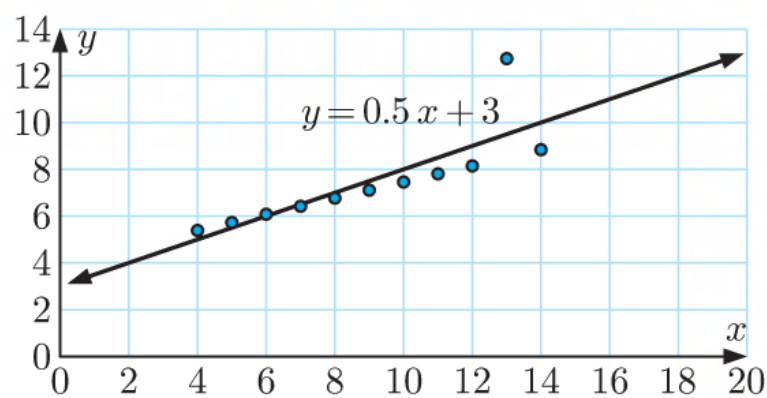
The regression line is  $y \approx 0.500x + 3.00$ .

**3** Data set A:

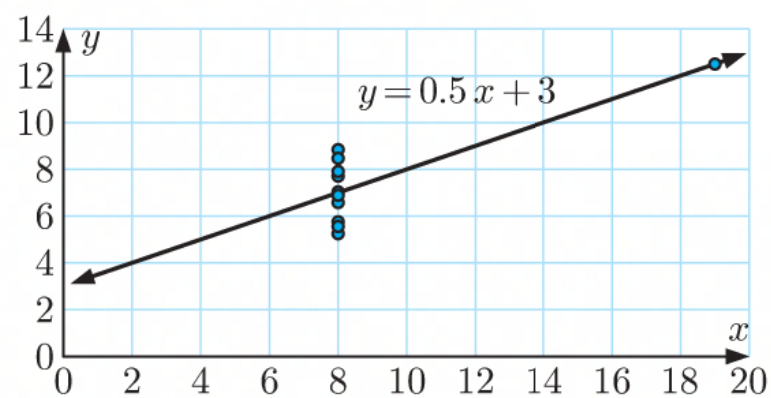
## Data set B:



## Data set C:



## Data set D:



- 4** Each data set has the same mean and variance for both variables, and the same regression line. However, we see that the scatter diagrams for each data set are wildly different from each other. A linear model is not necessarily appropriate for each data set.
- 5** A scatter diagram allows us to see patterns in data that cannot be conveyed with descriptive statistics alone.

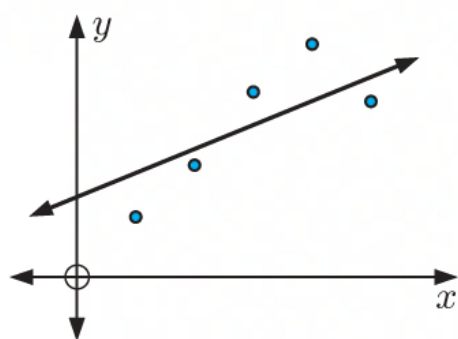


## ACTIVITY 3

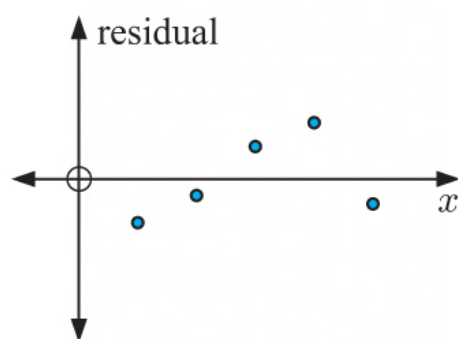
## RESIDUAL PLOTS

## PART 1: CONSTRUCTING A RESIDUAL PLOT

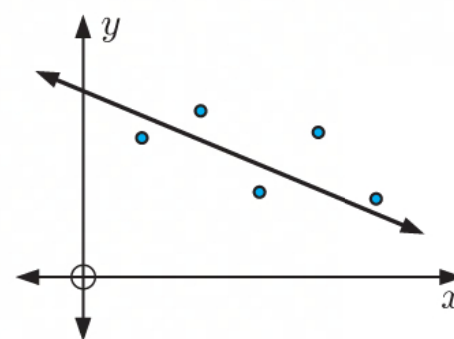
1 a



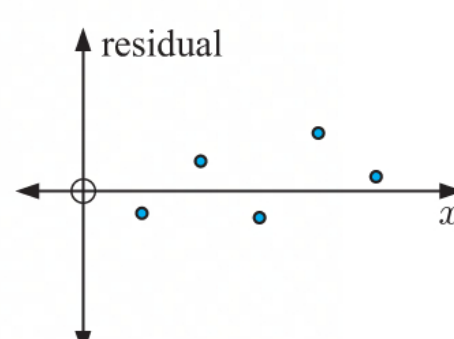
This scatter plot has 2 data points above the least squares regression line and 3 points below the regression line. This pattern is shown in the residual plot in **B**.



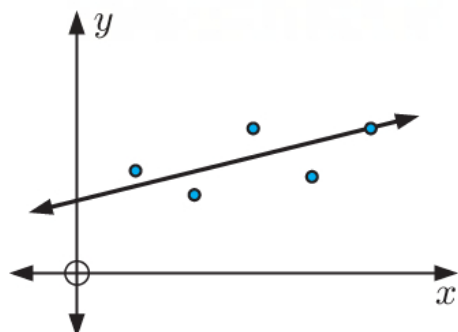
b



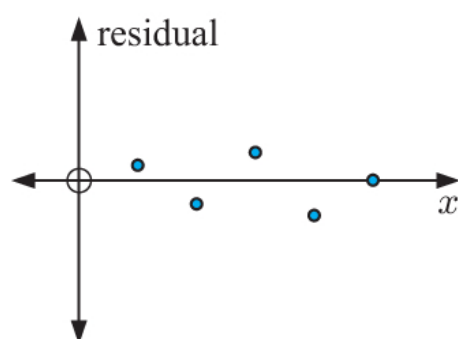
This scatter plot has 3 data points above the least squares regression line and 2 points below the regression line. This pattern is shown in the residual plot in **C**.

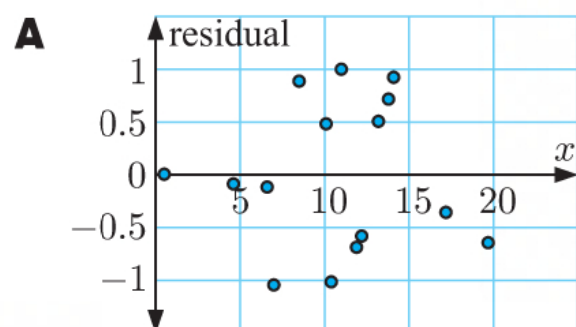
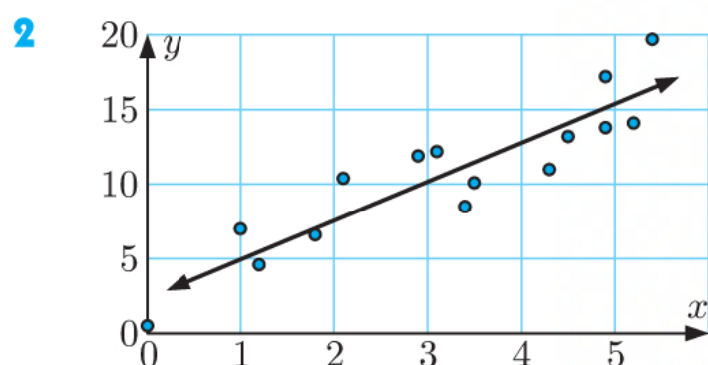


c



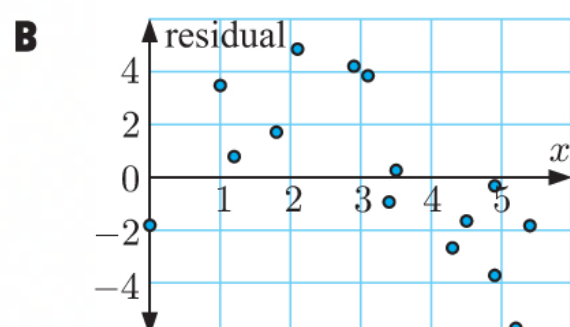
This scatter plot has 2 data points above the least squares regression line, 2 points below the regression line, and 1 point on the regression line. This pattern is shown in the residual plot in **A**.





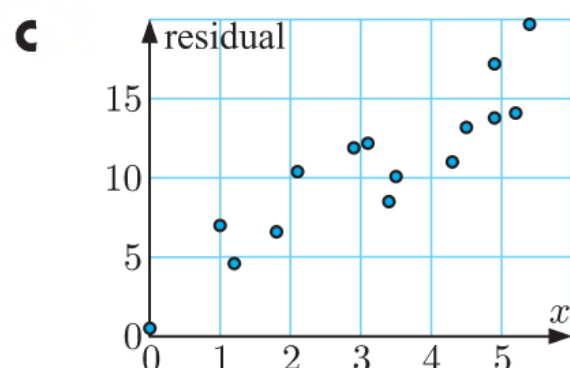
The values on the  $x$ -axis do not correspond to those on the scatter plot.

So **A** is not the correct residual plot.



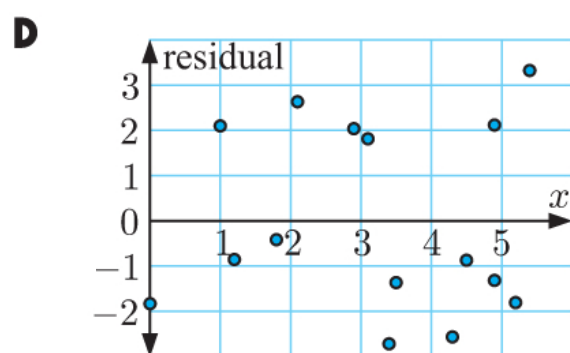
The residuals in this plot indicate that there are values which are more than 4 units from the regression line. The scatter plot however shows that the values are within about  $\pm 3$  of the regression line.

So **B** is not the correct residual plot.



The residuals in this plot indicate that all values are above the regression line. The scatter plot however shows that there are values below the regression line.

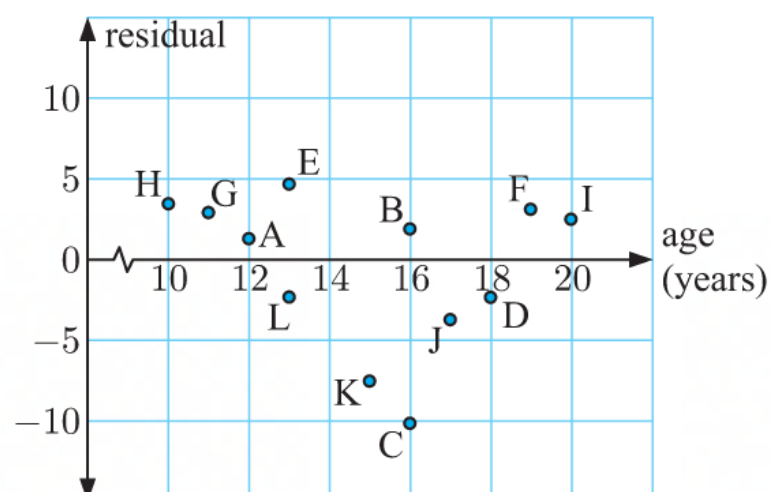
So **C** is not the correct residual plot.



The residuals in this plot indicate that all values are within about  $\pm 3$  of the regression line. The scatter plot shows that the values are within about  $\pm 3$  of the regression line.

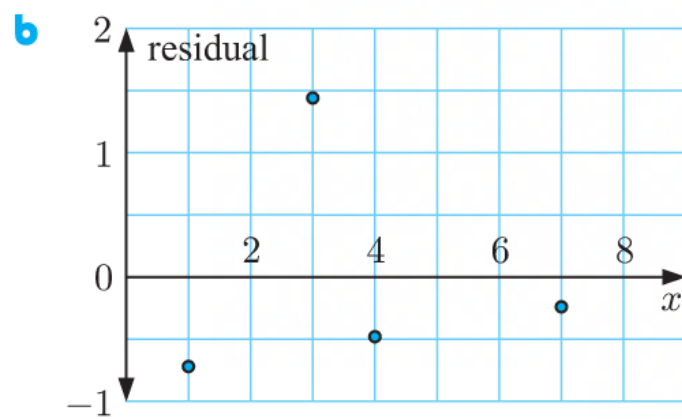
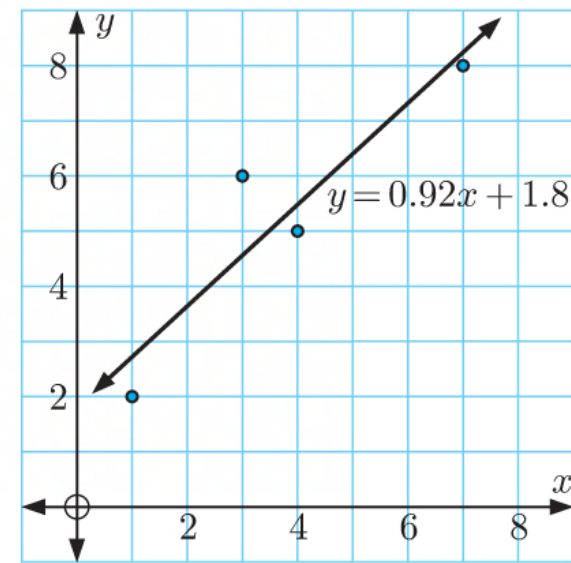
So **D** is the correct residual plot.

- 3**
- a** The athletes corresponding to the points above the  $x$ -axis threw the discus further than expected. These were athletes H, G, A, E, B, F, and I.
  - b** The athlete closest to the  $x$ -axis is A. So athlete A performed closest to what the linear model predicted.
  - c** No, it is not possible to determine which athlete threw the discus furthest. The residual plot only shows the difference between the actual distance and the predicted distance.



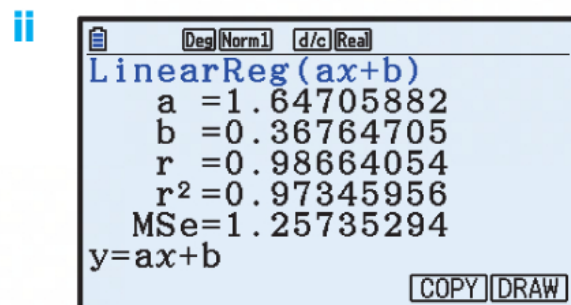
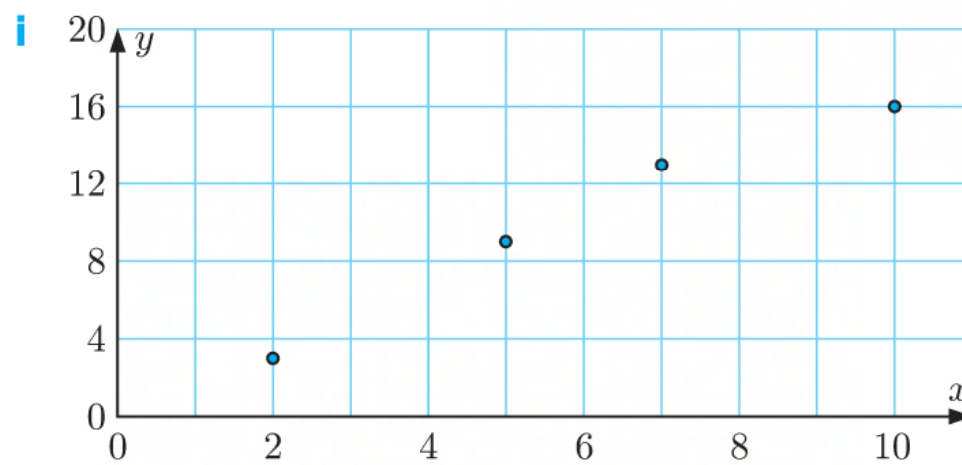
- 4 a** When  $x = 3$ ,  $y = 0.92(3) + 1.8 = 4.56$   
 When  $x = 4$ ,  $y = 0.92(4) + 1.8 = 5.48$   
 When  $x = 7$ ,  $y = 0.92(7) + 1.8 = 8.24$   
 So, the table is:

$x$	$y_{\text{obs}}$	$y_{\text{pred}}$	residual = $y_{\text{obs}} - y_{\text{pred}}$
1	2	2.72	-0.72
3	6	4.56	1.44
4	5	5.48	-0.48
7	8	8.24	-0.24



**5 a**

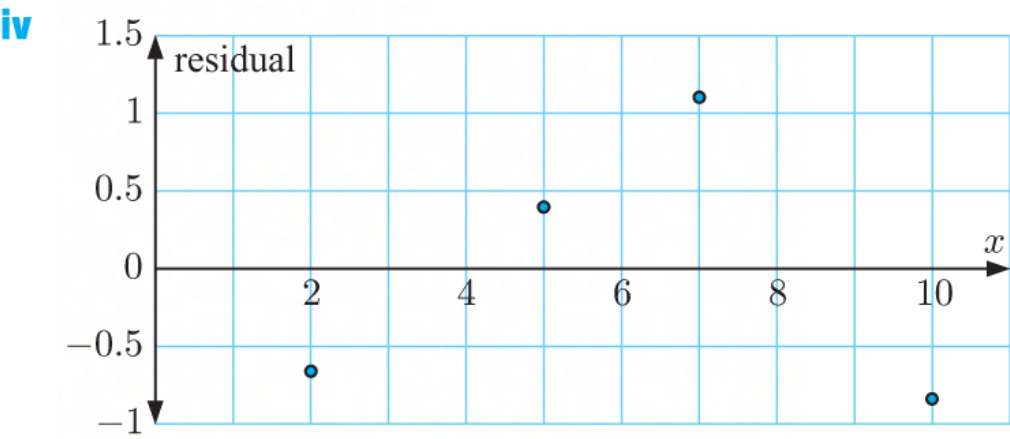
$x$	2	5	7	10
$y$	3	9	13	16



Using technology, the least squares regression line is  $y \approx 1.65x + 0.368$ .

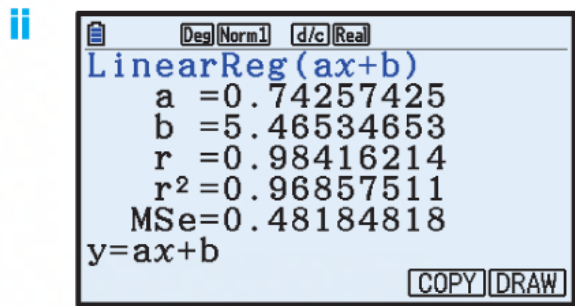
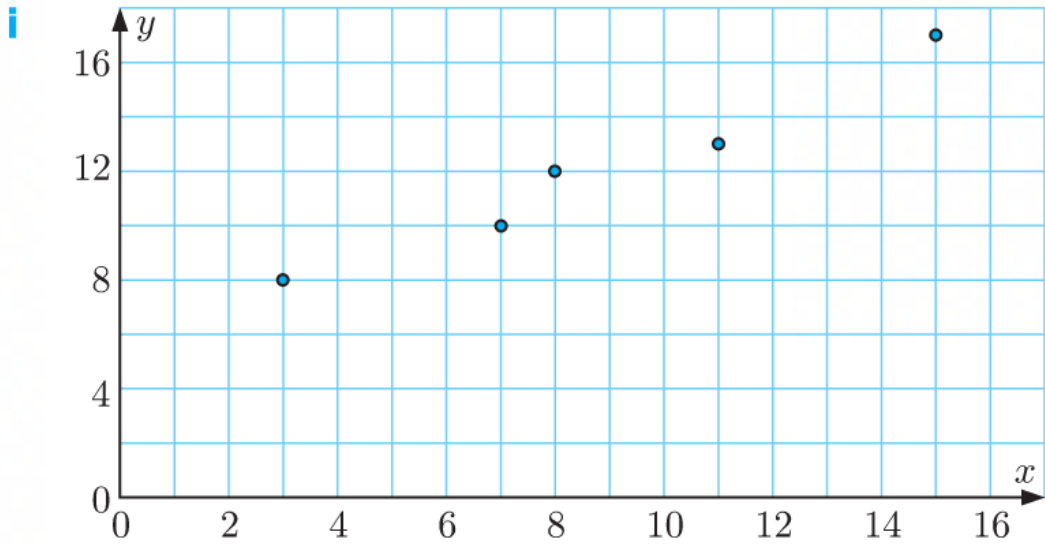
- iii** We find  $y_{\text{pred}}$  for each data point by evaluating  $y \approx 1.65x + 0.368$  for each of the  $x$ -values.

$x$	$y_{\text{obs}}$	$y_{\text{pred}}$	residual = $y_{\text{obs}} - y_{\text{pred}}$
2	3	3.66	-0.66
5	9	8.60	0.40
7	13	11.90	1.10
10	16	16.84	-0.84



**b**

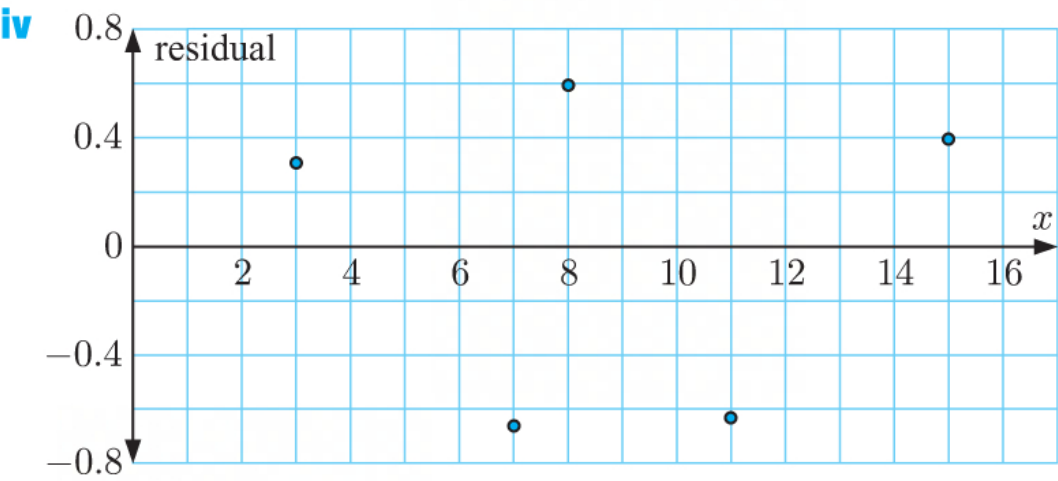
$x$	3	7	8	11	15
$y$	8	10	12	13	17



Using technology, the least squares regression line is  $y \approx 0.743x + 5.47$ .

**iii** We find  $y_{\text{pred}}$  for each data point by evaluating  $y \approx 0.743x + 5.47$  for each of the  $x$ -values.

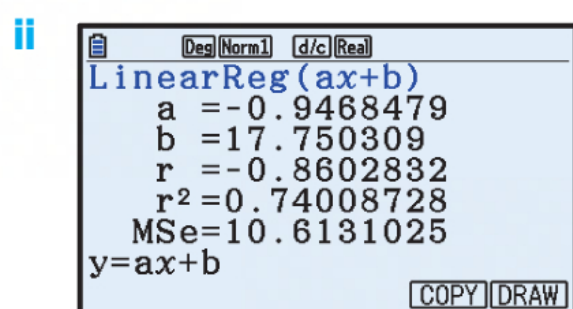
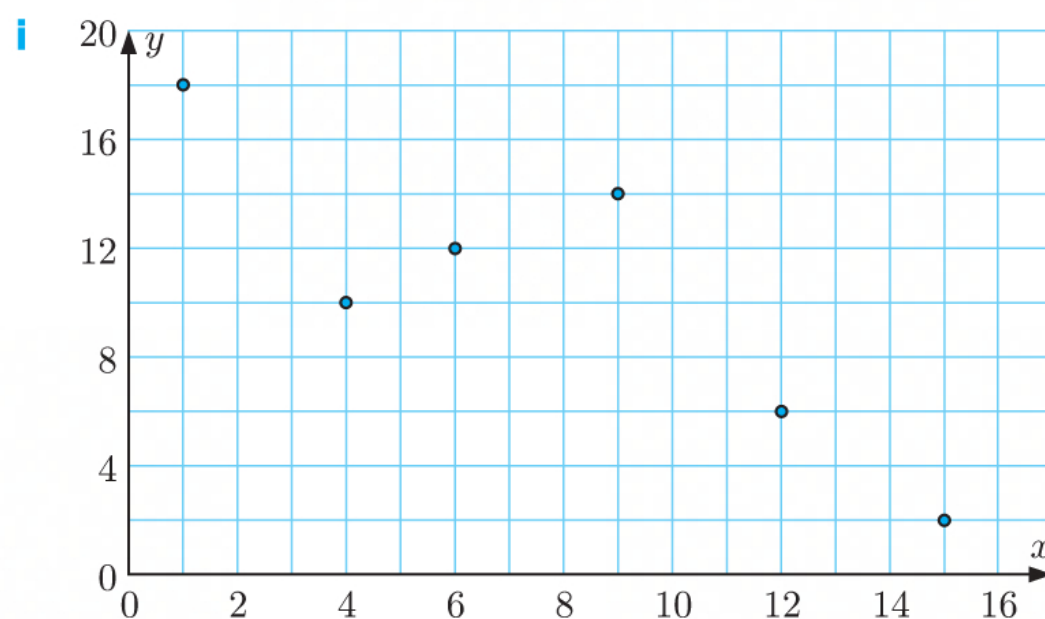
$x$	$y_{\text{obs}}$	$y_{\text{pred}}$	residual = $y_{\text{obs}} - y_{\text{pred}}$
3	8	7.69	0.31
7	10	10.66	-0.66
8	12	11.41	0.59
11	13	13.63	-0.63
15	17	16.60	0.40





**c**

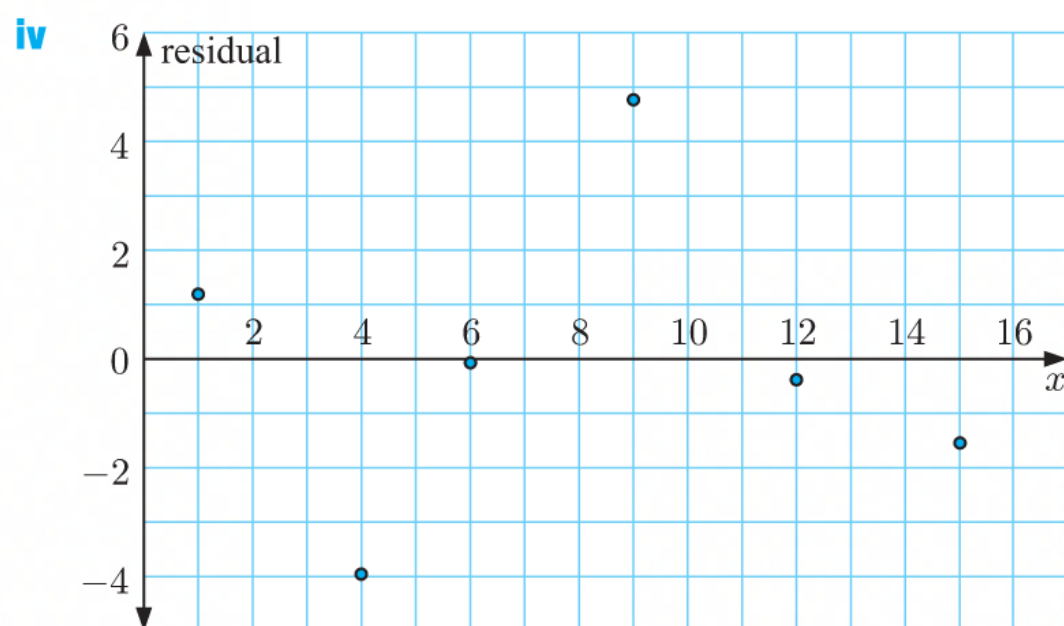
$x$	1	9	6	15	4	12
$y$	18	14	12	2	10	6

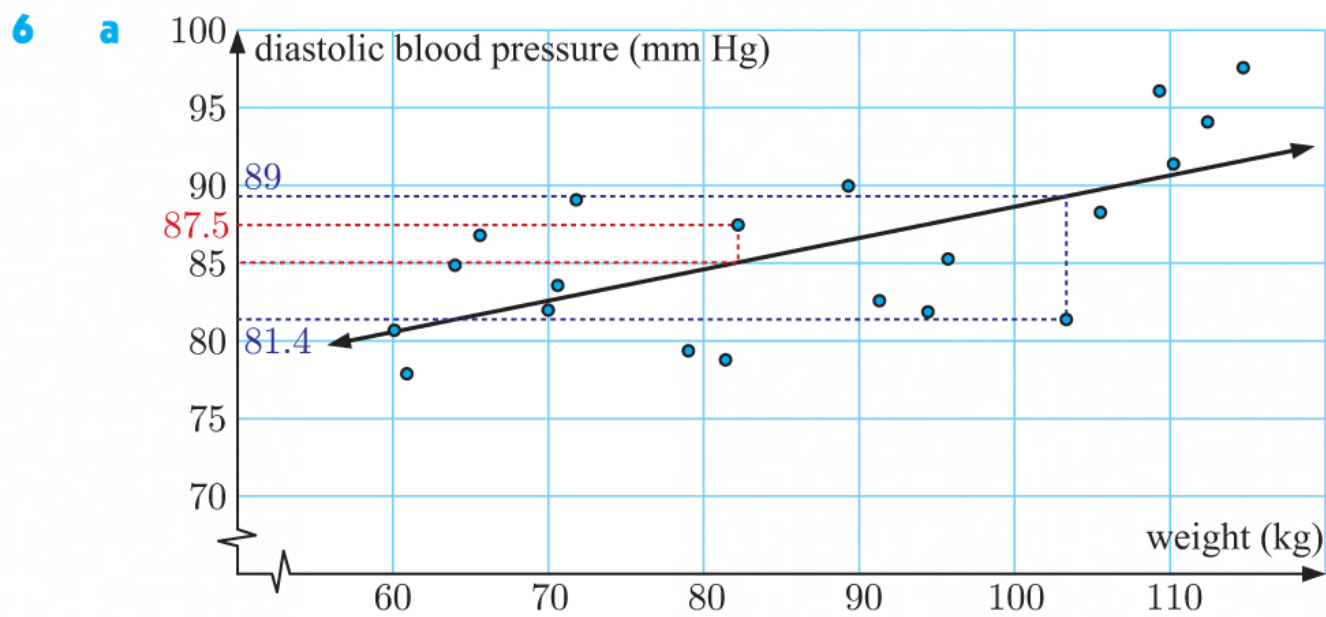


Using technology, the least squares regression line is  $y \approx -0.947x + 17.8$ .

- iii** We find  $y_{\text{pred}}$  for each data point by evaluating  $y \approx -0.947x + 17.8$  for each of the  $x$ -values.

$x$	$y_{\text{obs}}$	$y_{\text{pred}}$	residual = $y_{\text{obs}} - y_{\text{pred}}$
1	18	16.80	1.20
9	14	9.23	4.77
6	12	12.07	-0.07
15	2	3.55	-1.55
4	10	13.96	-3.96
12	6	6.39	-0.39





- b i** From the graph, the residual for the point (82, 87.5) is about 2.5.

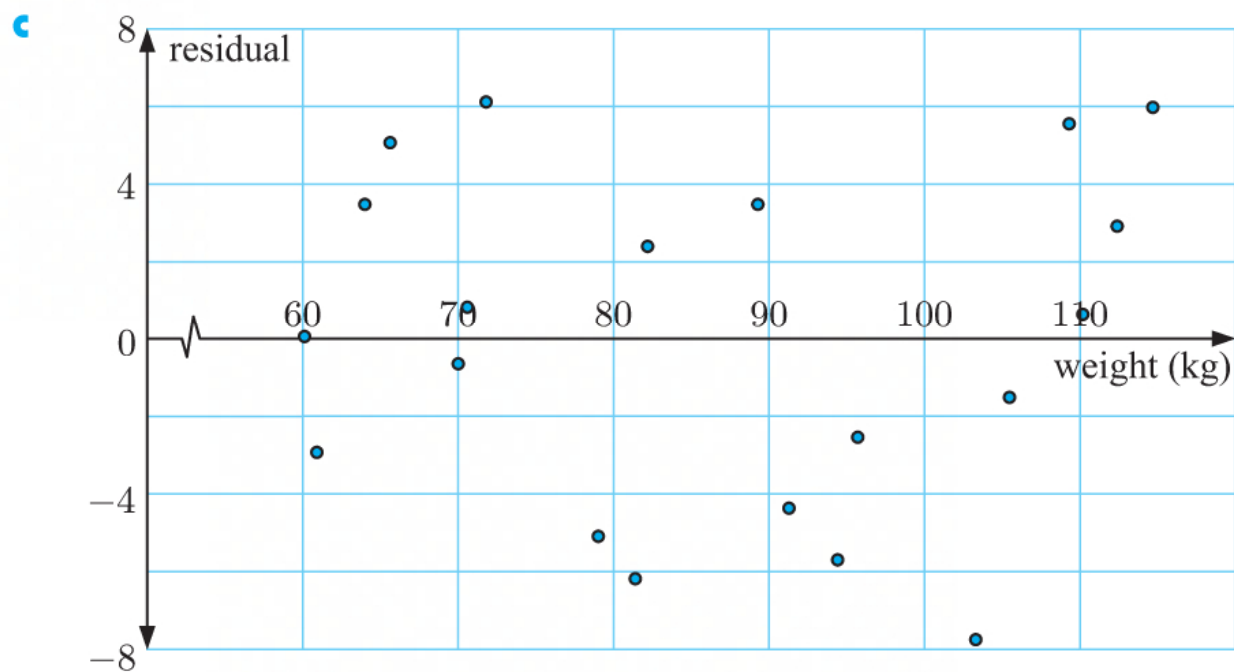
$$\begin{aligned} \text{From the equation, when } x = 82, \quad y_{\text{pred}} &= 68.5 + 0.2 \times 82 \\ &= 84.9 \end{aligned}$$

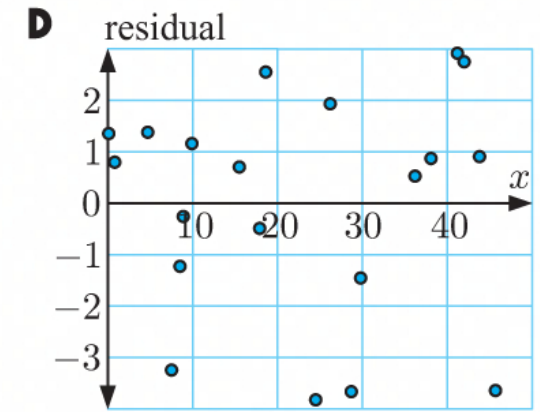
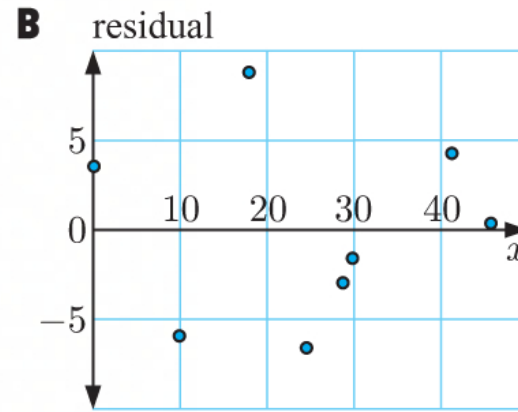
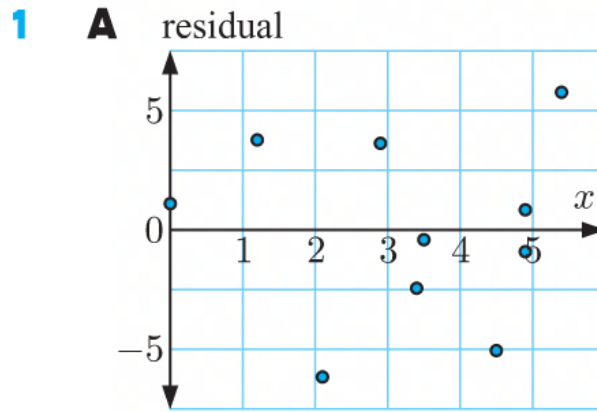
$$\begin{aligned} \therefore \text{ the residual} &= y_{\text{obs}} - y_{\text{pred}} \\ &= 87.5 - 84.9 \\ &= 2.6 \end{aligned}$$

- ii** From the graph, the residual for the point (103.3, 81.4) is about  $-7.6$ .

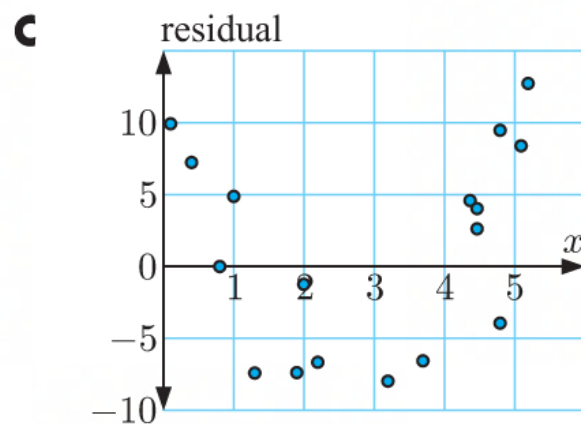
$$\begin{aligned} \text{From the equation, when } x = 103.3, \quad y_{\text{pred}} &= 68.5 + 0.2 \times 103.3 \\ &= 89.16 \end{aligned}$$

$$\begin{aligned} \therefore \text{ the residual} &= y_{\text{obs}} - y_{\text{pred}} \\ &= 81.4 - 89.16 \\ &= -7.76 \end{aligned}$$



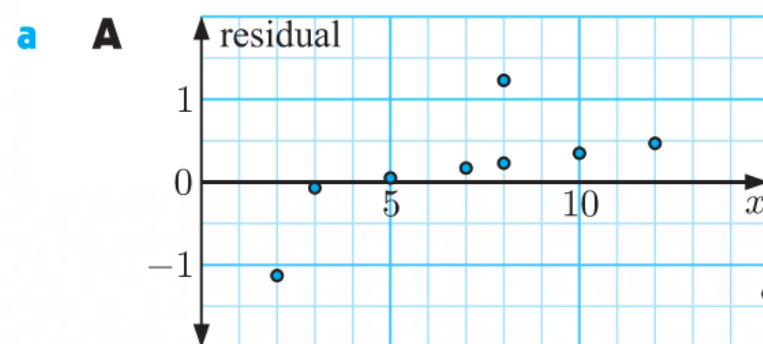
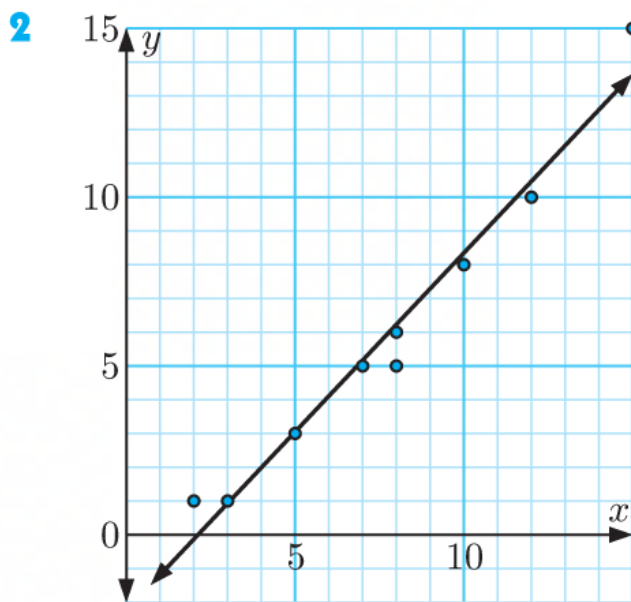
**PART 2: ANALYSING RESIDUAL PLOTS**

The residual plots for **A**, **B**, and **D** show points randomly scattered about the  $x$ -axis, with no obvious pattern.



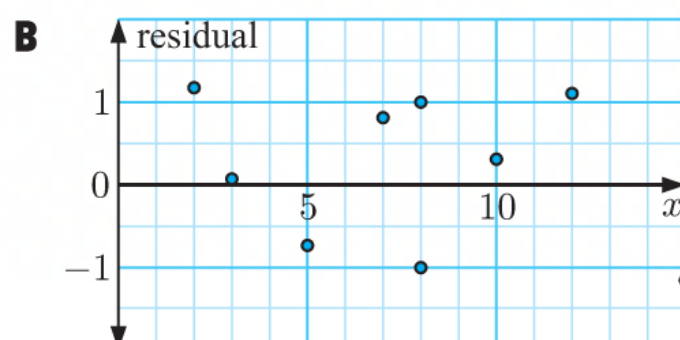
The residual plot for **C** however shows a clear, non-random pattern.

So the residual plot for **C** shows a regression line which is not a good fit for the data.



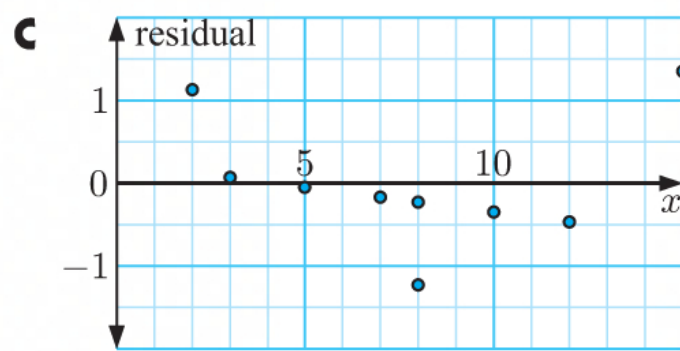
Most of the residuals in this plot are above the  $x$ -axis. The scatter plot however shows that only three data values are above the regression line.

So **A** is not the correct residual plot.



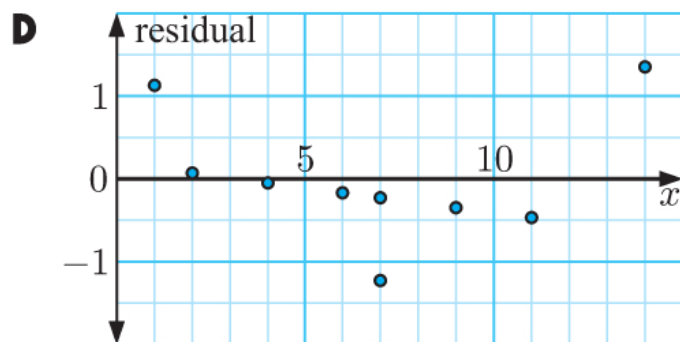
Most of the residuals in this plot are above the  $x$ -axis. The scatter plot however shows that only three data values are above the regression line.

So **B** is not the correct residual plot.



Most of the residuals in this plot are below the  $x$ -axis, with only three residuals above it. The scatter plot shows that only three data values are above the regression line, and the values on the  $x$ -axis correspond to those on the scatter diagram.

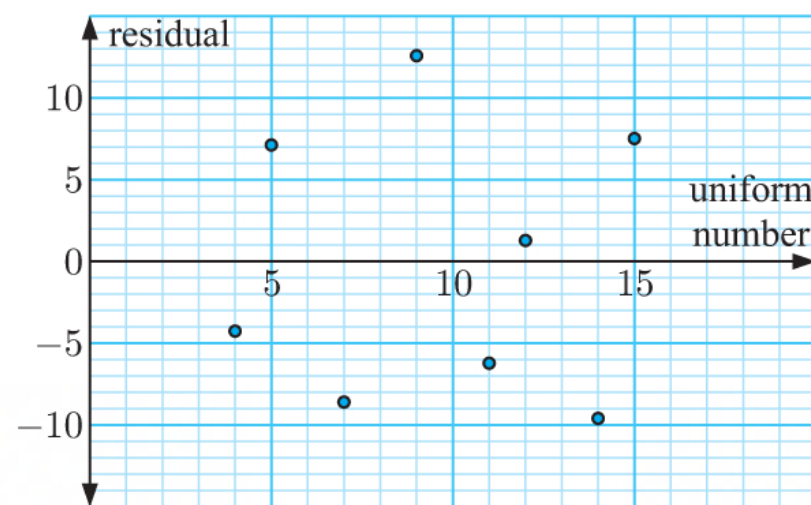
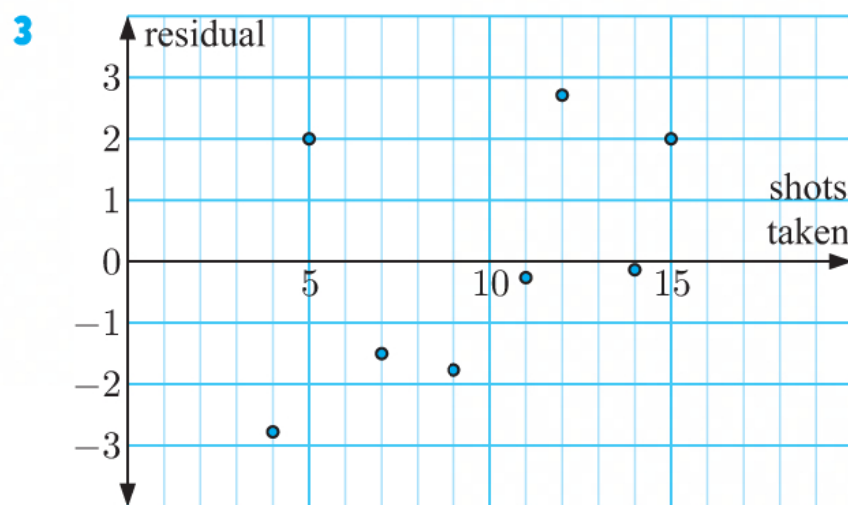
So **C** is the correct residual plot.



This is very similar to residual plot **C**, except the values on the  $x$ -axis do not correspond to those on the scatter diagram.

So **D** is not the correct residual plot.

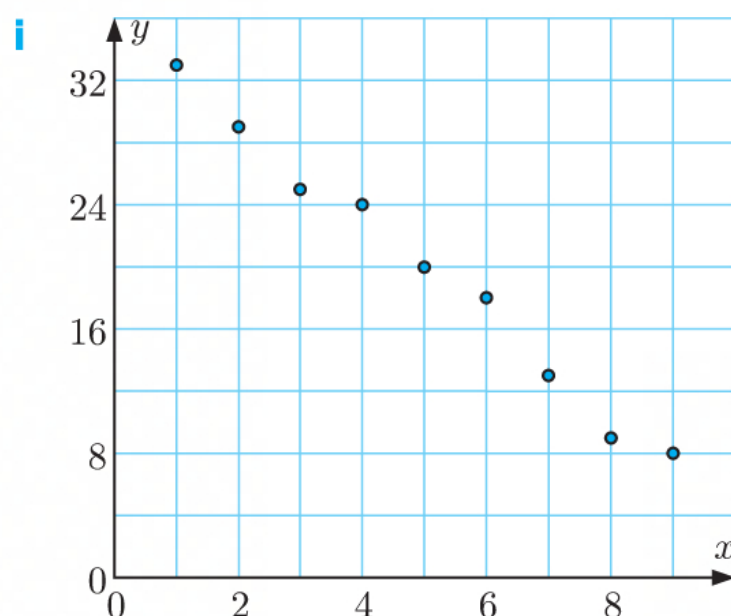
- b** The residual plot does not appear to be random, so a linear model may not be appropriate for the data.



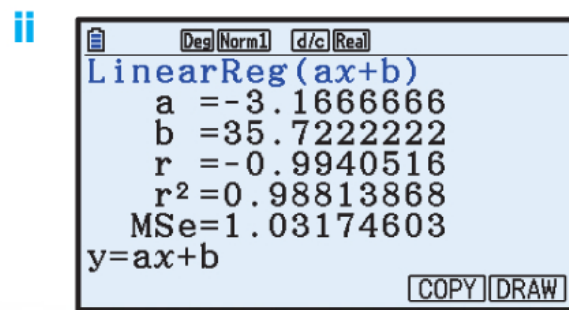
- a** Yes, the points in both plots appear to be randomly scattered.
- b** A linear model is most appropriate for the *points scored* vs *shots taken* data set. The residuals in this plot are generally smaller, which means that the points are generally closer to the least squares regression line.

**4 a**

$x$	1	2	3	4	5	6	7	8	9
$y$	33	29	25	24	20	18	13	9	8

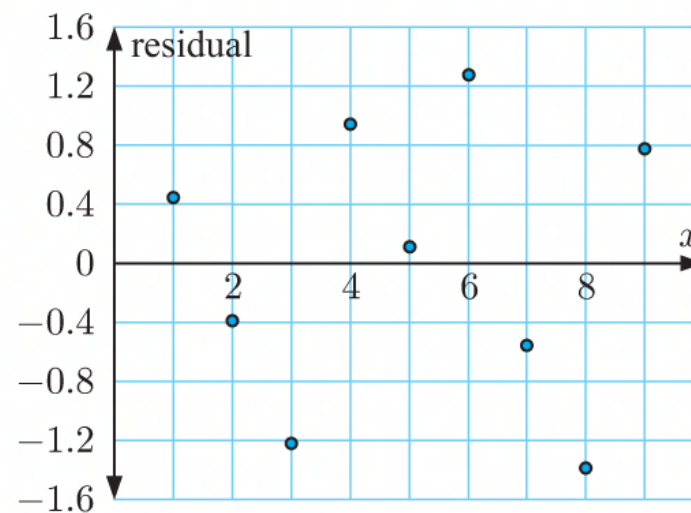






Using technology, the least squares regression line is  $y \approx -3.17x + 35.7$ , and  $r \approx -0.994$ .

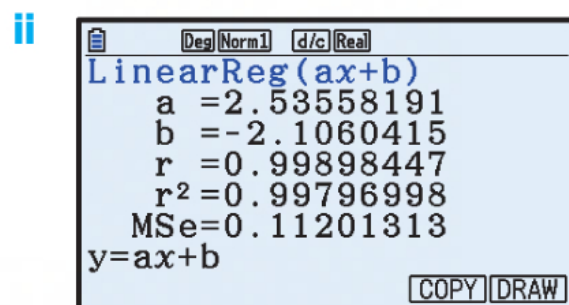
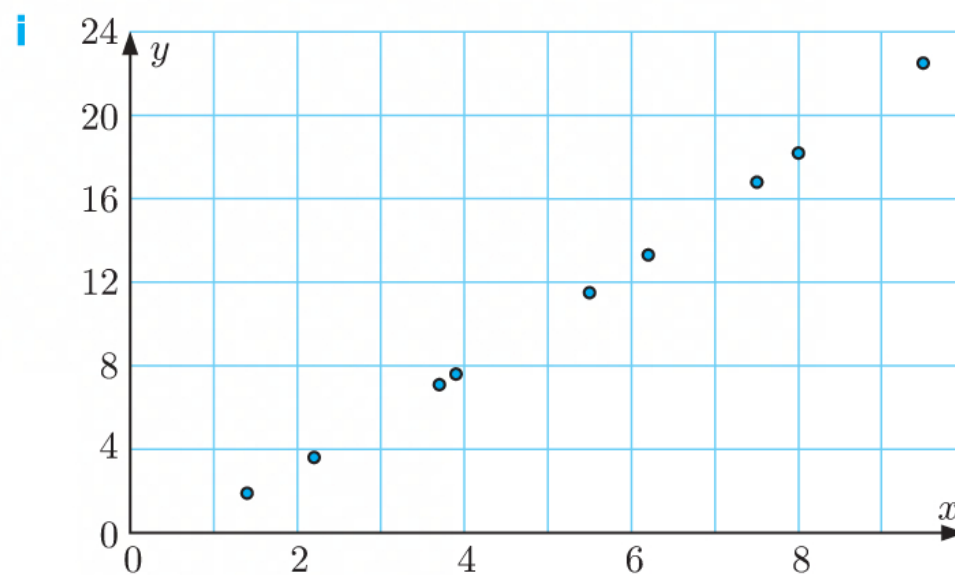
iii Using technology, the residual plot is:



iv There is very strong correlation and the residuals are randomly scattered, so the least squares regression line is appropriate to model the data.

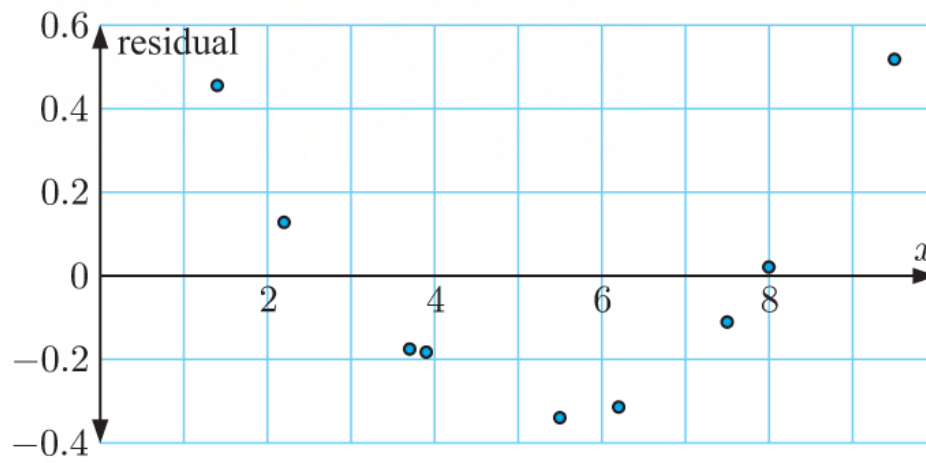
b

$x$	2.2	3.7	9.5	6.2	1.4	3.9	7.5	8	5.5
$y$	3.6	7.1	22.5	13.3	1.9	7.6	16.8	18.2	11.5



Using technology, the least squares regression line is  $y \approx 2.54x - 2.11$ , and  $r \approx 0.999$ .

iii Using technology, the residual plot is:

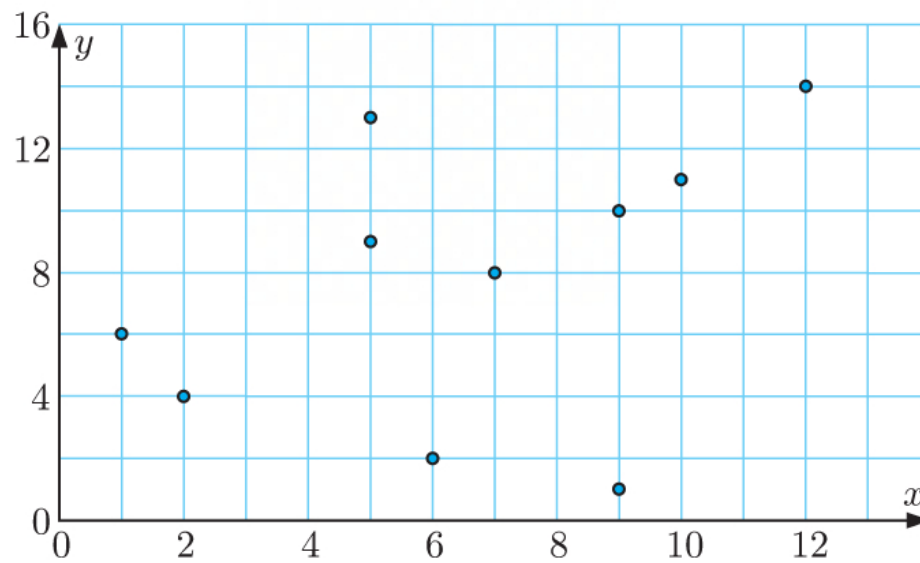


iv The residual plot shows a clear pattern and does not appear random. So the least squares regression line is not appropriate to model the data.

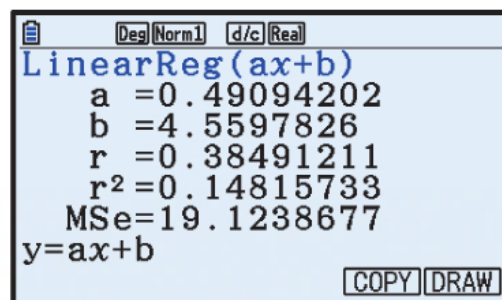
c

$x$	5	9	1	12	6	5	9	7	2	10
$y$	13	1	6	14	2	9	10	8	4	11

i

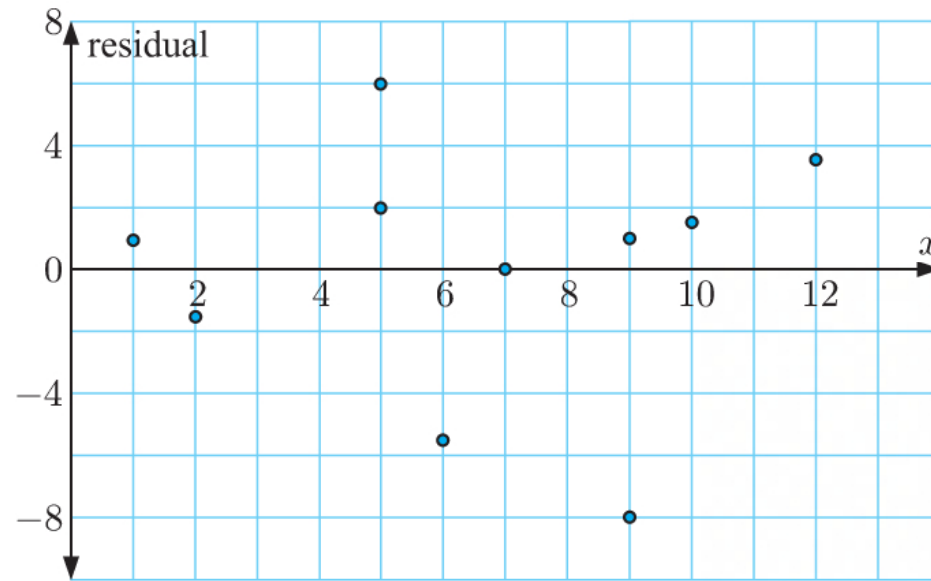


ii



Using technology, the least squares regression line is  $y \approx 0.491x + 4.56$ , and  $r \approx 0.385$ .

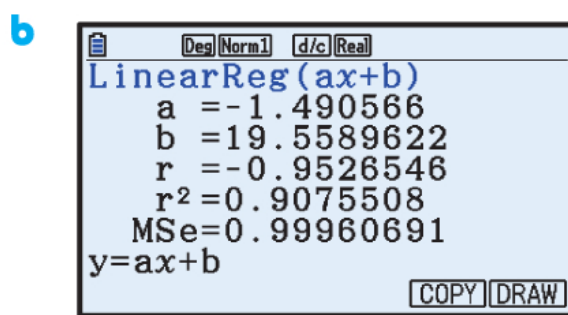
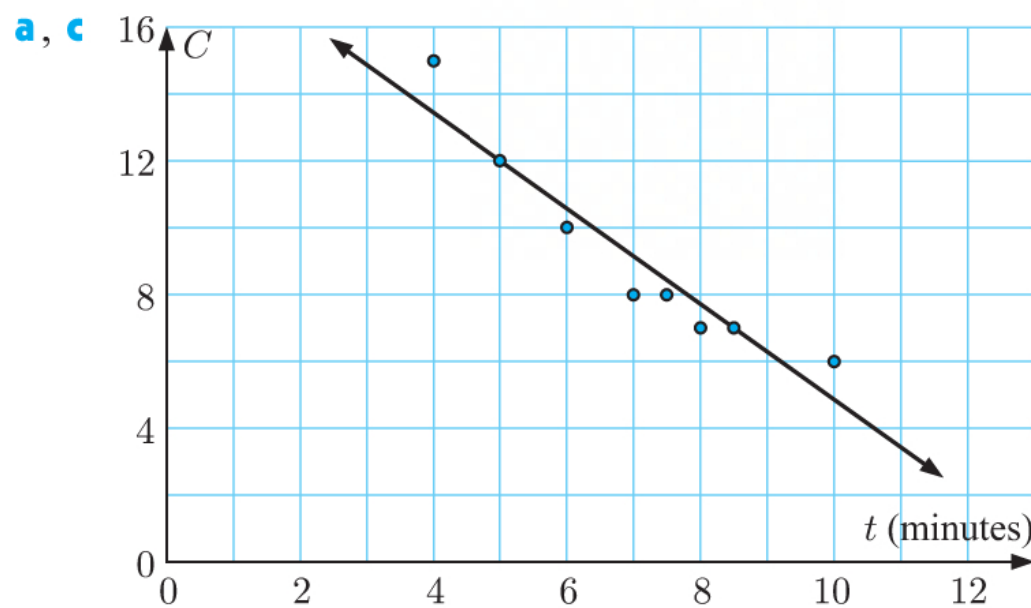
iii Using technology, the residual plot is:



iv The value of  $r$  is very small which indicates there is very weak correlation. So the least squares regression line is not appropriate to model the data.

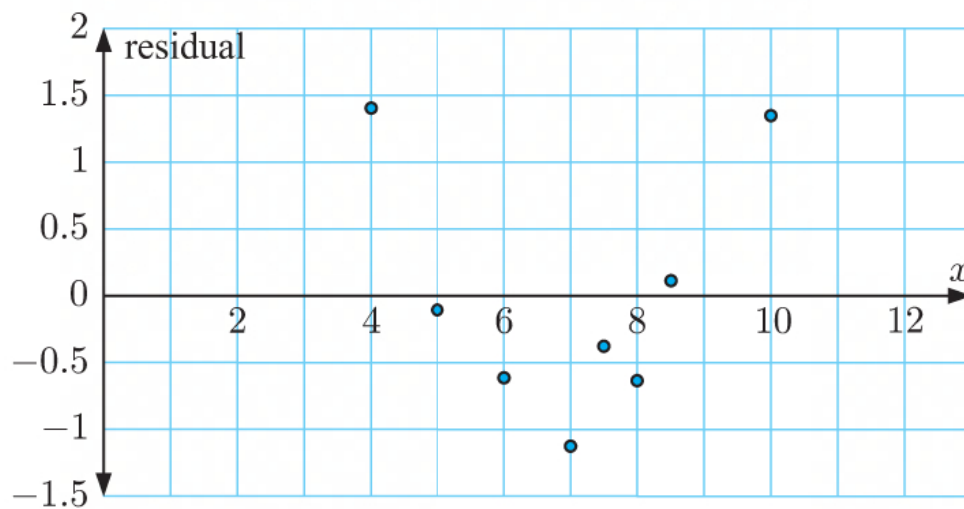
5

Time taken ( $t$ minutes)	6	8.5	4	5	8	7.5	10	7
Cranes made ( $C$ )	10	7	15	12	7	8	6	8



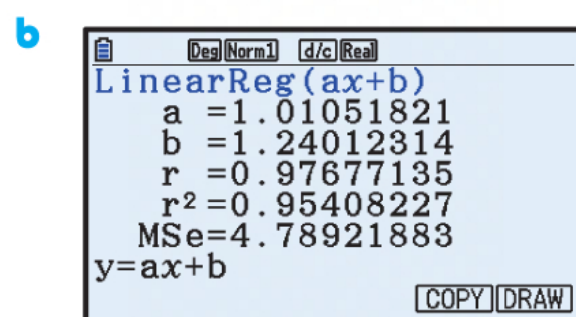
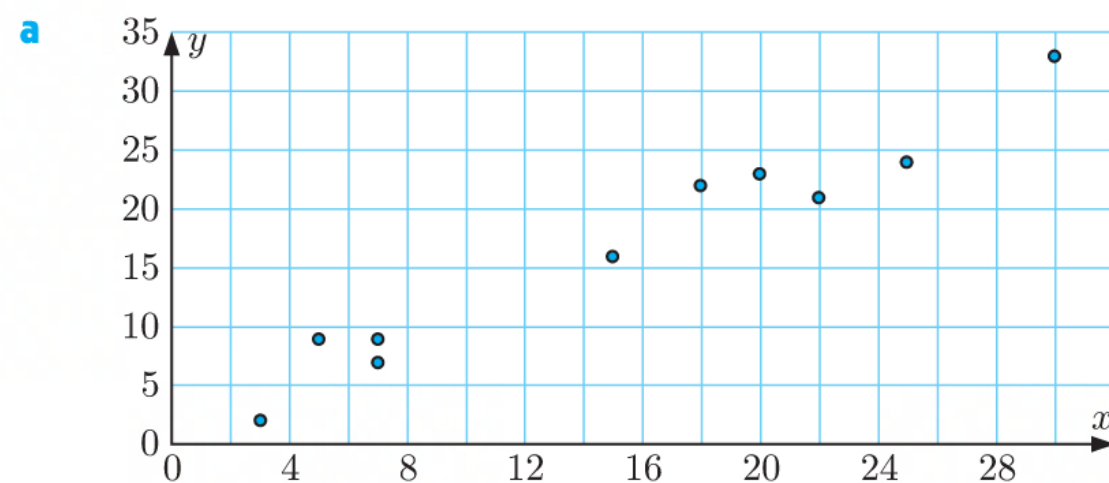
Using technology, the least squares regression line is  $y \approx -1.49x + 19.6$ , and  $r \approx -0.953$ .

- d Using technology, the residual plot is:



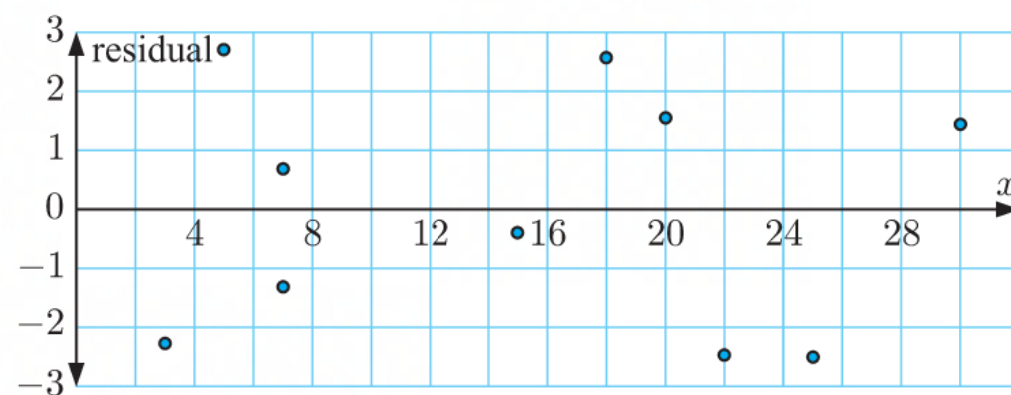
- e The residual plot shows a clear pattern, and does not appear random. So the least squares regression line is not appropriate to model the data.

6	Text messages sent ( $x$ )	18	3	7	22	15	5	20	30	7	25
	Text messages received ( $y$ )	22	2	9	21	16	9	23	33	7	24



Using technology, the least squares regression line is  $y \approx 1.01x + 1.24$ , and  $r \approx 0.977$ .

- c There is very strong, positive correlation between *text messages sent* and *text messages received*.
- d Using technology, the residual plot is:



- e There is very strong, positive correlation and the residuals are randomly scattered. So the least squares regression line is appropriate to model the data.



- f i** When  $y = 10$ ,  $10 \approx 1.01x + 1.24$   
 $\therefore 8.76 \approx 1.01x$   
 $\therefore x \approx 8.67$   
 $\approx 9$  {rounded to nearest integer}

So, we estimate Ted sent about 9 text messages.

- ii** As the estimate is an interpolation with strongly correlated data, it is fairly reliable.

## EXERCISE 26E

<b>1</b>	<i>Time on homemade meals (<math>x</math> hours)</i>	3.5	6.0	4.0	8.5	7.0	2.5	9.0	7.0	4.0	7.5
	<i>Money on fast food (\$<math>y</math>)</i>	85	0	60	0	27	100	15	40	59	29

- a** It is appropriate to use the regression line of  $x$  against  $y$  since the  $y$  variable, money spent on fast food, can be measured exactly. The  $x$  variable, time spent on homemade meals, will not be measured exactly.

**b**

Rad(Norm2) d/c(Real)

	List 1	List 2	List 3	List 4
SUB				
1	3.5	85		
2	6	0		
3	4	60		
4	8.5	0		

85

GRAPH1 GRAPH2 GRAPH3 SELECT SET

Rad(Norm2) d/c(Real)

LinearReg(ax+b)

a = -0.0575941

b = 8.29015603

r = -0.8702045

r<sup>2</sup> = 0.75725603

MSe = 1.39274347

y = ax + b

COPY DRAW

The regression line of  $x$  against  $y$  is  $x \approx -0.0576y + 8.29$ .

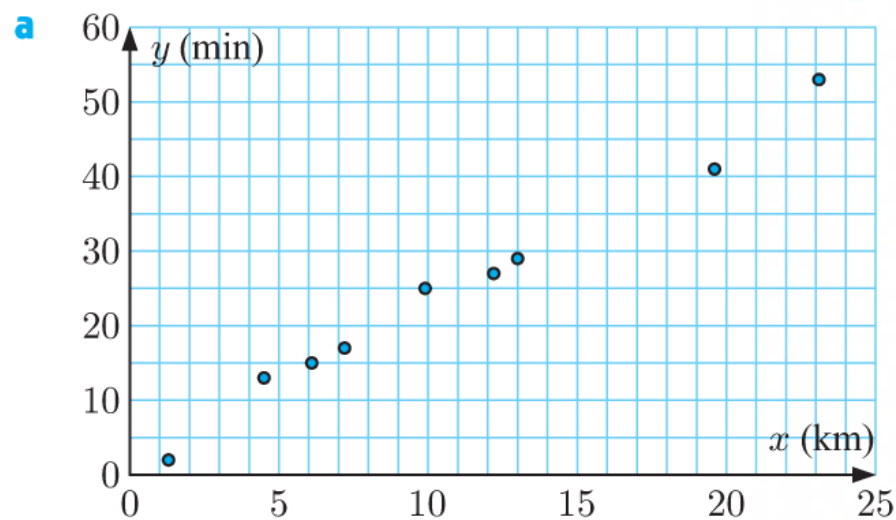
- c i** When  $y = 45$ ,  $x \approx -0.0576(45) + 8.29$   
 $\approx 5.70$

We expect a family that spends \$45 on fast food each week to spend about 5.70 hours each week preparing homemade meals.

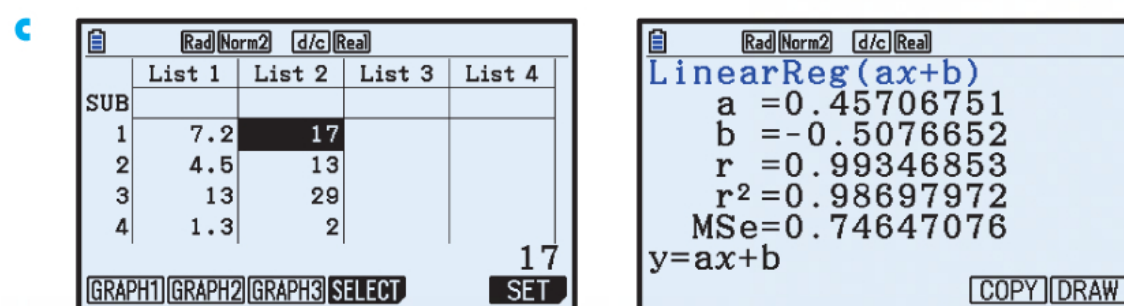
- ii** When  $x = 5$ ,  $5 \approx -0.0576y + 8.29$   
 $\therefore 0.0576y \approx 3.29$   
 $\therefore y \approx 57.13$

We expect a family that spends 5 hours each week preparing homemade meals to spend about \$57.13 on fast food each week.

<b>2</b>	Distance from school ( $x$ km)	7.2	4.5	13	1.3	9.9	12.2	19.6	6.1	23.1
	Time to travel to school ( $y$ min)	17	13	29	2	25	27	41	15	53



- b** We should use the regression line of  $x$  against  $y$ , since a student's time taken to travel to school can be more precisely measured than their distance from school.



The regression line of  $x$  against  $y$  is  $x \approx 0.457y - 0.508$ .

When  $x = 15$ ,  $15 \approx 0.457y - 0.508$

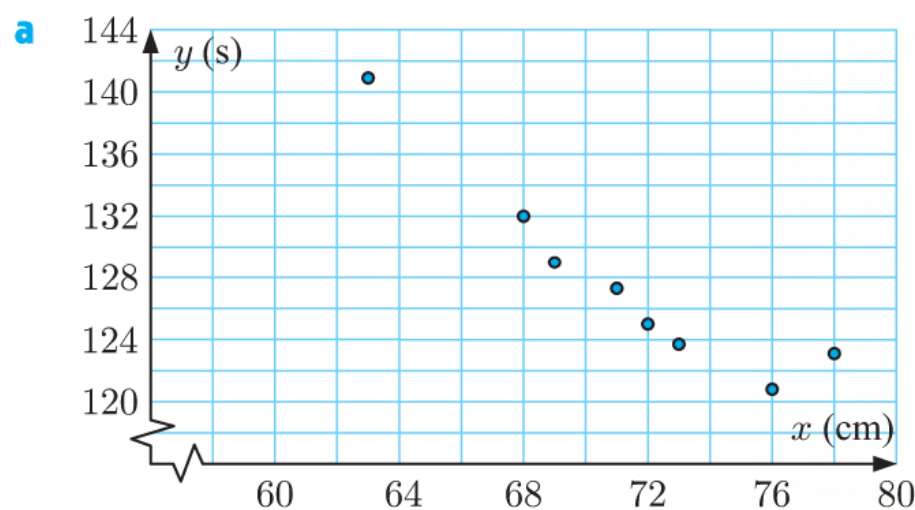
$$\therefore 0.457y \approx 15.508$$

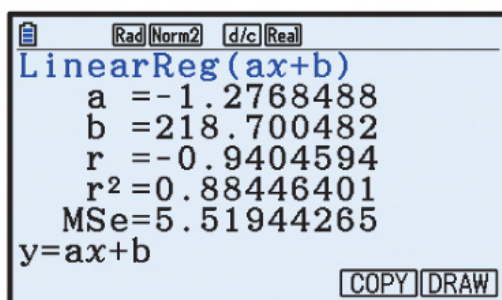
$$\therefore y \approx 33.9$$

We expect a student who lives 15 km from school will have a travel time of about 33.9 minutes.

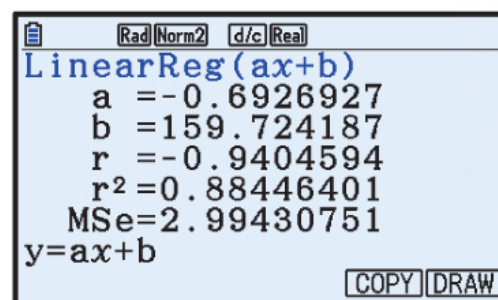
- d** The estimate in **c** is an interpolation, so it is likely to be reliable.

<b>3</b>	Length of arm ( $x$ cm)	78	73	71	68	76	72	63	69
	Breaststroke ( $y$ seconds)	123.1	123.7	127.3	132.0	120.8	125.0	140.9	129.0

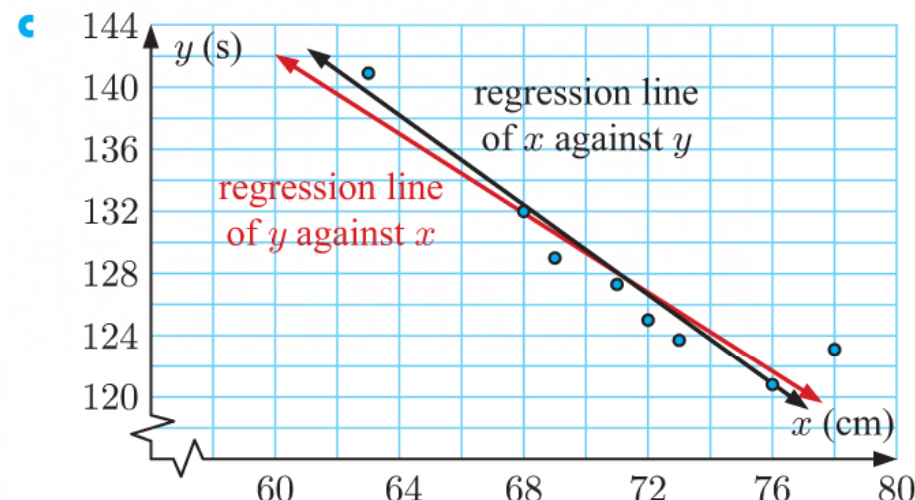


**b i**

The regression line of  $y$  against  $x$  is  
 $y \approx -1.28x + 219$ .

**ii**

The regression line of  $x$  against  $y$  is  
 $x \approx -0.693y + 160$ .



The two regression lines are very similar. The regression line of  $x$  against  $y$  is slightly steeper.

**4 a** The regression line of  $y$  against  $x$  is  $y = ax + b$  where  $a = \frac{\sum(x_i - \bar{x})(y_i - \bar{y})}{\sum(x_i - \bar{x})^2}$ .

The regression line of  $x$  against  $y$  is  $x = my + c$  where  $m = \frac{\sum(x_i - \bar{x})(y_i - \bar{y})}{\sum(y_i - \bar{y})^2}$ .

$$\begin{aligned} \therefore am &= \left[ \frac{\sum(x_i - \bar{x})(y_i - \bar{y})}{\sum(x_i - \bar{x})^2} \right] \left[ \frac{\sum(x_i - \bar{x})(y_i - \bar{y})}{\sum(y_i - \bar{y})^2} \right] \\ &= \frac{[\sum(x_i - \bar{x})(y_i - \bar{y})]^2}{\sum(x_i - \bar{x})^2 \sum(y_i - \bar{y})^2} \\ &= r^2 \quad \left\{ r = \frac{\sum(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum(x_i - \bar{x})^2 \sum(y_i - \bar{y})^2}} \right\} \end{aligned}$$

**b** The regression line of  $y$  against  $x$  is  $y = ax + b$ .

The regression line of  $x$  against  $y$  is  $y = \frac{1}{m}x - \frac{c}{m}$ .

$$\begin{aligned} \text{In the regression of } y \text{ against } x, \text{ when } x = \bar{x}, \quad y &= a\bar{x} + b \\ &= a\bar{x} + \bar{y} - a\bar{x} \\ &= \bar{y} \end{aligned}$$

$$\begin{aligned} \text{and in the regression of } x \text{ against } y, \text{ when } y = \bar{y}, \quad x &= m\bar{y} + c \\ &= m\bar{y} + \bar{x} - m\bar{y} \\ &= \bar{x} \end{aligned}$$

So, both lines pass through the mean point  $(\bar{x}, \bar{y})$ , which means the lines will be the same if and only if their gradients are equal.

$\therefore$  the regression lines will be the same if  $a = \frac{1}{m}$

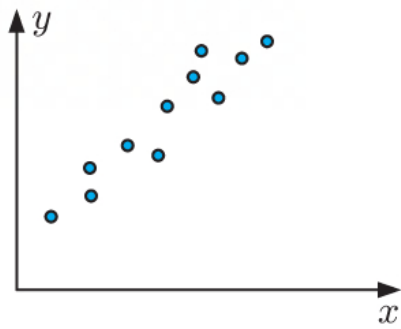
$$\therefore am = 1$$

$$\therefore r^2 = 1 \quad \{\text{using a}\}$$



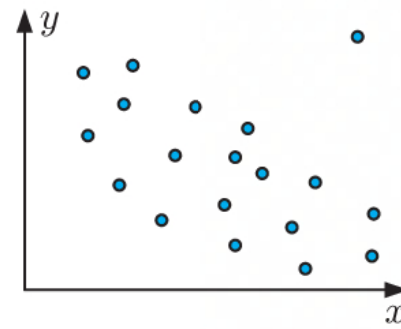
## REVIEW SET 26A

1 a



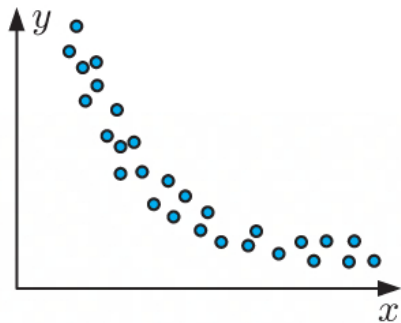
There is a strong, positive, linear correlation, with no outliers.

b



There is a weak, negative, linear correlation, with one outlier.

c



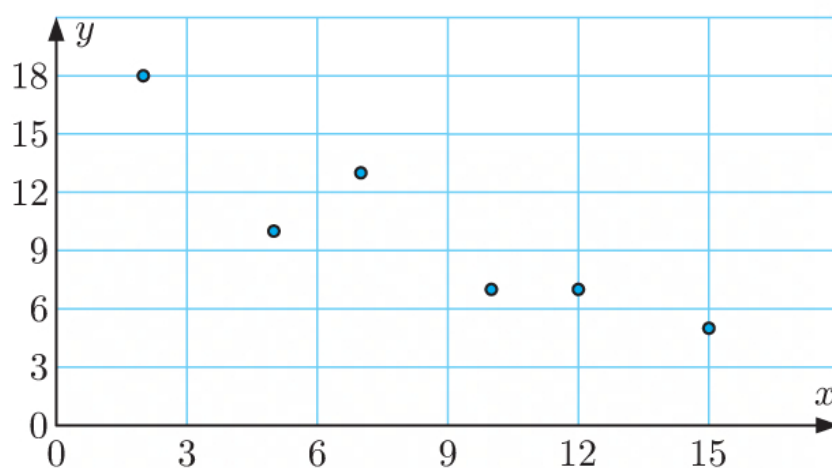
There is a strong, negative, non-linear correlation, with no outliers.

- 2 a The correlation between water bills and electricity bills is likely to be positive, as a household with a high water bill is also likely to have a high electricity bill, and vice versa.
- b No, there is not a causal relationship. Both variables mainly depend on the number of occupants in each house.

3

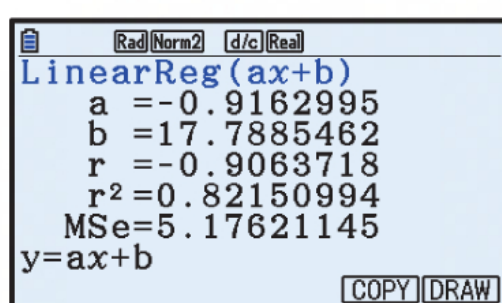
$x$	2	5	7	10	12	15
$y$	18	10	13	7	7	5

a



- b The correlation between the variables appears to be negative.

c

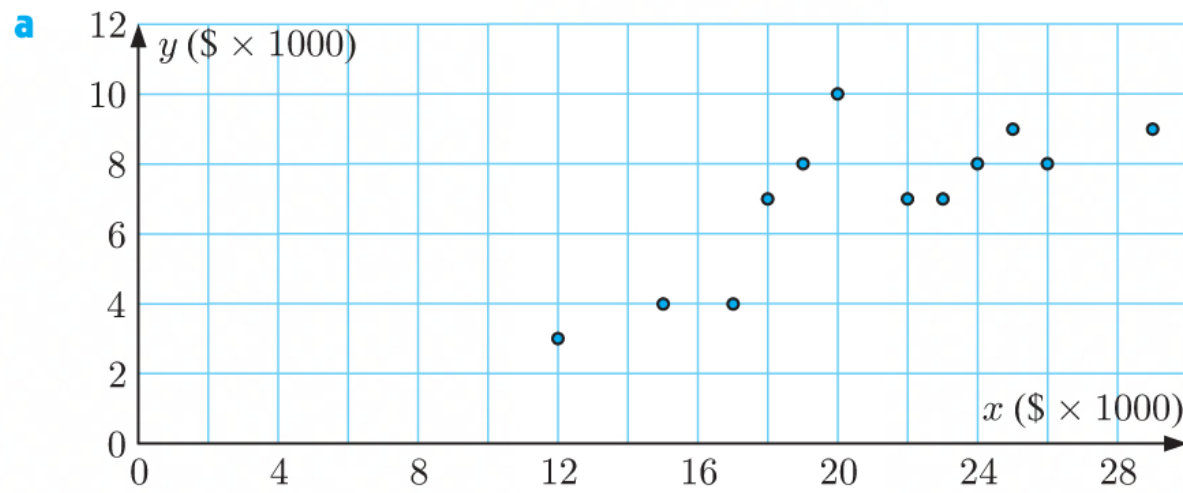


So,  $r \approx -0.906$ .

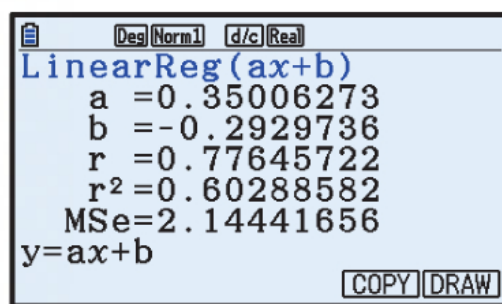


4

<i>Ticket sales</i> (\$ $x \times 1000$ )	25	22	15	19	12	17	24	20	18	23	29	26
<i>Beverage sales</i> (\$ $y \times 1000$ )	9	7	4	8	3	4	8	10	7	7	9	8



b



So,  $r \approx 0.776$ .

c There is a moderate, positive correlation between *ticket sales* and *beverage sales*.

5

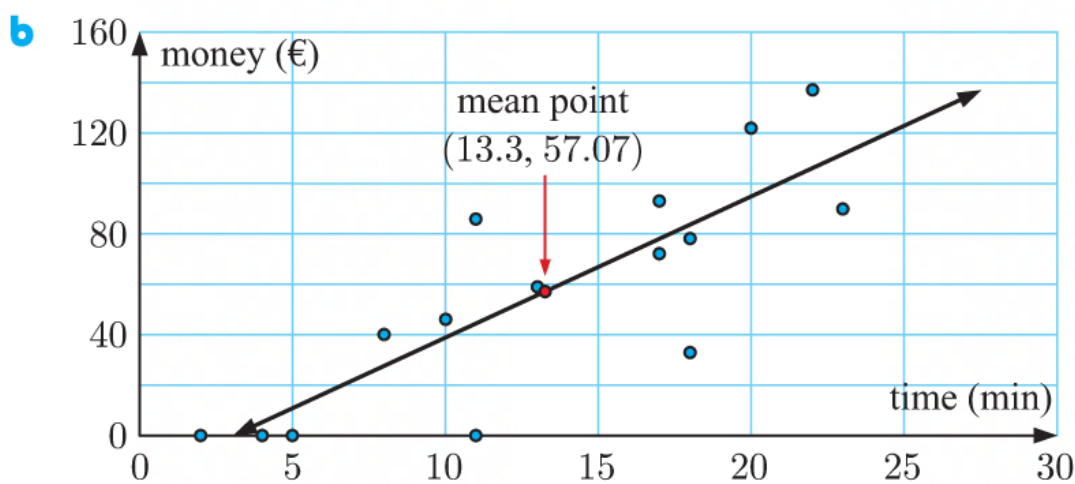
<i>Time</i> (min)	8	18	5	10	17	11	2	13	18	4	11	20	23	22	17
<i>Money</i> (€)	40	78	0	46	72	86	0	59	33	0	0	122	90	137	93

a

$$\bar{x} = \frac{8 + 18 + \dots + 22 + 17}{15} = \frac{199}{15} \approx 13.3$$

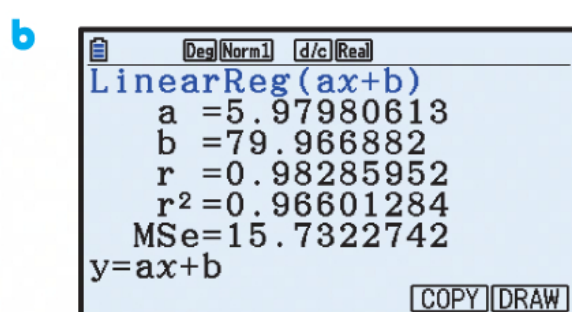
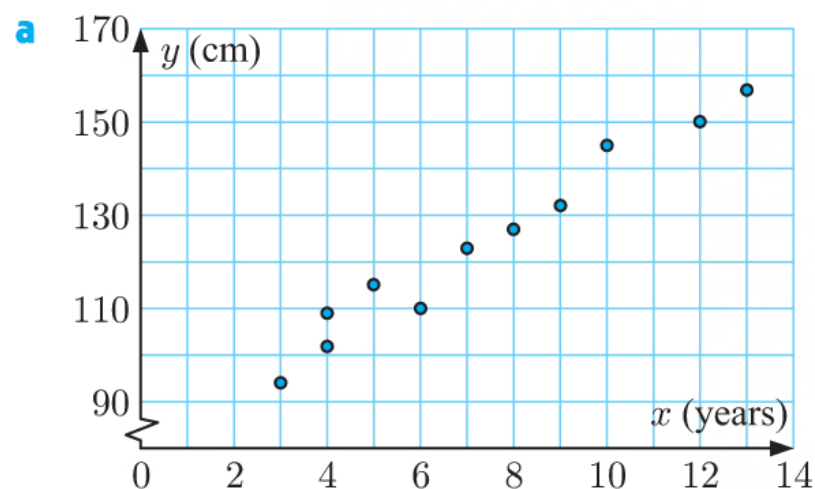
$$\bar{y} = \frac{40 + 78 + \dots + 137 + 93}{15} = \frac{856}{15} \approx 57.07$$

So the mean time is about 13.3 minutes, and the mean spending is about €57.07.



c There is a moderate, positive, linear correlation between *time in the store* and *money spent*.

<b>6</b>	Age ( $x$ years)	3	9	7	4	4	12	8	6	5	10	13
	Height ( $y$ cm)	94	132	123	102	109	150	127	110	115	145	157



Using technology, the least squares regression line is  $y \approx 5.98x + 80.0$ .

- c** The gradient of the least squares regression line  $\approx 5.98$ . This indicates that each year, a child grows taller by an average of 5.98 cm.

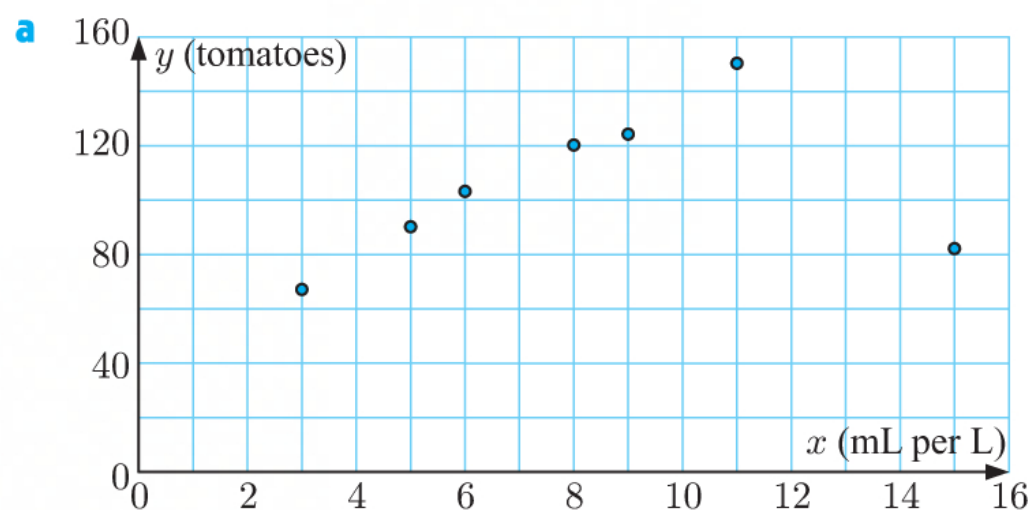
- d** When  $x = 5$ ,  $y \approx 5.98(5) + 80.0$   
 $\approx 110$

So, a 5 year old child would be approximately 110 cm tall.

- e** When  $y = 140$ ,  $140 \approx 5.98x + 80$   
 $5.98x \approx 60$   
 $x \approx 10.0$

A child would be expected to reach 140 cm in height at age 10 years.

<b>7</b>	Spray concentration ( $x$ mL per L)	3	5	6	8	9	11	15
	Yield of tomatoes per bush ( $y$ )	67	90	103	120	124	150	82



b

```

Des Norm1 d/c Real
LinearReg(ax+b)
a = 2.3938053
b = 85.6504424
r = 0.33958865
r^2 = 0.11532045
MSe = 851.567256
y = ax + b
COPY DRAW

```

So,  $r \approx 0.340$ .

There is a very weak, positive, linear correlation between spray concentration and yield.

c Yes,  $(15, 82)$  is an outlier which is affecting the correlation.

d

```

Des Norm1 d/c Real
LinearReg(ax+b)
a = 9.92857142
b = 39.5
r = 0.99427855
r^2 = 0.98858984
MSe = 11.9464285
y = ax + b
COPY DRAW

```

So,  $r \approx 0.994$ .

Yes it is now reasonable to draw a least squares regression line.

e

```

Des Norm1 d/c Real
LinearReg(ax+b)
a = 9.92857142
b = 39.5
r = 0.99427855
r^2 = 0.98858984
MSe = 11.9464285
y = ax + b
COPY DRAW

```

Using technology, the least squares regression line is  $y \approx 9.93x + 39.5$ .

f The gradient of the least squares regression line  $\approx 9.93$ . This indicates that for every additional mL per L the spray concentration increases, the yield of tomatoes per bush increases on average by 9.93 tomatoes.

The  $y$ -intercept of the least squares regression line  $\approx 39.5$ . This indicates that if the tomato bushes are not sprayed, the average yield per bush is approximately 39.5 tomatoes.

g i When  $x = 7$ ,  $y \approx 9.93(7) + 39.5$   
 $\approx 109$

If the spray concentration is 7 mL per L, the yield will be approximately 109 tomatoes per bush.

ii When  $y = 200$ ,  $200 \approx 9.93x + 39.5$   
 $9.93x \approx 160.5$   
 $x \approx 16.2$

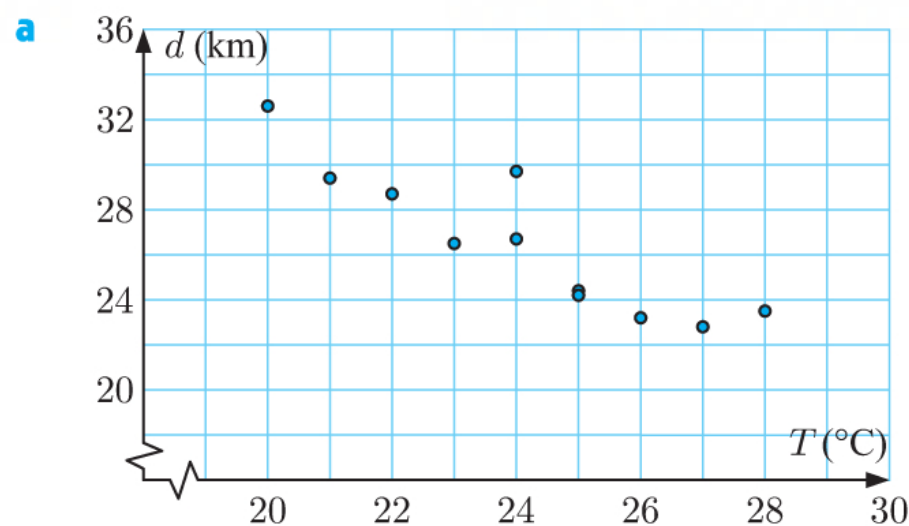
If the yield is 200 tomatoes per bush, the spray concentration would be approximately 16.2 mL per L.

h In g i, this is an interpolation, so this estimate is likely to be reliable.

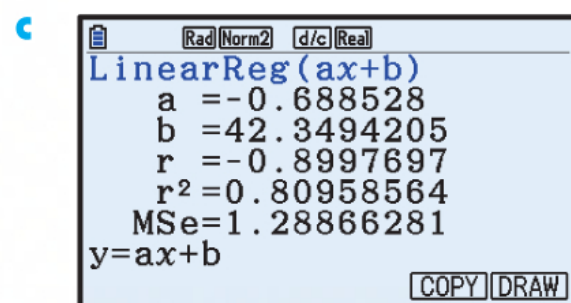
In g ii, this is an extrapolation, so this estimate may not be reliable.



8	Temperature ( $T$ °C)	23	24	25	27	28	20	22	21	25	26	24
	Distance ( $d$ km)	26.5	26.7	24.4	22.8	23.5	32.6	28.7	29.4	24.2	23.2	29.7



- b It is appropriate to use the regression line of  $T$  against  $d$  since the values for the distance travelled  $d$  are more precisely measured than the daily temperature, which Thomas is just estimating.



The regression line of  $T$  against  $d$  is  $T \approx -0.689d + 42.3$ .

- d When  $T = 30$ ,  $30 \approx -0.689d + 42.3$   
 $\therefore 0.689d \approx 12.3$   
 $\therefore d \approx 17.9$

On a 30°C day we would expect Thomas to ride about 17.9 km.

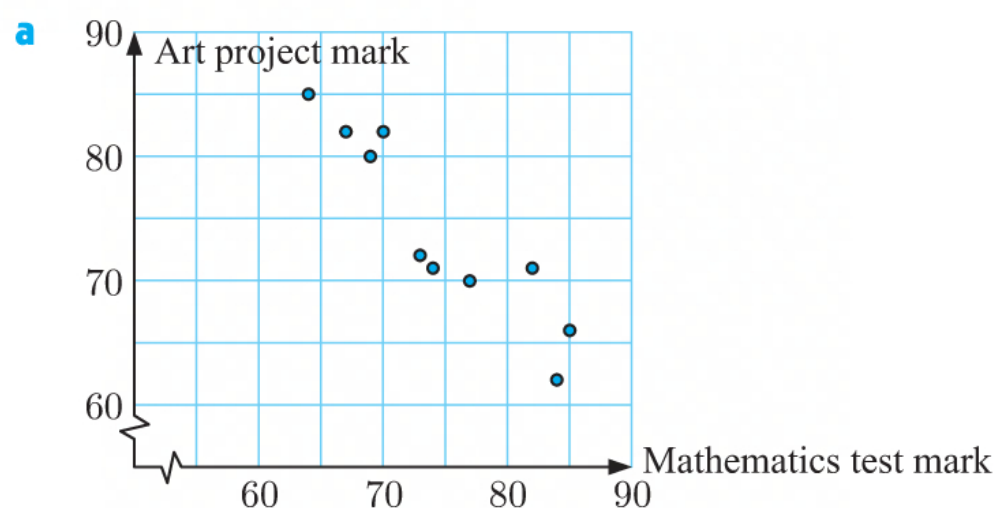
## REVIEW SET 26B

- 1 a The variables are likely to be negatively correlated, as prices increase, the number of tickets sold is likely to decrease.  
 This is a causal relationship as less people will be able to afford tickets as the prices increase.
- b The variables are likely to be positively correlated, as ice cream sales increase, the number of shark attacks is likely to increase.  
 This is not a causal relationship as both of these variables are dependent on the time of year.

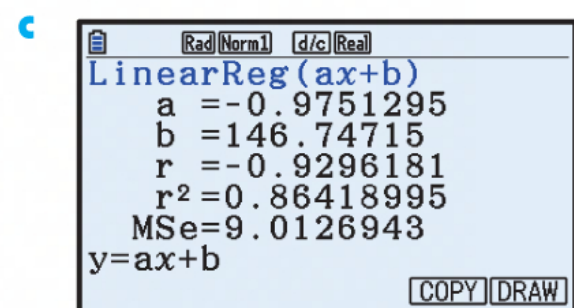


2

Student	A	B	C	D	E	F	G	H	I	J
Mathematics test	64	67	69	70	73	74	77	82	84	85
Art project	85	82	80	82	72	71	70	71	62	66



**b** There is a strong, negative, linear correlation between the Mathematics and Art marks.



So,  $r \approx -0.930$ .

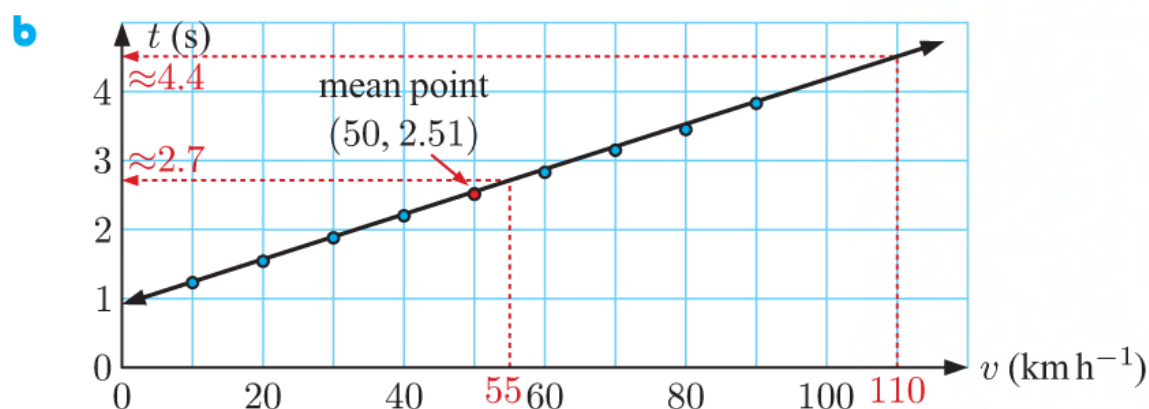
3

Speed ( $v$ km h <sup>-1</sup> )	10	20	30	40	50	60	70	80	90
Stopping time ( $t$ s)	1.23	1.54	1.88	2.20	2.52	2.83	3.15	3.45	3.83

**a**  $\bar{v} = \frac{10 + 20 + 30 + \dots + 80 + 90}{9}$   
 $= 50$

$\bar{t} = \frac{1.23 + 1.54 + 1.88 + \dots + 3.45 + 3.83}{9}$   
 $\approx 2.51$

$\therefore$  the mean point  $(\bar{v}, \bar{t})$  is  $(50, 2.51)$ .



- c**
- i** We estimate that the stopping time for a speed of 55 km h<sup>-1</sup> is about 2.7 seconds.
  - ii** We estimate that the stopping time for a speed of 110 km h<sup>-1</sup> is about 4.4 seconds.
- d** The estimate in **c i** is more likely to be reliable, since it is an interpolation.

4

$x$	2	3	6	8	13	16
$y$	12	17	32	41	50	61

a

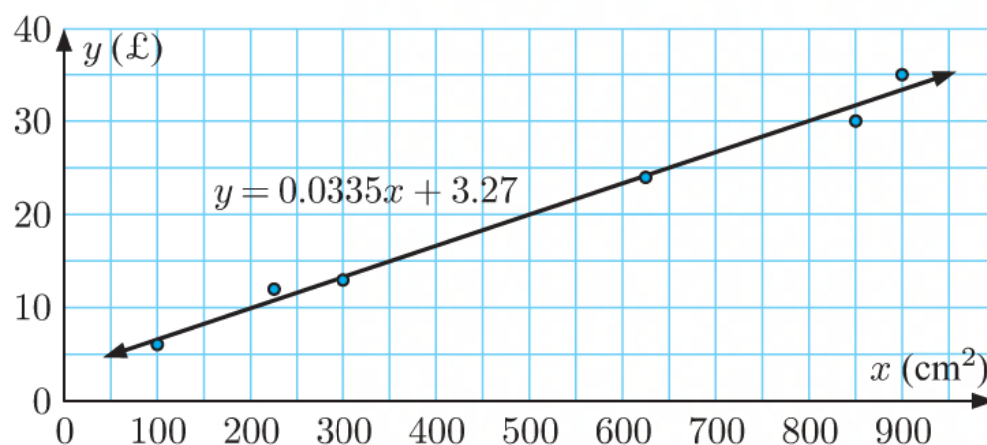
Rad	Norm2	d/c	Real
LinearReg(ax+b)			
a = 3.35714285			
b = 8.64285714			
r = 0.98264292			
r <sup>2</sup> = 0.96558712			
MSe = 15.4642857			
y = ax + b			
COPY		DRAW	

So,  $r \approx 0.983$ .b The regression line is  $y \approx 3.36x + 8.64$ .c When  $x = 10$ ,  $y \approx 3.36(10) + 8.64$   
 $\approx 42.2$ 

5

Area ( $x \text{ cm}^2$ )	100	225	300	625	850	900
Price (£ $y$ )	6	12	13	24	30	35

a, d



b

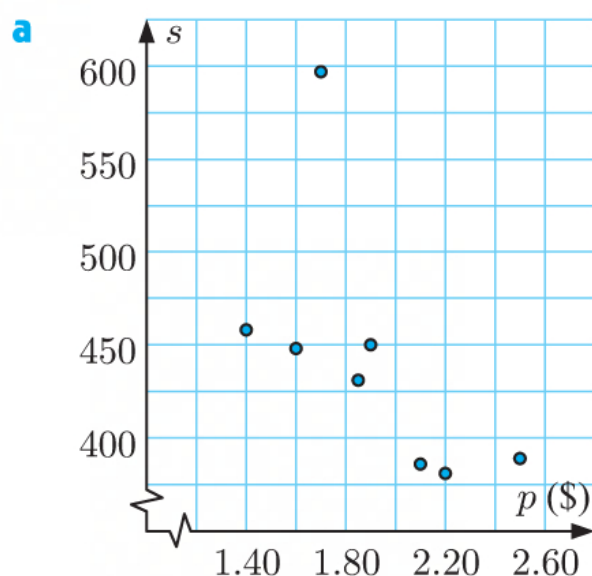
Des	Norm1	d/c	Real
LinearReg(ax+b)			
a = 0.03346405			
b = 3.26797385			
r = 0.99422161			
r <sup>2</sup> = 0.98847662			
MSe = 1.87254901			
y = ax + b			
COPY		DRAW	

So,  $r \approx 0.994$ .c There is a very strong, positive correlation between the *area* of a canvas and its *price*.d The regression line is  $y \approx 0.0335x + 3.27$ .e When  $x = 1200$ ,  $y \approx 0.0335(1200) + 3.27$   
 $\approx 43.42$ 

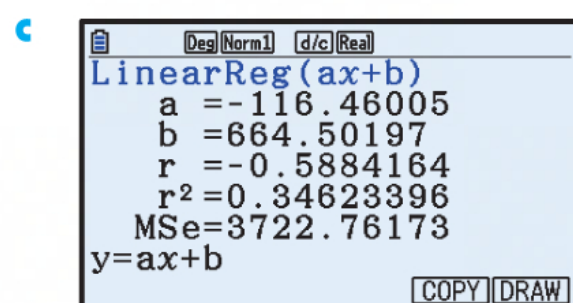
We estimate that a canvas with area  $1200 \text{ cm}^2$  will cost about £43.42. This is an extrapolation however, so it may be unreliable.

**6**

Price (\$p)	2.50	1.90	1.60	2.10	2.20	1.40	1.70	1.85
Sales (s)	389	450	448	386	381	458	597	431



- b** Yes, the point (1.70, 597) is an outlier. It should not be deleted as there is no evidence that it is a mistake.

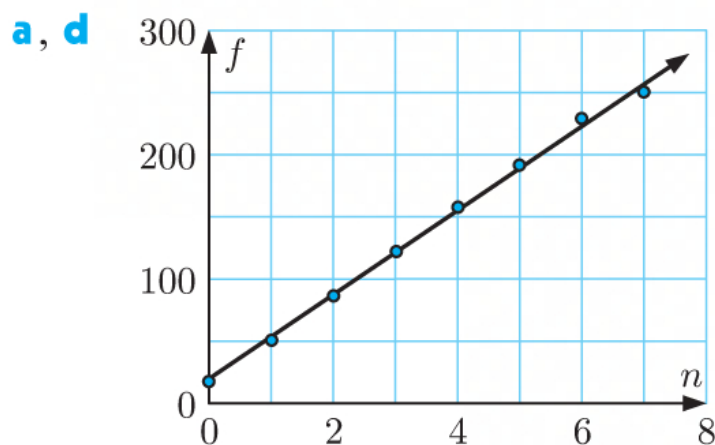


The regression line is  $s \approx -116p + 665$ .

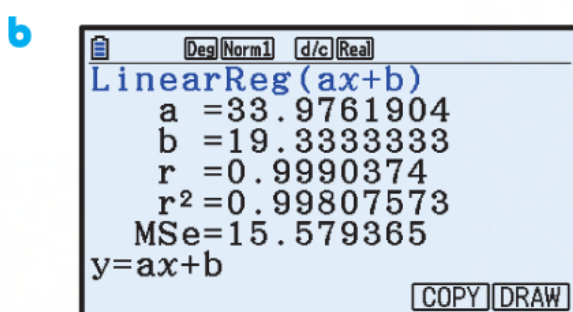
- d** The gradient of the least squares regression line  $\approx -116$ . This indicates that for every additional dollar the price increases by, the number of sales decreases by 116.
- e** No, the prediction of sales of Supa-fizz if it was priced at 50 cents would not be accurate, as it is an extrapolation well beyond the range of data values given.

**7**

Number of waterings (n)	0	1	2	3	4	5	6	7
Flowers produced (f)	18	52	86	123	158	191	228	250



There is a very strong, positive correlation between number of waterings and flowers produced.



The regression line is  $f \approx 34.0n + 19.3$ .



- c Yes, plants need water to grow. So it is expected that an increase in watering will result in an increase in flower production.

- e i 5 times a fortnight  $\equiv$  2.5 times a week

$$\text{When } n = 2.5, \quad f \approx 34.0(2.5) + 19.3 \\ \approx 104$$

Violet can expect about 104 flowers from this bed.

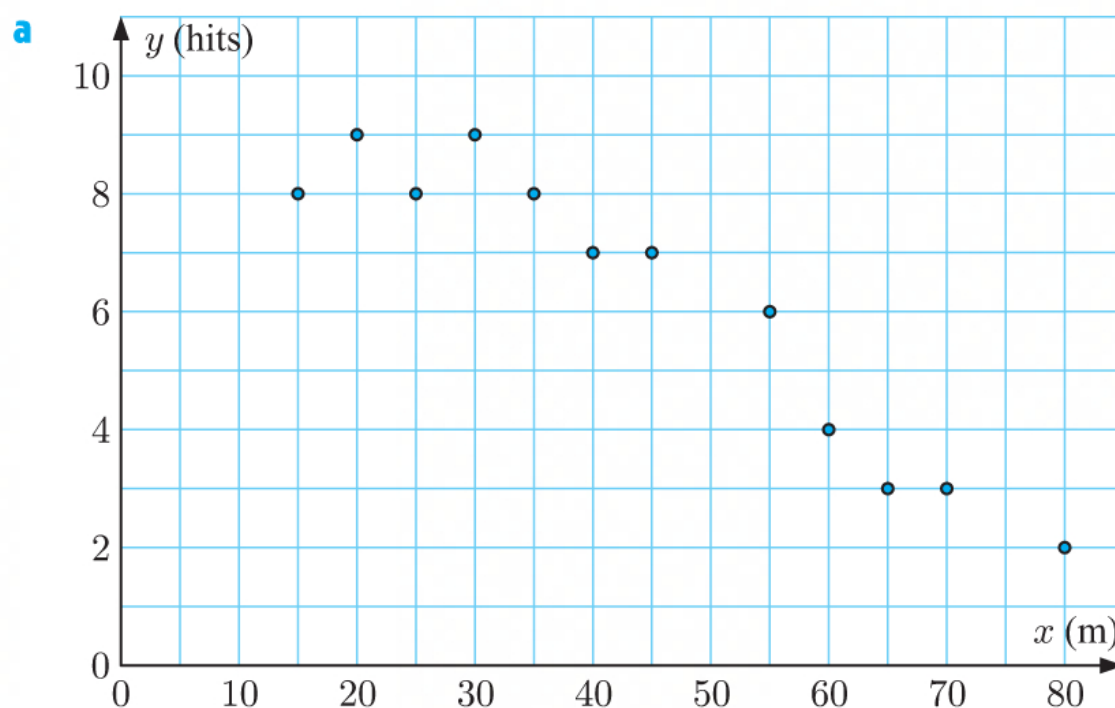
$$\text{When } n = 10, \quad f \approx 34.0(10) + 19.3 \\ \approx 359$$

Violet can expect about 359 flowers from this bed.

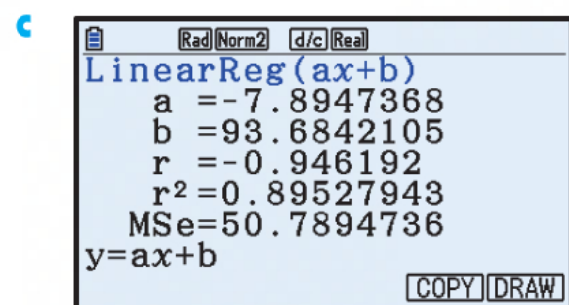
- ii The estimate for  $n = 2.5$  is reliable as it is an interpolation.

The estimate for  $n = 10$  is unreliable as it is an extrapolation and over-watering could be a problem.

8	Distance from target ( $x$ m)	20	25	15	35	40	55	30	45	60	80	65	70
	Hits ( $y$ )	9	8	8	8	7	6	9	7	4	2	3	3



- b It is appropriate to use the regression line of  $x$  against  $y$ , since the number of hits can be counted exactly, while the distance from the target will not be exact.



The regression line of  $x$  against  $y$  is  $x \approx -7.89y + 93.7$ .

- d When  $x = 100$ ,  $100 \approx -7.89y + 93.7$

$$\therefore 7.89y \approx -6.3$$

$$\therefore y \approx -0.800$$

Out of 10 shots fired at a distance of 100 m, we would expect about  $-0.8$  hits, but it is impossible to make a negative number of shots. This extrapolation is not valid.



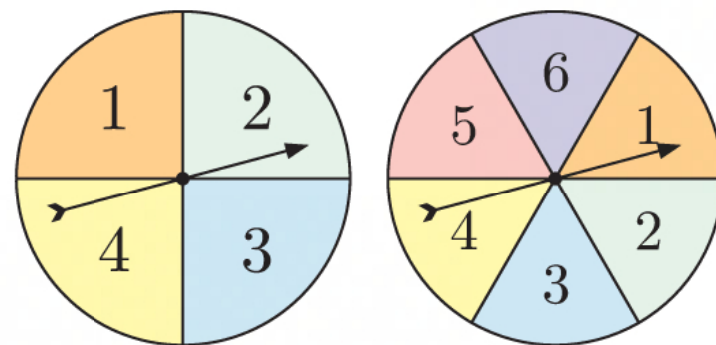
# Chapter 27

## DISCRETE RANDOM VARIABLES

### EXERCISE 27A

- 1
  - a The quantity of fat in a sausage is a continuous random variable.
  - b The mark out of 50 for a geography test is a discrete random variable.
  - c The weight of a Year 12 student is a continuous random variable.
  - d The volume of water in a cup of coffee is a continuous random variable.
  - e The number of trout in a lake is a discrete random variable.
  - f The number of the hairs on a cat is a discrete random variable.
  - g The length of a horse's mane is a continuous random variable.
  - h The height of a skyscraper is a continuous random variable.
  
- 2
  - a
    - i The random variable  $X$  is the height of water in the rain gauge.
    - ii The variable is a continuous random variable.
    - iii  $0 \leq X \leq 400$  mm
  - b
    - i The random variable  $X$  is the stopping distance.
    - ii The variable is a continuous random variable.
    - iii  $0 \leq X \leq 50$  m
  - c
    - i The random variable  $X$  is the number of times that the switch is turned off and on before it fails.
    - ii The variable is a discrete random variable.
    - iii  $X$  can be any integer  $\geq 1$

- 3
  - a  $X$  is the sum of a number from one spinner and a number from the other spinner. So  $X$  is a discrete random variable because  $X$  has a set of distinct possible values.
  - b  $X = 2, 3, 4, 5, 6, 7, 8, 9$ , or  $10$





- 4
  - a The two teams play against each other until one team wins 4 games (best out of 7).  
 $\therefore X = 4, 5, 6$ , or  $7$
  - b
    - i  $X = 5$
    - ii  $X = 6$  or  $7$
  
- 5
  - a There are four weighing devices and  $X$  is the number which are accurate.  
 $\therefore X = 0, 1, 2, 3$ , or  $4$

A	B	C	D	
✓	✓	✓	✓	$(X = 4)$
✓	✓	✓	✗	$(X = 3)$
✓	✓	✗	✓	
✓	✗	✓	✓	
✗	✓	✓	✓	
✗	✗	✓	✓	$(X = 2)$
✗	✓	✗	✓	
✗	✓	✓	✗	
✗	✗	✗	✓	$(X = 1)$
✗	✗	✓	✗	
✗	✗	✗	✗	$(X = 0)$

- c** **i** If exactly two devices are accurate, then  $X = 2$ .  
**ii** If at least two devices are accurate, then 2, 3, or 4 are accurate  $\therefore X = 2, 3, \text{ or } 4$ .

**6 a** If 3 coins are tossed then the number of heads  $X$  can be 0, 1, 2, or 3.

**b** Let H represent heads, and T represent tails.

HHH	HHT	TTH	TTT
	HTH	THT	
	THH	HTT	
$(X = 3)$	$(X = 2)$	$(X = 1)$	$(X = 0)$

**c**  $P(X = 0) = \frac{1}{8}, \quad P(X = 1) = \frac{3}{8}, \quad P(X = 2) = \frac{3}{8}, \quad P(X = 3) = \frac{1}{8}$

Since  $P(X = 2) \neq P(X = 3)$ , the possible values of  $X$  are not equally likely to occur.

## EXERCISE 27B

**1 a i**

$x$	1	2	3	4
$P(X = x)$	0.2	0.4	0.15	0.25

$$\sum_{x=1}^4 P(X = x) = 0.2 + 0.4 + 0.15 + 0.25 = 1$$

Since  $\sum_{x=1}^4 P(X = x) = 1$  and  $0 \leq P(X = x) \leq 1$  for all  $x$ , it is a valid probability distribution.

**ii**

$x$	0	1	2	3
$P(X = x)$	0.2	0.3	0.4	0.2

$$\sum_{x=0}^3 P(X = x) = 0.2 + 0.3 + 0.4 + 0.2 = 1.1$$

Since  $\sum_{x=0}^3 P(X = x) > 1$ , it is not a valid probability distribution.

**iii**

$x$	0	1	2	3	4
$P(X = x)$	0.2	0.2	0.2	0.2	0.2

$$\sum_{x=0}^4 P(X = x) = 0.2 + 0.2 + 0.2 + 0.2 + 0.2 = 1$$

Since  $\sum_{x=0}^4 P(X = x) = 1$  and  $0 \leq P(X = x) \leq 1$  for all  $x$ , it is a valid probability distribution.

**iv**

$x$	2	3	4	5
$P(X = x)$	0.3	0.4	0.5	-0.2

Since  $P(X = 5) = -0.2 < 0$ , it is not a valid probability distribution.

- b**  $X$  is a uniform random variable for the probability distribution in **a iii**, since  $p_i = 0.2$  for each value of  $i$ .

**2 a**

$x$	0	1	2
$P(X = x)$	0.3	$k$	0.5

$$\sum_{x=0}^2 P(X = x) = 1$$

$$\therefore 0.3 + k + 0.5 = 1$$

$$\therefore k = 0.2$$

**b**

$x$	0	1	2	3
$P(X = x)$	$k$	$2k$	$3k$	$k$

$$\sum_{x=0}^3 P(X = x) = 1$$

$$\therefore k + 2k + 3k + k = 1$$

$$\therefore 7k = 1$$

$$\therefore k = \frac{1}{7}$$

**3 a**

$$\sum_{x=0}^3 P(X = x) = 1$$

$$\therefore 0.1 + 0.25 + 0.45 + a = 1$$

$$\therefore a = 0.2$$

$x$	0	1	2	3
$P(X = x)$	0.1	0.25	0.45	$a$

**b** Since  $P(X = 0) \neq P(X = 1)$ , the probabilities of each outcome are not all equal, so  $X$  is not a uniform discrete random variable.

**c** Since  $P(X = 2)$  is the greatest probability, 2 is the mode of the distribution.

$$\begin{aligned} \mathbf{d} \quad P(X \geq 2) &= P(X = 2) + P(X = 3) \\ &= 0.45 + 0.2 \\ &= 0.65 \end{aligned}$$

**4**

$x$	0	1	2	3	4	5
$P(x)$	$a$	0.3333	0.1088	0.0084	0.0007	0.0000

**a**  $P(2) = 0.1088$  (from table)

**b** Since this is a probability distribution,  $\sum_{i=1}^n P(x_i) = 1$

$$\therefore a + 0.3333 + 0.1088 + 0.0084 + 0.0007 + 0.0000 = 1$$

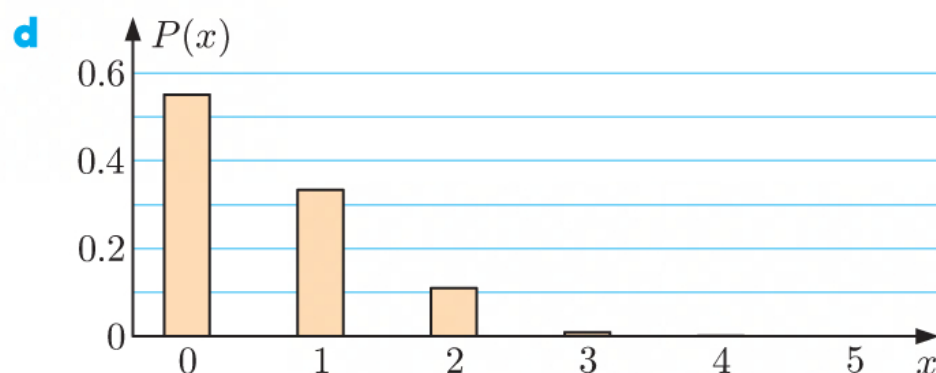
$$\therefore a + 0.4512 = 1$$

$$\therefore a = 0.5488$$

This is the probability that Jason does not hit a home run in a game.

$$\begin{aligned} \mathbf{c} \quad P(1) + P(2) + P(3) + P(4) + P(5) &= 0.3333 + 0.1088 + 0.0084 + 0.0007 + 0.0000 \\ &= 0.4512 \end{aligned}$$

This is the probability that Jason will hit one or more home runs in a game.



**e** Jason is most likely to score 0 home runs, so this is the mode of the distribution. Using **b**,  $P(0) = 0.5488 \geq 0.5$ , so the median is 0 home runs.



<b>5</b>	$x$	0	1	2	3	4
	$P(X = x)$	0.68	0.2	0.06	$k$	0.02

**a** 
$$\sum_{x=0}^4 P(X = x) = 1$$

$$\therefore 0.68 + 0.2 + 0.06 + k + 0.02 = 1$$

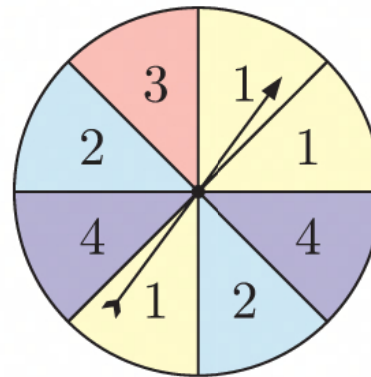
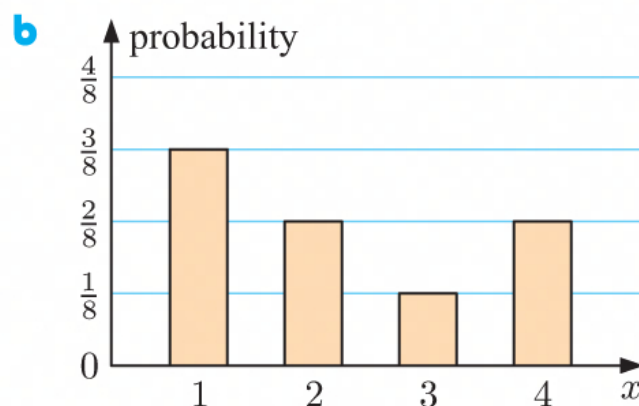
$$\therefore k = 0.04$$

**b** It is most likely that the number of tyres which needed replacing is 0, so the mode of the distribution is 0 tyres.

**c** 
$$\begin{aligned} P(X > 1) &= P(X = 2) + P(X = 3) + P(X = 4) \\ &= 0.06 + 0.04 + 0.02 \\ &= 0.12 \end{aligned}$$

This is the probability that more than 1 tyre will need replacing on a car being inspected.

<b>6</b>	<b>a</b>	$x$	1	2	3	4
		$P(X = x)$	$\frac{3}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	$\frac{2}{8}$



**c** The spinner is most likely to land on 1, so this is the mode of the distribution.

$$p_1 = \frac{3}{8} = 0.375$$

$$p_1 + p_2 = \frac{3}{8} + \frac{2}{8} = 0.625$$

Since  $p_1 + p_2 \geq 0.5$ , the median is 2.

**d** 
$$\begin{aligned} P(X \leq 3) &= P(X = 1) + P(X = 2) + P(X = 3) \\ &= \frac{3}{8} + \frac{2}{8} + \frac{1}{8} \\ &= \frac{6}{8} \\ &= \frac{3}{4} \end{aligned}$$

**7** **a**  $X = 1, 2, 3$ , or  $4$

**b**  $P(X = 1) = \frac{24}{100} = 0.24$

$$P(X = 2) = \frac{35}{100} = 0.35$$

$$P(X = 3) = \frac{27}{100} = 0.27$$

$$P(X = 4) = \frac{14}{100} = 0.14$$

$\therefore$  the probability table for  $X$  is

$x$	1	2	3	4
$P(X = x)$	0.24	0.35	0.27	0.14



- c It is most likely for a randomly selected person to have 2 bedrooms in their house, so this is the mode of the distribution.

$$p_1 = 0.24$$

$$p_1 + p_2 = 0.24 + 0.35 = 0.59$$

Since  $p_1 + p_2 \geq 0.5$ , the median is 2 bedrooms.

- 8 a  $X = 1, 2, 3$ , or 4

b  $P(X = 1) = \frac{12}{25} = 0.48$

$\therefore$  the probability table for  $X$  is

$$P(X = 2) = \frac{7}{25} = 0.28$$

$$P(X = 3) = \frac{2}{25} = 0.08$$

$$P(X = 4) = \frac{4}{25} = 0.16$$

$x$	1	2	3	4
$P(X = x)$	0.48	0.28	0.08	0.16

- c It is most likely for a randomly selected player to only need one shot to score a goal, so this is the mode of the distribution.

$$p_1 = 0.48$$

$$p_1 + p_2 = 0.48 + 0.28 = 0.76$$

Since  $p_1 + p_2 \geq 0.5$ , the median is 2 shots.

- 9 a  $P(x) = \frac{x+1}{10}$ ,  $x = 0, 1, 2, 3$

$$\therefore P(0) = \frac{1}{10}, \quad P(1) = \frac{2}{10}, \quad P(2) = \frac{3}{10}, \quad P(3) = \frac{4}{10}$$

$$0 \leq P(x_i) \leq 1 \text{ in each case, and } \sum_{i=1}^n P(x_i) = \frac{1}{10} + \frac{2}{10} + \frac{3}{10} + \frac{4}{10} = 1$$

$\therefore P(x)$  is a valid probability mass function.

b  $P(x) = \frac{6}{11x}$ ,  $x = 1, 2, 3$

$$\therefore P(1) = \frac{6}{11}, \quad P(2) = \frac{6}{22} = \frac{3}{11}, \quad P(3) = \frac{6}{33} = \frac{2}{11}$$

$$0 \leq P(x_i) \leq 1 \text{ in each case, and } \sum_{i=1}^n P(x_i) = \frac{6}{11} + \frac{3}{11} + \frac{2}{11} = 1$$

$\therefore P(x)$  is a valid probability mass function.

- 10 a  $P(x) = k(x+2)$ ,  $x = 1, 2, 3$

$$\therefore P(1) = 3k, \quad P(2) = 4k, \quad P(3) = 5k$$

$$\text{Since this is a probability distribution, } \sum_{i=1}^n P(x_i) = 1$$

$$\therefore 3k + 4k + 5k = 1$$

$$\therefore 12k = 1$$

$$\therefore k = \frac{1}{12}$$

$$\text{b } P(x) = \frac{k}{x+1}, \quad x = 0, 1, 2, 3$$

$$\therefore P(0) = k, \quad P(1) = \frac{k}{2}, \quad P(2) = \frac{k}{3}, \quad P(3) = \frac{k}{4}$$

Since this is a probability distribution,  $\sum_{i=1}^n P(x_i) = 1$

$$\therefore k + \frac{k}{2} + \frac{k}{3} + \frac{k}{4} = 1$$

$$\therefore \frac{12k + 6k + 4k + 3k}{12} = 1$$

$$\therefore \frac{25k}{12} = 1$$

$$\therefore k = \frac{12}{25}$$

$$\text{11 a } P(x) = \frac{4x - x^2}{a}, \quad x = 0, 1, 2, 3$$

$$\therefore P(0) = 0, \quad P(1) = \frac{3}{a}, \quad P(2) = \frac{4}{a}, \quad P(3) = \frac{3}{a}$$

Since this is a probability distribution,  $\sum_{i=1}^n P(x_i) = 1$

$$\therefore 0 + \frac{3}{a} + \frac{4}{a} + \frac{3}{a} = 1$$

$$\therefore \frac{10}{a} = 1$$

$$\therefore a = 10$$

$$\text{b } P(X = 1) = P(1) = \frac{3}{a} = \frac{3}{10}$$

$\text{c}$  Since  $P(X = 2) = P(2) = \frac{4}{10}$  is the greatest probability, the mode of the distribution is 2.

$$\text{12 } P(x) = a\left(\frac{1}{3}\right)^{x-1}, \quad x = 1, 2, 3, \dots$$

$$\begin{aligned} \text{a } \sum_{i=1}^{\infty} \left(\frac{1}{3}\right)^{i-1} &= \frac{1}{1 - \frac{1}{3}} \quad \{u_1 = 1, \quad r = \frac{1}{3}\} \\ &= \frac{1}{\frac{2}{3}} \\ &= \frac{3}{2} \end{aligned}$$

$\text{b}$  Since this is a probability distribution,  $\sum_{i=1}^{\infty} P(x_i) = 1$

$$\therefore \sum_{i=1}^{\infty} a\left(\frac{1}{3}\right)^{i-1} = 1$$

$$\therefore a \sum_{i=1}^{\infty} \left(\frac{1}{3}\right)^{i-1} = 1$$

$$\therefore a \times \frac{3}{2} = 1 \quad \{\text{using a}\}$$

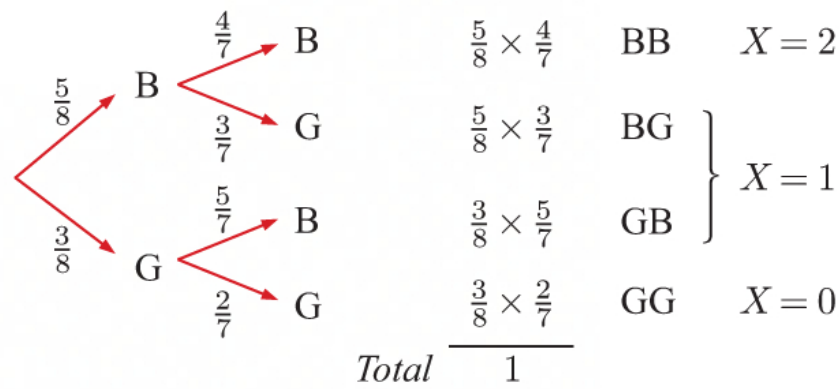
$$\therefore a = \frac{2}{3}$$

**13**  $P(x) = a\left(\frac{2}{5}\right)^x, \quad x = 0, 1, 2, 3, \dots$

$$\begin{aligned} \text{Now, } \sum_{i=0}^{\infty} \left(\frac{2}{5}\right)^i &= \frac{1}{1 - \frac{2}{5}} \quad \{u_1 = 1, \quad r = \frac{2}{5}\} \\ &= \frac{1}{\frac{3}{5}} \\ &= \frac{5}{3} \end{aligned}$$

$$\begin{aligned} \text{Since this is a probability distribution, } \sum_{i=0}^{\infty} P(x_i) &= 1 \\ \therefore \sum_{i=0}^{\infty} a\left(\frac{2}{5}\right)^i &= 1 \\ \therefore a \sum_{i=0}^{\infty} \left(\frac{2}{5}\right)^i &= 1 \\ \therefore a \times \frac{5}{3} &= 1 \\ \therefore a &= \frac{3}{5} \end{aligned}$$

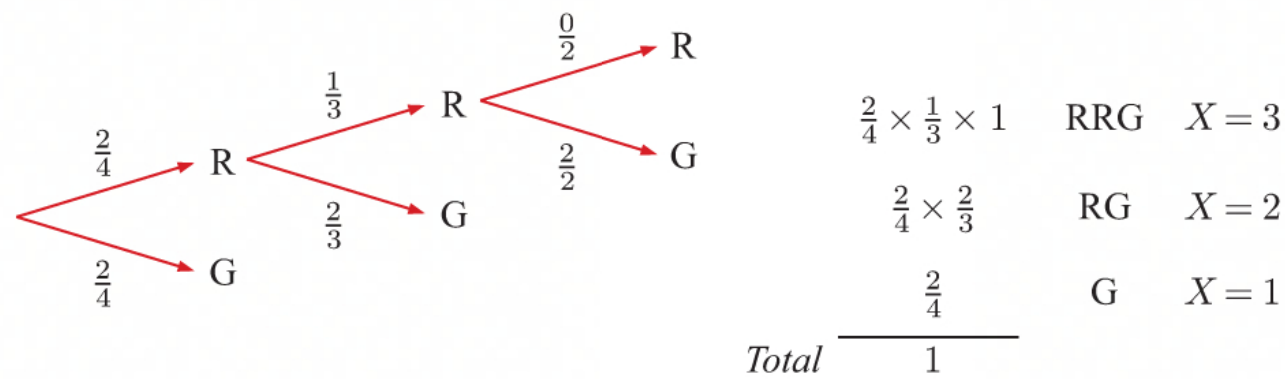
**14** 1st selection 2nd selection



$x$	0	1	2
$P(X = x)$	$\frac{3}{28}$	$\frac{15}{28}$	$\frac{10}{28}$

**15 a**  $X = 1, 2, \text{ or } 3$

**b** 1st selection 2nd selection 3rd selection



$x$	1	2	3
$P(X = x)$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$

**16 a**

		Die 2					
		1	2	3	4	5	6
Die 1	1	2	3	4	5	6	7
	2	3	4	5	6	7	8
	3	4	5	6	7	8	9
	4	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12

**b**

$s$	2	3	4	5	6	7	8	9	10	11	12
$P(S = s)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

**c** It is most likely to roll a sum of 7, so this is the mode of the distribution.**d**  $P(S \geq 10) = P(S = 10) + P(S = 11) + P(S = 12)$ 

$$\begin{aligned}
 &= \frac{3}{36} + \frac{2}{36} + \frac{1}{36} \\
 &= \frac{6}{36} \\
 &= \frac{1}{6}
 \end{aligned}$$

**17 a**  $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

$$\begin{aligned}
 \sum_{n=0}^{\infty} \frac{(0.2)^n e^{-0.2}}{n!} &= e^{-0.2} \sum_{n=0}^{\infty} \frac{(0.2)^n}{n!} \\
 &= e^{-0.2} \times e^{0.2} \quad \left\{ \text{since } e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \right\} \\
 &= e^0 \\
 &= 1
 \end{aligned}$$

**b i**  $0 \leq 0.2 \leq 1$ 

$$\therefore 0 \leq (0.2)^x \leq 1 \quad \text{for all } x = 0, 1, 2, \dots$$

$$\therefore 0 \leq (0.2)^x e^{-0.2} \leq 1 \quad \text{for all } x = 0, 1, 2, \dots$$

$$\therefore 0 \leq \frac{(0.2)^x e^{-0.2}}{x!} \leq 1 \quad \text{for all } x = 0, 1, 2, \dots$$

$$\therefore 0 \leq P(X = x) \leq 1 \quad \text{for all } x = 0, 1, 2, \dots$$

$$\begin{aligned}
 \sum_{x=0}^{\infty} P(X = x) &= \sum_{x=0}^{\infty} \frac{(0.2)^x e^{-0.2}}{x!} \\
 &= 1 \quad \{\text{from a}\}
 \end{aligned}$$

 $\therefore$  this is a valid probability distribution.

**ii**  $P(X = 0) = \frac{(0.2)^0 e^{-0.2}}{0!} = e^{-0.2} \approx 0.819$

$$P(X = 1) = \frac{(0.2)^1 e^{-0.2}}{1!} = 0.2e^{-0.2} \approx 0.164$$

$$P(X = 2) = \frac{(0.2)^2 e^{-0.2}}{2!} = 0.02e^{-0.2} \approx 0.0164$$



$$\begin{aligned}
\text{iii } P(\text{at least 3 cars will pass}) &= P(X \geq 3) \\
&= 1 - P(X \leq 2) \\
&= 1 - [P(X = 0) + P(X = 1) + P(X = 2)] \\
&= 1 - (e^{-0.2} + 0.2e^{-0.2} + 0.02e^{-0.2}) \quad \{\text{using b iii}\} \\
&\approx 0.00115
\end{aligned}$$

**EXERCISE 27C.1****1 a**

$x_i$	1	2	3
$p_i$	0.4	0.5	0.1

$$\begin{aligned}
E(X) &= \sum_{i=1}^n x_i p_i \\
&= 1(0.4) + 2(0.5) + 3(0.1) \\
&= 1.7
\end{aligned}$$

**b**

$x_i$	0	1	2	3	4
$p_i$	0.1	0.2	0.15	0.2	0.35

$$\begin{aligned}
E(X) &= \sum_{i=1}^n x_i p_i \\
&= 0(0.1) + 1(0.2) + 2(0.15) + 3(0.2) + 4(0.35) \\
&= 2.5
\end{aligned}$$

**c**

$x_i$	0	2	5	10
$p_i$	0.2	0.35	0.27	0.18

$$\begin{aligned}
E(X) &= \sum_{i=1}^n x_i p_i \\
&= 0(0.2) + 2(0.35) + 5(0.27) + 10(0.18) \\
&= 3.85
\end{aligned}$$

**d**

$x_i$	10	15	30	60
$p_i$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{12}$	$\frac{1}{3}$

$$\begin{aligned}
E(X) &= \sum_{i=1}^n x_i p_i \\
&= 10\left(\frac{1}{4}\right) + 15\left(\frac{1}{3}\right) + 30\left(\frac{1}{12}\right) + 60\left(\frac{1}{3}\right) \\
&= 30
\end{aligned}$$

**2 a** Since this is a probability distribution,  $\sum_{i=1}^n P(x_i) = 1$

$$\begin{aligned}
\therefore \frac{2}{5} + a + \frac{1}{10} &= 1 \\
\therefore a &= \frac{1}{2}
\end{aligned}$$

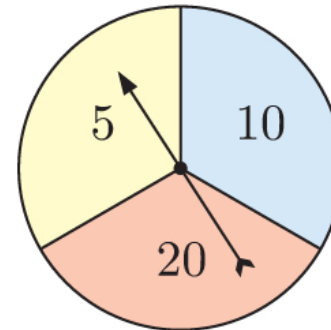
$x$	1	3	5
$P(X = x)$	$\frac{2}{5}$	$a$	$\frac{1}{10}$

**b** Since  $P(X = 3)$  is the greatest probability, 3 is the mode of the distribution.

$$\begin{aligned}\text{c } \mu = E(X) &= 1\left(\frac{2}{5}\right) + 3\left(\frac{1}{2}\right) + 5\left(\frac{1}{10}\right) \\ &= \frac{2}{5} + \frac{3}{2} + \frac{5}{10} \\ &= \frac{4}{10} + \frac{15}{10} + \frac{5}{10} \\ &= 2\frac{2}{5}\end{aligned}$$

**3** Each coloured region on the spinner has the same area.  
The probability table is:

<i>Number of points</i>	5	10	20
<i>Probability</i>	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$



$$\begin{aligned}E(X) &= \sum_{i=1}^n x_i p_i \\ &= 5\left(\frac{1}{3}\right) + 10\left(\frac{1}{3}\right) + 20\left(\frac{1}{3}\right) \\ &= \frac{35}{3} \\ &\approx 11.7 \text{ points}\end{aligned}$$

In the long term, we can expect to be awarded an average of about 11.7 points per spin.

**4**

<i>Number of fish</i>	0	1	2	3
<i>Probability</i>	0.17	0.28	0.36	0.19

$$\begin{aligned}E(X) &= \sum_{i=1}^n x_i p_i \\ &= 0(0.17) + 1(0.28) + 2(0.36) + 3(0.19) \\ &= 0.28 + 0.72 + 0.57 \\ &= 1.57 \text{ fish}\end{aligned}$$

On average, you would expect Ernie to catch 1.57 fish per trip.

**5**

<i>Number of books</i>	1	2	3	4	5
<i>Probability</i>	0.16	0.15	$a$	0.28	0.16

**a** Since this is a probability distribution,  $\sum_{i=1}^n P(x_i) = 1$

$$\therefore 0.16 + 0.15 + a + 0.28 + 0.16 = 1$$

$$\therefore a = 0.25$$

**b** Pam is most likely going to borrow 4 books when she visits the library, so this is the mode of the distribution.

$$\begin{aligned}\text{c } E(X) &= \sum_{i=1}^n x_i p_i \\ &= 1(0.16) + 2(0.15) + 3(0.25) + 4(0.28) + 5(0.16) \\ &= 0.16 + 0.30 + 0.75 + 1.12 + 0.80 \\ &= 3.13 \text{ books}\end{aligned}$$

On average, Pam borrows 3.13 books per visit.

**6**

Colour	Number of lollies
Red	4
Green	6
White	10

There are 5 red balls, 2 green balls, and 1 white ball, so in total there are  $5 + 2 + 1 = 8$  balls.

Number of lollies	4	6	10
Probability	$\frac{5}{8} = 0.625$	$\frac{2}{8} = 0.25$	$\frac{1}{8} = 0.125$

$$\begin{aligned}
 E(X) &= \sum_{i=1}^n x_i p_i \\
 &= 4(0.625) + 6(0.25) + 10(0.125) \\
 &= 2.5 + 1.5 + 1.25 \\
 &= 5.25 \text{ lollies}
 \end{aligned}$$

On average, Lachlan can expect to receive 5.25 lollies.

**7 a**  $P(\text{all ten pins}) = 1 - \frac{1}{3} - \frac{2}{5}$

$$\begin{aligned}
 &= \frac{15}{15} - \frac{5}{15} - \frac{6}{15} \\
 &= \frac{4}{15}
 \end{aligned}$$

**b**

Number of pins knocked down	8	9	10
Probability	$\frac{1}{3}$	$\frac{2}{5}$	$\frac{4}{15}$

$$\begin{aligned}
 E(X) &= \sum_{i=1}^n x_i p_i \\
 &= 8\left(\frac{1}{3}\right) + 9\left(\frac{2}{5}\right) + 10\left(\frac{4}{15}\right) \\
 &= \frac{40}{15} + \frac{54}{15} + \frac{40}{15} \\
 &= \frac{134}{15} \\
 &\approx 8.93 \text{ pins}
 \end{aligned}$$

On average, Jenna knocks down about 8.93 pins with her first bowl.

**8** Since this is a probability distribution,

$$\sum_{i=1}^n P(x_i) = 1$$

$$\therefore 0.3 + a + b + 0.2 = 1$$

$$\therefore b = 0.5 - a \quad \dots (*)$$

$$\text{Now, } E(X) = 2.5$$

$$\therefore 1(0.3) + 2(a) + 3(b) + 4(0.2) = 2.5$$

$$\therefore 0.3 + 2a + 3(0.5 - a) + 0.8 = 2.5 \quad \{\text{using } (*)\}$$

$$\therefore 2a + 1.5 - 3a = 1.4$$

$$\therefore a = 0.1 \text{ and } b = 0.4$$

$x$	1	2	3	4
$P(X = x)$	0.3	$a$	$b$	0.2



- 9 a When Brad's soccer team plays an offensive strategy,  $P(\text{draw}) = 1 - 0.3 - 0.55 = 0.15$   
 When Brad's soccer team plays a defensive strategy,  $P(\text{draw}) = 1 - 0.2 - 0.3 = 0.5$
- b Let  $X$  be the number of points awarded per game when Brad's soccer team plays an offensive strategy.

$$\begin{aligned} E(X) &= 3(0.3) + 1(0.15) + 0(0.55) \\ &= 0.9 + 0.15 \\ &= 1.05 \text{ points per game} \end{aligned}$$

Result	W	D	L
Points	3	1	0
Probability	0.3	0.15	0.55

Let  $Y$  be the number of points awarded per game when Brad's soccer team plays a defensive strategy.

$$\begin{aligned} E(Y) &= 3(0.2) + 1(0.5) + 0(0.3) \\ &= 0.6 + 0.5 \\ &= 1.1 \text{ points per game} \end{aligned}$$

Result	W	D	L
Points	3	1	0
Probability	0.2	0.5	0.3

- c It is better for the team to play a defensive strategy in the long run as the team is expected to gain more points per game.
- d If 4 points are awarded instead of 3 points for a win:

$$\begin{aligned} E(X) &= 4(0.3) + 1(0.15) + 0(0.55) \\ &= 1.2 + 0.15 \\ &= 1.35 \text{ points per game} \end{aligned}$$

Result	W	D	L
Points	4	1	0
Probability	0.3	0.15	0.55

and 
$$\begin{aligned} E(Y) &= 4(0.2) + 1(0.5) + 0(0.3) \\ &= 0.8 + 0.5 \\ &= 1.3 \text{ points per game} \end{aligned}$$

Result	W	D	L
Points	4	1	0
Probability	0.2	0.5	0.3

The team is expected to gain more points per game when they play an offensive strategy. The team should change their strategy.

- 10 a i car park B  
 ii car park A  
 iii car park B

Car park A

Time	Cost
0 - 1 hour	\$7
1 - 2 hours	\$12
2 - 3 hours	\$15
3 - 4 hours	\$19

Car park B

Time	Cost
0 - 1 hour	\$6.50
1 - 2 hours	\$11
2 - 3 hours	\$16
3 - 4 hours	\$18.50

- b Let  $\$X$  be the amount Zoe pays for parking.

When Zoe parks her car at car park A:

$$\begin{aligned} E(X) &= 7(0) + 12(0.2) + 15(0.7) + 19(0.1) \\ &= 2.4 + 10.5 + 1.9 \\ &= \$14.80 \end{aligned}$$

When Zoe parks her car at car park B:

$$\begin{aligned} E(X) &= 6.5(0) + 11(0.2) + 16(0.7) + 18.5(0.1) \\ &= 2.2 + 11.2 + 1.85 \\ &= \$15.25 \end{aligned}$$

$\therefore$  Zoe should choose car park A as it has the lower expected cost.



- 11** The probability of the ring not being stolen or lost is  $P(\text{ring is safe}) = 1 - 0.0025 - 0.03 = 0.9675$

Let  $X$  be the amount the insurance company pays the policy owner.

$$\begin{aligned} E(X) &= 0(0.9675) + 20\,000(0.0025) + 8000(0.03) \\ &= 50 + 240 \\ &= \$290 \text{ per policy} \end{aligned}$$

$\therefore$  the insurance company should charge \$390 per policy to have an expected return of \$100.

## EXERCISE 27C.2

- 1** Let  $X$  denote the return from one game.

<i>Number</i>	1	2	3	4	5	6
<i>Winnings</i>	\$3	\$1	\$3	\$1	\$3	\$1
<i>Probability</i>	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

$$\begin{aligned} E(X) &= 3 \times 3\left(\frac{1}{6}\right) + 3 \times 1\left(\frac{1}{6}\right) \\ &= \frac{9}{6} + \frac{3}{6} \\ &= \frac{12}{6} \\ &= 2 \end{aligned}$$

So, \$2 is the expected return.

Since the game costs \$2 to play, the expected gain = expected return – \$2  
 $= \$2 - \$2 = \$0$

Since the expected gain is zero, the game is fair.

- 2 a** Let  $X$  denote the return from each roll.

$$\begin{aligned} E(X) &= 1\left(\frac{1}{6}\right) + 2\left(\frac{1}{6}\right) + 3\left(\frac{1}{6}\right) + 4\left(\frac{1}{6}\right) + 5\left(\frac{1}{6}\right) + 6\left(\frac{1}{6}\right) \\ &= \frac{1}{6} \times 21 \\ &= \$3.50 \end{aligned}$$

- b** The expected gain is  $\$3.50 - \$4 = -\$0.50$

- c** The player should not play many games, as on average he would expect to lose \$0.50 with each roll.

- 3 a** Let  $X$  denote the return from each bet.

$$\begin{aligned} E(X) &= 2\left(\frac{18}{37}\right) + (-2)\left(\frac{19}{37}\right) \\ &= \frac{36}{37} - \frac{38}{37} \\ &= -\frac{2}{37} \\ &\approx -\$0.05 \end{aligned}$$

- b**  $100 \times \left(-\frac{2}{37}\right) \approx -\$5.41$

From 100 bets, I would expect to lose about \$5.41.

**4**

Result	Win
HH	\$10
HT or TH	\$3
TT	\$1

Let  $X$  be the gain from each game, and  
 $Y$  be the return from each game.

$$\begin{aligned}
 E(Y) &= 10\left(\frac{1}{4}\right) + 3\left(\frac{2}{4}\right) + 1\left(\frac{1}{4}\right) \\
 &= \frac{10}{4} + \frac{6}{4} + \frac{1}{4} \\
 &= \$4.25
 \end{aligned}$$

The expected return per game is \$4.25. It costs \$5.00 to play the game.

$$\begin{aligned}
 \text{So, the expected gain } E(X) &= E(Y) - \$5 \\
 &= \$4.25 - \$5.00 \\
 &= -\$0.75
 \end{aligned}$$

So we expect a loss of \$0.75 per game on average.

**5 a i**  $P(\text{win 5 tokens}) = \frac{6}{20}$  {there are 6 multiples of 3 between 1 and 20}

$$\begin{aligned}
 &= \frac{3}{10} \\
 &= 0.3
 \end{aligned}$$

**ii**  $P(\text{win 10 tokens}) = \frac{2}{20}$  {there are 2 multiples of 10 between 1 and 20}

$$\begin{aligned}
 &= \frac{1}{10} \\
 &= 0.1
 \end{aligned}$$

**b**  $E(X) = 0\left(\frac{12}{20}\right) + 5\left(\frac{6}{20}\right) + 10\left(\frac{2}{20}\right)$

$$\begin{aligned}
 &= \frac{30}{20} + \frac{20}{20} \\
 &= 2\frac{1}{2} \\
 &= 2.5 \text{ tokens}
 \end{aligned}$$

- c** It costs 3 tokens to play the game. So, the expected gain =  $2.5 - 3 = -0.5$  tokens.  
 We do not recommend playing the game many times as the player can expect to lose half a token on average per game.

**6 a**

Disc colour	Black	Blue	Gold
Winnings	\$1	\$5	\$20
Probability	$\frac{10}{15}$	$\frac{4}{15}$	$\frac{1}{15}$

Let  $X$  be the return from each game.

$$\begin{aligned}
 E(X) &= 1\left(\frac{10}{15}\right) + 5\left(\frac{4}{15}\right) + 20\left(\frac{1}{15}\right) \\
 &= \frac{10 + 20 + 20}{15} \\
 &= \frac{50}{15} \\
 &\approx \$3.33
 \end{aligned}$$

The expected return per game is \$3.33. It costs \$4.00 to play the game.

$$\begin{aligned}
 \text{So, the expected gain} &\approx \$3.33 - \$4.00 \\
 &\approx -\$0.67 \neq \$0, \text{ so the game is not fair.}
 \end{aligned}$$

- b** Let the new prize money for selecting the gold disc be \$ $x$ .

Now, for the game to be fair, the expected return must be equal to the cost of each game.

$$\therefore E(X) = 1\left(\frac{10}{15}\right) + 5\left(\frac{4}{15}\right) + x\left(\frac{1}{15}\right) = 4 \quad \{\text{the cost of the game is \$4}\}$$

$$\therefore \frac{10}{15} + \frac{20}{15} + \frac{x}{15} = 4$$

$$\therefore \frac{30+x}{15} = 4$$

$$\therefore 30+x = 60$$

$$\therefore x = 30$$

So, the new prize money for selecting the gold disc is \$30.

**7 a**

		Die 2					
		1	2	3	4	5	6
Die 1	1	2	3	4	5	6	7
	2	3	4	5	6	7	8
	3	4	5	6	7	8	9
	4	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12

36 possible results

$x$	2	3	4	5	6	7	8	9	10	11	12
$P(X = x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

$$\text{So, } P(X \leq 3) = P(X = 2) + P(X = 3) = \frac{1}{36} + \frac{2}{36} = \frac{1}{12}$$

$$P(4 \leq X \leq 6) = P(X = 4) + P(X = 5) + P(X = 6) = \frac{3}{36} + \frac{4}{36} + \frac{5}{36} = \frac{1}{3}$$

$$P(7 \leq X \leq 9) = P(X = 7) + P(X = 8) + P(X = 9) = \frac{6}{36} + \frac{5}{36} + \frac{4}{36} = \frac{5}{12}$$

$$P(X \geq 10) = P(X = 10) + P(X = 11) + P(X = 12) = \frac{3}{36} + \frac{2}{36} + \frac{1}{36} = \frac{1}{6}$$

- b** Let  $Y$  be the return on a game. The cost is \$ $a$ , so the expected gain is

$$\begin{aligned} E(Y) - a &= \frac{a}{3} [P(X \leq 3) + P(7 \leq X \leq 9)] + 7 [P(4 \leq X \leq 6)] + 21 [P(X \geq 10)] - a \\ &= \frac{a}{3} \left( \frac{1}{12} + \frac{5}{12} \right) + 7\left(\frac{1}{3}\right) + 21\left(\frac{1}{6}\right) - a \quad \{\text{using a}\} \\ &= \frac{a}{6} + \frac{7}{3} + \frac{21}{6} - a \\ &= \frac{35}{6} - \frac{5a}{6} \\ &= \frac{1}{6}(35 - 5a) \text{ dollars, as required.} \end{aligned}$$

- c** The game is fair when the expected gain is 0.

$$\therefore \frac{1}{6}(35 - 5a) = 0$$

$$\therefore 35 - 5a = 0$$

$$\therefore a = 7$$



d If  $a = 4$ , expected gain  $= \frac{1}{6} (35 - 5(4))$   
 $= \frac{15}{6} = \$2.50$

So, the people playing would expect to win \$2.50 per game, which means the organisers expect to lose \$2.50 per game.

e If  $a = 9$ , expected gain  $= \frac{1}{6} (35 - 5(9))$   
 $= -\frac{10}{6} \approx -\$1.33$

Expected gain from 2406 games  $= 2406 \times (-\frac{10}{6})$   
 $= -\$4010$

$\therefore$  the organisers would expect to gain \$4010.

8  $P(RRR) = \frac{5}{12} \times \frac{4}{11} \times \frac{3}{10} = \frac{60}{1320}$

$P(BBB) = \frac{4}{12} \times \frac{3}{11} \times \frac{2}{10} = \frac{24}{1320}$

$P(GGG) = \frac{3}{12} \times \frac{2}{11} \times \frac{1}{10} = \frac{6}{1320}$

$P(RBG) = P(RGB) = P(BRG) = P(BGR) = P(GRB) = P(GBR) = \frac{5}{12} \times \frac{4}{11} \times \frac{3}{10} = \frac{60}{1320}$

$P(\text{winning}) = P(\text{all the same colour or one of each})$   
 $= P(RRR) + P(BBB) + P(GGG) + P(RBG) + P(RGB)$   
 $\quad + P(BRG) + P(BGR) + P(GRB) + P(GBR)$   
 $= \frac{60}{1320} + \frac{24}{1320} + \frac{6}{1320} + \frac{60}{1320} \times 6$   
 $= \frac{60+24+6+360}{1320}$   
 $= \frac{450}{1320} = \frac{15}{44}$

The player expects to win  $11 \times \frac{15}{44} = \$3.75$

The organiser makes \$1 when the player loses \$1.

Now, the expected gain for the player = expected win – cost to play

$\therefore -\$1.00 = \$3.75 - \text{cost to play}$

$\therefore \text{cost to play} = \$4.75$

## EXERCISE 27D

1 a

$x$	1	2	3
$P(X = x)$	0.3	0.4	0.3

i  $\mu = \sum x_i p_i$   
 $= 1(0.3) + 2(0.4) + 3(0.3)$   
 $= 2$

ii  $\sigma^2 = \sum (x_i - \mu)^2 p_i$   
 $= (1 - 2)^2(0.3) + (2 - 2)^2(0.4) + (3 - 2)^2(0.3)$   
 $= 0.6$

iii  $\sigma = \sqrt{\sum (x_i - \mu)^2 p_i}$   
 $= \sqrt{0.6}$   
 $\approx 0.775$



**b**

$x$	0	1	2	3
$P(X = x)$	0.2	0.4	0.1	0.3

**i**  $\mu = \sum x_i p_i$   
 $= 0(0.2) + 1(0.4) + 2(0.1) + 3(0.3)$   
 $= 1.5$

**ii**  $\sigma^2 = \sum (x_i - \mu)^2 p_i$   
 $= (0 - 1.5)^2(0.2) + (1 - 1.5)^2(0.4) + (2 - 1.5)^2(0.1) + (3 - 1.5)^2(0.3)$   
 $= 1.25$

**iii**  $\sigma = \sqrt{\sum (x_i - \mu)^2 p_i}$   
 $= \sqrt{1.25}$   
 $\approx 1.12$

**2**

$x_i$	2	4	10	20
$p_i$	$k$	0.05	0.35	$3k$

**a** Since this is a probability distribution,  $\sum_{i=1}^n p_i = 1$   
 $\therefore k + 0.05 + 0.35 + 3k = 1$   
 $\therefore 4k + 0.4 = 1$   
 $\therefore 4k = 0.6$   
 $\therefore k = 0.15$

**b**  $3k = 0.45$   
 $\therefore$  the value 20 has the highest probability of occurring, so this is the mode of the distribution.

**c**  $\mu = \sum x_i p_i$   
 $= 2(0.15) + 4(0.05) + 10(0.35) + 20(0.45)$   
 $= 13$

**d**  $\sigma = \sqrt{\sum (x_i - \mu)^2 p_i}$   
 $= \sqrt{(2 - 13)^2(0.15) + (4 - 13)^2(0.05) + (10 - 13)^2(0.35) + (20 - 13)^2(0.45)}$   
 $= \sqrt{47.4}$   
 $\approx 6.88$

**3** Michelle:

Number of aces	0	1	2	3	4
Probability	0.1	0.15	0.45	0.25	0.05

Amanda:

Number of aces	0	1	2	3	4
Probability	0.2	0.1	0.35	0.2	0.15

**a** For Michelle,  $\mu = \sum x_i p_i$   
 $= 0(0.1) + 1(0.15) + 2(0.45) + 3(0.25) + 4(0.05)$   
 $= 2 \text{ aces}$

For Amanda,  $\mu = \sum x_i p_i$   
 $= 0(0.2) + 1(0.1) + 2(0.35) + 3(0.2) + 4(0.15)$   
 $= 2 \text{ aces}$

$\therefore$  each player is expected to serve an average of 2 aces per set.

**b** For Michelle,

$$\begin{aligned}\sigma^2 &= \sum (x_i - \mu)^2 p_i \\ &= (0 - 2)^2(0.1) + (1 - 2)^2(0.15) + (2 - 2)^2(0.45) + (3 - 2)^2(0.25) + (4 - 2)^2(0.05) \\ &= 1 \text{ ace}^2\end{aligned}$$

$\therefore \sigma = \sqrt{1} = 1 \text{ ace}$

For Amanda,

$$\begin{aligned}\sigma^2 &= \sum (x_i - \mu)^2 p_i \\ &= (0 - 2)^2(0.2) + (1 - 2)^2(0.1) + (2 - 2)^2(0.35) + (3 - 2)^2(0.2) + (4 - 2)^2(0.15) \\ &= 1.7 \text{ aces}^2\end{aligned}$$

$\therefore \sigma = \sqrt{1.7} \approx 1.30 \text{ aces}$

**c** From **b**, Amanda has a higher variance and standard deviation.

$\therefore$  Amanda has the greater variation in the number of aces served.

**4**

$x$	0	1	2	3	4	5	$> 5$
$P(X = x)$	0.54	0.26	0.15	$k$	0.01	0.01	0.00

**a** Since this is a probability distribution,  $\sum_{i=1}^n P(x_i) = 1$

$\therefore 0.54 + 0.26 + 0.15 + k + 0.01 + 0.01 + 0.00 = 1$

$\therefore k + 0.97 = 1$

$\therefore k = 0.03$

**b**  $\mu = \sum x_i p_i$

$= 0(0.54) + 1(0.26) + 2(0.15) + 3(0.03) + 4(0.01) + 5(0.01)$

$= 0.26 + 0.30 + 0.09 + 0.04 + 0.05$

$= 0.74$

So, over a long period the mean number of deaths per dozen crayfish is 0.74.

$$\begin{aligned}
 \text{c } \sigma &= \sqrt{\sum (x_i - \mu)^2 p_i} \\
 &= \sqrt{(0 - 0.74)^2(0.54) + (1 - 0.74)^2(0.26) + (2 - 0.74)^2(0.15) + \dots + (5 - 0.74)^2(0.01)} \\
 &= \sqrt{0.9924} \\
 &\approx 0.996
 \end{aligned}$$

**5 a**

$x$	1	2	3
$P(X = x)$	$\frac{2}{6}$	$\frac{1}{6}$	$\frac{3}{6}$

$$\begin{aligned}
 \text{b } \mu &= \sum x_i p_i \\
 &= 1\left(\frac{2}{6}\right) + 2\left(\frac{1}{6}\right) + 3\left(\frac{3}{6}\right) \\
 &= \frac{2}{6} + \frac{2}{6} + \frac{9}{6} \\
 &= \frac{13}{6} \\
 &\approx 2.17
 \end{aligned}$$

$$\begin{aligned}
 \text{c } \sigma &= \sqrt{\sum (x_i - \mu)^2 p_i} \\
 &= \sqrt{\left(1 - \frac{13}{6}\right)^2\left(\frac{2}{6}\right) + \left(2 - \frac{13}{6}\right)^2\left(\frac{1}{6}\right) + \left(3 - \frac{13}{6}\right)^2\left(\frac{3}{6}\right)} \\
 &= \sqrt{\frac{29}{36}} \\
 &\approx 0.898
 \end{aligned}$$

**6**  $P(x) = \frac{x^2 + x}{20}$  for  $x = 1, 2, 3$

$x$	1	2	3
$P(x)$	$\frac{2}{20} = 0.1$	$\frac{6}{20} = 0.3$	$\frac{12}{20} = 0.6$

**a** The most likely value of  $X$  is 3, so this is the mode of the distribution.

$$\begin{aligned}
 \text{b } p_1 &= 0.1 \\
 p_1 + p_2 &= 0.1 + 0.3 = 0.4 \\
 p_1 + p_2 + p_3 &= 0.4 + 0.6 = 1.0
 \end{aligned}$$

Since  $p_1 + p_2 + p_3 \geq 0.5$ , the median is 3.

$$\begin{aligned}
 \text{c } \mu &= \sum x_i p_i \\
 &= 1(0.1) + 2(0.3) + 3(0.6) \\
 &= 2.5
 \end{aligned}$$

$$\begin{aligned}
 \text{d } \sigma &= \sqrt{\sum (x_i - \mu)^2 p_i} \\
 &= \sqrt{(1 - 2.5)^2(0.1) + (2 - 2.5)^2(0.3) + (3 - 2.5)^2(0.6)} \\
 &= \sqrt{0.45} \\
 &\approx 0.671
 \end{aligned}$$

$$\begin{aligned}
7 \quad \sigma^2 &= E[(X - \mu)^2] \\
&= \sum (x_i - \mu)^2 p_i \\
&= (x_1 - \mu)^2 p_1 + (x_2 - \mu)^2 p_2 + \dots + (x_n - \mu)^2 p_n \\
&= (x_1^2 - 2x_1\mu + \mu^2)p_1 + (x_2^2 - 2x_2\mu + \mu^2)p_2 + \dots + (x_n^2 - 2x_n\mu + \mu^2)p_n \\
&= (x_1^2 p_1 + x_2^2 p_2 + x_3^2 p_3 + \dots + x_n^2 p_n) - 2\mu(x_1 p_1 + x_2 p_2 + \dots + x_n p_n) \\
&\quad + \mu^2(p_1 + p_2 + p_3 + \dots + p_n) \\
&= \sum x_i^2 p_i - 2\mu(\sum x_i p_i) + \mu^2(1) \quad \{p_1 + p_2 + \dots + p_n = 1\} \\
&= \sum x_i^2 p_i - 2\mu(\mu) + \mu^2 \quad \{\text{since } \sum x_i p_i = \mu\} \\
&= \sum x_i^2 p_i - \mu^2 \\
&= E(X^2) - (E(X))^2 \quad \text{as required.}
\end{aligned}$$

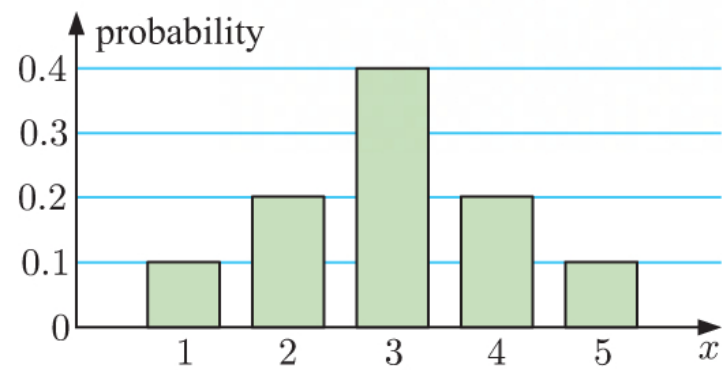
8 a

$x_i$	1	2	3	4	5
$p_i$	0.1	0.2	0.4	0.2	0.1

b

$$\begin{aligned}
\mu &= \sum x_i p_i \\
&= 1(0.1) + 2(0.2) + 3(0.4) + 4(0.2) + 5(0.1) \\
&= 3
\end{aligned}$$

$$\begin{aligned}
\sigma &= \sqrt{\sum (x_i - \mu)^2 p_i} \\
&= \sqrt{\sum x_i^2 p_i - \mu^2} \\
&= \sqrt{1^2(0.1) + 2^2(0.2) + 3^2(0.4) + 4^2(0.2) + 5^2(0.1) - 3^2} \\
&= \sqrt{1.2} \\
&\approx 1.10
\end{aligned}$$



c i

$$\begin{aligned}
&P(\mu - \sigma < X < \mu + \sigma) \\
&= P(3 - \sqrt{1.2} < X < 3 + \sqrt{1.2}) \\
&= P(1.90 < X < 4.10) \\
&= P(X = 2, 3, \text{ or } 4) \\
&= 0.2 + 0.4 + 0.2 \\
&= 0.8
\end{aligned}$$

ii

$$\begin{aligned}
&P(\mu - 2\sigma < X < \mu + 2\sigma) \\
&= P(3 - 2\sqrt{1.2} < X < 3 + 2\sqrt{1.2}) \\
&= P(0.81 < X < 5.19) \\
&= P(X = 1, 2, 3, 4, \text{ or } 5) \\
&= 0.1 + 0.2 + 0.4 + 0.2 + 0.1 \\
&= 1
\end{aligned}$$

- 9 a i Y would have the greater mean as the possible values for Y will be greater overall than the possible values for X.
- ii X would have the greater standard deviation as the probabilities of obtaining each outcome are more spread out.



**b**

$x$	1	2	3	4
$P(X = x)$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

For the probability distribution of  $Y$ , we can see that:

$$\begin{aligned}
 P(Y = 1) &= P(\text{first roll} = 1) \times P(\text{second roll} = 1) \\
 &= \frac{1}{4} \times \frac{1}{4} \\
 &= \frac{1}{16}
 \end{aligned}$$

$$\begin{aligned}
 P(Y = 2) &= P(\text{first roll} = 1) \times P(\text{second roll} = 2) \\
 &\quad + P(\text{first roll} = 2) \times P(\text{second roll} = 1) \\
 &\quad + P(\text{first roll} = 2) \times P(\text{second roll} = 2) \\
 &= \frac{1}{4} \times \frac{1}{4} + \frac{1}{4} \times \frac{1}{4} + \frac{1}{4} \times \frac{1}{4} \\
 &= \frac{1}{16} + \frac{1}{16} + \frac{1}{16} \\
 &= \frac{3}{16}
 \end{aligned}$$

$$\begin{aligned}
 P(Y = 3) &= P(\text{first roll} \leq 2) \times P(\text{second roll} = 3) \\
 &\quad + P(\text{first roll} = 3) \times P(\text{second roll} \leq 2) \\
 &\quad + P(\text{first roll} = 3) \times P(\text{second roll} = 3) \\
 &= \left(\frac{1}{4} + \frac{1}{4}\right) \times \frac{1}{4} + \frac{1}{4} \times \left(\frac{1}{4} + \frac{1}{4}\right) + \frac{1}{4} \times \frac{1}{4} \\
 &= \frac{2}{16} + \frac{2}{16} + \frac{1}{16} \\
 &= \frac{5}{16}
 \end{aligned}$$

$$\begin{aligned}
 P(Y = 4) &= P(\text{first roll} \leq 3) \times P(\text{second roll} = 4) \\
 &\quad + P(\text{first roll} = 4) \times P(\text{second roll} \leq 3) \\
 &\quad + P(\text{first roll} = 4) \times P(\text{second roll} = 4) \\
 &= \left(\frac{1}{4} + \frac{1}{4} + \frac{1}{4}\right) \times \frac{1}{4} + \frac{1}{4} \times \left(\frac{1}{4} + \frac{1}{4} + \frac{1}{4}\right) + \frac{1}{4} \times \frac{1}{4} \\
 &= \frac{3}{16} + \frac{3}{16} + \frac{1}{16} \\
 &= \frac{7}{16}
 \end{aligned}$$

$y$	1	2	3	4
$P(Y = y)$	$\frac{1}{16}$	$\frac{3}{16}$	$\frac{5}{16}$	$\frac{7}{16}$

- c** Since all possible outcomes of  $X$  are equally likely to occur,  $X$  is a uniform discrete random variable.

$$\begin{aligned}
\text{d } X: \quad \mu &= \sum x_i p_i \\
&= 1\left(\frac{1}{4}\right) + 2\left(\frac{1}{4}\right) + 3\left(\frac{1}{4}\right) + 4\left(\frac{1}{4}\right) \\
&= \frac{1}{4} + \frac{2}{4} + \frac{3}{4} + \frac{4}{4} \\
&= \frac{10}{4} \\
&= 2.5 \\
\sigma &= \sqrt{\sum x_i^2 p_i - \mu^2} \\
&= \sqrt{1^2\left(\frac{1}{4}\right) + 2^2\left(\frac{1}{4}\right) + 3^2\left(\frac{1}{4}\right) + 4^2\left(\frac{1}{4}\right) - \left(\frac{10}{4}\right)^2} \\
&= \sqrt{\frac{5}{4}} \\
&\approx 1.12
\end{aligned}$$

$$\begin{aligned}
Y: \quad \mu &= \sum x_i p_i \\
&= 1\left(\frac{1}{16}\right) + 2\left(\frac{3}{16}\right) + 3\left(\frac{5}{16}\right) + 4\left(\frac{7}{16}\right) \\
&= \frac{1}{16} + \frac{6}{16} + \frac{15}{16} + \frac{28}{16} \\
&= \frac{50}{16} \\
&= 3.125 \\
\sigma &= \sqrt{\sum x_i^2 p_i - \mu^2} \\
&= \sqrt{1^2\left(\frac{1}{16}\right) + 2^2\left(\frac{3}{16}\right) + 3^2\left(\frac{5}{16}\right) + 4^2\left(\frac{7}{16}\right) - \left(\frac{50}{16}\right)^2} \\
&= \sqrt{\frac{55}{64}} \\
&\approx 0.927
\end{aligned}$$

$$\begin{aligned}
\text{10 } \mu &= \sum x_i p_i \\
&= \sum_{i=1}^n \frac{i}{n} && \{X \text{ is a uniform discrete random variable and } x_i = i\} \\
&= \frac{1}{n} \sum_{i=1}^n i \\
&= \frac{1}{n} \times \frac{n(n+1)}{2} && \left\{1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}\right\} \\
&= \frac{n+1}{2}
\end{aligned}$$

$$\begin{aligned}
\text{So, } \sigma^2 &= \sum x_i^2 p_i - \mu^2 \\
&= \sum_{i=1}^n \frac{i^2}{n} - \left(\frac{n+1}{2}\right)^2 \\
&= \frac{1}{n} \sum_{i=1}^n i^2 - \left(\frac{n+1}{2}\right)^2 \\
&= \frac{1}{n} \left( \frac{n(n+1)(2n+1)}{6} \right) - \left(\frac{n+1}{2}\right)^2 \quad \left\{ 1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6} \right\} \\
&= \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4} \\
&= \frac{2n^2 + 3n + 1}{6} - \frac{n^2 + 2n + 1}{4} \\
&= \frac{4n^2 + 6n + 2 - (3n^2 + 6n + 3)}{12} \\
&= \frac{n^2 - 1}{12}
\end{aligned}$$

**INVESTIGATION 1****PROPERTIES OF  $aX + b$** **1**

$x$	1	2	3	4	5
$P(X = x)$	0.2	0.2	0.2	0.2	0.2

$$\begin{aligned}
\text{a } E(X) &= \sum x_i p_i \\
&= 1(0.2) + 2(0.2) + 3(0.2) + 4(0.2) + 5(0.2) \\
&= 3
\end{aligned}$$

$$\begin{aligned}
\text{Var}(X) &= \sum x_i^2 p_i - \mu^2 \\
&= 1^2(0.2) + 2^2(0.2) + 3^2(0.2) + 4^2(0.2) + 5^2(0.2) - 3^2 \\
&= 2
\end{aligned}$$

$$\text{b } Y = 2X + 3$$

$y$	5	7	9	11	13
$P(Y = y)$	0.2	0.2	0.2	0.2	0.2

$$\begin{aligned}
E(2X + 3) &= E(Y) \\
&= \sum y_i p_i \\
&= 5(0.2) + 7(0.2) + 9(0.2) + 11(0.2) + 13(0.2) \\
&= 9
\end{aligned}$$

$$\begin{aligned}
\text{Var}(2X + 3) &= \text{Var}(Y) \\
&= \sum y_i^2 p_i - \mu^2 \\
&= 5^2(0.2) + 7^2(0.2) + 9^2(0.2) + 11^2(0.2) + 13^2(0.2) - 9^2 \\
&= 8
\end{aligned}$$

**c i**  $Y = 3X - 2$

$y$	1	4	7	10	13
$P(Y = y)$	0.2	0.2	0.2	0.2	0.2

$$\begin{aligned} E(Y) &= \sum y_i p_i \\ &= 1(0.2) + 4(0.2) + 7(0.2) + 10(0.2) + 13(0.2) \\ &= 7 \end{aligned}$$

$$\begin{aligned} \text{Var}(Y) &= \sum y_i^2 p_i - \mu^2 \\ &= 1^2(0.2) + 4^2(0.2) + 7^2(0.2) + 10^2(0.2) + 13^2(0.2) - 7^2 \\ &= 18 \end{aligned}$$

**ii**  $Y = -2X + 5$

$y$	3	1	-1	-3	-5
$P(Y = y)$	0.2	0.2	0.2	0.2	0.2

$$\begin{aligned} E(Y) &= \sum y_i p_i \\ &= 3(0.2) + 1(0.2) + (-1)(0.2) + (-3)(0.2) + (-5)(0.2) \\ &= -1 \end{aligned}$$

$$\begin{aligned} \text{Var}(Y) &= \sum y_i^2 p_i - \mu^2 \\ &= 3^2(0.2) + 1^2(0.2) + (-1)^2(0.2) + (-3)^2(0.2) + (-5)^2(0.2) - (-1)^2 \\ &= 8 \end{aligned}$$

**iii**  $Y = \frac{X+1}{2}$

$y$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3
$P(Y = y)$	0.2	0.2	0.2	0.2	0.2

$$\begin{aligned} E(Y) &= \sum y_i p_i \\ &= 1(0.2) + \frac{3}{2}(0.2) + 2(0.2) + \frac{5}{2}(0.2) + 3(0.2) \\ &= 2 \end{aligned}$$

$$\begin{aligned} \text{Var}(Y) &= \sum y_i^2 p_i - \mu^2 \\ &= 1^2(0.2) + \left(\frac{3}{2}\right)^2(0.2) + 2^2(0.2) + \left(\frac{5}{2}\right)^2(0.2) + 3^2(0.2) - 2^2 \\ &= \frac{1}{2} \end{aligned}$$



$$\text{iv } Y = \frac{-X + 2}{3}$$

$y$	$\frac{1}{3}$	0	$-\frac{1}{3}$	$-\frac{2}{3}$	-1
$P(Y = y)$	0.2	0.2	0.2	0.2	0.2

$$\begin{aligned} E(Y) &= \sum y_i p_i \\ &= \frac{1}{3}(0.2) + 0(0.2) + \left(-\frac{1}{3}\right)(0.2) + \left(-\frac{2}{3}\right)(0.2) + (-1)(0.2) \\ &= -\frac{1}{3} \end{aligned}$$

$$\begin{aligned} \text{Var}(Y) &= \sum y_i^2 p_i - \mu^2 \\ &= \left(\frac{1}{3}\right)^2(0.2) + 0^2(0.2) + \left(-\frac{1}{3}\right)^2(0.2) + \left(-\frac{2}{3}\right)^2(0.2) + (-1)^2(0.2) - \left(-\frac{1}{3}\right)^2 \\ &= \frac{2}{9} \end{aligned}$$

$$\mathbf{3} \quad \mathbf{a} \quad E(aX + b) = aE(X) + b$$

$$\mathbf{b} \quad \text{Var}(aX + b) = a^2 \text{Var}(X)$$

$$\begin{aligned} \mathbf{4} \quad \sigma(aX + b) &= \sqrt{\text{Var}(aX + b)} \\ &= \sqrt{a^2 \text{Var}(X)} \quad \{\text{using } \mathbf{3} \text{ b}\} \\ &= \sqrt{a^2} \sqrt{\text{Var}(X)} \\ &= |a| \sigma(X) \end{aligned}$$

## EXERCISE 27E

$$\begin{aligned} \mathbf{1} \quad E(aX + b) &= E(aX) + E(b) && \{\text{using } E[g(X) + h(X)] = E[g(X)] + E[h(X)]\} \\ &= aE(X) + E(b) && \{\text{using } E(kX) = kE(X)\} \\ &= aE(X) + b && \{\text{using } E(k) = k, \quad k \text{ a constant}\} \end{aligned}$$

$$\begin{array}{lll} \mathbf{2} \quad \mathbf{a} \quad E(Y) = E(3X + 4) & \mathbf{b} \quad E(Y) = E(-2X + 1) & \mathbf{c} \quad E(Y) = E\left(\frac{4X - 2}{3}\right) \\ & = -2E(X) + 1 & = E\left(\frac{4}{3}X - \frac{2}{3}\right) \\ & = -2(3) + 1 & = \frac{4}{3}E(X) - \frac{2}{3} \\ & = -5 & = \frac{4}{3}(3) - \frac{2}{3} \\ & & = 3\frac{1}{3} \end{array}$$

$\mathbf{3}$   $X$  has mean 6 and standard deviation 2.

$$\begin{array}{ll} E(Y) = E(2X + 5) & \sigma(Y) = \sigma(2X + 5) \\ & = |2| \sigma(X) \\ & = 2(2) \\ & = 4 \\ & = 17 \end{array}$$

For  $Y$ , the mean is 17 and the standard deviation is 4.

**4**  $X$  has mean 5 and standard deviation 2.

$$\begin{aligned} \text{a} \quad E(Y) &= E(2X + 3) & \text{Var}(Y) &= \text{Var}(2X + 3) \\ &= 2E(X) + 3 & &= 2^2 \text{Var}(X) \\ &= 2(5) + 3 & &= 4 \times 2^2 \\ &= 13 & &= 16 \end{aligned}$$

$$\begin{aligned} \text{b} \quad E(Y) &= E(-5X + 3) & \text{Var}(Y) &= \text{Var}(-5X + 3) \\ &= -5E(X) + 3 & &= (-5)^2 \text{Var}(X) \\ &= -5(5) + 3 & &= 25 \times 2^2 \\ &= -22 & &= 100 \end{aligned}$$

$$\begin{aligned} \text{c} \quad Y &= \frac{X-5}{2} = \frac{1}{2}X - \frac{5}{2} \\ E(Y) &= E\left(\frac{1}{2}X - \frac{5}{2}\right) & \text{Var}(Y) &= \text{Var}\left(\frac{1}{2}X - \frac{5}{2}\right) \\ &= \frac{1}{2}E(X) - \frac{5}{2} & &= \left(\frac{1}{2}\right)^2 \text{Var}(X) \\ &= \frac{1}{2}(5) - \frac{5}{2} & &= \frac{1}{4} \times 2^2 \\ &= 0 & &= 1 \end{aligned}$$

**5**

$x_i$	1	2	3	4
$p_i$	0.4	0.3	0.2	0.1

$$\begin{aligned} \text{a} \quad E(X) &= \sum x_i p_i \\ &= 1(0.4) + 2(0.3) + 3(0.2) + 4(0.1) \\ &= 2 \end{aligned}$$

$$\begin{aligned} \text{b} \quad \text{Var}(X) &= \sum (x_i - \mu)^2 p_i \\ &= (1-2)^2(0.4) + (2-2)^2(0.3) + (3-2)^2(0.2) + (4-2)^2(0.1) \\ &= 1 \end{aligned}$$

$$\begin{aligned} \text{c} \quad \sigma(X) &= \sqrt{\text{Var}(X)} \\ &= \sqrt{1} \\ &= 1 \end{aligned}$$

$$\begin{aligned} \text{d} \quad E(X+1) &= E(X) + 1 \\ &= 2 + 1 \\ &= 3 \end{aligned}$$

$$\begin{aligned} \text{e} \quad \text{Var}(3X+1) &= 3^2 \text{Var}(X) \\ &= 9(1) \\ &= 9 \end{aligned}$$

$$\begin{aligned} \text{f} \quad \sigma(5-X) &= |-1| \sigma(X) \\ &= 1(1) \\ &= 1 \end{aligned}$$

$$\begin{aligned} \text{g} \quad E\left(\frac{2X+5}{3}\right) &= E\left(\frac{2}{3}X + \frac{5}{3}\right) \\ &= \frac{2}{3}E(X) + \frac{5}{3} \\ &= \frac{2}{3}(2) + \frac{5}{3} \\ &= 3 \end{aligned}$$

$$\begin{aligned} \text{h} \quad \text{Var}(20-4X) &= (-4)^2 \text{Var}(X) \\ &= 16 \times 1 \\ &= 16 \end{aligned}$$

**6**

$x$	1	2	3	4	5
$P(X = x)$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{12}$	$a$	$\frac{1}{6}$

**a** Since this is a probability distribution,  $\sum_{i=1}^n P(x_i) = 1$

$$\therefore \frac{1}{6} + \frac{1}{3} + \frac{1}{12} + a + \frac{1}{6} = 1$$

$$\therefore a + \frac{9}{12} = 1$$

$$\therefore a = \frac{1}{4}$$

**b i**  $E(X) = \sum x_i p_i$

$$= 1\left(\frac{1}{6}\right) + 2\left(\frac{1}{3}\right) + 3\left(\frac{1}{12}\right) + 4\left(\frac{1}{4}\right) + 5\left(\frac{1}{6}\right)$$

$$= \frac{1}{6} + \frac{2}{3} + \frac{1}{4} + 1 + \frac{5}{6}$$

$$= 2\frac{11}{12}$$

$$\text{Var}(X) = \sum x_i^2 p_i - \mu^2$$

$$= 1^2\left(\frac{1}{6}\right) + 2^2\left(\frac{1}{3}\right) + 3^2\left(\frac{1}{12}\right) + 4^2\left(\frac{1}{4}\right) + 5^2\left(\frac{1}{6}\right) - \left(2\frac{11}{12}\right)^2$$

$$= \frac{1}{6} + \frac{4}{3} + \frac{3}{4} + 4 + \frac{25}{6} - \frac{1225}{144}$$

$$= \frac{275}{144}$$

$$\approx 1.91$$

$$\sigma(X) = \sqrt{\text{Var}(X)}$$

$$= \sqrt{\frac{275}{144}}$$

$$\approx 1.38$$

**ii**  $E(X + 4) = E(X) + 4$

$$= 2\frac{11}{12} + 4$$

$$= 6\frac{11}{12}$$

$$\text{Var}(X + 4) = 1^2 \text{Var}(X)$$

$$= \text{Var}(X)$$

$$\approx 1.91$$

$$\sigma(X + 4) = |1| \sigma(X)$$

$$= \sigma(X)$$

$$\approx 1.38$$

**iii**  $E(3X - 1) = 3E(X) - 1$

$$= 3\left(2\frac{11}{12}\right) - 1$$

$$= 7\frac{3}{4}$$

$$\text{Var}(3X - 1) = 3^2 \text{Var}(X)$$

$$= 9 \times \frac{275}{144}$$

$$\approx 17.2$$

$$\sigma(3X - 1) = |3| \sigma(X)$$

$$= 3\sigma(X)$$

$$= 3 \times \sqrt{\frac{275}{144}}$$

$$\approx 4.15$$

<b>7</b>	$x$	0	1	2	3	4	5
	$P(X = x)$	0.17	0.25	0.3	0.15	0.1	0.03

**a**

**i**  $E(X) = \sum x_i p_i$   
 $= 0(0.17) + 1(0.25) + 2(0.3) + 3(0.15) + 4(0.1) + 5(0.03)$   
 $= 0.25 + 0.6 + 0.45 + 0.4 + 0.15$   
 $= 1.85$

**ii**  $\text{Var}(X) = \sum x_i^2 p_i - \mu^2$   
 $= 0^2(0.17) + 1^2(0.25) + 2^2(0.3) + 3^2(0.15) + 4^2(0.1) + 5^2(0.03) - (1.85)^2$   
 $= 1.7275$   
 $\approx 1.73$

**iii**  $\sigma(X) = \sqrt{\text{Var}(X)}$   
 $= \sqrt{1.7275}$   
 $\approx 1.31$

**b** Dominic earns \$100 plus \$25 for each sale made.

$$\therefore Y = 25X + 100$$

**c**

**i**  $E(Y) = E(25X + 100)$   
 $= 25E(X) + 100$   
 $= 25(1.85) + 100$   
 $= 146.25$

**ii**  $\text{Var}(Y) = \text{Var}(25X + 100)$   
 $= 25^2 \text{Var}(X)$   
 $= 625(1.7275)$   
 $\approx 1080$

**iii**  $\sigma(Y) = E(25X + 100)$   
 $= |25| \sigma(X)$   
 $= 25 \times \sqrt{1.7275}$   
 $\approx 32.9$

**8**  $\text{Var}(aX + b) = E((aX + b)^2) - (E(aX + b))^2$   
 $= E(a^2 X^2 + 2abX + b^2) - (aE(X) + b)^2$   
 $= a^2 E(X^2) + 2abE(X) + b^2 - [a^2(E(X))^2 + 2abE(X) + b^2]$   
 $= a^2 E(X^2) + 2abE(X) + b^2 - a^2(E(X))^2 - 2abE(X) - b^2$   
 $= a^2 (E(X^2) - (E(X))^2)$   
 $= a^2 \text{Var}(X) \quad \text{as required.}$

## EXERCISE 27F

- 1**
  - a** The binomial distribution applies, as tossing a coin has two possible outcomes (a head or a tail) and each toss is independent of every other toss.
  - b** The binomial distribution applies, as this is equivalent to tossing one coin 100 times.
  - c** The binomial distribution applies as we can draw out a red or a blue marble with the same chances each time.
  - d** The binomial distribution does not apply as the result of each draw is dependent upon the results of previous draws.



- e** The binomial distribution does not apply, assuming that ten bolts are drawn without replacement, as we do not have a repetition of independent trials. However, since there is such a large number of bolts in the bin, the trials are approximately independent, so the distribution is approximately binomial.

**2 a**  $(p + q)^4 = p^4 + 4p^3q + 6p^2q^2 + 4pq^3 + q^4$

**b i**  $P(4 \text{ heads})$   
 $= p^4$   
 $= \left(\frac{1}{2}\right)^4$   
 $= \frac{1}{16}$

**ii**  $P(3 \text{ heads})$   
 $= 4p^3q$   
 $= 4\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)$   
 $\quad \{\text{as } p = q = \frac{1}{2}\}$   
 $= \frac{1}{4}$

**iii**  $P(2 \text{ heads})$   
 $= 6p^2q^2$   
 $= 6\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2$   
 $= \frac{3}{8}$

**3 a**  $(p + q)^5 = p^5 + 5p^4q + 10p^3q^2 + 10p^2q^3 + 5pq^4 + q^5$

**b i**  $P(4H \text{ and } 1T)$   
 $= 5p^4q$   
 $= 5\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)$   
 $= \frac{5}{32}$

**ii**  $P(2H \text{ and } 3T)$   
 $= 10p^2q^3$   
 $= 10\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^3$   
 $= \frac{10}{32}$   
 $= \frac{5}{16}$

**iii**  $P(\text{HHHHT})$   
 $= \left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)$   
 $= \frac{1}{32}$

**4 a**  $\left(\frac{2}{3} + \frac{1}{3}\right)^4 = \left(\frac{2}{3}\right)^4 + 4\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right) + 6\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^2 + 4\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^3 + \left(\frac{1}{3}\right)^4$

- b** The probability of getting a strawberry cream is  $p = \frac{2}{3}$ .  
 Let  $X$  be the number of strawberry creams selected.

**i**  $P(\text{all strawberry creams}) = P(X = 4)$   
 $= \left(\frac{2}{3}\right)^4$   
 $= \frac{16}{81}$

**ii**  $P(\text{two of each}) = P(X = 2)$   
 $= 6\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^2$   
 $= \frac{8}{27}$

**iii**  $P(\text{at least 2 strawberry creams}) = P(X \geq 2)$   
 $= P(X = 2) + P(X = 3) + P(X = 4)$   
 $= \frac{16}{81} + 4\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right) + \frac{8}{27} \quad \{\text{using i and ii}\}$   
 $= \frac{8}{9}$

**5 a**  $\left(\frac{3}{4} + \frac{1}{4}\right)^5 = \left(\frac{3}{4}\right)^5 + 5\left(\frac{3}{4}\right)^4\left(\frac{1}{4}\right) + 10\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)^2 + 10\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^3 + 5\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^4 + \left(\frac{1}{4}\right)^5$

- b** The probability of getting a “normal” kiwi is  $p = \frac{3}{4}$ .  
 Let  $X$  be the number of “normal” kiwis selected.

**i**  $P(2 \text{ “flat backs”})$   
 $= P(3 \text{ “normal” kiwis})$   
 $= P(X = 3)$   
 $= 10\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)^2$   
 $= \frac{135}{512}$

**ii**  $P(\text{at least 3 “flat backs”})$   
 $= P(\text{at most 2 “normal” kiwis})$   
 $= P(X = 0) + P(X = 1) + P(X = 2)$   
 $= \left(\frac{1}{4}\right)^5 + 5\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^4 + 10\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^3$   
 $= \frac{53}{512}$

$$\begin{aligned}
 \text{iii } P(\text{at most 3 "normal" kiwis}) &= P(\text{at most 2 "normal" kiwis}) + P(3 \text{ "normal" kiwis}) \\
 &= \frac{135}{512} + \frac{53}{512} \quad \{\text{using i and ii}\} \\
 &= \frac{47}{128}
 \end{aligned}$$

$$\begin{aligned}
 6 \quad \text{a } \sum_{x=0}^n P(x) &= \sum_{x=0}^n \binom{n}{x} p^x (1-p)^{n-x} \\
 &= (p + (1-p))^n \quad \{\text{binomial theorem}\} \\
 &= 1^n \\
 &= 1
 \end{aligned}$$

$$\begin{aligned}
 \text{b } \binom{n}{x}, p^x, (1-p)^{n-x} &\geq 0 \quad \text{for all } x = 0, 1, \dots, n \\
 \therefore P(x) &\geq 0 \quad \text{for all } x
 \end{aligned}$$

$$\text{Now } \sum_{x=0}^n P(x) = 1 \quad \text{and} \quad P(x) \geq 0 \quad \text{for all } x$$

$$\therefore P(x) \leq 1 \quad \text{for all } x$$

$$\therefore 0 \leq P(x) \leq 1 \quad \text{for all } x = 0, 1, \dots, n. \quad \checkmark$$

c Our answers to a and b tell us that  $P(x)$  is a valid probability distribution.

## INVESTIGATION 2

## THE GRAPH OF A BINOMIAL DISTRIBUTION

$$1 \quad \text{a } X \sim B(n, p)$$

When  $n = 25$ ,  $p = 0.1$ , the mode of  $X$  is 2.

b The distribution is positively skewed.

2 When  $p = 0.5$ , the distribution is symmetric.

When  $p < 0.5$ , the distribution is positively skewed.

When  $p > 0.5$ , the distribution is negatively skewed.

3  $p = 0.1$ , and the value of  $n$  is free to change.

As  $n$  increases, the distribution becomes approximately symmetrical.

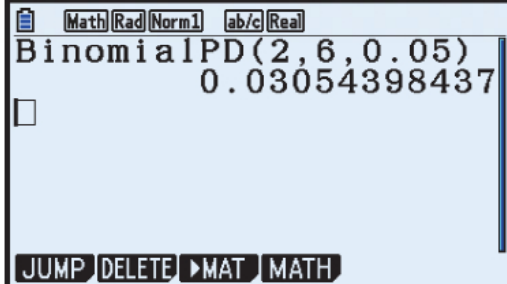
## EXERCISE 27G

1 Let  $X$  be the number of defective light bulbs.

$n = 6$ , so  $X = 0, 1, 2, 3, 4, 5$ , or  $6$ , and  $p = 5\% = 0.05$

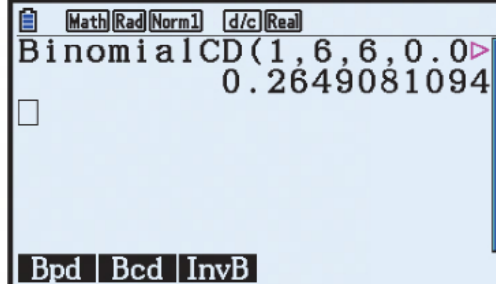
$$\therefore X \sim B(6, 0.05)$$

a



$$P(X = 2) \approx 0.0305$$

b



$$P(X \geq 1) \approx 0.265$$

- 2** Let  $X$  be the number of faulty items.

$n = 12$ , so  $X = 0, 1, 2, 3, \dots$ , or  $12$ , and  $p = 6\% = 0.06$

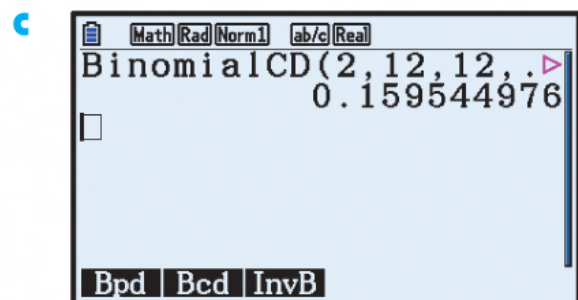
$\therefore X \sim B(12, 0.06)$

- a**  $P(\text{none will be faulty})$

$$\begin{aligned} &= P(X = 0) \\ &= \binom{12}{0} (0.06)^0 (0.94)^{12} \\ &\approx 0.476 \end{aligned}$$

- b**  $P(\text{at most one is faulty})$

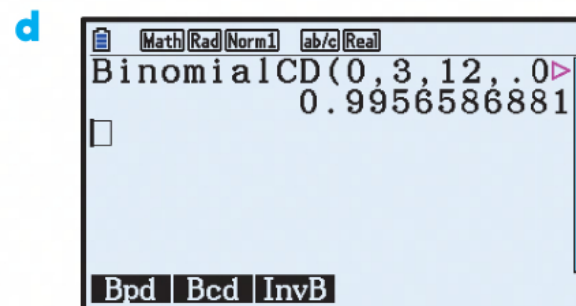
$$\begin{aligned} &= P(X \leq 1) \\ &= P(X = 0) + P(X = 1) \\ &\approx 0.476 + \binom{12}{1} (0.06)^1 (0.94)^{11} \\ &\approx 0.840 \end{aligned}$$



$$\begin{aligned} P(\text{at least two are faulty}) &= P(X \geq 2) \\ &\approx 0.160 \end{aligned}$$

or  $P(\text{at least two are faulty})$

$$\begin{aligned} &= 1 - P(\text{at most one is faulty}) \\ &\approx 1 - 0.840 \quad \{\text{from b}\} \\ &\approx 0.160 \end{aligned}$$



$$\begin{aligned} P(\text{less than four are faulty}) &= P(X < 4) \\ &= P(X \leq 3) \\ &\approx 0.996 \end{aligned}$$

- 3** Let  $X$  be the number of times in a week that the bus is on time.

Since it is late 2 in every 5 days, then it is on time 3 in every 5 days, so  $p = \frac{3}{5} = 0.6$ .

$n = 7$ , so  $X = 0, 1, 2, 3, 4, 5, 6$ , or  $7$ , and  $X \sim B(7, 0.6)$ .

**a**  $P(X = 7) = \binom{7}{7} (0.6)^7 (0.4)^0$

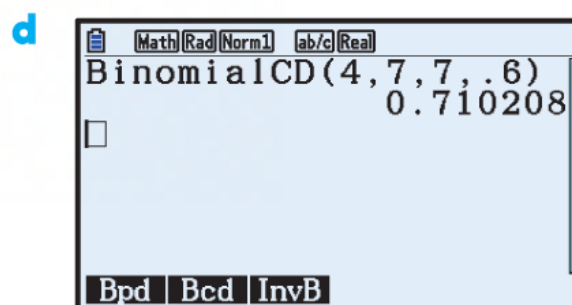
$$\approx 0.0280$$

**b**  $P(\text{on time only on Monday}) = (0.6)(0.4)^6$

$$\approx 0.00246$$

**c**  $P(X = 6) = \binom{7}{6} (0.6)^6 (0.4)$

$$\approx 0.131$$



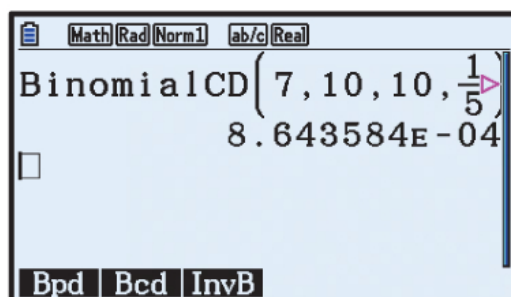
$$P(X \geq 4) \approx 0.710$$

- 4** Let  $X$  denote the number of questions Raj answers correctly.

$n = 10$ , so  $X = 0, 1, 2, \dots$ , or  $10$ , and  $p = \frac{1}{5}$

$\therefore X \sim B(10, \frac{1}{5})$

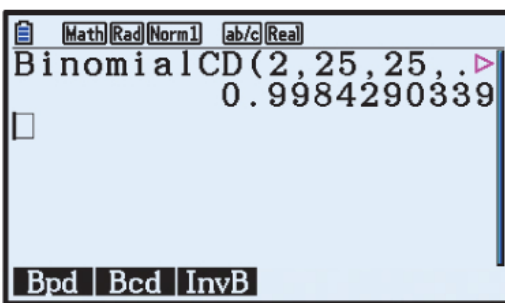
$$\begin{aligned} P(\text{Raj passes}) &= P(X \geq 7) \\ &\approx 0.000864 \end{aligned}$$





- 5 Let  $X$  be the number of students with the flu.  
 $n = 25$ , so  $X = 0, 1, 2, 3, \dots$ , or  $25$ , and  $p = 0.3$   
 $\therefore X \sim B(25, 0.3)$

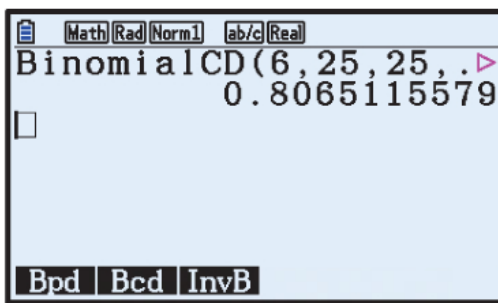
a



BinomialCD(2, 25, 25, .)  
 0.9984290339

$$P(X \geq 2) \approx 0.998$$

b



BinomialCD(6, 25, 25, .)  
 0.8065115579

$$20\% \text{ of } 25 = 5$$

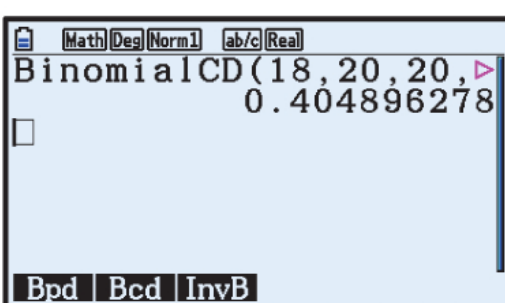
$$\therefore P(\text{test cancelled}) = P(X \geq 6)$$

$$\approx 0.807$$

- 6 Let  $X$  be the number of successful shots from the free throw line.  
 $n = 20$ , so  $X = 0, 1, 2, 3, \dots$ , or  $20$ , and  $p = 85\% = 0.85$   
 $\therefore X \sim B(20, 0.85)$

a  $P(X = 20) = \binom{20}{20} (0.85)^{20} (0.15)^0$   
 $\approx 0.0388$

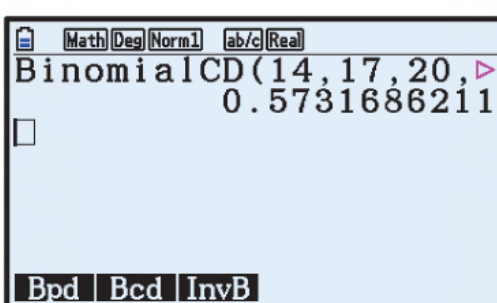
b



BinomialCD(18, 20, 20, .)  
 0.404896278

$$P(X \geq 18) \approx 0.405$$

c



BinomialCD(14, 17, 20, .)  
 0.5731686211

$$P(14 \leq X \leq 17) \approx 0.573$$

- 7 For Jelena to win a set of 6 games to 4, she must win 5 of the first 9 games, and then win the 10th game.

Let  $X$  be the number of games Jelena wins in the first 9 games.

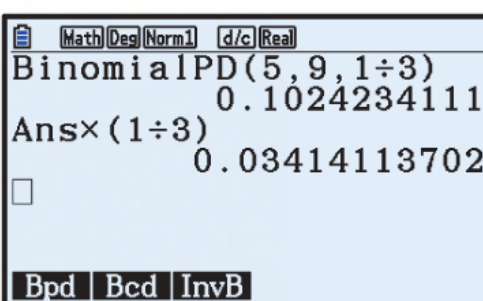
$n = 9$ , so  $X = 0, 1, 2, 3, \dots$ , or  $9$

Now, Martina beats Jelena in 2 games out of 3, so the probability of Jelena winning a game is

$$p = 1 - \frac{2}{3} = \frac{1}{3}$$

$$\therefore X \sim B(9, \frac{1}{3})$$

So,  $P(\text{J wins 6 games to 4})$   
 $= P(\text{J wins 5 of first 9 games}) \times P(\text{J wins 10th game})$   
 $= P(X = 5) \times \frac{1}{3}$   
 $\approx 0.1024 \times \frac{1}{3}$   
 $\approx 0.0341$



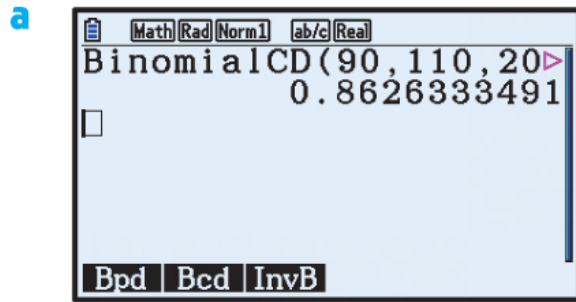
BinomialPD(5, 9, 1/3)  
 0.1024234111  
 Ans x (1/3)  
 0.03414113702



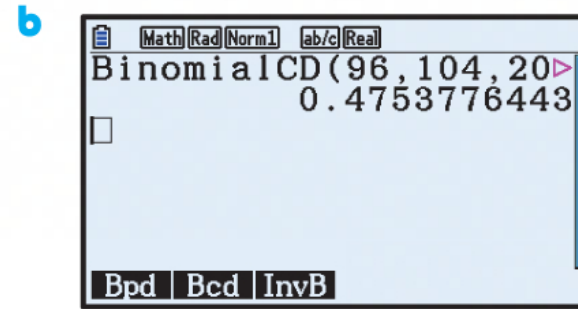
- 8 Let  $X$  be the number of heads.

$n = 200$ , so  $X = 0, 1, 2, 3, \dots$ , or  $200$ , and  $p = \frac{1}{2}$

$$\therefore X \sim B(200, \frac{1}{2})$$



$$P(90 \leq X \leq 110) \approx 0.863$$



$$P(95 < X < 105) = P(96 \leq X \leq 104) \approx 0.475$$

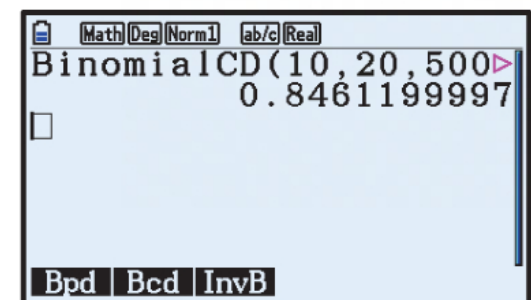
- 9 a  $P(\text{rolling double sixes}) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$

- b Let  $X$  be the number of double sixes rolled.

$n = 500$ , so  $X = 0, 1, 2, 3, \dots$ , or  $500$ , and  $p = \frac{1}{36}$

$$\therefore X \sim B(500, \frac{1}{36})$$

$$P(10 \leq X \leq 20) \approx 0.846$$

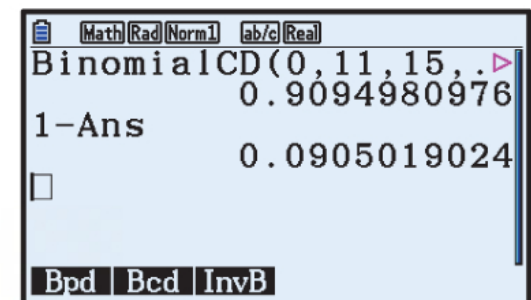


- 10 Let  $X$  be the number of traffic lights Shelley has stopped at.

$n = 15$ , so  $X = 0, 1, 2, 3, \dots$ , or  $15$ , and  $p = 0.6$

$$\therefore X \sim B(15, 0.6)$$

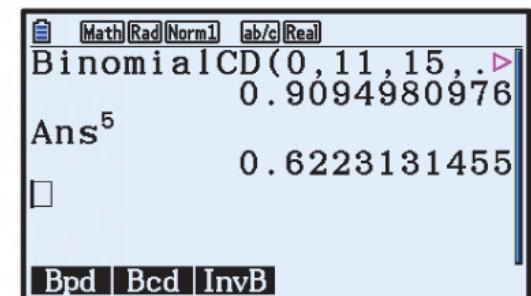
- a  $P(\text{Shelley will be late}) = P(X > 11)$   
 $= 1 - P(X \leq 11)$   
 $\approx 0.0905$



- b  $P(\text{Shelley will be on time}) = P(X \leq 11)$   
 $\approx 0.909$

$$P(\text{Shelley will be on time all 5 days}) = [P(X \leq 11)]^5$$

$$\approx 0.622$$

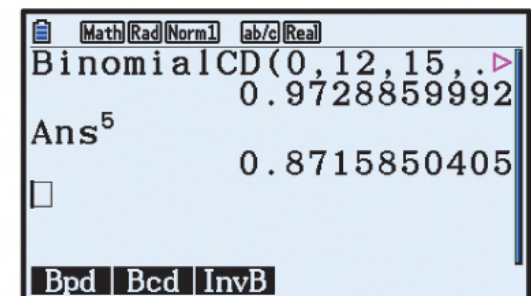


- c  $P(\text{Shelley will be on time}) = P(X \leq 12)$   
 $\approx 0.973$

$$P(\text{Shelley will be on time all 5 days}) = [P(X \leq 12)]^5$$

$$\approx 0.872$$

$\therefore$  yes, the probability that Shelley is on time for work each day of a 5 day week is now about 87.2%.



- 11 Let  $X$  be the number of solar components which fail.

$n = 20$ , so  $X = 0, 1, 2, 3, \dots$ , or  $20$ , and  $p = 0.85$

$$\therefore X \sim B(20, 0.85)$$

- a  $P(\text{hot water unit fails within one year}) = P(\text{all 20 components fail})$   
 $= P(X = 20)$   
 $= (0.85)^{20}$   
 $\approx 0.0388$

**b**  $P(\text{hot water unit with } n \text{ components fails within one year}) = (0.85)^n$

$\therefore P(\text{hot water unit with } n \text{ components is operating after one year}) = 1 - (0.85)^n$

$\therefore$  we need to find the smallest integer  $n$  such that  $1 - (0.85)^n \geq 0.98$

$$\therefore (0.85)^n \leq 0.02$$

$$\therefore n \log(0.85) \leq \log(0.02)$$

$$\therefore n \geq \frac{\log(0.02)}{\log(0.85)} \quad \{\log(0.85) < 0\}$$

$$\therefore n \geq 24.1$$

$\therefore$  at least 25 solar components are needed.

### INVESTIGATION 3

### THE MEAN AND STANDARD DEVIATION OF A BINOMIAL DISTRIBUTION

**1**  $X \sim B(30, 0.25)$

Consult the graphics calculator instructions by clicking on the icon in the Investigation box if you need help obtaining the result shown.

Symbol	Value
$\mu$	$\bar{x} = 7.5$
$\sigma$	$\sigma x = 2.371708245$

**2**

	$p = 0.1$	$p = 0.25$	$p = 0.5$	$p = 0.7$
$n = 10$	$\mu = 1$ $\sigma \approx 0.9487$	$\mu = 2.5$ $\sigma \approx 1.3693$	$\mu = 5$ $\sigma \approx 1.5811$	$\mu = 7$ $\sigma \approx 1.4491$
$n = 30$	$\mu = 3$ $\sigma \approx 1.6432$	$\mu = 7.5$ $\sigma \approx 2.3717$	$\mu = 15$ $\sigma \approx 2.7386$	$\mu = 21$ $\sigma \approx 2.5100$
$n = 50$	$\mu = 5$ $\sigma \approx 2.1213$	$\mu = 12.5$ $\sigma \approx 3.0619$	$\mu = 25$ $\sigma \approx 3.5355$	$\mu = 35$ $\sigma \approx 3.2404$

**3**

	$p = 0.1$	$p = 0.25$	$p = 0.5$	$p = 0.7$
$n = 10$	$np = 1$ $\sqrt{np(1-p)}$ $\approx 0.9487$	$np = 2.5$ $\sqrt{np(1-p)}$ $\approx 1.3693$	$np = 5$ $\sqrt{np(1-p)}$ $\approx 1.5811$	$np = 7$ $\sqrt{np(1-p)}$ $\approx 1.4491$
$n = 30$	$np = 3$ $\sqrt{np(1-p)}$ $\approx 1.6432$	$np = 7.5$ $\sqrt{np(1-p)}$ $\approx 2.3717$	$np = 15$ $\sqrt{np(1-p)}$ $\approx 2.7386$	$np = 21$ $\sqrt{np(1-p)}$ $\approx 2.5100$
$n = 50$	$np = 5$ $\sqrt{np(1-p)}$ $\approx 2.1213$	$np = 12.5$ $\sqrt{np(1-p)}$ $\approx 3.0619$	$np = 25$ $\sqrt{np(1-p)}$ $\approx 3.5355$	$np = 35$ $\sqrt{np(1-p)}$ $\approx 3.2404$

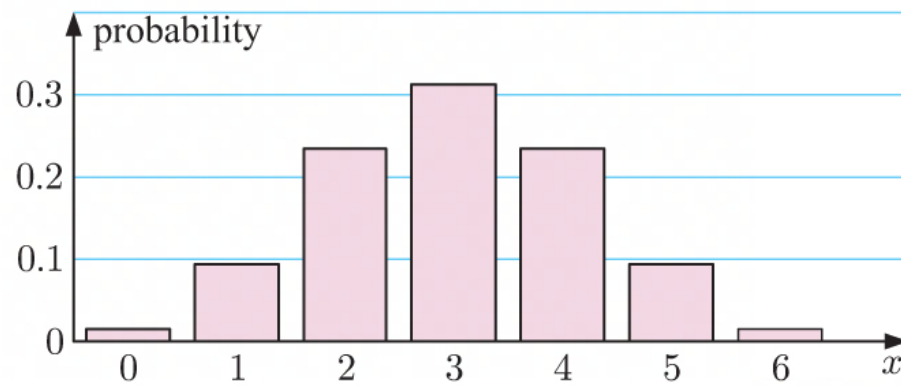
Our results in **2** and **3** agree with the formulae  $\mu = np$  and  $\sigma = \sqrt{np(1-p)}$ .

**EXERCISE 27H****1 a**  $X \sim B(6, 0.5)$ 

$$\begin{aligned}
 \text{i} \quad \mu &= np & \sigma &= \sqrt{np(1-p)} \\
 &= 6 \times 0.5 & &= \sqrt{6 \times 0.5 \times 0.5} \\
 &= 3 & &\approx 1.22
 \end{aligned}$$

$$\text{ii}$$

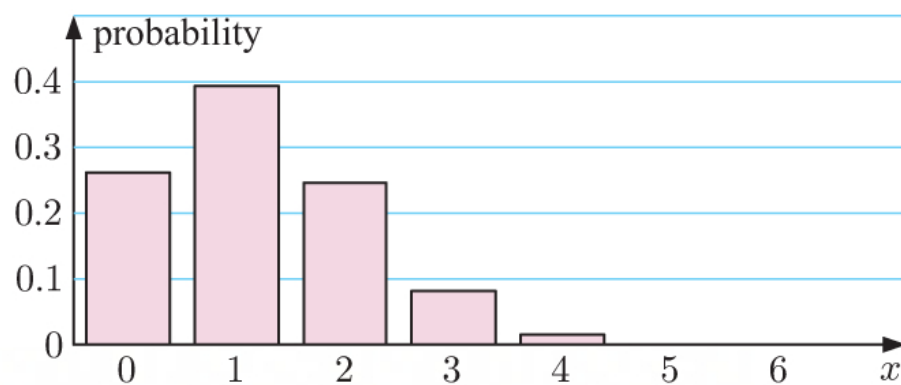
$x_i$	0	1	2	3	4	5	6
$P(x_i)$	0.0156	0.0938	0.2344	0.3125	0.2344	0.0938	0.0156

**iii** The distribution is symmetric.**b**  $X \sim B(6, 0.2)$ 

$$\begin{aligned}
 \text{i} \quad \mu &= np & \sigma &= \sqrt{np(1-p)} \\
 &= 6 \times 0.2 & &= \sqrt{6 \times 0.2 \times 0.8} \\
 &= 1.2 & &\approx 0.980
 \end{aligned}$$

$$\text{ii}$$

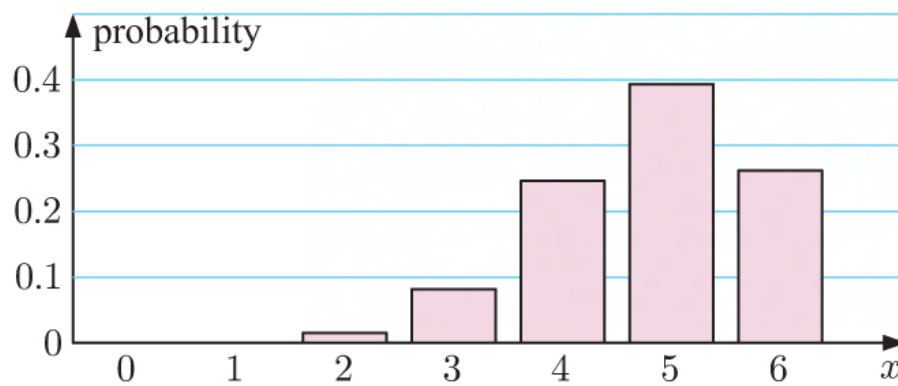
$x_i$	0	1	2	3	4	5	6
$P(x_i)$	0.2621	0.3932	0.2458	0.0819	0.0154	0.0015	0.0001

**iii** The distribution is positively skewed.**c**  $X \sim B(6, 0.8)$ 

$$\begin{aligned}
 \text{i} \quad \mu &= np & \sigma &= \sqrt{np(1-p)} \\
 &= 6 \times 0.8 & &= \sqrt{6 \times 0.8 \times 0.2} \\
 &= 4.8 & &\approx 0.980
 \end{aligned}$$



<b>ii</b>	$x_i$	0	1	2	3	4	5	6
	$P(x_i)$	0.0001	0.0015	0.0154	0.0819	0.2458	0.3932	0.2621



**iii** The distribution is negatively skewed, and is the exact reflection of the distribution in **b**.

**2**  $X \sim B(10, 0.5)$       mean  $\mu = np$       and      variance  $\sigma^2 = np(1-p)$   
 $\phantom{2} \phantom{X \sim B(10, 0.5)}$   $\phantom{mean} = 10 \times \frac{1}{2}$        $\phantom{and} \phantom{variance} = 10 \times \frac{1}{2} \times \frac{1}{2}$   
 $\phantom{2} \phantom{X \sim B(10, 0.5)}$   $\phantom{mean} = 5$        $\phantom{and} \phantom{variance} = 2.5$

**3 a**  $X \sim B(30, 0.04)$

$$\begin{aligned}\mu_X &= np \\ &= 30 \times 0.04 \\ &= 1.2\end{aligned}$$

$$\begin{aligned}\sigma_X &= \sqrt{np(1-p)} \\ &= \sqrt{30 \times 0.04 \times 0.96} \\ &\approx 1.07\end{aligned}$$

**b**  $Y \sim B(30, 0.96)$

$$\begin{aligned}\mu_Y &= np \\ &= 30 \times 0.96 \\ &= 28.8\end{aligned}$$

$$\begin{aligned}\sigma_Y &= \sqrt{np(1-p)} \\ &= \sqrt{30 \times 0.96 \times 0.04} \\ &\approx 1.07\end{aligned}$$

**4**  $X \sim B(30, 0.13)$

$$\begin{aligned}\mu &= np \\ &= 30 \times 0.13 \\ &= 3.9\end{aligned}$$

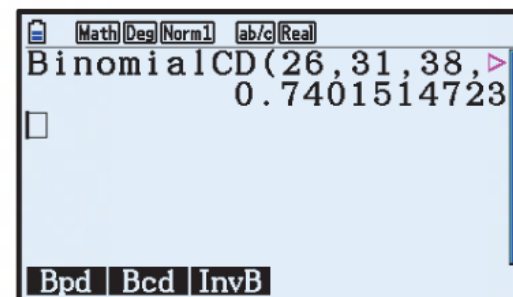
$$\begin{aligned}\sigma &= \sqrt{np(1-p)} \\ &= \sqrt{30 \times 0.13 \times 0.87} \\ &\approx 1.84\end{aligned}$$

**5**  $X \sim B(38, 0.75)$

**a**  $\mu = np$        $\sigma = \sqrt{np(1-p)}$   
 $\phantom{a} \phantom{\mu = np} = 38 \times 0.75$        $\phantom{a} \phantom{\sigma = \sqrt{np(1-p)}} = \sqrt{38 \times 0.75 \times 0.25}$   
 $\phantom{a} \phantom{\mu = np} = 28.5$        $\phantom{a} \phantom{\sigma = \sqrt{np(1-p)}} \approx 2.67$

**b**  $\mu - \sigma \approx 28.5 - 2.67$        $\mu + \sigma \approx 28.5 + 2.67$   
 $\phantom{b} \phantom{\mu - \sigma \approx 28.5 - 2.67} \approx 25.8$        $\phantom{b} \phantom{\mu + \sigma \approx 28.5 + 2.67} \approx 31.2$

$$\begin{aligned}\therefore P(\mu - \sigma < X < \mu + \sigma) &= P(26 \leq X \leq 31) \\ &\approx 0.740\end{aligned}$$



**6**  $X \sim (100, \frac{1}{2}), \quad Y \sim (300, \frac{1}{6})$

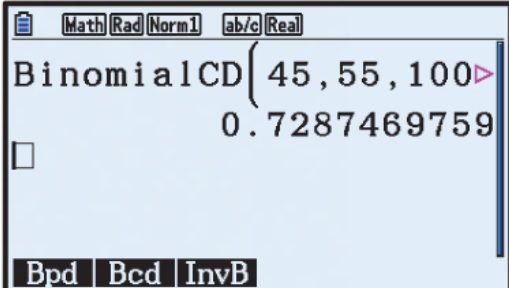
**a**  $\mu_X = np$        $\mu_Y = np$   
 $\phantom{a} \phantom{\mu_X = np} = 100 \times \frac{1}{2}$        $\phantom{a} \phantom{\mu_Y = np} = 300 \times \frac{1}{6}$   
 $\phantom{a} \phantom{\mu_X = np} = 50$        $\phantom{a} \phantom{\mu_Y = np} = 50$



$$\begin{aligned}
 \text{b } \sigma_X &= \sqrt{np(1-p)} & \sigma_Y &= \sqrt{np(1-p)} \\
 &= \sqrt{100 \times \frac{1}{2} \times \frac{1}{2}} & &= \sqrt{300 \times \frac{1}{6} \times \frac{5}{6}} \\
 &= \sqrt{25} & &\approx 6.45 \\
 &= 5
 \end{aligned}$$

- c  $X$  is more likely to lie between 45 and 55 inclusive because the standard deviation of  $X$  is lower than that of  $Y$ , which means there are more values of  $X$  which lie close to the mean.

d i

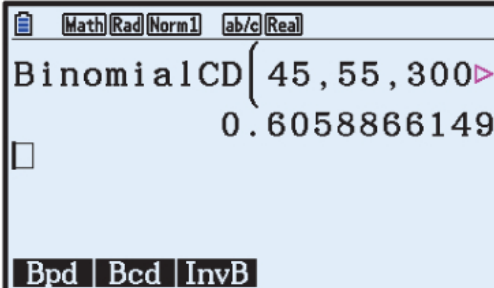


BinomialCD(45, 55, 100)  $\triangleright$   
0.7287469759

Bpd Bcd InvB

$$P(45 \leq X \leq 55) \approx 0.729$$

ii



BinomialCD(45, 55, 300)  $\triangleright$   
0.6058866149

Bpd Bcd InvB

$$P(45 \leq Y \leq 55) \approx 0.606$$

## REVIEW SET 27A

- 1 a The number of attempts to pass a driving test is a discrete random variable.  
 b The length of time before a phone loses its battery charge is a continuous random variable.  
 c The number of phone calls made before a salesperson has sold 3 products is a discrete random variable.

2 a i

$x$	1	2	3
$P(X = x)$	0.6	0.25	0.15

$$\sum_{x=1}^3 P(X = x) = 0.6 + 0.25 + 0.15 = 1$$

Since  $\sum_{x=1}^3 P(X = x) = 1$  and  $0 \leq P(X = x) \leq 1$  for all  $x$ , it is a valid probability distribution.

ii

$x$	0	2	5	10
$P(X = x)$	0.3	0.5	0.1	0.2

$$\sum p_i = 0.3 + 0.5 + 0.1 + 0.2 = 1.1$$

Since  $\sum p_i > 1$ , it is not a valid probability distribution.

iii

$x$	0	1	2	3
$P(X = x)$	0.4	-0.2	0.35	0.45

Since  $P(X = 1) = -0.2 < 0$ , this is not a valid probability distribution.

**iv**

$x$	2	3	4	5
$P(X = x)$	0.25	0.25	0.25	0.25

$$\sum_{x=2}^5 P(X = x) = 0.25 + 0.25 + 0.25 + 0.25 = 1$$

Since  $\sum_{x=2}^5 P(X = x) = 1$  and  $0 \leq P(X = x) \leq 1$  for all  $x$ , it is a valid probability distribution.

**v**

$x$	2	3
$P(X = x)$	0.7	0.3

$$\sum_{x=2}^3 P(X = x) = 0.7 + 0.3 = 1$$

Since  $\sum_{x=2}^3 P(X = x) = 1$  and  $0 \leq P(X = x) \leq 1$  for all  $x$ , it is a valid probability distribution.

**vi**

$x$	0	1
$P(X = x)$	0.28	0.72

$$\sum_{x=0}^1 P(X = x) = 0.28 + 0.72 = 1$$

Since  $\sum_{x=0}^1 P(X = x) = 1$  and  $0 \leq P(X = x) \leq 1$  for all  $x$ , it is a valid probability distribution.

**b** The distribution in **a iv** is a uniform discrete random variable because  $p_i = 0.25$  for each value of  $i$ .

**3 a**  $P(X = x) = \frac{a}{x^2 + 1}$  for  $x = 0, 1, 2, 3$

Since this is a probability mass function,

$$\sum_{i=1}^n P(x_i) = 1$$

$$\therefore a + \frac{a}{2} + \frac{a}{5} + \frac{a}{10} = 1$$

$$\therefore 10a + 5a + 2a + a = 10$$

$$\therefore 18a = 10$$

$$\therefore a = \frac{5}{9}$$

$x$	0	1	2	3
$P(X = x)$	$a$	$\frac{a}{2}$	$\frac{a}{5}$	$\frac{a}{10}$

**b**  $P(X \geq 1) = 1 - P(X = 0)$

$$= 1 - \frac{5}{9}$$

$$= \frac{4}{9}$$

4

$x$	0	1	2	3	4
$P(x)$	0.10	0.30	0.45	0.10	$k$

**a** Since this is a probability distribution, then  $\sum_{i=1}^n P(x_i) = 1$

$$\therefore 0.10 + 0.30 + 0.45 + 0.10 + k = 1$$

$$\therefore 0.95 + k = 1$$

$$\therefore k = 0.05$$

**b**  $P(X \geq 3) = P(X = 3) + P(X = 4)$   
 $= 0.10 + 0.05$   
 $= 0.15$

**c** Since  $P(X = 2)$  is the greatest probability, 2 is the mode of this distribution.

**d**  $E(X) = \sum_{i=1}^n x_i p_i$   
 $= 0(0.10) + 1(0.30) + 2(0.45) + 3(0.10) + 4(0.05)$   
 $= 0 + 0.3 + 0.9 + 0.3 + 0.2$   
 $= 1.7$

**5 a**  $X$  is a discrete random variable because it has a set of distinct possible values.

**b**  $X = 0, 1$ , or  $2$

**c**

1st draw	2nd draw	Outcome	$X$	Probability
	G	GG	2	$\frac{3}{5} \times \frac{2}{4} = \frac{3}{10}$
	Y	GY	1	$\frac{3}{5} \times \frac{2}{4} = \frac{3}{10}$
	G	YG	1	$\frac{2}{5} \times \frac{3}{4} = \frac{3}{10}$
	Y	YY	0	$\frac{2}{5} \times \frac{1}{4} = \frac{1}{10}$

$x$	0	1	2
$P(x)$	$\frac{1}{10}$	$\frac{3}{5}$	$\frac{3}{10}$

**d**  $E(X) = \sum_{i=1}^n x_i p_i$   
 $= 0\left(\frac{1}{10}\right) + 1\left(\frac{3}{5}\right) + 2\left(\frac{3}{10}\right)$   
 $= \frac{6}{5}$   
 $= 1.2 \text{ green balls}$

**6**  $X$  has probability table:

$x$	1	3	4	6
$P(x)$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{1}{6}$	$\frac{2}{6}$

$$E(X) = \sum_{i=1}^n x_i p_i$$

$$= 1\left(\frac{1}{6}\right) + 3\left(\frac{2}{6}\right) + 4\left(\frac{1}{6}\right) + 6\left(\frac{2}{6}\right)$$

$$= \frac{23}{6}$$

$$\approx 3.83$$

- 7 a** Let  $X$  denote the amount of money Lakshmi wins from one roll.  
 $X$  has probability table:

$x$	2	4	6	8	10	12
$P(x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

$$\begin{aligned} E(X) &= \sum_{i=1}^n x_i p_i \\ &= 2\left(\frac{1}{6}\right) + 4\left(\frac{1}{6}\right) + 6\left(\frac{1}{6}\right) + 8\left(\frac{1}{6}\right) + 10\left(\frac{1}{6}\right) + 12\left(\frac{1}{6}\right) \\ &= 7 \end{aligned}$$

$\therefore$  Lakshmi can expect to win \$7 from one roll of the die.

- b** Expected gain = \$7 - \$8 = -\$1.

So, Lakshmi should not play many games as she would lose \$1 per game in the long run.

**8 a**  $\mu = \sum x_i p_i$   
 $= 1(0.15) + 2(0.1) + 3(0.35) + 4(0.4)$   
 $= 3$

$x$	1	2	3	4
$P(X = x)$	0.15	0.1	0.35	0.4

**b**  $\sigma^2 = \sum x_i^2 p_i - \mu^2$   
 $= 1^2(0.15) + 2^2(0.1) + 3^2(0.35) + 4^2(0.4) - 3^2$   
 $= 1.1$

**c**  $\sigma = \sqrt{\sum x_i^2 p_i - \mu^2}$   
 $= \sqrt{1.1}$   
 $\approx 1.05$

**9**  $P(x) = a(x^2 - 8x)$  where  $x = 0, 1, 2, 3, \dots, 8$

- a**  $X$  has probability table:

$x$	0	1	2	3	4	5	6	7	8
$P(x)$	0	$-7a$	$-12a$	$-15a$	$-16a$	$-15a$	$-12a$	$-7a$	0

If this is a probability distribution then  $\sum_{i=1}^n P(x_i) = 1$

$$\therefore 0 + (-7a) + (-12a) + (-15a) + (-16a) + (-15a) + (-12a) + (-7a) + 0 = 1$$

$$\therefore -84a = 1$$

$$\therefore a = -\frac{1}{84}$$

**b**  $E(X) = 0(0) + 1\left(\frac{7}{84}\right) + 2\left(\frac{12}{84}\right) + 3\left(\frac{15}{84}\right) + 4\left(\frac{16}{84}\right) + 5\left(\frac{15}{84}\right) + 6\left(\frac{12}{84}\right) + 7\left(\frac{7}{84}\right) + 8(0)$   
 $= 4$  marsupials

**c**  $\sigma^2 = \sum x_i^2 p_i - \mu^2$   
 $= 0^2(0) + 1^2\left(\frac{7}{84}\right) + 2^2\left(\frac{12}{84}\right) + 3^2\left(\frac{15}{84}\right) + 4^2\left(\frac{16}{84}\right) + 5^2\left(\frac{15}{84}\right) + 6^2\left(\frac{12}{84}\right) + 7^2\left(\frac{7}{84}\right) + 8^2(0) - 4^2$   
 $= 3$



**10**  $Y = 4X + 3$

$$\begin{aligned} E(Y) &= E(4X + 3) & \sigma(Y) &= \sigma(4X + 3) \\ &= 4E(X) + 3 & &= |4| \sigma(X) \\ &= 4(6) + 3 & &= 4(2) \\ &= 27 & &= 8 \end{aligned}$$

For  $Y$ , the mean is 27 and the standard deviation is 8.

**11 a**  $\left(\frac{4}{5} + \frac{1}{5}\right)^5 = \left(\frac{4}{5}\right)^5 + 5\left(\frac{4}{5}\right)^4\left(\frac{1}{5}\right)^1 + 10\left(\frac{4}{5}\right)^3\left(\frac{1}{5}\right)^2 + 10\left(\frac{4}{5}\right)^2\left(\frac{1}{5}\right)^3 + 5\left(\frac{4}{5}\right)^1\left(\frac{1}{5}\right)^4 + \left(\frac{1}{5}\right)^5$

**b** The probability of Jack scoring a goal is  $p = \frac{4}{5}$ .

Let  $X$  = the number of goals scored,  $G$  represents scoring a goal

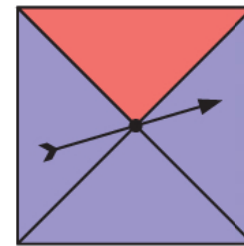
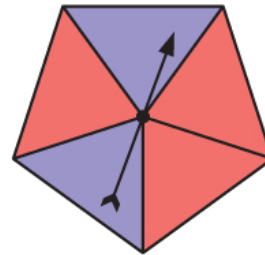
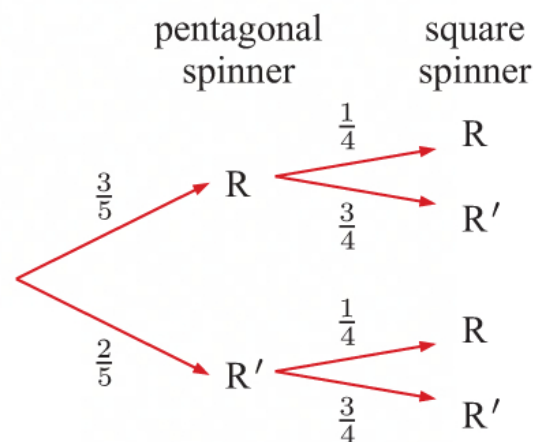
**i**  $P(3 \text{ goals then } 2 \text{ misses})$

$$\begin{aligned} &= P(GGGG'G') \\ &= \left(\frac{4}{5}\right)^3 \left(\frac{1}{5}\right)^2 \\ &= \frac{64}{3125} \\ &= 0.02048 \end{aligned}$$

**ii**  $P(3 \text{ goals and } 2 \text{ misses})$

$$\begin{aligned} &= P(X = 3) \\ &= 10\left(\frac{4}{5}\right)^3 \left(\frac{1}{5}\right)^2 \\ &= \frac{128}{625} \\ &= 0.2048 \end{aligned}$$

**12 a**



**b**  $P(\text{exactly one red}) = P(RR') + P(R'R)$

$$\begin{aligned} &= \frac{3}{5} \times \frac{3}{4} + \frac{2}{5} \times \frac{1}{4} \\ &= \frac{9}{20} + \frac{1}{10} \\ &= \frac{11}{20} \end{aligned}$$

**c i**  $X \sim B\left(10, \frac{11}{20}\right)$

**ii**  $n = 10, \quad p = \frac{11}{20}$

$$P(X = 1) = \binom{10}{1} \left(\frac{11}{20}\right)^1 \left(\frac{9}{20}\right)^9 \approx 0.00416$$

$$P(X = 9) = \binom{10}{9} \left(\frac{11}{20}\right)^9 \left(\frac{9}{20}\right)^1 \approx 0.0207$$

$\therefore$  it is more likely that exactly one red will occur 9 times.

**iii**  $\mu = np$

$$\begin{aligned} &= 10 \times \frac{11}{20} \\ &= 5.5 \end{aligned}$$

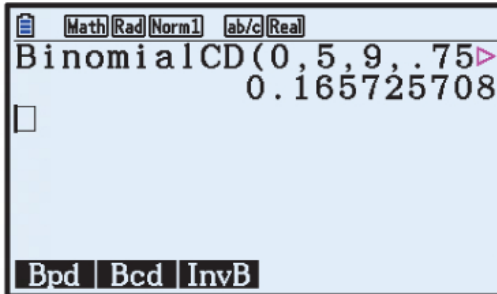
$$\begin{aligned} \sigma &= \sqrt{np(1-p)} \\ &= \sqrt{10 \times \frac{11}{20} \times \frac{9}{20}} \\ &\approx 1.57 \end{aligned}$$

**13** Let  $X$  denote the number of players who turn up to a game.

$n = 9$ , so  $X = 0, 1, 2, 3, \dots$ , or  $9$ , and  $p = 75\% = 0.75$

$\therefore X \sim B(9, 0.75)$

**a i**  $P(X = 9) = \binom{9}{9} (0.75)^9 (0.25)^0$   
 $\approx 0.0751$

**ii** 

$$\begin{aligned} P(\text{forfeit}) &= P(X < 6) \\ &= P(X \leq 5) \\ &\approx 0.166 \end{aligned}$$

**b** The team is expected to forfeit  $30 \times 0.1657 \approx 4.97$  games throughout the season.

## REVIEW SET 27B

**1**

$x$	0	1	2	3	4	5
$P(X = x)$	0.07	0.14	$k$	0.46	0.08	0.02

**a** The random variable  $X$  represents the number of hits that Sally has in a softball match.  
 $X = 0, 1, 2, 3, 4$ , or  $5$

**b i** Since this is a probability distribution,  $\sum_{i=1}^n P(x_i) = 1$   
 $\therefore 0.07 + 0.14 + k + 0.46 + 0.08 + 0.02 = 1$   
 $\therefore k + 0.77 = 1$   
 $\therefore k = 0.23$

**ii**  $P(X \geq 2) = 1 - P(X \leq 1)$   
 $= 1 - (0.07 + 0.14)$   
 $= 0.79$

**iii**  $P(1 \leq X \leq 3) = P(X = 1) + P(X = 2) + P(X = 3)$   
 $= 0.14 + 0.23 + 0.46$   
 $= 0.83$

**c** It is most likely that Sally will have 3 hits in one softball match, so 3 hits is the mode.

$$\begin{aligned} p_1 &= 0.07 \\ p_1 + p_2 &= 0.07 + 0.14 = 0.21 \\ p_1 + p_2 + p_3 &= 0.07 + 0.14 + 0.23 = 0.44 \\ p_1 + p_2 + p_3 + p_4 &= 0.07 + 0.14 + 0.23 + 0.46 = 0.9 \end{aligned}$$

Since  $p_1 + p_2 + p_3 + p_4 \geq 0.5$ , the median is 3 hits.

**2 a**  $P(x) = \frac{e^x}{1+e}, \quad x = 0, 1$

$$P(0) = \frac{1}{1+e}, \quad P(1) = \frac{e}{1+e}$$

Both of these obey  $0 \leq P(x_i) \leq 1$ , and  $\sum_{i=1}^n P(x_i) = \frac{1}{1+e} + \frac{e}{1+e} = 1$

$\therefore P(x)$  is a valid probability mass function.

**b**  $P(x) = \frac{x^2 + x}{40}, \quad x = 1, 2, 3, 4$

$$P(1) = \frac{1+1}{40} = \frac{2}{40}, \quad P(2) = \frac{4+2}{40} = \frac{6}{40}, \quad P(3) = \frac{9+3}{40} = \frac{12}{40}, \quad P(4) = \frac{16+4}{40} = \frac{20}{40}$$

All of these obey  $0 \leq P(x_i) \leq 1$ , and  $\sum_{i=1}^n P(x_i) = \frac{2}{40} + \frac{6}{40} + \frac{12}{40} + \frac{20}{40} = 1$

$\therefore P(x)$  is a valid probability mass function.

**c**  $P(x) = \log\left(\frac{x+1}{x}\right), \quad x = 1, 2, 3, \dots, 9$

$$P(1) = \log\left(\frac{1+1}{1}\right) = \log 2$$

$$P(2) = \log\left(\frac{2+1}{2}\right) = \log\left(\frac{3}{2}\right)$$

$$P(3) = \log\left(\frac{3+1}{3}\right) = \log\left(\frac{4}{3}\right)$$

$$P(4) = \log\left(\frac{4+1}{4}\right) = \log\left(\frac{5}{4}\right)$$

$$P(5) = \log\left(\frac{5+1}{5}\right) = \log\left(\frac{6}{5}\right)$$

$$P(6) = \log\left(\frac{6+1}{6}\right) = \log\left(\frac{7}{6}\right)$$

$$P(7) = \log\left(\frac{7+1}{7}\right) = \log\left(\frac{8}{7}\right)$$

$$P(8) = \log\left(\frac{8+1}{8}\right) = \log\left(\frac{9}{8}\right)$$

$$P(9) = \log\left(\frac{9+1}{9}\right) = \log\left(\frac{10}{9}\right)$$

$$\begin{aligned} \sum_{i=1}^n P(x_i) &= \log 2 + \log\left(\frac{3}{2}\right) + \log\left(\frac{4}{3}\right) + \log\left(\frac{5}{4}\right) + \log\left(\frac{6}{5}\right) + \log\left(\frac{7}{6}\right) + \log\left(\frac{8}{7}\right) \\ &\quad + \log\left(\frac{9}{8}\right) + \log\left(\frac{10}{9}\right) \\ &= \log\left(\cancel{2} \times \frac{\cancel{3}}{\cancel{2}} \times \frac{\cancel{4}}{\cancel{3}} \times \frac{\cancel{5}}{\cancel{4}} \times \frac{\cancel{6}}{\cancel{5}} \times \frac{\cancel{7}}{\cancel{6}} \times \frac{\cancel{8}}{\cancel{7}} \times \frac{\cancel{9}}{\cancel{8}} \times \frac{10}{\cancel{9}}\right) \quad \{\log A + \log B = \log AB\} \\ &= \log 10 \\ &= 1 \end{aligned}$$

All of these obey  $0 \leq P(x_i) \leq 1$ , and  $\sum_{i=1}^n P(x_i) = 1$

$\therefore P(x)$  is a valid probability mass function.

- 3 a** 2 has the highest probability of occurring, so this is the mode of the distribution.

**b**

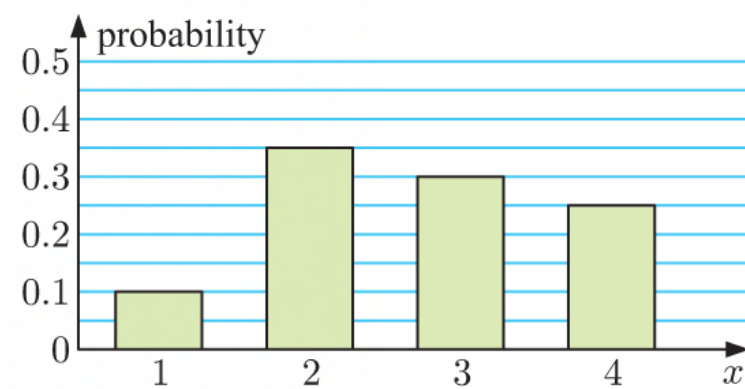
$$p_1 = 0.1$$

$$p_1 + p_2 = 0.1 + 0.35 = 0.45$$

$$p_1 + p_2 + p_3 = 0.1 + 0.35 + 0.3 = 0.75$$

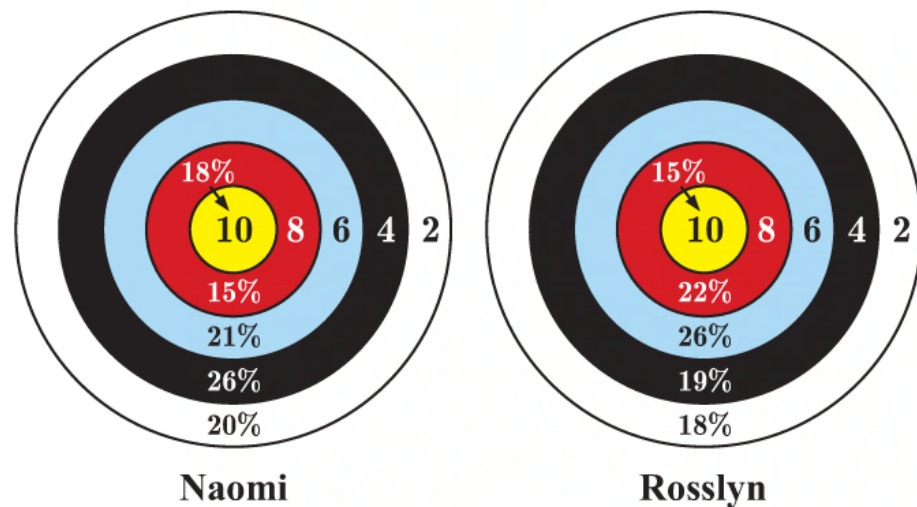
Since  $p_1 + p_2 + p_3 \geq 0.5$ , the median is 3.

**c**  $E(X) = 1(0.1) + 2(0.35) + 3(0.3) + 4(0.25)$   
 $= 2.7$



- 4** Let  $N$  be the number of points Naomi scores per shot.  
Let  $R$  be the number of points Rosslyn scores per shot.

- a i**  $P(N = 10) = 0.18$   
 $P(R = 10) = 0.15$   
 $\therefore$  Naomi is more likely to score 10 points on a single shot.



**ii**  $P(N \geq 6) = P(N = 6) + P(N = 8) + P(N = 10)$   
 $= 0.21 + 0.15 + 0.18$   
 $= 0.54$

$$P(R \geq 6) = P(R = 6) + P(R = 8) + P(R = 10)$$

$$= 0.26 + 0.22 + 0.15$$

$$= 0.63$$

$\therefore$  Rosslyn is more likely to score at least 6 points.

**b**  $E(N) = 2(0.2) + 4(0.26) + 6(0.21) + 8(0.15) + 10(0.18)$   
 $= 5.7$  points

$$E(R) = 2(0.18) + 4(0.19) + 6(0.26) + 8(0.22) + 10(0.15)$$

$$= 5.94$$
 points

$\therefore$  in the long run, Rosslyn is expected to score more points per shot.

- 5** Let  $X$  denote the number written on the ticket drawn.

**a i**  $P(\text{player wins \$3})$   
 $= P(X \text{ is even but not square})$   
 $= \frac{8}{20}$   
 $= \frac{2}{5}$

**ii**  $P(\text{player wins \$6})$   
 $= P(X \text{ is square but not even})$   
 $= \frac{2}{20}$   
 $= \frac{1}{10}$



$$\begin{aligned}
 \text{iii} \quad & P(\text{player wins \$9}) \\
 &= P(X \text{ is even and square}) \\
 &= \frac{2}{20} \\
 &= \frac{1}{10}
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & \text{The expected gain of one game is } E(X) = 0\left(\frac{8}{20}\right) + 3\left(\frac{2}{5}\right) + 6\left(\frac{1}{10}\right) + 9\left(\frac{1}{10}\right) \\
 &= \frac{27}{10} \\
 &= \$2.70 \text{ per game}
 \end{aligned}$$

To make the game fair, the game must cost the same as the expected gain, so \$2.70 should be charged each game.

$$\begin{aligned}
 \text{6 a} \quad & \text{Since this is a probability distribution, } \sum p_i = 1 \\
 & \therefore 0.2 + a + 0.3 + b = 1 \\
 & \therefore b = 0.5 - a \quad \dots (*)
 \end{aligned}$$

$x_i$	1	2	3	4
$p_i$	0.2	$a$	0.3	$b$

$$\begin{aligned}
 & \text{Now, } E(X) = 2.8 \\
 \therefore & (1 \times 0.2) + (2 \times a) + (3 \times 0.3) + (4 \times b) = 2.8 \\
 \therefore & 0.2 + 2a + 0.9 + 4(0.5 - a) = 2.8 \quad \{\text{using } (*)\} \\
 \therefore & 2a + 2 - 4a = 1.7 \\
 \therefore & -2a = -0.3 \\
 \therefore & a = 0.15 \text{ and } b = 0.35
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & \text{Var}(X) = \sum x_i^2 p_i - \mu^2 \\
 &= 1^2(0.2) + 2^2(0.15) + 3^2(0.3) + 4^2(0.35) - 2.8^2 \\
 &= 1.26
 \end{aligned}$$

7 a Let H be the event that the result is heads.

$$\begin{aligned}
 & P(H) \times P(H) = 0.64 \\
 \therefore & [P(H)]^2 = \frac{64}{100} \\
 \therefore & P(H) = \frac{8}{10} \quad \{P(H) \geq 0\} \\
 &= \frac{4}{5}
 \end{aligned}$$

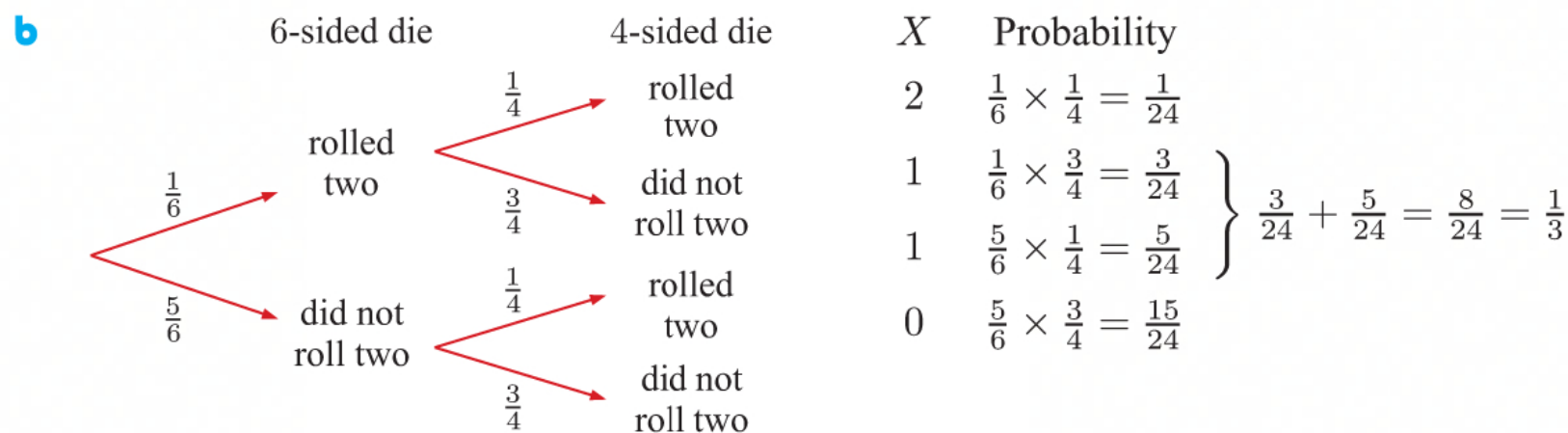
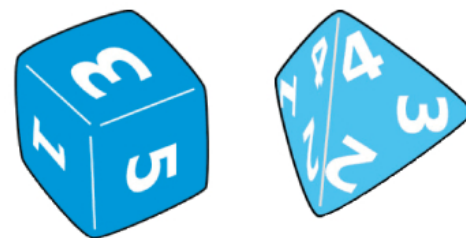
b Let  $X$  be the number of heads,  $X \sim B\left(10, \frac{4}{5}\right)$ .

$$\begin{aligned}
 \text{i} \quad & P(X = 6) = \binom{10}{6} \left(\frac{4}{5}\right)^6 \left(\frac{1}{5}\right)^4 \\
 & \approx 0.0881
 \end{aligned}$$

$$\begin{aligned}
 \text{ii} \quad & \text{BinomialCD}(6, 10, 10, 4) \\
 & 0.9672065024
 \end{aligned}$$

$$P(X \geq 6) \approx 0.967$$

- 8 a**  $X$  is not a binomial random variable because the probability of rolling a two is not the same for each die.



$x$	0	1	2
$P(X = x)$	$\frac{15}{24}$	$\frac{1}{3}$	$\frac{1}{24}$

**c i**  $E(X) = 0\left(\frac{15}{24}\right) + 1\left(\frac{1}{3}\right) + 2\left(\frac{1}{24}\right) = \frac{5}{12}$

**ii**  $\sigma(X) = \sqrt{\sum x_i^2 p_i - \mu^2}$

$$= \sqrt{0^2\left(\frac{15}{24}\right) + 1^2\left(\frac{1}{3}\right) + 2^2\left(\frac{1}{24}\right) - \left(\frac{5}{12}\right)^2}$$

$$\approx 0.571$$

**9**

$x$	0	1	2	3	4
$P(X = x)$	0.1	0.2	0.3	0.3	0.1

**a i**  $E(X) = 0(0.1) + 1(0.2) + 2(0.3) + 3(0.3) + 4(0.1)$

$$= 2.1$$

**ii**  $\text{Var}(X) = \sum x_i^2 p_i - \mu^2$

$$= 0^2(0.1) + 1^2(0.2) + 2^2(0.3) + 3^2(0.3) + 4^2(0.1) - (2.1)^2$$

$$= 1.29$$

**b i**  $Y = 4 - X$

**ii**  $E(Y) = E(4 - X)$

$$= 4 - E(X)$$

$$= 4 - 2.1$$

$$= 1.9$$

$\text{Var}(Y) = \text{Var}(4 - X)$

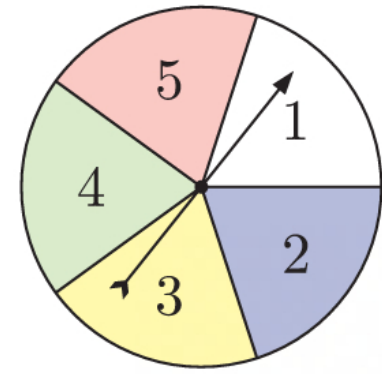
$$= (-1)^2 \text{Var}(X)$$

$$= \text{Var}(X)$$

$$= 1.29$$

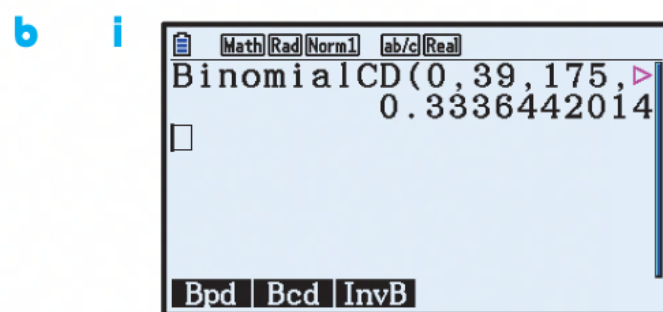
- 10 a** The probability of success (spinning a 3) is the same for each spin, and the number of spins is fixed.

$$\begin{aligned} \text{b } \mu &= np \\ &= 20 \times \frac{1}{5} \\ &= 4 \end{aligned} \quad \begin{aligned} \sigma &= \sqrt{np(1-p)} \\ &= \sqrt{20 \times \frac{1}{5} \times \frac{4}{5}} \\ &= \frac{4}{\sqrt{5}} \approx 1.79 \end{aligned}$$

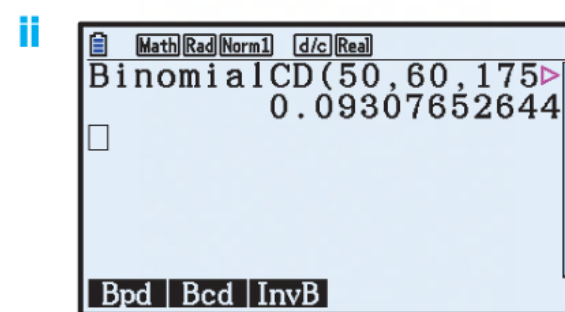


- 11** Let  $X$  be the number of visitors who make a voluntary donation upon entry.  
 $n = 175$ , so  $X = 0, 1, 2, 3, \dots$ , or  $175$ , and  $p = 24\% = 0.24$   
 $\therefore X \sim B(175, 0.24)$

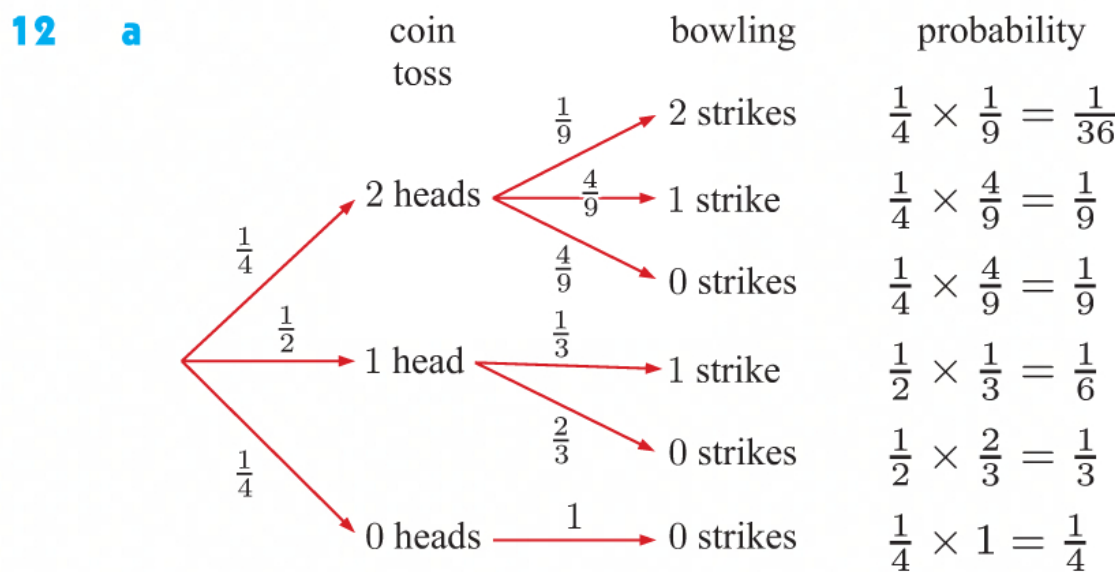
$$\begin{aligned} \text{a } E(X) &= \mu = np \\ &= 175 \times 0.24 \\ &= 42 \text{ donations} \end{aligned}$$



$$\begin{aligned} P(X < 40) &= P(X \leq 39) \\ &\approx 0.334 \end{aligned}$$



$$P(50 \leq X \leq 60) \approx 0.0931$$



$$\begin{aligned} \text{b } P(X = 0) &= \frac{1}{9} + \frac{1}{3} + \frac{1}{4} \\ &= \frac{25}{36} \end{aligned} \quad \begin{aligned} P(X = 1) &= \frac{1}{9} + \frac{1}{6} \\ &= \frac{5}{18} \end{aligned} \quad P(X = 2) = \frac{1}{36}$$

$x$	0	1	2
$P(X = x)$	$\frac{25}{36}$	$\frac{5}{18}$	$\frac{1}{36}$

- c** The expected return per game is  $E(X) = 0\left(\frac{25}{36}\right) + 10\left(\frac{5}{18}\right) + 20\left(\frac{1}{36}\right)$   
 $= \frac{10}{3}$  dollars  
 $\approx \$3.33$

- d** Suvi's expected gain per game  $\approx \$3.33 - \$5$   
 $\approx -\$1.67$

$\therefore$  Suvi should not play the game many times as she is expected to lose \$1.67 per game on average.

- 13 a** Let  $X$  denote the number of batteries that are defective.

$n = 20$ , so  $X = 0, 1, 2, 3, \dots$ , or  $20$ , and  $p = \frac{3}{100}$

$\therefore X \sim B(20, \frac{3}{100})$

$$\text{i } P(X = 0) = \binom{20}{0} \left(\frac{3}{100}\right)^0 \left(\frac{97}{100}\right)^{20} \\ \approx 0.544$$

$$\text{ii } P(X \geq 1) = 1 - P(X = 0) \\ \approx 1 - 0.544 \\ \approx 0.456$$

- b**  $X \sim B(n, \frac{3}{100})$

$$\text{i } P(X = 0) = \binom{n}{0} \left(\frac{3}{100}\right)^0 \left(\frac{97}{100}\right)^n \\ = (0.97)^n$$

$$\text{ii } P(X \geq 1) = 1 - P(X = 0) \\ = 1 - (0.97)^n$$

If  $P(X \geq 1) \geq 0.3$  then

$$1 - (0.97)^n \geq 0.3$$

$$\therefore (0.97)^n \leq 0.7$$

$$\therefore n \log(0.97) \leq \log(0.7)$$

$$\therefore n \geq \frac{\log(0.7)}{\log(0.97)} \quad \{\log(0.97) < 0\}$$

$$\therefore n \geq 11.7$$

$\therefore$  the smallest value of  $n$  such that  $P(X \geq 1) \geq 0.3$  is  $n = 12$ .



# Chapter 28

## CONTINUOUS RANDOM VARIABLES

### INVESTIGATION 1

### PROBABILITY DENSITY FUNCTIONS

- 1 a Consider  $x_1, x_2 \in [a, b]$  where  $x_1 < x_2$ .

$$\text{Now } P(X \leq x_2) = P(X \leq x_1) + P(x_1 \leq X \leq x_2)$$

$$\therefore P(X \leq x_2) \geq P(X \leq x_1) \quad \{0 \leq P(x_1 \leq X \leq x_2) \leq 1\}$$

$$\therefore F(x_2) \geq F(x_1)$$

$$\therefore F(x_1) \leq F(x_2)$$

$$\therefore F(x) \text{ is increasing for all } x.$$

$$\therefore F'(x) \geq 0 \text{ for all } x.$$

b i  $F(a) = P(X \leq a)$

$$= 0 \quad \{\text{there are no possible values less than } a\}$$

$$F(b) = P(X \leq b)$$

$$= 1 \quad \{\text{all possible values are less than or equal to } b\}$$

ii  $P(c \leq X \leq d) = P(X \leq d) - P(X \leq c)$   
 $= F(d) - F(c)$

c Using a and b i, the range of  $F(x)$  is  $\{y \mid 0 \leq y \leq 1\}$ .

2  $F(x) = \frac{1}{8}x^3, \quad 0 \leq x \leq 2$

a i  $P(1 \leq X \leq 2) = F(2) - F(1)$   
 $= \frac{1}{8}(2)^3 - \frac{1}{8}(1)^3$   
 $= 1 - \frac{1}{8}$   
 $= 0.875$

ii  $P(1 \leq X \leq 1.5) = F(1.5) - F(1)$   
 $= \frac{1}{8}(1.5)^3 - \frac{1}{8}(1)^3$   
 $= 0.296875$

iii  $P(1 \leq X \leq 1.25) = F(1.25) - F(1)$   
 $= \frac{1}{8}(1.25)^3 - \frac{1}{8}(1)^3$   
 $\approx 0.1191$

iv  $P(1 \leq X \leq 1.01) = F(1.01) - F(1)$   
 $= \frac{1}{8}(1.01)^3 - \frac{1}{8}(1)^3$   
 $\approx 0.003788$

v  $P(1 \leq X \leq 1.001) = F(1.001) - F(1)$   
 $= \frac{1}{8}(1.001)^3 - \frac{1}{8}(1)^3$   
 $\approx 3.754 \times 10^{-4}$

vi  $P(1 \leq X \leq 1.0001) = F(1.0001) - F(1)$   
 $= \frac{1}{8}(1.0001)^3 - \frac{1}{8}(1)^3$   
 $\approx 3.750 \times 10^{-5}$

b i  $\frac{P(1 \leq X \leq 2)}{1} = 0.875$

ii  $\frac{P(1 \leq X \leq 1.5)}{0.5} = 0.59375$

$$\text{iii} \quad \frac{P(1 \leq X \leq 1.25)}{0.25} \approx 0.4766$$

$$\text{iv} \quad \frac{P(1 \leq X \leq 1.01)}{0.01} \approx 0.3788$$

$$\text{v} \quad \frac{P(1 \leq X \leq 1.001)}{0.001} \approx 0.3754$$

$$\text{vi} \quad \frac{P(1 \leq X \leq 1.0001)}{0.0001} \approx 0.3750$$

The values appear to be converging to  $0.375 = \frac{3}{8}$ .

$$\text{c} \quad F(x) = \frac{1}{8}x^3$$

$$\therefore F'(x) = \frac{3}{8}x^2$$

$\therefore F'(1) = \frac{3}{8} = 0.375$ , which is the value that the values in **b** appear to be converging to.

$$\text{d} \quad \text{i} \quad P(1 \leq X \leq 1+h) = F(1+h) - F(1)$$

$$\begin{aligned} \text{ii} \quad \frac{1}{h} P(1 \leq X \leq 1+h) &= \frac{F(1+h) - F(1)}{h} \\ &\approx F'(1) \quad \text{for small values of } h \end{aligned}$$

**iii** From **ii**,  $P(1 \leq X \leq 1+h) \approx h F'(1)$  for small values of  $h$ .

$\therefore$  the probability that  $X$  is close to 1 is approximately  $h F'(1)$  for small values of  $h$ .

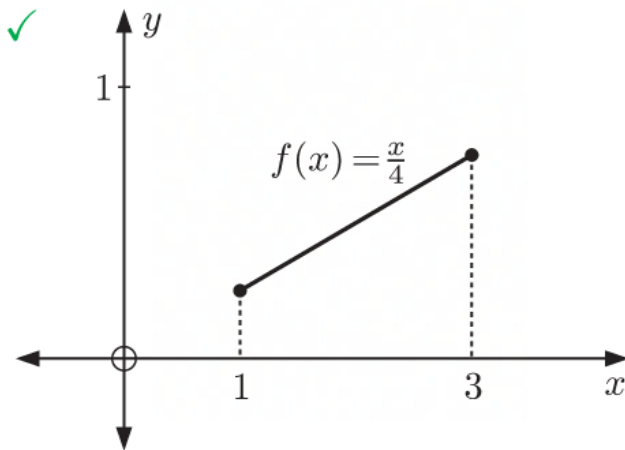
**e**  $F'(x)$  gives a measure of how “dense” the probability is for values near  $x$ .

**3** We find probabilities associated with a continuous random variable using definite integrals of the probability density function.

## EXERCISE 28A

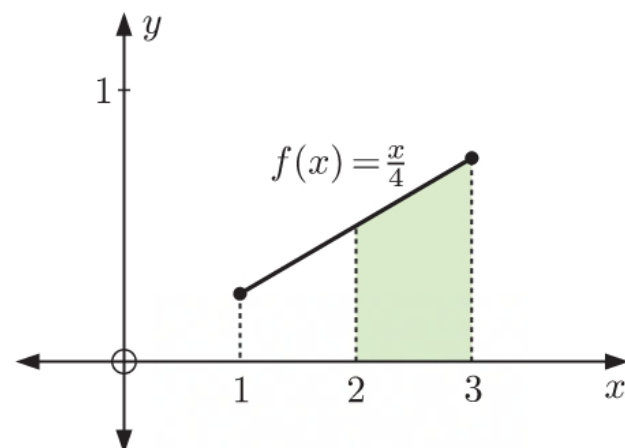
**1 a** • From the graph,  $f(x) \geq 0$  for all  $1 \leq x \leq 3$ . ✓

$$\begin{aligned} \bullet \quad \int_1^3 f(x) dx &= \int_1^3 \frac{x}{4} dx \\ &= \left[ \frac{x^2}{8} \right]_1^3 \\ &= \frac{9}{8} - \frac{1}{8} \\ &= 1 \quad \checkmark \end{aligned}$$

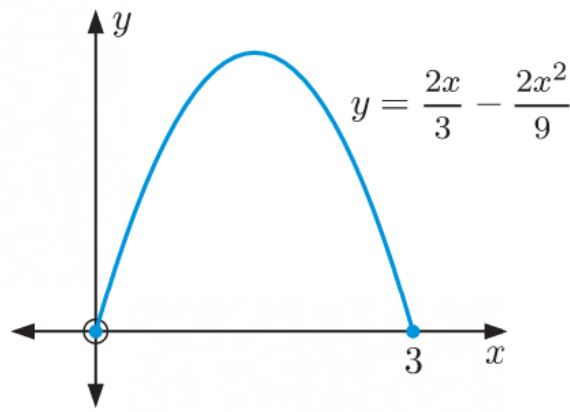


So,  $f(x)$  is a valid probability density function.

$$\begin{aligned} \text{b} \quad P(2 \leq X \leq 3) &= \int_2^3 \frac{x}{4} dx \\ &= \left[ \frac{x^2}{8} \right]_2^3 \\ &= \frac{9}{8} - \frac{4}{8} \\ &= \frac{5}{8} \end{aligned}$$



2 a

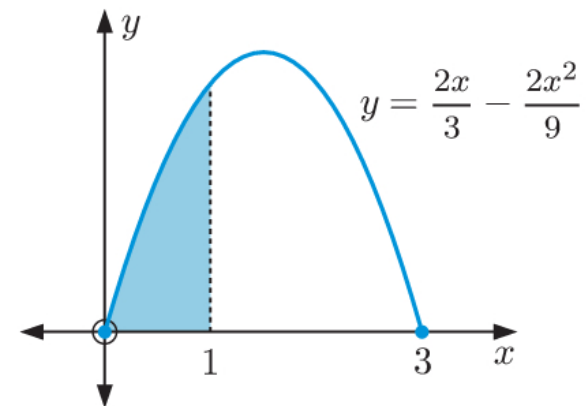


b • From the graph,  $f(x) \geq 0$  for all  $0 \leq x \leq 3$ . ✓

$$\begin{aligned} \bullet \int_0^3 f(x) dx &= \int_0^3 \left( \frac{2x}{3} - \frac{2x^2}{9} \right) dx \\ &= \left[ \frac{x^2}{3} - \frac{2x^3}{27} \right]_0^3 \\ &= \frac{9}{3} - \frac{54}{27} \\ &= 3 - 2 \\ &= 1 \quad \checkmark \end{aligned}$$

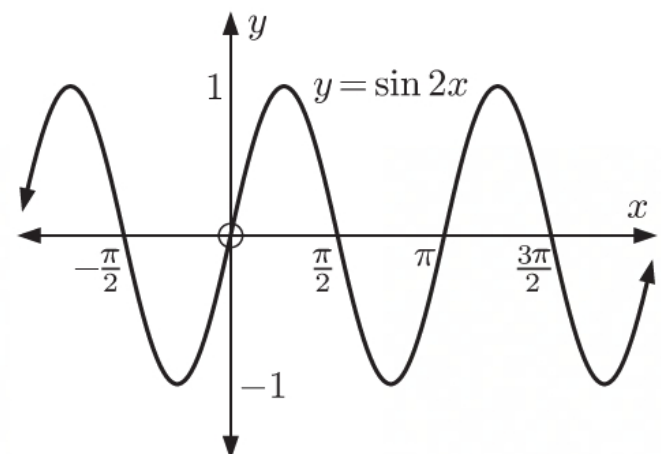
So,  $f(x)$  is a valid probability density function.

$$\begin{aligned} \text{c } P(0 \leq X \leq 1) &= \int_0^1 \left( \frac{2x}{3} - \frac{2x^2}{9} \right) dx \\ &= \left[ \frac{x^2}{3} - \frac{2x^3}{27} \right]_0^1 \\ &= \frac{1}{3} - \frac{2}{27} \\ &= \frac{7}{27} \end{aligned}$$



$$\begin{aligned} \text{3 a } \int_0^{\frac{3\pi}{2}} \sin 2x dx &= \left[ -\frac{1}{2} \cos 2x \right]_0^{\frac{3\pi}{2}} \\ &= -\frac{1}{2}(-1) - \left(-\frac{1}{2}\right)(1) \\ &= 1 \end{aligned}$$

b  $\sin 2x < 0$  for  $\frac{\pi}{2} < x < \pi$   
 $\therefore f(x) = \sin 2x$  is not a valid probability density function for  $0 \leq x \leq \frac{3\pi}{2}$ .

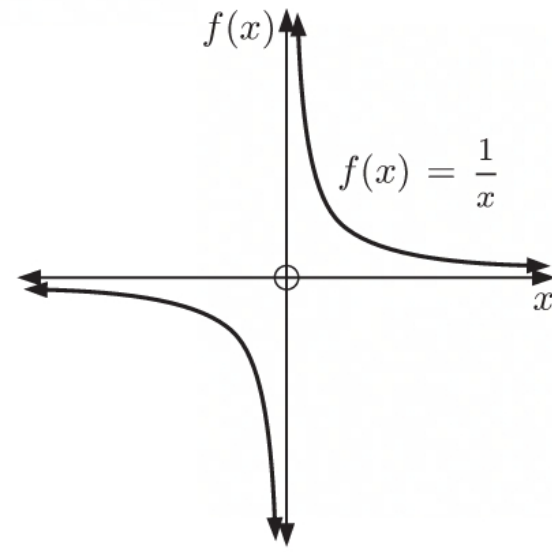


4 a If  $f(x) = a\sqrt{x}$  is a valid probability density function for  $0 \leq x \leq 4$ , then

$$\begin{aligned} \int_0^4 a\sqrt{x} dx &= 1 \\ \therefore \left[ \frac{2}{3}ax^{\frac{3}{2}} \right]_0^4 &= 1 \\ \therefore \frac{2}{3}a(8) &= 1 \\ \therefore a &= \frac{3}{16} \end{aligned}$$

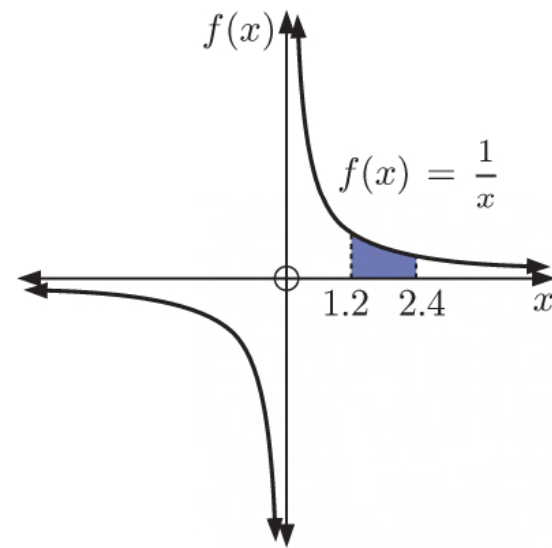
$$\begin{aligned} \text{b } P(0 \leq X \leq 1) &= \int_0^1 \frac{3}{16}\sqrt{x} dx \\ &= \left[ \frac{1}{8}x^{\frac{3}{2}} \right]_0^1 \\ &= \frac{1}{8}(1) \\ &= \frac{1}{8} \end{aligned}$$

- 5 a**  $f(x) < 0$  for  $x < 0$   
 $\therefore f(x) \not\geq 0$  for all  $x \in \mathbb{R}$   
 $\therefore f(x) = \frac{1}{x}$  is not a valid probability density function on the domain  $x \in \mathbb{R}$ .



- b** If  $f(x) = \frac{1}{x}$  is a valid probability density function for  $1 \leq x \leq k$ , then  $\int_1^k \frac{1}{x} dx = 1$   
 $\therefore [\ln |x|]_1^k = 1$   
 $\therefore \ln |k| - \ln 1 = 1$   
 $\therefore \ln k = 1 \quad \{k \geq 1\}$   
 $\therefore k = e$

- c**  $P(1.2 \leq X \leq 2.4) = \int_{1.2}^{2.4} \frac{1}{x} dx$   
 $= [\ln |x|]_{1.2}^{2.4}$   
 $= \ln 2.4 - \ln 1.2$   
 $= \ln 2 \quad \left\{ \ln a - \ln b = \ln \left( \frac{a}{b} \right) \right\}$   
 $\approx 0.693$



- 6 a**  $f(x) = ke^{-x}$  is a valid probability density function for  $0 \leq x \leq 1$  if  
 $\int_0^1 ke^{-x} dx = 1$   
 $\therefore [-ke^{-x}]_0^1 = 1$   
 $\therefore -ke^{-1} - (-k) = 1$   
 $\therefore -k + ke = 1$   
 $\therefore k(e - 1) = 1$   
 $\therefore k = \frac{1}{e - 1}$

- b**  $P(0.1 \leq X \leq 0.5)$   
 $= \int_{0.1}^{0.5} \left( \frac{e}{e - 1} e^{-x} \right) dx$   
 $= \left[ -\frac{e}{e - 1} e^{-x} \right]_{0.1}^{0.5}$   
 $= -\frac{e}{e - 1} e^{-0.5} - \left( -\frac{e}{e - 1} e^{-0.1} \right)$   
 $\approx 0.472$

- 7**  $P(X \leq \frac{2}{3}) = \frac{1}{243}$   
 $\therefore \int_0^{\frac{2}{3}} ax^4 dx = \frac{1}{243}$   
 $\therefore \left[ \frac{1}{5} ax^5 \right]_0^{\frac{2}{3}} = \frac{1}{243}$   
 $\therefore \frac{1}{5} a \times \left( \frac{2}{3} \right)^5 = \frac{1}{243}$   
 $\therefore \frac{1}{5} a \times \frac{32}{243} = \frac{1}{243}$   
 $\therefore a = \frac{5}{32}$

So,  $\int_0^k \frac{5}{32} x^4 dx = 1$   
 $\therefore \left[ \frac{1}{32} x^5 \right]_0^k = 1$   
 $\therefore \frac{1}{32} k^5 = 1$   
 $\therefore k^5 = 32$   
 $\therefore k = 2$



- 8 a** If  $f(x) = a(-x^2 + 30x - 144)$  is a valid probability density function for  $6 \leq x \leq 24$ ,

$$\text{then } \int_6^{24} a(-x^2 + 30x - 144) dx = 1$$

$$\therefore \left[ a\left(-\frac{1}{3}x^3 + 15x^2 - 144x\right) \right]_6^{24} = 1$$

$$\therefore (-4608 + 8640 - 3456) - (-72 + 540 - 864) = \frac{1}{a}$$

$$\therefore 576 + 396 = \frac{1}{a}$$

$$\therefore a = \frac{1}{972}$$

**b i**  $P(10 < X < 15) = \int_{10}^{15} \frac{1}{972}(-x^2 + 30x - 144) dx$

$$= \left[ \frac{1}{972} \left( -\frac{1}{3}x^3 + 15x^2 - 144x \right) \right]_{10}^{15}$$

$$= \frac{1}{972} [(-1125 + 3375 - 2160) - (-333\frac{1}{3} + 1500 - 1440)]$$

$$= \frac{1}{972} (90 + 273\frac{1}{3})$$

$$\approx 0.374$$

**ii**  $P(X > 20) = P(20 < X \leq 24)$

$$= \int_{20}^{24} \frac{1}{972}(-x^2 + 30x - 144) dx$$

$$= \left[ \frac{1}{972} \left( -\frac{1}{3}x^3 + 15x^2 - 144x \right) \right]_{20}^{24}$$

$$= \frac{1}{972} [(-4608 + 8640 - 3456) - (-2666\frac{2}{3} + 6000 - 2880)]$$

$$= \frac{1}{972} (576 - 453\frac{1}{3})$$

$$\approx 0.126$$

- 9 a** For  $f(x)$  to be a valid probability density function, we only require  $f(x) \geq 0$  for all

$$a \leq x \leq b \text{ and } \int_a^b f(x) dx = 1.$$

- b** If  $f(x) \leq 1$  for all  $a \leq x \leq b$ , then

$$\int_a^b f(x) dx \leq \int_a^b 1 dx \quad \{f(x) \geq 0\}$$

$$\text{Now } \int_a^b 1 dx = [x]_a^b = b - a$$

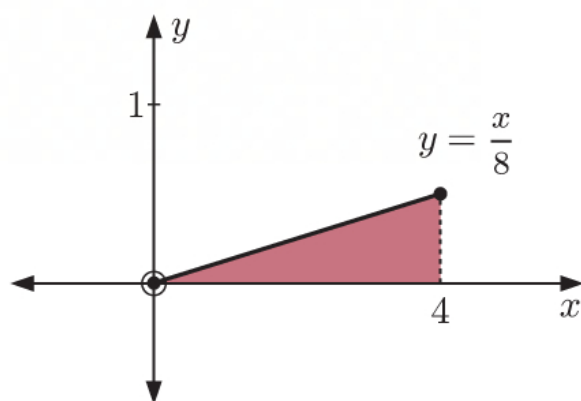
$$\therefore \int_a^b f(x) dx \leq b - a$$

$$\therefore 1 \leq b - a \quad \left\{ \int_a^b f(x) dx = 1 \right\}$$

So, it is possible that  $f(x) \leq 1$  for all  $a \leq x \leq b$  if  $b - a \geq 1$ .

## EXERCISE 28B

1 a



- From the graph,  $f(x) \geq 0$  for  $0 \leq x \leq 4$ . ✓

$$\begin{aligned} \bullet \text{ Area} &= \int_0^4 \frac{x}{8} dx \\ &= \left[ \frac{x^2}{16} \right]_0^4 \\ &= \frac{16}{16} \\ &= 1 \quad \checkmark \end{aligned}$$

So,  $f(x)$  is a valid probability density function.

- b i The maximum value of  $f(x)$  is when  $x = 4$   
 $\therefore$  the mode = 4.

- ii The median is the solution of

$$\int_0^m \frac{x}{8} dx = \frac{1}{2}$$

$$\therefore \left[ \frac{x^2}{16} \right]_0^m = \frac{1}{2}$$

$$\therefore \frac{m^2}{16} = \frac{1}{2}$$

$$\therefore m^2 = 8$$

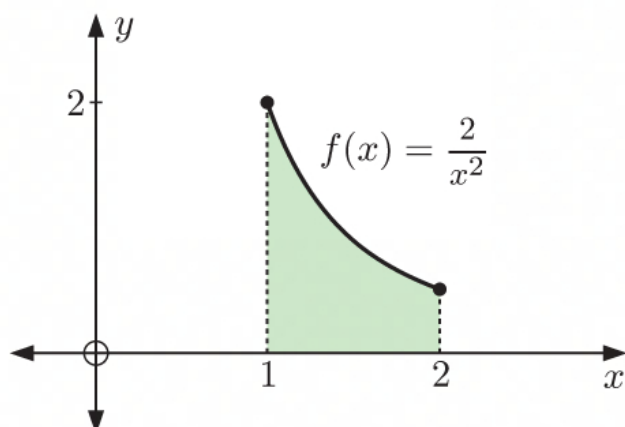
$$\therefore m = 2\sqrt{2} \quad \{\text{as } 0 \leq m \leq 4\}$$

$$\begin{aligned} \text{iii } \mu &= \int_0^4 x f(x) dx \\ &= \int_0^4 \frac{x^2}{8} dx \\ &= \left[ \frac{x^3}{24} \right]_0^4 \\ &= \frac{64}{24} \\ &= \frac{8}{3} = 2\frac{2}{3} \end{aligned}$$

$$\begin{aligned} \text{c } \text{Var}(X) &= \int_0^4 x^2 f(x) dx - \mu^2 \quad \text{and} \quad \sigma = \sqrt{\text{Var}(X)} \\ &= \int_0^4 \frac{x^3}{8} dx - \left(\frac{8}{3}\right)^2 \\ &= \left[ \frac{x^4}{32} \right]_0^4 - \frac{64}{9} \\ &= 8 - \frac{64}{9} \\ &= \frac{8}{9} \end{aligned}$$

$$\sigma = \frac{2\sqrt{2}}{3}$$

2 a



- From the graph,  $f(x) \geq 0$  for  $1 \leq x \leq 2$ . ✓

$$\begin{aligned} \bullet \text{ Area} &= \int_1^2 \frac{2}{x^2} dx \\ &= \left[ -\frac{2}{x} \right]_1^2 \\ &= -1 - (-2) \\ &= 1 \quad \checkmark \end{aligned}$$

So,  $f(x)$  is a valid probability density function.

$$\begin{aligned}
 \text{b } P(1 \leq X \leq 1.5) &= \int_1^{1.5} \frac{2}{x^2} dx \\
 &= \left[ -\frac{2}{x} \right]_1^{1.5} \\
 &= -\frac{4}{3} - (-2) \\
 &= \frac{2}{3}
 \end{aligned}$$

$$\begin{aligned}
 \text{c } \text{i } E(X) = \mu &= \int_1^2 x f(x) dx \\
 &= \int_1^2 \frac{2}{x} dx \\
 &= [2 \ln |x|]_1^2 \\
 &= 2 \ln 2
 \end{aligned}$$

$$\begin{aligned}
 \text{iii } \sigma(X) &= \sqrt{\text{Var}(X)} \\
 &= \sqrt{2 - 4(\ln 2)^2}
 \end{aligned}$$

$$\text{d } Y = 2X + 8$$

$$\begin{aligned}
 \text{i } E(Y) &= E(2X + 8) \\
 &= 2E(X) + 8 \\
 &= 2(2 \ln 2) + 8 \\
 &= 4 \ln 2 + 8
 \end{aligned}$$

$$\begin{aligned}
 \text{iii } \sigma(Y) &= \sigma(2X + 8) \\
 &= 2\sigma(X) \\
 &= 2\sqrt{2 - 4(\ln 2)^2}
 \end{aligned}$$

$$\begin{aligned}
 \text{ii } \text{Var}(X) &= \int_1^2 x^2 f(x) dx - \mu^2 \\
 &= \int_1^2 2 dx - \mu^2 \\
 &= [2x]_1^2 - \mu^2 \\
 &= (4 - 2) - (2 \ln 2)^2 \\
 &= 2 - 4(\ln 2)^2
 \end{aligned}$$

$$\begin{aligned}
 \text{ii } \text{Var}(Y) &= \text{Var}(2X + 8) \\
 &= 2^2 \text{Var}(X) \\
 &= 4(2 - 4(\ln 2)^2) \\
 &= 8 - 16(\ln 2)^2
 \end{aligned}$$

3 a  $f(x) = ax(x - 4)$ ,  $0 \leq x \leq 4$  is a probability density function.

$$\therefore \int_0^4 ax(x - 4) dx = 1$$

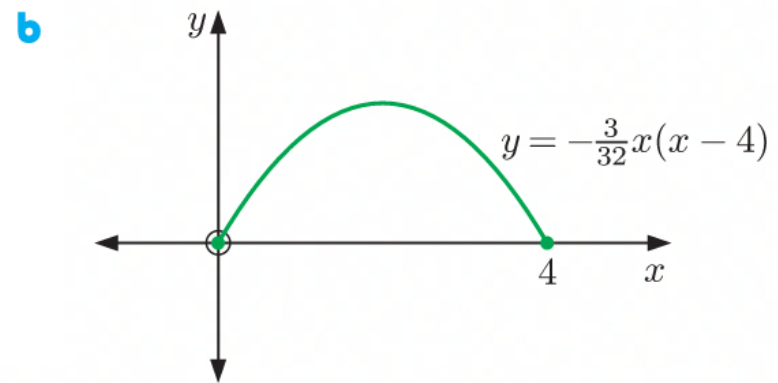
$$\therefore a \int_0^4 (x^2 - 4x) dx = 1$$

$$\therefore a \left[ \frac{1}{3}x^3 - 2x^2 \right]_0^4 = 1$$

$$\therefore a \left( \frac{64}{3} - 32 \right) = 1$$

$$\therefore a \left( -\frac{32}{3} \right) = 1$$

$$\therefore a = -\frac{3}{32}$$



$$\begin{aligned}
 \text{c i } \mu &= \int_0^4 x f(x) dx \\
 &= \int_0^4 -\frac{3}{32} x^2 (x-4) dx \\
 &= -\frac{3}{32} \int_0^4 (x^3 - 4x^2) dx \\
 &= -\frac{3}{32} \left[ \frac{1}{4} x^4 - \frac{4}{3} x^3 \right]_0^4 \\
 &= -\frac{3}{32} \left( 64 - \frac{256}{3} \right) \\
 &= -\frac{3}{32} \left( -\frac{64}{3} \right) \\
 &= 2
 \end{aligned}$$

ii The median is the solution of

$$\begin{aligned}
 \int_0^m -\frac{3}{32} x(x-4) dx &= \frac{1}{2} \\
 \therefore \int_0^m (x^2 - 4x) dx &= -\frac{16}{3} \\
 \therefore \left[ \frac{1}{3} x^3 - 2x^2 \right]_0^m &= -\frac{16}{3} \\
 \therefore \frac{1}{3} m^3 - 2m^2 &= -\frac{16}{3} \\
 \therefore m^3 - 6m^2 &= -16 \\
 \therefore m^3 - 6m^2 + 16 &= 0 \\
 \therefore (m-2)(m^2 - 4m - 8) &= 0 \\
 \therefore m = 2 \text{ or } \frac{4 \pm \sqrt{16 + 32}}{2} \\
 \therefore m = 2 \text{ or } 2 \pm 2\sqrt{3} \\
 \therefore m = 2 \text{ \{as } 0 < m < 4\}
 \end{aligned}$$

$$\begin{aligned}
 \text{iii } \text{Var}(X) &= \int_0^4 x^2 f(x) dx - \mu^2 \\
 &= \int_0^4 -\frac{3}{32} x^3 (x-4) dx - \mu^2 \\
 &= -\frac{3}{32} \int_0^4 (x^4 - 4x^3) dx - \mu^2 \\
 &= -\frac{3}{32} \left[ \frac{1}{5} x^5 - x^4 \right]_0^4 - \mu^2 \\
 &= -\frac{3}{32} \left( \frac{1024}{5} - 256 \right) - 2^2 \\
 &= -\frac{3}{32} \left( -\frac{256}{5} \right) - 4 \\
 &= \frac{24}{5} - 4 \\
 &= \frac{4}{5}
 \end{aligned}$$

$$\begin{aligned}
 \text{iv } \sigma &= \sqrt{\text{Var}(X)} \\
 &= \frac{2}{\sqrt{5}}
 \end{aligned}$$

4 a  $f(x) = \frac{1}{2x}$ ,  $1 \leq x \leq k$  is a probability density function.

$$\begin{aligned}
 \therefore \int_1^k \frac{1}{2x} dx &= 1 \\
 \therefore \left[ \frac{1}{2} \ln |x| \right]_1^k &= 1 \\
 \therefore \frac{1}{2} \ln |k| - \frac{1}{2} \ln 1 &= 1 \\
 \therefore \frac{1}{2} \ln k &= 1 \quad \{k \geq 1\} \\
 \therefore \ln k &= 2 \\
 \therefore k &= e^2
 \end{aligned}$$



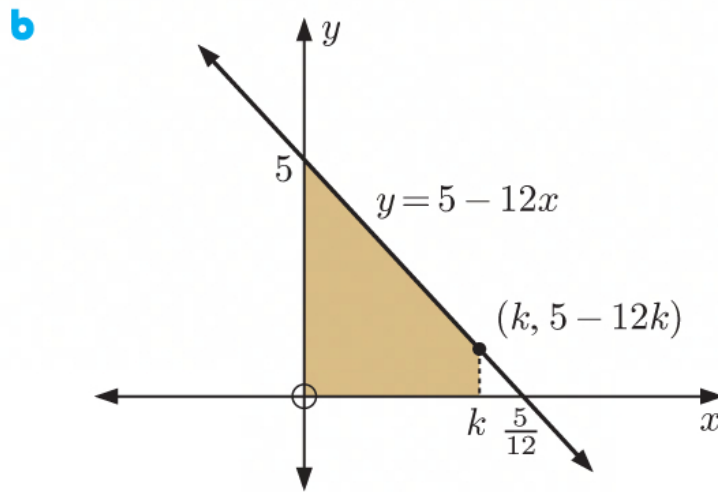
$$\begin{aligned}
 \text{b i } \mu &= \int_1^{e^2} x f(x) dx \\
 &= \int_1^{e^2} \frac{1}{2} dx \\
 &= \left[ \frac{1}{2}x \right]_1^{e^2} \\
 &= \frac{e^2 - 1}{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{iii } \sigma &= \sqrt{\text{Var}(X)} \\
 &= \sqrt{\frac{e^2 - 1}{2}}
 \end{aligned}$$

$$\begin{aligned}
 \text{c } \sigma(1 - \tfrac{1}{3}X) &= \left| -\tfrac{1}{3} \right| \sigma(X) \\
 &= \tfrac{1}{3} \sqrt{\frac{e^2 - 1}{2}}
 \end{aligned}$$

$$\begin{aligned}
 \text{ii } \text{Var}(X) &= \int_1^{e^2} x^2 f(x) dx - \mu^2 \\
 &= \int_1^{e^2} \frac{x}{2} dx - \left( \frac{e^2 - 1}{2} \right)^2 \\
 &= \left[ \frac{x^2}{4} \right]_1^{e^2} - \left( \frac{e^4 - 2e^2 + 1}{4} \right) \\
 &= \frac{e^4 - 1}{4} - \left( \frac{e^4 - 2e^2 + 1}{4} \right) \\
 &= \frac{2e^2 - 2}{4} \\
 &= \frac{e^2 - 1}{2}
 \end{aligned}$$

- 5 a  $5 - 12x < 0$  for  $\frac{5}{12} < x \leq \frac{1}{2}$   
 $\therefore f(x) \not\geq 0$  for all  $0 \leq x \leq \frac{1}{2}$   
 $\therefore f(x) = 5 - 12x$  is not a valid probability function for  $0 \leq x \leq \frac{1}{2}$   
 $\therefore k \neq \frac{1}{2}$



$$\begin{aligned}
 \int_0^k (5 - 12x) dx &= \text{shaded area} = 1 \\
 \therefore k \times \left( \frac{5 + (5 - 12k)}{2} \right) &= 1 \\
 \therefore k(5 - 6k) &= 1 \\
 \therefore 5k - 6k^2 &= 1 \\
 \therefore 6k^2 - 5k + 1 &= 0 \\
 \therefore (3k - 1)(2k - 1) &= 0 \\
 \therefore k &= \frac{1}{3} \text{ or } \frac{1}{2}
 \end{aligned}$$

From a,  $k \neq \frac{1}{2}$   $\therefore k = \frac{1}{3}$

$$\begin{aligned}
 \text{c } P(0 \leq X \leq \tfrac{1}{10}) &= \tfrac{1}{10} \times \left( \frac{5 + (5 - \frac{12}{10})}{2} \right) \\
 &= \frac{11}{25}
 \end{aligned}$$

$$\begin{aligned}
 \mu &= \int_0^{\frac{1}{3}} x f(x) dx \\
 &= \int_0^{\frac{1}{3}} (5x - 12x^2) dx \\
 &= \left[ \frac{5}{2}x^2 - 4x^3 \right]_0^{\frac{1}{3}} \\
 &= \frac{5}{18} - \frac{4}{27} \\
 &= \frac{7}{54} \\
 \therefore \text{mean} &= \frac{7}{54}
 \end{aligned}$$

The median is the solution of

$$\begin{aligned}
 \int_0^m (5 - 12x) dx &= \frac{1}{2} \\
 \therefore \left[ 5x - 6x^2 \right]_0^m &= \frac{1}{2} \\
 \therefore 5m - 6m^2 &= \frac{1}{2} \\
 \therefore 12m^2 - 10m + 1 &= 0 \\
 \therefore m &= \frac{10 \pm \sqrt{52}}{24}
 \end{aligned}$$

But  $m < \frac{1}{3}$ , so  $m = \frac{5 - \sqrt{13}}{12} \approx 0.116$

$$\therefore \text{median} = \frac{5 - \sqrt{13}}{12}$$

**6 a**  $f(x) = k$ ,  $a \leq x \leq b$  is a probability density function.

$$\begin{aligned}
 \therefore \int_a^b k dx &= 1 \\
 \therefore [kx]_a^b &= 1 \\
 \therefore bk - ak &= 1 \\
 \therefore k(b - a) &= 1 \\
 \therefore k &= \frac{1}{b - a}
 \end{aligned}$$

$$\begin{aligned}
 \mu &= \int_a^b x f(x) dx \\
 &= \int_a^b kx dx \\
 &= \frac{1}{b - a} \left[ \frac{1}{2}x^2 \right]_a^b \\
 &= \frac{1}{b - a} \left( \frac{b^2 - a^2}{2} \right) \\
 &= \frac{1}{2} \frac{(b - a)(b + a)}{(b - a)} \\
 \therefore \text{mean} &= \frac{a + b}{2}
 \end{aligned}$$

The median is the solution of

$$\begin{aligned}
 \int_a^m k dx &= \frac{1}{2} \\
 \therefore [kx]_a^m &= \frac{1}{2} \\
 \therefore \frac{m}{b - a} - \frac{a}{b - a} &= \frac{1}{2} \\
 \therefore m - a &= \frac{b - a}{2} \\
 \therefore m &= a + \frac{b - a}{2} \\
 &= \frac{a + b}{2} \\
 \therefore \text{median} &= \frac{a + b}{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{c } \int_a^b x^2 f(x) dx &= \int_a^b kx^2 dx \\
 &= \frac{1}{b-a} \left[ \frac{1}{3} x^3 \right]_a^b \\
 &= \frac{1}{b-a} \left( \frac{b^3 - a^3}{3} \right) \\
 &= \frac{1}{3} \frac{b^3 - a^3}{b-a} \\
 &= \frac{1}{3} \frac{(b-a)(b^2 + ab + a^2)}{(b-a)} \\
 &= \frac{b^2 + ab + a^2}{3}
 \end{aligned}$$

$$\begin{aligned}
 \text{Now, } \text{Var}(X) &= \int_a^b x^2 f(x) dx - \mu^2 \\
 &= \frac{b^2 + ab + a^2}{3} - \left( \frac{a+b}{2} \right)^2 \\
 &= \frac{4(b^2 + ab + a^2) - 3(a+b)^2}{12} \\
 &= \frac{4b^2 + 4ab + 4a^2 - 3a^2 - 6ab - 3b^2}{12} \\
 &= \frac{b^2 - 2ab + a^2}{12} \\
 &= \frac{(b-a)^2}{12}
 \end{aligned}$$

$$\text{and } \sigma = \sqrt{\frac{(b-a)^2}{12}} = \frac{b-a}{\sqrt{12}} \quad \{\text{as } b > a\}$$

- 7 a**  $f(x) = ke^{-x}$ ,  $0 \leq x \leq 3$  is a probability density function.

$$\therefore \int_0^3 ke^{-x} dx = 1$$

$$\therefore k \int_0^3 e^{-x} dx = 1$$

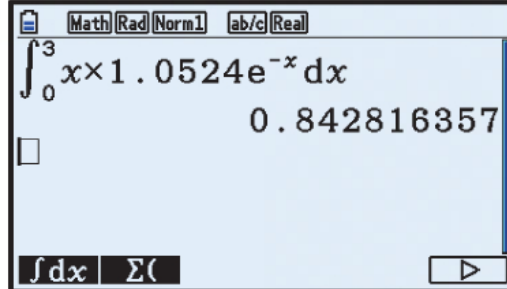
$$\therefore k [-e^{-x}]_0^3 = 1$$

$$\therefore k(-e^{-3} - (-1)) = 1$$

$$\therefore k(1 - e^{-3}) = 1$$

$$\therefore k = \frac{1}{1 - e^{-3}} \approx 1.0524$$

**b**



The calculator screen shows the expression  $\int_0^3 x \times 1.0524 e^{-x} dx$  and the result  $0.842816357$ . The mode is set to 'Math'.

The mean  $\approx 0.843$

- 8 a**  $f(x) = kx^2(x - 6)$ ,  $0 \leq x \leq 5$  is a probability density function.

$$\begin{aligned}\therefore \int_0^5 kx^2(x - 6) dx &= 1 \\ \therefore k \int_0^5 (x^3 - 6x^2) dx &= 1 \\ \therefore k \left[ \frac{1}{4}x^4 - 2x^3 \right]_0^5 &= 1 \\ \therefore k \left( \frac{625}{4} - 250 \right) &= 1 \\ \therefore k \left( -\frac{375}{4} \right) &= 1 \\ \therefore k &= -\frac{4}{375}\end{aligned}$$

**c** 
$$\begin{aligned}\mu &= \int_0^5 x f(x) dx \\ &= \int_0^5 -\frac{4}{375} x^3(x - 6) dx \\ &= \int_0^5 \left( -\frac{4}{375} x^4 + \frac{8}{125} x^3 \right) dx \\ &= \left[ -\frac{4}{1875} x^5 + \frac{2}{125} x^4 \right]_0^5 \\ &= 3\frac{1}{3}\end{aligned}$$

- b** The median is the solution of

$$\begin{aligned}\int_0^m -\frac{4}{375} x^2(x - 6) dx &= \frac{1}{2} \\ \therefore \int_0^m (x^3 - 6x^2) dx &= -\frac{375}{8} \\ \therefore \left[ \frac{1}{4}x^4 - 2x^3 \right]_0^m &= -\frac{375}{8} \\ \therefore \frac{1}{4}m^4 - 2m^3 &= -\frac{375}{8} \\ \therefore 2m^4 - 16m^3 + 375 &= 0 \\ \text{Using technology, } m &\approx 3.46\end{aligned}$$

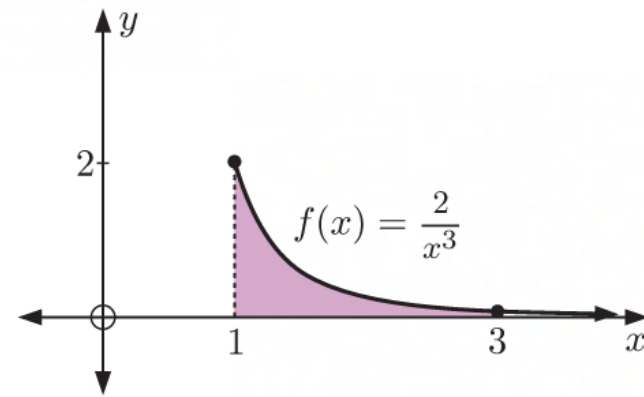
**d** 
$$\begin{aligned}\int_0^5 x^2 f(x) dx &= \int_0^5 -\frac{4}{375} x^4(x - 6) dx \\ &= \int_0^5 \left( -\frac{4}{375} x^5 + \frac{8}{125} x^4 \right) dx \\ &= \left[ -\frac{2}{1125} x^6 + \frac{8}{625} x^5 \right]_0^5 \\ &= 12\frac{2}{9}\end{aligned}$$

Now, 
$$\begin{aligned}\text{Var}(X) &= \int_0^5 x^2 f(x) dx - \mu^2 \\ &= 12\frac{2}{9} - \left( 3\frac{1}{3} \right)^2 \\ &= 1\frac{1}{9}\end{aligned}$$

**9 a** 
$$\begin{aligned}P(1 \leq X \leq 3) &= \int_1^3 \frac{2}{x^3} dx \\ &= \left[ -\frac{1}{x^2} \right]_1^3 \\ &= -\frac{1}{9} - (-1) \\ &= \frac{8}{9}\end{aligned}$$

**b** The median is the solution of 
$$\begin{aligned}\int_1^m \frac{2}{x^3} dx &= \frac{1}{2} \\ \therefore \left[ -\frac{1}{x^2} \right]_1^m &= \frac{1}{2} \\ \therefore -\frac{1}{m^2} - (-1) &= \frac{1}{2} \\ \therefore \frac{1}{m^2} &= \frac{1}{2} \\ \therefore m^2 &= 2 \\ \therefore m &= \sqrt{2} \quad \{\text{as } m > 1\}\end{aligned}$$

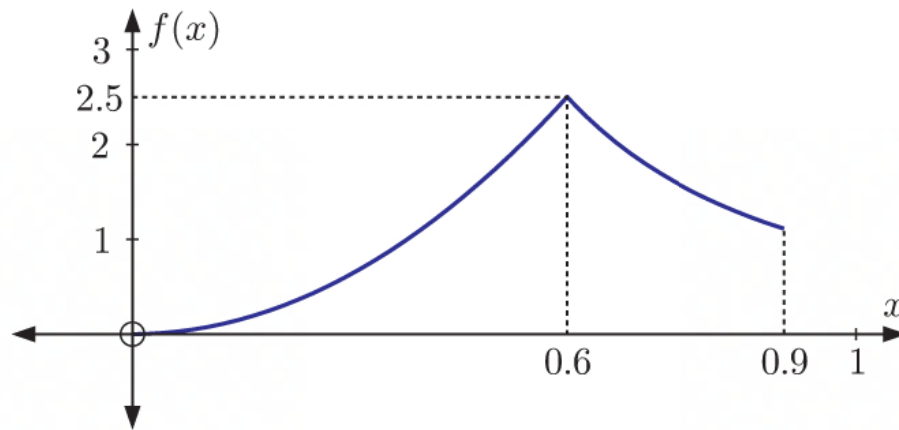
**c** 
$$\mu = \int_1^\infty x f(x) dx = \int_1^\infty \frac{2}{x^2} dx$$





$$\begin{aligned}
 \text{d } \mu &= \lim_{b \rightarrow \infty} \int_1^b \frac{2}{x^2} dx \\
 &= \lim_{b \rightarrow \infty} \left[ -\frac{2}{x} \right]_1^b \\
 &= \lim_{b \rightarrow \infty} \left( -\frac{2}{b} + \frac{2}{1} \right) \\
 &= -0 + 2 \\
 &= 2
 \end{aligned}$$

10 a



b • From the graph in a,  $f(x) \geq 0$  for all  $0 \leq x \leq 0.9$ . ✓

$$\begin{aligned}
 \bullet \text{ Area} &= \int_0^{0.6} \frac{125}{18} x^2 dx + \int_{0.6}^{0.9} \frac{9}{10x^2} dx \\
 &= \left[ \frac{125}{54} x^3 \right]_0^{0.6} + \left[ -\frac{9}{10x} \right]_{0.6}^{0.9} \\
 &= \frac{125}{54} \left( \frac{3}{5} \right)^3 + \left[ \left( -\frac{9}{9} \right) - \left( -\frac{9}{6} \right) \right] \\
 &= \frac{1}{2} - 1 + \frac{3}{2} \\
 &= 1 \quad \checkmark
 \end{aligned}$$

So,  $f(x)$  is a valid probability density function.

$$\begin{aligned}
 \text{c } \mu &= \int_0^{0.9} x f(x) dx \\
 &= \int_0^{0.6} \frac{125}{18} x^3 dx + \int_{0.6}^{0.9} \frac{9}{10x} dx \\
 &= \left[ \frac{125}{72} x^4 \right]_0^{0.6} + \left[ \frac{9}{10} \ln |x| \right]_{0.6}^{0.9} \\
 &= \frac{125}{72} (0.6)^4 + \frac{9}{10} \ln(0.9) - \frac{9}{10} \ln(0.6) \\
 &\approx 0.590
 \end{aligned}$$

∴ mean  $\approx 0.590$

$$\text{From part b, } \int_0^{0.6} f(x) dx = \int_{0.6}^{0.9} f(x) dx = \frac{1}{2}$$

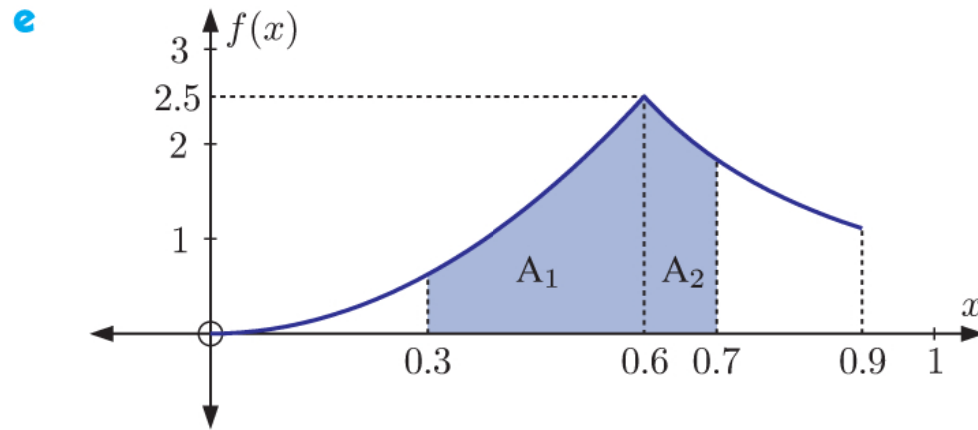
∴ median = 0.6

From the graph in a, the maximum value of  $f(x)$  is when  $x = 0.6$

∴ mode = 0.6

$$\begin{aligned}
 \text{d} \quad & \int_0^{0.9} x^2 f(x) dx \\
 &= \int_0^{0.6} \frac{125}{18} x^4 dx + \int_{0.6}^{0.9} \frac{9}{10} dx \\
 &= \left[ \frac{25}{18} x^5 \right]_0^{0.6} + \left[ \frac{9}{10} x \right]_{0.6}^{0.9} \\
 &= \frac{25}{18} \left( \frac{3}{5} \right)^5 + \left[ \frac{9}{10} \left( \frac{9}{10} \right) - \frac{9}{10} \left( \frac{3}{5} \right) \right] \\
 &= \frac{189}{500}
 \end{aligned}$$

$$\begin{aligned}
 \text{Now, } \text{Var}(X) &= \int_0^{0.9} x^2 f(x) dx - \mu^2 \\
 &\approx \frac{189}{500} - 0.590^2 \\
 &\approx 0.0300 \\
 \text{and } \sigma &= \sqrt{\text{Var}(X)} \\
 &\approx 0.173
 \end{aligned}$$



$$P(0.3 < X < 0.7) = \text{shaded area} = A_1 + A_2$$

$$\begin{aligned}
 &= \int_{0.3}^{0.6} \frac{125}{18} x^2 dx + \int_{0.6}^{0.7} \frac{9}{10x^2} dx \\
 &= \left[ \frac{125}{54} x^3 \right]_{0.3}^{0.6} + \left[ -\frac{9}{10x} \right]_{0.6}^{0.7} \\
 &= \frac{125}{54} ((0.6)^3 - (0.3)^3) + \left[ \left( -\frac{9}{7} \right) - \left( -\frac{9}{6} \right) \right] \\
 &\approx 0.652
 \end{aligned}$$

0.3 hours = 18 minutes, and 0.7 hours = 42 minutes

∴ the task can be performed in between 18 minutes and 42 minutes about 65.2% of the time.

$$\text{11} \quad f(x) = \begin{cases} \frac{\sin x}{2} & \text{if } 0 \leq x \leq \frac{\pi}{2} \\ k(x - \frac{\pi}{2}) + 0.5 & \text{if } \frac{\pi}{2} < x \leq \frac{\pi}{2} + 2 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned}
 \text{a} \quad P(X \leq \frac{\pi}{2}) &= \int_0^{\frac{\pi}{2}} f(x) dx \\
 &= \int_0^{\frac{\pi}{2}} \frac{\sin x}{2} dx \\
 &= \left[ \frac{-\cos x}{2} \right]_0^{\frac{\pi}{2}} \\
 &= \frac{-\cos \frac{\pi}{2}}{2} + \frac{\cos 0}{2} \\
 &= \frac{1}{2}
 \end{aligned}$$

- b**  $f(x)$  is a probability density function for  $0 \leq x \leq \frac{\pi}{2} + 2$ .

$$\therefore \int_0^{\frac{\pi}{2}+2} f(x) dx = 1$$

$$\therefore \int_0^{\frac{\pi}{2}} f(x) dx + \int_{\frac{\pi}{2}}^{\frac{\pi}{2}+2} f(x) dx = 1$$

$$\therefore \frac{1}{2} + \int_{\frac{\pi}{2}}^{\frac{\pi}{2}+2} f(x) dx = 1 \quad \{\text{using a}\}$$

$$\therefore \int_{\frac{\pi}{2}}^{\frac{\pi}{2}+2} f(x) dx = \frac{1}{2}$$

$$\therefore \int_{\frac{\pi}{2}}^{\frac{\pi}{2}+2} \left(k\left(x - \frac{\pi}{2}\right) + 0.5\right) dx = \frac{1}{2}$$

$$\therefore \left[\frac{k}{2}\left(x - \frac{\pi}{2}\right)^2 + 0.5x\right]_{\frac{\pi}{2}}^{\frac{\pi}{2}+2} = \frac{1}{2}$$

$$\therefore \frac{k}{2} \left(\cancel{\frac{\pi}{2}} + 2 - \cancel{\frac{\pi}{2}}\right)^2 + 0.5\left(\frac{\pi}{2} + 2\right) - \left[\frac{k}{2} \left(\cancel{\frac{\pi}{2}} - \cancel{\frac{\pi}{2}}\right)^2 + 0.5\left(\frac{\pi}{2}\right)\right] = \frac{1}{2}$$

$$\therefore 2k + 1 = \frac{1}{2}$$

$$\therefore 2k = -\frac{1}{2}$$

$$\therefore k = -\frac{1}{4}$$

- c** From **a**,  $P(X \leq \frac{\pi}{2}) = \frac{1}{2}$

$\therefore$  the median of  $X$  is  $\frac{\pi}{2}$ .

**d**  $\mu = \int_0^{\frac{\pi}{2}+2} x f(x) dx$

$$= \int_0^{\frac{\pi}{2}} x f(x) dx + \int_{\frac{\pi}{2}}^{\frac{\pi}{2}+2} x f(x) dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{x \sin x}{2} dx + \int_{\frac{\pi}{2}}^{\frac{\pi}{2}+2} x \left(-\frac{1}{4}\left(x - \frac{\pi}{2}\right) + 0.5\right) dx \quad \dots (*)$$

Using integration by parts,

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \frac{x \sin x}{2} dx &= \left[-\frac{x}{2} \cos x\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \frac{1}{2} \cos x dx & \begin{cases} u = \frac{x}{2} & v' = \sin x \\ u' = \frac{1}{2} & v = -\cos x \end{cases} \\ &= -\frac{\pi}{4} \cos \frac{\pi}{2} + 0 + \left[\frac{1}{2} \sin x\right]_0^{\frac{\pi}{2}} \\ &= \frac{1}{2} \sin \frac{\pi}{2} - \frac{1}{2} \sin 0 \\ &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned}
\text{Also } & \int_{\frac{\pi}{2}}^{\frac{\pi}{2}+2} x \left( -\frac{1}{4} \left( x - \frac{\pi}{2} \right) + 0.5 \right) dx \\
&= \int_{\frac{\pi}{2}}^{\frac{\pi}{2}+2} \left( -\frac{x^2}{4} + \left( \frac{\pi}{8} + 0.5 \right) x \right) dx \\
&= \left[ -\frac{x^3}{12} + \left( \frac{\pi}{16} + 0.25 \right) x^2 \right]_{\frac{\pi}{2}}^{\frac{\pi}{2}+2} \\
&= -\frac{\left( \frac{\pi}{2} + 2 \right)^3}{12} + \left( \frac{\pi}{16} + 0.25 \right) \left( \frac{\pi}{2} + 2 \right)^2 + \frac{\left( \frac{\pi}{2} \right)^3}{12} - \left( \frac{\pi}{16} + 0.25 \right) \left( \frac{\pi}{2} \right)^2 \\
&= -\left( \frac{\pi}{2} + 2 \right)^2 \left( \frac{\pi}{24} + \frac{1}{6} - \frac{\pi}{16} - \frac{1}{4} \right) - \left( \frac{\pi}{2} \right)^2 \left( -\frac{\pi}{24} + \frac{\pi}{16} + \frac{1}{4} \right) \\
&= -\left( \frac{\pi}{2} + 2 \right)^2 \left( -\frac{\pi}{48} - \frac{1}{12} \right) - \left( \frac{\pi}{2} \right)^2 \left( \frac{\pi}{48} + \frac{1}{4} \right) \\
&= \left[ \left( \frac{\pi}{2} \right)^2 + 2 \left( \frac{\pi}{2} \right) (2) + 4 \right] \left( \frac{\pi}{48} + \frac{1}{12} \right) - \left( \frac{\pi}{2} \right)^2 \left( \frac{\pi}{48} + \frac{1}{4} \right) \\
&= \left( \frac{\pi}{2} \right)^2 \left( \cancel{\frac{\pi}{48}} + \frac{1}{12} - \cancel{\frac{\pi}{48}} - \frac{1}{4} \right) + (2\pi + 4) \left( \frac{\pi}{48} + \frac{1}{12} \right) \\
&= \left( \frac{\pi}{2} \right)^2 \times \left( -\frac{1}{6} \right) + (\pi + 2) \left( \frac{\pi}{24} + \frac{1}{6} \right) \\
&= -\cancel{\frac{\pi^2}{24}} + \cancel{\frac{\pi^2}{24}} + \frac{\pi}{6} + \frac{\pi}{12} + \frac{1}{3} \\
&= \frac{\pi}{4} + \frac{1}{3}
\end{aligned}$$

$$\begin{aligned}
\text{Substituting into (*) gives } \mu &= \frac{1}{2} + \frac{\pi}{4} + \frac{1}{3} \\
&= \frac{5}{6} + \frac{\pi}{4} \\
&\approx 1.62
\end{aligned}$$

**12 a**  $\lambda > 0$ , and  $e^{-\lambda t} > 0$  for all  $t \geq 0$   
 $\therefore \lambda e^{-\lambda t} > 0$  for all  $t \geq 0$   
 $\therefore f(t) > 0$  for all  $t \geq 0$

**b**  $\int_0^\infty f(t) dt = \lim_{k \rightarrow \infty} \int_0^k f(t) dt$   
 $= \lim_{k \rightarrow \infty} \int_0^k \lambda e^{-\lambda t} dt$   
 $= \lim_{k \rightarrow \infty} [-e^{-\lambda t}]_0^k$   
 $= \lim_{k \rightarrow \infty} (-e^{-\lambda k} + e^0)$   
 $= \lim_{k \rightarrow \infty} (-e^{-\lambda k}) + 1$   
 $= 0 + 1$   
 $= 1$

**c i**  $E(T) = \int_0^\infty t f(t) dt$   
 $= \int_0^\infty \lambda t e^{-\lambda t} dt$



ii Using integration by parts,

$$\begin{aligned} \int_0^k \lambda t e^{-\lambda t} dt &= [-t e^{-\lambda t}]_0^k - \int_0^k -e^{-\lambda t} dt && \begin{cases} u = t & v' = \lambda e^{-\lambda t} \\ u' = 1 & v = -e^{-\lambda t} \end{cases} \\ &= -k e^{-\lambda k} + 0 - \left[ \frac{1}{\lambda} e^{-\lambda t} \right]_0^k \\ &= -k e^{-\lambda k} - \left( \frac{1}{\lambda} e^{-\lambda k} - \frac{1}{\lambda} e^0 \right) \\ &= -e^{-\lambda k} \left( k + \frac{1}{\lambda} \right) + \frac{1}{\lambda} \end{aligned}$$

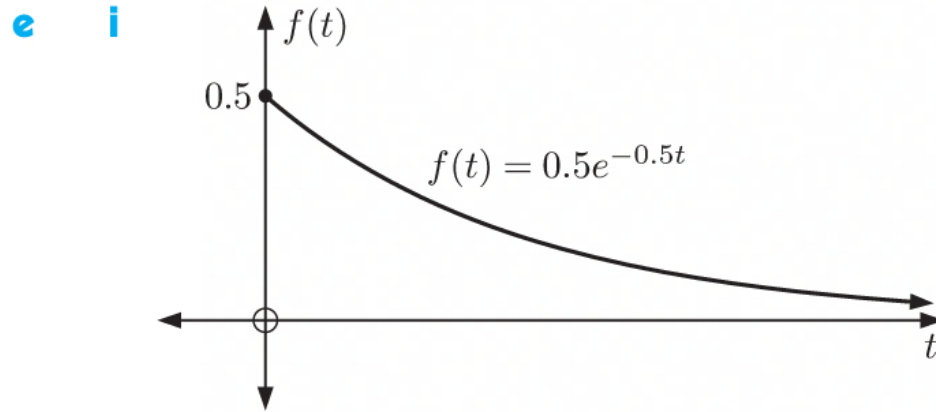
$$\begin{aligned} \text{iii } E(T) &= \lim_{k \rightarrow \infty} \int_0^k \lambda t e^{-\lambda t} dt \\ &= \lim_{k \rightarrow \infty} \left( -e^{-\lambda k} \left( k + \frac{1}{\lambda} \right) + \frac{1}{\lambda} \right) \quad \{\text{using ii}\} \\ &= \lim_{k \rightarrow \infty} \left( -k e^{-\lambda k} - \frac{1}{\lambda} e^{-\lambda k} + \frac{1}{\lambda} \right) \\ &= \lim_{k \rightarrow \infty} (-k e^{-\lambda k}) - \lim_{k \rightarrow \infty} \left( \frac{1}{\lambda} e^{-\lambda k} \right) + \frac{1}{\lambda} \\ &= \lim_{k \rightarrow \infty} \left( \frac{-k}{e^{\lambda k}} \right) - 0 + \frac{1}{\lambda} \\ &= \lim_{k \rightarrow \infty} \left( \frac{-1}{\lambda e^{\lambda k}} \right) + \frac{1}{\lambda} \quad \{\text{l'Hôpital}\} \\ &= 0 + \frac{1}{\lambda} \\ &= \frac{1}{\lambda} \end{aligned}$$

$$\begin{aligned} \text{d } \text{i } \text{Var}(T) &= \int_0^\infty t^2 f(t) dt - [E(T)]^2 \\ &= \int_0^\infty \lambda t^2 e^{-\lambda t} dt - \frac{1}{\lambda^2} \end{aligned}$$

ii Using integration by parts,

$$\begin{aligned} &\int_0^k \lambda t^2 e^{-\lambda t} dt \\ &= [-t^2 e^{-\lambda t}]_0^k - \int_0^k -2t e^{-\lambda t} dt \quad \leftarrow \begin{cases} u = t^2 & v' = \lambda e^{-\lambda t} \\ u' = 2t & v = -e^{-\lambda t} \end{cases} \\ &= -k^2 e^{-\lambda k} + 0 - \left( \left[ \frac{2t}{\lambda} e^{-\lambda t} \right]_0^k - \int_0^k \frac{2}{\lambda} e^{-\lambda t} dt \right) \quad \leftarrow \begin{cases} u = -2t & v' = e^{-\lambda t} \\ u' = -2 & v = -\frac{1}{\lambda} e^{-\lambda t} \end{cases} \\ &= -k^2 e^{-\lambda k} - \left( \frac{2k}{\lambda} e^{-\lambda k} - 0 \right) + \left[ -\frac{2}{\lambda^2} e^{-\lambda t} \right]_0^k \\ &= -k^2 e^{-\lambda k} - \frac{2k}{\lambda} e^{-\lambda k} + \left( -\frac{2}{\lambda^2} e^{-\lambda k} + \frac{2}{\lambda^2} e^0 \right) \\ &= -e^{-\lambda k} \left( k^2 + \frac{2k}{\lambda} + \frac{2}{\lambda^2} \right) + \frac{2}{\lambda^2} \end{aligned}$$

$$\begin{aligned}
\text{iii} \quad \text{Var}(T) &= \lim_{k \rightarrow \infty} \int_0^k \lambda t e^{-\lambda t} dt - \frac{1}{\lambda^2} \\
&= \lim_{k \rightarrow \infty} \left( -e^{-\lambda k} \left( k^2 + \frac{2k}{\lambda} + \frac{2}{\lambda^2} \right) + \frac{2}{\lambda^2} \right) - \frac{1}{\lambda^2} \\
&= \lim_{k \rightarrow \infty} \left( \frac{k^2 + \frac{2k}{\lambda} + \frac{2}{\lambda^2}}{-e^{\lambda k}} \right) + \frac{2}{\lambda^2} - \frac{1}{\lambda^2} \\
&= \lim_{k \rightarrow \infty} \left( \frac{2k + \frac{2}{\lambda}}{-\lambda e^{\lambda k}} \right) + \frac{1}{\lambda^2} \quad \{\text{l'Hôpital}\} \\
&= \lim_{k \rightarrow \infty} \left( \frac{2}{-\lambda^2 e^{\lambda k}} \right) + \frac{1}{\lambda^2} \quad \{\text{l'Hôpital}\} \\
&= 0 + \frac{1}{\lambda^2} \\
&= \frac{1}{\lambda^2}
\end{aligned}$$



$$\text{ii} \quad f(t) = 0.5e^{-0.5t} \quad \therefore \lambda = 0.5$$

$$\begin{aligned}
E(T) &= \frac{1}{0.5} \quad \{\text{using c iii}\} \\
&= 2
\end{aligned}$$

$$\begin{aligned}
\text{Var}(T) &= \frac{1}{(0.5)^2} \quad \{\text{using d iii}\} \\
&= \frac{1}{0.25} \\
&= 4
\end{aligned}$$

$$\begin{aligned}
\text{iii} \quad P(T < 1) &= \int_0^1 f(t) dt \\
&= \int_0^1 0.5e^{-0.5t} dt \\
&= [-e^{-0.5t}]_0^1 \\
&= -e^{-0.5} + 1 \\
&= 1 - e^{-0.5} \\
&\approx 0.393
\end{aligned}$$

**iv** The median is the solution of

$$\begin{aligned}
\int_0^m 0.5e^{-0.5t} dt &= \frac{1}{2} \\
\therefore [-e^{-0.5t}]_0^m &= \frac{1}{2} \\
\therefore -e^{-0.5m} + 1 &= \frac{1}{2} \\
\therefore e^{-0.5m} &= \frac{1}{2} \\
\therefore -0.5m &= \ln\left(\frac{1}{2}\right) \\
\therefore m &= -2\ln\left(\frac{1}{2}\right) \\
&= 2\ln 2 \text{ minutes}
\end{aligned}$$

$$\begin{aligned}
 \mathbf{13} \quad \mathbf{a} \quad \mathbb{E}(kX) &= \int_{\mathcal{D}} (kx) f(x) \, dx \\
 &= k \int_{\mathcal{D}} x f(x) \, dx \\
 &= k \mathbb{E}(X)
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad \mathbb{E}[g(X) + h(X)] &= \int_{\mathcal{D}} [g(x) + h(x)] f(x) \, dx \\
 &= \int_{\mathcal{D}} [g(x) f(x) + h(x) f(x)] \, dx \\
 &= \int_{\mathcal{D}} g(x) f(x) \, dx + \int_{\mathcal{D}} h(x) f(x) \, dx \\
 &= \mathbb{E}[g(X)] + \mathbb{E}[h(X)]
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad \text{Var}(X) &= \int_{\mathcal{D}} (x - \mathbb{E}(X))^2 f(x) \, dx \\
 &= \int_{\mathcal{D}} (x^2 - 2x \mathbb{E}(X) + [\mathbb{E}(X)]^2) f(x) \, dx \\
 &= \int_{\mathcal{D}} x^2 f(x) \, dx - 2 \mathbb{E}(X) \int_{\mathcal{D}} x f(x) \, dx + [\mathbb{E}(X)]^2 \int_{\mathcal{D}} f(x) \, dx \\
 &= \mathbb{E}(X^2) - 2 \mathbb{E}(X) \times \mathbb{E}(X) + [\mathbb{E}(X)]^2 \times 1 \\
 &= \mathbb{E}(X^2) - [\mathbb{E}(X)]^2
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{14} \quad \mathbf{a} \quad \mathbb{E}(aX + b) &= \int_{\mathcal{D}} (ax + b) f(x) \, dx \\
 &= a \int_{\mathcal{D}} x f(x) \, dx + b \int_{\mathcal{D}} f(x) \, dx \\
 &= a \mathbb{E}(X) + b(1) \\
 &= a \mathbb{E}(X) + b
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad \text{Var}(aX + b) &= \int_{\mathcal{D}} (ax + b - \mathbb{E}(aX + b))^2 f(x) \, dx \\
 &= \int_{\mathcal{D}} (ax + \cancel{b} - a \mathbb{E}(X) - \cancel{b})^2 f(x) \, dx \quad \{\text{using } \mathbf{a}\} \\
 &= \int_{\mathcal{D}} a^2 (x - \mathbb{E}(X))^2 f(x) \, dx \\
 &= a^2 \int_{\mathcal{D}} (x - \mathbb{E}(X))^2 f(x) \, dx \\
 &= a^2 \text{Var}(X)
 \end{aligned}$$

**15**  $Y = 2X + 1$ 

**a i**  $a \leq X \leq b$

$$\therefore 2a + 1 \leq 2X + 1 \leq 2b + 1$$

$$\therefore 2a + 1 \leq Y \leq 2b + 1$$

**ii**  $F_Y(y) = P(Y \leq y)$

$$= P(2X + 1 \leq y)$$

$$= P\left(X \leq \frac{y-1}{2}\right)$$

$$= F_X\left(\frac{y-1}{2}\right)$$

**iii**  $f_Y(y) = \frac{d}{dy} F_Y(y)$

$$= \frac{d}{dy} F_X\left(\frac{y-1}{2}\right) \quad \{\text{using ii}\}$$

$$= \frac{1}{2} F'_X\left(\frac{y-1}{2}\right) \quad \{\text{chain rule}\}$$

$$= \frac{1}{2} f_X\left(\frac{y-1}{2}\right)$$

**iv**  $E(Y) = \int_{2a+1}^{2b+1} y f_Y(y) dy$

$$= \int_{2a+1}^{2b+1} \frac{1}{2} y f_X\left(\frac{y-1}{2}\right) dy \quad \{\text{from iii}\}$$

Let  $y = 2x + 1$ , so  $1 = 2 \frac{dx}{dy} \therefore \frac{dx}{dy} = \frac{1}{2}$

When  $y = 2b + 1$ ,  $x = b$

When  $y = 2a + 1$ ,  $x = a$

$$\therefore E(Y) = \int_a^b (2x + 1) f_X(x) \frac{dx}{dy} dy$$

$$= \int_a^b (2x + 1) f_X(x) dx$$

$$= 2 \int_a^b x f_X(x) dx + \int_a^b f_X(x) dx$$

$$= 2E(X) + 1$$

$$\text{Var}(Y) = \int_{2a+1}^{2b+1} (y - E(Y))^2 f_Y(y) dy$$

$$= \int_{2a+1}^{2b+1} \frac{1}{2} (y - E(Y)) f_Y\left(\frac{y-1}{2}\right) dy \quad \{\text{from iii}\}$$

Let  $y = 2x + 1$ ,  $\therefore \frac{dx}{dy} = \frac{1}{2}$

When  $y = 2b + 1$ ,  $x = b$

When  $y = 2a + 1$ ,  $x = a$



$$\begin{aligned}
 \therefore \text{Var}(Y) &= \int_a^b (2x + 1 - E(Y))^2 f_X(x) \frac{dx}{dy} dy \\
 &= \int_a^b (2x + 1 - E(Y))^2 f_X(x) dx \\
 &= \int_a^b (2x + 1 - 2E(X) - 1)^2 f_X(x) dx \\
 &= \int_a^b 4(x - E(X))^2 f_X(x) dx \\
 &= 4 \int_a^b (x - E(X))^2 f_X(x) dx \\
 &= 4 \text{Var}(X)
 \end{aligned}$$

**b i** If  $g$  is increasing, then  $g^{-1}$  is also increasing.

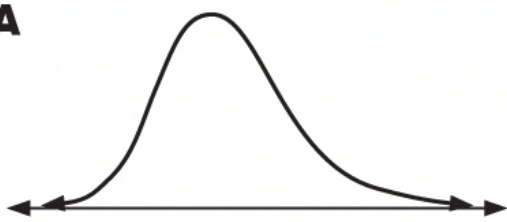
$$\begin{aligned}
 F_Y(y) &= P(Y \leq y) \\
 &= P(g(X) \leq y) \\
 &= P(X \leq g^{-1}(y)) \quad \{g^{-1} \text{ is increasing}\} \\
 &= F_X(g^{-1}(y))
 \end{aligned}$$

$$\begin{aligned}
 \text{ii } f_Y(y) &= \frac{d}{dy} F_Y(y) \\
 &= \frac{d}{dy} F_X(g^{-1}(y)) \quad \{\text{using i}\} \\
 &= F'_X(g^{-1}(y)) \times \frac{d}{dy} g^{-1}(y) \quad \{\text{chain rule}\} \\
 &= f_X(g^{-1}(y)) \times \frac{d}{dy} g^{-1}(y)
 \end{aligned}$$

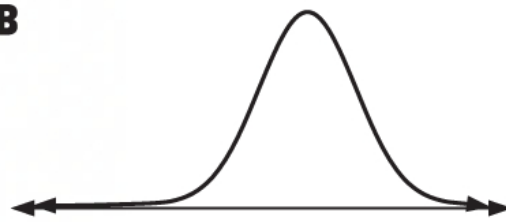
## EXERCISE 28C.1

**1**

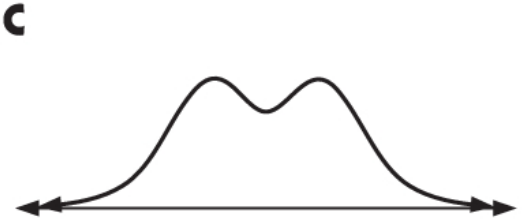
**A**



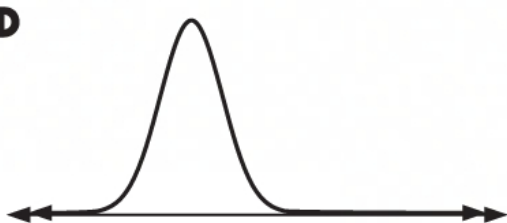
**B**



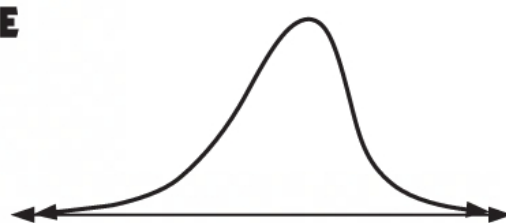
**C**



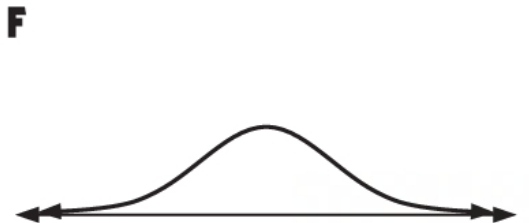
**D**



**E**



**F**



Distributions **B**, **D**, and **F** are symmetrical and bell-shaped.

$\therefore$  **B**, **D**, and **F** appear to be normally distributed.

**2** Most measurements in each situation will be centred about the mean, with random variation about the mean explained by some of the factors listed below.

**a** The diameters may be affected by:

- the type of lathe used
- the steadiness of the woodworker's hand
- the operating speed of the lathe.

**b** The scores may be affected by:

- the time spent studying
- natural ability (for example, memory, learning ability)
- general knowledge.

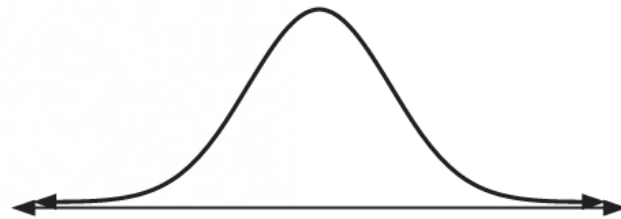
**c** The times may be affected by:

- the distance that the students live from their school
- walking speed
- physical fitness
- the terrain.

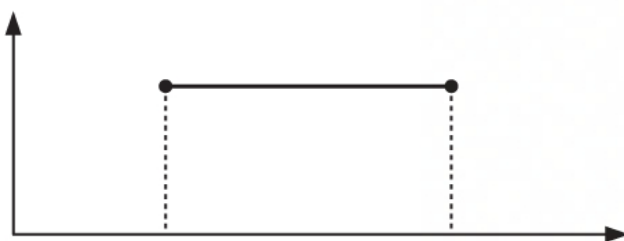
**3 a** The variable is not likely to be normally distributed as it is more likely that there would be more people younger than the mean age than there are older. The distribution may be positively skewed.



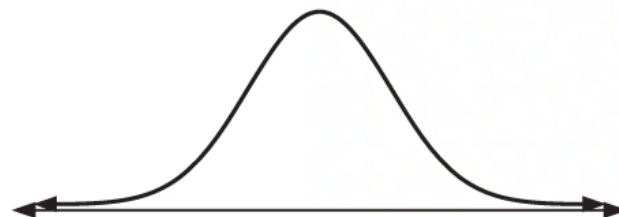
**b** The variable is likely to be normally distributed as the long jumper is likely to jump the same distance consistently, but it will vary due to factors such as the speed at which the long jumper runs before the jump, and the positioning of their body before hitting the sand.



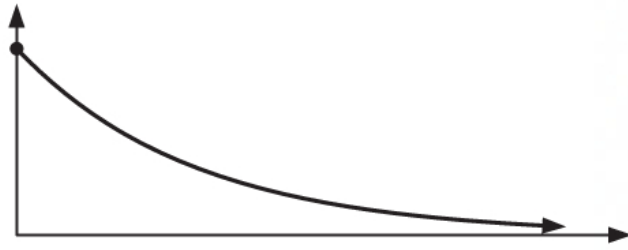
**c** The variable is not likely to be normally distributed as each number has the same chance of being drawn. The distribution should be uniform.



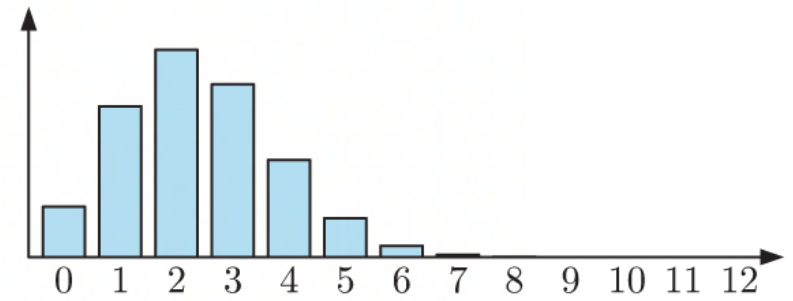
**d** The variable is likely to be normally distributed as the lengths of the carrots will be generally centred around the mean, but will vary due to factors such as soil quality, different weather conditions, harvest times, and so on.



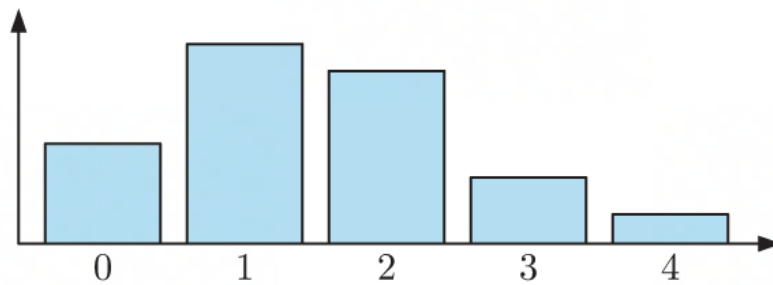
- e** The variable is not likely to be normally distributed. People are most likely to be served quite quickly. The distribution is likely to be negatively skewed.



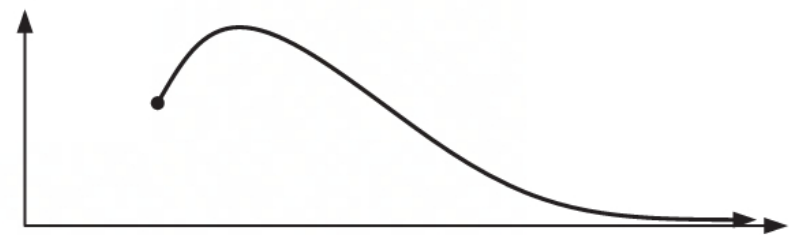
- f** The variable is not likely to be normal as it is a discrete variable. Each egg has the same probability of being brown, so the distribution is binomial.



- g** The variable is not likely to be normally distributed as it is a discrete variable. Most families will have 0 - 2 children, and there will be much fewer families with more than 2 children. The distribution will be positively skewed.



- h** The variable is not likely to be normally distributed as there will tend to be many more shorter buildings than tall buildings in a city. The distribution will be positively skewed.



## INVESTIGATION 2

## PROPERTIES OF THE NORMAL CURVE

- 1 a**  $\mu$  controls the vertical line that the curve is symmetrical about.  $\mu$  represents the mean, so it makes sense that the distribution is symmetrical about this value.  
 $\sigma$  controls the shape of the curve. As  $\sigma$  increases, the curve becomes flatter and more spread out, which is reasonable, since  $\sigma$  is the standard deviation and a measure of spread.
- b** The curve has a vertical line of symmetry  $x = \mu$ .
- c** The function is never negative. This is important because a probability density function can never be negative.
- d** As  $x \rightarrow \pm\infty$ , the curve approaches zero from above. The  $x$ -axis is a horizontal asymptote.
- e** The area under the curve should remain constant as we change  $\mu$  and  $\sigma$ , as the area under a probability density function must be 1.



**2 a** 
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$\begin{aligned}\therefore f'(x) &= \frac{1}{\sigma\sqrt{2\pi}} \times -\left(\frac{x-\mu}{\sigma}\right) \times \frac{1}{\sigma} \times e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \\ &= \frac{\mu-x}{\sigma^3\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \\ &= \frac{\mu-x}{\sigma^2} f(x)\end{aligned}$$

and  $f'(x) = 0$  when  $x = \mu$   $\{f(x) > 0\}$

So,  $f'(x)$  has sign diagram:

$\therefore$  the stationary point is  $(\mu, f(\mu))$  or  $\left(\mu, \frac{1}{\sigma\sqrt{2\pi}}\right)$  which is a maximum.

**b** 
$$f'(x) = \frac{\mu-x}{\sigma^2} f(x)$$

$$\begin{aligned}\therefore f''(x) &= -\frac{1}{\sigma^2} f(x) + \frac{\mu-x}{\sigma^2} f'(x) && \{\text{product rule}\} \\ &= -\frac{1}{\sigma^2} f(x) + \frac{\mu-x}{\sigma^2} \left(\frac{\mu-x}{\sigma^2}\right) f(x) && \{\text{from a}\} \\ &= -\frac{1}{\sigma^2} f(x) + \frac{(\mu-x)^2}{\sigma^4} f(x) \\ &= f(x) \left[ \frac{(\mu-x)^2}{\sigma^4} - \frac{1}{\sigma^2} \right]\end{aligned}$$

$$f''(x) = 0 \text{ when } \frac{(\mu-x)^2}{\sigma^4} - \frac{1}{\sigma^2} = 0 \quad \{f(x) > 0\}$$

$$\therefore \frac{(\mu-x)^2 - \sigma^2}{\sigma^4} = 0$$

$$\therefore (\mu-x)^2 - \sigma^2 = 0$$

$$\therefore (\mu-x)^2 = \sigma^2$$

$$\therefore \mu-x = \pm\sigma$$

$$\therefore x = \mu \pm \sigma$$

$\therefore$  the inflection points occur at  $x = \mu \pm \sigma$

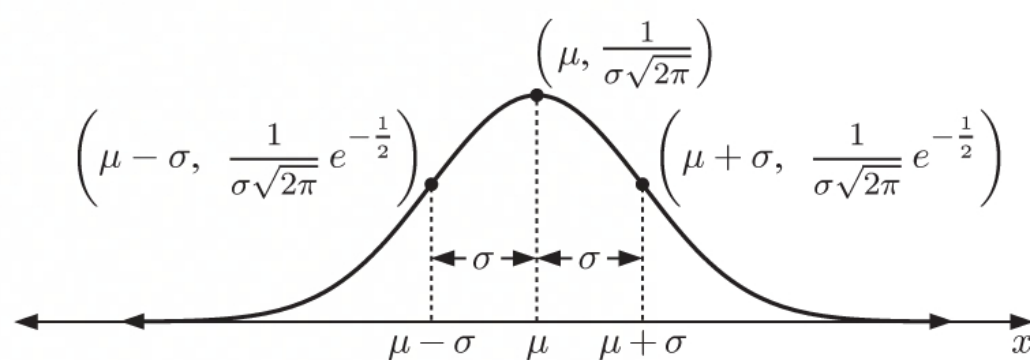
From **a**, the only stationary point is at  $x = \mu$ , so these are non-stationary inflection points.

$$\text{Now, } f(\mu + \sigma) = f(\mu - \sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}}$$

$\therefore$  the non-stationary inflection points are  $\left(\mu - \sigma, \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}}\right)$  and  $\left(\mu + \sigma, \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}}\right)$ .

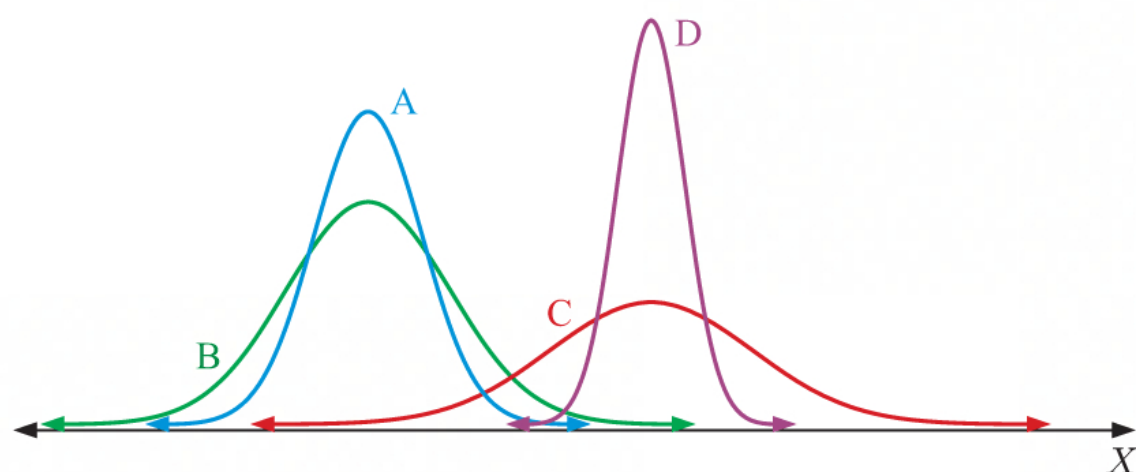


3



## EXERCISE 28C.2

1



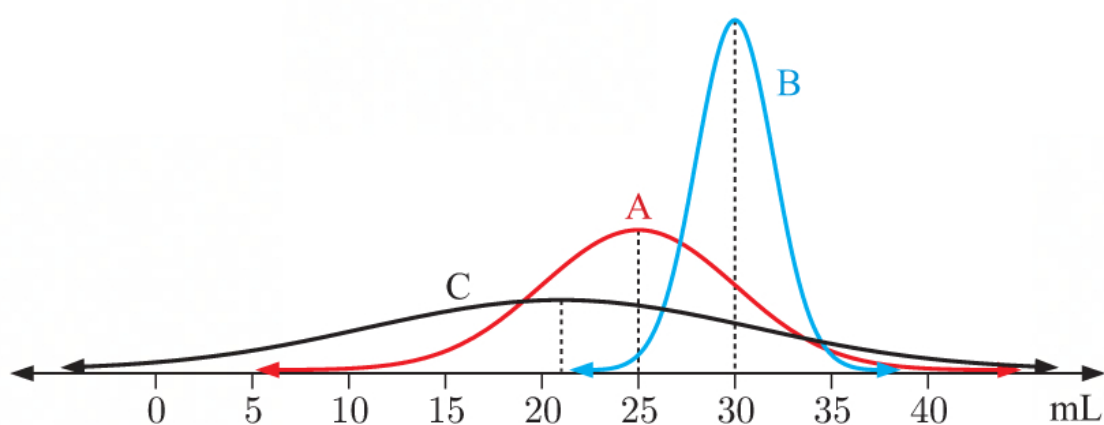
A and B both have  $\mu = 5$ , C and D both have  $\mu = 15$ .  
 B has a greater spread, and hence a larger standard deviation than A.  
 Similarly, C has a larger standard deviation than D.

- a**  $\mu = 5$ ,  $\sigma = 2$  corresponds to B  
**c**  $\mu = 5$ ,  $\sigma = 1$  corresponds to A

- b**  $\mu = 15$ ,  $\sigma = 0.5$  corresponds to D  
**d**  $\mu = 15$ ,  $\sigma = 3$  corresponds to C

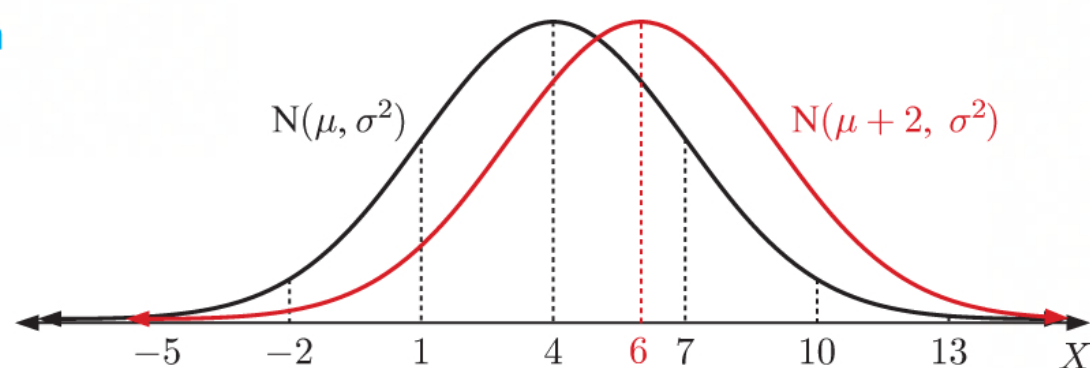
2

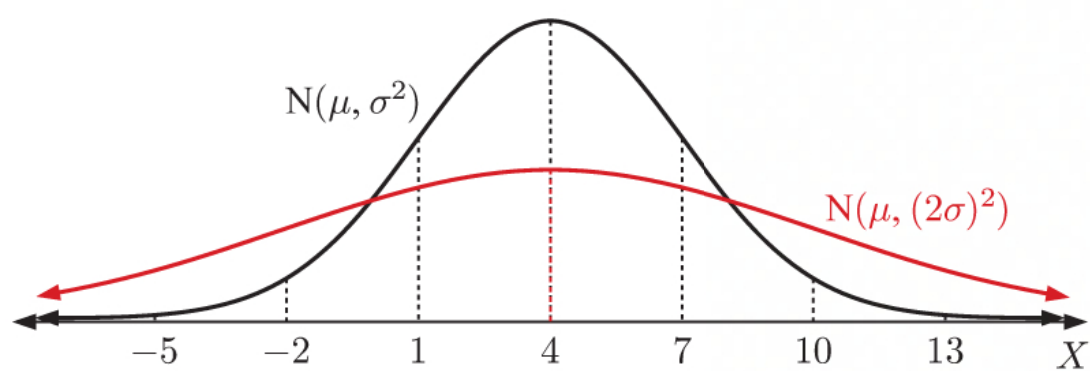
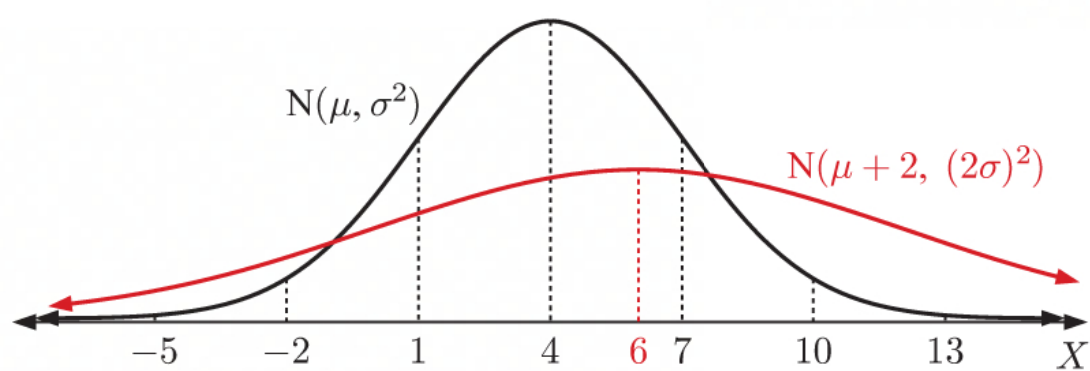
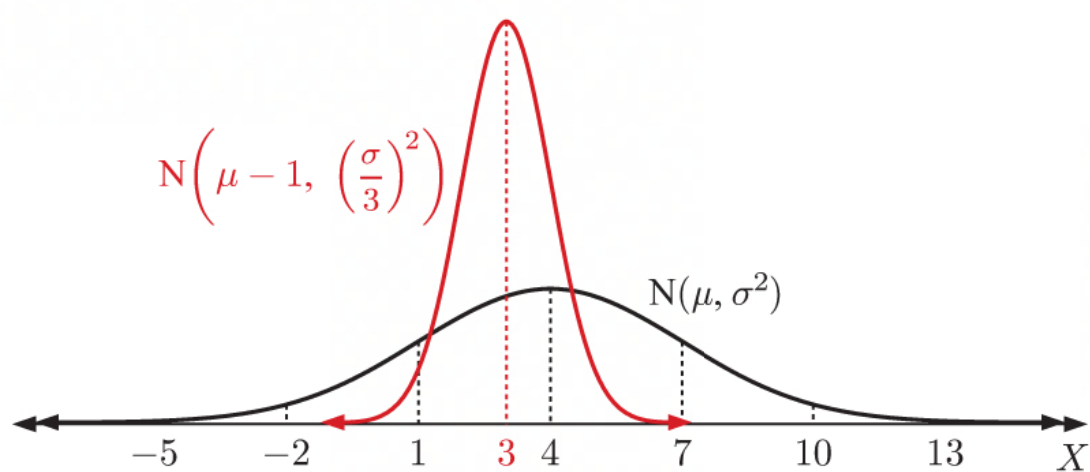
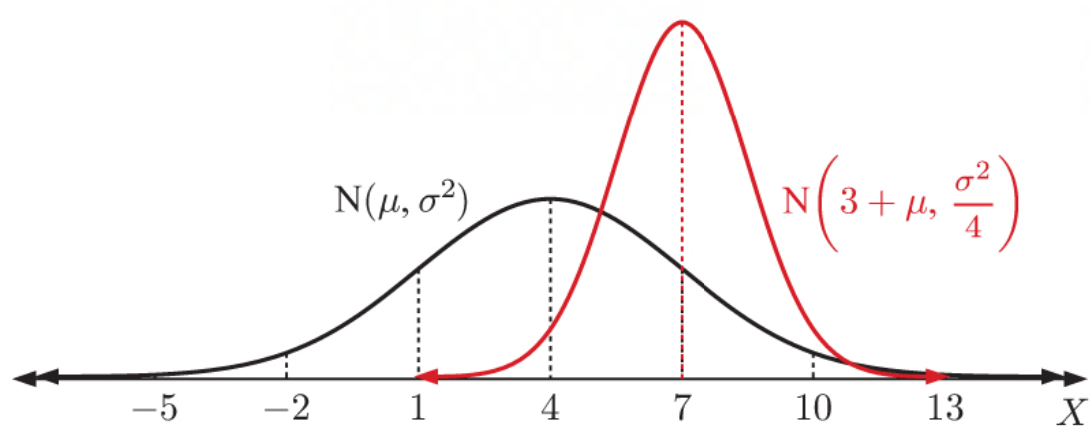
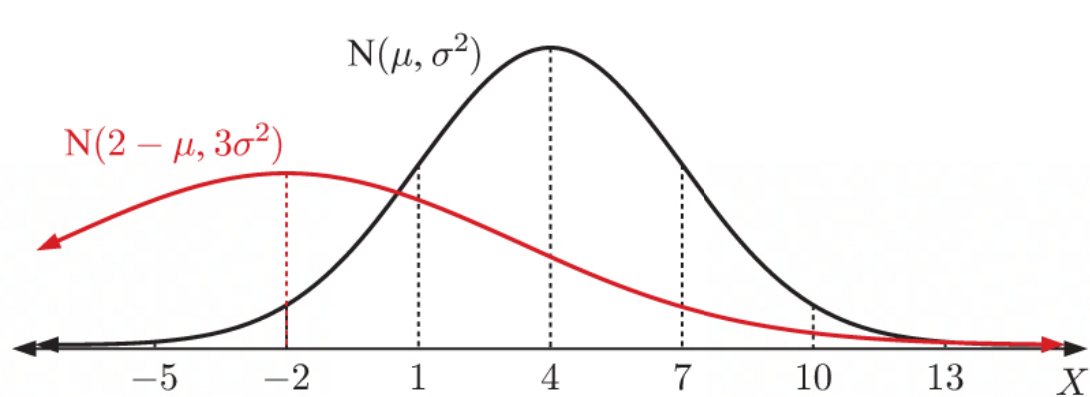
Distribution	Mean (mL)	Standard deviation (mL)
A	25	5
B	30	2
C	21	10



3

a



**b**

**c**

**d**

**e**

**f**


$$4 \quad \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\left(\frac{x-\mu}{\sqrt{2}\sigma}\right)^2} dx$$

$$\text{Let } z = \frac{x-\mu}{\sqrt{2}\sigma} \quad \therefore \quad \frac{dz}{dx} = \frac{1}{\sqrt{2}\sigma}$$

$$\text{As } x \rightarrow \infty, \quad z \rightarrow \infty$$

$$\text{As } x \rightarrow -\infty, \quad z \rightarrow -\infty$$

$$\begin{aligned} \therefore \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-z^2} \times \frac{dz}{dx} dx \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-z^2} dz \\ &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-z^2} dz \\ &= \frac{1}{\sqrt{\pi}} \times \sqrt{\pi} \quad \{\text{Laplace}\} \\ &= 1 \end{aligned}$$

$$5 \quad \text{a} \quad \text{The Maclaurin series of } e^{tX} = \sum_{k=0}^{\infty} \frac{(tX)^k}{k!}.$$

Suppose  $X$  has probability density function  $f(x)$  with domain  $a \leq x \leq b$ .

$$\therefore M_X(t) = E(e^{tX})$$

$$\begin{aligned} &= E\left(\sum_{k=0}^{\infty} \frac{(tX)^k}{k!}\right) \\ &= \sum_{k=0}^{\infty} E\left(\frac{t^k X^k}{k!}\right) \\ &= \sum_{k=0}^{\infty} \frac{t^k}{k!} E(X^k) \end{aligned}$$

$$\text{b} \quad \text{i} \quad M_X(t) = E(X^0) + \sum_{k=1}^{\infty} \frac{t^k}{k!} E(X^k)$$

$$\begin{aligned} \therefore M'_X(t) &= \sum_{k=1}^{\infty} \frac{kt^{k-1}}{k!} E(X^k) \\ &= \sum_{k=1}^{\infty} \frac{t^{k-1}}{(k-1)!} E(X^k) \\ &= E(X^1) + \sum_{k=2}^{\infty} \frac{t^{k-1}}{(k-1)!} E(X^k) \end{aligned}$$

$$\therefore M'_X(0) = E(X)$$

$$\begin{aligned}
\text{ii} \quad M_X''(t) &= \sum_{k=2}^{\infty} \frac{(k-1)t^{k-2}}{(k-1)!} E(X^k) \\
&= \sum_{k=2}^{\infty} \frac{t^{k-2}}{(k-2)!} E(X^k) \\
&= E(X^2) + \sum_{k=3}^{\infty} \frac{t^{k-2}}{(k-2)!} E(X^k) \\
\therefore M_X''(0) &= E(X^2)
\end{aligned}$$

$$\text{c} \quad P_n \text{ is: } M_X^{(n)}(t) = E(X^n) + \sum_{k=n+1}^{\infty} \frac{t^{k-n}}{(k-n)!} E(X^k) \quad \text{for all } n \in \mathbb{Z}^+.$$

**Proof:** (By the principle of mathematical induction)

(1)  $P_1$  is true from **b i**.

$$(2) \quad \text{If } P_j \text{ is true, then } M_X^{(j)}(t) = E(X^j) + \sum_{k=j+1}^{\infty} \frac{t^{k-j}}{(k-j)!} E(X^k)$$

$$\begin{aligned}
\therefore M_X^{(j+1)}(t) &= 0 + \sum_{k=j+1}^{\infty} \frac{(k-j)t^{k-(j+1)}}{(k-j)!} E(X^k) \\
&= \sum_{k=j+1}^{\infty} \frac{t^{k-(j+1)}}{(k-(j+1))!} E(X^k) \\
&= E(X^{j+1}) + \sum_{k=j+2}^{\infty} \frac{t^{k-(j+1)}}{(k-(j+1))!} E(X^k)
\end{aligned}$$

$\therefore P_{j+1}$  is true.

Since  $P_1$  is true, and  $P_{j+1}$  is true whenever  $P_j$  is true,

$P_n$  is true for all  $n \in \mathbb{Z}^+$ . {principle of mathematical induction}

$$\therefore M_X^{(n)}(t) = E(X^n) + \sum_{k=n+1}^{\infty} \frac{t^{k-n}}{(k-n)!} E(X^k) \quad \text{for all } n \in \mathbb{Z}^+.$$

$$\therefore M_X^{(n)}(0) = E(X^n) \quad \text{for all } n \in \mathbb{Z}^+.$$



**d i** If  $X \sim N(\mu, \sigma^2)$ , the probability density function of  $X$  is  $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ .

$$\therefore M_X(t) = E(e^{tX})$$

$$\begin{aligned} &= \int_{-\infty}^{\infty} e^{tx} \times \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx \\ &= \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{tx - \frac{1}{2\sigma^2}(x-\mu)^2} dx \\ &= \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x^2 - 2\mu x - 2t\sigma^2 x + \mu^2)} dx \\ &= \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x^2 - 2x(\mu + t\sigma^2) + \mu^2)} dx \\ &= \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}[(x - (\mu + t\sigma^2))^2 + \mu^2 - (\mu + t\sigma^2)^2]} dx \quad \{\text{completing the square}\} \\ &= \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}[(x - (\mu + t\sigma^2))^2 + \cancel{\mu^2} - \cancel{\mu^2} - 2\mu t\sigma^2 - t^2\sigma^4]} dx \\ &= \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x - (\mu + t\sigma^2)}{\sigma}\right)^2 + \mu t + \frac{1}{2}t^2\sigma^2} dx \\ &= e^{\mu t + \frac{1}{2}t^2\sigma^2} \int_{-\infty}^{\infty} \underbrace{\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x - (\mu + t\sigma^2)}{\sigma}\right)^2}}_{\text{probability density function of } N(\mu + t\sigma^2, \sigma^2)} dx \\ &= e^{\mu t + \frac{1}{2}t^2\sigma^2} \times 1 \\ &= e^{\mu t + \frac{1}{2}\sigma^2 t^2} \end{aligned}$$

**ii**  $M'_X(t) = (\mu + \sigma^2 t)e^{\mu t + \frac{1}{2}t^2\sigma^2}$

$$\therefore M'_X(0) = \mu e^0 = \mu$$

$$\therefore E(X) = M'_X(0) = \mu \quad \{\text{from b i}\}$$

$$M''_X(t) = \sigma^2 e^{\mu t + \frac{1}{2}\sigma^2 t^2} + (\mu + \sigma^2 t)^2 e^{\mu t + \frac{1}{2}\sigma^2 t^2} \quad \{\text{product rule}\}$$

$$= (\mu^2 + \sigma^2 + 2\mu\sigma^2 t + \sigma^4 t^2) e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

$$\therefore M''_X(0) = (\mu^2 + \sigma^2) e^0 = \mu^2 + \sigma^2$$

$$\therefore E(X^2) = M''_X(0) = \mu^2 + \sigma^2 \quad \{\text{from b ii}\}$$

$$\therefore \text{Var}(X) = E(X^2) - [E(X)]^2$$

$$= \mu^2 + \sigma^2 - \mu^2$$

$$= \sigma^2$$

**EXERCISE 28D.1**

1  $X \sim N(30, 5^2)$

a i The value which is 2 standard deviations above the mean  $= 30 + 2 \times 5$   
 $= 40$

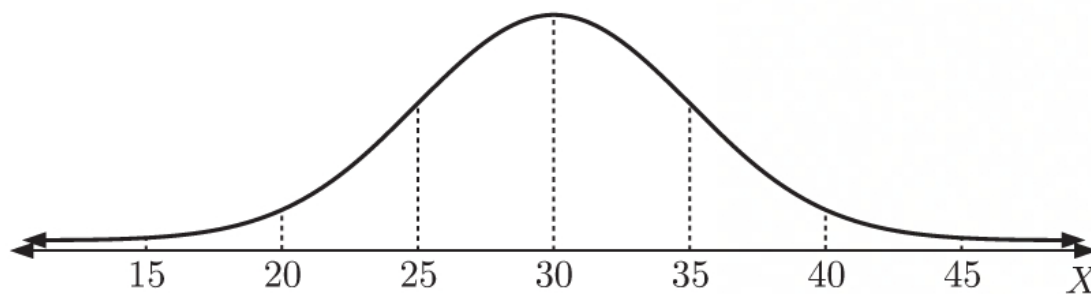
ii The value which is 1 standard deviation below the mean  $= 30 - 5$   
 $= 25$

b i  $35 = 30 + 5$   
 $\therefore 35$  is 1 standard deviation above the mean.

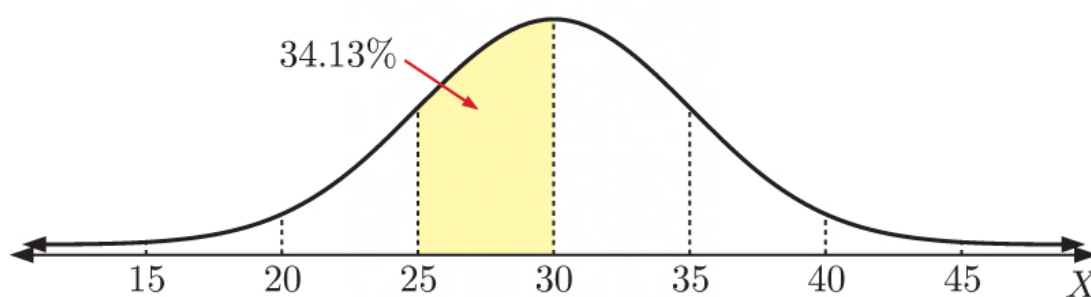
ii  $20 = 30 - 2 \times 5$   
 $\therefore 20$  is 2 standard deviations below the mean.

iii  $45 = 30 + 3 \times 5$   
 $\therefore 45$  is 3 standard deviations above the mean.

c

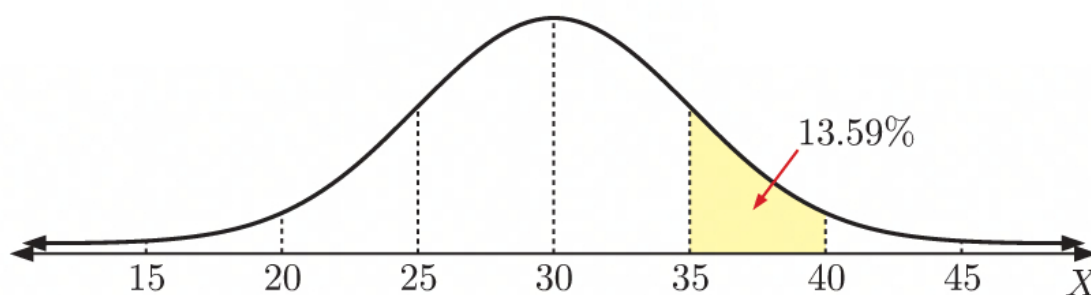


d



About 34.13% of the values of  $X$  are between 25 and 30.

e

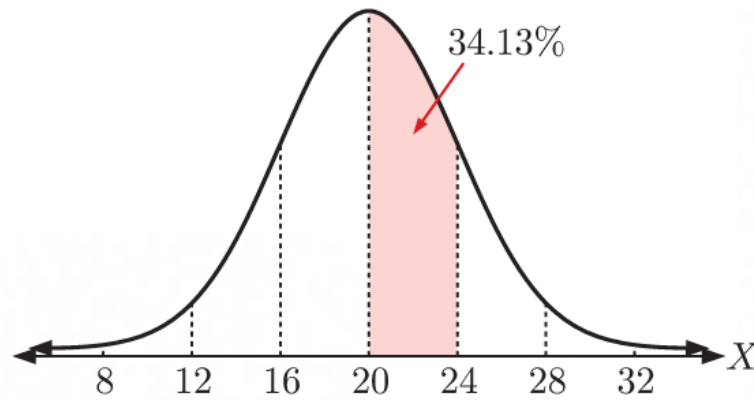


About 13.59% of the values of  $X$  are between 35 and 40.

$\therefore$  the probability that a randomly selected member of the population will measure between 35 and 40 is approximately 0.1359.

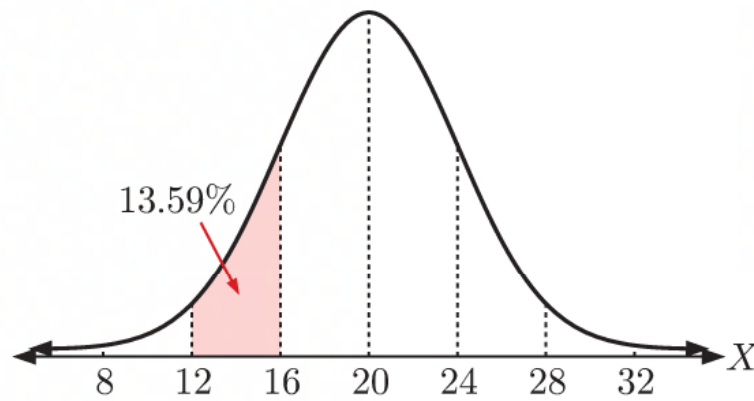
**2 a**  $\mu = 20, \sigma = 4$

**b i**



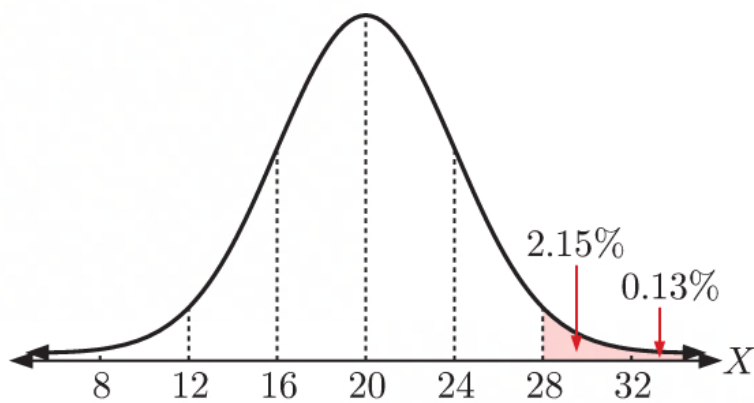
About 34.13% of  $X$  values are between 20 and 24.

**ii**



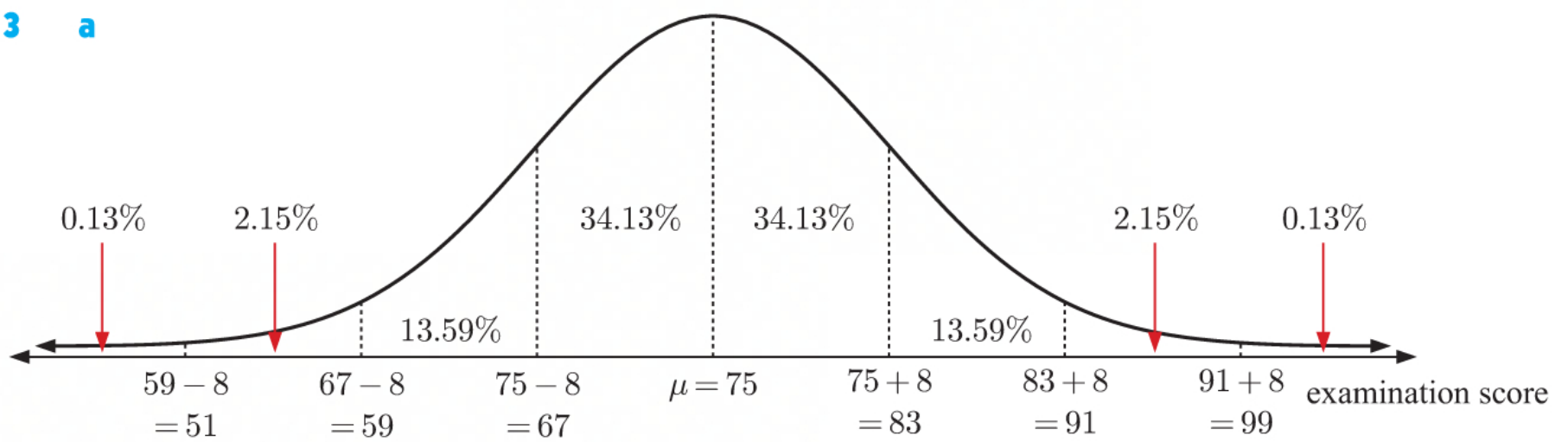
About 13.59% of  $X$  values are between 12 and 16.

**iii**

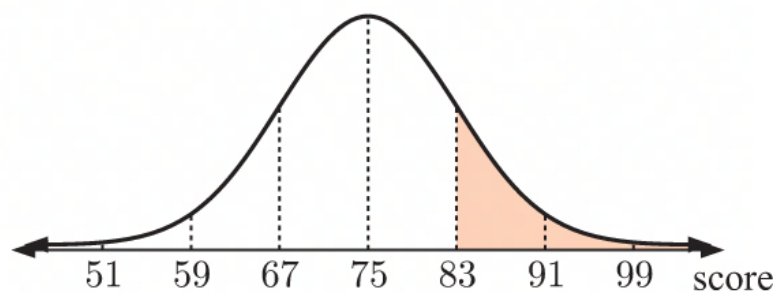


About  $2.15\% + 0.13\% = 2.28\%$  of  $X$  values are greater than 28.

**3 a**



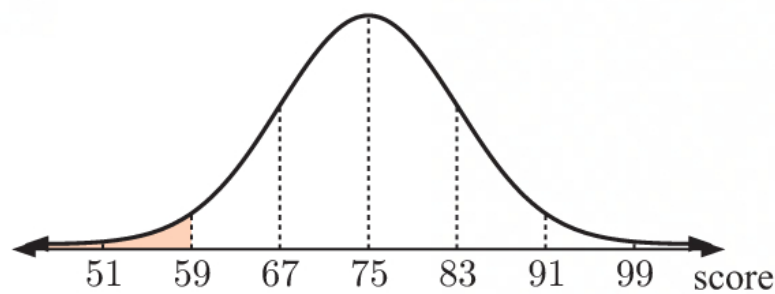
**b i**



About  $13.59\% + 2.15\% + 0.13\%$   
 $= 15.87\%$

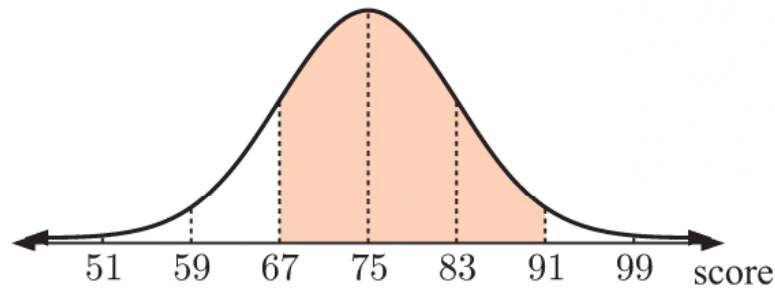
of students would be expected to have scored more than 83.

ii



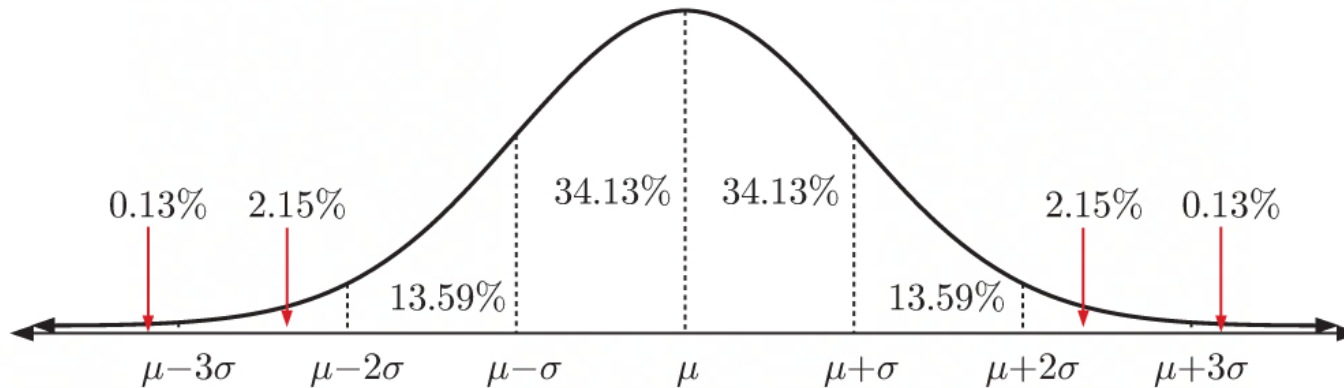
About  $2.15\% + 0.13\% = 2.28\%$  of students would be expected to have scored less than 59.

iii



About  $34.13\% + 34.13\% + 13.59\% = 81.85\%$  of students would be expected to have scored between 67 and 91.

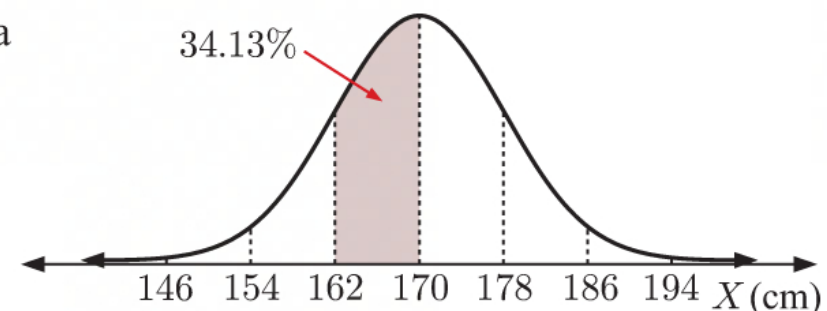
4



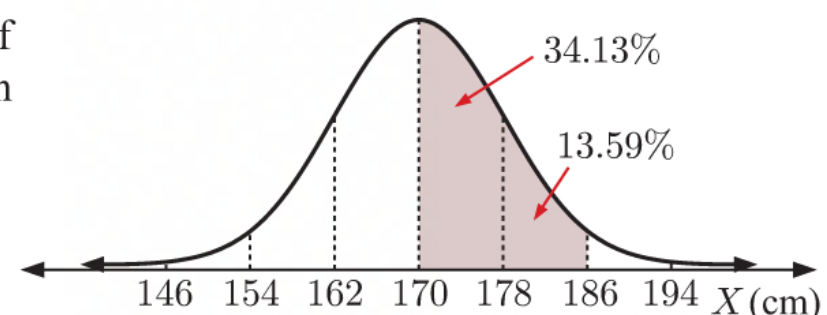
**a**  $P(\text{value between } \mu - \sigma \text{ and } \mu + \sigma)$   
 $\approx 0.3413 + 0.3413$   
 $\approx 0.6826$

**b**  $P(\text{value} > \mu + 2\sigma)$   
 $\approx 0.0215 + 0.0013$   
 $\approx 0.0228$

**5 a i** About 34.13% of female students have a height between 162 cm and 170 cm.

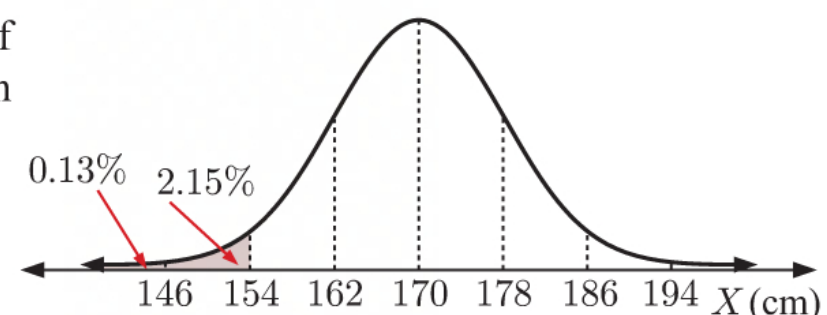


**ii** About  $34.13\% + 13.59\% = 47.72\%$  of female students have a height between 170 cm and 186 cm.



**b i** About  $2.15\% + 0.13\% = 2.28\%$  of female students have a height less than 154 cm.

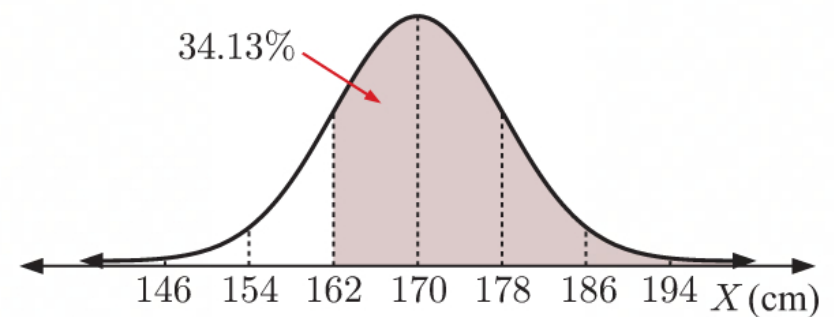
$\therefore P(\text{height is less than 154 cm})$   
 $\approx 0.0228$



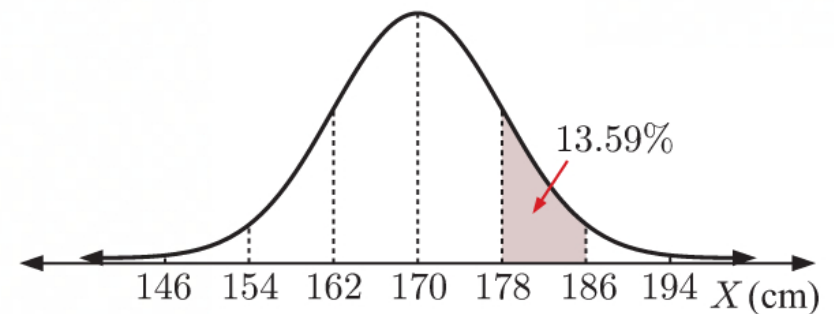


- ii About  $34.13\% + 50\% = 84.13\%$  of female students have a height greater than 162 cm.

$$\therefore P(\text{height is greater than } 162 \text{ cm}) \\ \approx 0.8413$$



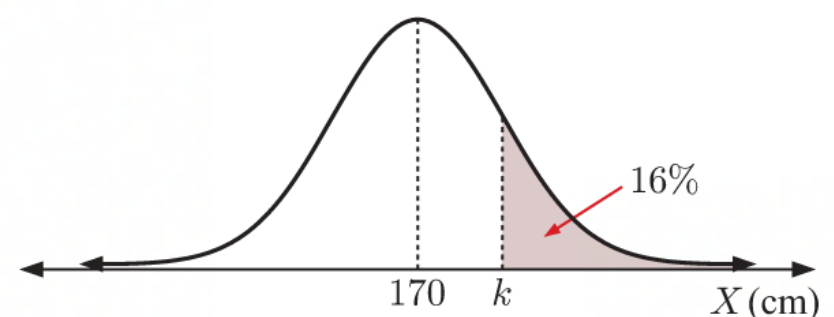
- c About 13.59% of the female students have a height between 178 cm and 186 cm. So, we would expect about 13.59% of  $500 \approx 68$  students to have a height between 178 cm and 186 cm.



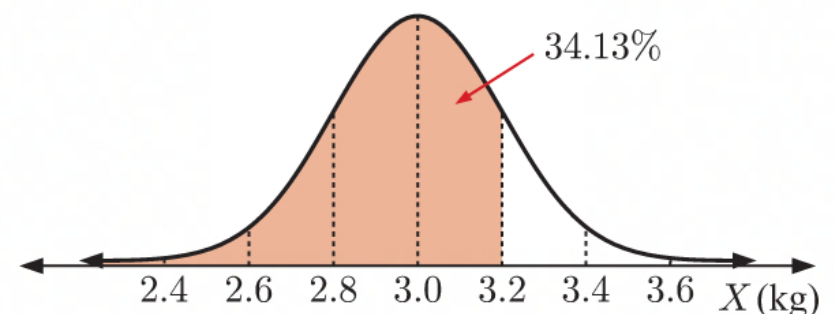
- d Approximately 16% of data lies more than one standard deviation above the mean.

$\therefore k$  is about  $\sigma$  above the mean  $\mu$

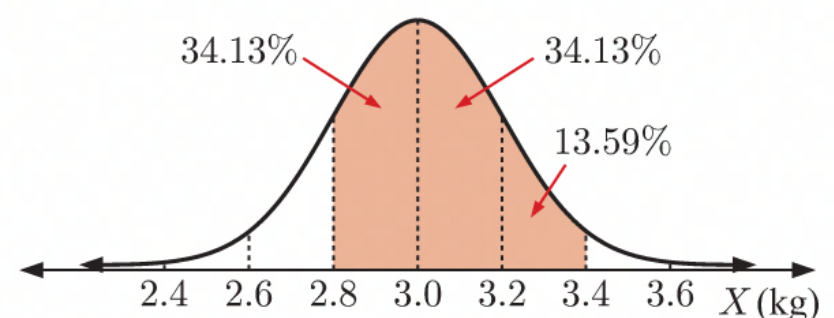
$$\therefore k \approx 170 + 8 \\ \approx 178$$



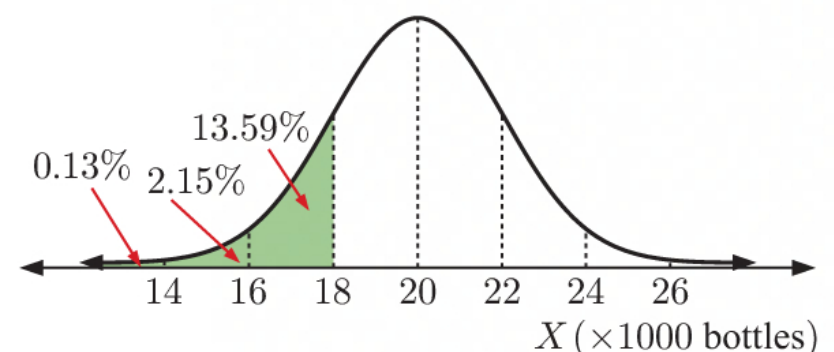
- 6 a About  $50\% + 34.13\% = 84.13\%$  of babies born weighed less than 3.2 kg. So, about  $84.13\% \times 545 \approx 459$  babies born weighed less than 3.2 kg.



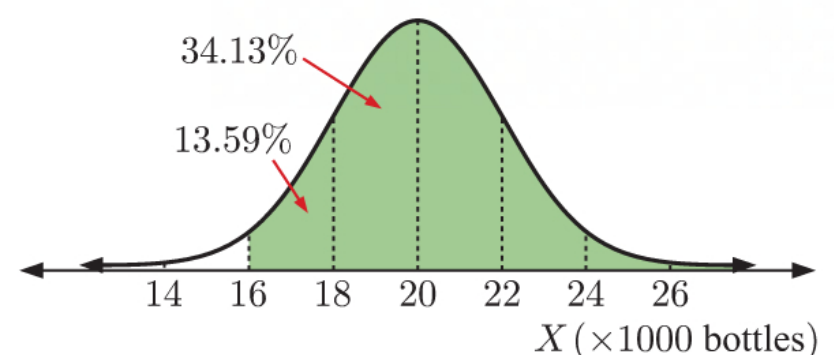
- b About  $34.13\% + 34.13\% + 13.59\% = 81.85\%$  of babies born weighed between 2.8 kg and 3.4 kg. So, about  $81.85\% \times 545 \approx 446$  babies born weighed between 2.8 kg and 3.4 kg.



- 7 a Under 18 000 bottles are filled on about  $0.13\% + 2.15\% + 13.59\% = 15.87\%$  of days. So, under 18 000 bottles are filled on about  $15.87\% \times 260 \approx 41$  days.

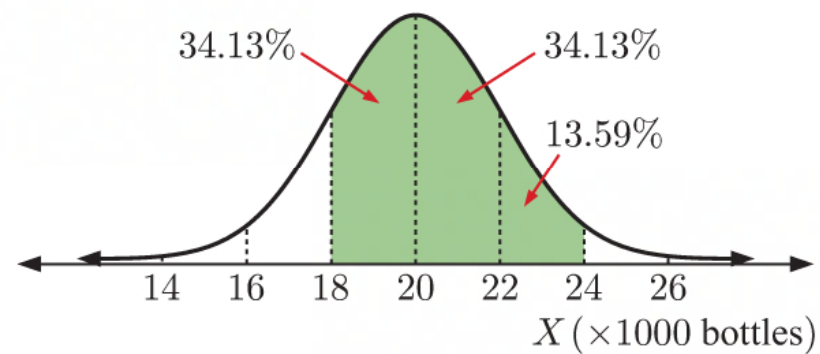


- b Over 16 000 bottles are filled on about  $13.59\% + 34.13\% + 50\% = 97.72\%$  of days. So, over 16 000 bottles are filled on about  $97.72\% \times 260 \approx 254$  days.



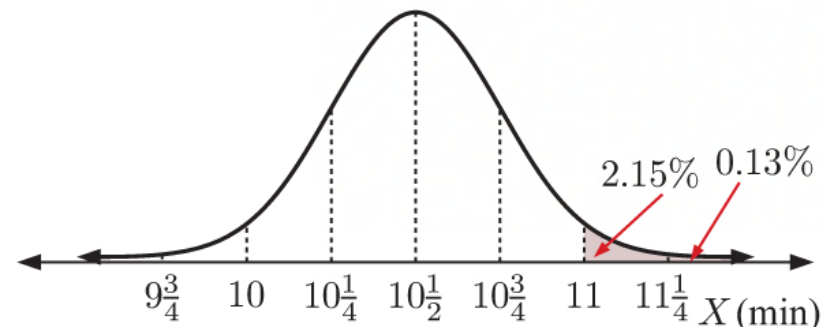
- c** Between 18 000 and 24 000 bottles are filled on about  $34.13\% + 34.13\% + 13.59\%$   
 $= 81.85\%$  of days.

So, between 18 000 and 24 000 bottles are filled on about  $81.85\% \times 260 \approx 213$  days.



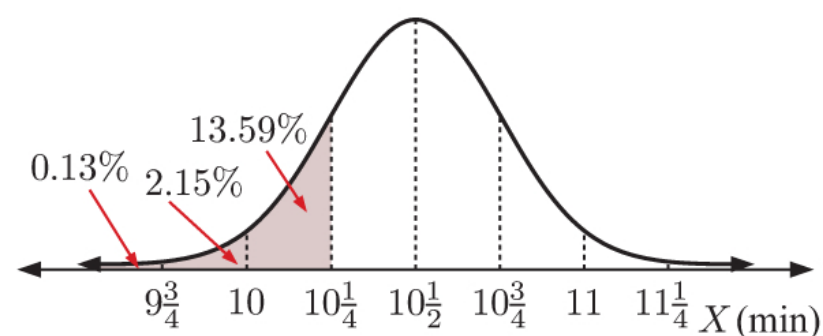
- 8 a** About  $2.15\% + 0.13\% = 2.28\%$  of competitors completed the race in a time longer than 11 minutes.

So, about  $2.28\% \times 200 \approx 5$  competitors completed the race in a time longer than 11 minutes.



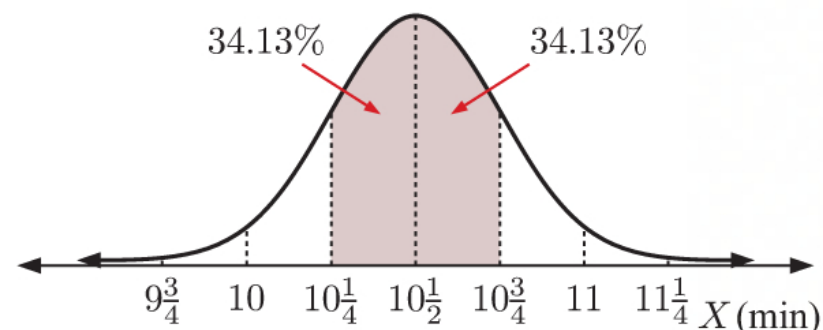
- b** About  $0.13\% + 2.15\% + 13.59\% = 15.87\%$  of competitors completed the race in a time less than 10 minutes 15 seconds.

So, about  $15.87\% \times 200 \approx 32$  competitors completed the race in a time less than 10 minutes 15 seconds.



- c** About  $34.13\% + 34.13\% = 68.26\%$  of competitors completed the race in a time between 10 minutes 15 seconds and 10 minutes 45 seconds.

So, about  $68.26\% \times 200 \approx 137$  competitors completed the race in a time between 10 minutes 15 seconds and 10 minutes 45 seconds.



- 9 a** Approximately 84% of data is more than one standard deviation below the mean.

$\therefore$  152 grams is about  $\sigma$  below the mean  $\mu$

$$\therefore \mu \approx 152 + \sigma \quad \dots (1)$$

Approximately 16% of data is more than one standard deviation above the mean.

$\therefore$  200 grams is  $\sigma$  above the mean  $\mu$

$$\therefore \mu \approx 200 - \sigma \quad \dots (2)$$

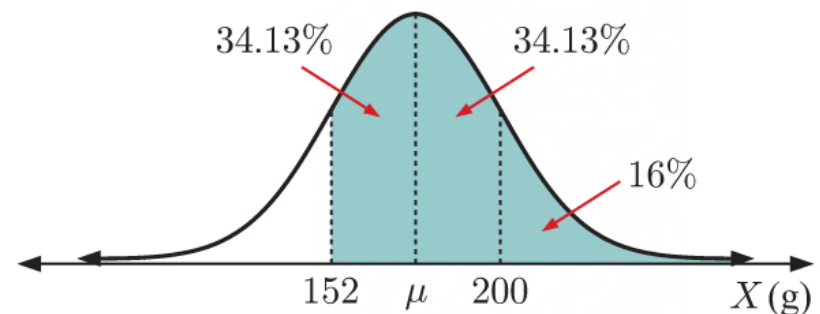
Equating (1) and (2) gives:  $152 + \sigma \approx 200 - \sigma$

$$\therefore 2\sigma \approx 48$$

$$\therefore \sigma \approx 24$$

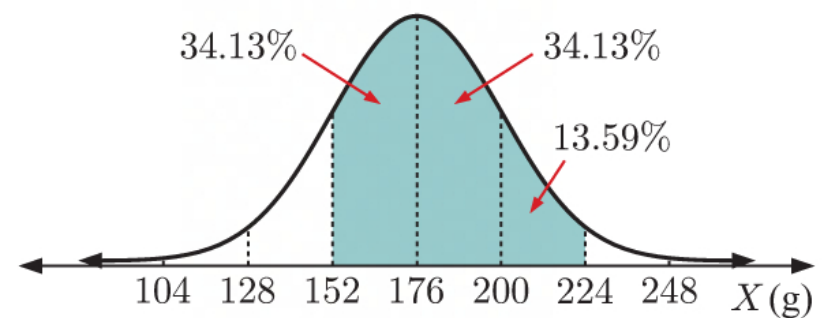
$$\text{and } \mu \approx 200 - 24 \quad \{\text{using (2)}\} \\ \approx 176$$

So,  $\mu \approx 176$  grams and  $\sigma \approx 24$  grams.

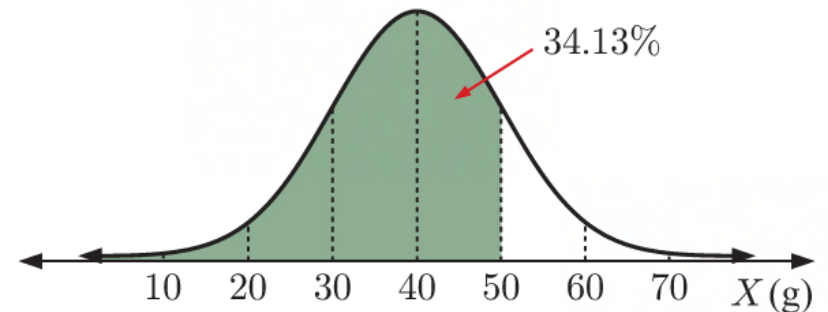




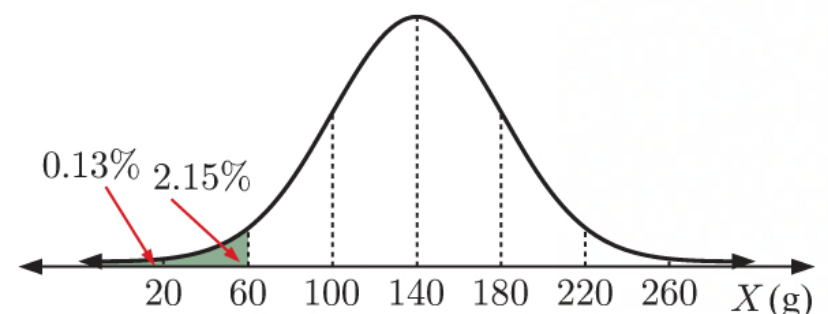
- b** About  $34.13\% + 34.13\% + 13.59\%$   
 $= 81.85\%$  of the oranges weigh  
 between 152 grams and 224 grams.



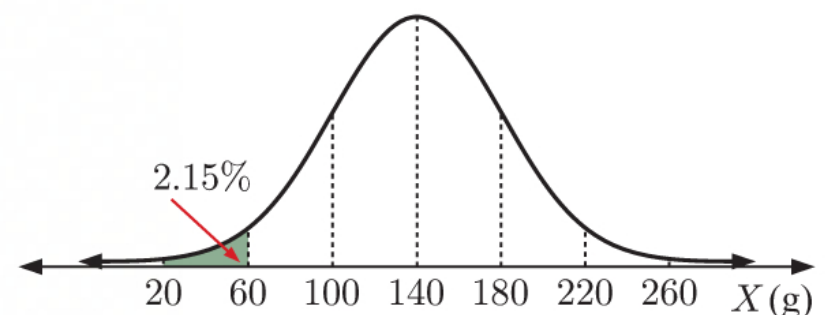
- 10 a i** About  $50\% + 34.13\% = 84.13\%$  of  
 radishes grown without fertiliser will  
 have weights less than 50 grams.



- ii** About  $0.13\% + 2.15\% = 2.28\%$  of  
 radishes grown with fertiliser will have  
 weights less than 60 grams.

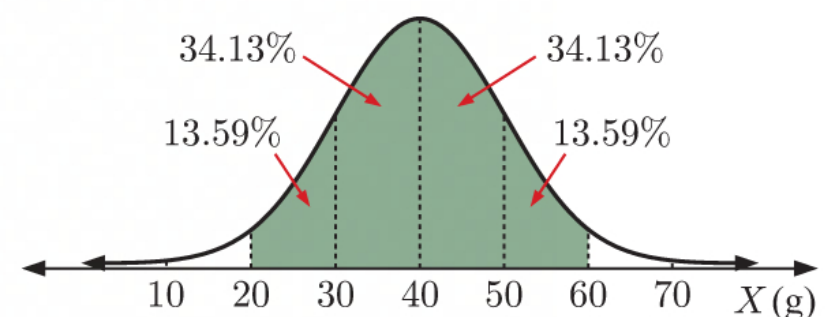


- b i** About 2.15% of radishes grown with  
 fertiliser will have weights between 20 g  
 and 60 g.



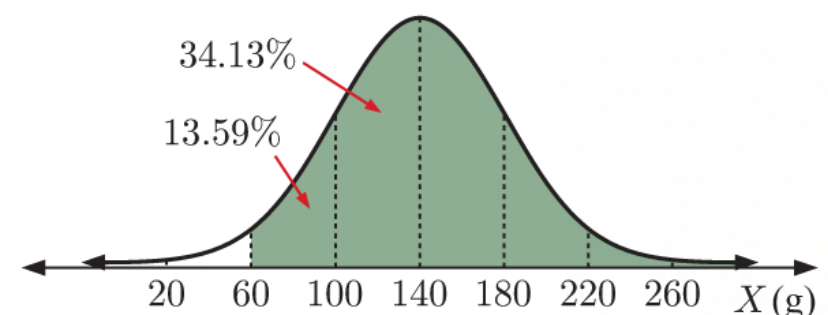
$\therefore P(\text{radish grown with fertiliser weighs between 20 g and 60 g}) \approx 0.0215$

- ii** About  
 $13.59\% + 34.13\% + 34.13\% + 13.59\%$   
 $= 95.44\%$   
 of radishes grown without fertiliser will  
 have weights between 20 g and 60 g.



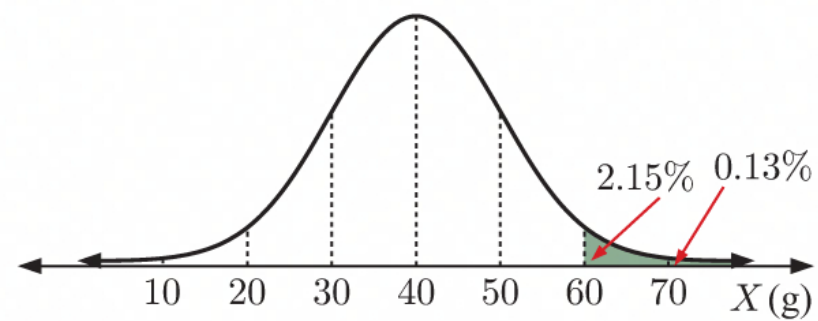
$\therefore P(\text{radish grown without fertiliser weighs between 20 g and 60 g}) \approx 0.9544$

- c** About  $13.59\% + 34.13\% + 50\% = 97.72\%$   
 of radishes grown with fertiliser will have  
 weights more than 60 g.



$\therefore P(\text{radish grown with fertiliser weighs more than 60 g}) \approx 0.9772$

About  $2.15\% + 0.13\% = 2.28\%$  of radishes grown without fertiliser will have weights more than 60 g.

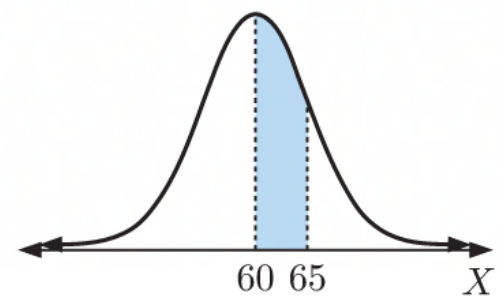
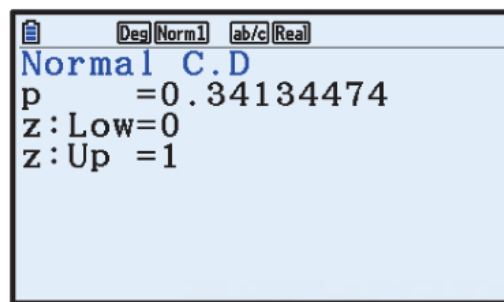
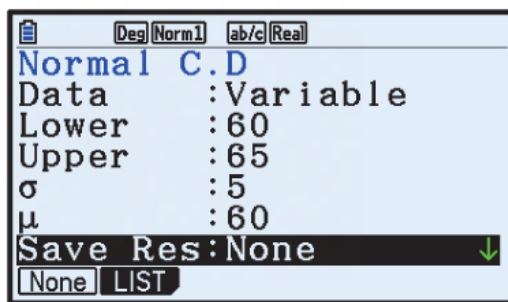


$\therefore P(\text{radish grown without fertiliser weighs more than 60 g}) \approx 0.0228$

$$\begin{aligned}
 &P(\text{both radishes weigh more than 60 g}) \\
 &= P(\text{radish grown with fertiliser weighs more than 60 g}) \\
 &\quad \times P(\text{radish grown without fertiliser weighs more than 60 g}) \\
 &\approx 0.9772 \times 0.0228 \\
 &\approx 0.0223
 \end{aligned}$$

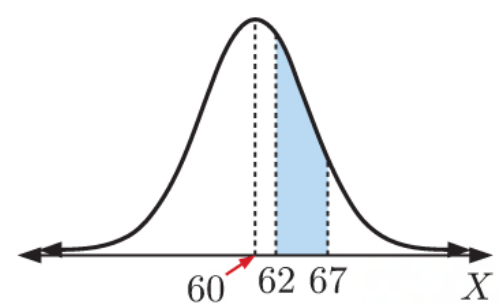
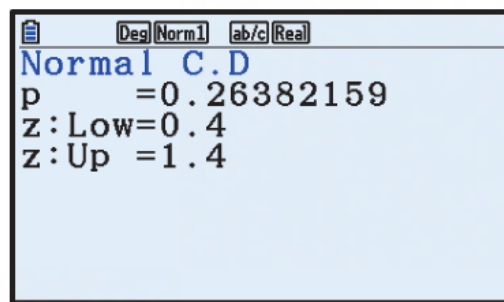
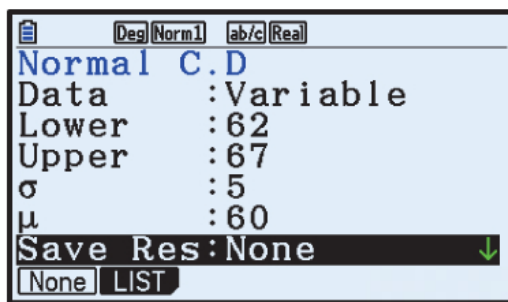
## EXERCISE 28D.2

- 1 a To find  $P(60 \leq X \leq 65)$ , we set the lower bound to 60 and the upper bound to 65.



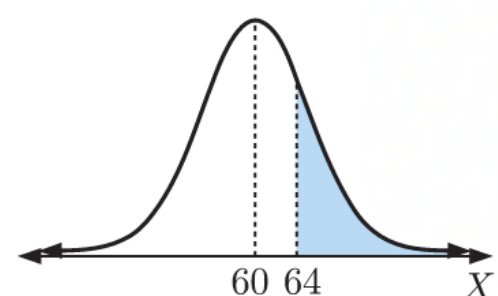
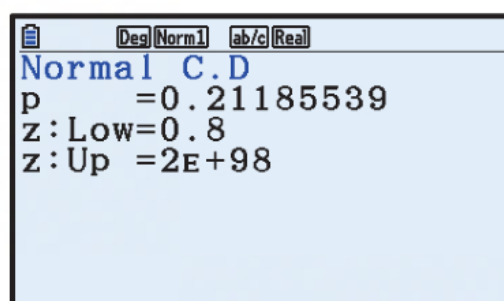
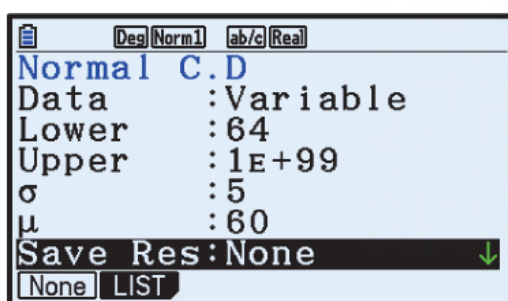
$$P(60 \leq X \leq 65) \approx 0.341$$

- b To find  $P(62 \leq X \leq 67)$ , we set the lower bound to 62 and the upper bound to 67.



$$P(62 \leq X \leq 67) \approx 0.264$$

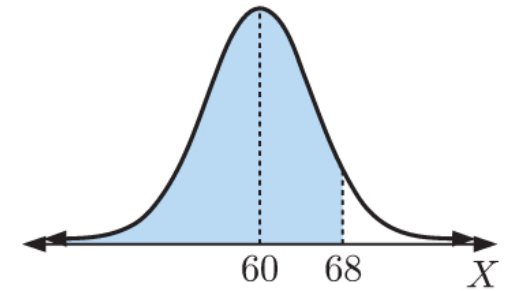
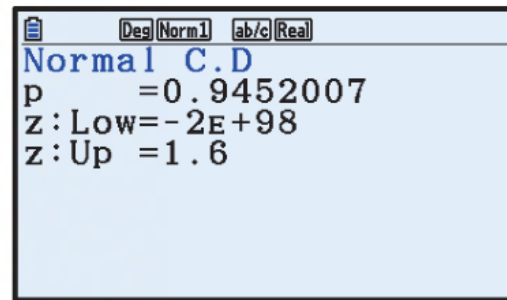
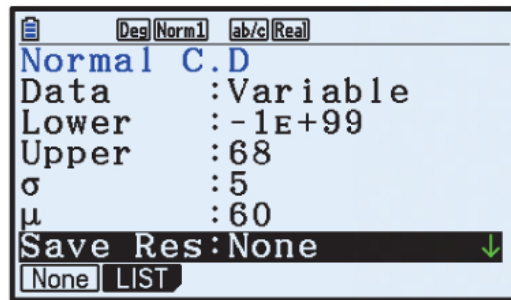
- c To find  $P(X \geq 64)$ , we use a very high value such as  $10^{99}$  to represent the upper bound.



$$P(X \geq 64) \approx 0.212$$

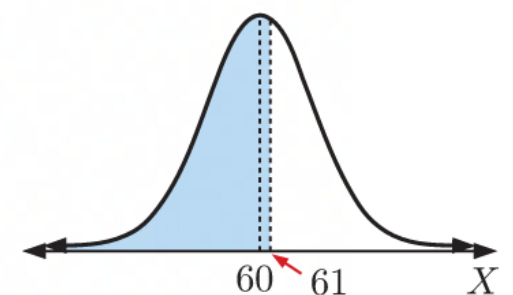
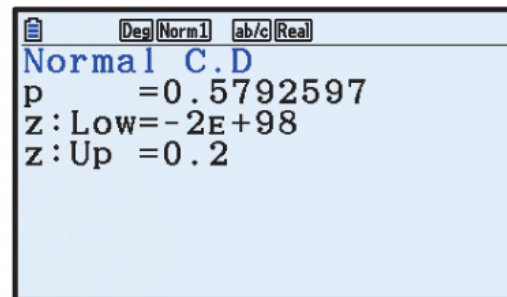
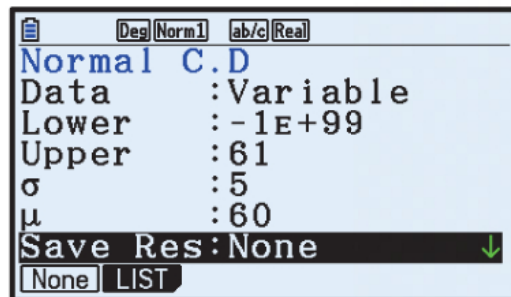


- d To find  $P(X \leq 68)$ , we use a very low value such as  $-10^{99}$  to represent the lower bound.



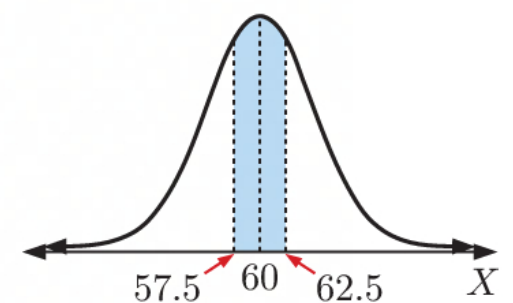
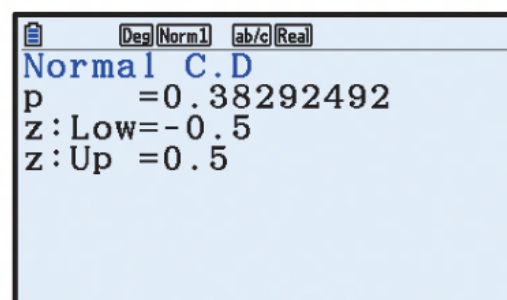
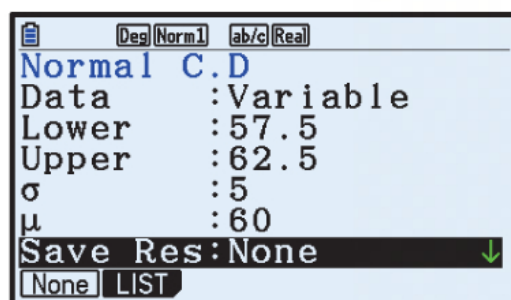
$$P(X \leq 68) \approx 0.945$$

- e To find  $P(X \leq 61)$ , we use a very low value such as  $-10^{99}$  to represent the lower bound.



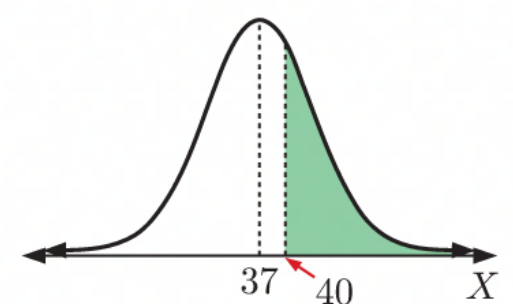
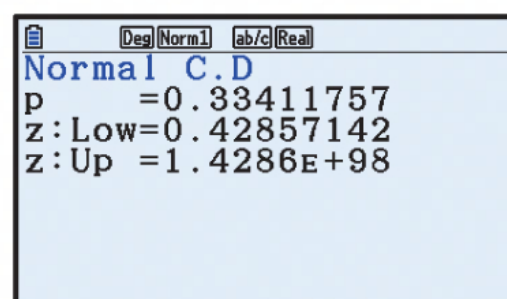
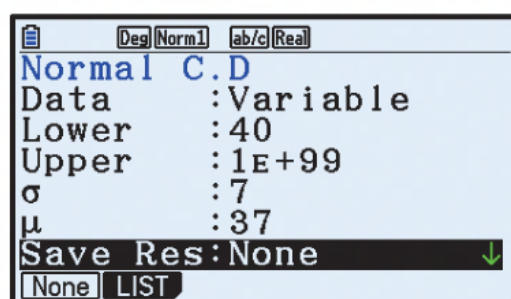
$$P(X \leq 61) \approx 0.579$$

- f To find  $P(57.5 \leq X \leq 62.5)$ , we set the lower bound to 57.5 and the upper bound to 62.5.



$$P(57.5 \leq X \leq 62.5) \approx 0.383$$

- 2 a To find  $P(X > 40)$ , we use a very high value such as  $10^{99}$  to represent the upper bound.

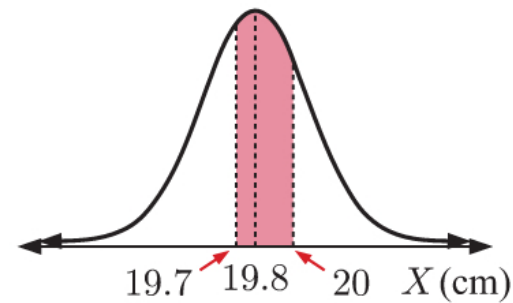
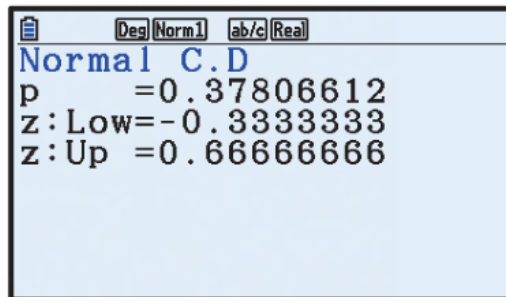
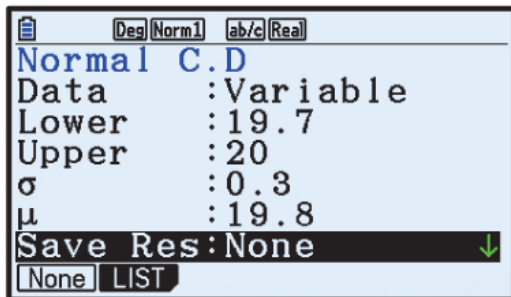


$$P(X > 40) \approx 0.334$$

- b Since the mean of the distribution is 37, then  $P(X > 37) = 0.5$

$$\begin{aligned} \therefore P(37 \leq X \leq 40) &= P(X > 37) - P(X > 40) \\ &\approx 0.5 - 0.334 \\ &\approx 0.166 \end{aligned}$$

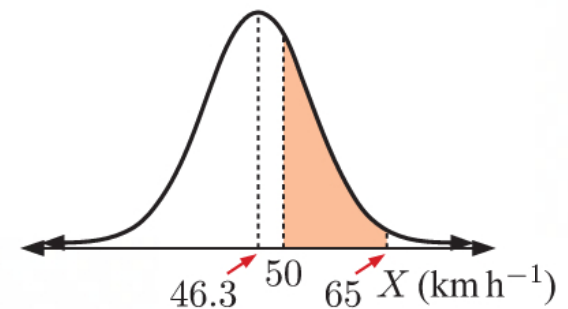
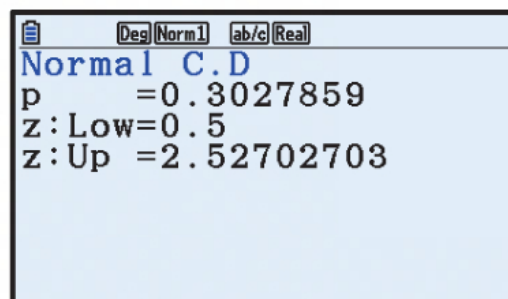
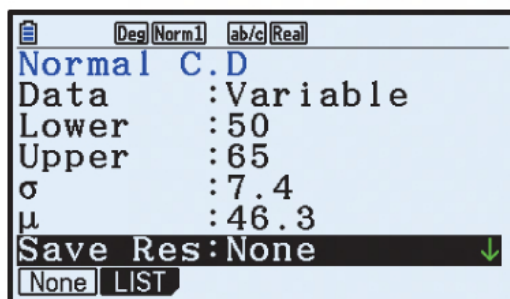
- 3 Let  $X$  cm be the length of a randomly selected bolt.



$$P(19.7 < X < 20) \approx 0.378$$

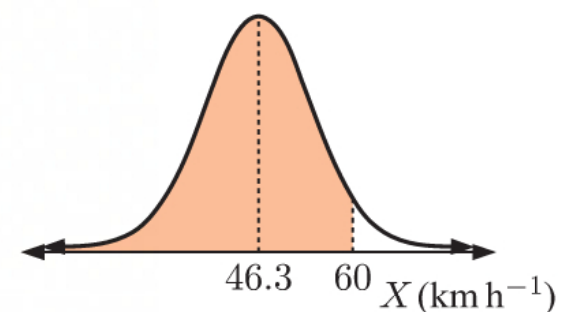
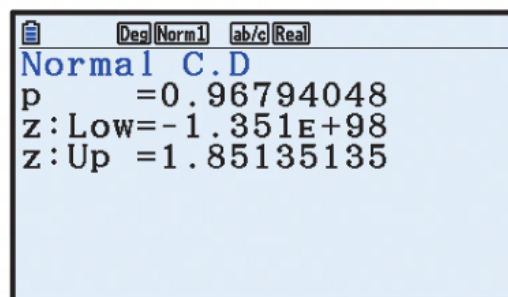
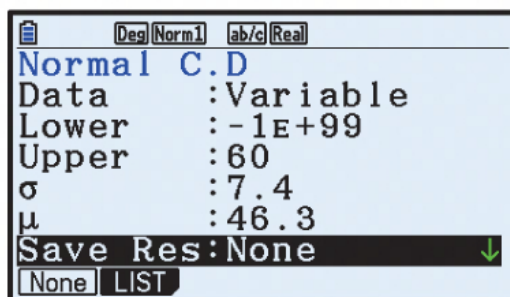
- 4 Let  $X$  km h<sup>-1</sup> be the speed of a randomly selected car.

a



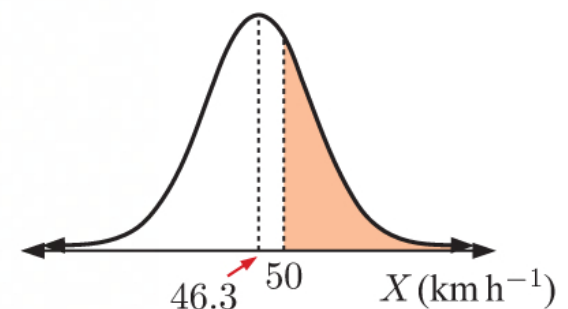
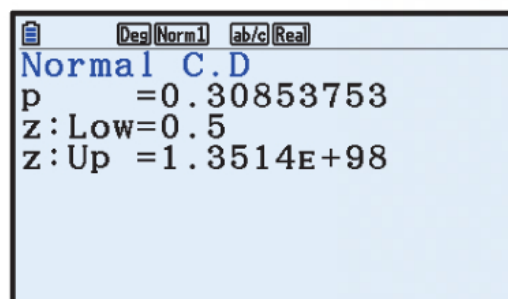
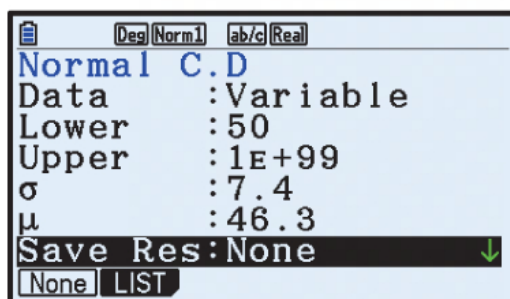
$$P(50 < X < 65) \approx 0.303$$

b



$$P(X < 60) \approx 0.968$$

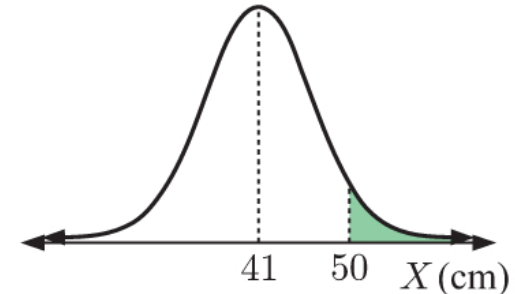
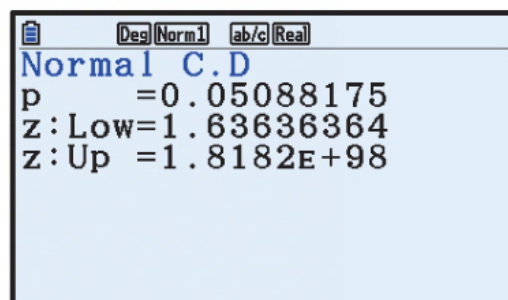
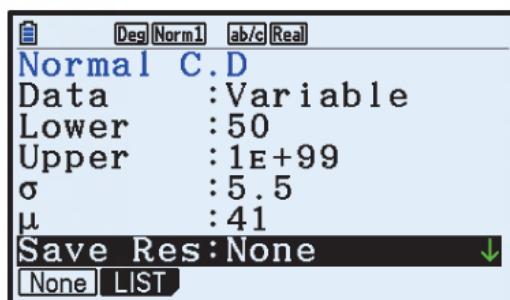
c



$$P(X > 50) \approx 0.309$$

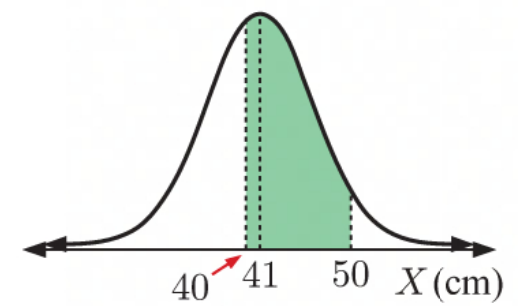
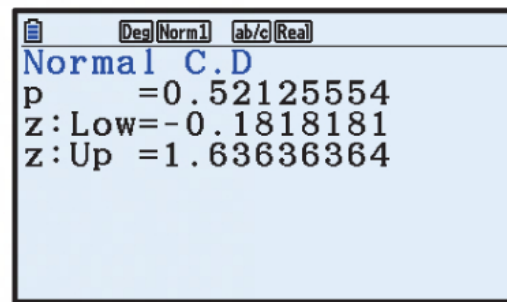
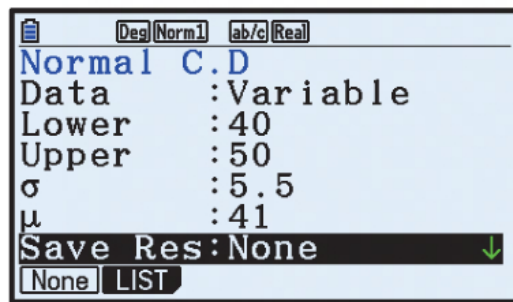
- 5 Let  $X$  cm be the length of a randomly selected eel.

a



$$P(X \geq 50) \approx 0.0509$$

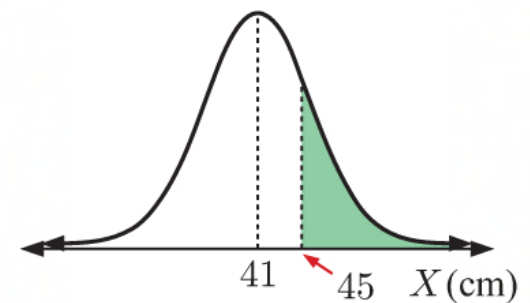
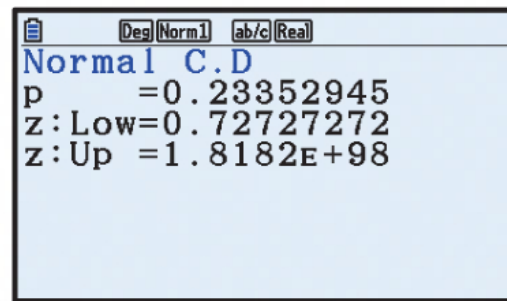
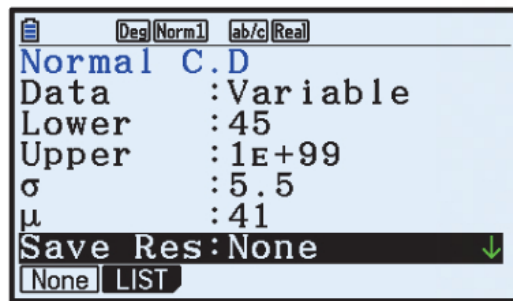
b



$$P(40 < X < 50) \approx 0.521$$

$\therefore$  about 52.1% of eels measure between 40 cm and 50 cm long.

c



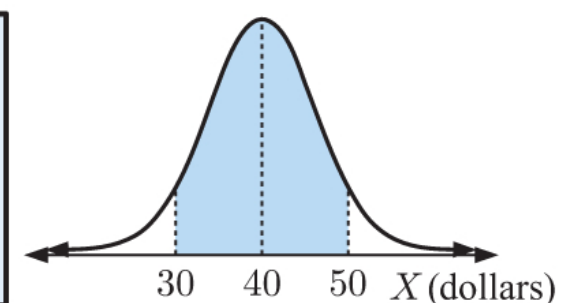
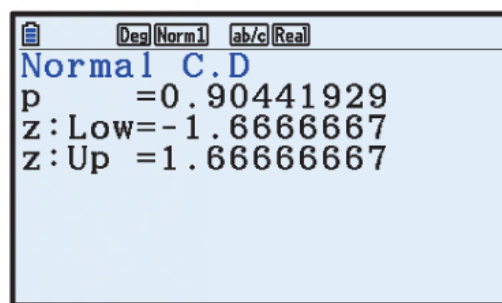
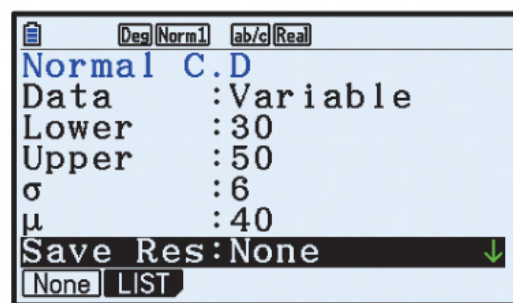
$$P(X \geq 45) \approx 0.234$$

$\therefore$  we would expect about  $0.234 \times 200 \approx 47$  eels to measure at least 45 cm in length.

6 Let  $X$  be the amount collected by Max in a randomly selected week.

a

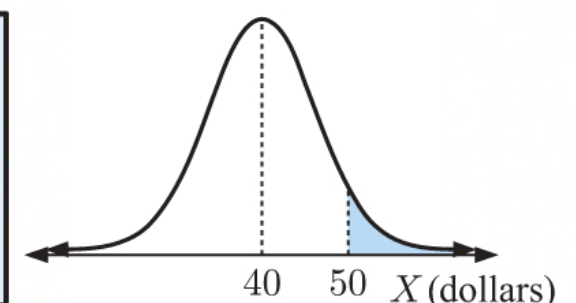
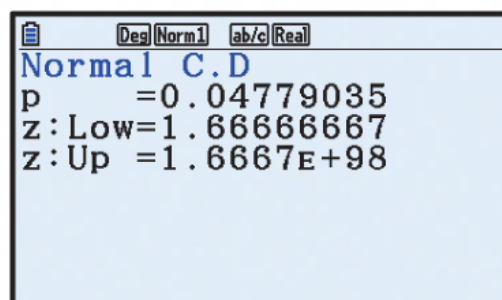
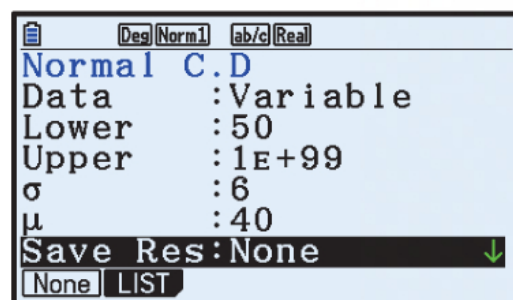
i



$$P(30 < X < 50) \approx 0.904$$

$\therefore$  Max would expect to collect between \$30 and \$50 in about 90.4% of weeks.

ii



$$P(X \geq 50) \approx 0.0478$$

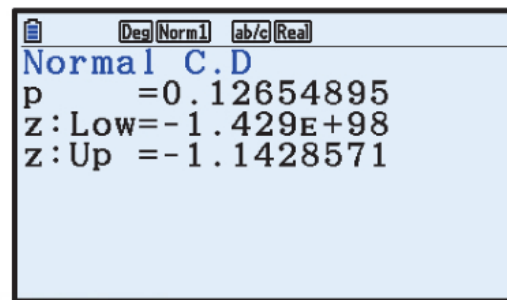
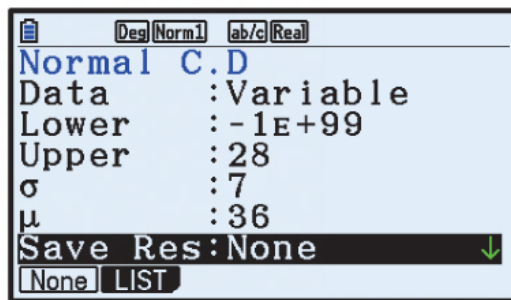
$\therefore$  Max would expect to collect at least \$50 in about 4.78% of weeks.

b There are about 52 weeks in a year, and the average weekly collection is \$40, so in 2 years we would expect Max to collect about  $2 \times 52 \times \$40 = \$4160$ .



7 Let  $X$  L be the amount of petrol bought by a randomly selected customer.

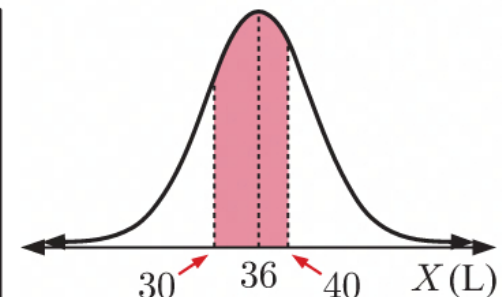
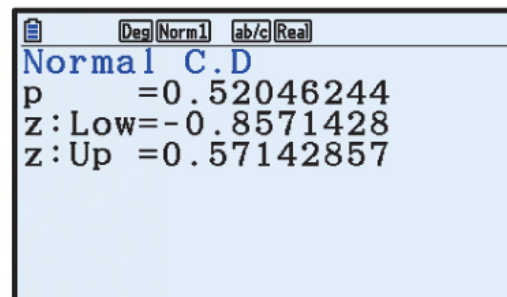
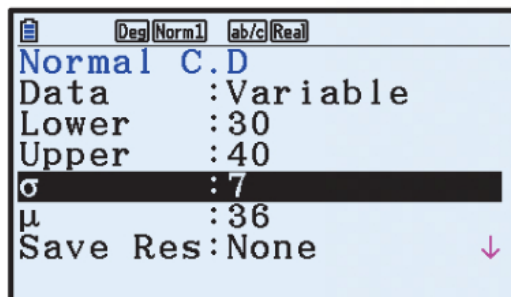
a i



$$P(X < 28) \approx 0.127$$

$\therefore$  about 12.7% of customers buy less than 28 L of petrol.

ii

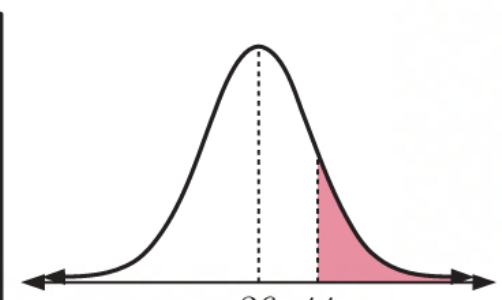
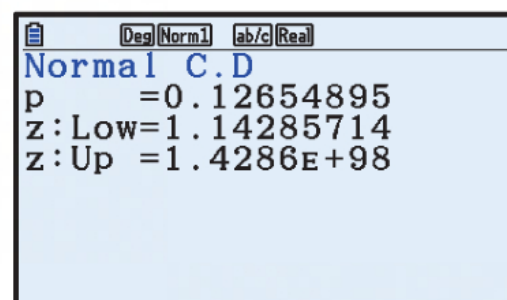
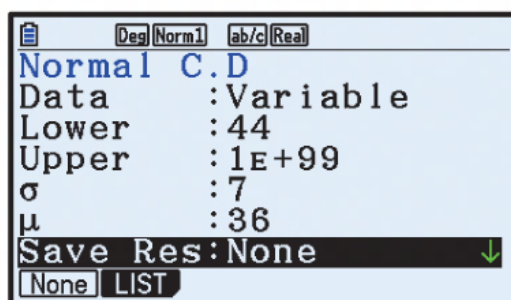


$$P(30 < X < 40) \approx 0.520$$

$\therefore$  about 52.0% of customers buy between 30 L and 40 L of petrol.

b i We would expect the petrol station to sell about  $36 \text{ L} \times 600 = 21.6 \text{ kL}$  of petrol.

ii

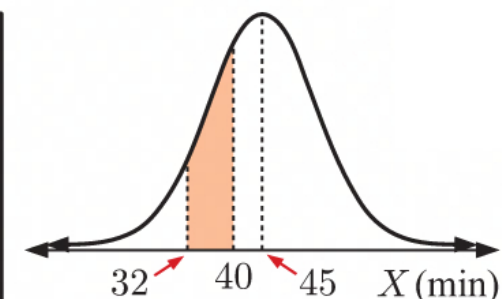
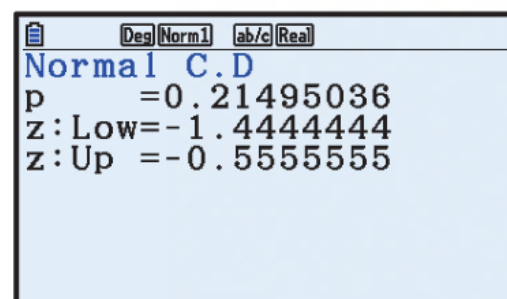
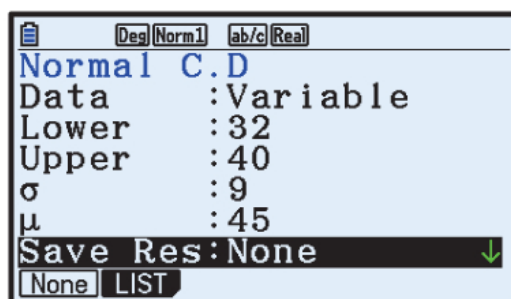


$$P(X \geq 44) \approx 0.127$$

$\therefore$  we would expect about  $0.127 \times 600 \approx 76$  customers to buy at least 44 L of petrol.

8 Let  $X$  minutes be the amount of time Enrique spends at the gym, and  $Y$  minutes be the amount of time Damien spends at the gym.

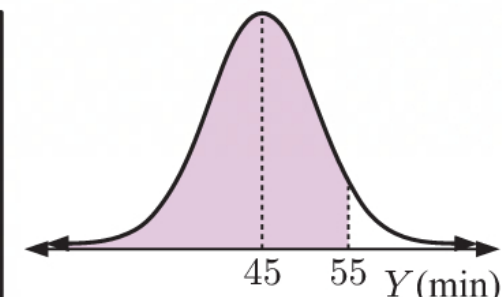
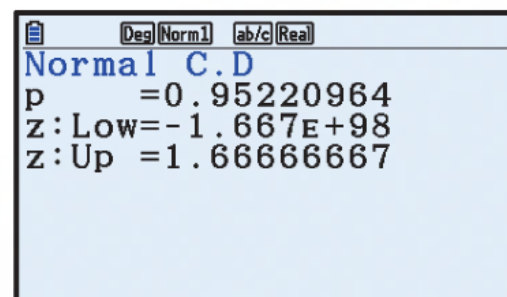
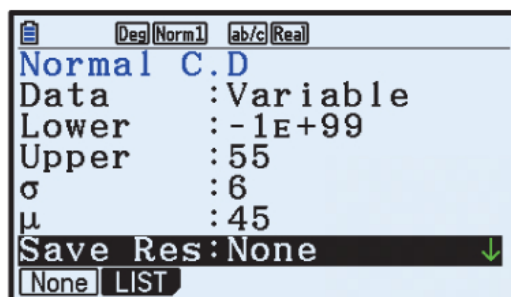
a i



$$P(32 < X < 40) \approx 0.215$$

$\therefore$  Enrique will spend between 32 and 40 minutes at the gym on about 21.5% of days.

ii



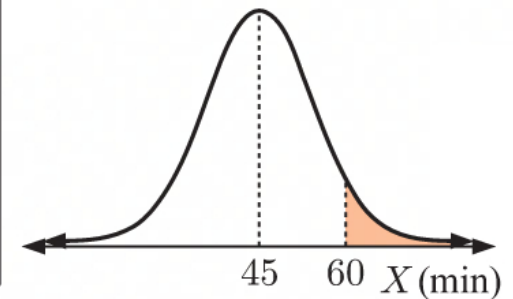
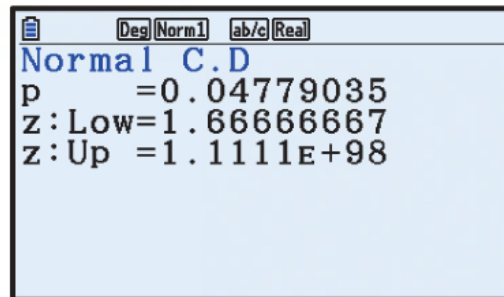
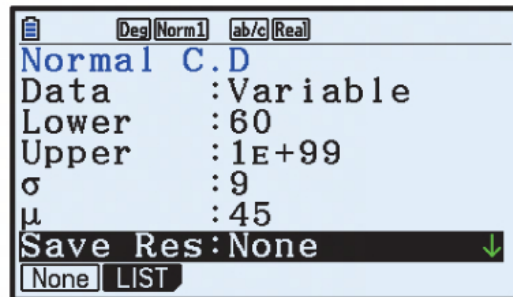
$$P(Y < 55) \approx 0.952$$

$\therefore$  Damien will spend less than 55 minutes at the gym on about 95.2% of days.



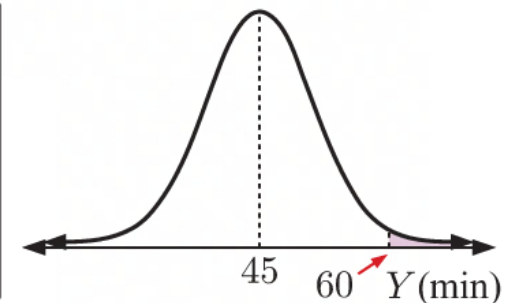
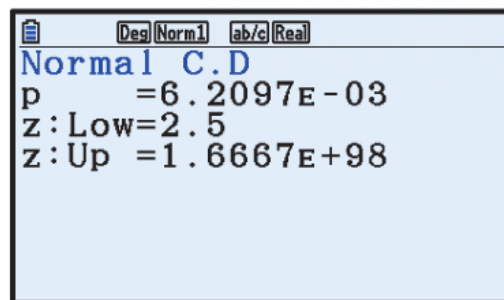
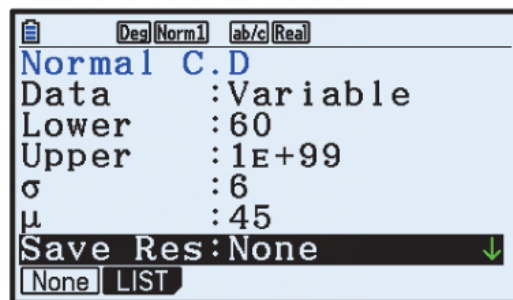
- b** **i** Enrique is more likely to spend at least 1 hour at the gym. The mean of both of their times is 45 minutes, but Enrique has a greater standard deviation, and so is more likely to exceed 1 hour.
- ii** Damien is more likely to spend between 40 and 50 minutes at the gym. Damien has the smaller standard deviation and is more likely to stay between 40 and 50 minutes, which is close to the mean of 45 minutes.

**c** **i** Enrique:



$$P(X > 60) \approx 0.0478$$

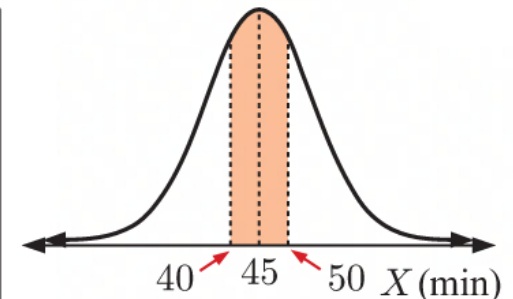
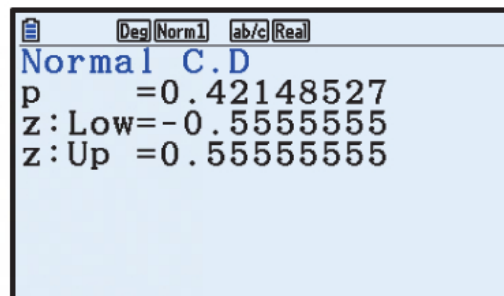
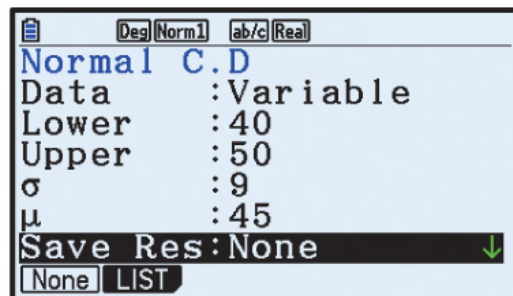
Damien:



$$P(Y > 60) \approx 0.00621$$

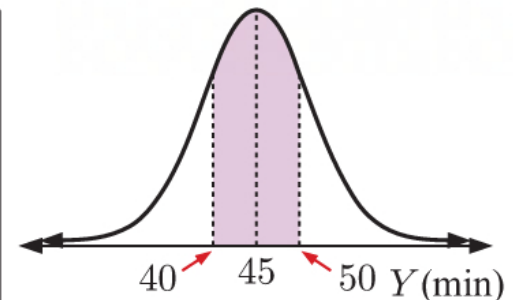
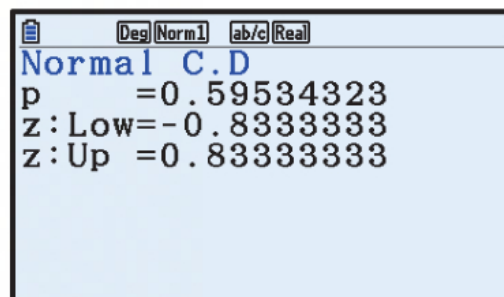
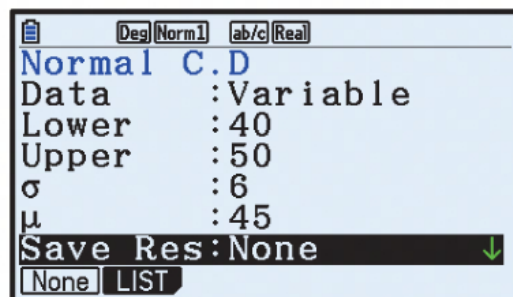
$\therefore$  Enrique is more likely to spend at least 1 hour at the gym.

**ii** Enrique:



$$P(40 < X < 50) \approx 0.421$$

Damien:



$$P(40 < Y < 50) \approx 0.595$$

$\therefore$  Damien is more likely to spend between 40 and 50 minutes at the gym.

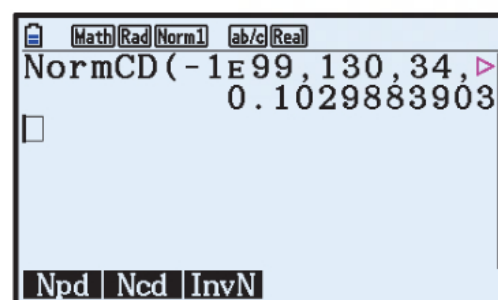
- 9 a Let  $X$  grams be the weight of a randomly selected apple.

$$X \sim N(173, 34^2)$$

$$\therefore P(X < 130) \approx 0.10299$$

$$\approx 0.103$$

$\therefore$  about 10.3% of apples from the crop were too small to sell.

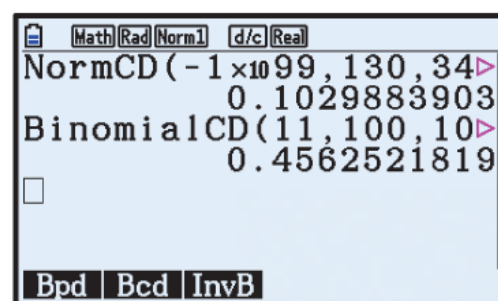


- b Let  $Y$  be the number of apples which were too small to sell.

$$Y \sim B(100, 0.10299)$$

$$\therefore P(Y > 10) = P(Y \geq 11)$$

$$\approx 0.456$$



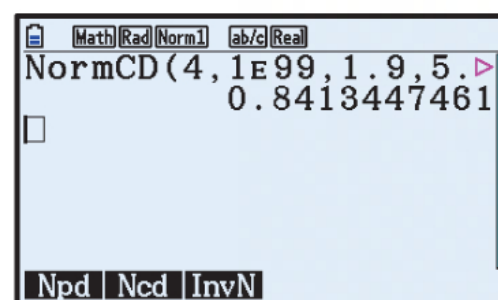
- 10 a Let  $X$  units be the drop in blood pressure of a randomly selected patient.

$$X \sim N(5.9, 1.9^2)$$

$$\therefore P(X > 4) \approx 0.84134$$

$$\approx 0.841$$

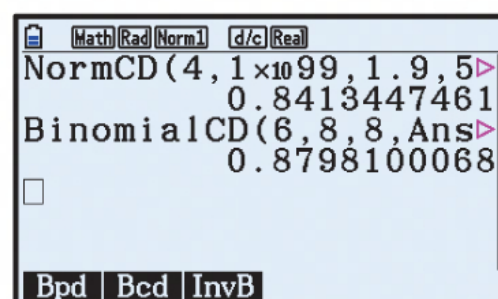
$\therefore$  about 84.1% of patients show a drop of more than 4 units.



- b Let  $Y$  be the number of patients with a drop of more than 4 units.

$$Y \sim B(8, 0.84134)$$

$$\therefore P(Y \geq 6) \approx 0.880$$

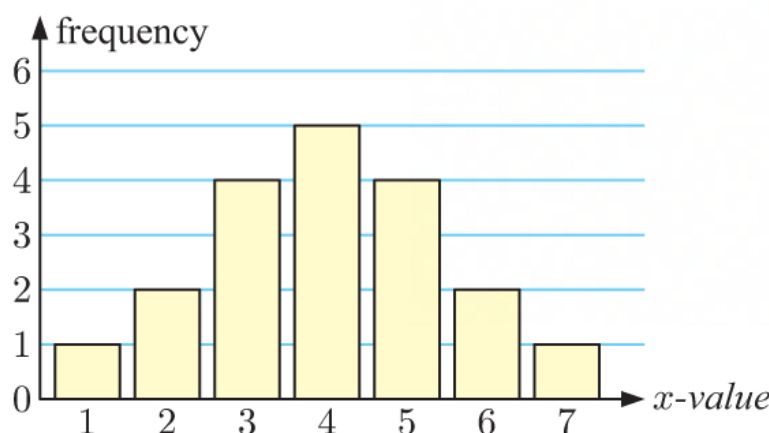


## INVESTIGATION 4

## $z$ -SCORES

- 1 a

$x$ -value	Frequency
1	1
2	2
3	4
4	5
5	4
6	2
7	1



- b

1-Variable	
$\bar{x}$	=4
$\Sigma x$	=76
$\Sigma x^2$	=346
$\sigma x$	=1.48678388
$sx$	=1.52752523
$n$	=19

So,  $\mu = 4$ ,  $\sigma \approx 1.49$

- c We calculate  $z = \frac{x - \mu}{\sigma}$  to 3 significant figures.

<i>x</i> -value	1	2	3	4	5	6	7
<i>z</i> -score	−2.02	−1.35	−0.673	0.00	0.673	1.35	2.02

d

<i>z</i> -score	Frequency
−2.02	1
−1.35	2
−0.673	4
0.00	5
0.673	4
1.35	2
2.02	1

Des Norm1	d/c Real
1-Variable	
$\bar{x}$	=0
$\Sigma x$	=0
$\Sigma x^2$	=19.0000021
$\sigma x$	=1.00000005
$sx$	=1.02740239
$n$	=19

So, the *z*-scores have mean  $\mu = 0$ , and standard deviation  $\sigma \approx 1$ .

- 2 c Both histograms are approximately normally distributed. The histogram of the *z*-scores appears to be normally distributed with mean 0 and standard deviation 1 for any sample.
- d If the original data is randomly generated from a normal distribution, the *z*-scores are also normally distributed with mean 0 and standard deviation 1.

If  $X \sim N(\mu, \sigma^2)$  and  $Z = \frac{X - \mu}{\sigma}$  then  $Z \sim N(0, 1^2)$ .

## EXERCISE 28E.1

- 1 a For English,  $z\text{-score} = \frac{48 - 40}{4.4} \approx 1.82$

For Mandarin,  $z\text{-score} = \frac{81 - 60}{9} \approx 2.33$

For Geography,  $z\text{-score} = \frac{84 - 55}{18} \approx 1.61$

For Biology,  $z\text{-score} = \frac{68 - 50}{20} = 0.9$

For Mathematics,  $z\text{-score} = \frac{84 - 50}{15} \approx 2.27$

Subject	Emma's score	$\mu$	$\sigma$
English	48	40	4.4
Mandarin	81	60	9
Geography	84	55	18
Biology	68	50	20
Mathematics	84	50	15

- b In order from best to worst: Mandarin, Mathematics, English, Geography, Biology.
- c It is reasonable to compare Emma's performances using *z*-scores as the scores in each of Emma's classes are normally distributed.



**2 a**

<i>Subject</i>	<i>Sergio's score</i>	$\mu$	$\sigma$
Physics	73%	78%	10.8%
Chemistry	77%	72%	11.6%
Mathematics	76%	74%	10.1%
German	91%	86%	9.6%
Biology	58%	62%	5.2%

For Physics,  $z\text{-score} = \frac{73 - 78}{10.8} \approx -0.463$

For Chemistry,  $z\text{-score} = \frac{77 - 72}{11.6} \approx 0.431$

For Mathematics,  $z\text{-score} = \frac{76 - 74}{10.1} \approx 0.198$

For German,  $z\text{-score} = \frac{91 - 86}{9.6} \approx 0.521$

For Biology,  $z\text{-score} = \frac{58 - 62}{5.2} \approx -0.769$

**b** In order from best to worst: German, Chemistry, Mathematics, Physics, Biology.

**3**

<i>Event</i>	<i>Time (seconds)</i>	$\mu$ (seconds)	$\sigma$ (seconds)
50 m freestyle	32.1	27.8	2.2
100 m backstroke	53.5	58.1	4.3
200 m breaststroke	140.0	143.7	6.4
100 m butterfly	59.6	57.7	5.5

**a** For 50 m freestyle,  $z\text{-score} = \frac{32.1 - 27.8}{2.2} \approx 1.95$

For 100 m backstroke,  $z\text{-score} = \frac{53.5 - 58.1}{4.3} \approx -1.07$

For 200 m breaststroke,  $z\text{-score} = \frac{140.0 - 143.7}{6.4} \approx -0.578$

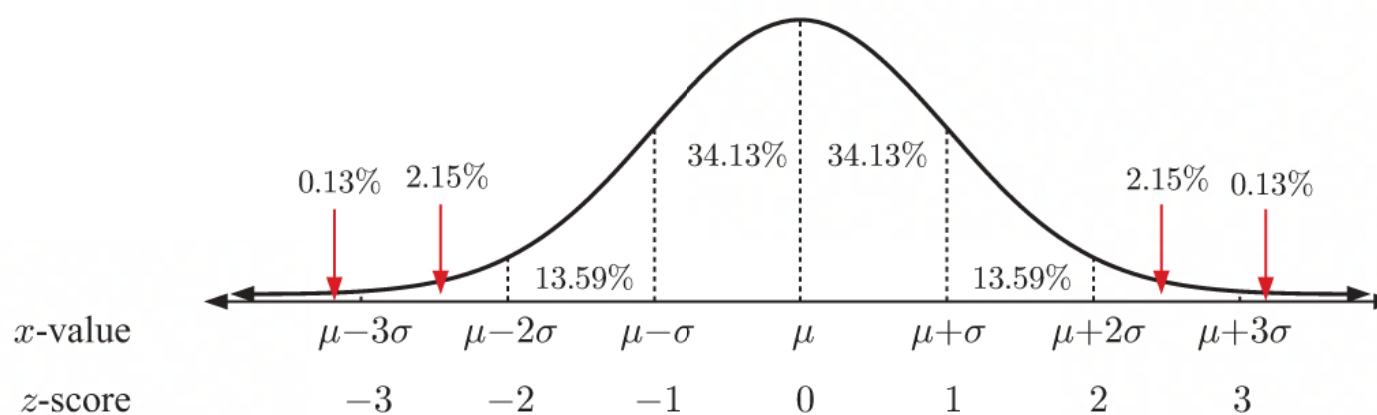
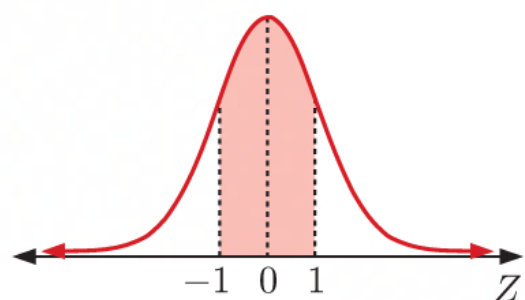
For 100 m butterfly,  $z\text{-score} = \frac{59.6 - 57.7}{5.5} \approx 0.345$

**b** A lower  $z$ -score is better as it indicates that the time is lower, and hence that Frederick swam faster.

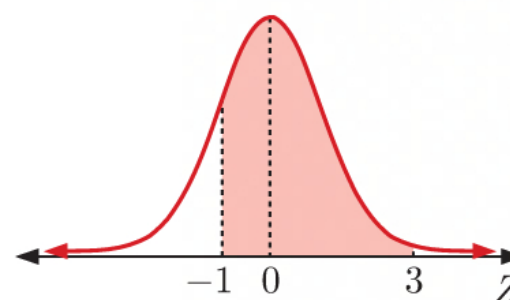
**c** In order from best to worst:

100 m backstroke, 200 m breaststroke, 100 m butterfly, 50 m freestyle.

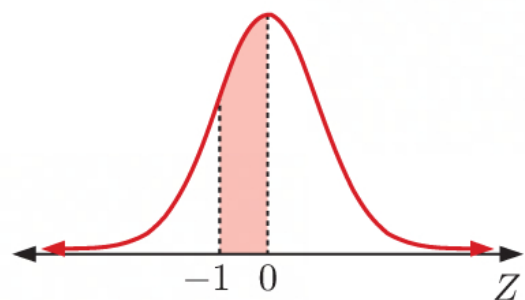


**EXERCISE 28E.2****1****a**

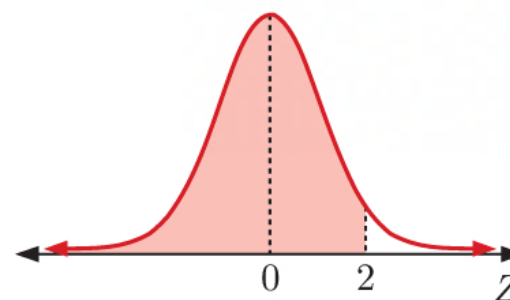
$$\begin{aligned}
 P(-1 < Z < 1) &\approx 34.13\% + 34.13\% \\
 &\approx 68.26\% \\
 &\approx 0.683
 \end{aligned}$$

**b**

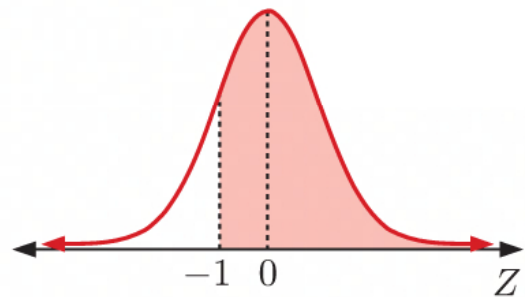
$$\begin{aligned}
 P(-1 \leq Z \leq 3) \\
 &\approx 34.13\% + 34.13\% + 13.59\% + 2.15\% \\
 &\approx 84.00\% \\
 &\approx 0.840
 \end{aligned}$$

**c**

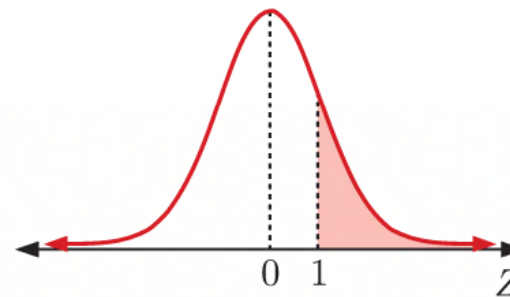
$$\begin{aligned}
 P(-1 < Z < 0) &\approx 34.13\% \\
 &\approx 0.341
 \end{aligned}$$

**d**

$$\begin{aligned}
 P(Z < 2) &\approx 50\% + 34.13\% + 13.59\% \\
 &\approx 97.72\% \\
 &\approx 0.977
 \end{aligned}$$

**e**

$$\begin{aligned}
 P(-1 < Z) &= P(Z > -1) \\
 &\approx 34.13\% + 50\% \\
 &\approx 84.13\% \\
 &\approx 0.841
 \end{aligned}$$

**f**

$$\begin{aligned}
 P(Z \geq 1) &\approx 13.59\% + 2.15\% + 0.13\% \\
 &\approx 15.87\% \\
 &\approx 0.159
 \end{aligned}$$

$$\begin{aligned}
 \text{2 a } E\left(\frac{X - \mu}{\sigma}\right) &= \frac{1}{\sigma} E(X - \mu) \\
 &= \frac{1}{\sigma} (E(X) - \mu) \\
 &= \frac{1}{\sigma} (\mu - \mu) \\
 &= 0
 \end{aligned}$$

$$\begin{aligned}
 \text{b } \text{Var}\left(\frac{X - \mu}{\sigma}\right) &= \frac{1}{\sigma^2} \text{Var}(X) \\
 &= \frac{1}{\sigma^2} \times \sigma^2 \\
 &= 1
 \end{aligned}$$

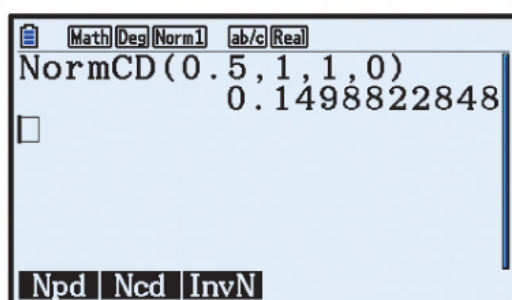
$$\begin{aligned}
 \text{3 a } \text{If } P(\mu - \sigma < X < \mu + 2\sigma) &= P(a < Z < b) \\
 \text{then } a &= \frac{(\mu - \sigma) - \mu}{\sigma} \quad \text{and} \quad b = \frac{(\mu + 2\sigma) - \mu}{\sigma} \\
 \therefore a &= \frac{-\sigma}{\sigma} \quad \therefore b = \frac{2\sigma}{\sigma} \\
 &= -1 \quad \quad \quad = 2 \\
 \therefore a &= -1, \quad b = 2
 \end{aligned}$$

$$\begin{aligned}
 \text{b } \text{If } P(\mu - 0.5\sigma < X < \mu) &= P(a < Z < b) \\
 \text{then } a &= \frac{(\mu - 0.5\sigma) - \mu}{\sigma} \quad \text{and} \quad b = \frac{\mu - \mu}{\sigma} \\
 \therefore a &= \frac{-0.5\sigma}{\sigma} \quad \therefore b = 0 \\
 &= -0.5 \\
 \therefore a &= -0.5, \quad b = 0
 \end{aligned}$$

$$\begin{aligned}
 \text{c } \text{If } P(0 \leq Z \leq 3) &= P(\mu - a\sigma \leq X \leq \mu + b\sigma) \\
 \text{then } \frac{(\mu - a\sigma) - \mu}{\sigma} &= 0 \quad \text{and} \quad \frac{(\mu + b\sigma) - \mu}{\sigma} = 3 \\
 \therefore \mu - a\sigma - \mu &= 0 \quad \therefore \mu + b\sigma - \mu = 3\sigma \\
 \therefore -a\sigma &= 0 \quad \therefore b\sigma = 3\sigma \\
 \therefore a &= 0 \quad \therefore b = 3 \\
 \therefore a &= 0, \quad b = 3
 \end{aligned}$$

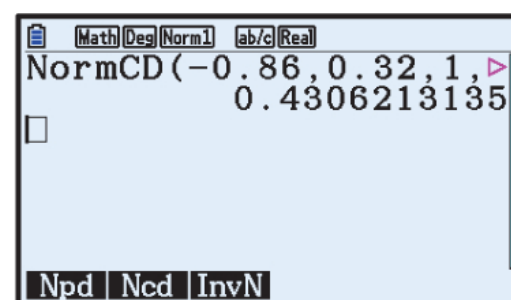
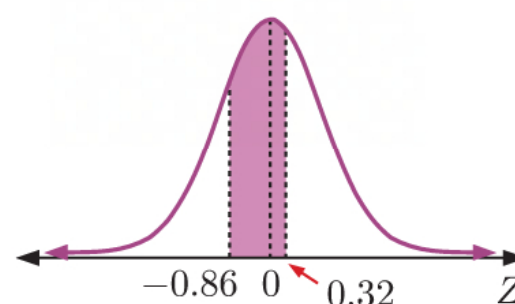
$$4 \quad Z \sim N(0, 1^2)$$

a

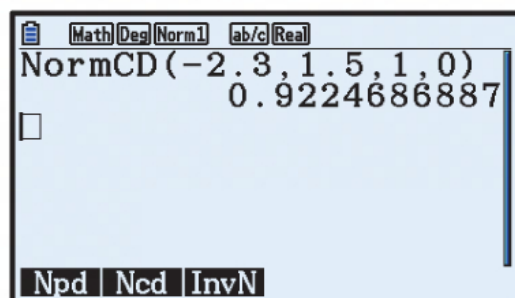
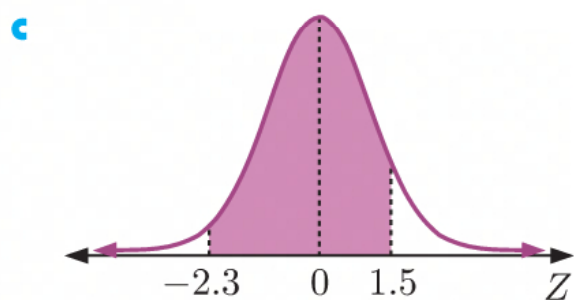


$$P(0.5 \leq Z \leq 1) \approx 0.150$$

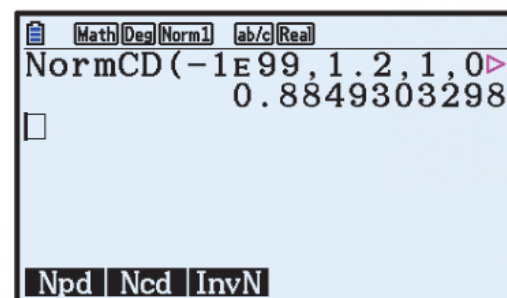
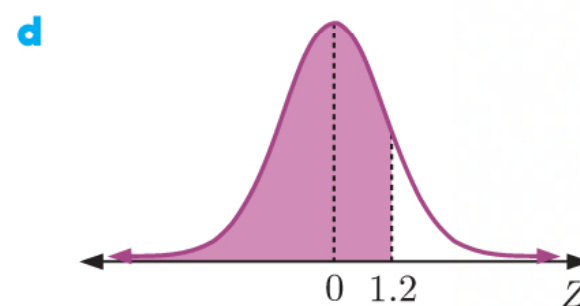
b



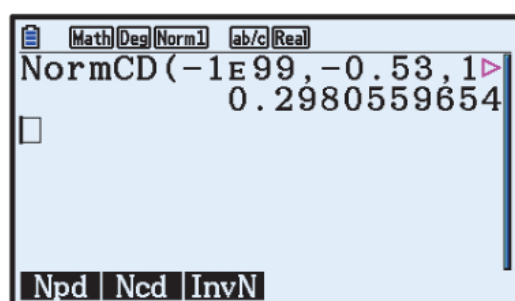
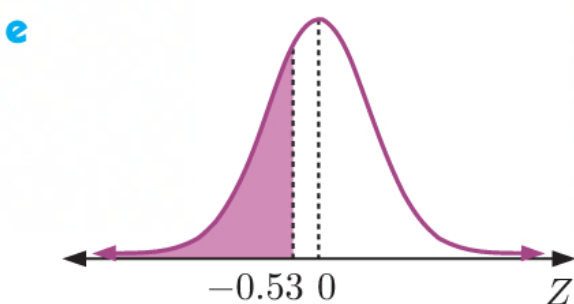
$$P(-0.86 \leq Z \leq 0.32) \approx 0.431$$



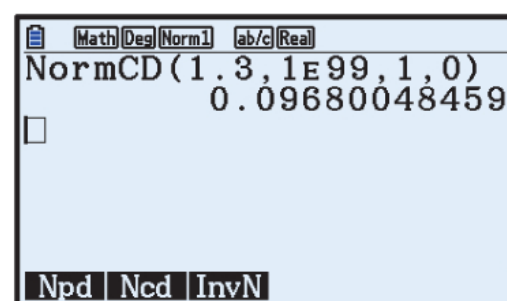
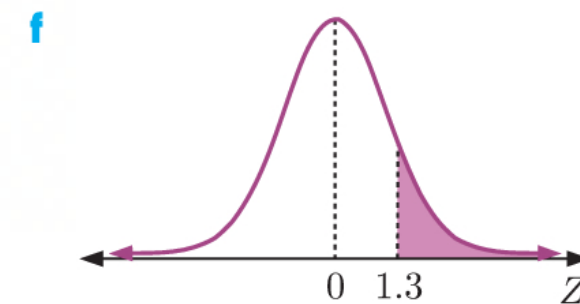
$$P(-2.3 \leq Z \leq 1.5) \approx 0.922$$



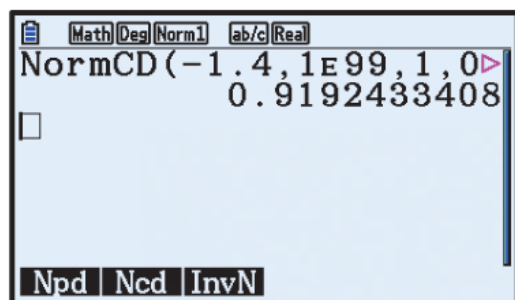
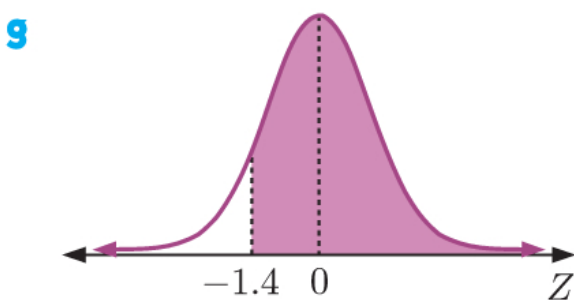
$$P(Z \leq 1.2) \approx 0.885$$



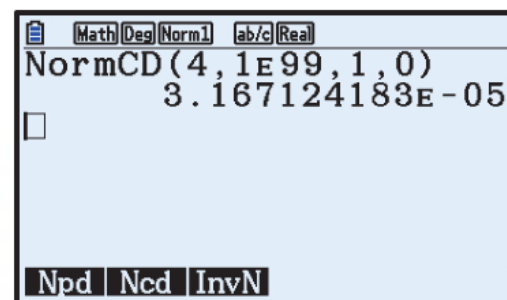
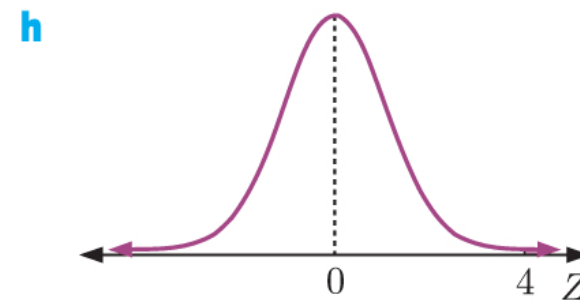
$$P(Z \leq -0.53) \approx 0.298$$



$$P(Z \geq 1.3) \approx 0.0968$$

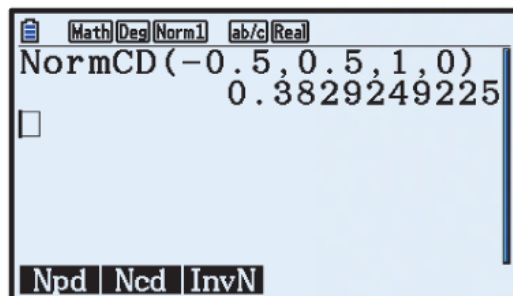
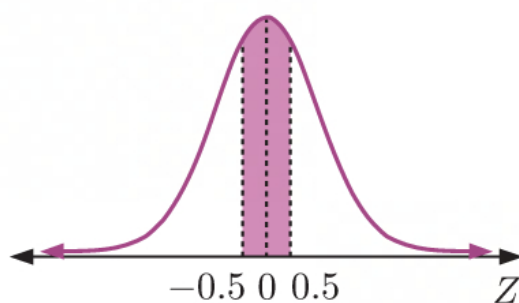


$$P(Z \geq -1.4) \approx 0.919$$



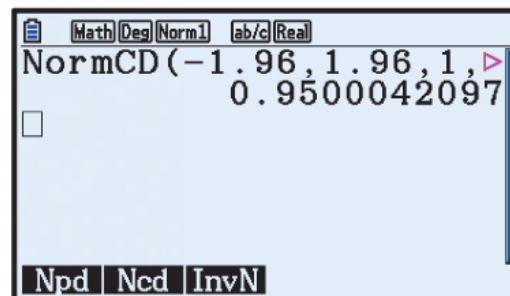
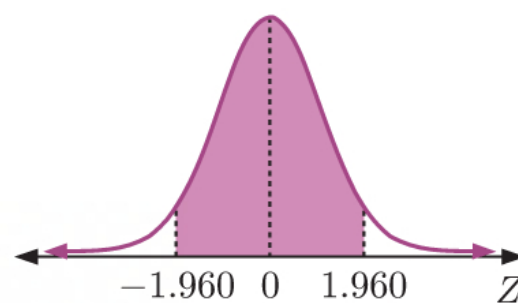
$$P(Z > 4) \approx 0.000\,031\,7 \quad (3.17 \times 10^{-5})$$

i



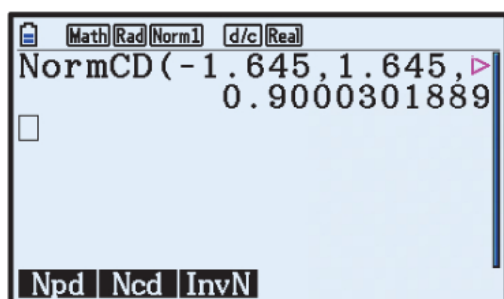
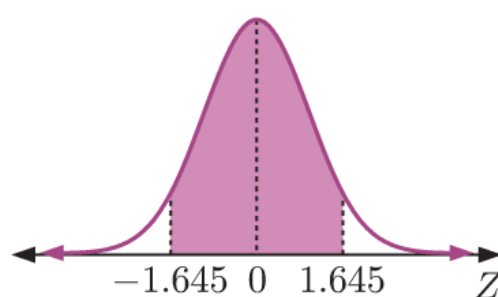
$$P(-0.5 < Z < 0.5) \approx 0.383$$

j



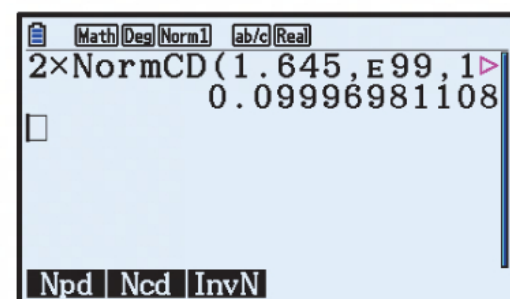
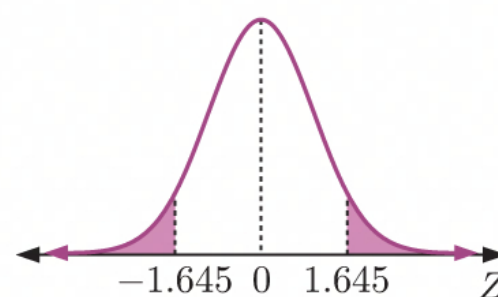
$$P(-1.960 \leq Z \leq 1.960) \approx 0.950$$

k



$$P(-1.645 \leq Z \leq 1.645) \approx 0.900$$

l



$$\begin{aligned} P(|Z| > 1.645) \\ &= P(Z < -1.645) + P(Z > 1.645) \\ &= 2 \times P(Z > 1.645) \\ &\approx 0.100 \end{aligned}$$

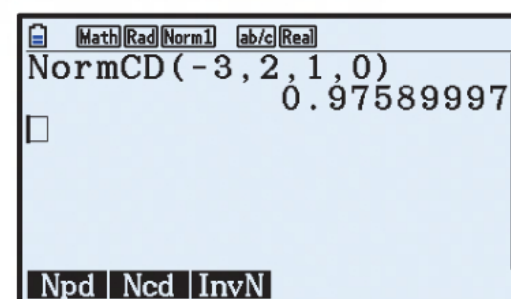
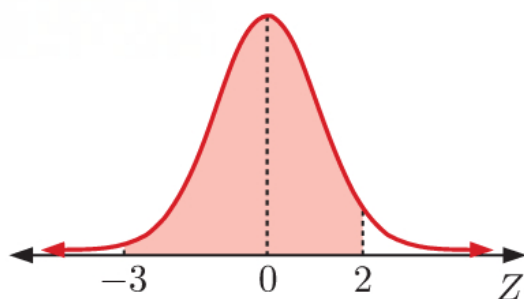
5 a i Since  $X \sim N(\mu, \sigma^2)$ , the  $Z$ -transformation of  $X$  is  $Z = \frac{X - \mu}{\sigma}$ .

$$\text{Now } P(\mu - 3\sigma < X < \mu + 2\sigma) = P(-3\sigma < X - \mu < 2\sigma)$$

$$= P\left(-3 < \frac{X - \mu}{\sigma} < 2\right)$$

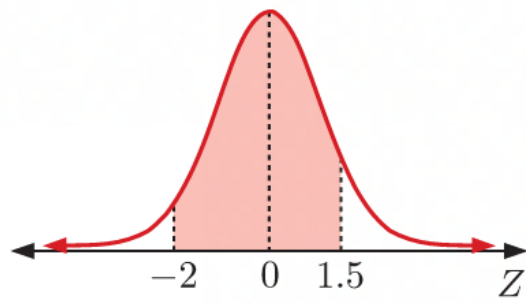
$$= P(-3 < Z < 2)$$

ii

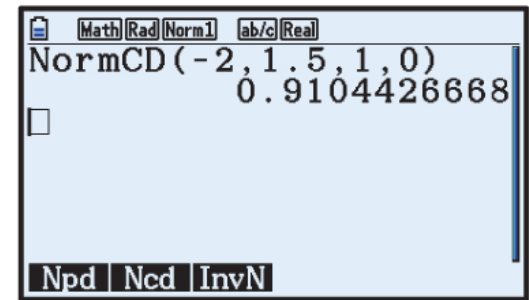
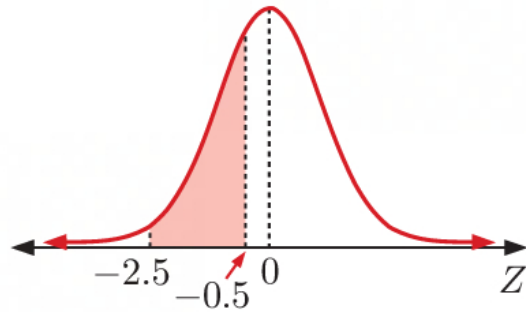


$$\begin{aligned} P(\mu - 3\sigma < X < \mu + 2\sigma) &= P(-3 < Z < 2) \\ &\approx 0.976 \end{aligned}$$

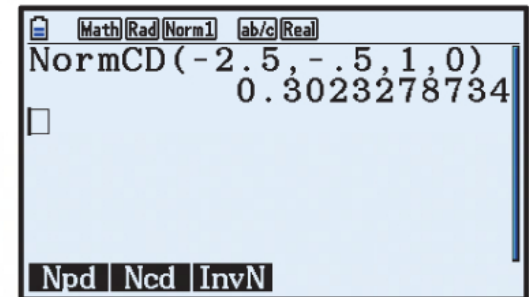


**b i**

$$P(\mu - 2\sigma < X < \mu + 1.5\sigma) = P(-2 < Z < 1.5) \\ \approx 0.910$$

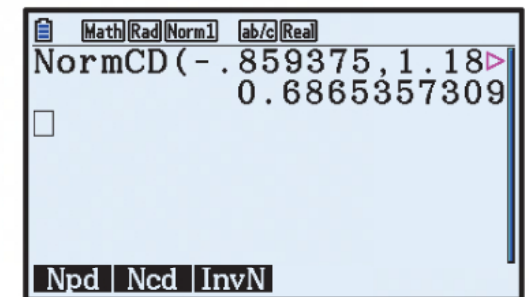
**ii**

$$P(\mu - 2.5\sigma < X < \mu - 0.5\sigma) = P(-2.5 < Z < -0.5) \\ \approx 0.302$$

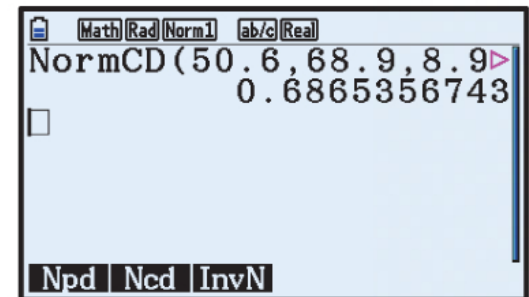
**6**  $X \sim N(58.3, 8.96^2)$ 

$$\begin{aligned} \text{a i } z_1 &= \frac{50.6 - 58.3}{8.96} & z_2 &= \frac{68.9 - 58.3}{8.96} \\ &= -0.859375 & &\approx 1.183036 \\ &\approx -0.859 & &\approx 1.18 \end{aligned}$$

$$\begin{aligned} \text{ii } Z &\sim N(0, 1) \\ P(z_1 \leq Z \leq z_2) &= P(-0.859375 \leq Z \leq 1.183036) \\ &\approx 0.687 \end{aligned}$$



**b** Using technology,  
 $P(50.6 \leq X \leq 68.9) \approx 0.687$  ✓

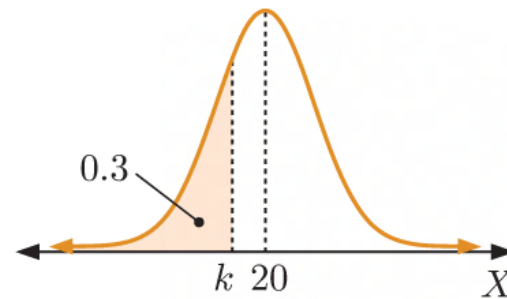


**EXERCISE 28F.1****1 a**

[Rad] [Norm1] [ab/c] [Real]	
Inverse Normal	
Data	:Variable
Tail	:Left
Area	:0.3
$\sigma$	:3
$\mu$	:20
Save Res:	None
[None] [LIST]	

[Rad] [Norm1] [ab/c] [Real]	
Inverse Normal	
xInv=18.4267985	

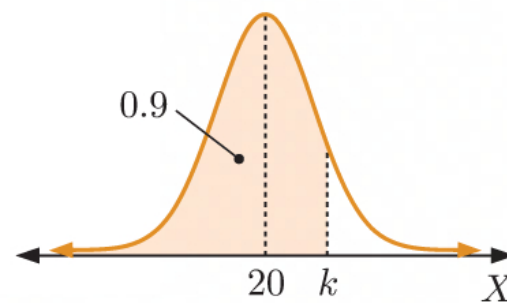
If  $P(X \leq k) = 0.3$   
then  $k \approx 18.4$

**b**

[Rad] [Norm1] [ab/c] [Real]	
Inverse Normal	
Data	:Variable
Tail	:Left
Area	:0.9
$\sigma$	:3
$\mu$	:20
Save Res:	None
[None] [LIST]	

[Rad] [Norm1] [ab/c] [Real]	
Inverse Normal	
xInv=23.8446547	

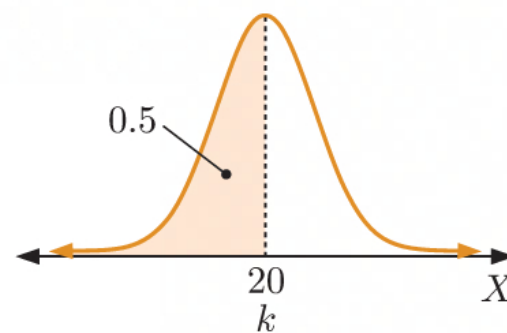
If  $P(X \leq k) = 0.9$   
then  $k \approx 23.8$

**c**

[Rad] [Norm1] [ab/c] [Real]	
Inverse Normal	
Data	:Variable
Tail	:Left
Area	:0.5
$\sigma$	:3
$\mu$	:20
Save Res:	None
[None] [LIST]	

[Rad] [Norm1] [ab/c] [Real]	
Inverse Normal	
xInv=20	

If  $P(X \leq k) = 0.5$   
then  $k = 20$

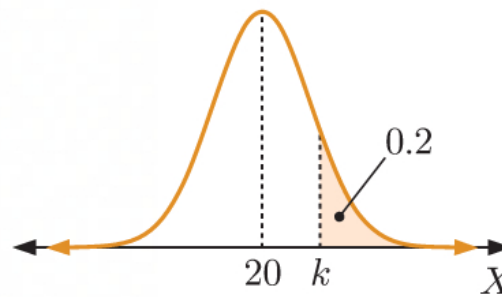


d

Rad Norm1 ab/c Real	
Inverse Normal	
Data	:Variable
Tail	:Right
Area	:0.2
$\sigma$	:3
$\mu$	:20
Save Res:None	
None	LIST

Rad Norm1 ab/c Real	
Inverse Normal	
xInv=22.5248637	

If  $P(X > k) = 0.2$   
 then  $k \approx 22.5$

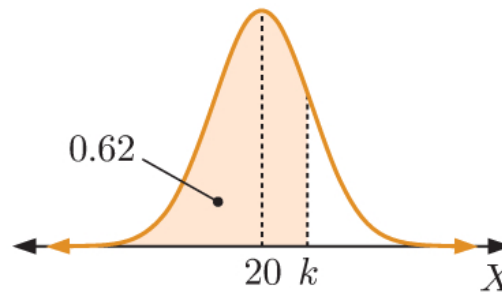


e

Rad Norm1 ab/c Real	
Inverse Normal	
Data	:Variable
Tail	:Left
Area	:0.62
$\sigma$	:3
$\mu$	:20
Save Res:None	
None	LIST

Rad Norm1 ab/c Real	
Inverse Normal	
xInv=20.9164424	

If  $P(X < k) = 0.62$   
 then  $k \approx 20.9$

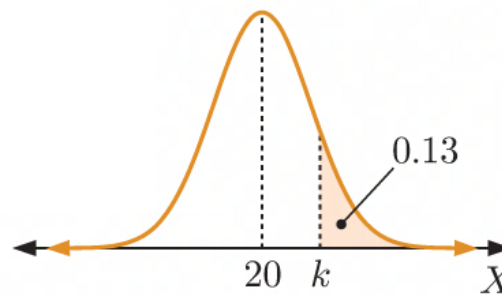


f

Rad Norm1 ab/c Real	
Inverse Normal	
Data	:Variable
Tail	:Right
Area	:0.13
$\sigma$	:3
$\mu$	:20
Save Res:None	
None	LIST

Rad Norm1 ab/c Real	
Inverse Normal	
xInv=23.3791734	

If  $P(X \geq k) = 0.13$   
 then  $k \approx 23.4$

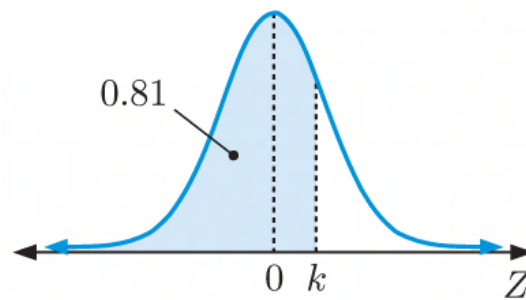


**2 a**

Rad Norm1 ab/c Real	
Inverse Normal	
Data	:Variable
Tail	:Left
Area	:0.81
$\sigma$	:1
$\mu$	:0
Save Res:None	
None	LIST

Rad Norm1 ab/c Real	
Inverse Normal	
xInv=0.87789629	

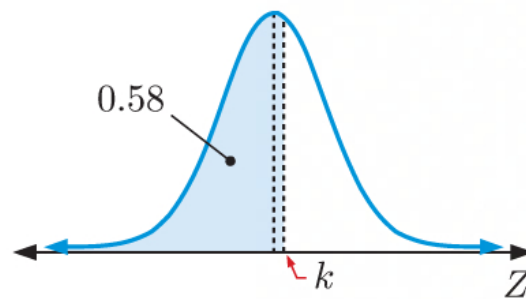
If  $P(Z \leq k) = 0.81$   
 then  $k \approx 0.878$

**b**

Rad Norm1 ab/c Real	
Inverse Normal	
Data	:Variable
Tail	:Left
Area	:0.58
$\sigma$	:1
$\mu$	:0
Save Res:None	
None	LIST

Rad Norm1 ab/c Real	
Inverse Normal	
xInv=0.20189347	

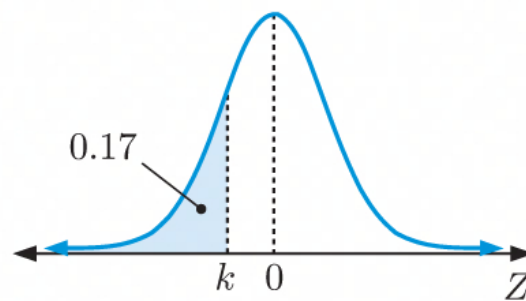
If  $P(Z \leq k) = 0.58$   
 then  $k \approx 0.202$

**c**

Rad Norm1 ab/c Real	
Inverse Normal	
Data	:Variable
Tail	:Left
Area	:0.17
$\sigma$	:1
$\mu$	:0
Save Res:None	
None	LIST

Rad Norm1 ab/c Real	
Inverse Normal	
xInv=-0.9541652	

If  $P(Z \leq k) = 0.17$   
 then  $k \approx -0.954$



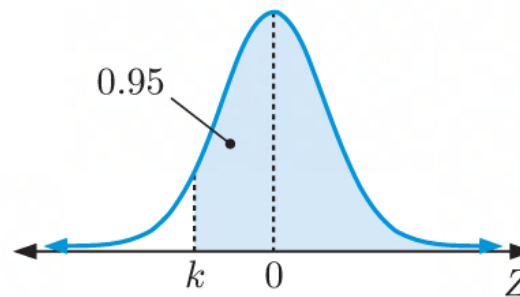


d

Rad Norm1 ab/c Real	
Inverse Normal	
Data	:Variable
Tail	:Right
Area	:0.95
$\sigma$	:1
$\mu$	:0
Save Res:None	
None	LIST

Rad Norm1 ab/c Real	
Inverse Normal	
xInv=-1.6448536	

If  $P(Z \geq k) = 0.95$   
 $\therefore k \approx -1.64$

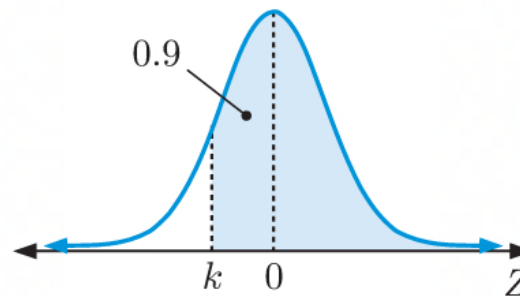


e

Rad Norm1 ab/c Real	
Inverse Normal	
Data	:Variable
Tail	:Right
Area	:0.9
$\sigma$	:1
$\mu$	:0
Save Res:None	
None	LIST

Rad Norm1 ab/c Real	
Inverse Normal	
xInv=-1.2815516	

If  $P(Z \geq k) = 0.9$   
 $\therefore k \approx -1.28$

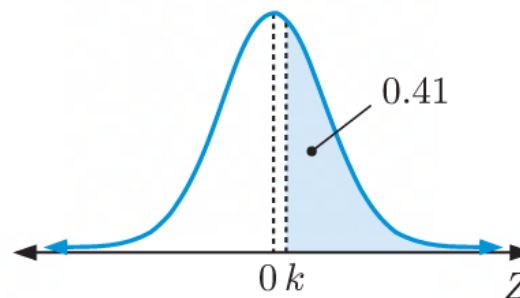


f

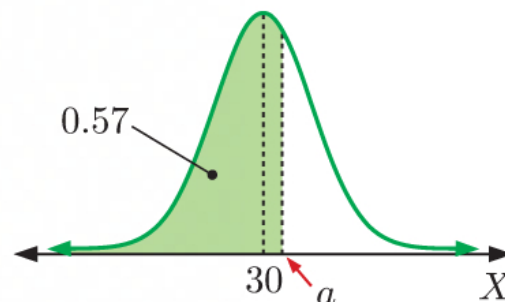
Rad Norm1 ab/c Real	
Inverse Normal	
Data	:Variable
Tail	:Right
Area	:0.41
$\sigma$	:1
$\mu$	:0
Save Res:None	
None	LIST

Rad Norm1 ab/c Real	
Inverse Normal	
xInv=0.22754497	

If  $P(Z \geq k) = 0.41$   
 $\therefore k \approx 0.228$

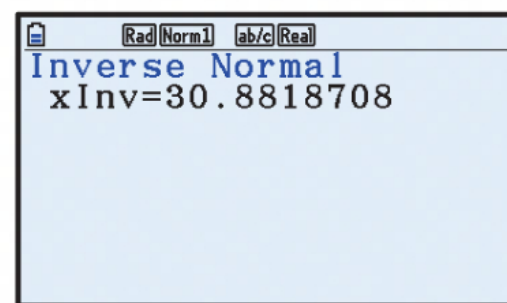
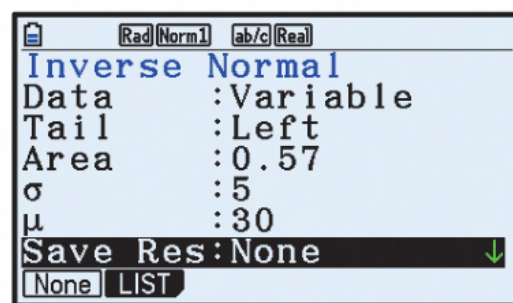


3 a  $X \sim N(30, 5^2)$



$\therefore a > 30$

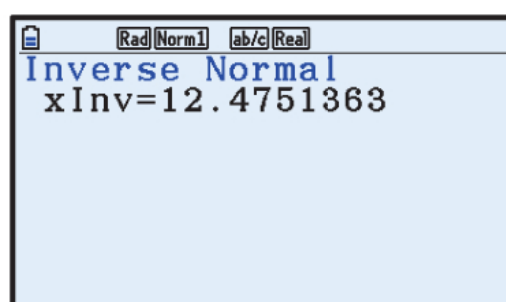
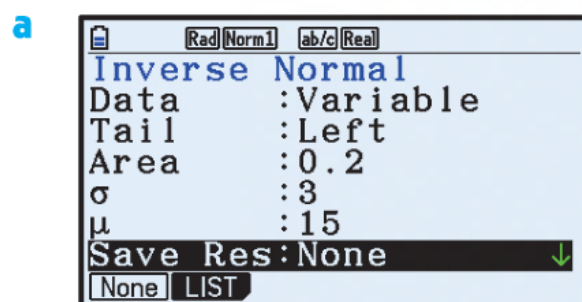
**b**  $P(X \leq a) = 0.57$   
 $\therefore a \approx 30.9$



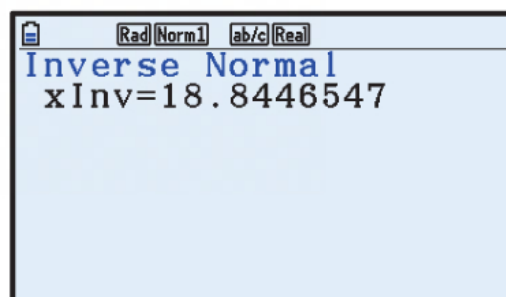
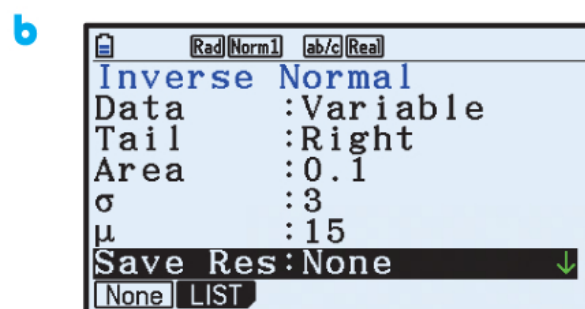
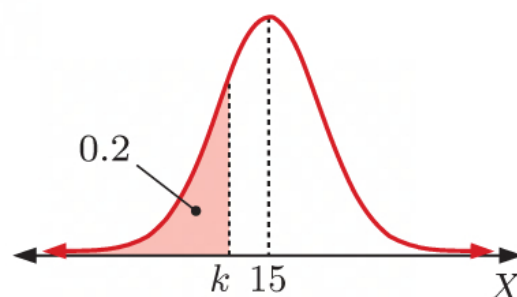
**c i**  $P(X \geq a) = 1 - P(X \leq a)$   
 $= 1 - 0.57$   
 $= 0.43$

**ii**  $P(30 \leq X \leq a) = P(X \leq a) - P(X \leq 30)$   
 $= 0.57 - 0.5$   
 $= 0.07$

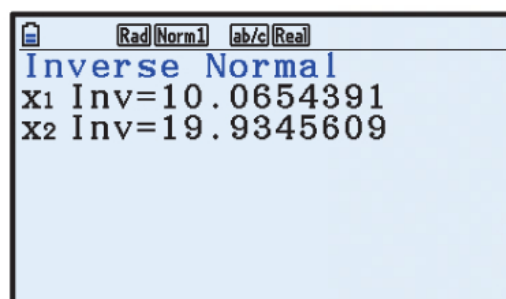
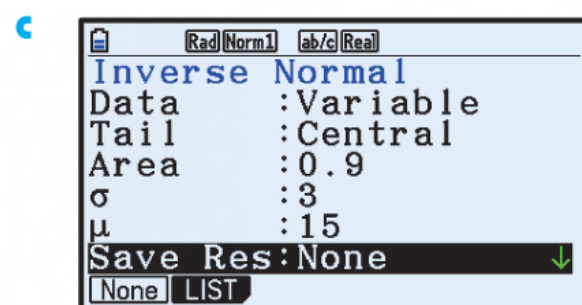
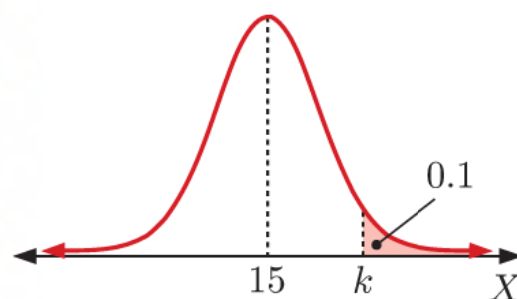
**4**  $X \sim N(15, 3^2)$



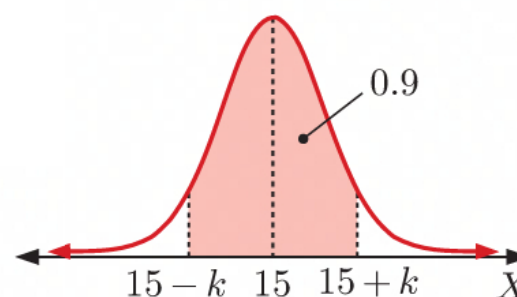
If  $P(X < k) = 0.2$   
 then  $k \approx 12.5$



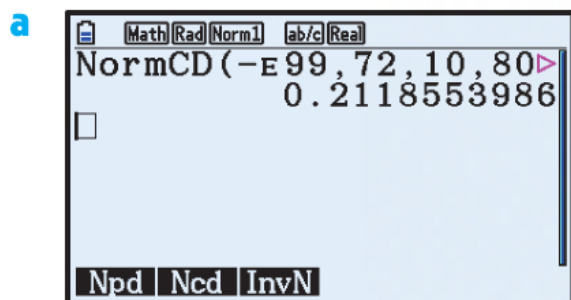
If  $P(X > k) = 0.1$   
 then  $k \approx 18.8$



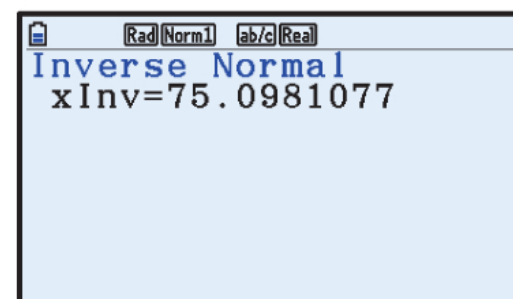
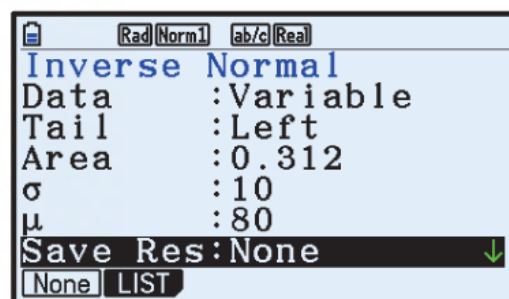
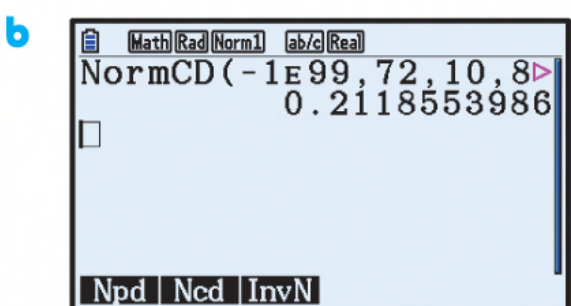
If  $P(15 - k < X < 15 + k) = 0.9$   
 then  $15 - k \approx 10.07$   
 or  $15 + k \approx 19.93$   
 $\therefore k \approx 4.93$



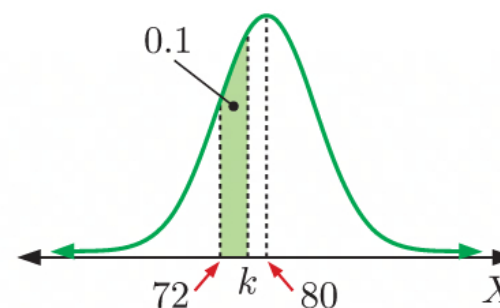
5  $X \sim N(80, 10^2)$



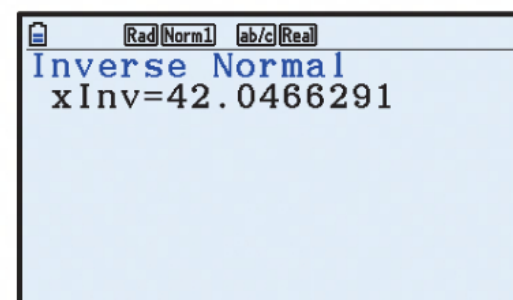
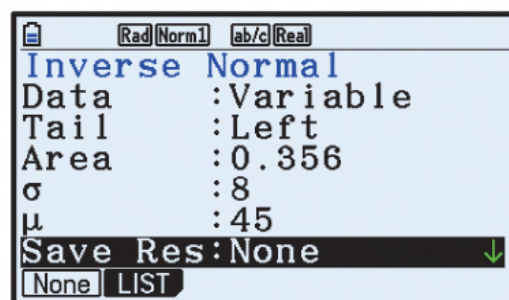
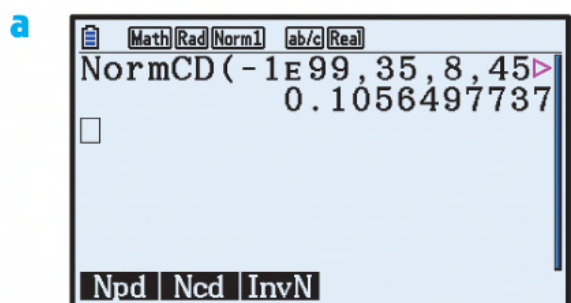
$$P(X \leq 72) \approx 0.212$$



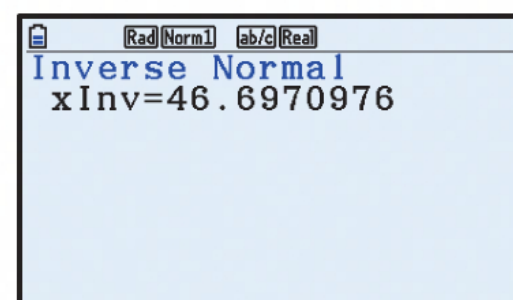
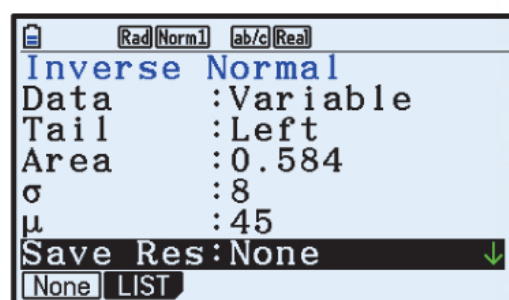
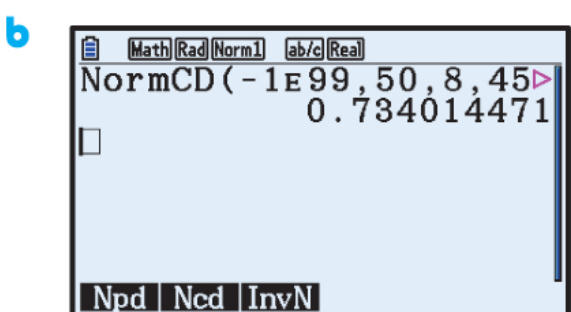
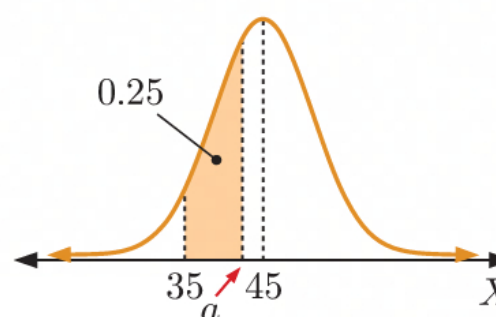
$$\begin{aligned} P(72 \leq X \leq k) &= 0.1 \\ \therefore P(X \leq k) - P(X \leq 72) &= 0.1 \\ \therefore P(X \leq k) - 0.212 &\approx 0.1 \\ \therefore P(X \leq k) &\approx 0.312 \\ \therefore k &\approx 75.1 \end{aligned}$$



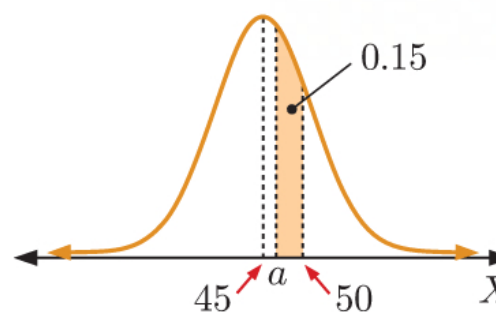
6  $X \sim N(45, 8^2)$



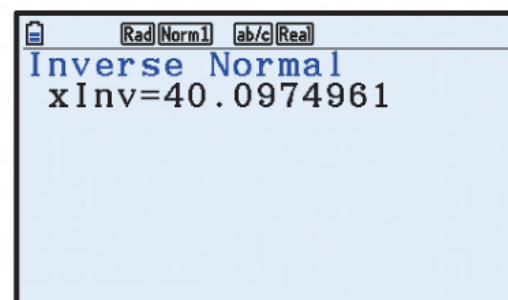
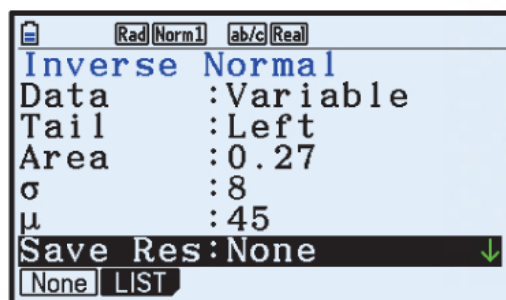
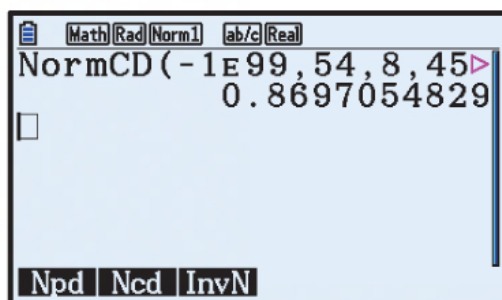
$$\begin{aligned} P(35 \leq X \leq a) &= 0.25 \\ \therefore P(X \leq a) - P(X \leq 35) &= 0.25 \\ \therefore P(X \leq a) - 0.106 &\approx 0.25 \\ \therefore P(X \leq a) &\approx 0.356 \\ \therefore a &\approx 42.0 \end{aligned}$$



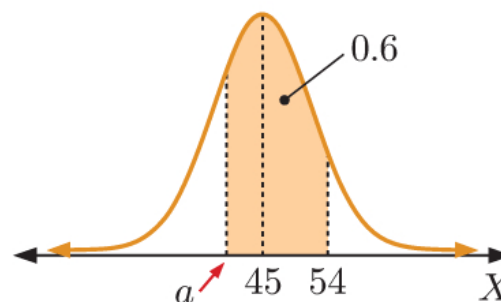
$$\begin{aligned} P(a \leq X \leq 50) &= 0.15 \\ \therefore P(X \leq 50) - P(X \leq a) &= 0.15 \\ \therefore 0.734 - P(X \leq a) &\approx 0.15 \\ \therefore P(X \leq a) &\approx 0.584 \\ \therefore a &\approx 46.7 \end{aligned}$$



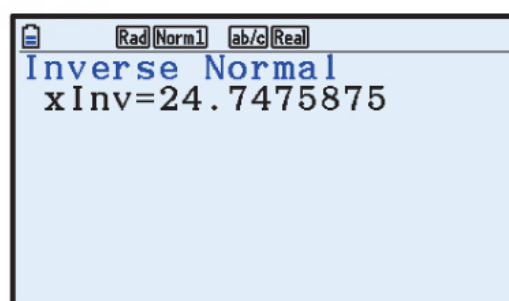
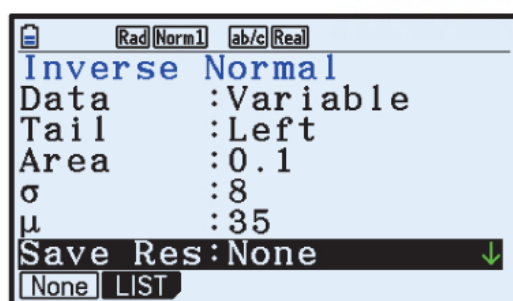
c



$$\begin{aligned} P(a \leq X \leq 54) &= 0.6 \\ \therefore P(X \leq 54) - P(X \leq a) &= 0.6 \\ \therefore 0.870 - P(X \leq a) &\approx 0.6 \\ \therefore P(X \leq a) &\approx 0.27 \\ \therefore a &\approx 40.1 \end{aligned}$$

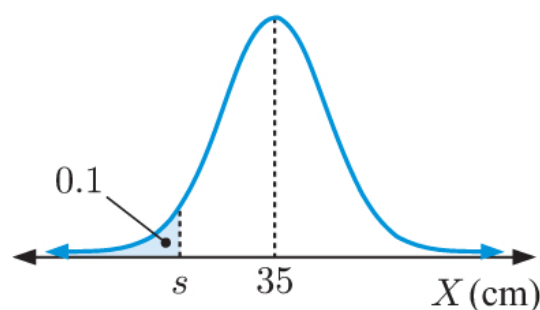


7

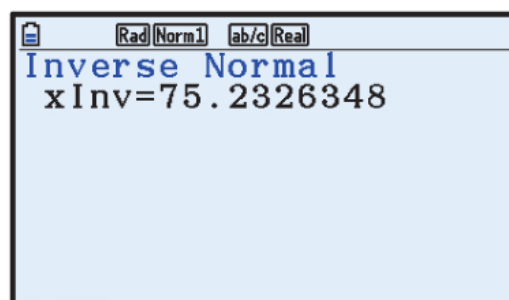
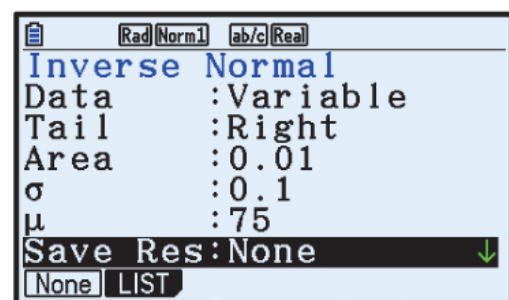


$$\begin{aligned} P(X \leq s) &= 0.1 \\ \therefore s &\approx 24.7 \end{aligned}$$

$\therefore$  the size of the smallest fish that can be harvested is about 24.7 cm.

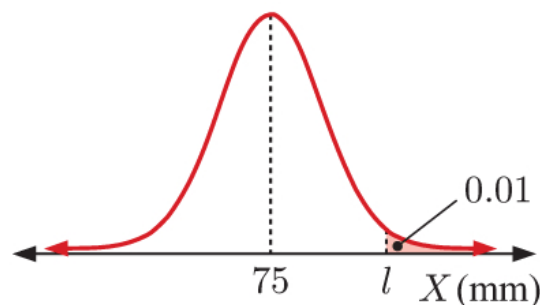


8



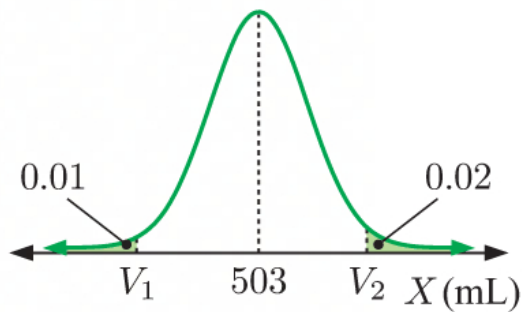
$$\begin{aligned} P(X \geq l) &= 0.01 \\ \therefore l &\approx 75.2 \end{aligned}$$

$\therefore$  the length of the smallest screw to be rejected is about 75.2 mm.



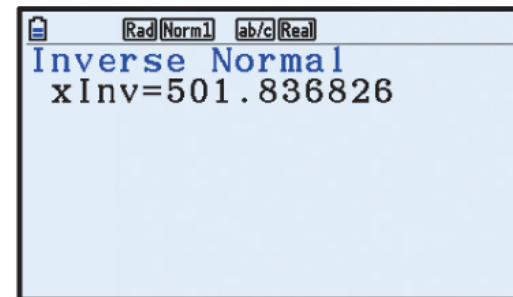
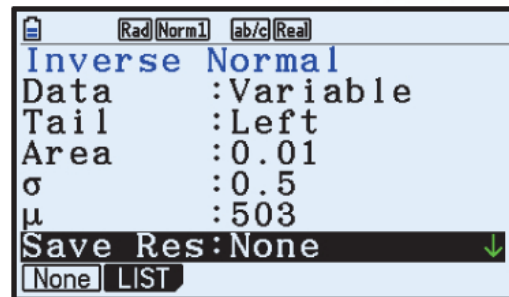


9



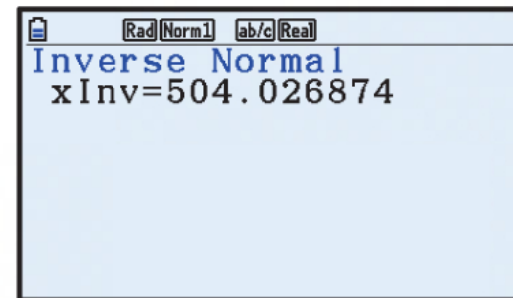
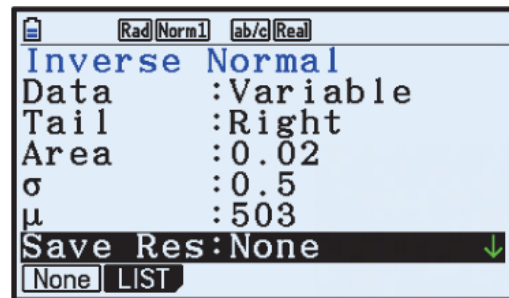
$$P(X \leq V_1) = 0.01$$

$$\therefore V_1 \approx 501.8$$



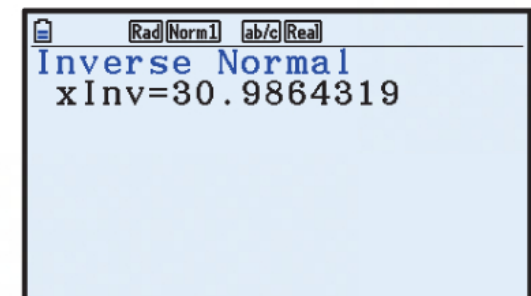
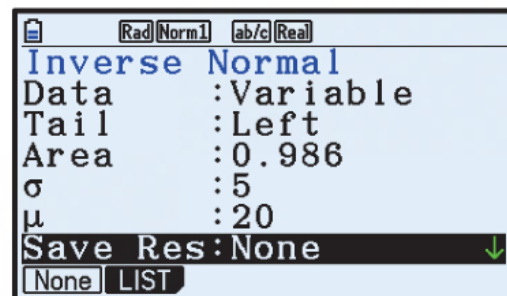
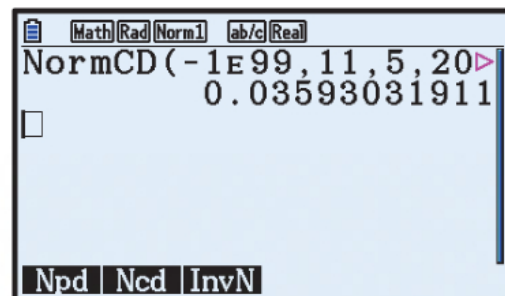
$$P(X \geq V_2) = 0.02$$

$$\therefore V_2 \approx 504.0$$



$\therefore$  the bottles which are kept have volumes ranging from about 501.8 mL to 504.0 mL.

10



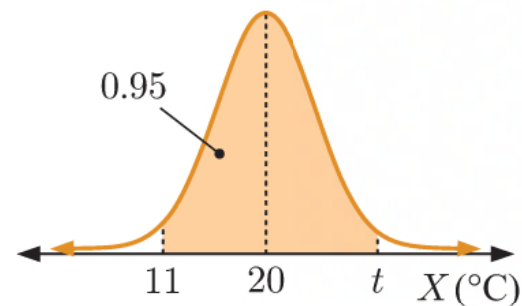
$$P(11 \leq X \leq t) = 0.95$$

$$\therefore P(X \leq t) - P(X \leq 11) = 0.95$$

$$\therefore P(X \leq t) - 0.0359 \approx 0.95$$

$$\therefore P(X \leq t) \approx 0.986$$

$$\therefore t \approx 31.0$$



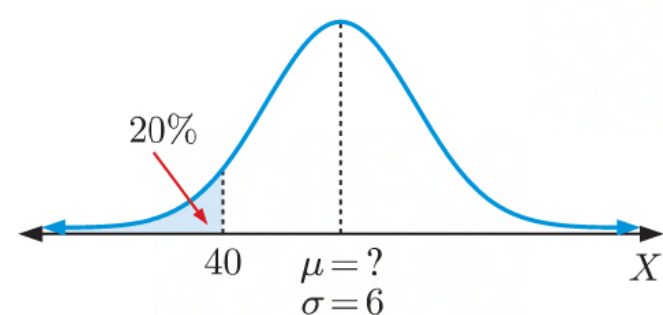
$\therefore$  the upper limit of Abbey's walking temperatures is about 31.0°C.

## EXERCISE 28F.2

1 a  $P(X < 40) = 0.2$

So, data values which are less than 40 make up only 20% of all values.

$\therefore$  the mean of  $X$  would be greater than 40.



**b**  $X \sim N(\mu, 6^2)$

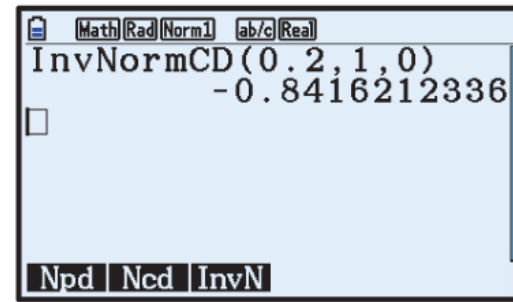
$$P(X < 40) = 0.2$$

$$\therefore P\left(Z \leq \frac{40 - \mu}{6}\right) = 0.2$$

$$\therefore \frac{40 - \mu}{6} \approx -0.8416$$

$$\therefore 40 - \mu \approx -5.0496$$

$$\therefore \mu \approx 45.0$$



**2**  $X \sim N(15, \sigma^2)$

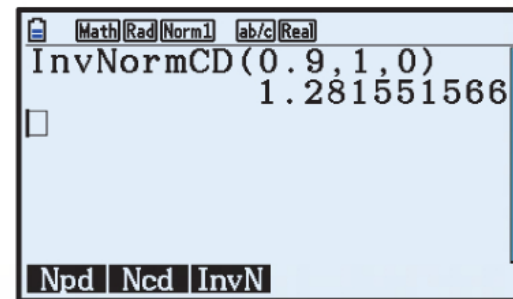
$$P(X > 20) = 0.1$$

$$\therefore P(X < 20) = 0.9$$

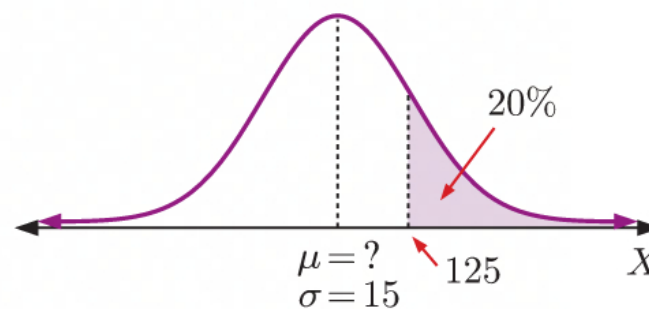
$$\therefore P\left(Z < \frac{20 - 15}{\sigma}\right) = 0.9$$

$$\therefore \frac{5}{\sigma} \approx 1.2816$$

$$\therefore \sigma \approx 3.90$$



- 3** Let the mean IQ of students at the school be  $\mu$ .  
If  $X$  is the IQ of a student at the school, then  
 $X \sim N(\mu, 15^2)$ .



$$P(X \geq 125) = 0.2$$

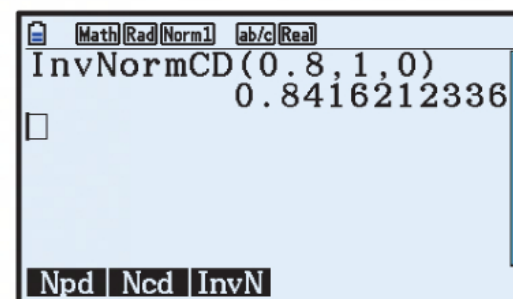
$$\therefore P(X < 125) = 0.8$$

$$\therefore P\left(Z < \frac{125 - \mu}{15}\right) = 0.8$$

$$\therefore \frac{125 - \mu}{15} \approx 0.8416$$

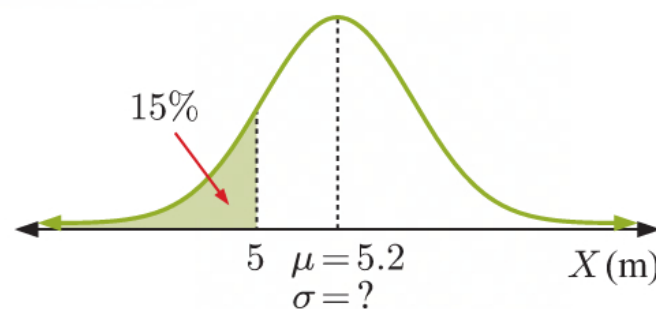
$$\therefore 125 - \mu \approx 12.624$$

$$\therefore \mu \approx 112.4$$



So, the mean IQ of students at the school is approximately 112.

- 4** Let the standard deviation of the distances jumped be  $\sigma$  m.  
If  $X$  is the distance jumped by the athlete, then  $X \sim N(5.2, \sigma^2)$ .

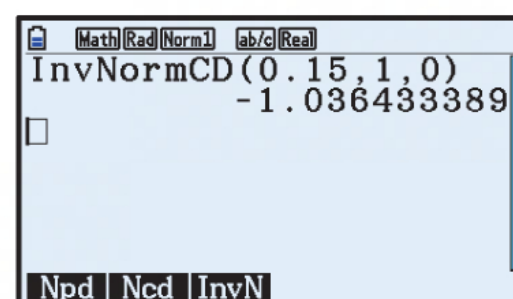


$$P(X < 5) = 0.15$$

$$\therefore P\left(Z < \frac{5 - 5.2}{\sigma}\right) = 0.15$$

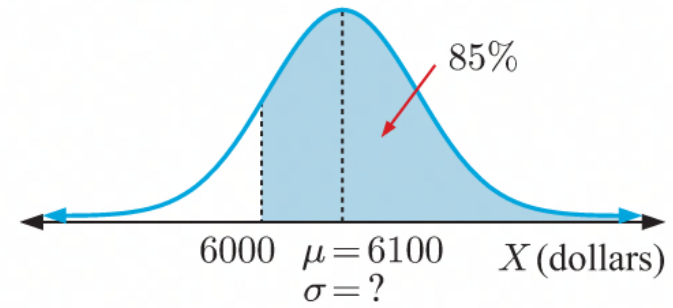
$$\therefore -\frac{0.2}{\sigma} \approx -1.036$$

$$\therefore \sigma \approx 0.193$$

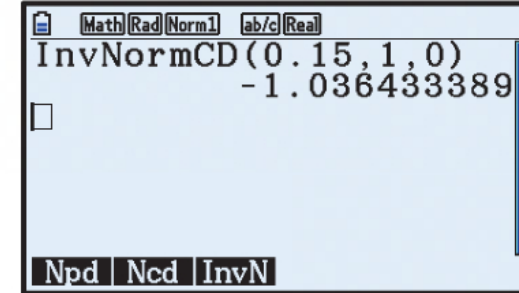


So, the standard deviation of the distances jumped is approximately 0.193 m.

- 5 Let the standard deviation of the weekly income be  $\text{€}\sigma$ .  
If  $X$  denotes the weekly income of the bakery, then  $X \sim N(6100, \sigma^2)$ .

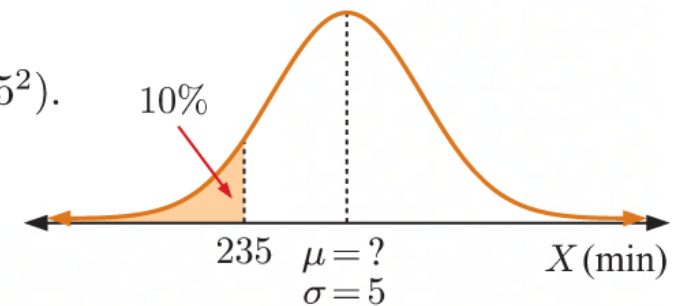


$$\begin{aligned} P(X \geq 6000) &= 0.85 \\ \therefore P(X < 6000) &= 0.15 \\ \therefore P\left(Z < \frac{6000 - 6100}{\sigma}\right) &= 0.15 \\ \therefore \frac{-100}{\sigma} &\approx -1.0364334 \\ \therefore \sigma &\approx 96.48 \end{aligned}$$

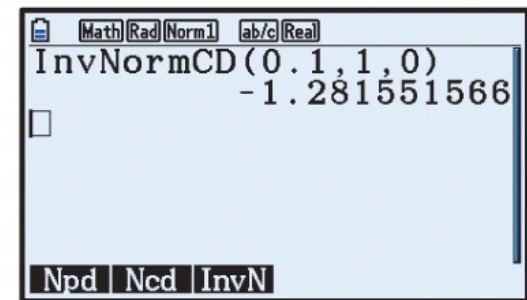


So, the standard deviation is approximately  $\text{€}96.48$ .

- 6 Let the mean arrival time be  $\mu$  minutes after midday.  
If  $X$  denotes the arrival time of a bus, then  $X \sim N(\mu, 5^2)$ .  
3:55 pm is  $3 \times 60 + 55 = 235$  minutes after midday.



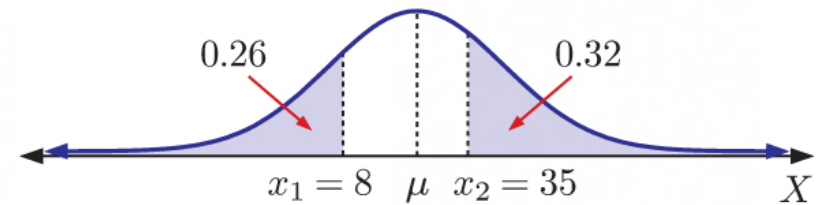
$$\begin{aligned} \text{So, } P(X \leq 235) &= 0.1 \\ \therefore P\left(Z \leq \frac{235 - \mu}{5}\right) &= 0.1 \\ \therefore \frac{235 - \mu}{5} &\approx -1.2815516 \\ \therefore 235 - \mu &\approx -6.407758 \\ \therefore \mu &\approx 241.407758 \end{aligned}$$



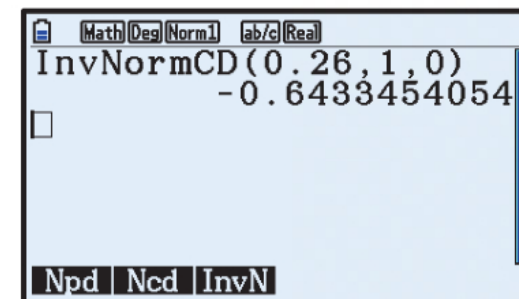
241.407758 minutes  $\approx$  4 h 1 m 24 s

$\therefore$  the mean arrival time of buses at the depot is about 4:01:24 pm.

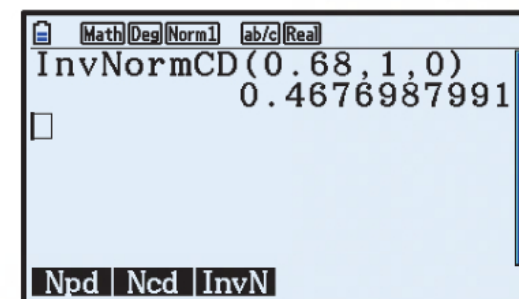
- 7  $X \sim N(\mu, \sigma^2)$  where we have to find  $\mu$  and  $\sigma$ .  
We start by finding  $z_1$  and  $z_2$  which correspond to  $x_1 = 8$  and  $x_2 = 35$ .



$$\begin{aligned} \text{Now } P(X \leq x_1) &= 0.26 \\ \therefore P\left(Z \leq \frac{8 - \mu}{\sigma}\right) &= 0.26 \\ \therefore z_1 = \frac{8 - \mu}{\sigma} &\approx -0.6433 \\ \therefore 8 - \mu &\approx -0.6433\sigma \quad \dots (1) \end{aligned}$$



$$\begin{aligned} \text{and } P(X \geq x_2) &= 0.32 \\ \therefore P(X < x_2) &= 0.68 \\ \therefore P\left(Z < \frac{35 - \mu}{\sigma}\right) &= 0.68 \\ \therefore z_2 = \frac{35 - \mu}{\sigma} &\approx 0.4677 \\ \therefore 35 - \mu &\approx 0.4677\sigma \quad \dots (2) \end{aligned}$$



Solving (1) and (2) simultaneously,  $\mu \approx 23.6$  and  $\sigma \approx 24.3$ .



- 8 a**  $X \sim N(\mu, \sigma^2)$  where we have to find  $\mu$  and  $\sigma$ .

We start by finding  $z_1$  and  $z_2$  which correspond to  $x_1 = 30$  and  $x_2 = 80$ .

$$\text{Now } P(X \leq x_1) = 0.15$$

$$\therefore P\left(Z \leq \frac{30 - \mu}{\sigma}\right) = 0.15$$

$$\therefore z_1 = \frac{30 - \mu}{\sigma} \approx -1.0364$$

$$\therefore 30 - \mu \approx -1.0364\sigma \quad \dots (1)$$

$$\text{and } P(X \geq x_2) = 0.1$$

$$\therefore P(X < x_2) = 0.9$$

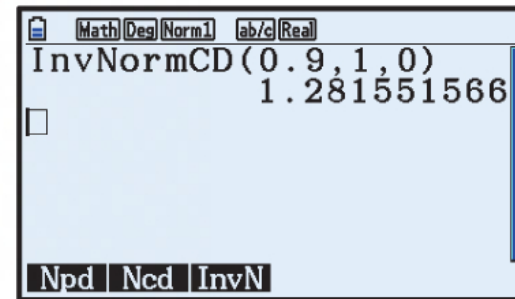
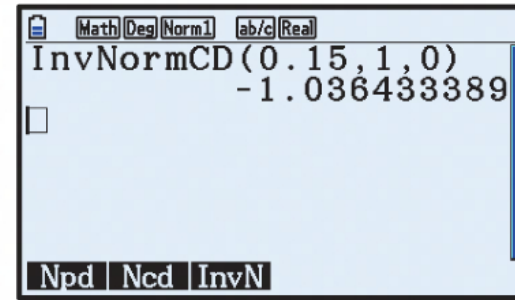
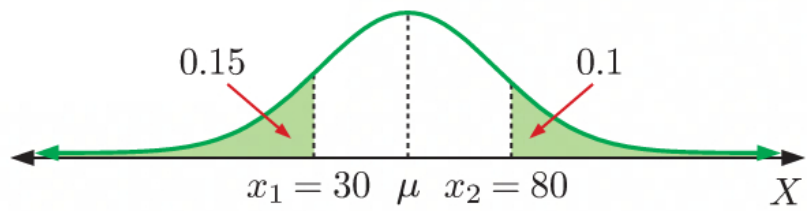
$$\therefore P\left(Z < \frac{80 - \mu}{\sigma}\right) = 0.9$$

$$\therefore z_2 = \frac{80 - \mu}{\sigma} \approx 1.2816$$

$$\therefore 80 - \mu \approx 1.2816\sigma \quad \dots (2)$$

Solving (1) and (2) simultaneously,  $\mu \approx 52.35548$  and  $\sigma \approx 21.57032$

$$\therefore \mu \approx 52.4 \quad \text{and} \quad \sigma \approx 21.6$$



- b** Let  $X$  be the result of the Mathematics examination.

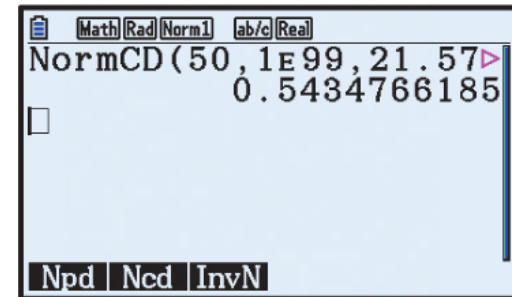
$$X \sim N(\mu, \sigma^2)$$

We know that  $P(X \geq 80) = 0.1$  and  $P(X \leq 30) = 0.15$ .

So, from **a**,  $\mu \approx 52.35548$  and  $\sigma \approx 21.57032$ .

$$P(X > 50) \approx 0.543$$

So, approximately 54.3% of students scored more than 50.



- 9 a**  $X \sim N(\mu, \sigma^2)$  where we have to find  $\mu$  and  $\sigma$ .

We start by finding  $z_1$  and  $z_2$  which correspond to  $x_1 = 3.994$  and  $x_2 = 4.006$ .

$$\text{Now } P(X \leq x_1) = 0.04$$

$$\therefore P\left(Z \leq \frac{3.994 - \mu}{\sigma}\right) = 0.04$$

$$\therefore \frac{3.994 - \mu}{\sigma} \approx -1.750686$$

$$\therefore 3.994 - \mu \approx -1.750686\sigma \quad \dots (1)$$

$$\text{and } P(X \geq x_2) = 0.05$$

$$\therefore P(X < x_2) = 0.95$$

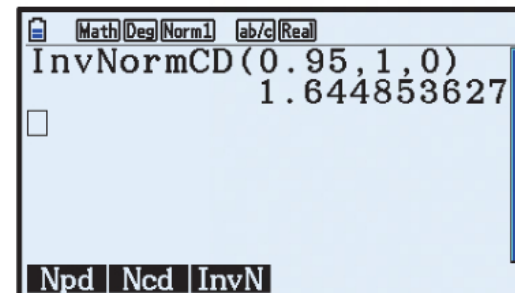
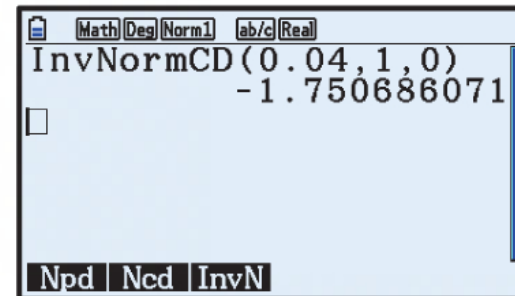
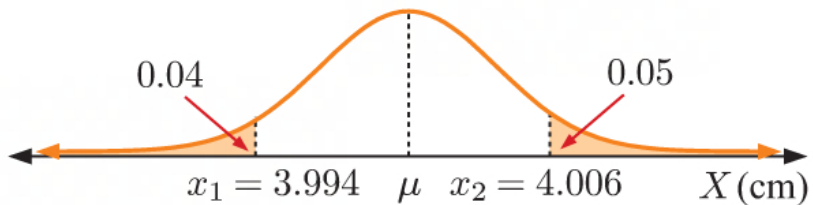
$$\therefore P\left(Z \leq \frac{4.006 - \mu}{\sigma}\right) = 0.95$$

$$\therefore \frac{4.006 - \mu}{\sigma} \approx 1.644854$$

$$\therefore 4.006 - \mu \approx 1.644854\sigma \quad \dots (2)$$

Solving (1) and (2) simultaneously,  $\mu \approx 4.000187$  and  $\sigma \approx 0.003534$

$$\therefore \mu \approx 4.00 \text{ cm} \quad \text{and} \quad \sigma \approx 0.00353 \text{ cm}$$



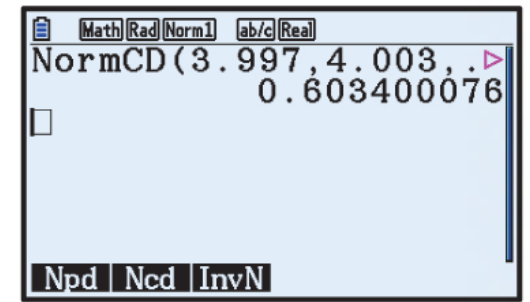


- b** From **a**,  $\mu \approx 4.000187$  and  $\sigma \approx 0.003534$

$$\therefore X \sim N(4.000187, 0.003534^2)$$

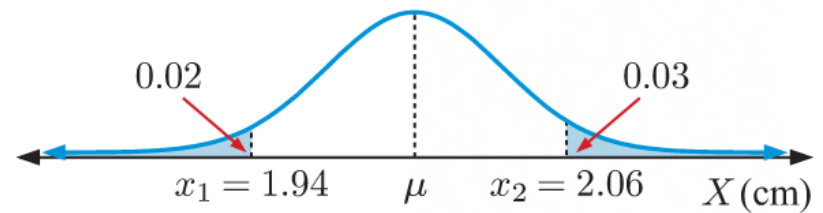
$$\therefore P(3.997 \leq X \leq 4.003) \approx 0.603$$

So, the probability that a randomly chosen piston has diameter between 3.997 cm and 4.003 cm is approximately 0.603.



- 10 a**  $X \sim N(\mu, \sigma^2)$  where we have to find  $\mu$  and  $\sigma$ .

We start by finding  $z_1$  and  $z_2$  which correspond to  $x_1 = 1.94$  and  $x_2 = 2.06$ .



$$\text{Now } P(X < x_1) = 0.02$$

$$\therefore P\left(Z < \frac{1.94 - \mu}{\sigma}\right) = 0.02$$

$$\therefore z_1 = \frac{1.94 - \mu}{\sigma} \approx -2.053749$$

$$\therefore 1.94 - \mu \approx -2.053749\sigma \quad \dots (1)$$

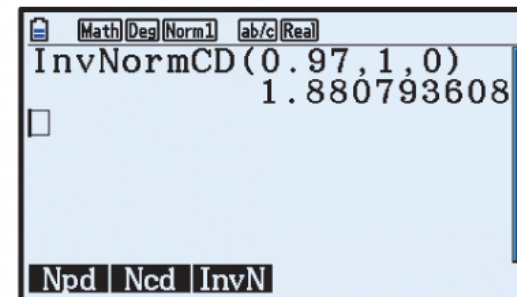
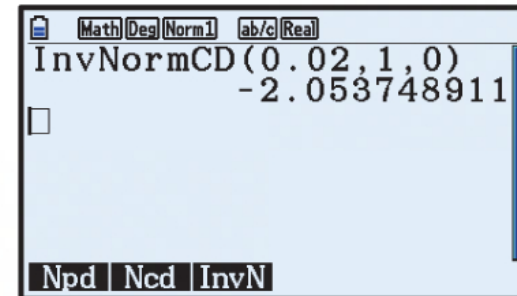
$$\text{and } P(X > x_2) = 0.03$$

$$\therefore P(X < 2.06) = 0.97$$

$$\therefore P\left(Z \leq \frac{2.06 - \mu}{\sigma}\right) = 0.97$$

$$\therefore z_2 = \frac{2.06 - \mu}{\sigma} \approx 1.880794$$

$$2.06 - \mu \approx 1.880794\sigma \quad \dots (2)$$

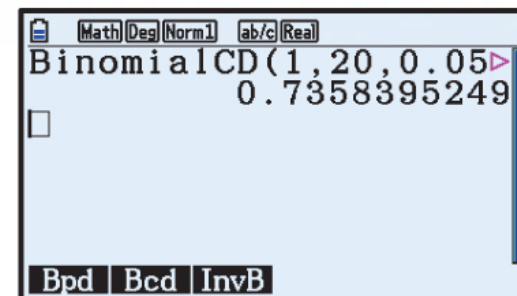


Solving (1) and (2) simultaneously,  $\mu \approx 2.002637$  and  $\sigma \approx 0.030499$

$$\therefore \mu \approx 2.00 \text{ cm} \quad \text{and} \quad \sigma \approx 0.0305 \text{ cm.}$$

- b** Let  $Y$  be the number of tokens which will not operate the machine. This is a binomial situation with the probability  $p = 0.02 + 0.03 = 0.05$  of failure to operate and  $n = 20$ . So,  $Y \sim B(20, 0.05)$ .

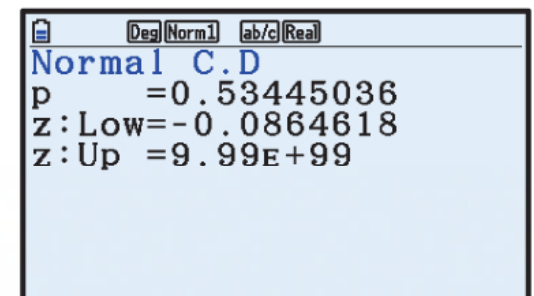
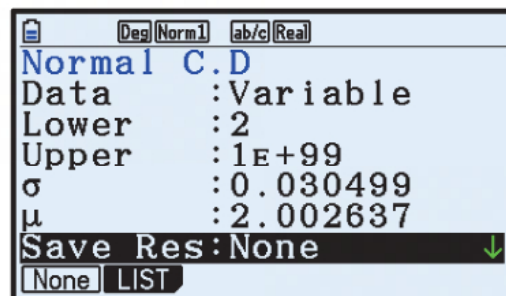
$$\therefore P(\text{at most one will not operate}) = P(Y \leq 1) \approx 0.736$$



- c**  $P(X > 2) \approx 0.5345$

The probability of one token being greater than 2 cm is approximately 0.5345.

$$\therefore P(Y = 3) \approx (0.5345)^3 \approx 0.153$$



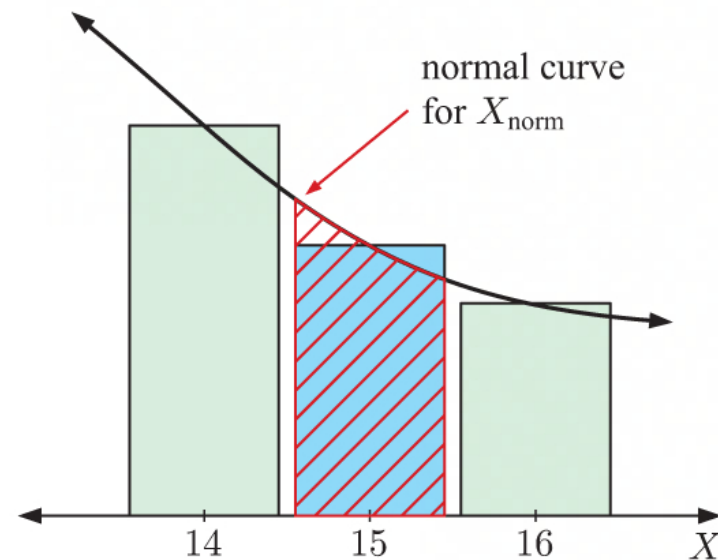
**INVESTIGATION 5****THE NORMAL APPROXIMATION TO THE BINOMIAL DISTRIBUTION**

- 1
  - a As  $n$  increases, the distribution of  $X$  approaches that of the normal distribution.
  - b As  $n$  increases, the distribution of  $X$  approaches that of the normal distribution for all values of  $p$  used.
  - c It is reasonable to approximate the binomial distribution with a normal distribution as long as  $n$  is sufficiently large. The distribution should be symmetrical about the most commonly occurring value.
  - d We expect that  $\mu = np$  and  $\sigma = \sqrt{np(1-p)}$ , like that of the binomial distribution.

- 2
  - a
 
$$\begin{aligned}\mu &= np \\ &= 50 \times 0.2 \\ &= 10\end{aligned}$$

$$\begin{aligned}\sigma &= \sqrt{np(1-p)} \\ &= \sqrt{50 \times 0.2 \times 0.8} \\ &= \sqrt{8} \\ &\approx 2.83\end{aligned}$$

- b The blue shaded area represents  $P(X = 15)$ .  
The red shaded area represents  $P(14.5 \leq X_{\text{norm}} \leq 15.5)$ .  
The two areas are approximately equal.  
 $\therefore P(X = 15) \approx P(14.5 \leq X_{\text{norm}} \leq 15.5)$ .



- c  $X_{\text{norm}} \sim N(10, (\sqrt{8})^2)$ 
  - i  $P(X \leq 10) \approx P(X_{\text{norm}} \leq 10.5)$
  - ii  $P(X < 25) = P(X \leq 24) \approx P(X_{\text{norm}} \leq 24.5)$
  - iii  $P(10 \leq X < 25) = P(10 \leq X \leq 24) \approx P(9.5 \leq X_{\text{norm}} \leq 24.5)$

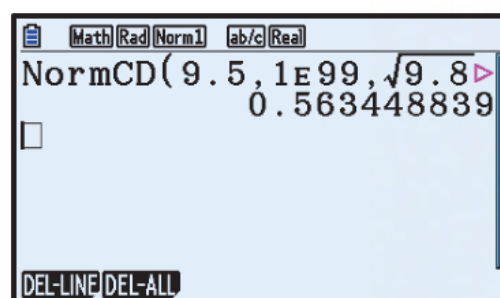
- 3  $X \sim B(500, 0.02)$ 

$$\begin{aligned}\mu &= np \\ &= 500 \times 0.02 \\ &= 10\end{aligned}$$

$$\begin{aligned}\sigma &= \sqrt{np(1-p)} \\ &= \sqrt{500 \times 0.02 \times 0.98} \\ &= \sqrt{9.8} \\ &\approx 3.13\end{aligned}$$

$$X_{\text{norm}} \sim N(10, (\sqrt{9.8})^2)$$

$$\begin{aligned}P(X \geq 10) &\approx P(X_{\text{norm}} \geq 9.5) \\ &\approx 0.563\end{aligned}$$



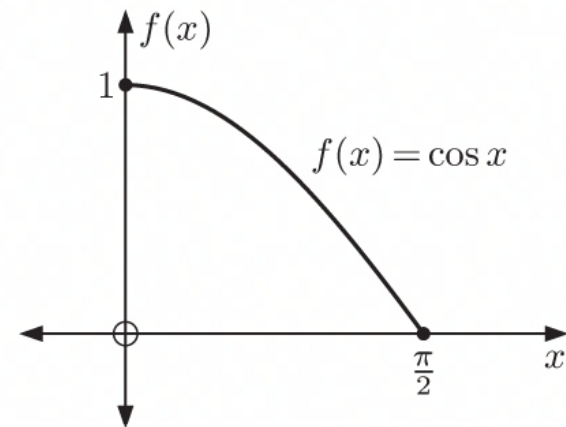
$\therefore$  we estimate that the probability that at least 10 tyres in the sample will be unfit for sale is approximately 0.563.

## REVIEW SET 28A

- 1 a • From the graph,  $f(x) \geq 0$  for all  $0 \leq x \leq \frac{\pi}{2}$ . ✓

$$\begin{aligned} \bullet \int_0^{\frac{\pi}{2}} \cos x \, dx &= \left[ \sin x \right]_0^{\frac{\pi}{2}} \\ &= 1 - 0 \\ &= 1 \quad \checkmark \end{aligned}$$

So,  $f(x)$  is a valid probability density function.



$$\begin{aligned} \text{b } P(0 \leq x \leq \frac{\pi}{4}) &= \int_0^{\frac{\pi}{4}} \cos x \, dx \\ &= \left[ \sin x \right]_0^{\frac{\pi}{4}} \\ &= \sin \frac{\pi}{4} - \sin 0 \\ &= \frac{1}{\sqrt{2}} \end{aligned}$$

- c The median is the solution of

$$\begin{aligned} \int_0^m \cos x \, dx &= \frac{1}{2} \\ \therefore \left[ \sin x \right]_0^m &= \frac{1}{2} \\ \therefore \sin m &= \frac{1}{2} \\ \therefore m &= \frac{\pi}{6} \quad \{0 \leq x \leq \frac{\pi}{2}\} \end{aligned}$$

$$\begin{aligned} \text{2 a } f(x) &= a(x+1)x(x-1)(x-2), \quad 0 \leq x \leq 1 \\ &= a(x^2+x)(x^2-3x+2) \\ &= a(x^4-2x^3-x^2+2x) \end{aligned}$$

$$\text{Now } \int_0^1 f(x) \, dx = 1$$

$$\therefore a \int_0^1 (x^4 - 2x^3 - x^2 + 2x) \, dx = 1$$

$$\therefore a \left[ \frac{1}{5}x^5 - \frac{1}{2}x^4 - \frac{1}{3}x^3 + x^2 \right]_0^1 = 1$$

$$\therefore a \left[ \frac{1}{5} - \frac{1}{2} - \frac{1}{3} + 1 \right] = 1$$

$$\therefore a \left( \frac{11}{30} \right) = 1$$

$$\therefore a = \frac{30}{11}$$

$$\text{b } P(0 \leq X \leq 0.2)$$

$$= a \int_0^{0.2} (x^4 - 2x^3 - x^2 + 2x) \, dx$$

$$= \frac{30}{11} \left[ \frac{1}{5}x^5 - \frac{1}{2}x^4 - \frac{1}{3}x^3 + x^2 \right]_0^{0.2}$$

$$= \frac{30}{11} \left[ \frac{1}{15625} - \frac{1}{1250} - \frac{1}{375} + \frac{1}{25} \right]$$

$$= \frac{30}{11} \left[ \frac{3431}{93750} \right]$$

$$\approx 0.0998$$

$$\text{c } E(X) = \int_0^1 x f(x) \, dx$$

$$= \frac{30}{11} \int_0^1 (x^5 - 2x^4 - x^3 + 2x^2) \, dx$$

$$= \frac{30}{11} \left[ \frac{1}{6}x^6 - \frac{2}{5}x^5 - \frac{1}{4}x^4 + \frac{2}{3}x^3 \right]_0^1$$

$$= \frac{30}{11} \left[ \frac{1}{6} - \frac{2}{5} - \frac{1}{4} + \frac{2}{3} \right]$$

$$= \frac{30}{11} \left[ \frac{11}{60} \right]$$

$$= \frac{1}{2}$$



$$\begin{aligned}
 \text{d} \quad Y &= 5 - \frac{X}{2} \\
 \therefore E(Y) &= E\left(5 - \frac{X}{2}\right) \\
 &= 5 - \frac{E(X)}{2} \\
 &= 5 - \frac{\frac{1}{2}}{2} \quad \{\text{from c}\} \\
 &= 5 - \frac{1}{4} \\
 &= \frac{19}{4}
 \end{aligned}$$

- 3 a  $f(x) = ax(4 - x^2)$ ,  $0 \leq x \leq 2$  is a probability density function.

$$\therefore \int_0^2 ax(4 - x^2) dx = 1$$

$$\therefore a \int_0^2 (4x - x^3) dx = 1$$

$$\therefore a \left[ 2x^2 - \frac{1}{4}x^4 \right]_0^2 = 1$$

$$\therefore a(8 - 4) = 1$$

$$\therefore a = \frac{1}{4}$$

$$\begin{aligned}
 \text{c} \quad \mu &= \int_0^2 x f(x) dx \\
 &= \frac{1}{4} \int_0^2 (4x^2 - x^4) dx \\
 &= \frac{1}{4} \left[ \frac{4}{3}x^3 - \frac{1}{5}x^5 \right]_0^2 \\
 &= \frac{1}{4} \left( \frac{32}{3} - \frac{32}{5} \right) \\
 &= \frac{1}{4} \left( \frac{64}{15} \right) \\
 &= \frac{16}{15}
 \end{aligned}$$

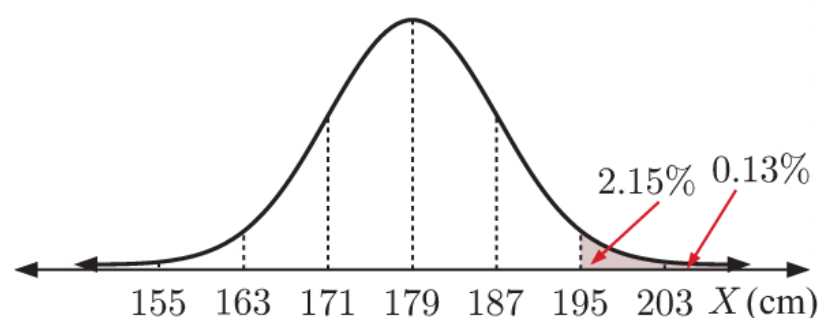
$$\begin{aligned}
 \text{b} \quad P(0 \leq X \leq 1) &= \frac{1}{4} \int_0^1 (4x - x^3) dx \\
 &= \frac{1}{4} \left[ 2x^2 - \frac{1}{4}x^4 \right]_0^1 \\
 &= \frac{1}{4} \left( 2 - \frac{1}{4} \right) \\
 &= \frac{1}{4} \left( \frac{7}{4} \right) \\
 &= \frac{7}{16}
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad \text{Var}(X) &= \int_0^2 x^2 f(x) dx - \mu^2 \\
 &= \frac{1}{4} \int_0^2 (4x^3 - x^5) dx - \left( \frac{16}{15} \right)^2 \\
 &= \frac{1}{4} \left[ x^4 - \frac{1}{6}x^6 \right]_0^2 - \frac{256}{225} \\
 &= \frac{1}{4} \left( 16 - \frac{32}{3} \right) - \frac{256}{225} \\
 &= \frac{1}{4} \left( \frac{16}{3} \right) - \frac{256}{225} \\
 &= \frac{4}{3} - \frac{256}{225} \\
 &= \frac{44}{225} \\
 \therefore \sigma &= \sqrt{\text{Var}(X)} \\
 &= \frac{\sqrt{44}}{15}
 \end{aligned}$$

- 4 Let  $X$  be the height of a 17 year old boy.

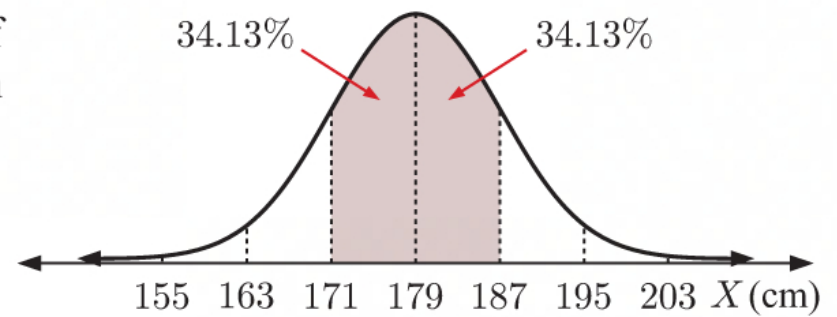
$$X \sim N(179, 8^2)$$

- a About  $2.15\% + 0.13\% = 2.28\%$  of 17 year old boys have a height more than 195 cm.

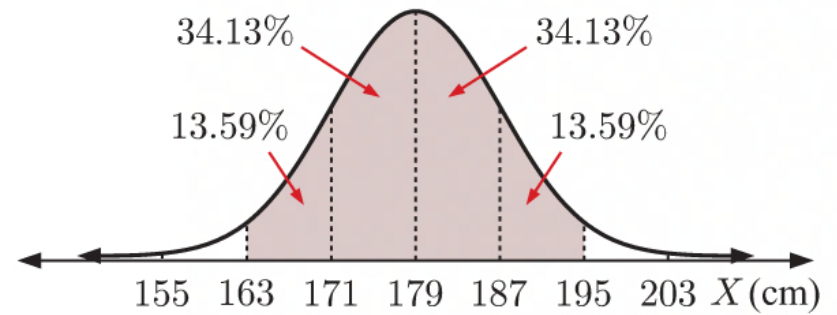




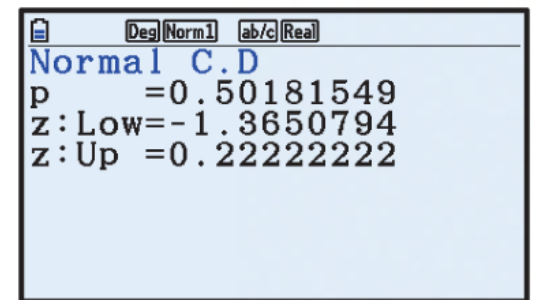
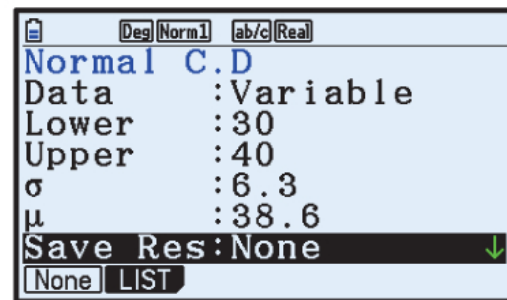
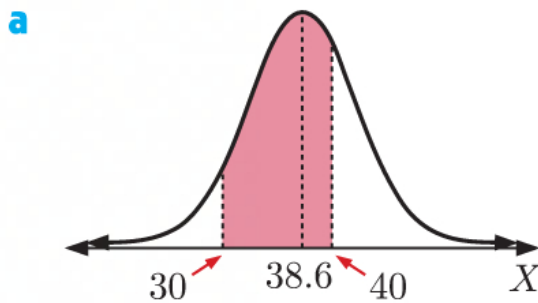
- b** About  $34.13\% + 34.13\% = 68.26\%$  of 17 year old boys have a height between 171 cm and 187 cm.



- c** About  $13.59\% + 34.13\% + 34.13\% + 13.59\% = 95.44\%$  of 17 year old boys have a height between 163 cm and 195 cm.

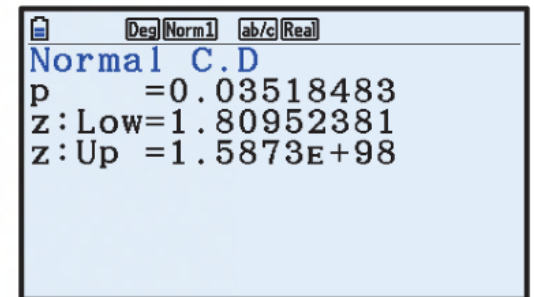
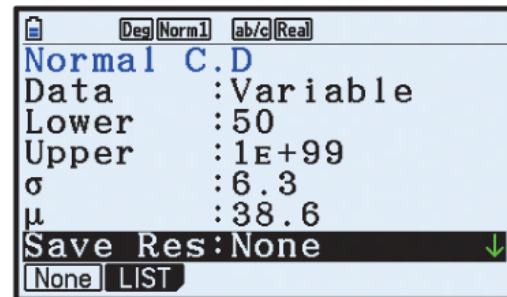
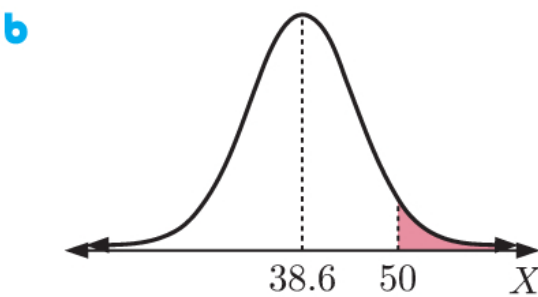


- 5** Let  $X$  grams be the weight of the edible part of a randomly selected Coffin Bay oyster.



$$P(30 < X < 40) \approx 0.502 \approx 50.2\%$$

About 50.2% of oysters have an edible part that weighs between 30 g and 40 g.

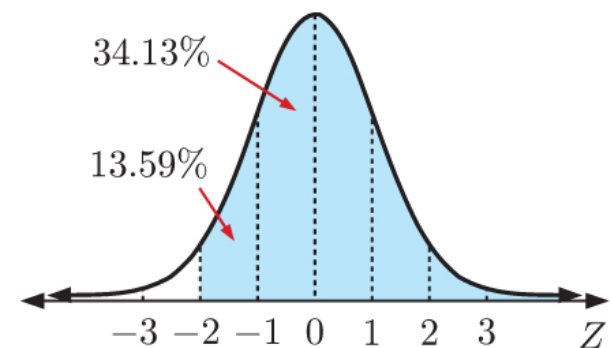


$$P(X > 50) \approx 0.0352$$

$\therefore$  we would expect about  $0.0352 \times 200 \approx 7$  oysters to have an edible part that weighs more than 50 g.

- 6 a** A  $z$ -score of  $-2$  indicates that Harri's score is 2 standard deviations below the mean.

- b** About  $13.59\% + 34.13\% + 50\% = 97.72\%$  of students obtained a better score than Harri.

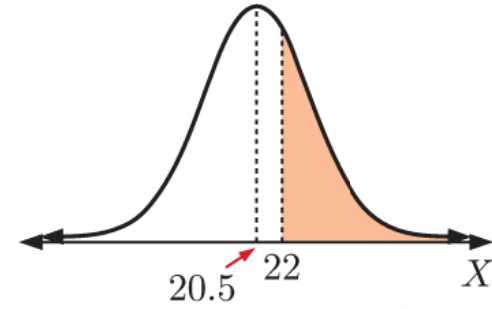
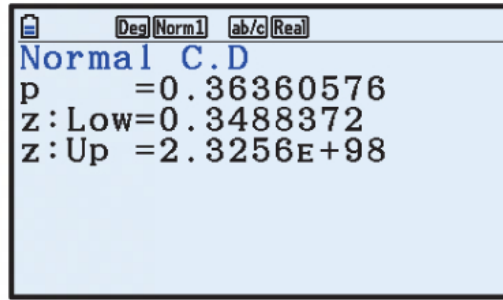
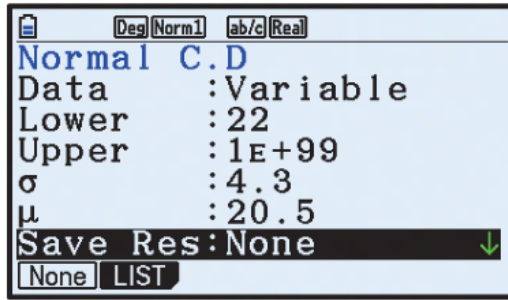


- c**  $\mu = 61$  and  $\mu - 2\sigma = 47$   
 $\therefore 61 - 2\sigma = 47$   
 $\therefore 2\sigma = 14$   
 $\therefore \sigma = 7$

The standard deviation of the test scores was 7.

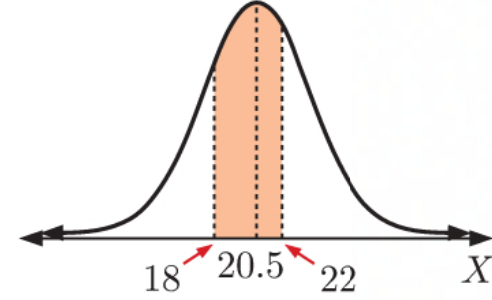
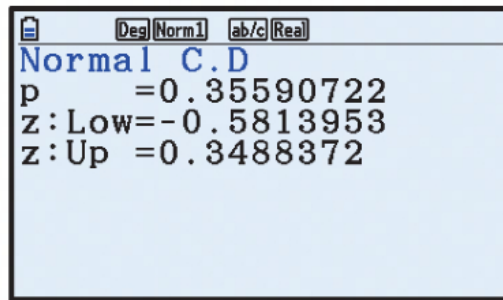
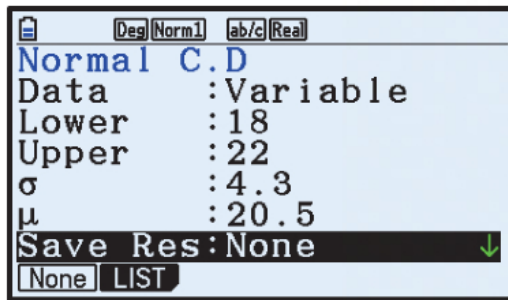
7  $X \sim N(20.5, 4.3^2)$

a



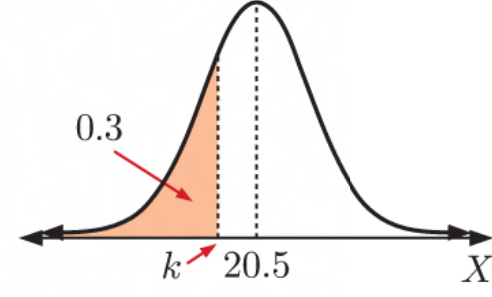
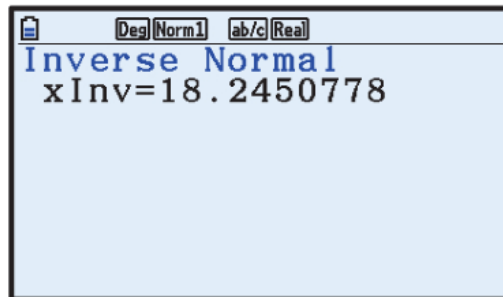
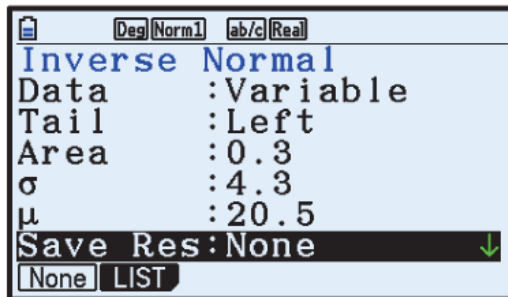
$$P(X \geq 22) \approx 0.364$$

b



$$P(18 \leq X \leq 22) \approx 0.356$$

c



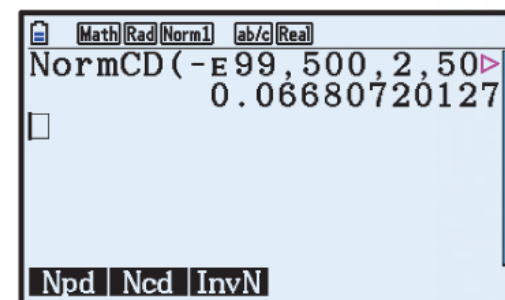
If  $P(X \leq k) = 0.3$   
then  $k \approx 18.2$

8 a  $X \sim N(503, 2^2)$

$$P(X < 500) \approx 0.066\,807$$

$$\approx 0.0668$$

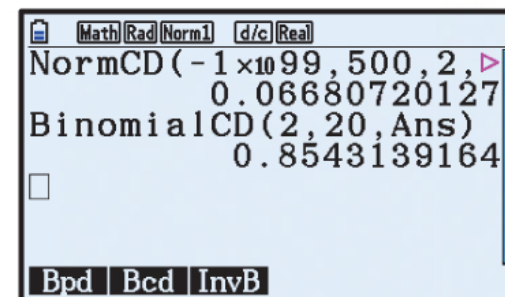
$\therefore$  approximately 6.68% of the bags are underweight.



b Let  $Y$  be the number of bags which are underweight.

$$Y \sim B(20, 0.066\,807)$$

$$\therefore P(Y \leq 2) \approx 0.854$$



9

a

```

Rad|Norm1|ab/c|Real
Inverse Normal
Data :Variable
Tail :Left
Area :0.7
σ :6
μ :25
Save Res:None
[None] LIST

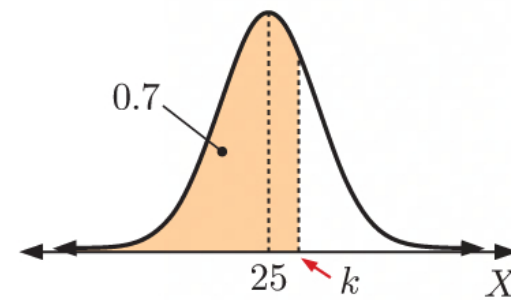
```

```

Rad|Norm1|ab/c|Real
Inverse Normal
xInv=28.1464031

```

If  $P(X \leq k) = 0.7$   
then  $k \approx 28.1$



b

```

Rad|Norm1|ab/c|Real
Inverse Normal
Data :Variable
Tail :Right
Area :0.4
σ :6
μ :25
Save Res:None
[None] LIST

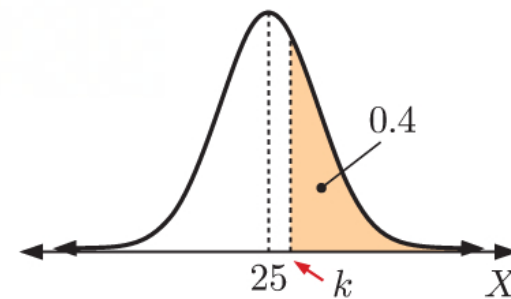
```

```

Rad|Norm1|ab/c|Real
Inverse Normal
xInv=26.5200826

```

If  $P(X \geq k) = 0.4$   
then  $k \approx 26.5$



c

```

Math|Rad|Norm1|ab/c|Real
NormCD(-1E99,20,6,25▶
0.202328381
[None] LIST

```

```

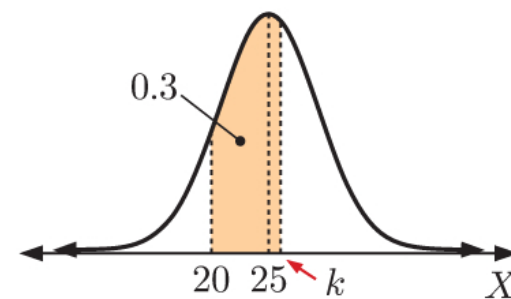
Rad|Norm1|ab/c|Real
Inverse Normal
Data :Variable
Tail :Left
Area :0.502
σ :6
μ :25
Save Res:None
[None] LIST

```

```

Rad|Norm1|ab/c|Real
Inverse Normal
xInv=25.0300797

```

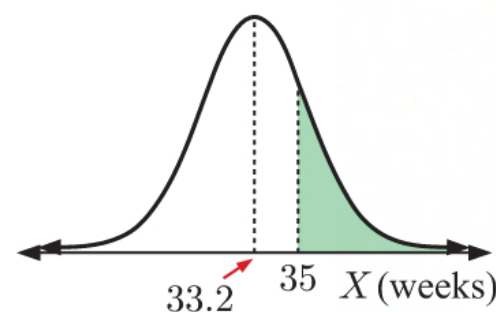
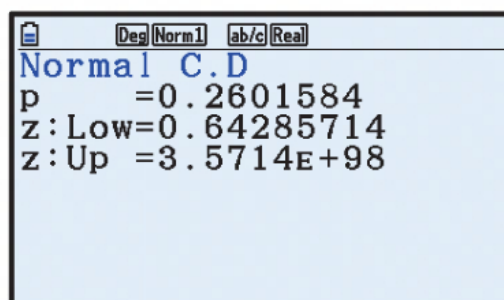
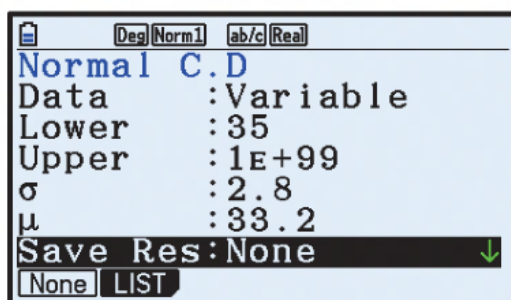
$$\begin{aligned}
 P(20 \leq X \leq k) &= 0.3 \\
 \therefore P(X \leq k) - P(X \leq 20) &= 0.3 \\
 \therefore P(X \leq k) - 0.202 &\approx 0.3 \\
 \therefore P(X \leq k) &\approx 0.502 \\
 \therefore k &\approx 25.0
 \end{aligned}$$




- 10** Let  $X$  be the life of a battery in weeks.

$$X \sim N(33.2, 2.8^2)$$

**a**



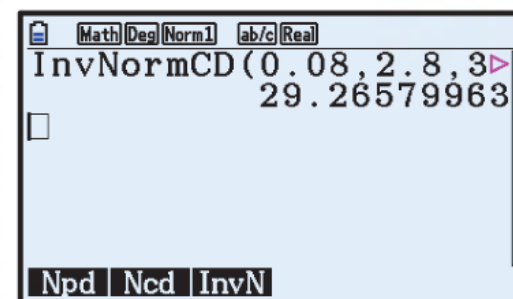
$$P(X \geq 35) \approx 0.260$$

- b** We need to find  $k$  such that

$$P(X \leq k) = 0.08$$

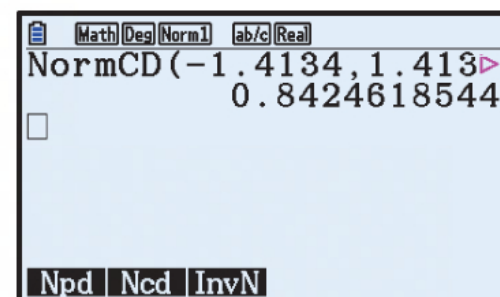
$$\therefore k \approx 29.3$$

So, the manufacturer can expect the batteries to last about 29.3 weeks before 8% of them fail.



- 11**  $X \sim N(\mu, 2.83^2)$

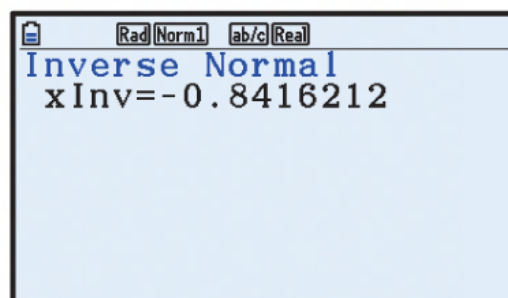
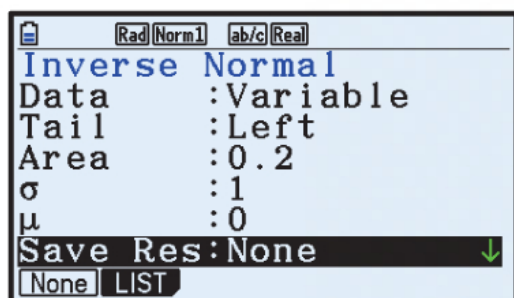
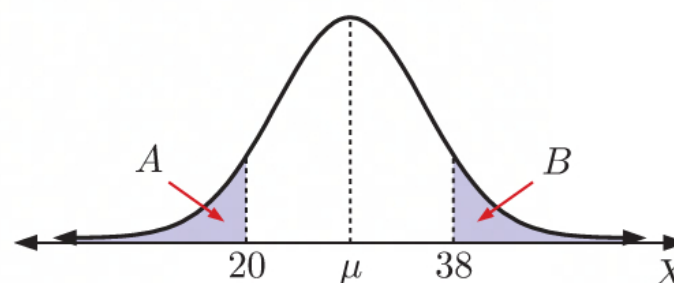
$$\begin{aligned} \therefore P(-4 < X - \mu < 4) &= P\left(\frac{-4}{2.83} < \frac{X - \mu}{2.83} < \frac{4}{2.83}\right) \\ &\approx P(-1.4134 < Z < 1.4134) \\ &\approx 0.842 \end{aligned}$$



- 12 a** Since Area  $A$  = Area  $B$ , 20 and 38 must be equal distances away from the mean  $\mu$ , because of the symmetry of the normal distribution.

$\therefore \mu$  is halfway between 20 and 38, so

$$\mu = \frac{20 + 38}{2} = 29$$



$$\text{Now } P(X \leq 20) = 0.2$$

$$\therefore P\left(Z \leq \frac{20 - 29}{\sigma}\right) = 0.2$$

$$\therefore P\left(Z \leq -\frac{9}{\sigma}\right) = 0.2$$

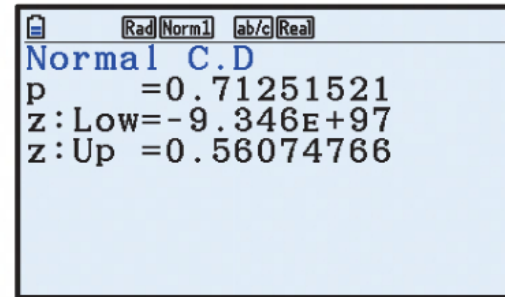
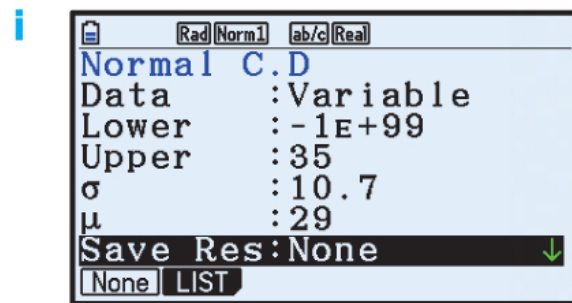
$$\therefore -\frac{9}{\sigma} \approx -0.8416$$

$$\therefore \sigma \approx 10.69$$

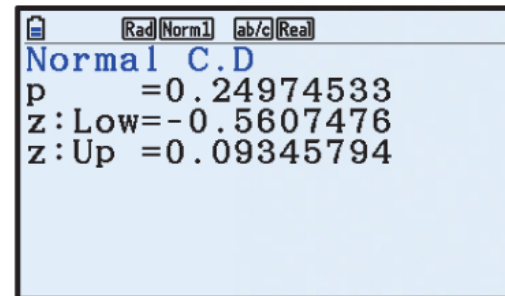
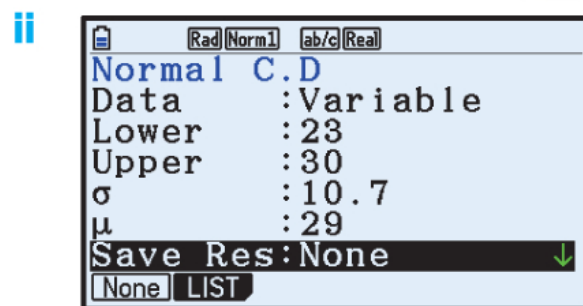
$$\therefore \mu = 29, \quad \sigma \approx 10.7$$



**b** Using the values obtained for  $\mu$  and  $\sigma$  in **a**:



$$P(X \leq 35) \approx 0.713$$

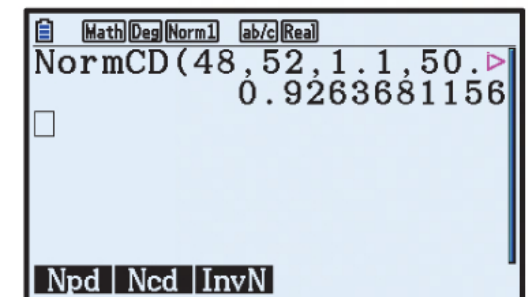


$$P(23 \leq X \leq 30) \approx 0.250$$

**13 a i**  $X_A \sim N(50.2, 1.1^2)$

$$P(48 \leq X_A \leq 52) \approx 0.9264$$

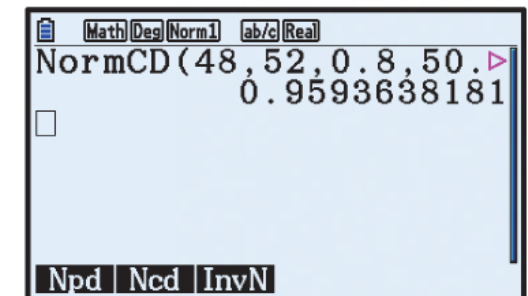
$\therefore$  the probability that a nail from machine A needs to be rejected is about  $1 - 0.9264 \approx 0.0736$ .



**ii**  $X_B \sim N(50.6, 0.8^2)$

$$P(48 \leq X_B \leq 52) \approx 0.9594$$

$\therefore$  the probability that a nail from machine B needs to be rejected is about  $1 - 0.9594 \approx 0.0406$ .



$$\begin{aligned} \mathbf{b} \quad P(\text{made by machine A} \mid \text{rejected}) &= \frac{P(\text{made by machine A} \cap \text{rejected})}{P(\text{rejected})} \\ &\approx \frac{0.5 \times 0.0736}{0.5 \times 0.0736 + 0.5 \times 0.0406} \\ &\approx 0.644 \end{aligned}$$

The probability that the nail was made by machine A *given* that it should be rejected is approximately 0.644.

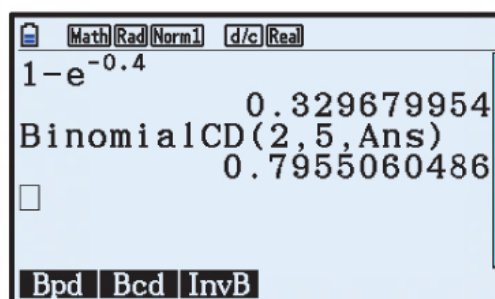
- 14 a** Let  $T$  be the lifetime in years of a solar cell component.

$$\begin{aligned}
 \therefore P(T \leq 1) &= \int_0^1 0.4e^{-0.4t} dt \\
 &= [-e^{-0.4t}]_0^1 \\
 &= -e^{-0.4} - (-e^0) \\
 &= 1 - e^{-0.4} \\
 &\approx 0.32968 \\
 &\approx 0.330
 \end{aligned}$$

- b** Let  $X$  be the number of components not working after one year.

$$X \sim B(5, 0.32968)$$

$$\begin{aligned}
 \therefore P(\text{solar cell still operates}) \\
 &= P(X \leq 2) \quad \{\text{at least 3 work}\} \\
 &\approx 0.796
 \end{aligned}$$



## REVIEW SET 28B

- 1 a** • From the graph,  $f(x) \geq 0$  for all  $1 \leq x \leq 4$ . ✓

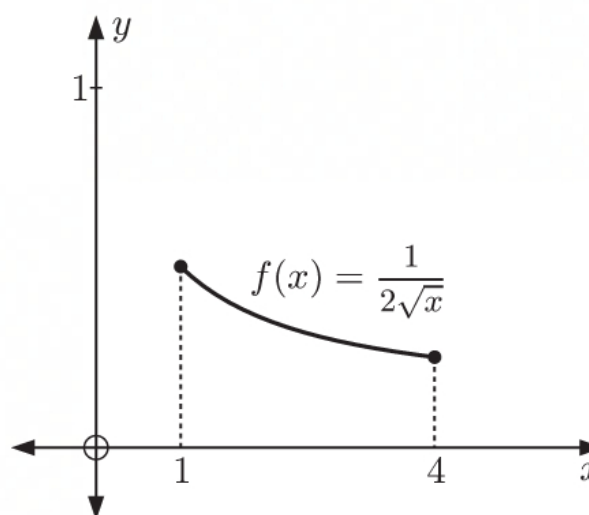
$$\begin{aligned}
 \bullet \int_1^4 \frac{1}{2\sqrt{x}} dx &= \int_1^4 \frac{1}{2} x^{-\frac{1}{2}} dx \\
 &= [\sqrt{x}]_1^4 \\
 &= 2 - 1 \\
 &= 1 \quad \checkmark
 \end{aligned}$$

So,  $f(x)$  is a valid probability density function.

$$\begin{aligned}
 \text{b } P(1 \leq X \leq 2) &= \int_1^2 \frac{1}{2\sqrt{x}} dx \\
 &= [\sqrt{x}]_1^2 \\
 &= \sqrt{2} - 1 \\
 &\approx 0.414
 \end{aligned}$$

- c** The median is the value of  $m$  such that  $\int_1^m f(x) dx = 0.5$ .

If  $\int_1^2 \frac{1}{2\sqrt{x}} dx \approx 0.414$ , then the median must be greater than 2.



**d** The median is the solution of  $\int_0^m \frac{1}{2\sqrt{x}} dx = \frac{1}{2}$

$$\therefore \left[ \sqrt{x} \right]_1^m = \frac{1}{2}$$

$$\therefore \sqrt{m} - 1 = \frac{1}{2}$$

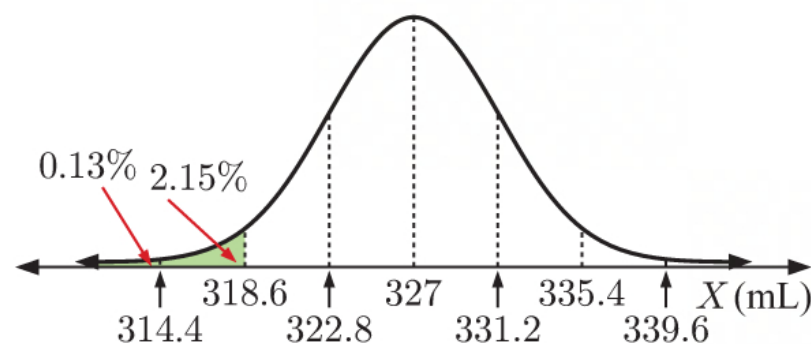
$$\therefore \sqrt{m} = \frac{3}{2}$$

$$\therefore m = \frac{9}{4}$$

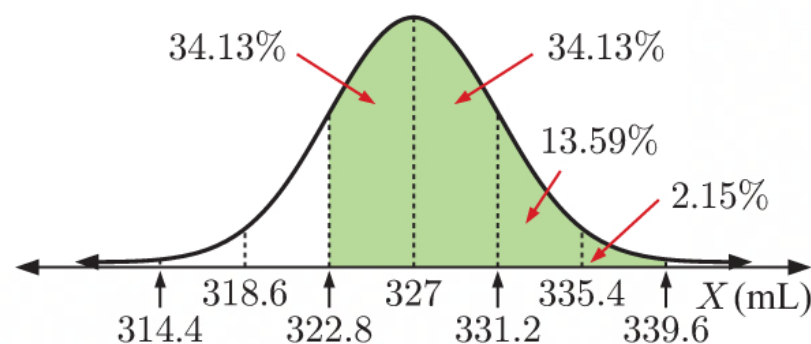
**2** Let  $X$  mL be the contents of the container.

$$X \sim N(327, 4.2^2)$$

**a i** About  $0.13\% + 2.15\% = 2.28\%$  of cans have contents less than 318.6 mL.

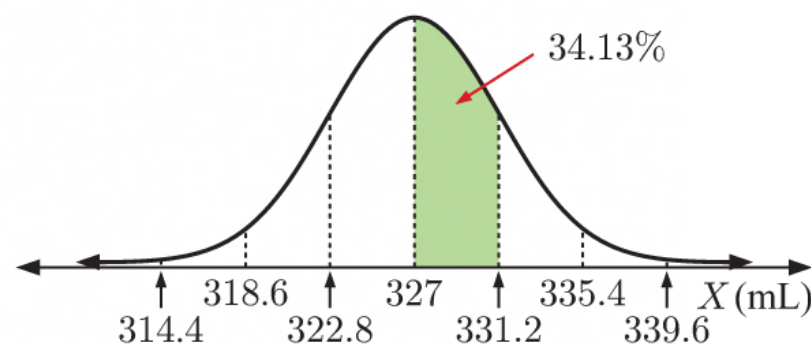


**ii** About  $34.13\% + 34.13\% + 13.59\% + 2.15\% = 84.0\%$  of cans have contents between 322.8 mL and 339.6 mL.



**b** About 34.13% of cans have contents between 327 mL and 331.2 mL.

$$P(\text{contents between 327 mL and 331.2 mL}) \approx 0.3413$$



**3 a**  $f(x) = ax(x - 3)$ ,  $0 \leq x \leq 2$  is a probability density function.

$$\therefore \int_0^2 ax(x - 3) dx = 1$$

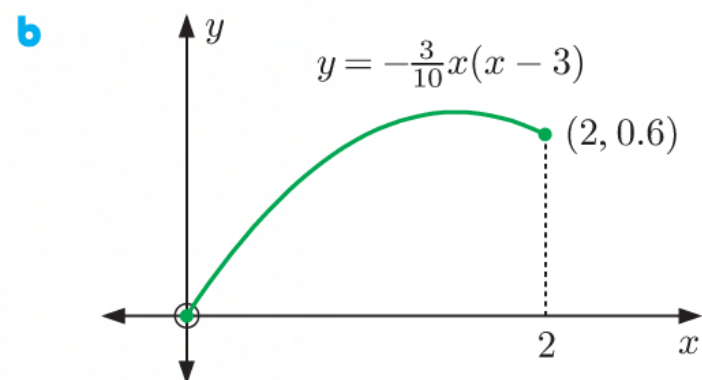
$$\therefore a \int_0^2 (x^2 - 3x) dx = 1$$

$$\therefore a \left[ \frac{1}{3}x^3 - \frac{3}{2}x^2 \right]_0^2 = 1$$

$$\therefore a \left[ \frac{8}{3} - 6 \right] = 1$$

$$\therefore a \left( -\frac{10}{3} \right) = 1$$

$$\therefore a = -\frac{3}{10}$$



$$\begin{aligned}
 \text{c i } \mu &= \int_0^2 x f(x) dx \\
 &= \int_0^2 -\frac{3}{10} x^2 (x-3) dx \\
 &= -\frac{3}{10} \int_0^2 (x^3 - 3x^2) dx \\
 &= -\frac{3}{10} \left[ \frac{1}{4} x^4 - x^3 \right]_0^2 \\
 &= -\frac{3}{10} (4 - 8) \\
 &= -\frac{3}{10} (-4) \\
 &= \frac{6}{5}
 \end{aligned}$$

$$\begin{aligned}
 \text{ii } &\int_0^2 x^2 f(x) dx \\
 &= \int_0^2 -\frac{3}{10} x^3 (x-3) dx \\
 &= -\frac{3}{10} \int_0^2 (x^4 - 3x^3) dx \\
 &= -\frac{3}{10} \left[ \frac{1}{5} x^5 - \frac{3}{4} x^4 \right]_0^2 \\
 &= -\frac{3}{10} \left( \frac{32}{5} - 12 \right) \\
 &= \frac{42}{25} \\
 \text{Now, } \text{Var}(X) &= \int_0^2 x^2 f(x) dx - \mu^2 \\
 &= \frac{42}{25} - \left( \frac{6}{5} \right)^2 \\
 &= \frac{6}{25}
 \end{aligned}$$

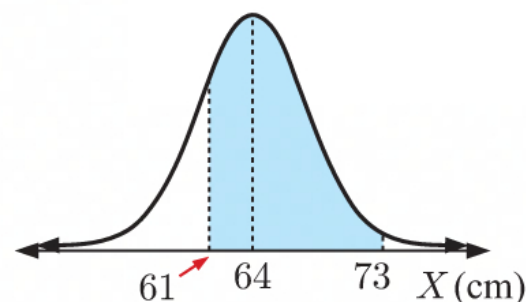
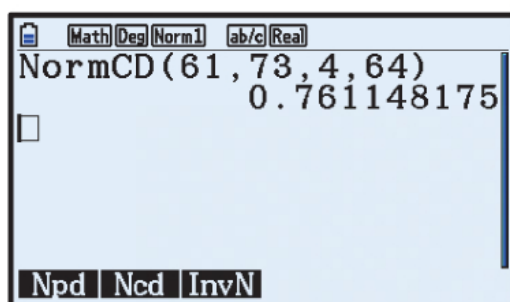
$$\begin{aligned}
 \text{iii } \sigma &= \sqrt{\text{Var}(X)} \\
 &= \sqrt{\frac{6}{25}} \\
 &= \frac{\sqrt{6}}{5}
 \end{aligned}$$

$$\begin{aligned}
 \text{d } P(1 \leq X \leq 2) &= \int_1^2 -\frac{3}{10} x(x-3) dx \\
 &= -\frac{3}{10} \int_1^2 (x^2 - 3x) dx \\
 &= -\frac{3}{10} \left[ \frac{1}{3} x^3 - \frac{3}{2} x^2 \right]_1^2 \\
 &= -\frac{3}{10} \left( \frac{8}{3} - 6 - \frac{1}{3} + \frac{3}{2} \right) \\
 &= \frac{13}{20}
 \end{aligned}$$

4 Let  $X$  cm be the arm length of a randomly selected 18 year old female.

$$X \sim N(64, 4^2)$$

a i

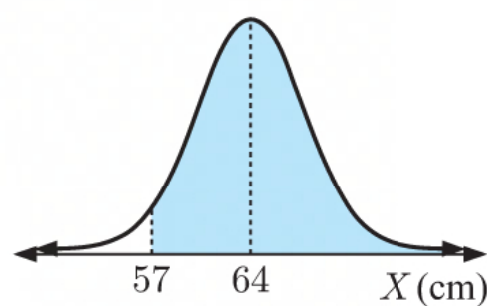
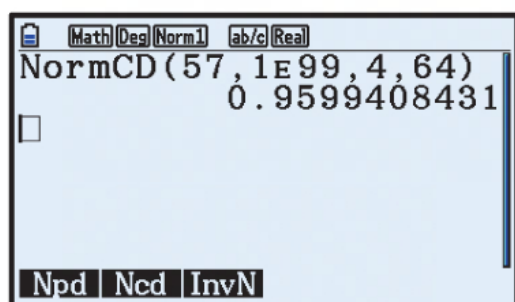


$$P(61 < X < 73) \approx 0.761$$

$\therefore$  approximately 76.1% of 18 year old females have an arm length between 61 cm and 73 cm.



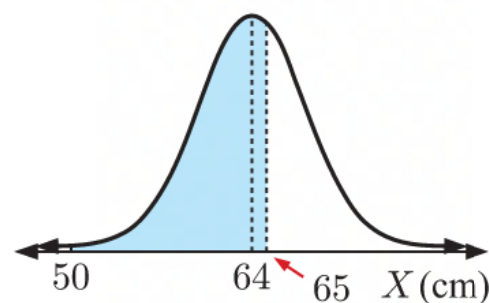
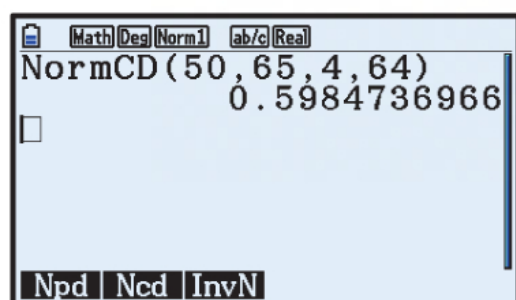
ii



$$P(X > 57) \approx 0.960$$

$\therefore$  approximately 96.0% of 18 year old females have an arm length greater than 57 cm.

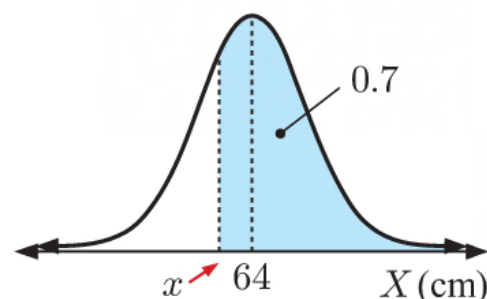
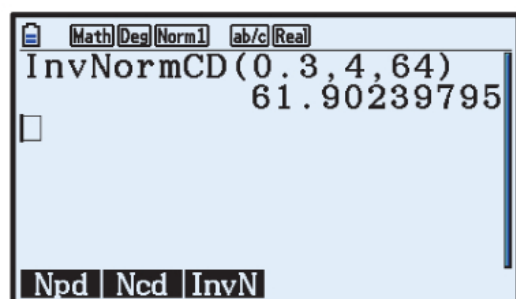
b



$$P(50 < X < 65) \approx 0.598$$

$\therefore$  the probability that an 18 year old female has an arm length in the range 50 cm to 65 cm is approximately 0.598.

c



$$P(X > x) = 0.7$$

$$\therefore P(X < x) = 0.3$$

$$\therefore x \approx 61.9$$

5

$$\text{If } P(-k \leq Z \leq k) = 0.95$$

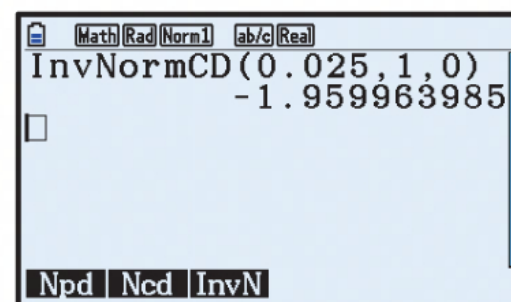
$$\therefore 1 - P(Z \leq -k) - P(Z \geq k) = 0.95$$

$$\therefore 1 - 2P(Z \leq -k) = 0.95 \quad \{\text{symmetry of the normal distribution}\}$$

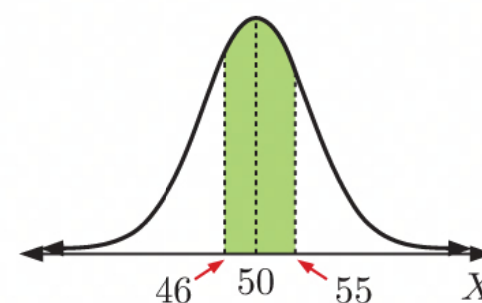
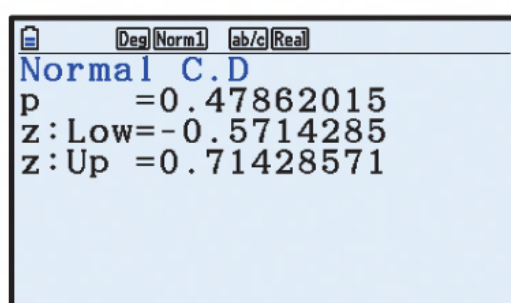
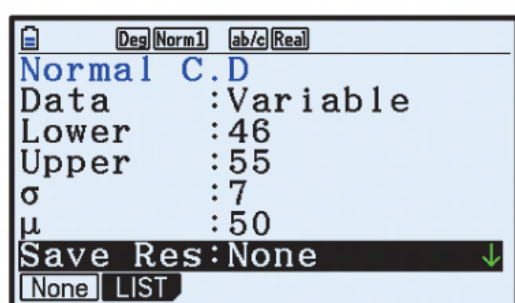
$$\therefore 2P(Z \leq -k) = 0.05$$

$$\therefore P(Z \leq -k) = 0.025$$

$$\therefore k \approx 1.96$$

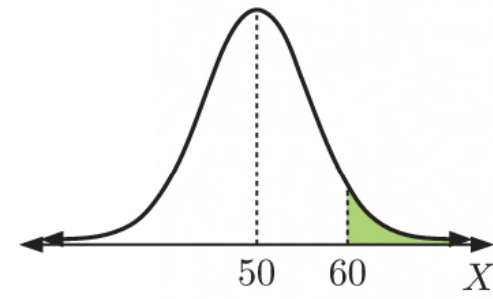
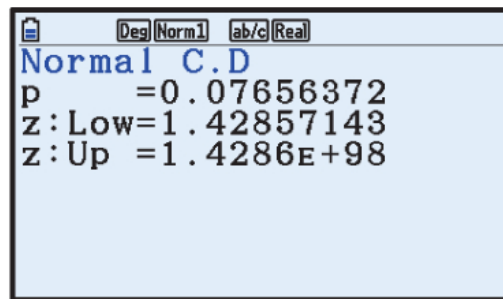
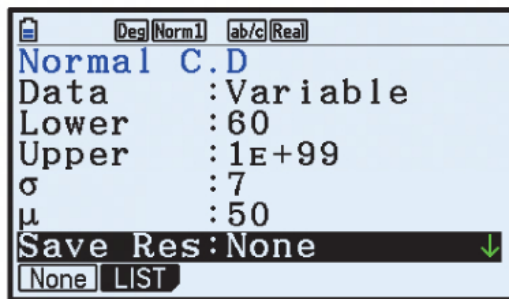
6  $X \sim N(50, 7^2)$ 

a



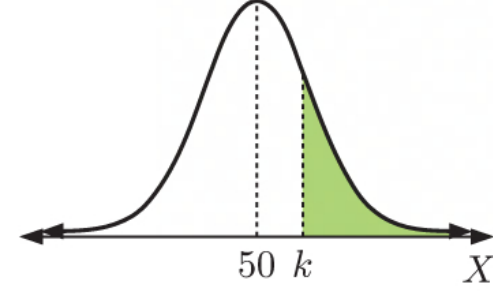
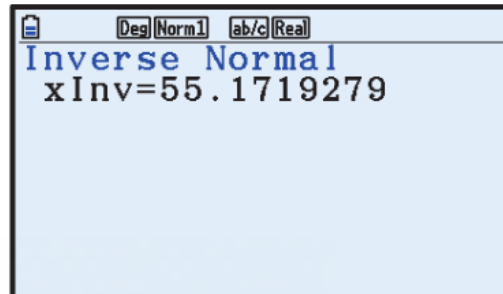
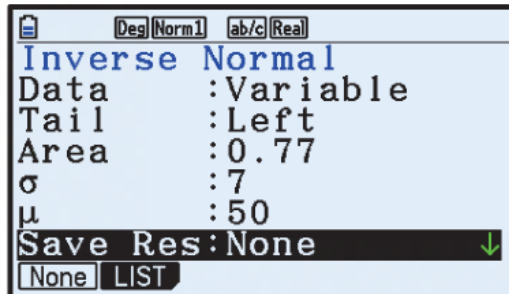
$$P(46 \leq X \leq 55) \approx 0.479$$

b



$$P(X \geq 60) \approx 0.0766$$

c



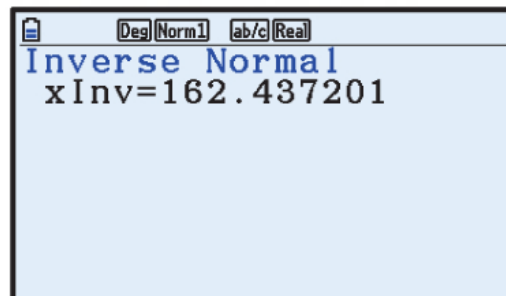
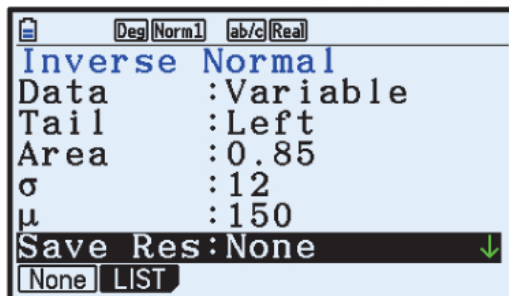
$$\text{If } P(X > k) = 0.23$$

$$\therefore P(X < k) = 0.77$$

$$\therefore k \approx 55.2$$

7 Let  $X$  seconds be the time a contestant holds their breath.

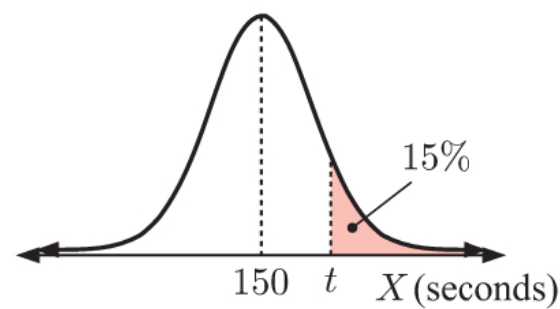
$$X \sim N(150, 12^2)$$



$$P(X > t) = 0.15$$

$$\therefore P(X < t) = 0.85$$

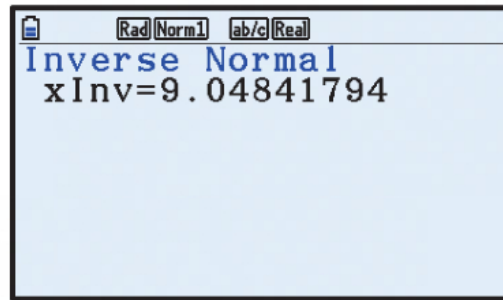
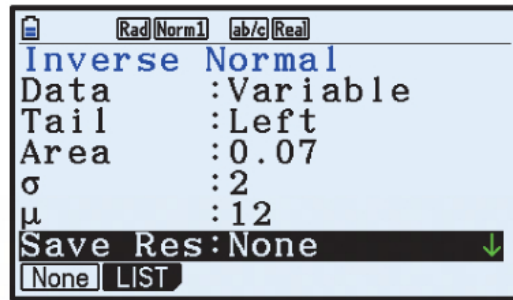
$$\therefore t \approx 162.4$$



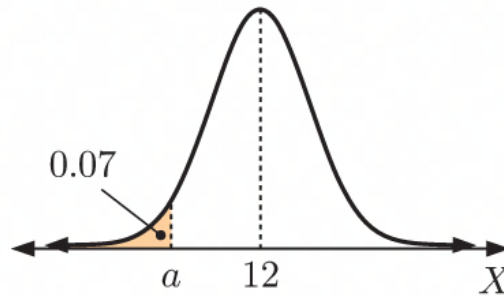
To advance to the final round, a contestant would need to hold their breath for about 162 seconds.

8  $X \sim N(12, 2^2)$

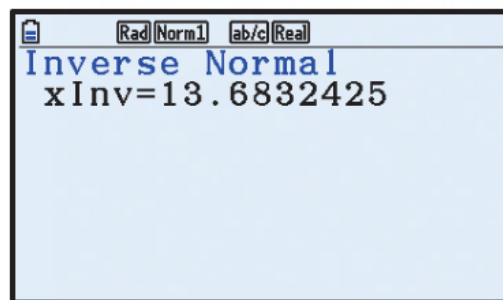
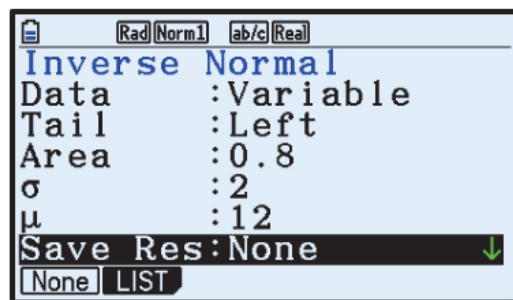
a



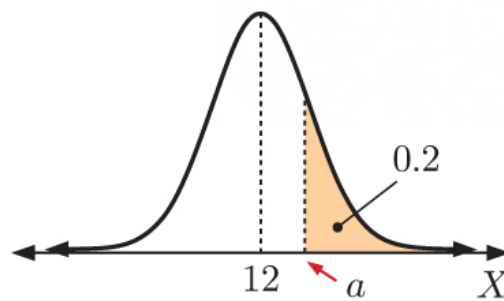
If  $P(X < a) = 0.07$   
then  $a \approx 9.05$



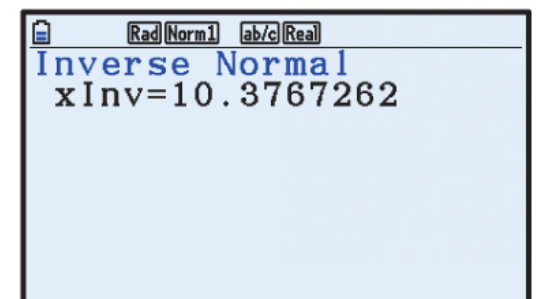
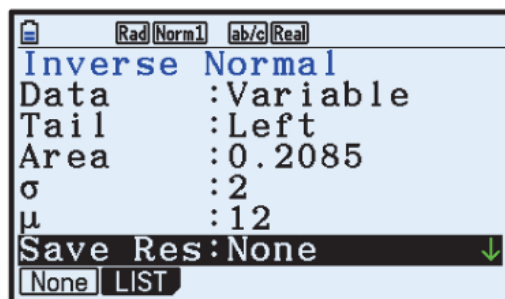
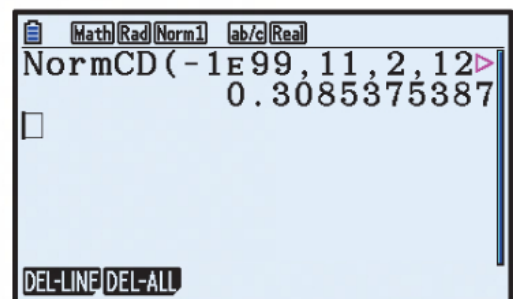
b



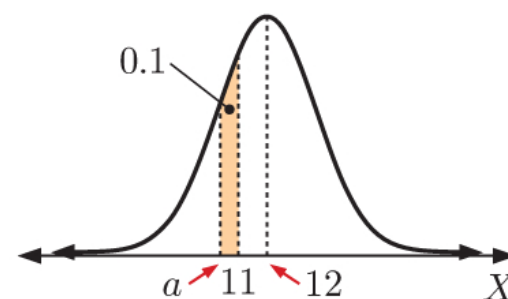
If  $P(X > a) = 0.2$   
 $\therefore P(X < a) = 0.8$   
 $\therefore a \approx 13.7$



c



If  $P(a \leq X \leq 11) = 0.1$   
 $\therefore P(X \leq 11) - P(X \leq a) = 0.1$   
 $\therefore 0.3085 - P(X \leq a) \approx 0.1$   
 $\therefore P(X \leq a) \approx 0.2085$   
 $\therefore a \approx 10.4$



9  $X \sim N(\mu, 2.1^2), \quad Z \sim N(0, 1^2)$

$$\begin{aligned} P(Z > -1.7) &= P(X > 5.4) \\ &= P\left(\frac{X - \mu}{2.1} > \frac{5.4 - \mu}{2.1}\right) \\ &= P\left(Z > \frac{5.4 - \mu}{2.1}\right) \end{aligned}$$

$$\therefore \frac{5.4 - \mu}{2.1} = -1.7$$

$$\therefore 5.4 - \mu = -3.57$$

$$\therefore \mu = 8.97$$

10  $P(X < 90) \approx 0.975$

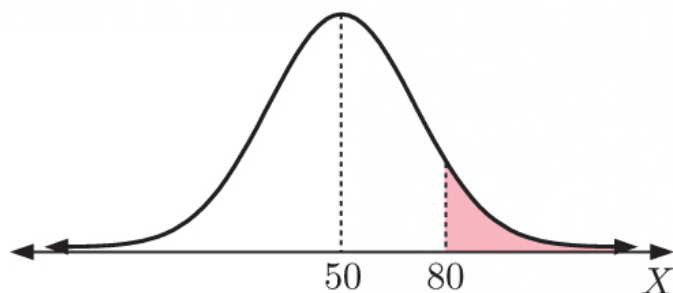
$$\therefore P\left(Z < \frac{90 - 50}{\sigma}\right) \approx 0.975$$

$$\therefore P\left(Z < \frac{40}{\sigma}\right) \approx 0.975$$

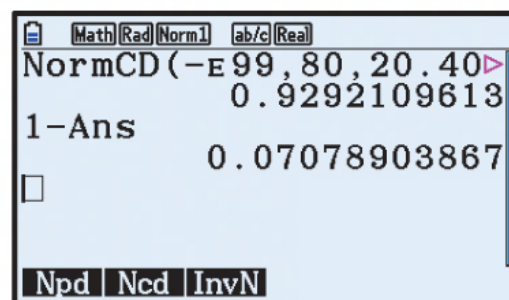
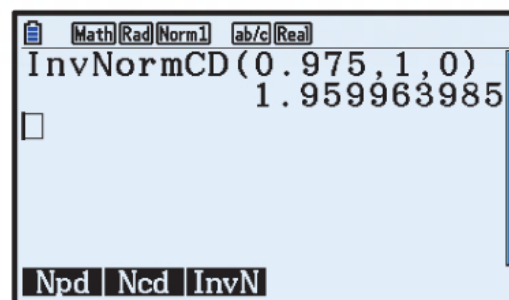
$$\therefore \frac{40}{\sigma} \approx 1.95996$$

$$\therefore \sigma \approx 20.409$$

So,  $X \sim N(50, 20.409^2)$



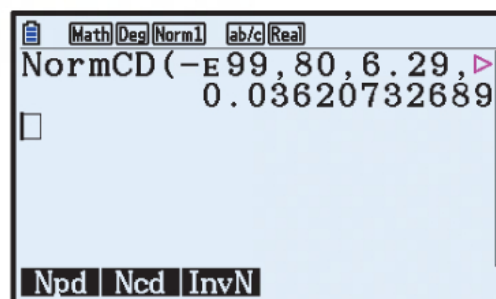
$$\begin{aligned} \text{Now, the shaded area} &= P(X \geq 80) \\ &= 1 - P(X < 80) \\ &\approx 1 - 0.9292 \\ &\approx 0.0708 \text{ units}^2 \end{aligned}$$



- 11 a Let  $X_F$  be the weight in kilograms of a female ostrich and  $X_M$  be the weight in kilograms of a male ostrich.

$$X_F \sim N(78.6, 5.03^2) \quad \text{and} \quad X_M \sim N(91.3, 6.29^2)$$

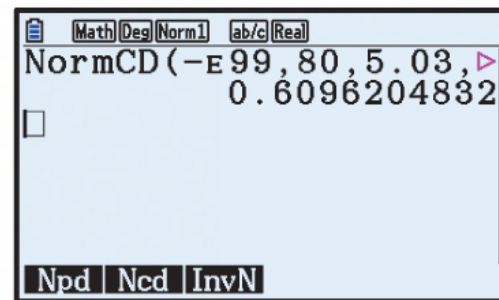
i  $P(X_M < 80) \approx 0.0362$



The probability that a randomly selected male ostrich will weigh less than 80 kg is about 0.0362.

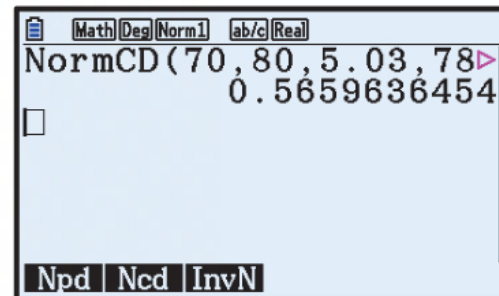


ii  $P(X_F < 80) \approx 0.610$



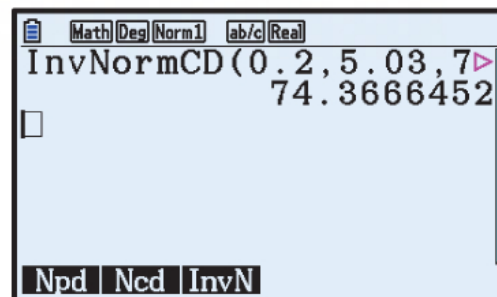
The probability that a randomly selected female ostrich will weigh less than 80 kg is about 0.610.

iii  $P(70 < X_F < 80) \approx 0.566$



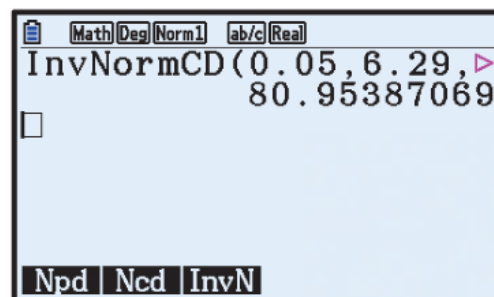
The probability that a randomly selected female ostrich will weigh between 70 and 80 kg is about 0.566.

b  $P(X_F < k) = 0.2$   
then  $k \approx 74.4$

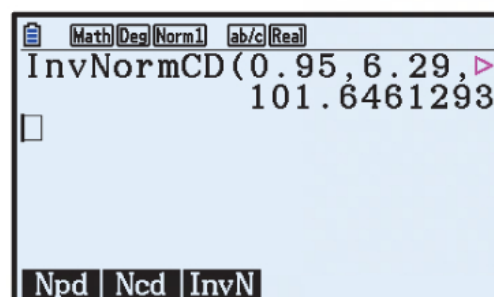


c We need to find  $a$  and  $b$  such that  $P(X_M < a) = 0.05$  and  $P(X_M > b) = 0.05$ .

If  $P(X_M < a) = 0.05$   
then  $a \approx 81.0$



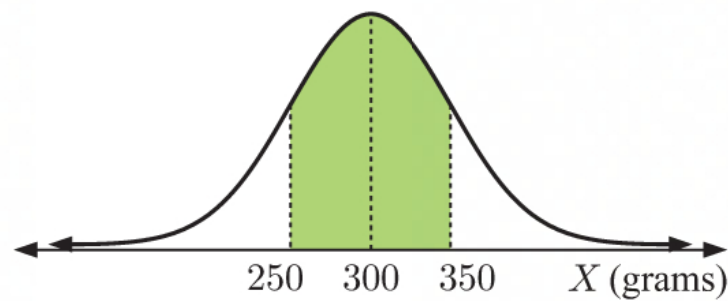
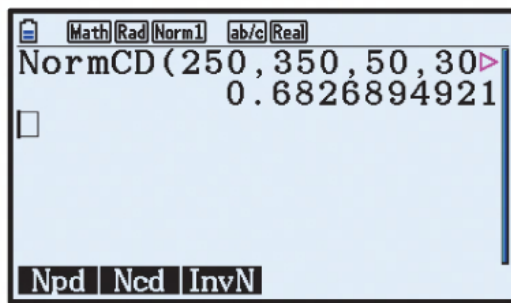
If  $P(X_M > b) = 0.05$   
then  $P(X_M < b) = 0.95$   
 $\therefore b \approx 102$



d Probability that the ostrich weighs less than 80 kg  
 $= P(\text{ostrich is female} \cap \text{less than 80 kg}) + P(\text{ostrich is male} \cap \text{less than 80 kg})$   
 $\approx 0.82 \times 0.610 + 0.18 \times 0.0362$   
 $\approx 0.506$

- 12 a** Let  $X$  grams be the weight of an apple.

$$X \sim N(300, 50^2)$$



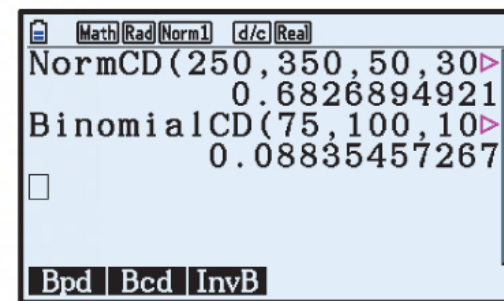
$$P(250 \leq X \leq 350) \approx 0.682689$$

So, approximately 68.3% of apples are fit for sale.

- b** Let  $Y$  be the number of apples that are fit for sale.

$$Y \sim B(100, 0.682689)$$

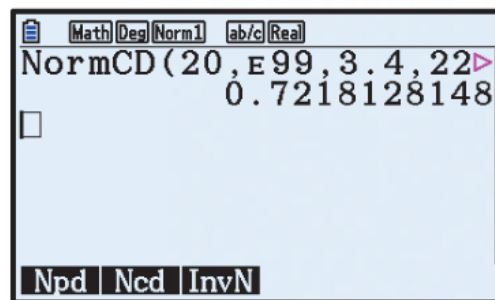
$$P(Y \geq 75) \approx 0.0884$$



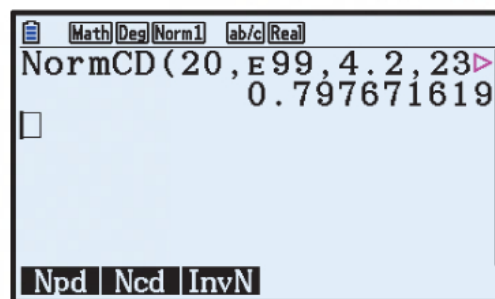
- 13 a** Let  $X_G$  be the length in cm of a carrot from Giovanni's farm, and  $X_B$  be the length in cm of a carrot from Beppe's farm.

$$X_G \sim N(22, 3.4^2) \text{ and } X_B \sim N(23.5, 4.2^2)$$

**i**  $P(X_G > 20) \approx 0.722$



**ii**  $P(X_B > 20) \approx 0.798$



- b** Probability that neither carrot is longer than 20 cm =  $P(X_G < 20) \times P(X_B < 20)$   
 $\approx (1 - 0.722) \times (1 - 0.798)$   
 $\approx 0.0563$

- 14 a**  $f(x)$  is a probability density function for  $0 \leq x \leq k$ .

$$\begin{aligned}\therefore \int_0^k f(x) dx &= 1 \\ \therefore \int_0^2 \frac{x}{5} dx + \int_2^k \frac{8}{5x^2} dx &= 1 \\ \therefore \left[ \frac{x^2}{10} \right]_0^2 + \left[ -\frac{8}{5x} \right]_2^k &= 1 \\ \therefore \frac{4}{10} + \left[ \left( -\frac{8}{5k} \right) - \left( -\frac{8}{10} \right) \right] &= 1 \\ \therefore -\frac{8}{5k} &= -\frac{2}{10} \\ \therefore 10k &= 80 \\ \therefore k &= 8\end{aligned}$$

- b** If  $m$  is the median of  $X$ , then  $\int_0^m f(x) dx = \frac{1}{2}$

$$\begin{aligned}\therefore \text{since } \int_0^2 \frac{x}{5} dx &= \frac{4}{10} < \frac{1}{2}, \quad \int_0^2 \frac{x}{5} dx + \int_2^m \frac{8}{5x^2} dx = \frac{1}{2} \\ \therefore \frac{4}{10} + \left[ -\frac{8}{5x} \right]_2^m &= \frac{1}{2} \\ \therefore \frac{4}{10} + \left[ \left( -\frac{8}{5m} \right) - \left( -\frac{8}{10} \right) \right] &= \frac{1}{2} \\ \therefore -\frac{8}{5m} &= -\frac{7}{10} \\ \therefore 35m &= 80 \\ \therefore m &= \frac{16}{7} = 2\frac{2}{7}\end{aligned}$$

$$\begin{aligned}\text{c } \mu = E(X) &= \int_0^8 x f(x) dx \\ &= \int_0^2 \frac{x^2}{5} dx + \int_2^8 \frac{8}{5x} dx \\ &= \left[ \frac{x^3}{15} \right]_0^2 + \left[ \frac{8}{5} \ln |x| \right]_2^8 \\ &= \frac{8}{15} + \frac{8}{5} \ln 8 - \frac{8}{5} \ln 2 \\ &\approx 2.751 \\ &\approx 2.75\end{aligned}$$

$$\begin{aligned}\int_0^8 x^2 f(x) dx &= \int_0^2 \frac{x^3}{5} dx + \int_2^8 \frac{8}{5} dx \\ &= \left[ \frac{x^4}{20} \right]_0^2 + \left[ \frac{8}{5} x \right]_2^8 \\ &= \frac{16}{20} + \left( \frac{64}{5} - \frac{16}{5} \right) \\ &= \frac{52}{5}\end{aligned}$$

$$\begin{aligned}\text{Now, } \text{Var}(X) &= \int_0^8 x^2 f(x) dx - \mu^2 \\ &\approx \frac{52}{5} - 2.751^2 \\ &\approx 2.83\end{aligned}$$

- d**  $Y = 2.4X - 1.2$

$$\begin{aligned}E(Y) &= E(2.4X - 1.2) \\ &= 2.4 E(X) - 1.2 \\ &\approx 2.4(2.751) - 1.2 \\ &\approx 5.40\end{aligned}$$

$$\begin{aligned}\text{Var}(Y) &= \text{Var}(2.4X - 1.2) \\ &= 2.4^2 \text{Var}(X) \\ &\approx 5.76(2.83) \\ &\approx 16.3\end{aligned}$$